TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 360, Number 4, April 2008, Pages 2041–2075
S 0002-9947(07)04281-X
Article electronically published on October 17, 2007

SYMMETRIC MARKOV CHAINS ON $\mathbb{Z}^d$ WITH UNBOUNDED RANGE

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ABSTRACT. We consider symmetric Markov chains on $\mathbb{Z}^d$ where we do **not** assume that the conductance between two points must be zero if the points are far apart. Under a uniform second moment condition on the conductances, we obtain upper bounds on the transition probabilities, estimates for exit time probabilities, and certain lower bounds on the transition probabilities. We show that a uniform Harnack inequality holds if an additional assumption is made, but that without this assumption such an inequality need not hold. We establish a central limit theorem giving conditions for a sequence of normalized symmetric Markov chains to converge to a diffusion on $\mathbb{R}^d$ corresponding to an elliptic operator in divergence form.

1. Introduction

Let X_n be a symmetric Markov chain on $\mathbb{Z}^d$. We say that X_n has *bounded* range if there exists $K > 0$ such that $\mathbb{P}(X_{n+1} = y \mid X_n = x) = 0$ whenever $|y - x| \geq K$. The range is *unbounded* if for every K there exists x and y (depending on K) with $|x - y| > K$ such that $\mathbb{P}(X_{n+1} = y \mid X_n = x) > 0$. There is a great deal known about Markov chains on graphs when the chains have bounded range. The purpose of this paper is to obtain results for Markov chains on $\mathbb{Z}^d$ that have unbounded range.

Suppose C_{xy} is the conductance between x and y. We impose a condition on C_{xy} (see (A3) below) which essentially says that the C_{xy} satisfy a uniform second moment condition. Let Y_t be the continuous time Markov chain on $\mathbb{Z}^d$ determined by the C_{xy}, while X_n is the discrete time Markov chain determined by these conductances. The transition probabilities for the Markov chain X are defined by

$$\mathbb{P}^x(X_1 = y) = \frac{C_{xy}}{\sum_z C_{xz}},$$

while the process Y_t is the Markov chain that has the same jumps as X but where the times between jumps are independent exponential random variables. When (A3) holds, together with two very mild regularity conditions, we obtain upper bounds on the transition probabilities of the form

$$\mathbb{P}(Y_t = y \mid Y_0 = x) \leq ct^{-d/2}$$

Received by the editors August 30, 2005 and, in revised form, February 16, 2006.
2000 *Mathematics Subject Classification.* Primary 60J10; Secondary 60F05, 60J27.
The first author's research was partially supported by NSF grant DMS0244737.
The second author's research was partially supported by Ministry of Education, Japan, Grant-in-Aid for Scientific Research for Young Scientists (B) 16740052.

and some corresponding lower bounds when x and y are not too far apart. Unlike the case of bounded range, reasonable universal bounds of Gaussian type need not hold when the range is unbounded. We also obtain bounds on the exit probabilities $\mathbb{P}(\sup_{s\le t}|Y_s - x| > \lambda t^{1/2})$.

We say a uniform Harnack inequality holds for X if whenever h is nonnegative and harmonic for the Markov chain X in the ball $B(x_0, R)$ of radius $R > 1$ about a point x_0, then

$$h(x) \le Ch(y), \qquad |x - x_0|, |y - x_0| < R/2,$$

where C is independent of R. Even when X_n is a random walk, i.e., the increments $X_n - X_{n-1}$ form an independent identically distributed sequence, a uniform Harnack inequality need not hold. However, if we impose an additional strong assumption (see (A4)) on the conductances, then we can prove such a Harnack inequality.

We prove that if we have Markov chains $X^{(n)}$ on $\mathbb{Z}^d$ satisfying assumption (A3) uniformly in n, the sequence of processes $X_t^{(n)} = X_{[n^2t]}/n$ is tight in the space $D[0,\infty)$ of right continuous, left limit functions, and all subsequential limit points are continuous processes. Under an additional condition on the conductances (A5) (different than the one needed for the Harnack inequality), we then show that the $X_\cdot^{(n)}$ converge weakly as processes to the law of the diffusion corresponding to an elliptic operator

$$\mathcal{L}f(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}(\cdot)\frac{\partial f}{\partial x_j}(\cdot)\Big)(x)$$

in divergence form. The exact statement is given by Theorem 6.1.

In the case of bounded range Markov chains on $\mathbb{Z}^d$ some of our estimates have been obtained by [SZ], and we obviously owe a debt to that paper. Not all of their methods extend to the unbounded case, however. In particular,

(1) New techniques were needed to obtain the exit probability estimates.
(2) A new method was needed to obtain lower bounds for the process killed on exiting a ball. This method should apply in many other instances, and is of interest in itself.
(3) Harnack inequalities in the case of unbounded range are quite a bit more subtle, and this section is all new.
(4) In the proof of the central limit theorem, new methods were needed to handle the case of unbounded range. Moreover, even in the bounded range case our result covers more general situations.

There are many versions of central limit theorems that investigate the asymptotic behavior of $\sum_{i=1}^n f(X_i)$ when X_n is a symmetric Markov chain on a graph. These are quite different from the central limit theorem of this paper. Our formulation has much more in common with the work of Stroock and Varadhan [SV], Chapter 11. There they consider certain nonsymmetric chains and show convergence to the law of a diffusion corresponding to an operator in nondivergence form:

$$\mathcal{L}f(x) = \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial f}{\partial x_i}(x).$$

Our result is the analogue for symmetric chains and operators in divergence form.

The next section sets up the notation and framework and states the assumptions we need. Section 3 has the exit time and hitting time estimates, Section 4 has the lower bounds, and Section 5 discusses the Harnack inequality. Our central limit theorem is proved in Section 6.

The letter c with or without subscripts and primes will denote finite positive constants whose exact value is unimportant and which may change from line to line.

2. Framework

We let $|\cdot|$ be the Euclidean norm and $B(x,r) := \{y \in \mathbb{Z}^d : |x-y| < r\}$. We sometimes write $|A|$ for the cardinality of a set $A \subset \mathbb{Z}^d$.

For each $x, y \in \mathbb{Z}^d$ with $x \neq y$, let $C_{xy} \in [0,\infty)$ be such that $C_{xy} = C_{yx}$. We call C_{xy} the *conductance* between x and y. Throughout the paper, we assume the following:

(A1) There exist $c_1, c_2 > 0$ such that

$$c_1 \leq \nu_x := \sum_{y\in\mathbb{Z}^d} C_{xy} \leq c_2 \text{ for all } \ x \in \mathbb{Z}^d.$$

(A2) There exist $M_0 \geq 1, \delta > 0$ such that the following holds: for any $x, y \in \mathbb{Z}^d$ with $|x-y| = 1$, there exist $N \geq 2$ and $x_1, \dots, x_N \in B(x, M_0)$ such that $x_1 = x$, $x_N = y$ and $C_{x_i x_{i+1}} \geq \delta$ for $i = 1, \dots, N-1$.

(A3) There exists a decreasing function $\varphi : \mathbb{N} \to \mathbb{R}_+$ with $\sum_{i=1}^{\infty} i^{d+1}\varphi(i) < \infty$ such that

$$C_{xy} \leq \varphi(|x-y|) \qquad \text{for all } \ x, y \in \mathbb{Z}^d.$$

Note that (A1) and (A2) are very mild regularity conditions. (A1) prevents degeneracies, while (A2) says, roughly speaking, that the chain is locally irreducible in a uniform way. (A3) is the substantive assumption and says that the C_{xy} satisfy a uniform finite second moment condition. In fact, (A3) implies the following: there exists $C_0 > 0$ such that

$$\sup_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} |x-y|^2 C_{xy} \leq C_0. \tag{2.1}$$

To see this,

$$\begin{aligned}\sum_{y\in\mathbb{Z}^d} |x-y|^2 C_{xy} &\leq \sum_{y\in\mathbb{Z}^d} |x-y|^2 \varphi(|x-y|) \\ &= \sum_{i=0}^{\infty} \sum_{i<|x-y|\leq i+1} |x-y|^2 \varphi(|x-y|) \\ &\leq c_3 \sum_i (i+1)^2 \varphi(i)(i+1)^{d-1} < \infty\end{aligned} \tag{2.2}$$

for all $x \in \mathbb{Z}^d$, where (A3) is used in the last inequality.

Define a symmetric Markov chain by

$$\mathbb{P}^x(X_1 = y) = \frac{C_{xy}}{\nu_x} \text{ for all } \ x, y \in \mathbb{Z}^d.$$

Define $p_n(x,y) := \mathbb{P}^x(X_n = y)$ and $\bar{p}_n(x,y) = p_n(x,y)/\nu_y$. Note that $\bar{p}_n(x,y) = \bar{p}_n(y,x)$. By (A1), the ratio of $p_n(x,y)$ to $\bar{p}_n(x,y)$ is bounded above and below by positive constants.

Let $\mu_x \equiv 1$ for all $x \in \mathbb{Z}^d$ and for each $A \subset \mathbb{Z}^d$, define $\mu(A) = \sum_{y\in A} \mu_y = |A|$ and $\nu(A) = \sum_{y\in A} \nu_y$. Note that $L^2(\mathbb{Z}^d, \mu) = L^2(\mathbb{Z}^d, \nu)$ by (A1). Now, for each $f \in L^2(\mathbb{Z}^d, \mu)$, define

$$\begin{aligned}\mathcal{E}(f,f) &= \tfrac{1}{2} \sum_{x,y\in\mathbb{Z}^d} (f(x) - f(y))^2 C_{xy},\\ \mathcal{F} &= \{f \in L^2(\mathbb{Z}^d, \mu) : \mathcal{E}(f,f) < \infty\}.\end{aligned}$$

It is easy to check $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{Z}^d, \mu)$ and the generator is

$$\sum_{y\in\mathbb{Z}^d} (f(y) - f(x)) C_{xy}.$$

Let Y_t be the corresponding continuous time μ-symmetric Markov chain on $\mathbb{Z}^d$. Let $\{U_i^x : i \in \mathbb{N}, x \in \mathbb{Z}^d\}$ be an independent sequence of exponential random variables, where the parameter for U_i^x is ν_x, and that is independent of X_n, and define $T_0 = 0, T_n = \sum_{k=1}^n U_k^{X_{k-1}}$. Set $\widetilde{Y}_t = X_n$ if $T_n \le t < T_{n+1}$; it is well known that the laws of $\widetilde{Y}$ and Y are the same, and hence $\widetilde{Y}$ is a realization of the continuous time Markov chain corresponding to (a time change of) X_n. Note that by (A1), the mean exponential holding time at each point for $\widetilde{Y}$ can be controlled uniformly from above and below by a positive constant. Let $p(t, x, y)$ be the transition density for Y_t with respect to μ.

