SECOND PHASE SPINODAL DECOMPOSITION FOR THE CAHN-HILLIARD-COOK EQUATION

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ABSTRACT. We consider the dynamics of a nonlinear partial differential equation perturbed by additive noise. Assuming that the underlying deterministic equation has an unstable equilibrium, we show that the nonlinear stochastic partial differential equation exhibits essentially linear dynamics far from equilibrium. More precisely, we show that most trajectories starting at the unstable equilibrium are driven away in two stages. After passing through a cylindrical region, most trajectories diverge from the deterministic equilibrium through a cone-shaped region which is centered around a finite-dimensional subspace corresponding to strongly unstable eigenfunctions of the linearized equation, and on which the influence of the nonlinearity is surprisingly small.

This abstract result is then applied to explain spinodal decomposition in the stochastic Cahn-Hilliard-Cook equation on a domain G . This equation depends on a small interaction parameter $\varepsilon > 0$, and one is generally interested in asymptotic results as $\varepsilon \rightarrow 0$. Specifically, we show that linear behavior dominates the dynamics up to distances from the deterministic equilibrium which can reach $\varepsilon^{-2+\dim G/2}$ with respect to the $H^2(G)$ -norm.

1. INTRODUCTION

The formation of complex patterns is ubiquitous throughout the applied sciences. Sometimes the generated patterns are fairly robust in the sense that they can be reproduced almost exactly if only the underlying experiment or simulation is performed carefully enough. However, there are many pattern formation processes which do not exhibit this robustness of the generated patterns. One such example is spinodal decomposition, which is an important phenomenon in the study of certain metal alloys and can be described as follows. If a high-temperature homogeneous mixture of two suitable metallic components is rapidly cooled below a certain temperature, a process of phase separation may set in, during which the mixture becomes inhomogeneous. It forms a fine-grained structure, more or less alternating between the components, which exhibits characteristic snake-like patterns with a specific length scale. If this experiment is repeated even with the greatest care to ensure almost identical initial conditions, the observed pattern still exhibits the same broad characteristic features, yet its detail structure will be significantly different. In other words, the mechanism responsible for the generation of the pattern includes an element of randomness or stochasticity. In this paper, we will provide

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an explanation of spinodal decomposition based on a stochastic partial differential equation.

In order to describe spinodal decomposition and other phase separation processes in metal alloys, Cahn and Hilliard [5, 6] proposed a deterministic partial differential equation which describes the evolution of the alloy composition using a mean field approach. Their model has been widely used to explain various phase separation mechanisms in binary alloys, and it has become one of the fundamental equations for studying conservative pattern formation phenomena. Yet, as a deterministic model it completely neglects random effects that are present in any real material, such as for example fluctuations due to thermal noise.

Such random effects have been incorporated into an extension of the Cahn-Hilliard model by Cook [7] and Langer [18], leading to the fourth-order parabolic stochastic partial differential equation

$$(1) \quad \begin{aligned} \partial_t u &= -\Delta (\varepsilon^2 \Delta u + f(u)) + \sigma_\varepsilon \cdot \xi & \text{in } G, \\ \frac{\partial u}{\partial \nu} &= \frac{\partial \Delta u}{\partial \nu} = 0 & \text{on } \partial G. \end{aligned}$$

In this formulation, the order parameter u describes the local alloy composition in the sense that $u \approx \pm 1$ corresponds to the pure materials, and values in between correspond to respective mixtures of these materials. The set G is a bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary, for $d \in \{1, 2, 3\}$. The nonlinearity f is the negative derivative of a double-well potential with equal minima at ± 1 , the standard example being the cubic function $f(u) = u - u^3$. Furthermore, the parameter ε is small and positive, and it models interaction length. The stochastic noise process ξ is conservative [7, 18] and is the generalized derivative of a Hilbert space valued Wiener process, i.e., it is a Gaussian process which is white in time. The real parameter σ_ε denotes the noise strength. For $\sigma_\varepsilon = 0$ the Cahn-Hilliard-Cook equation (1) reduces to the deterministic Cahn-Hilliard model. Since the noise is conservative, the total mass $\int_G u \, dx$ is constant. A more detailed description and discussion of the noise process will be given in Section 3.1, in particular in Remarks 3.3 and 3.4.

In recent years, the phenomenon of spinodal decomposition in the deterministic Cahn-Hilliard model has received considerable attention. For this model, every constant function $u_o \equiv m$ is a stationary solution. This equilibrium is unstable if the initial mass m is contained in the so-called *spinodal interval*, which is the (usually connected) set of all $m \in \mathbb{R}$ for which $f'(m) > 0$. Thus, if m lies in the spinodal interval, any orbit originating near u_o is likely to be driven away, and understanding exactly how this occurs in the underlying nonlinear model is fundamental for explaining spinodal decomposition.

The first mathematical result on spinodal decomposition in the deterministic model is due to Grant [15], who exclusively considered the case of one-dimensional domains. The study of the higher-dimensional case was initiated by Maier-Paape and Wanner [19, 20], who described the early stages of the decomposition process. Using techniques from the theory of infinite-dimensional dynamical systems, in particular invariant manifold theory, it was shown in [20] that most solutions of the Cahn-Hilliard equation originating near $u_o \equiv m$ leave a neighborhood of the equilibrium close to a finite-dimensional *dominating subspace* which is determined by strongly unstable eigenmodes. Functions in this dominating subspace exhibit

the complicated snake-like patterns observed in experiments. In addition, one can describe the characteristic wavelength ε of the patterns. For more details we refer the reader to [19, Section 4]. The later stages of the decomposition process were addressed by Sander and Wanner [22, 23], who showed that even far from equilibrium, the dynamics of most orbits starting close to the homogeneous state u_o are linearly driven. This surprising result relies on the fact that the nonlinearity of the Cahn-Hilliard equation is extremely small in a cone around the dominating subspace, even at large distances from the equilibrium. Unfortunately, the nonlinearity estimates in [23] exhibit a dependence on the dimension of the underlying domain, which resulted in worse results in higher dimensions. Only recently, Wanner [25] could remove this dimension-dependence using probabilistic methods (but for the deterministic equation only), thereby describing the complete decomposition process for the first time.

Despite the apparent success of the deterministic model, the above results leave many fundamental questions about spinodal decomposition unanswered — and all of these are related to the stochastic aspects mentioned in the beginning of the section.

- In the deterministic model, the individuality of the observed patterns is a consequence of the specific choice of initial condition. Yet, the model does not provide any selection mechanism. In addition, it was shown in [25] that there are initial conditions which lead to nonphysical patterns. Why are they avoided?
- A recent numerical study of the pattern complexity evolution during spinodal decomposition by Gameiro, Mischaikow, and Wanner [14] has shown a surprising complexity increase along trajectories for the deterministic model, whose significance depends on the closeness of the considered initial condition to the homogeneous state. This complexity increase seems to be absent in experiments, and can be attributed to the fact that for initial conditions which are too close to the equilibrium, the resulting dynamics stay close to the stationary state for extremely long times.
- As a consequence of the different approaches in [20, 23, 25], it is somewhat involved to conclusively connect the dynamics of the different stages. While this could be done between the early stages described in [20] and the subsequent stages described in [23], linking them with the probabilistic results of [25] remains open.

It is our belief that these issues can only be resolved by studying the stochastic Cahn-Hilliard-Cook model (1), and preliminary results do exist. In [3] we considered the first stage of the decomposition process. While formally furnishing results analogous to [20], the approach in [3] does include an intrinsic selection mechanism for the pattern individuality. In the stochastic case, the homogeneous state is no longer an equilibrium. The stochastic fluctuations drive the dynamics quickly away from the homogeneous state and thereby establish the resulting patterns. In fact, one can derive a deterministic lower bound on the time necessary for the development of these patterns, making it impossible for solutions to stay too close to the equilibrium for exceedingly long times — which was mentioned above as a reason for the surprising pattern complexity increase. Yet, the results of [3] were not strong enough to allow for an explanation of the subsequent decomposition stage.

In the present paper we will address the second phase of the decomposition process. This will be accomplished by strengthening the results of [3] and combining them with the strong nonlinearity estimates of [23]. In contrast to the deterministic situation described above, we will be able to conclusively link the two stages of the underlying dynamics. Our results will show that spinodal decomposition in (1) is a linearly driven phenomenon, even far from the homogeneous state. More precisely, we show that for most solution trajectories of (1) linear behavior prevails until the $H^2(G)$ -norm of the solution reaches a large distance R_ε from the homogeneous state. For the nonlinearity $f(u) = u - u^{1+\kappa}$ with $\kappa \geq 1$ and $m = 0$ we can choose R_ε proportional to ε^{φ_R} , where

$$\varphi_R \gtrsim \left(-1 + \frac{1}{\kappa} + \frac{1}{\kappa^2} \right) \cdot \left(2 - \frac{\dim G}{2} \right).$$

Although this order is slightly worse than the one obtained in the deterministic case [23], the deviation becomes less significant with larger κ . In fact, for large values of κ we obtain the same limiting value of $\varphi_R \approx -2 + \dim G/2$.

Our results are applicable for small noise intensities, whose specific size will be described in more detail later. Nevertheless, our method does not involve large deviation techniques as in Freidlin, Wentzell [13], since we couple the noise strength to the instability of the equation. Furthermore, we have to make sure that the spatial regularity of the noise is not too large. The instability occurs in fairly high modes, and therefore these modes have to be exposed to sufficient random forcing.

The remainder of the paper is organized as follows. In Section 2 we begin our study of spinodal decomposition in the Cahn-Hilliard-Cook model with a detailed study of the underlying dynamics. Similar to the deterministic setting discussed in [19, 20, 22, 23] the decomposition process is divided into two stages, during which the linearization of (1) determines the dynamics of the Cahn-Hilliard-Cook equation. For the first stage this is a consequence of the closeness of the solution to the homogeneous state. Yet, the linear behavior during the second stage is due to sharp nonlinearity estimates in a suitable region of phase space. This region extends to large distances from the homogeneous state. The main results of Section 2 are presented in a fairly general setting and are based on the introduction of two suitable stochastic events. Section 3 provides tools to estimate the probability of the two events arising in the previous section. The final section, Section 4, combines the results of the other two sections and applies them to the Cahn-Hilliard-Cook equation. In this way, the linearly driven dynamics will allow us to provide an explanation of spinodal decomposition which addresses both the dynamics and the morphology of the observed patterns.

2. DYNAMICS OF THE DECOMPOSITION PROCESS

In this section we address the dynamics of the two decomposition stages. Rather than restricting ourselves to the specific setting of the Cahn-Hilliard-Cook model, the results are formulated in a general framework involving nonlinear semigroups in Hilbert spaces. This will isolate the crucial arguments more clearly.

2.1. Basic definitions and assumptions. We study initial value problems for abstract stochastic partial differential equations of the form

$$(2) \quad \partial_t u = Au + F(u) + \partial_t W, \quad u(0) = 0,$$

considered on a suitable Hilbert space. In this formulation, the linear part of the evolution equation is given by the operator A and the nonlinearity is given by F . The additive stochastic term is the generalized time-derivative of a Q -Wiener process W . Our basic assumptions regarding (2) are as follows.

Assumption 2.1. *Let X denote a Hilbert space with norm $\|\cdot\|$, and let A be a linear operator on X such that $-A$ is sectorial. Let $\{e^{tA}\}_{t \geq 0}$ denote the analytic semigroup of bounded linear operators on X generated by A .*

In addition to the space X , we consider a fractional power space of A . This Hilbert space is denoted by $X^s \subset X$ with norm $\|\cdot\|_s$, where $0 \leq s < 1$. Finally, let $F : X^s \rightarrow X$ be a sufficiently smooth nonlinear mapping with $F(0) = 0$.

As for the noise process, we assume that W is a (cylindrical) Q -Wiener process over a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ taking values in X . Furthermore, we assume that the stochastic convolution

$$(3) \quad W_A(t) = \int_0^t e^{(t-\tau)A} dW(\tau)$$

takes values in the Hilbert space X^s .

In our applications of the abstract results, the Hilbert space X^s will be chosen as a phase space for the evolution equation (2). For more information on analytic semigroups and fractional power spaces we refer the reader to [17]; an introduction to Q -Wiener processes and stochastic convolutions can be found in [10].

In the situation of Assumption 2.1 one can prove the existence of a unique local mild solution u of (2) by standard arguments [9, 10]. If we assume that $u : [0, T_{ex}) \rightarrow X^s$ is the unique mild solution on its maximal existence interval, then it satisfies the variation of constants formula

$$(4) \quad u(t) = \int_0^t e^{(t-\tau)A} F(u(\tau)) d\tau + W_A(t)$$

for all $0 \leq t < T_{ex}$. Here T_{ex} is the random stopping time denoting the maximal interval of existence, and W_A denotes the stochastic convolution defined in (3).

The above variation of constants formula will be used extensively below. In addition, we need the following translated version of (4). Fix a nonrandom time $t_0 > 0$, and let

$$(5) \quad \widetilde{W}(t) = W(t + t_0) - W(t_0), \quad \text{for all } t \geq 0,$$

denote the shifted Q -Wiener process starting at t_0 . Then for all $0 \leq t < T_{ex} - t_0$ we have the identity

$$(6) \quad u(t + t_0) = e^{tA} u(t_0) + \int_0^t e^{(t-\tau)A} F(u(\tau + t_0)) d\tau + \widetilde{W}_A(t),$$

where $\widetilde{W}_A(t) = \int_0^t e^{(t-\tau)A} d\widetilde{W}(\tau) = \int_0^t e^{(t-\tau)A} dW(\tau + t_0)$. In other words, $\widetilde{W}_A$ is the stochastic convolution with respect to the shifted Q -Wiener process $\widetilde{W}$. Since the time t_0 is deterministic, it can readily be seen that the processes W_A and $\widetilde{W}_A$ have the same distribution.

Assumption 2.1 establishes the basic framework for our abstract results. The occurrence of spinodal decomposition in equations of the form (2) is a consequence of additional assumptions on both the linear and the nonlinear part of the evolution equation. The assumptions on the linear part are as follows.

Assumption 2.2. *In the situation of Assumption 2.1, assume that there exists an orthogonal splitting $X^s = X^+ \oplus X^-$ into A -invariant subspaces of X^s . Let $A^\pm$ denote the restriction of A to $X^\pm$, and for every element $w \in X^s$ denote its orthogonal projection onto $X^\pm$ by $w^\pm$.*

We assume that for every $t \geq 0$ the operator $e^{tA^+} : X^+ \rightarrow X^+$ is onto, and that there are constants $\lambda > 0$, $b > 1$, $0 < \gamma < 1$, and $M_1 > 0$ such that the following estimates hold:

$$(7) \quad \|e^{tA} w\|_s \leq M_1 \cdot t^{-s} \cdot e^{t\lambda b} \cdot \|w\| \quad \text{for all } t > 0 \quad \text{and } w \in X,$$

$$(8) \quad \|e^{tA} w\|_s \leq e^{t\lambda} \cdot \|w\|_s \quad \text{for all } t \geq 0 \quad \text{and } w \in X^s,$$

$$(9) \quad \|e^{tA^+} w^+\|_s \geq e^{t\lambda\gamma} \cdot \|w^+\|_s \quad \text{for all } t \geq 0 \quad \text{and } w^+ \in X^+,$$

$$(10) \quad \|e^{tA^-} w^-\|_s \leq e^{t\lambda\gamma} \cdot \|w^-\|_s \quad \text{for all } t \geq 0 \quad \text{and } w^- \in X^-.$$

Notice that due to (9) the restriction $e^{tA^+} : X^+ \rightarrow X^+$ is an isomorphism for all $t \geq 0$.

The validity of estimates (7) and (8) in Assumption 2.2 is a direct consequence of the fact that $-A$ is a sectorial operator. The remaining two estimates are satisfied if the splitting $X^s = X^+ \oplus X^-$ is induced by a suitable splitting of the spectrum of A . If we assume that the spectrum of A^+ lies to the right of the line $\{z \in \mathbb{C} : \operatorname{Re} z = \lambda\gamma\}$, and the spectrum of A^- lies to the left of this line, then estimates (9) and (10) are derived in [17].

In contrast to the above assumptions on the linear part, the assumptions on the nonlinearity F are more restrictive, as they reflect the specific properties necessary for establishing spinodal decomposition. As we mentioned in the Introduction, we distinguish two stages of the decomposition process. Each of these stages is associated with a specific region in phase space, and we need to assume separate nonlinearity estimates associated with these regions. Before presenting the assumptions on the nonlinearity in detail, we define the regions of interest.

Definition 2.3. For given positive constants η , R , q^- , and q^+ we define the following three subsets of the phase space X^s :

$$\mathcal{C}_\eta := \{w \in X^s : \|w^-\|_s \leq \eta \cdot \|w^+\|_s\},$$

$$\mathcal{K}_{\eta,R} := \{w \in X^s : w \in \mathcal{C}_\eta \quad \text{and} \quad \|w\|_s \leq R\},$$

$$\mathcal{Z}_{q^-,q^+} := \{w \in X^s : \|w^-\|_s \leq q^- \quad \text{and} \quad \|w^+\|_s \leq q^+\}.$$

The regions defined above are illustrated in Figure 1 for specific order relations between and scalings of the involved parameters. The set $\mathcal{C}_\eta$ is an unbounded cone around the space X^+ with opening angle of $\arctan \eta$. Intersecting this cone with the ball of radius R centered at the origin yields the bounded cone $\mathcal{K}_{\eta,R}$. Finally, the set $\mathcal{Z}_{q^-,q^+}$ is a cylindrical region.

The two stages of spinodal decomposition will be associated with the bounded sets $\mathcal{Z}_{q^-,q^+}$ and $\mathcal{K}_{\eta,R}$, respectively. On these regions, we require the following nonlinearity estimates.

Assumption 2.4. *In the situation of Assumptions 2.1 and 2.2 consider four constants*

$$(11) \quad \delta > 0 \quad \text{and} \quad 0 < r < r_0 < R.$$

FIGURE 1. The sets $\mathcal{Z}_{\delta r, r_0}$, $\mathcal{K}_{\delta, R}$, and $\mathcal{C}_\delta$.