We now introduce several processes related to Y_t, needed in what follows. For each $D \ge 1$, let $\mathcal{S} = D^{-1}\mathbb{Z}^d$ and define the rescaled process as $V_t = D^{-1} Y_{D^2 t}$. Let μ^D be a measure on $\mathcal{S}$ defined by $\mu^D(A) = D^{-d}\mu(DA) = D^{-d}|A|$ for $A \subset \mathcal{S}$. We can easily show that the Dirichlet form corresponding to V_t is

$$\mathcal{E}^D(f,f) = \tfrac{1}{2} \sum_{x,y\in\mathcal{S}} (f(x) - f(y))^2 D^{2-d} C_{Dx,Dy},$$

and the infinitesimal generator of V_t is

$$\mathcal{A}^D f(x) = \sum_{y\in\mathcal{S}} (f(y) - f(x)) C_{Dx,Dy} D^2 = \sum_{y\in\mathcal{S}} (f(y) - f(x)) \frac{C_{Dx,Dy} D^{2-d}}{\mu_x^D},$$

for each $f \in L^2(\mathcal{S}, \mu^D)$, where we denote $\mu_x^D := \mu^D(\{x\}) = D^{-d}$ for each $x \in \mathcal{S}$. The heat kernel $p^D(t, x, y)$ for V_t with respect to μ^D can be expressed as

$$p^D(t,x,y) = D^d p(D^2 t, Dx, Dy) \text{ for all } x, y \in \mathcal{S}, t > 0. \tag{2.3}$$

For $\lambda \ge 1$, let W_t^λ be a process on $\mathcal{S}$ with the large jumps of V_t removed. More precisely, W_t^λ is a process whose Dirichlet form and infinitesimal generator are

$$\begin{aligned}\mathcal{E}^{D,\lambda}(f,f) &= \tfrac{1}{2} \sum_{\substack{x,y\in\mathcal{S}\\ |x-y|\le\lambda^{1/2}}} (f(x) - f(y))^2 D^{2-d} C_{Dx,Dy},\\ \mathcal{A}^\lambda f(x) &= \sum_{\substack{y\in\mathcal{S}\\ |x-y|\le\lambda^{1/2}}} (f(y) - f(x)) \frac{C_{Dx,Dy} D^2}{\mu_x^D}\end{aligned}$$

for each $f \in L^2(\mathcal{S}, \mu^D)$. We denote the heat kernel for W_t^λ by $p^{D,\lambda}(t, x, y)$, $x, y \in \mathcal{S}$.

3. Heat kernel estimates

3.1. **Nash inequality.** For $f \in L^2(\mathbb{Z}^d, \mu)$, let

$$\mathcal{E}_{NN}(f,f) = \tfrac{1}{2} \sum_{x,y \in \mathbb{Z}^d; |x-y|=1} (f(x) - f(y))^2,$$

which is the Dirichlet form for the simple symmetric random walk in $\mathbb{Z}^d$. We will prove the following Nash inequality.

Proposition 3.1. *There exists $c_1 > 0$ such that for any $f \in L^2(\mathbb{Z}^d, \mu)$,*

$$\|f\|_2^{2(1+2/d)} \le c_1 \mathcal{E}(f,f) \|f\|_1^{4/d}. \tag{3.1}$$

In particular,

$$p(t,x,y) \le c_1 t^{-d/2} \quad \text{for all} \quad x, y \in \mathbb{Z}^d, t > 0, \tag{3.2}$$

$$p^D(t,x,y) \le c_1 t^{-d/2} \quad \text{for all} \quad x, y \in \mathcal{S}, t > 0. \tag{3.3}$$

Remark 3.2. Since $p(t,x,y) = \mathbb{P}^x(Y_t = y)/\mu_y$, we have $p(t,x,y) \le 1/\mu_y$, so (3.2) is a crude estimate for small t. However, we will continue to use it since we are mainly interested in the large time asymptotics.

Proof. Note that the equivalence of (3.1) and (3.2) is a well-known fact (see [CKS]).

The Markov chain corresponding to $\mathcal{E}_{NN}$ is a (continuous time) simple random walk; let r_t be its transition probabilities. Since, as is well known, we have $r_t(x,x) \le ct^{-d/2}$, then by [CKS] we have

$$\|f\|_2^{2(1+2/d)} \le c_1 \mathcal{E}_{NN}(f,f) \|f\|_1^{4/d} \text{ for all } f \in L^2(\mathbb{Z}^d, \mu).$$

See also [SZ]. By (A2), there exists $c_2 > 0$ such that

$$\mathcal{E}_{NN}(f,f) \le c_2 \mathcal{E}(f,f) \text{ for all } f \in L^2(\mathbb{Z}^d, \mu).$$

Using these facts and (2.3), we have the desired result. □

3.2. **Exit time probability estimates.** In this subsection, we will obtain some exit time estimates. The argument presented here was first established in [BL1] and then extended and simplified in [CK], [HK].

Lemma 3.3. *There exists $c_1 > 0$ such that*

$$p^{D,\lambda}(t,x,y) \le c_1 \, t^{-\frac{d}{2}} \, \exp\left(-\lambda^{-\frac{1}{2}} |x-y|\right) \tag{3.4}$$

for all $t \in (0,1]$, $x, y \in \mathcal{S}$ and $\lambda \ge M_0^2$, where M_0 is given in (A2).

Proof. Since $\lambda \ge M_0^2$, by (A2), we have $\mathcal{E}_{NN}(f,f) \le c\mathcal{E}^{D,\lambda}(f,f)$ for all $f \in L^2(\mathbb{Z}^d, \mu)$. So we have (3.1) where $\mathcal{E}(f,f)$ is replaced by $\mathcal{E}^{1,\lambda}(f,f)$, and by a scaling argument we have

$$p^{D,\lambda}(t,x,y) \le c_1 t^{-d/2} \text{ for all } x, y \in \mathcal{S}, t > 0.$$

Thus by Theorem (3.25) of [CKS], we have

$$p^{D,\lambda}(t,x,y) \le c_1 \, t^{-\frac{d}{2}} \, \exp\left(-E(2t,x,y)\right) \tag{3.5}$$

for all $t \le 1$ and $x, y \in \mathcal{S}$, where

$$\begin{aligned} E(t,x,y) &= \sup\{|\psi(y) - \psi(x)| - t\,\Lambda(\psi)^2 : \Lambda(\psi) < \infty\}, \\ \Lambda(\psi)^2 &= \|e^{-2\psi}\Gamma_\lambda[e^\psi]\|_\infty \vee \|e^{2\psi}\Gamma_\lambda[e^{-\psi}]\|_\infty, \end{aligned}$$

and Γ_λ is defined by

$$\Gamma_\lambda[v](\xi) = \sum_{\substack{\eta\in\mathcal{S}\\ |\xi-\eta|\le\lambda^{1/2}}} (v(\eta)-v(\xi))^2 \frac{C_{D\eta,D\xi}D^2}{\mu_{D\xi}}, \qquad \xi\in\mathcal{S}. \tag{3.6}$$

Now let $\psi(\xi) = \lambda^{-1/2}(|\xi-x|\wedge|x-y|)$. Then, $|\psi(\eta)-\psi(\xi)|\le\lambda^{-1/2}|\eta-\xi|$, so that

$$(e^{\psi(\eta)-\psi(\xi)}-1)^2 \le |\psi(\eta)-\psi(\xi)|^2 e^{2|\psi(\eta)-\psi(\xi)|} \le c\lambda^{-1}|\eta-\xi|^2$$

for $\eta,\xi\in\mathcal{S}$ with $|\eta-\xi|\le\lambda^{1/2}$. Hence

$$\begin{aligned} e^{-2\psi(\xi)}\Gamma_\lambda[e^\psi](\xi) &= \sum_{\substack{\eta\in\mathcal{S}\\ |\xi-\eta|\le\lambda^{1/2}}} (e^{\psi(\eta)-\psi(\xi)}-1)^2 \frac{C_{D\eta,D\xi}D^2}{\mu_{D\xi}} \\ &\le \lambda^{-1} \sum_{\substack{\eta'\in\mathbb{Z}^d\\ |\xi'-\eta'|\le D\lambda^{1/2}}} |\eta'-\xi'|^2 \frac{C_{\eta',\xi'}}{\mu_{\xi'}} \le C' \end{aligned}$$

for all $\xi\in\mathcal{S}$ where (2.1) is used in the last inequality. We have the same bound when ψ is replaced by $-\psi$, so $\Lambda(\psi)^2\le C'^2$. Noting that $|\psi(y)-\psi(x)|\le\lambda^{-\frac12}\,|x-y|$, we see that (3.4) follows from (3.5). $\square$

We now prove the following exit time estimate for the process. For $A\subset\mathbb{Z}^d$ and a process Z_t on $\mathbb{Z}^d$, let

$$\tau=\tau_A(Z):=\inf\{t\ge0: Z_t\notin A\}, \qquad T_A=T_A(Z):=\inf\{t\ge0: Z_t\in A\}.$$