Let $\eta = 21\delta$, $q^- = 4\delta r$, and $q^+ = 2r_0$ in Definition 2.3. Moreover, assume there are positive constants κ , C_F , and C_δ such that the nonlinearity $F : X^s \rightarrow X$ satisfies the estimates

$$(12) \quad \|F(u)\| \leq C_F \cdot (\delta r_0)^\kappa \cdot \|u\|_s \quad \text{for all } u \in \mathcal{Z}_{4\delta r, 2r_0},$$

$$(13) \quad \|F(u)\| \leq C_\delta \cdot \|u\|_s^{1+\kappa} \quad \text{for all } u \in \mathcal{K}_{21\delta, R}.$$

In Assumption 2.4, the constants κ and C_F depend generally only on F , and C_δ depends only on δ and F . This will be particularly true in our application to the Cahn-Hilliard-Cook equation. We would also like to point out that estimates such as (12) and (13) are more or less automatically satisfied on a full neighborhood of 0, provided $F : X^s \rightarrow X$ is sufficiently smooth and has vanishing Frechet derivatives up to order κ at 0. However, it will become clear in the remainder of this section that precise information on the size of the constants C_F and C_δ is necessary to establish spinodal decomposition. In particular, it will turn out to be crucial to show that C_δ can be chosen extremely small on the cone $\mathcal{K}_{21\delta, R}$ — even though it would be large for a full neighborhood of 0. These estimates will be established in Section 4.

2.2. First decomposition stage. We begin our study of the dynamics of spinodal decomposition by describing the initial stage of the separation process. During this stage, most solution trajectories of (2) are driven away from the initial state $u(0) = 0$ through the combined effects of the additive noise term, the linearized dynamics, and the smallness of the nonlinearity in a neighborhood of 0. The end of this stage is reached after a deterministic time t_0 . At this time, the solution trajectories have traveled a certain minimum distance from 0, while still being close to the subspace X^+ introduced in Assumption 2.2.

Before we state and prove the main result, we have to introduce some notation. Recalling that the probability space underlying the stochastic partial differential

FIGURE 2. A sketch of three typical trajectories of u on $\mathcal{N}_1$.

equation (2) is given by $(\Omega, \mathcal{A}, \mathbb{P})$, consider the following two subsets of Ω :

$$(14) \quad \Omega_1 := \left\{ W_A(t) \in \mathcal{Z}_{\delta r, r_0} \text{ for all } t \in [0, t_0], \quad \|W_A^+(t_0)\|_s \geq r \right\},$$

$$(15) \quad \mathcal{N}_1 := \left\{ u(t) \in \mathcal{Z}_{4\delta r, 2r_0} \text{ for all } t \in [0, t_0], \quad \|u^+(t_0)\|_s \geq r/2 \right\}.$$

In these definitions, t_0 is some deterministic positive time, W_A denotes the stochastic convolution defined in (3), and u is the solution process of (2). As usual we do not explicitly mention the ω -dependence of W_A and u . The remaining constants in (14) and (15) are positive and will be chosen later. The cylinder used in the definition of Ω_1 is sketched in Figure 1; typical trajectories of the solution u on $\mathcal{N}_1$ are shown in Figure 2.

The set Ω_1 consists of all $\omega \in \Omega$ for which the corresponding solution trajectory of the stochastic convolution W_A remains in a certain cylinder on the whole time interval $[0, t_0]$, and has traveled a minimum distance in the X^+ -direction at time t_0 . The set $\mathcal{N}_1$ is analogous, yet for the solution u of the nonlinear equation and a different, larger cylinder. In this sense, the sets Ω_1 and $\mathcal{N}_1$ describe very similar dynamical behavior of the solution W_A of the linearized equation and the solution u of the nonlinear equation. Moreover, in our application to the Cahn-Hilliard-Cook equation the constant δ will be extremely small, thus enforcing that at the end of the first decomposition stage the solution u is extremely close to the subspace X^+ . This fact will be crucial for the description of the complicated patterns observed during spinodal decomposition.

The following result relates the sets Ω_1 and $\mathcal{N}_1$. Under suitable conditions on the constants involved in the definition of these sets it is shown that $\Omega_1 \subset \mathcal{N}_1$. In other words, assuming a certain linear dynamical behavior, we will show that the solution of the nonlinear problem behaves similarly. The result improves existing results on the first decomposition stage which were presented in [3] by deriving an additional upper bound on the solution.

Theorem 2.5. *Suppose that Assumptions 2.1, 2.2, and 2.4 are satisfied. Moreover, assume that $0 < \delta < 1/2$, and choose $t_0 > 0$ in such a way that*

$$(16) \quad \frac{2\sqrt{2}C_F M_1}{(1-s)b^{1-s}\lambda^{1-s}} \cdot r_0^{\kappa+1} \cdot \delta^\kappa \cdot e^{t_0 b \lambda} \leq \delta r.$$

Then $\Omega_1 \subset \mathcal{N}_1$, i.e., the set Ω_1 defined in (14) is contained in the set $\mathcal{N}_1$ defined in (15).

Proof. Let $T_z := \inf\{t > 0 : u(t) \notin \mathcal{Z}_{4\delta r, 2r_0}\}$ denote the exit time of u from the cylinder $\mathcal{Z}_{4\delta r, 2r_0}$. (As is usual in probability theory, $T_z = T_z(\omega)$ is a random variable, and we omit the ω -dependence to simplify the notation.) Then for all $0 \leq t \leq T_z$ the choice of the constants in (11) and $\delta < 1/2$ furnish

$$\|u(t)\|_s \leq 2\sqrt{2} \cdot r_0 ,$$

and (12) implies for all $0 \leq t \leq T_z$ the estimate

$$\|F(u(t))\| \leq C_F \cdot (\delta r_0)^\kappa \cdot \|u(t)\|_s \leq 2\sqrt{2}C_F \cdot \delta^\kappa \cdot r_0^{\kappa+1} .$$

Together with (4) and Assumption 2.2 we obtain

$$\begin{aligned} \|u(t) - W_A(t)\|_s &\leq M_1 \cdot \int_0^t (t - \tau)^{-s} \cdot e^{(t-\tau)b\lambda} \cdot \|F(u(\tau))\| d\tau \\ &\leq 2\sqrt{2}C_F M_1 \cdot \delta^\kappa \cdot r_0^{\kappa+1} \cdot e^{tb\lambda} \cdot \int_0^t (t - \tau)^{-s} e^{-\tau b\lambda} d\tau \\ &\leq \frac{2\sqrt{2}C_F M_1}{(1-s)b^{1-s}\lambda^{1-s}} \cdot \delta^\kappa \cdot r_0^{\kappa+1} \cdot e^{tb\lambda} , \end{aligned}$$

where the last inequality follows as in [23, Lemma 2.4]. Note also that we used both the assumption $\lambda > 0$ and $b > 1$. Hence, for all $0 \leq t \leq \min\{t_0, T_z\}$ we have

$$\|u(t) - W_A(t)\|_s \leq \frac{2\sqrt{2}C_F M_1}{(1-s)b^{1-s}\lambda^{1-s}} \cdot \delta^\kappa \cdot r_0^{\kappa+1} \cdot e^{t_0 b\lambda} \leq \delta r$$

according to (16). It can now be readily verified that on Ω_1 we must have $t_0 < T_z$, as well as $\|u^+(t_0)\|_s \geq r/2$. This implies $u(t) \in \mathcal{Z}_{4\delta r, 2r_0}$ for all $0 \leq t \leq t_0$ and all $\omega \in \Omega_1$, i.e., we have $\Omega_1 \subset \mathcal{N}_1$. $\square$

At this point it is far from obvious that one can actually choose a time $t_0 > 0$ satisfying (16). In our application to the Cahn-Hilliard-Cook equation, we will choose the constant $\delta > 0$ extremely small. For $\kappa > 1$ this implies that the factor δ^κ appearing on the left-hand side of (16) is considerably smaller than the factor δ on the right-hand side, which guarantees that (16) is satisfied for sufficiently small $t_0 > 0$. Moreover, it is unclear how large the sets Ω_1 and $\mathcal{N}_1$ are with respect to the probability measure $\mathbb{P}$. This will be addressed in Section 4, building on the estimates of Section 3.

2.3. Second decomposition stage. At the end of the first decomposition stage, the solution of (2) is contained in the dark region depicted in Figure 2, provided $\omega \in \Omega_1 \subset \mathcal{N}_1$. This happens at the deterministic time t_0 , and according to Definition 2.3 we also have $u(t_0) \in \mathcal{C}_{8\delta}$ on $\mathcal{N}_1$. In other words, the first decomposition stage drives most solution trajectories of (2) into a cone-shaped region around the subspace X^+ . The second decomposition stage will describe the journey of solution trajectories through this cone.

From a mathematical point of view the main ingredient for the second phase of spinodal decomposition is the new nonlinearity estimate (13). As we have mentioned before, this estimate shows that on certain cones around the subspace X^+ the nonlinearity is extremely small, even far from the initial state 0. Using this

fact, we show in the following that the dynamics of $u(t)$ for $t > t_0$ is essentially driven by the linearization A .

Similar to our description of the first decomposition stage, the main result is achieved by relating the dynamics of the solution u of the nonlinear equation to the linearized stochastic dynamics. This time, however, it is not sufficient to relate u to the dynamics of the stochastic convolution W_A . The latter solves the initial value problem

$$\partial_t v = Av + \partial_t W, \quad v(0) = 0,$$

and the deviation of $v(t_0) = W_A(t_0)$ from $u(t_0)$ is too large to take advantage of (13). In order to explain the second decomposition phase, we have to relate the dynamics of $u(t)$ for $t > t_0$ to the solution of the linearized system originating at $u(t_0)$. To this end, recall that if we define the shifted Q -Wiener process $\widetilde{W}$ as in (5), then u satisfies the translated variation of constants formula (6). The term $v(t) = e^{tA}u(t_0) + \widetilde{W}_A(t)$ appearing on the right-hand side of (6) satisfies the abstract initial value problem

$$(17) \quad \partial_t v = Av + \partial_t \widetilde{W}, \quad v(0) = u(t_0),$$

and it is the dynamics of v which are responsible for the dynamics of $u(t + t_0)$ for $t \geq 0$.

As in the description of the first phase, we introduce two probabilistic events — one based on the linear dynamics, and the other one based on the solution u of the nonlinear problem. Using (11) and the notation of Assumption 2.2, the linear event is defined as

$$(18) \quad \Omega_2 := \left\{ \left\| \widetilde{W}_A(t) \right\|_s \leq \frac{r_0}{2} \cdot e^{t\lambda}, \quad \left\| e^{-tA^+} \widetilde{W}_A^+(t) \right\|_s \leq \frac{r}{4}, \right. \\ \left. \text{and} \quad \left\| \widetilde{W}_A^-(t) \right\|_s \leq \frac{r\delta}{2} \cdot e^{t\lambda\gamma} \text{ for all } t \geq 0 \right\}.$$

The set Ω_2 corresponds to all solution trajectories of the stochastic convolution $\widetilde{W}_A$ satisfying certain exponential growth conditions. In order to describe the relevant solution trajectories of the nonlinear equation, we consider the exit time

$$(19) \quad T_c := \inf \{t > 0 : u(t + t_0) \notin \mathcal{K}_{21\delta, R}\}$$

for the solution u of (2). Then the nonlinear event is defined as

$$(20) \quad \mathcal{N}_2 := \{u(t + t_0) \in \mathcal{K}_{21\delta, R} \text{ for all } 0 \leq t \leq T_c, \text{ and } \|u(T_c + t_0)\|_s = R\}.$$

The set $\mathcal{N}_2$ corresponds to all solution trajectories of u which stay in a certain cone around X^+ until their norm reaches size R .

The following theorem is the main result for the abstract second decomposition stage. It basically shows that on the set $\Omega_1 \cap \Omega_2$, the trajectories $u(t + t_0)$ stay inside the cone $\mathcal{C}_{21\delta}$ for $0 \leq t \leq T_c$, and for $t = T_c$ they reach norm R . We will see in Sections 3 and 4 that in our application the probability of the set $\Omega_1 \cap \Omega_2$ is large for suitably chosen parameters. In Figure 3, typical trajectories of u are indicated, all of them exhibiting both stages of the decomposition process.

Theorem 2.6. *Suppose that Assumptions 2.1, 2.2, and 2.4 are satisfied, as well as (16) and $0 < \delta < 1/18$. Let $\vartheta > 0$ be such that*

$$(21) \quad M_1 \cdot C_\delta \cdot R^\kappa \cdot \Gamma(1-s) \leq \lambda^{1-s} \cdot \vartheta^{1-s},$$

FIGURE 3. A sketch of three typical trajectories on $\Omega_1 \cap \Omega_2$ for $0 \leq t \leq T_c$.

where Γ denotes the Γ -function. Finally, assume that

$$(22) \quad \frac{16c(s)}{(1-s)(b-1)^{1-s}} \cdot \frac{M_1 C_\delta R^\kappa r_0}{\lambda^{1-s} r} \cdot \left(\frac{8R}{r} \right)^{(b+\vartheta-\gamma)/\gamma} \leq \delta,$$

where $c(s)$ denotes an s -dependent constant with $c(1/2) = 2$.

Then $\Omega_1 \cap \Omega_2 \subset \mathcal{N}_1 \cap \mathcal{N}_2$, i.e., the intersection $\Omega_1 \cap \Omega_2$ of the events defined in (14) and (18) is contained in the intersection $\mathcal{N}_1 \cap \mathcal{N}_2$ of the sets defined in (15) and (20).

Proof. Theorem 2.5 immediately implies $\Omega_1 \cap \Omega_2 \subset \mathcal{N}_1$. To obtain the assertion concerning $\mathcal{N}_2$ we derive an upper bound on the difference w between the solution $u(t+t_0)$ and the solution of (17). According to (6) and our discussion above, this difference is given by

$$w(t) := u(t+t_0) - e^{tA}u(t_0) - \widetilde{W}_A(t) = \int_0^t e^{(t-\tau)A} F(u(\tau+t_0)) d\tau \quad \text{for } t \geq 0.$$

On the set Ω_2 we have $\|\widetilde{W}_A(t)\|_s \leq r_0/2 \cdot e^{t\lambda}$. In addition, on the intersection $\Omega_1 \cap \Omega_2$ Theorem 2.5 furnishes $u(t_0) \in \mathcal{Z}_{4\delta r, 2r_0}$, and therefore $\|u(t_0)\|_s \leq 2\sqrt{2} \cdot r_0$. Together with (8) these estimates immediately imply

$$\left\| e^{tA}u(t_0) + \widetilde{W}_A(t) \right\|_s < 4r_0 \cdot e^{t\lambda} \quad \text{for all } t \geq 0$$

on the set $\Omega_1 \cap \Omega_2$. In combination with (6), (7), and (13) we further obtain

$$\begin{aligned}
 \|w(t)\|_s &\leq M_1 \cdot \int_0^t (t-\tau)^{-s} \cdot e^{(t-\tau)\lambda b} \cdot \|F(u(\tau+t_0))\| d\tau \\
 &\leq M_1 C_\delta \cdot R^\kappa \cdot \int_0^t (t-\tau)^{-s} \cdot e^{(t-\tau)\lambda b} \cdot \|u(\tau+t_0)\|_s d\tau \\
 &\leq M_1 C_\delta \cdot R^\kappa \cdot \int_0^t (t-\tau)^{-s} \cdot e^{(t-\tau)\lambda b} \cdot \|w(\tau)\|_s d\tau \\
 &\quad + 4M_1 C_\delta \cdot R^\kappa \cdot r_0 \cdot \int_0^t (t-\tau)^{-s} \cdot e^{(t-\tau)\lambda b} \cdot e^{\tau\lambda} d\tau
 \end{aligned}$$

for all $0 \leq t \leq T_c$ on the intersection $\Omega_1 \cap \Omega_2$. Similar to [23, Lemma 2.4] an application of the generalized Gronwall inequality implies with (21) the bound

$$\begin{aligned}
 \|w(t)\|_s &\leq \frac{4c(s)M_1 C_\delta \cdot R^\kappa \cdot r_0}{(1-s)(\lambda(b-1))^{1-s}} \cdot \exp\left(t \cdot \left(\lambda b + (M_1 C_\delta \cdot R^\kappa \cdot \Gamma(1-s))^{1/(1-s)}\right)\right) \\
 (23) \quad &\leq \frac{4c(s)}{(1-s)(b-1)^{1-s}} \cdot \frac{M_1 C_\delta R^\kappa r_0}{\lambda^{1-s}} \cdot e^{t\lambda \cdot (b+\vartheta)}
 \end{aligned}$$

for all $0 \leq t \leq T_c$. The constant $c(s)$ depends only on s and is due to the application of the generalized Gronwall inequality. For more details we refer the reader to [17, Lemma 7.1.1].

The next step in the proof of Theorem 2.6 is the derivation of a lower growth bound for the solution of the linearized problem (17), which in turn will be used to establish the positive invariance of a certain cone with respect to this linear solution. On the set Ω_2 we have $\|e^{-tA^+} \widetilde{W}_A^+(t)\|_s \leq r/4$ for all $t \geq 0$, and on Ω_1 Theorem 2.5 implies the inequality $\|u^+(t_0)\|_s \geq r/2$. Together with (9) these estimates furnish

$$(24) \quad \left\| e^{tA} u(t_0) + \widetilde{W}_A(t) \right\|_s \geq \left\| e^{tA} u^+(t_0) + \widetilde{W}_A^+(t) \right\|_s \geq \frac{r}{4} \cdot e^{t\lambda\gamma}$$

for all $t \geq 0$ on the intersection $\Omega_1 \cap \Omega_2$. Due to Theorem 2.5 we have $\|u^-(t_0)\|_s \leq 4\delta r$, and together with (10), (18), and (24) this estimate yields

$$(25) \quad \frac{\left\| e^{tA} u^-(t_0) + \widetilde{W}_A^-(t) \right\|_s}{\left\| e^{tA} u^+(t_0) + \widetilde{W}_A^+(t) \right\|_s} \leq 18\delta \quad \text{for all } t \geq 0 \quad \text{on } \Omega_1 \cap \Omega_2.$$

In other words, the $\Omega_1 \cap \Omega_2$ -trajectories of the solution of (17) are contained in the cone $\mathcal{C}_{18\delta}$ for all $t \geq 0$.