Proposition 3.4. *For $A>0$ and $0<B<1$, there exist $\gamma_i=\gamma_i(A,B)\in(0,1)$, $i=1,2$, such that for every $D>0$ and $x\in\mathbb{Z}^d$,*

$$\mathbb{P}^x\left(\tau_{B(x,\,AD)}(Y)<\gamma_1\,D^2\right) \le B, \tag{3.7}$$

$$\mathbb{P}^x\left(\tau_{B(x,\,AD)}(X)<\gamma_2\,D^2\right) \le B. \tag{3.8}$$

Proof. It follows from Lemma 3.3 that for $t\in[1/4,\,1]$ and $r>0$,

$$\mathbb{P}^x\left(|W_t^\lambda-x|\ge r\right) = \sum_{y\in\mathcal{S}:\,|y-x|>r} p^{D,\lambda}(t,x,y)\mu_y^D \le c_1\,I_{r,\lambda}, \tag{3.9}$$

where $I_{r,\lambda}:=e^{-\frac{r}{2}\lambda^{-\frac12}}$. Define $\sigma_r:=\inf\{t\ge0: |W_t^\lambda-W_0^\lambda|\ge r\}$. Then by (3.9) and the strong Markov property of W^λ at time σ_r,

$$\begin{aligned} \mathbb{P}^x(\sigma_r\le1/2) &\le \mathbb{P}^x\left(\sigma_r\le1/2 \text{ and } |W_1^\lambda-x|\le r/2\right)+\mathbb{P}^x\left(|W_1^\lambda-x|>r/2\right) \\ &\le \mathbb{P}^x\left(\sigma_r\le1/2 \text{ and } |W_1^\lambda-W_{\sigma_r}^\lambda|>r/2\right)+c_1\,I_{r/2,\lambda} \\ &= \mathbb{P}^x\left(1_{\{\sigma_r\le1/2\}}\mathbb{P}^{W_{\sigma_r}^\lambda}\left(|W_{1-\sigma_r}^\lambda-W_0^\lambda|>r/2\right)\right)+c_1\,I_{r/2,\lambda} \\ &\le \sup_{y\in B(x,r)^c}\sup_{s\le1/2}\mathbb{P}^y\left(|W_{1-s}^\lambda-y|>r/2\right)+c_1\,I_{r/2,\lambda}. \end{aligned}$$

Here in the second and the last inequalities, we used (3.9). By the strong Markov property of W^λ, for every $r > 0$,

$$\begin{aligned}
\mathbb{P}^x\left(\sup_{s\le 1}|W^\lambda_s - W^\lambda_0| > r\right) &\le \mathbb{P}^x(\sigma_r \le 1/2) + \mathbb{P}^x(1/2 < \sigma_r \le 1)\\
&\le c_2\, I_{r/2,\lambda} + \mathbb{P}^x(\sigma_{r/2} \le 1/2) + \mathbb{P}^x(\sigma_{r/2} > 1/2, \sigma_r \le 1)\\
&\le c_2\, I_{r/2,\lambda} + \mathbb{P}^x(\sigma_{r/2} \le 1/2) + \mathbb{E}^x\left[\mathbb{P}^{W^\lambda_{1/2}}(\sigma_{r/2} \le 1/2)\right]\\
&\le c_3\, I_{r/4,\lambda}. \qquad (3.10)
\end{aligned}$$

The constants $c_1, c_2, c_3 > 0$ above are independent of $D \ge 1$, $x \in \mathcal{S}$ and $\lambda \ge M_0^2$.

Now, define B^λ to be the infinitesimal generator of V_t with small jumps removed:

$$B^\lambda v(\xi) = \sum_{\substack{\eta\in\mathcal{S}\\ |\eta-\xi|>\lambda^{1/2}}} (f(\eta) - f(\xi))\frac{C_{D\eta,D\xi}D^2}{\mu_{D\xi}}. \qquad (3.11)$$

Recall that $\mathcal{A}^\lambda$ is the generator of W^λ. We see that $\mathcal{A}^\lambda + B^\lambda$ is the generator for V_t. Hence, if Q^V_t and $Q^{W^\lambda}_t$ are the semigroups associated with V_t and W^λ_t respectively, we have that

$$Q^V_t v = Q^{W^\lambda}_t v + \sum_{k=1}^\infty S^\lambda_k(t) v, \qquad v \in L^\infty(\mathcal{S}, \mu^D), \qquad (3.12)$$

where

$$S^\lambda_k(t) v = \int_0^t S^\lambda_{k-1}(s) B^\lambda Q^{W^\lambda}_{t-s} v\, ds, \qquad k \ge 1, \qquad (3.13)$$

with $S^\lambda_0(t) := Q^{W^\lambda}_t$ (see, for example, Theorem 2.2 in [Le]). Note that the series in (3.12) defines a bounded linear operator on $L^\infty(\mathcal{S}, \mu^D)$ for each $t > 0$; this can be seen as follows. First, by (2.2) and a simple calculation, we have

(3.14)

$$\sum_{\substack{\eta\in\mathcal{S}\\ |\eta-\xi|>\lambda^{1/2}}} \frac{C_{D\eta,D\xi}D^2}{\mu_{D\xi}} \le c_4 \sum_{\substack{y\in\mathbb{Z}^d\\ |y-x|>D\lambda^{1/2}}} C_{x,y}D^2 \le \frac{c_5}{D^2\lambda}\sum_{y\in\mathbb{Z}^d}|x-y|^2\varphi(|x-y|)D^2 \le \frac{c_8}{\lambda}.$$

Using this, we see that there exists $c_7 > 0$ independent of λ such that

$$\|B^\lambda v\|_\infty \le \frac{c_7}{\lambda}\|v\|_\infty.$$

Noting that $\|Q^{W^\lambda}_t v\|_\infty \le \|v\|_\infty$, by induction we have from (3.13) that

$$\|S^\lambda_k(t) v\|_\infty \le \frac{(c_8\lambda^{-1}\, t)^k}{k!}\|v\|_\infty, \quad t > 0,\ k \ge 1, \qquad (3.15)$$

and so the series above is bounded from $L^\infty(\mathcal{S}, \mu^D)$ to $L^\infty(\mathcal{S}, \mu^D)$ for each $t > 0$.

We will apply the above with $\lambda = M_0^2$. By (3.15), for any bounded function f on $\mathcal{S}$, we have

$$\|Q^V_t f - Q^{W^\lambda}_t f\|_\infty \le \sum_{k=1}^\infty \frac{(c_8\lambda^{-1}\, t)^k}{k!}\,\|f\|_\infty \le c_9\, t\, e^{c_9 t}\,\|f\|_\infty.$$

Applying this with f equal to the indicator of $(\overline{B(\xi,r)})^c$, it follows that there is a constant $c_{10}>0$ that is independent of $D\geq 1$ such that for every $\xi\in\mathcal{S}$ and every $t\leq 1$,

$$\mathbb{P}^\xi\left(|V_t-\xi|>r\right)\leq \mathbb{P}^\xi\left(|W_t^{M_0^2}-\xi|>r\right)+c_{10}\,t. \tag{3.16}$$

Applying the same argument we used in deriving (3.10), we conclude there are positive constants c_{11},c_{12} such that for $\xi\in\mathcal{S}$,

$$\mathbb{P}^\xi\left(\sup_{s\leq t}|V_s-\xi|>r\right)\leq c_{11}e^{-c_{12}r}+c_{11}\,t \qquad \text{for every } r>0 \text{ and } t\leq 1. \tag{3.17}$$

This implies that for every $x\in\mathbb{Z}^d$, $D'\geq 1$ and $r>0$,

$$\mathbb{P}^x\left(\sup_{s\leq D'^2\,t}|Y_s-x|>r\,D'\right)\leq c_{11}e^{-c_{12}r}+c_{11}\,t \qquad \text{for every } r>0 \text{ and } t\leq 1. \tag{3.18}$$

For $A>0$ and $B\in(0,1)$, we choose r_0 and t_0 so that $c_{11}e^{-c_{12}r_0}+c_{11}\,t_0<B$ and take $D=r_0D'/A$. Then, by (3.18),

$$\mathbb{P}^x\left(\sup_{s\leq\gamma_1\,D^2}|Y_s-x|\geq A\,D\right)\leq B \qquad \text{for every } D\geq r_0/A,$$

where $\gamma_1=(A/r_0)^2t_0$. For $D<r_0/A$, we have

$$\mathbb{P}(U_1>\gamma_1\frac{r_0^2}{A^2})\leq\mathbb{P}(U_1>\gamma_1D^2)\leq\mathbb{P}^x\left(\sup_{s\leq\gamma_1\,D^2}|Y_s-x|<A\,D\right), \tag{3.19}$$

where U_1 is an exponential random variable with parameter 1. By (A1), the left hand side of (3.19) is greater than $1-B$ if γ_1 is taken to be small. Thus, (3.7) is proved.

Now (3.8) can be proved in the same way as Theorem 2.8 in [BL1]. $\square$

4. Lower bounds and regularity for the heat kernel

We now introduce the space-time process $Z_s:=(U_s,V_s)$, where $U_s=U_0+s$. The filtration generated by Z satisfying the usual conditions will be denoted by $\{\widetilde{\mathcal{F}}_s;\,s\geq 0\}$. The law of the space-time process $s\mapsto Z_s$ starting from (t,x) will be denoted as $\mathbb{P}^{(t,x)}$. We say that a nonnegative Borel measurable function $q(t,x)$ on $[0,\infty)\times\mathcal{S}$ is *parabolic* in a relatively open subset B of $[0,\infty)\times\mathcal{S}$ if for every relatively compact open subset B_1 of B, $q(t,x)=\mathbb{E}^{(t,x)}\left[q(Z_{\tau_{B_1}})\right]$ for every $(t,x)\in B_1$, where $\tau_{B_1}=\inf\{s>0:\,Z_s\notin B_1\}$.

We denote by $\gamma:=\gamma(1/2,1/2)<1$ the constant in (3.7) corresponding to $A=B=1/2$. For $t\geq 0$ and $r>0$, we define

$$Q^D(t,x,r):=[t,t+\gamma r^2]\times(B(x,r)\cap\mathcal{S}),$$

where $B(x,r)=\{y\in\mathbb{R}^d:|x-y|\leq r\}$.

It is easy to see the following (see, for example, Lemma 4.5 in [CK] for the proof).

Lemma 4.1. *For each $t_0>0$ and $x_0\in\mathbb{Z}^d$, $q^D(t,x):=p^D(t_0-t,x,x_0)$ is parabolic on $[0,t_0)\times\mathcal{S}$.*

The next proposition provides a lower bound for the heat kernel and is the key step for the proof of the Hölder continuity of $p^D(t,x,y)$.