Now we are in a position to relate the dynamics of the solution u of the nonlinear problem to the dynamics of (17). This will be accomplished by controlling the relative distance between these solutions. Define the stopping time T_q on the set Ω_1 by

$$T_q := \inf \left\{ t > 0 : \frac{\|w(t)\|_s}{\left\| e^{tA} u(t_0) + \widetilde{W}_A(t) \right\|_s} \geq \frac{1}{2} \right\} > 0.$$

As long as $0 \leq t \leq T_q$, a straightforward calculation shows that (24) implies

$$\|u(t+t_0)\|_s \geq \frac{1}{2} \cdot \left\| e^{tA} u(t_0) + \widetilde{W}_A(t) \right\|_s \geq \frac{r}{8} \cdot e^{t\lambda\gamma}.$$

Due to $\|u(T_c + t_0)\|_s \leq R$, this last estimate immediately implies

$$(26) \quad \min\{T_c, T_q\} \leq \frac{1}{\lambda\gamma} \cdot \ln\left(\frac{8R}{r}\right).$$

For all $0 \leq t \leq \min\{T_c, T_q\}$, the estimates (23), (24), and (26), in combination with the assumption in (22), can now be used to establish

$$(27) \quad \begin{aligned} \frac{\|w(t)\|_s}{\|e^{tA}u(t_0) + \widetilde{W}_A(t)\|_s} &\leq \frac{16c(s)}{(1-s)(b-1)^{1-s}} \cdot \frac{M_1 C_\delta R^\kappa r_0}{\lambda^{1-s} r} \cdot e^{t\lambda \cdot (b+\vartheta-\gamma)} \\ &\leq \frac{16c(s)}{(1-s)(b-1)^{1-s}} \cdot \frac{M_1 C_\delta R^\kappa r_0}{\lambda^{1-s} r} \cdot \left(\frac{8R}{r}\right)^{(b+\vartheta-\gamma)/\gamma} \\ &\leq \delta \end{aligned}$$

on the intersection $\Omega_1 \cap \Omega_2$. Notice that the last estimate immediately furnishes $T_q > T_c$, according to the definition of T_q and $\delta < 1/2$. In other words, the estimate (27) is satisfied for all $0 \leq t \leq T_c$ on $\Omega_1 \cap \Omega_2$.

In order to complete the proof of Theorem 2.6 we only have to verify the identity $\|u(T_c + t_0)\|_s = R$, i.e., the solution u stays in the cone $\mathcal{C}_{21\delta}$ (at least) until its norm reaches the value R .

Recall that if v denotes the solution of (17), then (6) implies $v(t) = u(t+t_0) - w(t)$ for all $t \geq 0$. According to (25) and (27) we therefore have

$$v(t) \in \mathcal{C}_{18\delta} \quad \text{and} \quad \frac{\|u(t+t_0) - v(t)\|_s}{\|v(t)\|_s} \leq \delta, \quad \text{for all } 0 \leq t \leq T_c$$

on $\Omega_1 \cap \Omega_2$. In other words, the linear solution v is contained in the cone $\mathcal{C}_{18\delta}$ around X^+ and the relative distance between $u(t+t_0)$ and $v(t)$ is bounded. Thus, the solution of the nonlinear problem has to be contained in a slightly larger cone. In fact, [23, Lemma 2.8] and the assumption $0 < \delta < 1/18$ furnish for all $0 \leq t \leq T_c$ the inclusion

$$u(t+t_0) \in \mathcal{C}_{\tilde{\delta}}, \quad \text{where } \tilde{\delta} = 18\delta + \frac{\delta \cdot (1 + 324\delta^2)}{\sqrt{1 - \delta^2} - 18\delta^2} < 21\delta,$$

on the intersection $\Omega_1 \cap \Omega_2$. This implies that $u(T_c + t_0)$ is contained in the interior of $\mathcal{C}_{21\delta}$, and therefore (19) yields $\|u(T_c + t_0)\|_s = R$. This completes the proof of the theorem. $\square$

3. PROBABILITY ESTIMATES

In the last section specific dynamical behavior of an abstract stochastic partial differential equation was related to two probabilistic events Ω_1 and Ω_2 defined in terms of the linearized dynamics. These results will be used in Section 4 to explain the occurrence of spinodal decomposition in the Cahn-Hilliard-Cook model. There we will show that the dynamical behavior derived in Section 2 corresponds to this phase separation mechanism, and that most solution trajectories in fact exhibit the two stage dynamics described before. Yet, in order to estimate the probabilities of the events Ω_1 and Ω_2 we need additional results, which will be presented in this section.

Recalling the definitions of Ω_1 and Ω_2 in (14) and (18), respectively, one can see that these events are defined purely in terms of exponential growth estimates. In order to estimate the probabilities of the associated events we need to establish large

deviation type results for the stochastic convolutions W_A and $\widetilde{W}_A$. Many results of this type exist in the literature. For example, results similar to Lemma 3.8 below are well-known and can be found in [4, 21, 24], as well as the references therein. For our applications in Section 4, however, we need to establish the precise dependence of the constants in these results on the various underlying parameters. These parameters include the underlying time interval, as well as the eigenvalues of the covariance operator of the noise process. In order to keep the proof effort reasonable, we will sometimes accept suboptimal results, as long as they suffice for our purposes.

3.1. Problem reduction. Despite the fact that formally the definitions of Ω_1 and Ω_2 are different, we can reduce our considerations to a very specific situation. We begin by recalling a simple yet useful fact, which was mentioned earlier.

Remark 3.1. Consider the situation of Assumption 2.1, let $t_0 > 0$ be a deterministic time, and define the stochastic process $\widetilde{W}$ as in (5). Then the stochastic convolutions W_A and $\widetilde{W}_A$ have the same distribution. In other words, in order to estimate the probabilities of Ω_1 and Ω_2 it suffices to study exponential estimates for W_A .

In addition, we assume that the stochastic convolution W_A can be written as a generalized Fourier series on some general Hilbert space H . For this, the following assumption is necessary.

Assumption 3.2. *Let H denote a real Hilbert space and let A be a self-adjoint operator on H . We assume that there exists a complete orthonormal system $\{f_k\}_{k \in \mathbb{N}}$ in H which consists of eigenfunctions of A with corresponding eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}}$, i.e., we assume that*

$$Af_k = \lambda_k f_k \quad \text{for all } k \in \mathbb{N}.$$

Furthermore, suppose that the stochastic convolution W_A has the generalized Fourier series expansion

$$(28) \quad W_A(t) = \sum_{k \in \mathbb{N}} \alpha_k \cdot \int_0^t e^{(t-\tau)\lambda_k} d\beta_k(\tau) \cdot f_k \in H \quad \text{for all } t \geq 0,$$

where $\{\alpha_k\}_{k \in \mathbb{N}}$ is a family of (deterministic) real numbers which, without loss of generality, are chosen as nonnegative numbers $\alpha_k \geq 0$, and $\{\beta_k\}_{k \in \mathbb{N}}$ is a family of independent real-valued standard Brownian motions. In other words, we assume that A and Q have a joint set of eigenfunctions and that the Q -Wiener process W is (formally) given by

$$W(t) = \sum_{k \in \mathbb{N}} \alpha_k \cdot \beta_k(t) \cdot f_k.$$

Assuming a joint set of eigenfunctions for A and Q might appear to be fairly restrictive at first sight. However, as is pointed out in the following remark, this assumption could be removed at the expense of considerably more complicated formulations of the results.

Remark 3.3. The results of this paper can in principle also be derived without assuming the existence of a joint set of eigenfunctions for A and Q . However, doing this will result in numerous trace conditions which involve the covariance operator Q and the semigroup e^{tA} . For example, bounds on $W_A(t)$ in $L^2(G)$ require control over the trace of the operator $\int_0^t e^{sA} Q e^{sA} ds$.

The main reason for our above assumption is to explicitly evaluate all these traces, and give conditions in terms of just a few coefficients, instead of sums over all the eigenvalues of Q . In this way we can concentrate on the two main parameters, namely the noise intensity and the regularity of the noise, and we can establish how precisely they affect the pattern formation process.

In other words, assuming the existence of a joint set of eigenfunctions for A and Q is done mostly to keep the formulation of our main results as simple as possible. Yet, even in the situation of Assumption 3.2 we can easily realize physically realistic noise processes, as is explained in more detail in the following remark, where we discuss smooth approximations of space-time white noise, which on a qualitative level behaves very similar to space-time white noise.

An in-depth discussion of the implications of Assumption 3.2 can be found in the recent paper by Blömker [2], and we summarize some of the relevant statements below.

Remark 3.4. For a given Q -Wiener process W as in Assumption 3.2 one can define a noise process ξ by considering $\xi = \partial_t W$ in the sense of generalized stochastic processes. Exactly such a process ξ is used in the stochastic Cahn-Hilliard model (1). From an applied perspective, the mean value and correlation functionals of the process ξ are of central importance, since they will effectively describe length scales for the mass transport that has to take place in the conservative case. In our situation it can be shown that the mean value and correlation functionals of ξ are given by

$$\mathbb{E}\xi(t, x) = 0, \quad \mathbb{E}\xi(t, x)\xi(s, y) = \delta(t - s) \cdot q(x, y),$$

where δ denotes the Dirac delta distribution and q is a positive semidefinite and symmetric kernel on $G \times G$. The above identities imply that the noise process ξ is completely uncorrelated in time, but — as expected — there are spatial correlations as described by the correlation kernel q .

It was shown in [2, Theorem 3.3] that the covariance operator Q in Assumption 3.2 is given by the Hilbert-Schmidt operator $Q : L^2(G) \rightarrow L^2(G)$ with kernel $q \in L^2(G \times G)$ defined by

$$(Qf)(x) := \int_G q(x, y)f(y) dy \quad \text{for all } f \in L^2(G).$$

Using this explicit connection between the covariance operator Q and the spatial correlation kernel q it is then possible to characterize all correlation kernels for which Q and A have the same set of eigenfunctions. In our situation, the operator A has the same set of eigenfunctions as the Laplacian, and for the case of an interval and homogeneous Neumann boundary conditions, Blömker [2, Theorem 4.8] obtains the following result:

Let $G = (0, 1)$ be an interval, and consider a symmetric, positive semidefinite, and sufficiently smooth kernel q , say $q \in C^2(G \times G)$. Then Q commutes with Δ subject to homogeneous Neumann boundary conditions if and only if there exists a 2-periodic even function R such that $q(x, y) = R(x - y) + R(x + y)$.

Functions R of this type which approximate the Dirac delta distribution can easily be constructed. For example, if we consider for fixed $M > 0$ and large values of

FIGURE 4. A positive semidefinite correlation kernel $q(x, y) = R(x - y) + R(x + y)$, whose covariance operator Q commutes with the Laplacian subject to Neumann boundary conditions.

$N \in \mathbb{N}$ the function

$$R_{M,N}(x) = \frac{M}{N} \cdot \sum_{k=1}^N \cos k\pi x ,$$

then one obtains a function which is close to zero everywhere, except for a sharp peak at the origin; see for example the left graph in Figure 4 for $M = 1$ and $N = 50$. The resulting correlation kernel is depicted in the right diagram of the same figure, and it shows that correlations exist only in a small neighborhood of the diagonal $x = y$, but that the correlation is close to zero everywhere else. Notice in particular the peaks in the correlation kernel q for (x, y) close to $(0, 0)$ or $(1, 1)$, which are expected consequences of boundary effects. Moreover, notice that we have

$$\lim_{N \rightarrow \infty} R_{M,N}(0) = M , \quad \text{and} \quad \lim_{N \rightarrow \infty} R_{M,N}(x) = 0 \quad \text{for all} \quad 0 < x < 2 ,$$

i.e., the amplitude of the oscillations which are visible in Figure 4 will converge to zero. By adding suitable small tail expressions to the above definition of $R_{M,N}$, we can maintain the qualitative form of $R_{M,N}$ while at the same time achieving the decay condition in Assumption 4.1 below.

In other words, even though Assumption 3.2 does put restrictions on the possible correlation kernels q , physically relevant kernels can easily be constructed. Yet, using Assumption 3.2 makes the formulations of this paper considerably more readable. We would also like to point out that in many physical applications cut-off noise is used exclusively, i.e., noise which acts only on finitely many Fourier modes. This type of noise corresponds exactly to kernels of the type shown in Figure 4.

While the setting of Assumption 3.2 only considers stochastic convolutions which take values in a general Hilbert space H , this framework suffices to treat fractional power spaces in Section 4. The precise connection is established in the following remark.

Remark 3.5. The setting of Assumption 3.2 suffices to estimate probabilistic events involving stochastic convolutions in fractional power spaces. Specifically, let L denote a positive self-adjoint operator on H having the same eigenfunctions as A , i.e., assume that $Lf_k = \mu_k f_k$ for all $k \in \mathbb{N}$. Then the domain $D(L)$ (equipped with

the graph norm) is given by all $u \in H$ satisfying

$$u = \sum_{k=1}^{\infty} \gamma_k f_k \quad \text{and} \quad \sum_{k=1}^{\infty} \gamma_k^2 \mu_k^2 < +\infty,$$

and for such u the graph norm on $D(L)$ is given by

$$\|u\|_{D(L)} = \|Lu\| = \left(\sum_{k=1}^{\infty} \gamma_k^2 \mu_k^2 \right)^{1/2}.$$

Thus, we can think of the $D(L)$ -norm of the stochastic convolution W_A in (28) as a weighted norm and incorporate the weights into the coefficients α_k . In this way we obtain the identity

$$\|W_A(t)\|_{D(L)} = \|\hat{W}_A(t)\|_H, \quad \text{where} \quad \hat{W}_A(t) = \sum_{k \in \mathbb{N}} (\alpha_k \mu_k) \cdot \int_0^t e^{(t-\tau)\lambda_k} d\beta_k(\tau) \cdot f_k.$$

In our application to the Cahn-Hilliard-Cook equation in Section 4 this remark will be used with L being the negative Laplacian, $H = L^2(G)$, and A being the linearized Cahn-Hilliard operator. This will be described in more detail later.

Before presenting our probability estimates, we would like to indicate one last reduction. Most of the exponential estimates contained in (14) and (18) consider components of the stochastic convolution with respect to an orthogonal decomposition of the underlying Hilbert space. Moreover, the few estimates that consider the complete convolution can easily be reduced to these component estimates. Thus, it suffices to study the two components of the stochastic convolution W_A .

3.2. Estimates for the strongly unstable component. We begin by studying the strongly unstable component associated with the subspace X^+ in Section 2. The following assumption clarifies how the strongly unstable subspace of H is related to the spectrum of the operator A . Roughly speaking, we assume that the eigenvalues λ_k of A which correspond to eigenfunctions in X^+ are positive and large. The indices of these eigenvalues are collected in the index set Λ^+ below. Notice that since this section is concerned with the growth behavior of the stochastic convolution W_A , in particular with both lower and upper bounds on the growth, we only consider terms in the expansion (28) for which $\alpha_k \neq 0$. More precisely, we assume the following.

Assumption 3.6. *In the situation of Assumption 3.2, suppose that $\Lambda^+ \subset \mathbb{N}$ denotes a nonempty finite subset such that*

$$(29) \quad \lambda_k > 0 \quad \text{and} \quad \alpha_k \neq 0 \quad \text{holds for all} \quad k \in \Lambda^+.$$

Then the strongly unstable subspace of H is defined as $H^+ := \text{span}\{f_k : k \in \Lambda^+\}$. (Notice that Λ^+ will in general not contain all $k \in \mathbb{N}$ for which $\lambda_k > 0$ and $\alpha_k \neq 0$ is satisfied, but only a subset of these!)

It is clear from Assumption 3.2 that H^+ is A -invariant, and we denote the restriction of A to H^+ by A^+ . Finally, let $W_A^+(t)$ denote the orthogonal projection of $W_A(t)$ onto the finite-dimensional subspace H^+ .

For the later results of this section, we need more detailed information on the set Λ^+ . In these cases, we will assume that there are constants $\lambda > 0$ and $0 < \gamma < 1$

such that

$$(30) \quad \gamma \leq \frac{\lambda_k}{\lambda} \leq 1 \quad \text{for all} \quad k \in \Lambda^+.$$

Finally, let $\gamma < \gamma^+ < 1$ and assume that the set

$$(31) \quad \Lambda^{++} := \left\{ k \in \Lambda^+ : \gamma^+ \leq \frac{\lambda_k}{\lambda} \leq 1 \right\} \subset \Lambda^+$$

is not empty. As before, we denote the subspace of H spanned by the eigenfunctions f_k for which $k \in \Lambda^{++}$ by H^{++} , and indicate orthogonal projections onto this subspace by the appropriate superscript.

Not all of these assumptions are necessary for each of the following results. Thus, we will indicate in each case which assumptions are used.

In the first result of this section we relate the growth of the stochastic convolution W_A^+ to the growth of the deterministic semigroup e^{tA^+} . This result basically shows that the second estimate in the definition of Ω_2 is satisfied with high probability; see also (18). Notice also that for the formulation of this result it is essential to only consider terms in the expansion (28) for which $\alpha_k \neq 0$.

Lemma 3.7. *Suppose that Assumption 3.2 and (29) of Assumption 3.6 are satisfied. Then for all $r > 0$ the estimate*

$$\mathbb{P} \left(\sup_{t \geq 0} \|e^{-tA^+} W_A^+(t)\| \geq r \right) \leq (\sqrt{2} \cdot e)^{|\Lambda^+|} \cdot \exp \left(-\frac{r^2}{2} \cdot \min_{k \in \Lambda^+} \frac{\lambda_k}{\alpha_k^2} \right)$$

holds, where $|\Lambda^+|$ denotes the number of elements of Λ^+ .

Proof. It is well-known that for a real-valued standard Brownian motion β the stochastic processes $\int_0^t e^{(t-\tau)\lambda} d\beta(\tau)$ and $e^{t\lambda} \cdot \beta(\int_0^t e^{-2\tau\lambda} d\tau)$ are versions of the same process. This immediately implies that the stochastic processes

$$e^{-tA^+} W_A^+(t) = \sum_{k \in \Lambda^+} \alpha_k \cdot \int_0^t e^{-\tau\lambda_k} d\beta_k(\tau) \cdot f_k$$

and

$$\sum_{k \in \Lambda^+} \alpha_k \cdot \beta_k \left(\int_0^t e^{-2\tau\lambda_k} d\tau \right) \cdot f_k$$

have the same distribution. This fact, together with Chebyshev's inequality and the independence of the processes β_k furnishes for arbitrary $h > 0$ the estimate

$$\begin{aligned} \mathbb{P} \left(\sup_{t \geq 0} \|e^{-tA^+} W_A^+(t)\| \geq r \right) &= \mathbb{P} \left(\sup_{t \geq 0} \sum_{k \in \Lambda^+} \alpha_k^2 \cdot \beta_k^2 \left(\int_0^t e^{-2\tau\lambda_k} d\tau \right) \geq r^2 \right) \\ &\leq \mathbb{P} \left(\sum_{k \in \Lambda^+} \alpha_k^2 \cdot \sup_{0 \leq t < 1/(2\lambda_k)} \beta_k^2(t) \geq r^2 \right) \\ &\leq e^{-hr^2} \cdot \mathbb{E} \prod_{k \in \Lambda^+} \exp \left(\alpha_k^2 \cdot h \cdot \sup_{0 \leq t < 1/(2\lambda_k)} \beta_k^2(t) \right) \\ (32) \quad &= e^{-hr^2} \cdot \prod_{k \in \Lambda^+} \mathbb{E} \sup_{0 \leq t < 1/(2\lambda_k)} e^{\alpha_k^2 \cdot h \cdot \beta_k^2(t)}. \end{aligned}$$

Furthermore, it can easily be shown that for any real-valued centered Gaussian random variable Y we have

$$(33) \quad \mathbb{E} e^{\eta \cdot Y^2} = \frac{1}{\sqrt{1 - 2\eta \cdot \mathbb{E} Y^2}} \quad \text{for all} \quad \eta < \frac{1}{2 \cdot \mathbb{E} Y^2}.$$

Thus, the random variable $\exp(\alpha_k^2 \cdot h \cdot \beta_k^2(t))$ is integrable for all $0 \leq t < 1/(2\lambda_k)$, as long as $h \leq \lambda_k/\alpha_k^2$. As a Brownian motion the process β_k is a continuous martingale, and together with the just established integrability we conclude that for all $h \leq \lambda_k/\alpha_k^2$ the process $\exp(\alpha_k^2 \cdot h \cdot \beta_k^2(t))$, $0 \leq t < 1/(2\lambda_k)$, is a continuous submartingale. The maximal inequality [16, Satz 3.28(ii)] now implies

$$\mathbb{E} \sup_{0 \leq t < 1/(2\lambda_k)} e^{\alpha_k^2 \cdot h \cdot \beta_k^2(t)} \leq \left(\frac{p}{p-1} \right)^p \cdot \sup_{0 \leq t < 1/(2\lambda_k)} \mathbb{E} e^{\alpha_k^2 \cdot h \cdot \beta_k^2(t)}$$

for all $p > 1$ and $h \leq \lambda_k/\alpha_k^2$. By passing to the limit $p \rightarrow \infty$, applying (33), and assuming the stronger condition $h \leq \lambda_k/(2\alpha_k^2)$ we obtain

$$\mathbb{E} \sup_{0 \leq t < 1/(2\lambda_k)} e^{\alpha_k^2 \cdot h \cdot \beta_k^2(t)} \leq e \cdot \sup_{0 \leq t < 1/(2\lambda_k)} \frac{1}{\sqrt{1 - 2\alpha_k^2 \cdot h \cdot t}} \leq e \cdot \sqrt{2}.$$

If we now choose $h = \min_{k \in \Lambda^+} (\lambda_k/(2\alpha_k^2)) > 0$ and apply the last inequality to the right-hand side of (32), then the proof is complete. $\square$

The next lemma is concerned with explicit upper bounds on the strongly unstable component of the stochastic convolution W_A over a compact time interval. In addition, we will also derive an explicit lower bound for $W_A^+(T)$ for sufficiently large times T . The lemma is formulated in such a way that it can be combined easily with the results of the last section. In other words, in our application of the lemma, the constants γ and λ will have the same meaning as in Section 2. For now, they are just positive constants.

Lemma 3.8. *Suppose that Assumption 3.2 is satisfied, as well as (29) and (30) of Assumption 3.6. Then for every sufficiently large $\eta > 0$ there exists a constant $c_\eta > 0$ (which depends only on η) such that for all $r > 0$ the estimates*

$$\mathbb{P} \left(\sup_{t \in [0, T]} \|W_A^+(t)\| \geq r \right) \leq e^{-c_\eta \cdot |\Lambda^+|} \text{ for } 0 < T \leq \frac{1}{2\lambda} \cdot \ln \left(1 + \frac{r^2}{\eta \cdot |\Lambda^+|} \cdot \min_{k \in \Lambda^+} \frac{\lambda_k}{\alpha_k^2} \right),$$

$$\mathbb{P} (\|W_A^+(T)\| \leq r) \leq e^{-c_\eta \cdot |\Lambda^+|} \text{ for } T \geq \frac{1}{2\gamma\lambda} \cdot \ln \left(1 + \frac{\eta \cdot r^2}{|\Lambda^+|} \cdot \max_{k \in \Lambda^+} \frac{\lambda_k}{\alpha_k^2} \right)$$

are satisfied.

Proof. We begin by verifying the second estimate. Using Chebyshev's inequality and (33) we obtain for arbitrary $h > 0$ the estimate

$$\begin{aligned}
& \mathbb{P}(\|W_A^+(T)\| \leq r) \\
& \leq e^{hr^2} \cdot \prod_{k \in \Lambda^+} \mathbb{E} \exp \left(-h \cdot \alpha_k^2 \cdot \left(\int_0^T e^{(T-\tau)\lambda_k} d\beta_k(\tau) \right)^2 \right) \\
& = e^{hr^2} \cdot \prod_{k \in \Lambda^+} \left(1 + h \cdot \frac{\alpha_k^2}{\lambda_k} \cdot (e^{2T\lambda_k} - 1) \right)^{-1/2} \\
& \leq e^{hr^2} \cdot \left(1 + h \cdot \min_{k \in \Lambda^+} \frac{\alpha_k^2}{\lambda_k} \cdot (e^{2T\lambda_\gamma} - 1) \right)^{-|\Lambda^+|/2} \\
& = \exp \left(|\Lambda^+| \cdot \left(1 - \frac{1}{2} \cdot \ln \left(1 + \frac{|\Lambda^+|}{r^2} \cdot \min_{k \in \Lambda^+} \frac{\alpha_k^2}{\lambda_k} \cdot (e^{2T\lambda_\gamma} - 1) \right) \right) \right),
\end{aligned}$$

where for the last identity we choose $h = |\Lambda^+|/r^2$. The assertion now follows easily from the bound on T stated in the formulation of the lemma. Furthermore, for the above estimates to hold we need to assume $\eta > e^2 - 1$, and the constant c_η is given by $c_\eta = (1/2) \cdot \ln(1 + \eta) - 1$.

The first estimate of the lemma can be derived similar to the proof of Lemma 3.7 by using a maximal inequality. Thus, we only provide a brief sketch of the proof. For small enough $h > 0$ we have

$$\begin{aligned}
& \mathbb{P} \left(\sup_{t \in [0, T]} \|W_A^+(t)\| \geq r \right) \\
& \leq e^{-hr^2} \cdot \prod_{k \in \Lambda^+} \mathbb{E} \exp \left(h \cdot \alpha_k^2 \cdot e^{2T\lambda_k} \cdot \sup_{t \in [0, T]} \left(\int_0^t e^{-\tau\lambda_k} d\beta_k(\tau) \right)^2 \right) \\
& \leq e^{|\Lambda^+| - hr^2} \cdot \prod_{k \in \Lambda^+} \left(1 - h \cdot \frac{\alpha_k^2}{\lambda_k} \cdot (e^{2T\lambda_k} - 1) \right)^{-1/2} \\
& \leq e^{|\Lambda^+| - hr^2} \cdot \left(1 - h \cdot \max_{k \in \Lambda^+} \frac{\alpha_k^2}{\lambda_k} \cdot (e^{2T\lambda} - 1) \right)^{-|\Lambda^+|/2} \\
& = \exp \left(|\Lambda^+| \cdot \ln(\sqrt{2} \cdot e) - \frac{r^2}{2} \cdot \min_{k \in \Lambda^+} \frac{\lambda_k}{\alpha_k^2} \cdot (e^{2T\lambda} - 1)^{-1} \right),
\end{aligned}$$

provided we choose $h = (2 \cdot (e^{2T\lambda} - 1))^{-1} \cdot \min_{k \in \Lambda^+} (\lambda_k / \alpha_k^2)$. The lower bound on T stated in the formulation of the lemma now readily furnishes the desired estimate. Furthermore, for the above estimates to hold we need to assume $\eta > \ln(2e^2)$, and the constant c_η is given by $c_\eta = \eta/2 - \ln(\sqrt{2} \cdot e)$. $\square$

The two estimates of Lemma 3.8 are essential for determining the probability of the set Ω_1 defined in (14). Notice, however, that they have to be combined in the following sense. On the time interval $[0, t_0]$ we require an upper bound for the stochastic convolution, and at the end of this time interval we need to establish a certain lower bound. In other words, we need to combine the estimates of Lemma 3.8 for $T = t_0$, and this necessitates that it is actually possible to find a time which simultaneously satisfies both conditions. This issue will be resolved in

the following corollary. For technical reasons which will become clear in Section 4, we need to employ a further splitting of the set Λ^+ , which was introduced in (31) of Assumption 3.6.

Corollary 3.9. *Suppose that Assumptions 3.2 and 3.6 are satisfied. Then for all sufficiently large $\eta > 0$ there exists a constant $c_\eta > 0$ such that for arbitrary $r > 0$ the estimate*

$$\mathbb{P} \left(\|W_A^+(t_0)\| < r \quad \text{or} \quad \sup_{t \in [0, t_0]} \|W_A^+(t)\| > r_0 \right) \leq 2e^{-c_\eta \cdot |\Lambda^{++}|}$$

holds, where $r_0 > 0$ is defined by

$$(34) \quad r_0^2 = \left(\left(1 + \frac{\eta \cdot r^2}{|\Lambda^{++}|} \cdot \max_{k \in \Lambda^{++}} \frac{\lambda_k}{\alpha_k^2} \right)^{1/\gamma^+} - 1 \right) \cdot |\Lambda^+| \cdot \eta \cdot \max_{k \in \Lambda^+} \frac{\alpha_k^2}{\lambda_k},$$

and $t_0 > 0$ is defined by

$$(35) \quad t_0 = \frac{1}{2\gamma^+ \lambda} \cdot \ln \left(1 + \frac{\eta \cdot r^2}{|\Lambda^{++}|} \cdot \max_{k \in \Lambda^{++}} \frac{\lambda_k}{\alpha_k^2} \right).$$

Proof. We want to apply the first estimate of Lemma 3.8 with r replaced by r_0 . The definition of r_0 implies the equality of the right-hand sides of the T -constraints in Lemma 3.8 — provided in the second T -constraint we replace γ and Λ^+ by γ^+ and Λ^{++} , respectively. Furthermore, the time t_0 is defined as the common value of these T -constraints. The corollary now follows immediately by noting that

$$\mathbb{P} (\|W_A^+(t_0)\| < r) \leq \mathbb{P} (\|W_A^{++}(t_0)\| \leq r) \leq e^{-c_\eta \cdot |\Lambda^{++}|}$$

and

$$\mathbb{P} \left(\sup_{t \in [0, t_0]} \|W_A^+(t)\| > r_0 \right) \leq \mathbb{P} \left(\sup_{t \in [0, t_0]} \|W_A^+(t)\| \geq r_0 \right) \leq e^{-c_\eta \cdot |\Lambda^+|} < e^{-c_\eta \cdot |\Lambda^{++}|},$$

where the last inequality is due to $\Lambda^{++} \subset \Lambda^+$. $\square$

3.3. Estimates for the complementary component. In the last section we considered the growth behavior of the stochastic convolution in the space X^+ , which was spanned by (all or some of) the eigenfunctions whose eigenvalues satisfy $\lambda_k \geq \gamma\lambda > 0$. In this section we will obtain growth estimates involving the complementary condition $\lambda_k < \gamma\lambda$, i.e., we determine the growth behavior of the stochastic convolution component W_A^- . The resulting growth bounds are necessary for determining the probabilities of the sets Ω_1 and Ω_2 . We begin by formulating the main assumptions concerning the complementary component.

Assumption 3.10. *In the situation of Assumption 3.2, consider two constants $\lambda > 0$ and $0 < \gamma < 1$. Let $\Lambda^- \subset \mathbb{N}$ denote a nonempty subset such that*

$$(36) \quad \lambda_k < \gamma \cdot \lambda \quad \text{for all } k \in \Lambda^-.$$

We define the subspace H^- of H as the closure of $\text{span} \{f_k : k \in \Lambda^-\}$, and let $W_A^-(t)$ denote the orthogonal projection of $W_A(t)$ onto the subspace H^- . (We would like to point out that Λ^- does not have to contain all indices for which $\lambda_k < \gamma\lambda$, but in general only a subset of these. Also, in contrast to Assumption 3.2, we do not have to require that $\alpha_k \neq 0$, since we are only interested in upper growth bounds.)

For the second result of this section, we need to introduce a further splitting of H^- . To this end, let $\zeta > 0$ and define

$$(37) \quad \gamma^- := \frac{\gamma}{1+3\zeta} < \gamma.$$

The splitting of H^- is induced by a splitting of Λ^- into two nonempty sets

$$\begin{aligned} \Lambda^{-+} &= \{k \in \Lambda^- : \gamma^- \cdot \lambda \leq \lambda_k < \gamma \cdot \lambda\} \quad \text{and} \\ \Lambda^{--} &= \{k \in \Lambda^- : \lambda_k < \gamma^- \cdot \lambda\}, \end{aligned}$$

i.e., the subspaces $H^{-\pm}$ denote the closures of the linear hulls of the eigenfunctions associated with $\Lambda^{-\pm}$. Moreover, the orthogonal projections of W_A onto these subspaces are denoted by the respective superscripts.

Not all of these assumptions are necessary for each of the following two results. Thus, we will indicate in each case which assumptions are used.

The first result concerning the complementary component W_A^- addresses the remaining estimate in the definition of Ω_1 . The lemma also provides a preliminary version of the exponential growth bound in the definition of Ω_2 . This latter bound, however, is suboptimal and has to be refined in a second step.

Lemma 3.11. *Suppose that Assumption 3.2 is satisfied, as well as (36) of Assumption 3.6. Let $0 < \alpha < 1/2$ and $p > 1/(2\alpha)$ be arbitrary. Then there exists a positive constant $c_{\alpha,p}$ (depending only on α and p) such that for all $\delta > 0$, $\zeta > 0$, and $T > 0$ the following holds. If we define*

$$C^* := \frac{c_{\alpha,p}}{(\zeta\lambda\gamma)^{2\alpha p} \cdot \delta^{2p}} \cdot \left(\sum_{k \in \Lambda^-} \frac{\alpha_k^2}{(\lambda\gamma(1+\zeta) - \lambda_k)^{1-2\alpha}} \right)^p,$$

then both

$$\mathbb{P} \left(\sup_{t \in [0, T]} \|W_A^-(t)\| > \delta \right) \leq C^* \cdot e^{2pT\lambda\gamma(1+3\zeta)}$$

and

$$\mathbb{P} \left(\sup_{t \geq 0} e^{-t\lambda\gamma(1+3\zeta)} \cdot \|W_A^-(t)\| > \delta \right) \leq C^*$$

are satisfied.

Proof. We begin by verifying the first probability estimate. It can easily be shown that for arbitrary $\xi > 0$ and $0 < \kappa < 1$ we have

$$(38) \quad \int_0^t (t-\tau)^{\kappa-1} \cdot e^{-\tau\xi} d\tau \leq \frac{1}{\kappa \cdot \xi^\kappa} \quad \text{for all } t > 0;$$

see for example [23, Lemma 2.4]. Our proof is based on the factorization method of DaPrato and Zabczyk as described in [3, 10]. For this, let

$$\begin{aligned} Y_\alpha(t) &= \int_0^t (t-\tau)^{-\alpha} \cdot e^{(t-\tau)A} dW^-(\tau) \\ (39) \quad &= \sum_{k \in \Lambda^-} \alpha_k \cdot \int_0^t (t-\tau)^{-\alpha} \cdot e^{(t-\tau)\lambda_k} d\beta_k(\tau) \cdot f_k. \end{aligned}$$

Then due to [10, Equation (5.18)] we have

$$W_A^-(t) = \frac{\sin(\pi\alpha)}{\pi} \cdot \int_0^t (t-\tau)^{\alpha-1} \cdot e^{(t-\tau)A} Y_\alpha(\tau) d\tau.$$

This furnishes for all $\zeta \geq 0$ the estimate

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} \|W_A^-(t)\|^{2p} &\leq C \cdot \mathbb{E} \sup_{t \in [0, T]} \left(\int_0^t (t-\tau)^{\alpha-1} \cdot \|e^{(t-\tau)A} Y_\alpha(\tau)\| d\tau \right)^{2p} \\ (40) \quad &\leq C \cdot \mathbb{E} \sup_{t \in [0, T]} \left(\int_0^t (t-\tau)^{\alpha-1} \cdot e^{(t-\tau)\lambda\gamma(1+3\zeta)} \cdot \|Y_\alpha(\tau)\| d\tau \right)^{2p}. \end{aligned}$$

In the above estimate, as well as for the remainder of this proof, constants which depend only on α and p are denoted by C , and the specific value of C may change from line to line. Notice also that (40) is actually true if $\zeta = 0$, i.e., the introduction of ζ seems completely artificial at this point. Nevertheless, its introduction will simplify our proofs considerably. In particular, it allows us to avoid terms such as $1/(\lambda\gamma - \lambda_k)$. Terms of this form are difficult to deal with in our application to the Cahn-Hilliard-Cook equation and would necessitate the discussion of spectral gaps.