Proposition 4.2. *There exist $c_1 > 0$ and $\theta \in (0,1)$ such that if $|x-x_0|, |y-x_0| \le t^{1/2}$, $x, y, x_0 \in \mathbb{Z}^d$ and $r \ge t^{1/2}/\theta$, then*

$$\mathbb{P}^x(Y_t = y, \tau_{B(x_0,r)} > t) \ge c_1 t^{-d/2}.$$

To prove this we first need some preliminary propositions. A version of the following weighted Poincaré inequality can be found in Lemma 1.19 of [SZ]; we give an alternate proof.

Lemma 4.3. *For $D \ge 1$ and $l \in \mathbb{Z}^d$, let*

$$g_D(l) = c_1 \prod_{i=1}^d e^{-|l_i|/D},$$

where c_1 is determined by the equation $\sum_{l\in\mathbb{Z}^d} g_D(l) = D^d$. Then there exists $c_2 > 0$ such that

$$c_2 \Big\langle (f - \langle f\rangle_{g_D})^2 \Big\rangle_{g_D} |1 \le D^{2-d} \sum_{l\in\mathbb{Z}^d} g_D(l) \sum_{i=1}^d (f(D^{-1}(l+e^i)) - f(D^{-1}l))^2, \qquad f \in L^2(\mathcal{S}),$$

where

$$\langle f\rangle_{g_D} = D^{-d} \sum_{l\in\mathbb{Z}^d} f(D^{-1}l) g_D(l)$$

and e^i is the element of $\mathbb{Z}^d$ whose j-th component is 1 if $j = i$ and 0 otherwise.

Proof. A scaling argument shows that it suffices to consider only the $D = 1$ case. Because of the product structure, it is enough to consider the case when $d = 1$.

The weighted Poincaré inequality restricted to integers in $[-10, 10]$, i.e., where the sums are restricted to being over $\{-10, \dots, 10\}$, follows easily from the usual Poincaré inequality. We will prove our weighted Poincaré inequality for positive k and the same argument works for negative k. These facts together with the weighted Poincaré inequality on $[-10, 10]$ and standard techniques as in [Je] give us the weighted Poincaré inequality for all of $\mathbb{Z}$. So we restrict our attention to nonnegative k. Therefore all our sums below are over nonnegative integers.

Let $c_3 = (\sum_k e^{-k})^{-1}$, $\overline{f} := c_3 \sum_{l\in\mathbb{N}\cup\{0\}} f(l)e^{-l}$, and define

$$\begin{aligned}
I &= c_3^2 \sum_{k,\ell} (f(k) - f(\ell))^2 e^{-k} e^{-\ell}, \\
J_k &= c_3 \sum_{\ell>k} \sum_{m=k}^{\ell-1} \sum_{n=k}^{\ell-1} [f(m+1) - f(m)]\,[f(n+1) - f(n)] e^{-\ell}, \\
K &= c_3 \sum_n [f(n+1) - f(n)]^2 e^{-n}.
\end{aligned}$$

Note

$$I = 2c_3 \sum_k (f(k) - \overline{f})^2 e^{-k},$$

so we need to show $I \le c_4 K$. We have, since $f(k) - f(\ell) = 0$ when $k = \ell$,

$$\begin{aligned} I/c_3^2 &= 2\sum_k \sum_{\ell>k} (f(k) - f(\ell))^2 e^{-k} e^{-\ell} \\ &= 2\sum_k \sum_{\ell>k} \Big(\sum_{m=k}^{\ell-1} [f(m+1) - f(m)] \Big) \Big(\sum_{n=k}^{\ell-1} [f(n+1) - f(n)] \Big) e^{-k} e^{-\ell} \\ &= 2\sum_k J_k e^{-k}/c_3. \end{aligned}$$

We see that

$$\begin{aligned} J_k/c_3 &= \sum_{m\ge k} \sum_{n\ge k} \sum_{\ell > m\vee n} e^{-\ell}[f(m+1) - f(m)]\,[f(n+1) - f(n)] \\ &\le \sum_{m\ge k} \sum_{n\ge k} e^{-m\vee n}[f(m+1) - f(m)]\,[f(n+1) - f(n)] \\ &\le 2\sum_{m\ge k} \sum_{n\ge m} e^{-n}[f(m+1) - f(m)]\,[f(n+1) - f(n)] \\ &= 2\sum_{n\ge k} \sum_{m=k}^{n-1} e^{-n}[f(m+1) - f(m)]\,[f(n+1) - f(n)] \\ &\qquad +2\sum_{n\ge k} e^{-n}[f(n+1) - f(n)]^2 \\ &\le 2\sum_{n\ge k} e^{-n}[f(n+1) - f(n)]\,(f(n) - f(k)) + 2K. \end{aligned}$$

Hence

$$\begin{aligned} I &\le c_5\Big(\sum_k \sum_{n\ge k} e^{-n}[f(n+1) - f(n)]\,(f(n) - f(k))e^{-k} + \sum_k 2e^{-k}K \Big) \\ &\le c_6\Big(\sum_k \sum_{n\ge k} e^{-n}e^{-k}[f(n+1) - f(n)]^2 \Big)^{1/2} \Big(\sum_k \sum_{n\ge k} e^{-n}[f(n) - f(k)]^2 e^{-k} \Big)^{1/2} \\ &\qquad +c_6 K \\ &\le c_7 K^{1/2} I^{1/2} + c_6 K. \end{aligned}$$

This implies

$$I \le c_8 K$$

as required. □

The proof of the following lemma is similar to that of (1.16) in [SZ], but since we need some modifications, we will give the proof.

Lemma 4.4. *There is an $\varepsilon > 0$ such that*

$$p^D(t, D^{-1}k, D^{-1}m) \ge \varepsilon t^{-d/2}, \tag{4.1}$$

for all $D \ge 1$, $(t, k, m) \in (D^{-1}, \infty) \times \mathcal{S} \times \mathcal{S}$ with $|D^{-1}k - D^{-1}m| \le 2t^{1/2}$.

Proof. First, note that it is enough to prove the following: there is an $\varepsilon > 0$ such that

$$D^{-d} \sum_{l\in\mathbb{Z}^d} \log \Big(p^D(\tfrac{1}{2}, D^{-1}k, D^{-1}(l+m)) \Big) g_D(l) \ge \tfrac{1}{2}\log\varepsilon, \tag{4.2}$$

for all $D \geq 1$ and $k, m \in \mathbb{Z}^d$ with $|D^{-1}(k-m)| \leq 2$. Indeed, by the Chapman-Kolmogorov equation, symmetry, and the fact $g_D(j) \leq 1$ for all $j \in \mathbb{Z}^d$,

$$\begin{aligned} &p^D(1, D^{-1}k, D^{-1}m) \\ &\quad \geq D^{-d} \sum_j p^D(\tfrac{1}{2}, D^{-1}k, D^{-1}(j+k)) p^D(\tfrac{1}{2}, D^{-1}m, D^{-1}(j+k)) g_D(j). \end{aligned}$$

Thus, by Jensen's inequality, (4.2) gives

$$p^D(1, D^{-1}k, D^{-1}l) \geq \varepsilon, \qquad D \geq 1, |D^{-1}k - D^{-1}l| \leq 2.$$

By a simple scaling argument, this gives (4.1).

So we will prove (4.2). Set $u_t(l) = p^D(t, D^{-1}k, D^{-1}(l+m))$ and let

$$G(t) = D^{-d} \sum_{l \in \mathbb{Z}^d} \log(u_t(l)) g_D(l).$$

By Jensen's inequality, we see that $G(t) \leq 0$. Further,

$$G'(t) = D^{-d} \sum_{l \in \mathbb{Z}^d} \frac{\partial u}{\partial t}(l) \frac{g_D(l)}{u_t(l)} = -\mathcal{E}^D(u_t(D\,\cdot), \frac{g_D(D\,\cdot)}{u_t(D\,\cdot)}).$$

Next, note that the following elementary inequality holds (see (1.23) of [SZ] for the proof):

$$\Big(\frac{d}{b} - \frac{c}{a}\Big)(b-a) \leq -\frac{c \wedge d}{2}(\log b - \log a)^2 + \frac{(d-c)^2}{2(c \wedge d)}, \qquad a, b, c, d > 0.$$

Hence

$$\begin{aligned} G'(t) &= -\frac{D^{2-d}}{2} \sum_{l \in \mathbb{Z}^d} \sum_{e \in \mathbb{Z}^d} \Big(\frac{g_D(l+e)}{u_t(l+e)} - \frac{g_D(l)}{u_t(l)}\Big)\Big(u_t(l+e) - u_t(l)\Big) C_{l,l+e} \\ &\geq \frac{D^{2-d}}{2} \sum_{l \in \mathbb{Z}^d} \sum_{e \in \mathbb{Z}^d} \frac{g_D(l+e) \wedge g_D(l)}{2} \Big(\log u_t(l+e) - \log u_t(l)\Big)^2 C_{l,l+e} \\ &\quad -\frac{D^{2-d}}{2} \sum_{l \in \mathbb{Z}^d} \sum_{e \in \mathbb{Z}^d} \frac{|g_D(l+e) - g_D(l)|^2}{2(g_D(l+e) \wedge g_D(l))} C_{l,l+e} \\ &\geq cD^{2-d} \sum_{l \in \mathbb{Z}^d} \sum_{j=1}^d (g_D(l+e^j) \wedge g_D(l)) \Big(\log u_t(l+e^j) - \log u_t(l)\Big)^2 \\ &\quad -D^{2-d} \sum_{l \in \mathbb{Z}^d} \sum_{e \in \mathbb{Z}^d} \frac{|g_D(l+e) - g_D(l)|^2}{4(g_D(l+e) \wedge g_D(l))} C_{l,l+e}, \end{aligned}$$

where the last inequality is due to (A2) and the definition of g_D (here recall that e^i is in the element of $\mathbb{Z}^d$ whose j-th component is 1 if $j = i$ and 0 otherwise). Note $|g_D(l+e) - g_D(l)| \leq c_1 D^{-1}|e|(g_D(l+e) \wedge g_D(l))$. Thus

$$\begin{aligned} &D^{2-d} \sum_{l \in \mathbb{Z}^d} \sum_{e \in \mathbb{Z}^d} \frac{|g_D(l+e) - g_D(l)|^2}{4(g_D(l+e) \wedge g_D(l))} C_{l,l+e} \\ &\quad \leq c_2 D^{-d} \sum_l \sum_e C_{l,l+e}|e|^2 (g_D(l+e) \wedge g_D(l)) \\ &\quad \leq c_3 \Big(\sup_l \sum_e C_{l,l+e}|e|^2\Big) \cdot D^{-d} \sum_l g_D(l) = c_3 \Big(\sup_l \sum_e C_{l,l+e}|e|^2\Big) < c_4, \end{aligned}$$

where we used (A3) in the last inequality. Note also $\min_{1\le i\le d} g_D(l+e^i) \ge c_5 g_D(l)$. Combining these, we have

$$\begin{aligned} G'(t) &\ge c_6 D^{2-d} \sum_{l\in\mathbb{Z}^d}\sum_{j=1}^{d} \Big(\log u_t(l+e^j) - \log u_t(l)\Big)^2 g_D(l) - c_4 \\ &\ge c_7 D^{-d} \sum_l (\log u_t(l) - G(t))^2 g_D(l) - c_4, \end{aligned}$$

where we used Lemma 4.3 in the last inequality.