Using Hölder's inequality, $2p > 1/\alpha$, and (38) we now obtain for arbitrary $\zeta > 0$ the estimate

$$\begin{aligned} &\left(\int_0^t (t-\tau)^{\alpha-1} \cdot e^{-\tau\lambda\gamma(1+3\zeta)} \cdot \|Y_\alpha(\tau)\| d\tau \right)^{2p} \\ &\leq \left(\int_0^t ((t-\tau)^{\alpha-1} \cdot e^{-\tau\lambda\gamma\zeta})^{2p/(2p-1)} d\tau \right)^{2p-1} \cdot \int_0^t e^{-2p\tau\lambda\gamma(1+2\zeta)} \\ &\quad \cdot \|Y_\alpha(\tau)\|^{2p} d\tau \\ (41) \quad &\leq C \cdot (\zeta\lambda\gamma)^{1-2p\alpha} \cdot \int_0^\infty e^{-2p\tau\lambda\gamma(1+2\zeta)} \cdot \|Y_\alpha(\tau)\|^{2p} d\tau, \end{aligned}$$

and combining both (40) and (41) furnishes

$$\begin{aligned} &\mathbb{E} \sup_{t \in [0, T]} \|W_A^-(t)\|^{2p} \\ (42) \quad &\leq C \cdot e^{2pT\lambda\gamma(1+3\zeta)} \cdot (\zeta\lambda\gamma)^{1-2p\alpha} \cdot \int_0^\infty e^{-2p\tau\lambda\gamma(1+2\zeta)} \cdot \mathbb{E} \|Y_\alpha(\tau)\|^{2p} d\tau. \end{aligned}$$

From the series expansion in (39) we deduce for all $t > 0$ the estimate

$$\begin{aligned} e^{-2t\lambda\gamma(1+\zeta)} \cdot \mathbb{E} \|Y_\alpha(t)\|^2 &= e^{-2t\lambda\gamma(1+\zeta)} \cdot \sum_{k \in \Lambda^-} \alpha_k^2 \cdot \mathbb{E} \left(\int_0^t (t-\tau)^{-\alpha} \cdot e^{(t-\tau)\lambda_k} d\beta_k(\tau) \right)^2 \\ &= e^{-2t\lambda\gamma(1+\zeta)} \cdot \sum_{k \in \Lambda^-} \alpha_k^2 \cdot \int_0^t \tau^{-2\alpha} \cdot e^{2\tau\lambda_k} d\tau \\ &\leq C \cdot \sum_{k \in \Lambda^-} \frac{\alpha_k^2}{(\lambda\gamma(1+\zeta) - \lambda_k)^{1-2\alpha}}. \end{aligned}$$

Notice that for the last inequality we used the fact that for all $\lambda^* > 0$ with $\lambda^* > \lambda_k$ we have

$$e^{-2t\lambda^*} \cdot \int_0^t \tau^{-2\alpha} e^{2\tau\lambda_k} d\tau \leq \int_0^t \tau^{-2\alpha} e^{-2\tau(\lambda^* - \lambda_k)} d\tau \leq \frac{C}{(\lambda^* - \lambda_k)^{1-2\alpha}},$$

since we assumed $0 < \alpha < 1/2$. Estimating the $2p$ -th moment of $Y_\alpha(t)$ by its second moment using [10, Corollary 2.17], (42) finally implies

$$\mathbb{E} \sup_{t \in [0, T]} \|W_A^-(t)\|^{2p} \leq C \cdot e^{2pT\lambda\gamma(1+3\zeta)} \cdot (\zeta\lambda\gamma)^{-2p\alpha} \cdot \left(\sum_{k \in \Lambda^-} \frac{\alpha_k^2}{(\lambda\gamma(1+\zeta) - \lambda_k)^{1-2\alpha}} \right)^p,$$

where we also used the estimate $\int_0^\infty e^{-2p\tau\lambda\gamma\zeta} d\tau \leq C/(\zeta\lambda\gamma)$. Chebyshev's inequality now furnishes the first probability estimate of Lemma 3.11.

The second probability estimate can be proved analogously. By following the arguments from the first part of the proof we obtain

$$\begin{aligned} & \mathbb{E} \left(\sup_{t \geq 0} e^{-t\lambda\gamma(1+3\zeta)} \cdot \|W_A^-(t)\| \right)^{2p} \\ & \leq C \cdot \mathbb{E} \left(\sup_{t \geq 0} \int_0^t (t-\tau)^{\alpha-1} \cdot e^{-\tau\lambda\gamma(1+3\zeta)} \cdot \|Y_\alpha(\tau)\| d\tau \right)^{2p} \\ & \leq C \cdot \left((\zeta\lambda\gamma)^{-2\alpha} \cdot \sum_{k \in \Lambda^-} \frac{\alpha_k^2}{(\lambda\gamma(1+\zeta) - \lambda_k)^{1-2\alpha}} \right)^p, \end{aligned}$$

and again the result follows from Chebyshev's inequality. $\square$

With Lemma 3.11 we have provided all the ingredients to determine the probability of the event Ω_1 . Yet, in its present form the lemma cannot be used for Ω_2 defined in (18). The lemma only allows us to establish an exponential growth bound of the form $e^{t\lambda\gamma(1+3\zeta)}$, while in (18) we require a stronger bound of the form $e^{t\lambda\gamma}$. Fortunately, it is possible to derive the stronger bound by using the additional splitting of H^- introduced in Assumption 3.10, and combining Lemmas 3.7 and 3.11.

Corollary 3.12. *Suppose that Assumptions 3.2 and 3.10 are satisfied. Let $0 < \alpha < 1/2$ and $p > 1/(2\alpha)$ be arbitrary. Furthermore, assume that $\alpha_k \neq 0$ for all $k \in \Lambda^{++}$. Then there exists a positive constant $c_{\alpha,p}$ (depending only on α and p) such that for all $\delta > 0$ and $\zeta > 0$ we have*

$$\begin{aligned} \mathbb{P} \left(\sup_{t \geq 0} e^{-t\lambda\gamma} \cdot \|W_A^-(t)\| > \delta \right) & \leq \left(\sqrt{2} \cdot e \right)^{|\Lambda^{++}|} \cdot \exp \left(-\frac{\delta^2}{8} \cdot \min_{k \in \Lambda^{++}} \frac{\lambda_k}{\alpha_k^2} \right) \\ & + c_{\alpha,p} \cdot \left(\frac{4}{(\zeta\lambda\gamma^-)^{2\alpha} \cdot \delta^2} \cdot \sum_{k \in \Lambda^{--}} \frac{\alpha_k^2}{(\lambda\gamma^-(1+\zeta) - \lambda_k)^{1-2\alpha}} \right)^p, \end{aligned}$$

where $\gamma^- = \gamma/(1+3\zeta)$ was defined in (37).

Proof. It can easily be verified that

$$(43) \quad \mathbb{P} \left(\sup_{t \geq 0} e^{-t\lambda\gamma} \cdot \|W_A^-(t)\| > \delta \right) \leq \mathbb{P} \left(\sup_{t \geq 0} e^{-t\lambda\gamma} \cdot \|W_A^{-+}(t)\| > \frac{\delta}{2} \right)$$

$$(44) \quad + \mathbb{P} \left(\sup_{t \geq 0} e^{-t\lambda\gamma^-(1+3\zeta)} \cdot \|W_A^{--}(t)\| > \frac{\delta}{2} \right).$$

The probability on the right-hand side of (44) can be estimated directly by applying Lemma 3.11. In addition, the probability in (43) can be estimated by applying Lemma 3.7. One only has to note that the representation of the stochastic convolution in (28) implies $e^{-t\lambda\gamma} \cdot \|W_A^{-+}(t)\| \leq \|e^{-tA}W_A^{-+}(t)\|$ for all $t \geq 0$. This completes the proof of the corollary. $\square$

4. APPLICATION TO THE CAHN-HILLIARD-COOK MODEL

In this final section of the paper we combine the abstract results of Section 2 with the probability estimates of Section 3 to provide an explanation of spinodal decomposition in the Cahn-Hilliard-Cook model. Throughout this section, we will consider the dynamical behavior of solution trajectories of (1) starting at the deterministic homogeneous state $u(0) \equiv m$. We assume that the constant m is contained in the spinodal interval, i.e., that it satisfies the inequality $f'(m) > 0$. Only in this situation can one expect the occurrence of spinodal decomposition.

4.1. Functional-analytic setting. We begin by reformulating (1) in an abstract functional-analytic setting which allows for the direct application of the results of Section 2. Since this setting has been considered in previous papers on the subject, our discussion will be brief. For more details we refer the reader to [3, 15, 19, 20, 23, 25].

In order to simplify the presentation, the abstract results of Section 2 were formulated for solution trajectories starting at the origin of the underlying phase space. Rather than considering (1) subject to the condition $u(0) \equiv m$, we therefore perform a change of variables and consider instead the initial value problem

$$(45) \quad \begin{aligned} \partial_t u &= -\Delta (\varepsilon^2 \Delta u + f(m+u)) + \sigma_\varepsilon \cdot \xi \quad \text{in} \quad G, \\ \frac{\partial u}{\partial \nu} &= \frac{\partial \Delta u}{\partial \nu} = 0 \quad \text{on} \quad \partial G, \\ u(0) &\equiv 0 \quad \text{and} \quad \int_G u \, dx = 0. \end{aligned}$$

If u is the solution of (45), then the sum $m+u$ solves the original Cahn-Hilliard-Cook equation (1) and originates at the homogeneous state $u_0 \equiv m$. For the remainder of this paper we will consider the transformed problem (45), except in the formulation of our main result Theorem 4.13, which will be stated for the original equation (1).

The abstract results of Section 2 were based on Assumptions 2.1, 2.2, and 2.4, which in turn were formulated for a stochastic evolution equation of the form (2). In order to recast (45) into this form, we define the ε -dependent linear operator A_ε by

$$(46) \quad A_\varepsilon u := -\Delta (\varepsilon^2 \Delta u + f'(m)u),$$

and the nonlinearity F by

$$(47) \quad F(u) := -\Delta g(u), \quad \text{where} \quad g(u) := f(m+u) - f'(m)u - f(m).$$

Then (45) is formally of the form (2).

We now turn our attention to the underlying phase space. It was mentioned in the Introduction that the Cahn-Hilliard-Cook equation is conservative in the sense that the total concentration $\int_G u(t, x) \, dx$ remains constant as a function of $t \geq 0$. Rather than considering the Hilbert space $L^2(G)$ we therefore introduce the closed subspace

$$X := \left\{ v \in L^2(G) : \int_\Omega v \, dx = 0 \right\}$$

equipped with the Hilbert norm $\|\cdot\| = \|\cdot\|_{L^2(G)}$. The linear operator A_ε defined in (46) is considered as an operator on X with domain

$$D(A_\varepsilon) := \left\{ v \in X \cap H^4(G) : \frac{\partial v}{\partial \nu}(x) = \frac{\partial \Delta v}{\partial \nu}(x) = 0 \text{ for all } x \in \partial G \right\}.$$

It is well-known that in this setting the operator A_ε is self-adjoint. In order to describe its spectrum, let $0 < \mu_1 \leq \mu_2 \leq \dots \rightarrow +\infty$ denote the eigenvalues of the negative Laplacian $-\Delta : X \rightarrow X$ subject to homogeneous Neumann boundary conditions, and let $\psi_1, \psi_2, \dots$ denote the complete $L^2(G)$ -orthonormal set of corresponding eigenfunctions. Then the spectrum of A_ε consists of the real eigenvalues

$$(48) \quad \lambda_{k,\varepsilon} = \mu_k \cdot (f'(m) - \varepsilon^2 \cdot \mu_k), \quad k \in \mathbb{N},$$

and the eigenfunction associated with $\lambda_{k,\varepsilon}$ is given by ψ_k , for $k \in \mathbb{N}$.

For our application it is essential to have precise knowledge of the asymptotic behavior of the eigenvalues μ_k . For the case of domains G with piece-wise C^1 -boundary this behavior is well-known. According to [8, p. 442] or [12] we have

$$(49) \quad \lim_{k \rightarrow \infty} \mu_k \cdot k^{-2/\dim G} = C_G > 0,$$

where the positive constant C_G depends only on the domain G .

While the space X forms the basis for the linear operator A_ε , it is not suitable as phase space for the nonlinear evolution equation. In order to define the actual phase space, recall that we assumed $f'(m) > 0$ and let

$$(50) \quad \lambda_\varepsilon^{\max} := \max \{ \mu \cdot (f'(m) - \varepsilon^2 \mu) : \mu > 0 \} = \frac{f'(m)^2}{4\varepsilon^2}.$$

If we now define $c_\varepsilon = 4\lambda_\varepsilon^{\max}$, then the operator $c_\varepsilon I - A_\varepsilon$ is positive and sectorial, i.e., we can consider its associated fractional power spaces as in [17]. In particular, consider

$$X^{1/2} = D \left((c_\varepsilon I - A_\varepsilon)^{1/2} \right), \quad \text{equipped with the norm } \|\cdot\|_* = \|\Delta \cdot\|.$$

Notice that even though the definition of $X^{1/2}$ depends on ε , the resulting space turns out to be independent of ε . In fact, we have $X^{1/2} \subset H^2(G)$. The norm $\|\cdot\|_*$ defined above is not the canonical (graph) norm on the fractional power space $X^{1/2}$, but an equivalent one. One of the main reasons for choosing $\|\cdot\|_*$ over the standard $X^{1/2}$ -norm is the identity

$$(51) \quad \|u\|_* = \left(\sum_{k=1}^{\infty} \gamma_k^2 \cdot \mu_k^2 \right)^{1/2} \quad \text{for all } u = \sum_{k=1}^{\infty} \gamma_k \cdot \psi_k \in X^{1/2},$$

i.e., we can think of $\|u\|_*$ as a weighted $L^2(G)$ -norm. Furthermore, a function u with an $L^2(G)$ -expansion as in (51) is contained in $X^{1/2}$ if and only if $\sum_{k=1}^{\infty} \gamma_k^2 \cdot \mu_k^2 < +\infty$. For more details we refer the reader to [20, Lemma 3.2]. Due to these results, we can employ Remark 3.5 with $L = -\Delta$ for studying the $\|\cdot\|_*$ -norm of stochastic convolutions.

Using the framework described so far we can now establish the validity of Assumption 2.1. The necessary conditions are collected in the following assumption.

Assumption 4.1. *Let $G \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, be a bounded domain with piece-wise C^1 -boundary, assume $f'(m) > 0$, and adopt the notation introduced above. In*

addition, suppose that the noise process ξ in (45) is given as a generalized derivative $\xi = \partial_t \mathcal{W}$ of a Q -Wiener process $\mathcal{W} = \{\mathcal{W}(t)\}_{t \geq 0}$ with

$$\mathcal{W}(t) = \sum_{k=1}^{\infty} a_k \cdot \beta_k(t) \cdot \psi_k,$$

where $\{\beta_k\}_{k \in \mathbb{N}}$ denotes a family of independent real-valued standard Brownian motions, and $\{\psi_k\}_{k \in \mathbb{N}}$ is the $L^2(G)$ -orthonormal set of eigenfunctions of the negative Laplacian on X subject to homogeneous Neumann boundary conditions. Finally, assume that the real coefficients $\{a_k\}_{k \in \mathbb{N}}$ satisfy

$$\lim_{k \rightarrow \infty} a_k^2 \cdot \mu_k^{s_{\mathcal{R}}} = C > 0 \quad \text{with} \quad s_{\mathcal{R}} > \frac{d}{2}.$$

It can easily be shown that Assumption 4.1 implies the validity of Assumption 2.1 for the Cahn-Hilliard-Cook model (45) with $s = 1/2$. In particular, the assumed decay of the coefficients a_k and the regularity condition $s_{\mathcal{R}} > \dim G/2$ ensure that the associated stochastic convolution takes values in the fractional power space $X^{1/2} \subset H^2(G)$. Notice also that we consider conservative noise as in [7, 18]. The assumption that $\mathcal{W}$ is given as a series with respect to the eigenfunctions of the Laplacian has already been discussed in Remarks 3.3 and 3.4. While it is putting restrictions on the admissible noise processes, the results of [2] demonstrate that physically relevant correlation functions can easily be achieved.

In the situation of Assumption 4.1 it is well-known that the Cahn-Hilliard-Cook equation (45) has a unique global mild solution [9]. Moreover, this solution satisfies the variation of constants formula (4), provided we replace A by A_{ε} , as well as $W_A(t)$ by $\mathcal{W}_{A_{\varepsilon}}(t)$. The stochastic convolution is now defined as

$$(52) \quad \mathcal{W}_{A_{\varepsilon}}(t) = \sigma_{\varepsilon} \cdot \int_0^t e^{(t-\tau)A_{\varepsilon}} d\mathcal{W}(\tau) \quad \text{for all} \quad t \geq 0,$$

i.e., we incorporate the noise strength σ_{ε} into its definition. After analogous replacements, (6) determines the mild solution of (45) for $t \geq t_0$.

Next we address the specific assumptions for the linearization A_{ε} . It was already mentioned in Section 2.1 that the growth conditions of Assumption 2.2 are an easy consequence of a suitable splitting of the spectrum of A_{ε} . This splitting will be introduced in the following assumption.

Assumption 4.2. Let $0 < \gamma < \gamma^+ < 1$ denote two constants which will be specified in more detail later, and assume the situation of Assumption 4.1. Using the definition (50) of $\lambda_{\varepsilon}^{\max}$ we then define three closed subspaces of $X^{1/2}$ by

$$\begin{aligned} X_{\varepsilon}^{++} &:= \text{span} \{ \psi_k : \lambda_{k,\varepsilon} \geq \gamma^+ \cdot \lambda_{\varepsilon}^{\max} \}, \\ X_{\varepsilon}^{+} &:= \text{span} \{ \psi_k : \lambda_{k,\varepsilon} \geq \gamma \cdot \lambda_{\varepsilon}^{\max} \}, \quad \text{and} \\ X_{\varepsilon}^{-} &:= \text{closure of span} \{ \psi_k : \lambda_{k,\varepsilon} < \gamma \cdot \lambda_{\varepsilon}^{\max} \}, \end{aligned}$$

and we denote the corresponding index sets by $\Lambda_{\varepsilon}^{++}$, $\Lambda_{\varepsilon}^{+}$, and $\Lambda_{\varepsilon}^{-}$, respectively. The orthogonal projections of a function $u \in X^{1/2}$ onto these spaces are denoted by the respective superscripts.

It is clear from (48) and the above assumption that both X_{ε}^{++} and X_{ε}^{+} are finite-dimensional subspaces of $X^{1/2}$, which consist of superpositions of strongly

unstable eigenfunctions of A_ε . In fact, the asymptotic behavior of the eigenvalues of the Laplacian given in (49) readily implies

$$(53) \quad \dim X_\varepsilon^{++} \sim \varepsilon^{-\dim G} \quad \text{and} \quad \dim X_\varepsilon^+ \sim \varepsilon^{-\dim G} \quad \text{for } \varepsilon \rightarrow 0.$$

In addition, the definition of the Hilbert norm $\|\cdot\|_*$, combined with the fact that the functions ψ_k are $L^2(G)$ -orthonormal eigenfunctions of the Laplacian, immediately furnishes the orthogonality of X_ε^+ and X_ε^- with respect to the inner product in $X^{1/2}$, i.e., we have the orthogonal splitting $X^{1/2} = X_\varepsilon^+ \oplus X_\varepsilon^-$.

Despite the fact that Assumption 4.2 only introduces a splitting of the spectrum of the operator A_ε , this definition suffices to establish the necessary assumptions for the linear part. To this end, recall that the operator $-A_\varepsilon$ is sectorial. Using standard results from [17] one can then immediately verify that Assumption 2.2 is satisfied with

$$(54) \quad s = \frac{1}{2}, \quad M_1 = \frac{C}{\varepsilon}, \quad \lambda = \lambda_\varepsilon^{\max} = \frac{f'(m)^2}{4\varepsilon^2}, \quad \text{and} \quad b = 1 + \rho,$$

where $\rho > 0$ is a small constant which will be specified in more detail later. For the derivation of the ε -dependence of M_1 we refer the reader to [23, Lemma 3.1].

As a final step we have to establish the validity of Assumption 2.4. We begin by describing the nonlinearity in (47) further.

Assumption 4.3. *Consider the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined in (47). We assume that there exists a constant $\kappa \geq 1$ such that $g(u) = u^{1+\kappa} \cdot \tilde{g}(u)$, and that $\tilde{g}$ is a C^2 -function on an open interval containing 0.*

This assumption basically states that g consists only of higher-order terms, which in view of (47) is not very restrictive. Yet, we also need the following assumption for the underlying domain G .

Assumption 4.4. *Assume that $G \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, is a bounded domain with piece-wise C^1 -boundary. In addition, suppose there are positive constants C_1 and C_2 such that for all $k \in \mathbb{N}$ we have both*

$$(55) \quad \|\psi_k\|_{L^\infty(G)} \leq C_1 \quad \text{and} \quad \|\nabla \psi_k\|_{L^\infty(G)} \leq C_2 \cdot \sqrt{\mu_k},$$

where $\{\psi_k\}_{k \in \mathbb{N}}$ denotes the complete $L^2(G)$ -orthonormal set of eigenfunctions of the negative Laplacian on X subject to homogeneous Neumann boundary conditions, with associated eigenvalues $\{\mu_k\}_{k \in \mathbb{N}}$.

For arbitrary one-dimensional domains G this assumption can easily be verified. However, in two and three space dimensions (55) is rather restrictive, and it is not known which geometric conditions on G imply these estimates. Nevertheless, based on the explicit eigenfunction knowledge, one can show that Assumption 4.4 is satisfied for rectangular, hexagonal, and triangular domains. It is known that the condition does not hold for the case of disks; see for example the extensive discussion in [11]. In fact, disks exhibit the worst-case growth behavior of the maximum norms of the eigenfunction. Using formal calculations, it was conjectured in [1] that for generic domains, the estimates above have to be replaced by estimates of the form

$$(56) \quad \|\psi_k\|_{L^\infty(G)} \leq C_1 \cdot \ln \mu_k \quad \text{and} \quad \|\nabla \psi_k\|_{L^\infty(G)} \leq C_2 \cdot \sqrt{\mu_k} \cdot \ln \mu_k,$$

but establishing this bound rigorously is still an open problem. For more details we refer the reader to [1, 11], as well as the references in these papers. We would like to point out, however, that the validity of (56) would not affect our main

results, since the additional logarithmic term can be absorbed into faster growing polynomial terms with exponents close to zero.

Besides being crucial for our results on spinodal decomposition, Assumption 4.4 can also be used to establish the regularity of $\mathcal{W}_{A_\varepsilon}$ in spaces of continuous functions. For more details we refer the reader to [10, p. 139]. In our application to the Cahn-Hilliard-Cook model, Assumption 4.4 is responsible for the following result.

Lemma 4.5. *Suppose that Assumptions 4.2, 4.3 and 4.4 are satisfied, let F be defined as in (47), let $\delta_0 > 0$ be arbitrary, define*

$$(57) \quad \delta_\varepsilon := \delta_0 \cdot \varepsilon^{2-d/2} ,$$

and consider the sets of Definition 2.3 with respect to the splitting $X^{1/2} = X_\varepsilon^+ \oplus X_\varepsilon^-$. Then there exist positive constants K_1 and C_F such that for all $0 < \varepsilon < 1$ and arbitrary

$$0 < R \leq K_1 \cdot \varepsilon^{-2+d/2}$$

the following holds. The nonlinearity $F : X^{1/2} \rightarrow X$ satisfies the estimate

$$(58) \quad \|F(u)\| \leq C_F \cdot \delta_\varepsilon^\kappa \cdot \|u\|_*^{1+\kappa} \quad \text{for all} \quad u \in \mathcal{K}_{21\delta_\varepsilon, R} ,$$

and for any choice of

$$0 < r < r_0 \leq K_1 \cdot \varepsilon^{-2+d/2}$$

we further have

$$(59) \quad \|F(u)\| \leq C_F \cdot (\delta_\varepsilon \cdot r_0)^\kappa \cdot \|u\|_* \quad \text{for all} \quad u \in \mathcal{Z}_{4\delta_\varepsilon r, 2r_0} .$$

The constants K_1 and C_F depend only on δ_0 , G , γ , and g .

Proof. The first estimate (58) is a direct consequence of [23, Lemma 3.3]. Similarly, the second estimate (59) can be easily verified using the results of [23, Lemma 3.3]. In this lemma it was shown that we have both

$$\|u^+\|_{L^\infty(G)} \leq C \cdot \delta_\varepsilon \cdot \|u^+\|_* \quad \text{and} \quad \|\nabla u^+\|_{L^4(G)} \leq C \cdot \delta_\varepsilon^{1/2} \cdot \|u^+\|_*$$

for all $u^+ \in X_\varepsilon^+$. Furthermore, Sobolev's embedding theorem furnishes both

$$\|u^-\|_{L^\infty(G)} \leq C \cdot \|u^-\|_* \quad \text{and} \quad \|\nabla u^-\|_{L^4(G)} \leq C \cdot \|u^-\|_*$$

for all $u^- \in X_\varepsilon^-$. Thus, for all $u \in \mathcal{Z}_{4\delta_\varepsilon r, 2r_0}$ we have $\|u\|_{L^\infty(G)} \leq C \cdot \delta_\varepsilon \cdot r_0$, as well as

$$\|\nabla u^-\|_{L^4(G)}^2 \leq C \cdot \|u^-\|_*^2 \leq C \cdot \delta_\varepsilon \cdot r_0 \cdot \|u\|_* , \quad \text{and} \quad \|\nabla u\|_{L^4(G)}^2 \leq C \cdot \delta_\varepsilon \cdot r_0 \cdot \|u\|_* .$$

Finally, due to $r_0 \leq K_1 \cdot \varepsilon^{-2+d/2}$ we can use the proof of [23, Lemma 3.3] one more time to show that

$$\|F(u)\|_{L^2(G)} \leq C \cdot \|u\|_{L^\infty(G)}^\kappa \cdot \|u\|_* + C \cdot \|u\|_{L^\infty(G)}^{\kappa-1} \cdot \|\nabla u\|_{L^4(G)}^2 \quad \text{for all} \quad u \in \mathcal{Z}_{4\delta_\varepsilon r, 2r_0} ,$$

which immediately implies (59). $\square$

Lemma 4.5 shows that the Cahn-Hilliard-Cook nonlinearity satisfies Assumption 2.4. Notice in particular that the constant C_δ in (13) is given by $C_\delta = C_F \cdot \delta_\varepsilon^\kappa$, i.e., it is extremely small for small $\varepsilon > 0$. This will turn out to be crucial in the following.

4.2. Probabilistic considerations. The setting of Section 4.1 allows us to apply Theorems 2.5 and 2.6 to the Cahn-Hilliard-Cook model (45). However, we still have to make sure that the specific dynamical behavior described in these results is in fact likely to occur. In other words, we have to apply the probabilistic tools of Section 3 to the setting described above — and this is the subject of the present section.

The Cahn-Hilliard-Cook model depends on the small parameter $\varepsilon > 0$. Thus, all quantities arising in the abstract results of Sections 2 and 3 will depend on ε as well, and we are interested in the asymptotic behavior of these quantities as $\varepsilon \rightarrow 0$. To simplify our notation, we make use of the following abbreviations.

Definition 4.6. Let f_ε and g_ε denote two real quantities depending on the small parameter $\varepsilon > 0$. Then we say that g_ε is of order f_ε , provided there are (ε -independent) positive constants c and C such that

$$c \cdot f_\varepsilon \leq g_\varepsilon \leq C \cdot f_\varepsilon \quad \text{for all sufficiently small } \varepsilon > 0.$$

If this is the case we write $g_\varepsilon = \mathcal{O}(f_\varepsilon)$. If on the other hand we have

$$\lim_{\varepsilon \rightarrow 0} \frac{g_\varepsilon}{f_\varepsilon} = 0,$$

then we write $g_\varepsilon = o(f_\varepsilon)$. Finally, we write $g_\varepsilon \in \mathcal{P}_\varepsilon$ if and only if there are positive constants c and C such that

$$|g_\varepsilon| \leq C \cdot \varepsilon^c \quad \text{for all sufficiently small } \varepsilon > 0.$$

We begin by considering the probability of the set Ω_1 defined in (14). As mentioned above, the quantities appearing in this definition will now depend on ε , i.e., we have to consider the set

$$(60) \quad \Omega_{1,\varepsilon} = \left\{ \mathcal{W}_{A_\varepsilon}(t) \in \mathcal{Z}_{\delta_\varepsilon r_\varepsilon, r_{0,\varepsilon}} \text{ for all } t \in [0, t_{0,\varepsilon}] , \quad \|\mathcal{W}_{A_\varepsilon}^+(t_{0,\varepsilon})\|_* \geq r_\varepsilon \right\},$$

with ε -dependent quantities δ_ε , r_ε , $r_{0,\varepsilon}$, and $t_{0,\varepsilon}$. The stochastic convolution $\mathcal{W}_{A_\varepsilon}(t)$ was defined in (52).

One of the crucial tools for determining the probability of the set $\Omega_{1,\varepsilon}$ is Corollary 3.9, and this result imposes restrictions on the time $t_{0,\varepsilon}$, as well as on the radii $r_{0,\varepsilon}$ and r_ε . In order to simplify our discussion of these conditions, we introduce the following assumption.

Assumption 4.7. Let $s_{\mathcal{R}}$ be the regularity exponent from Assumption 4.1, let d denote the dimension of the underlying domain G , and define

$$D := 2s_{\mathcal{R}} - d - 2.$$

Then we assume that

$$(61) \quad \Theta_\varepsilon := \frac{r_\varepsilon^2}{\sigma_\varepsilon^2 \cdot \varepsilon^D} \rightarrow \infty$$

in the limit $\varepsilon \rightarrow 0$. Notice also that the condition on $s_{\mathcal{R}}$ given in Assumption 4.1 is equivalent to $D > -2$.

According to (34) and (35) the time $t_{0,\varepsilon}$ and the radius $r_{0,\varepsilon}$ are related to r_ε via the identities

$$t_{0,\varepsilon} = \frac{1}{2\gamma^+ \lambda_\varepsilon^{\max}} \cdot \ln \left(1 + \frac{\eta \cdot r_\varepsilon^2}{|\Lambda_\varepsilon^{++}|} \cdot \max_{k \in \Lambda_\varepsilon^{++}} \frac{\lambda_{k,\varepsilon}}{\alpha_k^2} \right)$$

and

$$r_{0,\varepsilon}^2 = \left(\left(1 + \frac{\eta \cdot r_\varepsilon^2}{|\Lambda_\varepsilon^{++}|} \cdot \max_{k \in \Lambda_\varepsilon^{++}} \frac{\lambda_{k,\varepsilon}}{\alpha_k^2} \right)^{1/\gamma^+} - 1 \right) \cdot |\Lambda_\varepsilon^+| \cdot \eta \cdot \max_{k \in \Lambda_\varepsilon^+} \frac{\alpha_k^2}{\lambda_{k,\varepsilon}},$$

where $\eta > 0$ is sufficiently large. Notice also that in view of Remark 3.5, (51), and (52) we have $\alpha_k = \sigma_\varepsilon \cdot a_k \cdot \mu_k$.

Using Assumptions 4.1 and 4.7 we can now determine the asymptotic behavior of both $t_{0,\varepsilon}$ and $r_{0,\varepsilon}$ for $\varepsilon \rightarrow 0$. One only has to note that (48), (50), and Assumption 4.2 imply $\mu_k = \mathcal{O}(\varepsilon^{-2})$ uniformly for $k \in \Lambda_\varepsilon^+$. Combined with the asymptotic growth of the μ_k given in (49) this also furnishes $|\Lambda_\varepsilon^+| = \mathcal{O}(\varepsilon^{-d})$, as well as $|\Lambda_\varepsilon^{++}| = \mathcal{O}(\varepsilon^{-d})$. Thus, it can easily be verified that we have both

$$(62) \quad t_{0,\varepsilon} = \frac{1}{2\gamma^+ \lambda_\varepsilon^{\max}} \cdot \ln \mathcal{O}(\Theta_\varepsilon)$$

and

$$(63) \quad \begin{aligned} r_{0,\varepsilon} &= r_\varepsilon \cdot \left(\mathcal{O}(\Theta_\varepsilon)^{1/\gamma^+} - 1 \right)^{1/2} \cdot \mathcal{O} \left(\Theta_\varepsilon^{-1/2} \right) \\ &= r_\varepsilon \cdot \mathcal{O} \left(\Theta_\varepsilon^{(1-\gamma^+)/(2\gamma^+)} \right). \end{aligned}$$

In order to determine the probability of the set $\Omega_{1,\varepsilon}$ we need the following lemma. It will allow us to estimate the sum appearing in Lemma 3.11 in terms of ε .

Lemma 4.8. *Suppose that Assumptions 4.1, 4.2, and 4.7 are satisfied. In addition, assume that*

$$D > -2, \quad 0 < \alpha < \frac{1}{2}, \quad \text{and} \quad \alpha < \frac{D+2}{8}.$$

Then for every $\zeta > 0$ there exists an ε -independent constant $C > 0$ such that for all sufficiently small $\varepsilon > 0$ we have

$$\sum_{k \in \Lambda_\varepsilon^-} \frac{a_k^2 \mu_k^2}{(\gamma \lambda_\varepsilon^{\max} \cdot (1 + \zeta) - \lambda_{k,\varepsilon})^{1-2\alpha}} \leq C \cdot \varepsilon^{2-4\alpha} \cdot \Phi_\varepsilon,$$

where we use the abbreviation

$$\Phi_\varepsilon := \begin{cases} 1 & \text{for } D > 2, \\ \ln(\varepsilon^{-1}) & \text{for } D = 2, \\ \varepsilon^{D-2} & \text{for } D < 2. \end{cases}$$

Proof. We estimate the sum in two parts. More precisely, let Λ_ε^{--} consist of all $k \in \Lambda_\varepsilon^-$ satisfying $\lambda_{k,\varepsilon} < -\gamma \lambda_\varepsilon^{\max}$, and let $\Lambda_\varepsilon^{-+} = \Lambda_\varepsilon^- \setminus \Lambda_\varepsilon^{--}$. Then (48) and (49) readily imply $\max \Lambda_\varepsilon^{-+} = \min \Lambda_\varepsilon^{--} - 1 = \mathcal{O}(\varepsilon^{-d})$. The first part of the sum corresponds to Λ_ε^{-+} and contains large wavelength eigenfunctions. In this situation,

$0 < \alpha < 1/2$ furnishes

$$\begin{aligned}
 \sum_{k \in \Lambda_{\varepsilon}^{-+}} \frac{a_k^2 \mu_k^2}{(\gamma \lambda_{\varepsilon}^{\max} \cdot (1 + \zeta) - \lambda_{k,\varepsilon})^{1-2\alpha}} &\leq C \cdot (\gamma \lambda_{\varepsilon}^{\max})^{2\alpha-1} \cdot \sum_{k \in \Lambda_{\varepsilon}^{-+}} \mu_k^{2-s_{\mathcal{R}}} \\
 &\leq C \cdot \varepsilon^{2-4\alpha} \cdot \sum_{k=1}^{\max \Lambda_{\varepsilon}^{-+}} k^{2 \cdot (2-s_{\mathcal{R}})/d} \\
 &\leq C \cdot \varepsilon^{2-4\alpha} \cdot \left(1 + \int_1^{C \cdot \varepsilon^{-d}} \tau^{(4-2s_{\mathcal{R}})/d} d\tau \right) \\
 &\leq C \cdot \varepsilon^{2-4\alpha} \cdot \Phi_{\varepsilon},
 \end{aligned}$$

where we again used the asymptotics of the eigenvalues μ_k given in (49).