Next, for $\sigma > 0$, set $A_t(\sigma) = \{l \in \mathbb{Z}^d : u_t(l) \ge e^{-\sigma}\}$. Then, writing f^+ and f^- for the positive and negative parts of f, we have for each $\sigma > 0$,

$$\begin{aligned} D^{-d}\sum_l (\log u_t(l) - G(t))^2 g_D(l) &\ge D^{-d}\sum_l (-(\log u_t)^-(l) - G(t))^2 g_D(l) \\ &\ge \frac{G(t)^2}{2D^d} \sum_{l\in A_t(\sigma)} g_D(l) - \sigma^2, \end{aligned}$$

where we used the elementary inequality $(A+B)^2 \ge (A^2/2) - B^2$, $A, B \in \mathbb{R}$, in the last inequality. Thus, we have

$$G'(t) \ge c_8 I_{t,\sigma} G(t)^2 - (c_4 + \sigma^2), \tag{4.3}$$

where we let $I_{t,\sigma} = D^{-d}\sum_{l\in A_t(\sigma)} g_D(l)$. On the other hand, by (3.7) and scaling, we can find $r_0 > 2$ such that

$$D^{-d} \sum_{|D^{-1}l|\le r_0} p^D(t, D^{-1}k, D^{-1}(l+m)) \ge 1/2$$

for $D \ge 1, t \le 1,$ and $|D^{-1}(k-m)| \le 2$. In particular, if β is the smallest value of $g_D(\cdot)$ on $[-r_0, r_0]$, then for each $t \in [1/4, 1]$,

$$1/2 \le D^{-d} \sum_{|D^{-1}l|\le r_0} u_t(l) \le e^{-\sigma} r_0^d + (\sup_k |u_t(k)|) \cdot \frac{I_{t,\sigma}}{\beta}.$$

Thus by taking $\sigma = \log(4r_0^d)$ and using (3.3), we obtain $I_{t,\sigma} \ge c\beta$. Combining this with (4.3), there exists $0 < \delta < 1$ such that

$$G'(t) \ge \delta G(t)^2 - \delta^{-1}, \qquad D \ge 1, t \in [1/4, 1], \text{ and } |D^{-1}(k-m)| \le 2. \tag{4.4}$$

Now, by (4.4) and the mean value theorem,

$$G(1/2) - G(t) \ge -(4\delta)^{-1}, \qquad t \in [1/4, 1]. \tag{4.5}$$

We may assume $G(1/2) \le -5/(2\delta)$, since otherwise (4.2) is clear. Then, by (4.5) we have $G(t) \le -2\delta^{-1}$. So $\delta G(t)^2/2 - \delta^{-1} \ge \delta^{-1} > 0$. So, by (4.4) again,

$$G'(t) \ge \delta G(t)^2/2, \qquad t \in [1/4, 1].$$

But this means that

$$G(\tfrac12)^{-1} \le G(\tfrac12)^{-1} - G(\tfrac14)^{-1} = -\int_{1/4}^{1/2} \frac{G'(s)}{G^2(s)} ds \le -\frac{\delta}{8},$$

and therefore $G(1/2) \ge -8\delta^{-1}$. Thus (4.2) holds with $\varepsilon^{1/2} = \frac12 \exp(-8\delta^{-1})$. $\square$

Lemma 4.5. *Given $\delta > 0$ there exists κ such that if $x, y \in \mathbb{Z}^d$ and $C \subset \mathbb{Z}^d$ with* $\operatorname{dist}(x, C)$ *and* $\operatorname{dist}(y, C)$ *both larger than $\kappa t^{1/2}$, then*

$$\mathbb{P}^x(Y_t = y, T_C \le t) \le \delta t^{-d/2}.$$

Proof. By the strong Markov property we have

$$\begin{aligned}\mathbb{P}^x(Y_t = y, T_C \le t/2) &= \mathbb{P}^x(1_{\{T_C \le t/2\}}\mathbb{P}^{Y_{T_C}}(Y_{t-T_C} = y)) \\ &\le c_1(t/2)^{-d/2}\mathbb{P}^x(T_C \le t/2).\end{aligned}$$

In Proposition 3.4 let us choose $A = 1$ and $B = \delta/(4c_1 2^{d/2})$. If we take $\kappa > (2\gamma_1)^{-1/2}$, then Proposition 3.4 tells us that

$$\mathbb{P}^x(T_C \le t/2) \le \mathbb{P}^x(\tau_{B(x,\kappa t^{1/2})} \le t/2) \le B,$$

and then

$$\mathbb{P}^x(Y_t = y, T_C \le t/2) \le \frac{\delta}{2} t^{-d/2}. \tag{4.6}$$

We now consider $\mathbb{P}^x(Y_t = y, t/2 \le T_C \le t)$. If the first hitting time of C occurs between time $t/2$ and time t, then the last hitting time of C before time t happens after time $t/2$. So if $S_C = \sup\{s \le t : Y_s \in C\}$, then

$$\mathbb{P}^x(Y_t = y, t/2 \le T_C \le t) \le \mathbb{P}^x(Y_t = y, t/2 \le S_C \le t).$$

We claim that by time reversal,

$$\mathbb{P}^x(Y_t = y, t/2 \le S_C \le t) = \mathbb{P}^y(Y_t = x, T_C \le t/2). \tag{4.7}$$

To see this, observe by the symmetry of the heat kernel p, we have that if $t_i = (t/2) + it/(2n)$, then

$$\begin{aligned}&\mathbb{P}^x(Y_{t_k} = z_k, \ldots, Y_{t_{n-1}} = z_{n-1}, Y_{t_n} = y) \\ &= p(t_k, x, z_k)p(t/(2n), z_k, z_{k+1}) \cdots p(t/(2n), z_{n-1}, y) \\ &= \mathbb{P}^y(Y_{t/(2n)} = z_{n-1}, \ldots, Y_{t-t_k} = z_k, Y_t = x).\end{aligned}$$

If we sum over $z_k \in C$ and $z_{k+1}, \ldots, z_{n-1} \notin C$, we have

$$\begin{aligned}&\mathbb{P}^x(Y_{t_k} \in C, Y_{t_{k+1}} \notin C, \ldots, Y_{t_{n-1}} \notin C, Y_t = y) \\ &\quad = \mathbb{P}^y(Y_{t/(2n)} \notin C, \ldots, Y_{t-t_{k+1}} \notin C, Y_{t-t_k} \in C, Y_t = x).\end{aligned}$$

If we sum over k, this yields

$$\mathbb{P}^x(t/2 \le S_n' \le t, Y_t = y) = \mathbb{P}^y(0 \le T_n' \le t/2, Y_t = x),$$

where $S_n' = \sup\{t_k : Y_{t_k} \in C\}$ and $T_n' = \inf\{t_k : Y_{t_k} \in C\}$. Letting $n \to \infty$ proves (4.7).

Arguing as in the first part of the proof,

$$\mathbb{P}^y(Y_t = x, T_C \le t/2) \le \frac{\delta}{2} t^{-d/2}.$$

Therefore

$$\mathbb{P}^x(Y_t = y, t/2 \le T_C \le t) \le \frac{\delta}{2} t^{-d/2},$$

and combining with (4.6) proves the proposition. $\square$

Proof of Proposition 4.2. We have from Lemma 4.4 that there exists ε such that

$$p(t,x,y) \geq \varepsilon t^{-d/2}$$

if $|x-y| \leq 2t^{1/2}$. If we take $\delta = \varepsilon/2$ in Lemma 4.5, then provided $r > (\kappa+1)t^{1/2}$, we have

$$\mathbb{P}^x(Y_t = y, \tau_{B(x_0,r)} \leq t) \leq \frac{\varepsilon}{2} t^{-d/2}.$$

Subtracting,

$$\mathbb{P}^x(Y_t = y, \tau_{B(x_0,r)} > t) \geq \frac{\varepsilon}{2} t^{-d/2}$$

if $|x-y| \leq t^{1/2}$, which is equivalent to what we want. □

As a corollary of Proposition 4.2 we have

Corollary 4.6. *For each* $0 < \varepsilon < 1$, *there exists* $\theta = \theta(\varepsilon) \in (0,1)$ *with the following property: if* $D \geq 1$, $x, y \in \mathcal{S}$ *with* $|x-y| < t^{1/2}$, $r > cM_0/D$, $t \in [0, (\theta r)^2)$, *and* $\Gamma \subset B(y, t^{1/2}) \cap \mathcal{S}$ *satisfies* $\mu^D(\Gamma) t^{-d/2} \geq \varepsilon$, *then*

$$\mathbb{P}^x(V_t \in \Gamma \text{ and } \tau_{B(x,r)} > t) \geq c_1 \varepsilon. \tag{4.8}$$