The second part of the sum corresponds to the index set $\Lambda_{\varepsilon}^{--}$ and consists exclusively of short wavelength eigenfunctions. In fact, a straightforward calculation shows that for all $k \in \Lambda_{\varepsilon}^{--}$ we have $\lambda_{k,\varepsilon} \leq -C \cdot \varepsilon^2 \cdot \mu_k^2$, and this implies

$$\begin{aligned}
 \sum_{k \in \Lambda_{\varepsilon}^{--}} \frac{a_k^2 \mu_k^2}{(\gamma \lambda_{\varepsilon}^{\max} \cdot (1 + \zeta) - \lambda_{k,\varepsilon})^{1-2\alpha}} &\leq C \cdot \varepsilon^{4\alpha-2} \cdot \sum_{k \in \Lambda_{\varepsilon}^{--}} a_k^2 \mu_k^{4\alpha} \\
 &\leq C \cdot \varepsilon^{4\alpha-2} \cdot \int_{C \cdot \varepsilon^{-d}}^{\infty} \tau^{(8\alpha-2s_{\mathcal{R}})/d} d\tau \\
 &\leq C \cdot \varepsilon^{D-4\alpha} \leq C \cdot \varepsilon^{2-4\alpha} \cdot \Phi_{\varepsilon}.
 \end{aligned}$$

Notice that the exponent of the above integrand is strictly less than -1 due to our assumption $\alpha < (D+2)/8$. This completes the proof of the lemma. $\square$

We are finally in a position to estimate the probability of the set $\Omega_{1,\varepsilon}$ for small $\varepsilon > 0$. The following proposition shows that its probability converges to 1 as $\varepsilon \rightarrow 0$.

Proposition 4.9. *Suppose that Assumptions 4.1, 4.2, and 4.7 are satisfied, let $\Omega_{1,\varepsilon}$ be defined as in (60), and let δ_{ε} be as in (57). Furthermore, suppose that*

$$-2 < D \leq 2 \quad \text{and} \quad \delta_{\varepsilon}^{-2} \cdot \Theta_{\varepsilon}^{-1+(1+3\zeta) \cdot \gamma/\gamma^+} \in \mathcal{P}_{\varepsilon}$$

for some small $\zeta > 0$. Then for every $q > 0$ there exists a constant $C > 0$ such that

$$\mathbb{P}(\Omega_{1,\varepsilon}) \geq 1 - C \cdot \varepsilon^q$$

for all sufficiently small $\varepsilon > 0$.

Proof. According to the definition of $\Omega_{1,\varepsilon}$ given in (60) we have

$$(64) \quad \mathbb{P}(\Omega_{1,\varepsilon}^c) \leq \mathbb{P}\left(\sup_{t \in [0, t_{0,\varepsilon}]} \|\mathcal{W}_{A_{\varepsilon}}^+(t)\|_* > r_{0,\varepsilon} \quad \text{or} \quad \|\mathcal{W}_{A_{\varepsilon}}^+(t_{0,\varepsilon})\|_* < r_{\varepsilon}\right)$$

$$(65) \quad + \mathbb{P}\left(\sup_{t \in [0, t_{0,\varepsilon}]} \|\mathcal{W}_{A_{\varepsilon}}^-(t)\|_* > \delta_{\varepsilon} \cdot r_{\varepsilon}\right).$$

The term on the right-hand side of (64) can be estimated using Corollary 3.9, provided we use $\alpha_k = \sigma_{\varepsilon} \cdot a_k \cdot \mu_k$. For more details we refer the reader to Remark 3.5 and the discussion of (51) and (52). Together with (53), Corollary 3.9 then furnishes an upper bound of the form $2 \cdot e^{-C \cdot \varepsilon^{-d}}$.

The term in (65) can be estimated using Lemma 3.11 with $T = t_{0,\varepsilon}$, $\alpha_k = \sigma_\varepsilon \cdot a_k \cdot \mu_k$, and $\lambda = \lambda_\varepsilon^{\max}$. Together with Lemma 4.8, we obtain for arbitrary $p > 1/(2\alpha)$ and $\zeta > 0$ the estimate

$$\begin{aligned} & \mathbb{P} \left(\sup_{t \in [0, t_{0,\varepsilon}]} \|\mathcal{W}_{A_\varepsilon}^-(t)\|_* > \delta_\varepsilon \cdot r_\varepsilon \right) \\ & \leq C \cdot \left(\frac{e^{2t_{0,\varepsilon} \cdot \lambda_\varepsilon^{\max} \cdot \gamma \cdot (1+3\zeta)}}{(\zeta \gamma \lambda_\varepsilon^{\max})^{2\alpha} \cdot \delta_\varepsilon^2 \cdot r_\varepsilon^2} \cdot \sum_{k \in \Lambda_\varepsilon} \frac{\sigma_\varepsilon^2 \cdot a_k^2 \cdot \mu_k^2}{(\lambda_\varepsilon^{\max} \gamma (1 + \zeta) - \lambda_{k,\varepsilon})^{1-2\alpha}} \right)^p \\ & \leq C \cdot \left(\frac{\sigma_\varepsilon^2}{\delta_\varepsilon^2 \cdot r_\varepsilon^2} \cdot \Theta_\varepsilon^{(1+3\zeta)\gamma/\gamma^+} \cdot \varepsilon^2 \cdot \Phi_\varepsilon \right)^p \\ & \leq C \cdot \left(\delta_\varepsilon^{-2} \cdot \Theta_\varepsilon^{-1+(1+3\zeta)\gamma/\gamma^+} \cdot \varepsilon^{2-D} \cdot \Phi_\varepsilon \right)^p. \end{aligned}$$

If we choose p sufficiently large and combine this estimate with the one for the term in (64) derived above, then the result follows immediately. $\square$

We now turn our attention to the nonlinear Cahn-Hilliard-Cook model (45). Analogously to (15) we consider the event

$$(66) \quad \mathcal{N}_{1,\varepsilon} := \left\{ u(t) \in \mathcal{Z}_{4\delta_\varepsilon r_\varepsilon, 2r_{0,\varepsilon}} \quad \text{for all } t \in [0, t_{0,\varepsilon}] , \quad \|u^+(t_{0,\varepsilon})\|_* \geq \frac{r_\varepsilon}{2} \right\}.$$

This event contains all $\omega \in \Omega$ corresponding to solution trajectories of (45) which exhibit the first stage of spinodal decomposition. By combining Theorem 2.5 with Proposition 4.9 we can show that the probability of the event $\mathcal{N}_{1,\varepsilon}$ is close to 1 as well.

Corollary 4.10. *Suppose that Assumptions 4.1, 4.2, 4.3, 4.4, and 4.7 are satisfied, let $\mathcal{N}_{1,\varepsilon}$ be defined as in (66), and let δ_ε be as in (57). Furthermore, suppose that with K_1 as in Lemma 4.5 we have*

$$-2 < D \leq 2, \quad r_\varepsilon < r_{0,\varepsilon} < \frac{K_1}{2\delta_\varepsilon}, \quad \text{and} \quad \delta_\varepsilon^{-2} \cdot \Theta_\varepsilon^{-1+(1+3\zeta)\gamma/\gamma^+} \in \mathcal{P}_\varepsilon$$

for some small $\zeta > 0$, as well as

$$(67) \quad \lim_{\varepsilon \rightarrow 0} r_\varepsilon^\kappa \cdot \delta_\varepsilon^{\kappa-1} \cdot \Theta_\varepsilon^{(1+\rho+(\kappa+1)(1-\gamma^+))/(2\gamma^+)} = 0.$$

Then for every $q > 0$ there exists a constant $C > 0$ such that

$$\mathbb{P}(\mathcal{N}_{1,\varepsilon}) \geq 1 - C \cdot \varepsilon^q$$

for all sufficiently small $\varepsilon > 0$.

Proof. Assumptions 4.1 through 4.4, in combination with Lemma 4.5, provide the necessary functional-analytic setting for Theorem 2.5. Moreover, the assumptions of Corollary 4.10 are such that we can apply Proposition 4.9 directly. In order to combine the latter result with Theorem 2.5, we only have to establish the validity of (16). Due to (54) this condition is equivalent to

$$\frac{C \cdot \varepsilon^{-1}}{((1+\rho) \cdot \lambda_\varepsilon^{\max})^{1/2}} \cdot r_{0,\varepsilon}^{\kappa+1} \cdot e^{(1+\rho) \cdot t_{0,\varepsilon} \cdot \lambda_\varepsilon^{\max}} \leq \delta_\varepsilon^{1-\kappa} \cdot r_\varepsilon,$$

and according to (62) this estimate is satisfied if and only if

$$C \cdot r_{0,\varepsilon}^{\kappa+1} \cdot \Theta_\varepsilon^{(1+\rho)/(2\gamma^+)} \leq \delta_\varepsilon^{1-\kappa} \cdot r_\varepsilon.$$

This last estimate, however, follows immediately from (63) and (67) for sufficiently small $\varepsilon > 0$. This completes the proof of the corollary. $\square$

With the above corollary the discussion of the first decomposition stage is complete. In order to study the second phase of spinodal decomposition, we have to consider the ε -dependent counterparts of the events defined in (18) and (20). Define the ε -dependent shifted Wiener process

$$\widetilde{\mathcal{W}}_\varepsilon(t) = \mathcal{W}(t + t_{0,\varepsilon}) - \mathcal{W}(t_{0,\varepsilon}) \quad \text{for all } t \geq 0,$$

and denote its associated stochastic convolution by

$$\widetilde{\mathcal{W}}_{A_\varepsilon}(t) = \sigma_\varepsilon \cdot \int_0^t e^{(t-\tau)A_\varepsilon} d\widetilde{\mathcal{W}}_\varepsilon(\tau).$$

See also (5) and (52). Then the solution u of (45) for $t \geq t_{0,\varepsilon}$ can be represented using the variation of constants formula (6). Moreover, we can introduce the analogue of the event Ω_2 from Section 2.3, which now takes the form

$$(68) \quad \begin{aligned} \Omega_{2,\varepsilon} = & \left\{ \left\| \widetilde{\mathcal{W}}_{A_\varepsilon}(t) \right\|_* \leq \frac{r_{0,\varepsilon}}{2} \cdot e^{t \cdot \lambda_\varepsilon^{\max}}, \quad \left\| e^{-tA_\varepsilon^+} \widetilde{\mathcal{W}}_{A_\varepsilon}^+(t) \right\|_* \leq \frac{r_\varepsilon}{4}, \right. \\ & \left. \text{and } \left\| \widetilde{\mathcal{W}}_{A_\varepsilon}^-(t) \right\|_* \leq \frac{r_\varepsilon \cdot \delta_\varepsilon}{2} \cdot e^{t \cdot \gamma \lambda_\varepsilon^{\max}} \text{ for all } t \geq 0 \right\}. \end{aligned}$$

The probability of this event will be determined in the following result.

Proposition 4.11. *Suppose that Assumptions 4.1, 4.2, and 4.7 are satisfied, let $\Omega_{2,\varepsilon}$ be defined as in (68), and let δ_ε be as in (57). Furthermore, suppose that*

$$-2 < D < 2, \quad \text{and} \quad \delta_\varepsilon^{-2} \cdot \Theta_\varepsilon^{-1} \in \mathcal{P}_\varepsilon.$$

Then for every $q > 0$ there exists a constant $C > 0$ such that

$$\mathbb{P}(\Omega_{2,\varepsilon}) \geq 1 - C \cdot \varepsilon^q$$

for all sufficiently small $\varepsilon > 0$.

Proof. As we mentioned in Remark 3.1, the stochastic convolutions $\mathcal{W}_{A_\varepsilon}$ and $\widetilde{\mathcal{W}}_{A_\varepsilon}$ induce the same probability measures. Thus, we can use the results from Section 3 directly, and we will drop the tilde for the remainder of the proof. Furthermore, due to Assumption 4.7 and (63) we have $r_\varepsilon < r_{0,\varepsilon}$ for sufficiently small $\varepsilon > 0$. Together with the semigroup estimates of Assumption 2.2, which are a consequence

of Assumptions 4.1 and 4.2, we obtain

$$\begin{aligned}
 \mathbb{P}(\Omega_{2,\varepsilon}^c) &\leq \mathbb{P}\left(\sup_{t \geq 0} e^{-t\lambda_\varepsilon^{\max}} \|\mathcal{W}_{A_\varepsilon}(t)\|_* > \frac{r_{0,\varepsilon}}{2}\right) + \mathbb{P}\left(\sup_{t \geq 0} \|e^{-tA_\varepsilon^+} \mathcal{W}_{A_\varepsilon}^+(t)\|_* > \frac{r_\varepsilon}{4}\right) \\
 &\quad + \mathbb{P}\left(\sup_{t \geq 0} e^{-t\gamma\lambda_\varepsilon^{\max}} \|\mathcal{W}_{A_\varepsilon}^-(t)\|_* > \frac{\delta_\varepsilon r_\varepsilon}{2}\right) \\
 &\leq \mathbb{P}\left(\sup_{t \geq 0} e^{-t\lambda_\varepsilon^{\max}} \|\mathcal{W}_{A_\varepsilon}^+(t)\|_* > \frac{r_{0,\varepsilon}}{4}\right) \\
 &\quad + \mathbb{P}\left(\sup_{t \geq 0} e^{-t\lambda_\varepsilon^{\max}} \|\mathcal{W}_{A_\varepsilon}^-(t)\|_* > \frac{r_{0,\varepsilon}}{4}\right) \\
 &\quad + \mathbb{P}\left(\sup_{t \geq 0} \|e^{-tA_\varepsilon^+} \mathcal{W}_{A_\varepsilon}^+(t)\|_* > \frac{r_\varepsilon}{4}\right) \\
 &\quad + \mathbb{P}\left(\sup_{t \geq 0} e^{-t\gamma\lambda_\varepsilon^{\max}} \|\mathcal{W}_{A_\varepsilon}^-(t)\|_* > \frac{\delta_\varepsilon r_\varepsilon}{2}\right) \\
 (69) \quad &\leq 2 \cdot \mathbb{P}\left(\sup_{t \geq 0} \|e^{-tA_\varepsilon^+} \mathcal{W}_{A_\varepsilon}^+(t)\|_* > \frac{r_\varepsilon}{4}\right) \\
 (70) \quad &\quad + 2 \cdot \mathbb{P}\left(\sup_{t \geq 0} e^{-t\gamma\lambda_\varepsilon^{\max}} \|\mathcal{W}_{A_\varepsilon}^-(t)\|_* > \frac{\delta_\varepsilon r_\varepsilon}{4}\right)
 \end{aligned}$$

for every sufficiently small $\varepsilon > 0$. Notice that for the last inequality we have to assume that $0 < \delta_\varepsilon < 1$, which is automatically satisfied for small ε due to $d \in \{1, 2, 3\}$.

The expression in (69) can easily be estimated. As in the proof of Proposition 4.9 we set $\alpha_k = \sigma_\varepsilon \cdot a_k \cdot \mu_k$. Then an application of Lemma 3.7 furnishes

$$(69) \leq 2 \cdot e^{|\Lambda_\varepsilon^+| \cdot (1 + (\ln 2)/2 - \mathcal{O}(\Theta_\varepsilon))} \leq 2 \cdot e^{-C \cdot \varepsilon^{-d}}$$

for sufficiently small $\varepsilon > 0$.