Lemma 4.7. *For each* $0 < \delta < 1$, *there exists* $\gamma = \gamma_\delta \in (0,1)$ *such that for* $t > 0$, $r > cM_0/D$ *and* $x \in \mathcal{S}$, *if* $A \subset Q^D_\gamma(t,x,r) := [t, t+\gamma_\delta r^2] \times (B(x,r) \cap \mathcal{S})$ *satisfies* $m \otimes \mu^D(A)/m \otimes \mu^D(Q^D_\gamma(t,x,r)) \geq \delta$, *then*

$$\mathbb{P}^{(t,x)}(T_A(Z) < \tau_{Q^D_\gamma(t,x,r)}(Z)) \geq c_1 \delta.$$

Proof. For each $\delta > 0$, take $\gamma = \theta(\delta/4)^2$. Note that there exists $s = s_r \in [t + \delta\gamma r^2/4, t+\gamma r^2)$ such that

$$\mu^D(A_s) \geq \delta r^d/4 \ \geq \frac{\delta}{4}\Big(\frac{s-t}{\gamma}\Big)^{d/2} \geq \frac{\delta}{4}(s-t)^{d/2}, \tag{4.9}$$

where $A_s = \{(s,z) \in [0,\infty) \times \mathcal{S} : (s,z) \in A\}$. Indeed, if not, then

$$m \otimes \mu^D(A) \leq \delta\gamma r^{2+d}/4 + (\gamma - \delta\gamma/4)\cdot(\delta/4)\cdot r^{2+d} \leq \delta\gamma r^{2+d}/2,$$

which contradicts $m \otimes \mu^D(A) \geq \delta m \otimes \mu^D(Q^D_\gamma(t,x,r)) = \delta\gamma r^{2+d}$. Now, using this fact and Corollary 4.6 (with $\varepsilon = \delta/4$), we have

$$\begin{aligned} \mathbb{P}^{(t,x)}(T_A(Z) < \tau_{Q^D_\gamma(t,x,r)}(Z)) &\geq \mathbb{P}^{(t,x)}(V_{s-t}\circ\theta_t \in A_s \text{ and } \tau_{B(x,r)}\circ\theta_t > s-t) \\ &\geq c_1\delta/4, \end{aligned}$$

which completes the proof. □

We will also use the following Lévy system formula for Y (cf. Lemma 4.7 in [CK]).

Lemma 4.8. *Let* f *be a nonnegative measurable function on* $\mathbb{R}_+ \times \mathcal{S} \times \mathcal{S}$, *vanishing on the diagonal. Then for every* $t \geq 0$, $x \in \mathcal{S}$ *and a stopping time* T *of* $\{\mathcal{F}_t\}_{t\geq 0}$,

$$\mathbb{E}^x\left[\sum_{s\leq T} f((s, V_{s-}, V_s))\right] = \mathbb{E}^x\left[\int_0^T \sum_{y\in\mathcal{S}} f((s, V_s, y)) \frac{D^2 C_{DV_s, Dy}}{\mu_{Y_{D^2 s}}}\, ds\right].$$

Now we prove that the heat kernel $p^D(t,x,y)$ is Hölder continuous in (t,x,y), uniformly over D. For $(t,x)\in[0,\infty)\times\mathcal{S}$ and $r>0$ let $Q^D(t,x,r):=[t,\,t+\gamma r^2]\times(B(x,r)\cap\mathcal{S})$, where $\gamma:=\gamma(1/2,1/2)\wedge\gamma_{1/3}<1$. Here $\gamma(1/2,1/2)$ is the constant in (3.7) corresponding to $A=B=1/2$ and $\gamma_{1/3}$ is the constant in Lemma 4.7 corresponding to $\delta=1/3$.

The following theorem can be proved similarly to Theorem 4.1 in [BL2] and Theorem 4.14 in [CK]. We will write down the proof for completeness.

Theorem 4.9. *There are constants $c>0$ and $\beta>0$ (independent of R,D) such that for every $0<R$, every $D\geq 1$, and every bounded parabolic function q in $Q^D(0,x_0,4R)$,*

$$|q(s,x)-q(t,y)|\leq c\,\|q\|_{\infty,R}\,R^{-\beta}\,\Big(|t-s|^{1/2}+|x-y|\Big)^{\beta} \tag{4.10}$$

holds for $(s,x),\,(t,y)\in Q^D(0,x_0,R)$, where $\|q\|_{\infty,R}:=\sup_{(t,y)\in[0,\,\gamma(4R)^2]\times\mathcal{S}}|q(t,y)|$. In particular, for the transition density function $p^D(t,x,y)$ of V,

(4.11)
$$|p^D(s,x_1,y_1)-p^D(t,x_2,y_2)|\leq c\,t_0^{-(d+\beta)/2}\Big(|t-s|^{1/2}+|x_1-x_2|+|y_1-y_2|\Big)^{\beta},$$

for any $0<t_0<1$, $t,\,s\in[t_0,\,\infty)$ and $(x_i,y_i)\in\mathcal{S}\times\mathcal{S}$ with $i=1,2$.

Proof. Recall that $Z_s=(U_s,V_s)$ is the space-time process of V, where $U_s=U_0+s$. In the following, we suppress the superscript D from $Q^D(\cdot,\cdot,\cdot)$. Without loss of generality, assume that $0\leq q(z)\leq\|q\|_{\infty,R}=1$ for $z\in[0,\,\gamma\,(4R)^2]\times\mathcal{S}$. By Lemma 4.7, there is a constant $c_1>0$ such that if $x\in\mathcal{S}$, $0<r<1$ and $A\subset Q(t,x,r/2)$ with $\frac{m\otimes\mu^D(A)}{m\otimes\mu^D(Q(t,x,r/2))}\geq 1/3$, then

$$\mathbb{P}^{(t,x)}(T_A(Z)<\tau_r(Z))\geq c_1, \tag{4.12}$$

where $\tau_r:=\tau_{Q(t,x,r)}$. By Lemma 4.8 with $f(s,y,z)=1_{B(x,r)}(y)\,1_{\mathcal{S}\setminus B(x,s)}(z)$ and $T=\tau_r$, there is a constant $c_2>0$ such that if $s\geq 2r$,

(4.13)
$$\mathbb{P}^{(t,x)}(V_{\tau_r}\notin B(x,s))=\mathbb{E}^{\,(t,x)}\left[\int_0^{\tau_r}\sum_{y\in\mathcal{S}\setminus\overline{B(x,s)}}\frac{D^2C_{DV_v,Dy}}{\mu_{Y_{D^2v}}}\,dv\right]\leq\frac{c_2}{s^2}\mathbb{E}^{\,(t,x)}[\tau_r]\leq\frac{c_2r^2}{s^2}.$$

The first inequality of (4.13) is due to the following computation:

$$\begin{aligned}
&\sup_{z\in B(x,r)\cap\mathcal{S}}D^2\sum_{y\in\mathcal{S}\setminus\overline{B(x,s)}}C_{Dz,Dy}\\
&\quad\leq\sup_{z'\in B(Dx,Dr)}D^2\sum_{|z'-y'|\geq Ds/2}C_{z'y'}\leq D^2\sum_{i>Ds/2}\varphi(i)i^{d-1}\\
&\quad\leq\frac{4}{s^2}\sum_i\varphi(i)i^{d+1}\leq\frac{c}{s^2},
\end{aligned}$$

where (A3) is used in the last inequality. The last inequality of (4.13) is due to the fact $\mathbb{E}^{\,(t,x)}[\tau_r]\leq r^2$; this is clearly true since the time interval for $Q(t,x,r)$ is γr^2, which is less than r^2. ($\mathbb{E}^{\,x}\tau_{B(x_0,r)}\leq c_1r^2$ is also true; see Lemma 5.2 (a).) Let

$$\eta=1-\frac{c_1}{4}\quad\text{and}\quad\rho=\tfrac12\wedge\left(\frac{\eta}{2}\right)^{1/2}\wedge\left(\frac{c_1\,\eta}{8\,c_2}\right)^{1/2}.$$

Note that for every $(t,x)\in Q(0,x_0,R)$, q is parabolic in $Q(t,x,R)\subset Q(0,\,x_0,\,2R)$. We will show that

$$\sup_{Q(t,x,\rho^k R)} q - \inf_{Q(t,x,\rho^k R)} q \le \eta^k \qquad \text{for all } k. \tag{4.14}$$

For notational convenience, we write Q_i for $Q(t,x,\rho^i R)$ and τ_i for $\tau_{Q(t,x,\rho^i R)}$. Define

$$a_i = \inf_{Q_i} q \quad \text{and} \quad b_i = \sup_{Q_i} q.$$

Clearly $b_i - a_i \le 1 \le \eta^i$ for all $i\le 0$. Now suppose that $b_i - a_i \le \eta^i$ for all $i\le k$ and we are going to show that $b_{k+1}-a_{k+1}\le \eta^{k+1}$. Observe that $Q_{k+1}\subset Q_k$ and so $a_k\le q\le b_k$ on Q_{k+1}. Define

$$A' := \{z\in Q_{k+1} :\, q(z)\le (a_k+b_k)/2\}.$$

We may suppose $\frac{m\otimes\mu^D(A')}{m\otimes\mu^D(Q_{k+1})}\ge 1/2$, for if not, we use $1-q$ instead of q. Let A be a compact subset of A' such that $\frac{m\otimes\mu^D(A)}{m\otimes\mu^D(Q_{k+1})}\ge 1/3$. For any given $\varepsilon>0$, pick $z_1,z_2\in Q_{k+1}$ so that $q(z_1)\ge b_{k+1}-\varepsilon$ and $q(z_2)\le a_{k+1}+\varepsilon$. Then by (4.12)-(4.14),

$$\begin{aligned}
b_{k+1}-a_{k+1}-2\varepsilon &\le q(z_1)-q(z_2)\\
&= \mathbb{E}^{z_1}\left[q(Z_{T_A\wedge\tau_{k+1}})-q(z_2)\right]\\
&= \mathbb{E}^{z_1}\left[q(Z_{T_A})-q(z_2);\, T_A<\tau_{k+1}\right]\\
&\qquad +\mathbb{E}^{z_1}\big[q(Z_{\tau_{k+1}})-q(z_2);\, T_A>\tau_{k+1},\\
&\qquad\qquad Z_{\tau_{k+1}}\in Q_k\big]\\
&\qquad +\sum_{i=1}^{\infty}\mathbb{E}^{z_1}\big[q(Z_{\tau_{k+1}})-q(z_2);\, T_A>\tau_{k+1},\\
&\qquad\qquad Z_{\tau_{k+1}}\in Q_{k-i}\setminus Q_{k+1-i}\big]\\
&\le \left(\frac{a_k+b_k}{2}-a_k\right)\mathbb{P}^{z_1}(T_A<\tau_{k+1})\\
&\qquad +(b_k-a_k)\mathbb{P}^{z_1}(T_A>\tau_{k+1})\\
&\qquad +\sum_{i=1}^{\infty}(b_{k-i}-a_{k-i})\mathbb{P}^{z_1}(Z_{\tau_{k+1}}\notin Q_{k+1-i})\\
&\le (b_k-a_k)\left(1-\frac{\mathbb{P}^{z_1}(T_A<\tau_{k+1})}{2}\right)+\sum_{i=1}^{\infty}c_2\,\eta^k(\rho^2/\eta)^i\\
&\le (1-\frac{c_1}{2})\,\eta^k+2c_2\eta^{k-1}\rho^2\\
&\le (1-\frac{c_1}{2})\eta^k+\frac{c_1}{4}\eta^k\\
&= \eta^{k+1}.
\end{aligned}$$

Since ε is arbitrary, we have $b_{k+1}-a_{k+1}\le\eta^{k+1}$, and this proves (4.14).

For $z=(s,x)$ and $w=(t,y)$ in $Q(0,x_0,R)$ with $s\le t$, let k be the largest integer such that $|z-w|:=(\gamma^{-1}|t-s|)^{1/2}+|x-y|\le\rho^k R$. Then $\log(|z-w|/R)\ge$

$(k+1)\log\rho$, $w \in Q(s,x,\rho^k R)$ and

$$|q(z)-q(w)| \le \eta^k = e^{k\log\eta} \le c_3 \left(\frac{|z-w|}{R}\right)^{\log\eta/\log\rho}.$$

This proves (4.10) with $\beta = \log\eta/\log\rho$.

By (3.2) and Lemma 4.1, for every $0 < t_0 < 1$, $T_0 \ge 2$ and $y \in \mathcal{S}$, $q(t,x) := p^D(T_0 - t, x, y)$ is a parabolic function on $[0, T_0 - \frac{t_0}{2}] \times \mathcal{S}$ bounded above by $c_4\, t_0^{-d/2}$.

For each fixed $t_0 \in (0,1)$ and $T_0 \ge 2$, take R such that $\gamma R^2 = t_0/2$. Let $s,t \in [t_0, T_0]$ with $s > t$ and $x_1, x_2 \in \mathcal{S}$. Assume first that

$$|s-t|^{1/2} + |x_1 - x_2| < \gamma^{1/2} R = (t_0/2)^{1/2} \tag{4.15}$$

and so $(T_0 - t, x_2) \in Q(T_0 - s, x_1, R) \subset [0, T_0 - \frac{t_0}{2}) \times \mathcal{S}$. Applying (4.10) to the parabolic function $q(t,x)$ with $(T_0 - s, x_1)$, $(T_0 - t, x_2)$ and $Q(T_0 - s, x_1, R)$ in place of (s,x), (t,y) and $Q(0, x_0, R)$ there respectively, we have

$$|p^D(s,x_1,y) - p^D(t,x_2,y)| \le c\, t_0^{-(d+\beta)/2} (|t-s|^{1/2} + |x_1 - x_2|)^\beta. \tag{4.16}$$

By (3.3), the inequality (4.16) is true when (4.15) does not hold. So (4.16) holds for every $t, s \in [t_0, T_0]$ and $x_1, x_2 \in \mathcal{S}$ for all $T_0 \ge 2$. Inequality (4.11) now follows from (4.16) by the symmetry of $p(t,x,y)$ in x and y. □

5. Harnack inequality

A function h defined on $\mathbb{Z}^d$ is harmonic on a subset A of $\mathbb{Z}^d$ with respect to the Markov chain X if

$$\sum_z h(z)\mathbb{P}^x(X_1 = z) = h(x), \qquad x \in A.$$

Because the Markov chain may not have bounded range, h must be defined on all of $\mathbb{Z}^d$. In order to avoid h possibly being infinite in A, we will assume that h is bounded on $\mathbb{Z}^d$, but in what follows, the constants do not depend at all on the L^∞ bound on h. We say h is harmonic with respect to Y if $h(Y_{t\wedge\tau_A})$ is a $\mathbb{P}^x$-martingale for each $x \in \mathbb{Z}^d$, where $\tau_A = \inf\{t : Y_t \notin A\}$. It is not hard to see that a function is harmonic for X if and only if it is harmonic for Y, since the hitting probabilities of X and Y are the same. Also, because the state space is discrete, it is routine to see that a function is harmonic in a domain A if and only if $\mathcal{E}(h,f) = 0$ for all bounded f supported in A; we will not use this latter fact.

In this section we first give an example of a symmetric random walk, i.e., where $\{X_{n+1} - X_n\}$ are symmetric i.i.d. random variables, for which a uniform Harnack inequality fails. Note that the Harnack inequality does hold for each ball of radius n, but not with a constant independent of n. Our example is similar to one in [LP]. Let e^j be the unit vector in the x_j direction, $j = 1, \ldots, d$.

Let $b_n = n^{n^n}$ (or any other quickly growing sequence), and let a_n be a sequence of positive numbers tending to 0, subject only to $\sum a_n \le 1/32$ and $\sum_n a_n b_n^2 < \infty$. Let $\varepsilon = 2\sum a_n$. Let ξ_i be an i.i.d. sequence of random vectors on $\mathbb{Z}^d$ with

$$\mathbb{P}^0(\xi_1 = \pm e^j) = (1-\varepsilon)/(2d).$$

Let $\mathbb{P}^0(\xi_1 = \pm b_n e^1) = a_n$. Let $X_n = \sum_{i=1}^n \xi_i$.

Now let $\delta \in (0,1)$, $r_n = (1-\delta)b_n$, $z_n = (b_n, 0)$, $B_n = B(0, r_n)$, $\tau_n = \min\{k : X_k \notin B_n\}$, and $T_0 = \min\{k : X_k = 0\}$. Define

$$h_n(x) = \mathbb{P}^x(X_{\tau_n} = z_n).$$

Each h_n is a harmonic function in B_n. If a uniform Harnack inequality were to hold, there would exist C not depending on n such that

$$h_n(0)/h_n(y) \le C, \qquad y \in B(0, r_n/2).$$

Since $\delta b_n \gg b_{n-1}$ for n large, the only way X_{τ_n} can equal z_n is if the random walk jumps from 0 to z_n. So for $y_n \in B_n$, $y_n \neq 0$,

$$h_n(y_n) = \mathbb{P}^{y_n}(T_0 < \tau_n) h_n(0).$$

But we claim that if $y_n \sim r_n/4$, then $\mathbb{P}^{y_n}(T_0 < \tau_n)$ will tend to 0 when $n \to \infty$, and then $h_n(0)/h_n(y_n) \to \infty$. So no uniform Harnack inequality exists.

The claim is true in all dimensions greater than or equal to 2, but is easier to prove when $d \ge 3$, so we concentrate on this case. We have

$$\begin{aligned} \mathbb{P}^{y_n}(T_0 < \tau_n) &\le \mathbb{P}^{y_n}(T_0 < \infty) = \mathbb{P}^{y_n}(T_0 < r_n^{1/4}) + \mathbb{P}^{y_n}(T_0 \ge r_n^{1/4}) \\ &\le \mathbb{P}^{y_n}\big(\max_{i \le r_n^{1/4}} |X_i - X_0| \ge |y_n|\big) + \sum_{i=[r_n^{1/4}]}^{\infty} \mathbb{P}^{y_n}(X_i = 0). \end{aligned}$$

The first term on the last line goes to 0 by Doob's inequality (applied to each (X_i, e^j), $j = 1, \dots, d$). By Spitzer [Sp], p. 75, the sum above is bounded by

$$c \sum_{i=[r_n^{1/4}]}^{\infty} \frac{1}{i^{d/2}} \le c'(r_n^{1/4})^{1-(d/2)},$$

which goes to 0 as $n \to \infty$.

Note that by taking a_n tending to 0 fast enough, ξ_1 can be made to be sub-Gaussian, or have even better tails.

As this example shows, a uniform Harnack inequality need not hold when the range is unbounded, so an additional assumption is needed to handle this case. The assumption is modeled after [BK] and the proof is similar to the one in [BL2]. We assume

(A4) There exists a constant c_1 such that $C_{xy} \le c_1 C_{xy'}$ whenever $|y - y'| \le |x - y|/3$.

Lawler [Law] proved that the Harnack inequality holds for a class of symmetric random walks with bounded range and also for a class of Markov chains with bounded range which are in general not reversible. See also [LP] for some results concerning random walks with unbounded range. Theorem 5.1 says that the Harnack inequality continues to be true for symmetric Markov chains with bounded range and for symmetric Markov chains satisfying (A4).

Theorem 5.1. *Suppose (A1)–(A3) hold. Suppose either (A4) holds or else the Markov chain has bounded range. Suppose $x_0 \in \mathbb{Z}^d$ and $R > M_0$, where M_0 is defined in (A2). There exists a constant c_1 such that if h is nonnegative and bounded on $\mathbb{Z}^d$ and harmonic on $B(x_0, 2R)$, then*

$$h(x) \le c_1 h(y), \qquad x, y \in B(x_0, R). \tag{5.1}$$

Before proving Theorem 5.1 we prove a lemma. Note that (A4) is not needed for this lemma.