We now turn our attention to the term in (70). For this, let $\zeta > 0$, $0 < \alpha < 1/2$, as well as $p > 1/(2\alpha)$ be arbitrary. Then Corollary 3.12 implies with $\delta = \delta_\varepsilon \cdot r_\varepsilon/4$ the estimate

$$\begin{aligned}
 (71) \quad (70) &\leq 2 \cdot \exp\left(|\Lambda_\varepsilon^{-+}| \cdot \left(1 + \frac{\ln 2}{2} - C \cdot \frac{\delta_\varepsilon^2 \cdot r_\varepsilon^2}{|\Lambda_\varepsilon^{-+}|} \cdot \min_{k \in \Lambda_\varepsilon^{-+}} \frac{\lambda_{k,\varepsilon}}{\alpha_k^2}\right)\right) \\
 (72) \quad &\quad + C \cdot \left(\frac{\sigma_\varepsilon^2}{\varepsilon^{-4\alpha} \cdot \delta_\varepsilon^2 \cdot r_\varepsilon^2} \cdot \sum_{k \in \Lambda_\varepsilon^{--}} \frac{a_k^2 \cdot \mu_k^2}{(\lambda_\varepsilon^{\max} \cdot \gamma^- \cdot (1 + \zeta) - \lambda_{k,\varepsilon})^{1-2\alpha}}\right)^p.
 \end{aligned}$$

Recall that for Corollary 3.12 we had to introduce a further splitting of the set Λ_ε^- into Λ_ε^{-+} and Λ_ε^{--} , which is determined by the constant $\gamma^- = \gamma/(1+3\zeta)$. For more details we refer the reader to Assumption 3.10. According to this assumption, the set Λ_ε^{-+} corresponds to a small band of eigenvalues. Nevertheless, its size remains of the order $\mathcal{O}(\varepsilon^{-d})$. Moreover, one can easily verify that the eigenvalues μ_k and $\lambda_{k,\varepsilon}$ on the set Λ_ε^{-+} exhibit the same asymptotics as on Λ_ε^+ . Thus, we immediately obtain

$$(71) \leq 2 \cdot \exp\left(\mathcal{O}(\varepsilon^{-d}) \cdot \left(1 + \frac{\ln 2}{2} - \mathcal{O}(\delta_\varepsilon^2 \cdot \Theta_\varepsilon)\right)\right) \leq 2 \cdot e^{-C \cdot \varepsilon^{-d}}.$$

The term in (72) can be estimated using Lemma 4.8. Together with $-2 < D < 2$ this lemma furnishes

$$(72) \leq C \cdot \left(\frac{\sigma_\varepsilon^2}{\delta_\varepsilon^2 \cdot r_\varepsilon^2} \cdot \varepsilon^2 \cdot \Phi_\varepsilon \right)^p \leq C \cdot (\delta_\varepsilon^{-2} \cdot \Theta_\varepsilon^{-1})^p .$$

Since we assumed $\delta_\varepsilon^{-2} \cdot \Theta_\varepsilon^{-1} \in \mathcal{P}_\varepsilon$, this completes the proof of Proposition 4.11. $\square$

In order to complete our discussion of spinodal decomposition, we still have to apply the abstract Theorem 2.6. For this, let

$$(73) \quad T_{c,\varepsilon} = \inf \{ t > 0 : u(t + t_{0,\varepsilon}) \notin \mathcal{K}_{21\delta_\varepsilon, R_\varepsilon} \} ,$$

where u denotes the solution of (45), and the radius R_ε of the used cone is from now on an ε -dependent quantity. In addition, consider the event

$$(74) \quad \mathcal{N}_{2,\varepsilon} = \{ u(t + t_{0,\varepsilon}) \in \mathcal{K}_{21\delta_\varepsilon, R_\varepsilon} \text{ for } 0 \leq t \leq T_{c,\varepsilon} , \text{ and } \|u(T_{c,\varepsilon} + t_{0,\varepsilon})\|_* = R_\varepsilon \} .$$

The following result relates this event to $\Omega_{1,\varepsilon}$ and $\Omega_{2,\varepsilon}$.

Proposition 4.12. *Suppose that Assumptions 4.1, 4.2, 4.3, 4.4, and 4.7 are satisfied, and let δ_ε be as in (57). Furthermore, suppose that with K_1 as in Lemma 4.5 and $r_{0,\varepsilon}$ as in (63) we have*

$$(75) \quad r_{0,\varepsilon} < R_\varepsilon ,$$

and that (67), as well as

$$(76) \quad \lim_{\varepsilon \rightarrow 0} \delta_\varepsilon \cdot R_\varepsilon = 0$$

and

$$(77) \quad \lim_{\varepsilon \rightarrow 0} \delta_\varepsilon^{\kappa-1} \cdot R_\varepsilon^\kappa \cdot \Theta_\varepsilon^{(1-\gamma^+)/(2\gamma^+)} \cdot \left(\frac{R_\varepsilon}{r_\varepsilon} \right)^{(1+\rho+\vartheta-\gamma)/\gamma} = 0$$

are satisfied, for some small $\rho > 0$ and $\vartheta > 0$. Then for all sufficiently small $\varepsilon > 0$ we have

$$\Omega_{1,\varepsilon} \cap \Omega_{2,\varepsilon} \subset \mathcal{N}_{1,\varepsilon} \cap \mathcal{N}_{2,\varepsilon} ,$$

where $\Omega_{1,\varepsilon}$, $\mathcal{N}_{1,\varepsilon}$, $\Omega_{2,\varepsilon}$, and $\mathcal{N}_{2,\varepsilon}$ are defined as in (60), (66), (68), and (74), respectively.

Proof. Assumptions 4.1 through 4.4, in combination with (75) and Lemma 4.5, provide the necessary functional-analytic setting for Theorem 2.6. Notice that the assumption (76) furnishes $R_\varepsilon < K_1/(2\delta_\varepsilon)$ for sufficiently small $\varepsilon > 0$. Furthermore, the validity of (67) implies (16), as was already shown in Corollary 4.10. Thus, in order to apply Theorem 2.6, we only have to verify (21) and (22). The former estimate holds, provided we can find a small $\vartheta > 0$ such that

$$M_1 \cdot C_F \cdot \delta_\varepsilon^\kappa \cdot R_\varepsilon^\kappa \cdot \Gamma \left(\frac{1}{2} \right) \leq (\lambda_\varepsilon^{\max} \cdot \vartheta)^{1/2} ,$$

which is a consequence of (54) and (76). Similarly, the estimate (22) reduces in our setting to

$$C \cdot \delta_\varepsilon^\kappa \cdot R_\varepsilon^\kappa \cdot r_{0,\varepsilon} \cdot r_\varepsilon^{-1} \cdot \left(\frac{R_\varepsilon}{r_\varepsilon} \right)^{(1+\rho+\vartheta-\gamma)/\gamma} \leq \delta_\varepsilon .$$

Because of (63), this last estimate is a consequence of (77). This completes the proof of the proposition. $\square$

By combining Propositions 4.9, 4.11, and 4.12 one can see that the event $\mathcal{N}_{1,\varepsilon} \cap \mathcal{N}_{2,\varepsilon}$ has probability close to 1 for sufficiently small $\varepsilon > 0$. This event corresponds to trajectories of the solution u of (45) which leave the homogeneous state 0 through the thin cylinder $\mathcal{Z}_{4\delta_\varepsilon r_\varepsilon, 2r_{0,\varepsilon}}$, and then stay in the truncated cone $\mathcal{K}_{21\delta_\varepsilon R_\varepsilon}$ until they reach norm R_ε . In other words, this event characterizes both stages of spinodal decomposition.

4.3. Spinodal decomposition. In this final section we will combine and summarize the above results, and discuss the somewhat involved set of necessary conditions. We begin by stating the main result which establishes spinodal decomposition for the Cahn-Hilliard-Cook equation (1). For the remainder of this paper, u denotes a solution of (1) with mass $\int_G u \, dx / |G| = m$, and is no longer a solution of (45).

Theorem 4.13. *Consider the Cahn-Hilliard-Cook equation (1) and let u denote its mild solution satisfying*

$$(78) \quad u(0) = m ,$$

where m denotes an initial mass such that $f'(m) > 0$ holds. Suppose that Assumptions 4.1, 4.2, 4.3, and 4.4 are satisfied, and that the regularity exponent $s_{\mathcal{R}}$ satisfies

$$(79) \quad \frac{d}{2} < s_{\mathcal{R}} < 2 + \frac{d}{2} , \quad \text{or equivalently,} \quad D = 2s_{\mathcal{R}} - d - 2 \in (-2, 2) .$$

Let ζ , ρ , and ϑ be small positive constants such that

$$\begin{aligned} \tilde{\gamma} &:= \gamma \cdot (1 + 3\zeta) < \gamma^+ , \\ \tilde{\kappa} &:= \frac{\rho + (\kappa + 2) \cdot (1 - \gamma^+)}{2\gamma^+} > 0 , \quad \text{as well as} \\ \tilde{\rho} &:= \frac{1 + \rho + \vartheta - \gamma}{\gamma} > 0 \end{aligned}$$

holds. Let r_ε , R_ε , and σ_ε be positive ε -dependent constants, define $\Theta_\varepsilon = r_\varepsilon^2 \cdot \sigma_\varepsilon^{-2} \cdot \varepsilon^{-D}$, and assume that the following conditions hold with $\delta_\varepsilon = \delta_0 \cdot \varepsilon^{2-d/2}$ for some $\delta_0 > 0$:

$$(80) \quad \delta_\varepsilon^{-2} \cdot \Theta_\varepsilon^{-1+\tilde{\gamma}/\gamma^+} \in \mathcal{P}_\varepsilon ,$$

$$(81) \quad \lim_{\varepsilon \rightarrow 0} r_\varepsilon^\kappa \cdot \delta_\varepsilon^{\kappa-1} \cdot \Theta_\varepsilon^{\tilde{\kappa}+1/2} = 0 ,$$

$$(82) \quad \lim_{\varepsilon \rightarrow 0} R_\varepsilon \cdot \delta_\varepsilon = 0 ,$$

$$(83) \quad \lim_{\varepsilon \rightarrow 0} r_\varepsilon^{\tilde{\rho}} \cdot \delta_\varepsilon^{\kappa-1} \cdot R_\varepsilon^{\kappa+\tilde{\rho}} \cdot \Theta_\varepsilon^{(1-\gamma^+)/(2\gamma^+)} = 0 ,$$

$$(84) \quad \lim_{\varepsilon \rightarrow 0} r_\varepsilon \cdot R_\varepsilon^{-1} \cdot \Theta_\varepsilon^{(1-\gamma^+)/(2\gamma^+)} = 0 .$$

Then the solution of (1) satisfying (78) exhibits spinodal decomposition in the following sense. Using the notation from Sections 2.1 and 4.1, in particular (62)

and (63), let $\mathcal{S}_\varepsilon$ denote the subset of Ω for which u satisfies

$$u(t) \in m + \mathcal{Z}_{4\delta_\varepsilon r_\varepsilon, 2r_{0,\varepsilon}} \quad \text{for all } t \in [0, t_{0,\varepsilon}] , \quad \|u^+(t_{0,\varepsilon}) - m\|_* \geq \frac{r_\varepsilon}{2} ,$$

as well as

$$u(t+t_{0,\varepsilon}) \in m + \mathcal{K}_{21\delta_\varepsilon, R_\varepsilon} \quad \text{for all } 0 \leq t \leq T_{c,\varepsilon} , \quad \text{and} \quad \|u(T_{c,\varepsilon} + t_{0,\varepsilon}) - m\|_* = R_\varepsilon ,$$

where $T_{c,\varepsilon}$ denotes the (positive) first exit time of $u(\cdot + t_{0,\varepsilon})$ from the cone $m + \mathcal{K}_{21\delta_\varepsilon, R_\varepsilon}$. Then for any $q > 0$ there exists an ε -independent constant $C > 0$ such that

$$\mathbb{P}(\mathcal{S}_\varepsilon) \geq 1 - C \cdot \varepsilon^q$$

for all sufficiently small $\varepsilon > 0$.

Proof. The assertion of the theorem follows immediately from Propositions 4.9, 4.11, and 4.12, once we have established their assumptions. To this end, notice first that (61) is a consequence of (80), since $-1 + \tilde{\gamma}/\gamma^+ < 0$, which in turn follows from $\tilde{\gamma} < \gamma^+$. Thus, Assumption 4.7 is satisfied.

The assumptions of Proposition 4.9 follow immediately from (79) and (80). Since the latter condition also implies $\delta_\varepsilon^{-2} \cdot \Theta_\varepsilon^{-1} \in \mathcal{P}_\varepsilon$, this already establishes the requirements of Proposition 4.11.

We now turn our attention to Proposition 4.12. The inequality in (75) is an immediate consequence of (84), given (61) and the identity in (63), and we even have $r_\varepsilon < r_{0,\varepsilon} < R_\varepsilon$. Moreover, (67), (76), and (77) are immediate consequences of (81), (82), and (83), respectively. $\square$

Remark 4.14. We would like to add a few comments on the size of the parameters $\tilde{\gamma}$, $\tilde{\kappa}$, and $\tilde{\rho}$ in Theorem 4.13. In order to explain the typical solution patterns observed during spinodal decomposition we need to assume that both γ and γ^+ are close to 1; for more details, see [19, Section 4]. This implies that $\tilde{\gamma}$ has to be close to 1 as well. Furthermore, the parameter $\tilde{\rho}$ will be small. On the other hand, the parameter $\tilde{\kappa}$ could take small or large positive values, depending on the size of κ .

The conditions of Theorem 4.13 are fairly involved, and therefore we will spend the remainder of the paper discussing their implications for various values of κ . To simplify matters, let us assume that the relevant ε -dependent quantities are given as powers of ε in the form

$$r_\varepsilon = \varepsilon^{\varphi_r} , \quad R_\varepsilon = \varepsilon^{\varphi_R} , \quad \text{and} \quad \sigma_\varepsilon = \varepsilon^{\varphi_\sigma} .$$

Then the auxiliary variable Θ_ε defined in the theorem satisfies $\Theta_\varepsilon = \varepsilon^{\varphi_\Theta}$, where

$$\varphi_\Theta = 2\varphi_r - 2\varphi_\sigma - 2s_{\mathcal{R}} + d + 2 = 2\varphi_r - 2\varphi_\sigma - D .$$

Using the definition of δ_ε given in the theorem, one can now easily verify that (80) through (84) are equivalent to

$$(85) \quad \varphi_\Theta < -\frac{2\gamma^+}{\gamma^+ - \tilde{\gamma}} \cdot \left(2 - \frac{d}{2}\right),$$

$$(86) \quad \varphi_r > \frac{1 - \kappa}{\kappa} \cdot \left(2 - \frac{d}{2}\right) - \frac{1 + 2\tilde{\kappa}}{2\kappa} \cdot \varphi_\Theta,$$

$$(87) \quad \varphi_R > -\left(2 - \frac{d}{2}\right),$$

$$(88) \quad \varphi_R > \frac{1 - \kappa}{\kappa + \tilde{\rho}} \cdot \left(2 - \frac{d}{2}\right) + \frac{\tilde{\rho}}{\kappa + \tilde{\rho}} \cdot \varphi_r - \frac{1 - \gamma^+}{2\gamma^+(\kappa + \tilde{\rho})} \cdot \varphi_\Theta,$$

$$(89) \quad \varphi_R < \varphi_r + \frac{1 - \gamma^+}{2\gamma^+} \cdot \varphi_\Theta.$$

In terms of explaining spinodal decomposition, we are mainly interested in the exponents φ_R , φ_σ , and s_R . While the exponent φ_r does arise in the statement and the proof of Theorem 4.13, it is only an auxiliary quantity. From the form of estimates (85) through (89) it is not hard to see that one could eliminate the variable φ_r completely, and thereby obtain conditions exclusively for φ_R and the sum $\varphi_\sigma + s_R$. We will not pursue this idea further, since the resulting conditions are fairly complicated.

Rather, we use the estimates in the above form, which will enable us to choose the parameters one after another. Let us assume that $m = 0$ and $f(u) = u - u^{1+\kappa}$ in Theorem 4.13. Then the optimal values of φ_R can be determined as follows:

- Using (85), choose and fix the quantity φ_Θ . In order to achieve the best possible value for φ_R , we have to choose φ_Θ as large as possible, i.e.,

$$\varphi_\Theta \lesssim -\frac{2\gamma^+}{\gamma^+ - \tilde{\gamma}} \cdot \left(2 - \frac{d}{2}\right),$$

which generally will be a large negative number.

- Next, we use (86) to choose and fix the auxiliary exponent φ_r , which has to be chosen as small as possible. Thus, we have

$$\varphi_r \gtrsim \left(\frac{1 - \kappa}{\kappa} + \frac{\gamma^+(1 + 2\tilde{\kappa})}{\kappa(\gamma^+ - \tilde{\gamma})} \right) \cdot \left(2 - \frac{d}{2}\right),$$

which is a large positive number.

- The remaining three estimates (87), (88), and (89) then determine the possible range for φ_R . Ideally, we would like to be able to choose φ_R close to $-2 + d/2$ as given by (87), especially in light of the deterministic results in [23]. However, the lower bound given in (88) prevents this choice. One can easily show that the optimal choice of φ_R implied by (88) is given by

$$\begin{aligned} \varphi_R \gtrsim & \left(\frac{1 - \kappa}{\kappa + \tilde{\rho}} + \frac{\tilde{\rho}\gamma^+(1 + 2\tilde{\kappa})}{\kappa(\kappa + \tilde{\rho})(\gamma^+ - \tilde{\gamma})} + \frac{\tilde{\rho}(1 - \kappa)}{\kappa(\kappa + \tilde{\rho})} + \frac{1 - \gamma^+}{(\gamma^+ - \tilde{\gamma})(\kappa + \tilde{\rho})} \right) \\ & \cdot \left(2 - \frac{d}{2}\right). \end{aligned}$$

The upper bound in (89) usually does not pose a serious restriction and can easily be satisfied by choosing the involved constants appropriately.

- Using the above expressions, we can also determine the order of magnitude for the noise intensity exponent φ_σ , which is given by

$$\varphi_\sigma \gtrsim -\frac{D}{2} + \left(\frac{1-\kappa}{\kappa} + \frac{\gamma^+(1+\kappa+2\tilde{\kappa})}{\kappa(\gamma^+-\tilde{\gamma})} \right) \cdot \left(2 - \frac{d}{2} \right).$$

Considering that we assumed $|D| < 2$, this expression is generally a large positive number, which effectively limits the allowable intensity of the noise process.

By choosing the parameters $\gamma < \tilde{\gamma} < \gamma^+ < 1$ close enough to 1, as well as choosing $\tilde{\kappa} > 0$ and $\tilde{\rho} > 0$ sufficiently small, one can greatly simplify the above expression for φ_R . It can easily be shown that the last two terms in parentheses can be made arbitrarily small. Moreover, due to the definition of $\tilde{\rho}$ in Theorem 4.13, the expression $\tilde{\rho}\gamma^+ / (\gamma^+ - \tilde{\gamma})$ is always bigger than 1, yet it can be chosen arbitrarily close to 1. Altogether, we can obtain

$$\varphi_R \gtrsim \left(-1 + \frac{1}{\kappa} + \frac{1}{\kappa^2} \right) \cdot \left(2 - \frac{d}{2} \right).$$

This order is slightly worse than the one obtained in the deterministic case discussed in [23], which furnished the right-hand side of $(-1 + 1/\kappa) \cdot (2 - d/2)$. However, for large values of κ the deviation is very small. In addition, our stochastic result provides the first description of the complete decomposition process, including a satisfactory explanation of the transition between the two stages.

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