Lemma 5.2. *(a) $\mathbb{E}^x \tau_{B(x_0,r)} \le c_1 r^2$.*

(b) There exist $\theta \in (0, 1/2)$ and $c_2, c_3 > 0$ such that if $r > M_0/\theta$, then $\mathbb{P}^x(\tau_{B(x_0,r)} \ge r^2) \ge c_2$ and $\mathbb{E}^x \tau_{B(x_0,r)} \ge c_3 r^2$ if $x \in B(x_0, \theta r)$.

Proof. If $p(t,x,y)$ denotes the transition densities for Y_t, we know

$$p(t,x,y) \le c_4 t^{-d/2}.$$

So if we take $t = c_5 r^2$ for large enough c_5, then

$$\mathbb{P}^x(Y_t \in B(x_0,r)) = \sum_{z\in B(x_0,r)} p(t,x,z) \le c_6 t^{-d/2}|B(x_0,r)| \le \tfrac{1}{2}.$$

This implies

$$\mathbb{P}^x(\tau_{B(x_0,r)} > t) \le \tfrac{1}{2}.$$

By the Markov property, for m a positive integer

$$\begin{aligned}\mathbb{P}^x(\tau_{B(x_0,r)} > (m+1)t) &\le \mathbb{E}^x[\mathbb{P}^{Y_{mt}}(\tau_{B(x_0,r)} > t); \tau_{B(x_0,r)} > mt]\\ &\le \tfrac{1}{2}\mathbb{P}^x(\tau_{B(x_0,r)} > mt).\end{aligned}$$

By induction,

$$\mathbb{P}^x(\tau_{B(x_0,r)} > mt) \le 2^{-m},$$

and (a) follows.

We also know by Proposition 4.2 that there exists $\kappa > 1$ such that

$$\mathbb{P}^x(Y_t = y, \tau_{B(x_0,r)} > t) \ge c_7 t^{-d/2}$$

if $|x-x_0|, |y-x_0| \le t^{1/2}$ and $r \ge \kappa t^{1/2}$. Therefore taking $t = r^2/\kappa^2$,

$$\mathbb{P}^x(\tau_{B(x_0,r)} > t) \ge \mathbb{P}^x(Y_t \in B(x_0,t^{1/2}), \tau_{B(x_0,r)} > t) \ge c_7 t^{-d/2}|B(x_0,t^{1/2})| \ge c_8$$

if $x \in B(x_0, r/\kappa)$. Let $\theta = 1/\kappa$. So $\mathbb{E}^x \tau_{B(x_0,r)} \ge t\mathbb{P}^x(\tau_{B(x_0,r)} > t) \ge c_8 r^2$, which proves (b). $\square$

Proof of Theorem 5.1. Let κ and θ be as in Lemma 5.2. Since we have (A1) and (A2), it is easy to check that a Harnack inequality holds for each finite R, provided $R \le 32M_0/\theta$. So it suffices to assume $R > 32M_0/\theta$. If the Markov chain has bounded range, choose L so that $C_{xy} = 0$ whenever $|x-y| \ge L$ and assume $R > (32M_0/\theta) \vee (2L)$.

First of all, if $z_1 \in \mathbb{Z}^d$ and $w \notin B(z_1, 2r)$, by the Lévy system formula,

$$\mathbb{E}^x \sum_{s \le \tau_{B(z_1,r)}\wedge t} 1_{(Y_{s-}\in B(z_1,r), Y_s = w)} = \mathbb{E}^x \int_0^{\tau_{B(z_1,r)}\wedge t} C_{Y_s,w}\, ds.$$

Letting $t \to \infty$, we have

$$\mathbb{P}^x(Y_{\tau_{B(z_1,r)}} = w) = \mathbb{E}^x \int_0^{\tau_{B(z_1,r)}} C_{Y_s,w}\, ds.$$

If (A4) holds, $c_2 C_{z_1,w} \le C_{Y_s,w} \le c_3 C_{z_1,w}$ when $Y_s \in B(z_1,r)$. So the right hand side is bounded above by the quantity $c_3 C_{z_1 w}\mathbb{E}^x \tau_{B(z_1,r)}$ and below by the quantity $c_2 C_{z_1 w}\mathbb{E}^x\tau_{B(z_1,r)}$. By Lemma 5.2, if $x,y \in B(z_1,\theta r)$, then $\mathbb{E}^x\tau_{B(z_1,r)} \le c_4 \mathbb{E}^y \tau_{B(z_1,r)}$. We conclude

$$\mathbb{P}^x(Y_{\tau_{B(z_1,r)}} = w) \le c_5 \mathbb{P}^y(Y_{\tau_{B(z_1,r)}} = w).$$

Taking linear combinations, if H is a bounded function supported in $B(z_1,2r)^c$, then

$$\mathbb{E}^x H(Y_{\tau_{B(z_1,r)}}) \le c_5 \mathbb{E}^y H(Y_{\tau_{B(z_1,r)}}), \qquad x,y \in B(z_1,\theta r). \tag{5.2}$$

Choose $r_0 = 16M_0/\theta$.

If, on the other hand, the Markov chain has bounded range and $r \geq L$, then (5.2) again holds because both sides are zero. In the bounded range case set $r_0 = (16M_0/\theta) \vee L$.

If $r \geq r_0$, then setting $t = r^2/\kappa^2$,

$$\mathbb{P}^x(Y_t = y, \tau_{B(z_1,r)} > t) \geq c_6 t^{-d/2}, \qquad x, y \in B(z_1, \theta r).$$

Summing over $A \subset B(z_1, \theta r)$, we see that

$$\begin{aligned} \mathbb{P}^x(T_A < \tau_{B(z_1,r)}) &\geq \mathbb{P}^x(Y_t \in A, \tau_{B(z_1,r)} > t) \\ &\geq c_6 |A| t^{-d/2} = c_6 |A| r^{-d}, \qquad x \in B(z_1, \theta r). \end{aligned} \tag{5.3}$$

In particular, note that if $C \subset B(z_1, \theta r)$ and $|C|/|B(z_1, \theta r)| \geq 1/3$, then

$$\mathbb{P}^x(T_C < \tau_{B(z_1,r)}) \geq c_7, \qquad x \in B(z_1, \theta r). \tag{5.4}$$

Next suppose $x, y \in B(z_1, \theta r_0)$. In view of (A2)

$$\mathbb{P}^x(T_{\{y\}} < \tau_{B(z_1,r_0)}) \geq c_8.$$

By optional stopping,

$$\begin{aligned} h(x) &\geq \mathbb{E}^x[h(Y_{T_{\{y\}}}); T_{\{y\}} < \tau_{B(z_1,r_0)}] \\ &= h(y) \mathbb{P}^x(T_{\{y\}} < \tau_{B(z_1,r_0)}) \\ &\geq c_8 h(y). \end{aligned}$$

By looking at a constant multiple of h, we may assume $\inf_{B(x_0,\theta R/2)} h = 1$. Choose $z_0 \in B(x_0, \theta R/2)$ such that $h(z_0) = 1$. We want to show that h is bounded above in $B(x_0, \theta R/2)$ by a constant not depending on h. This will show

$$h(y) \leq c_9 h(x), \qquad x, y \in B(x_0, \theta R/2). \tag{5.5}$$

Once we have (5.5) a standard chain of balls argument yields our theorem.

Let

$$\eta = \frac{c_7}{3}, \qquad \zeta = \frac{1}{3} \wedge (c_5^{-1} \eta) \wedge c_8. \tag{5.6}$$

Now suppose there exists $x \in B(x_0, \theta R/2)$ with $h(x) = K$ for some K large. Let r be chosen so that

$$2R^d/(c_6 \zeta K) \leq |B(x_0, \theta r)| \leq 4R^d/(c_6 \zeta K). \tag{5.7}$$

Note this implies

$$r \leq c_{10} K^{-1/d} R. \tag{5.8}$$

Without loss of generality we may assume K is large enough that $r \leq \theta R/4$. Let

$$A = \{w \in B(x, \theta r) : h(w) \geq \zeta K\}. \tag{5.9}$$

By (5.3) and optional stopping,

$$\begin{aligned} 1 \geq h(z_0) &\geq \mathbb{E}^{z_0}[h(Y_{T_A \wedge \tau_{B(x_0,2R)}}); T_A < \tau_{B(x_0,2R)}] \\ &\geq \zeta K \mathbb{P}^{z_0}(T_A < \tau_{B(x_0,2R)}) \\ &\geq c_6 \zeta K |A| R^{-d}, \end{aligned}$$

hence

$$\frac{|A|}{|B(x, \theta r)|} \leq \frac{R^d}{c_6 \zeta K |B(x, \theta r)|} \leq \frac{1}{2}. \tag{5.10}$$

Let C be a set contained in $B(x,\theta r)\setminus A$ such that

$$\frac{|C|}{|B(x,\theta r)|}\ge\frac{1}{3}. \tag{5.11}$$

Let $H=h1_{B(x,2r)^c}$. We claim

$$\mathbb{E}^x[h(Y_{\tau_{B(x,r)}});Y_{\tau_{B(x,r)}}\notin B(x,2r)]\le\eta K.$$

If not

$$\mathbb{E}^x H(Y_{\tau_{B(x,r)}})>\eta K,$$

and by (5.2), for all $y\in B(x,\theta r)$,

$$\begin{aligned}h(y)&\ge\mathbb{E}^y h(Y_{\tau_{B(x,r)}})\ge\mathbb{E}^y[h(Y_{\tau_{B(x,r)}});Y_{\tau_{B(x,r)}}\notin B(x,2r)]\\&\ge c_5^{-1}\mathbb{E}^x H(Y_{\tau_{B(x,r)}})\ge c_5^{-1}\eta K\\&\ge\zeta K,\end{aligned}$$

contradicting (5.11) and the definition of A.

Let $N=\sup_{B(x,2r)}h(z)$. We then have

$$\begin{aligned}K=h(x)&=\mathbb{E}^x[h(Y_{T_C});T_C<\tau_{B(x,r)}]\\&\qquad+\mathbb{E}^x[h(Y_{\tau_{B(x,r)}});\tau_{B(x,r)}<T_C,Y_{\tau_{B(x,r)}}\in B(x,2r)]\\&\qquad+\mathbb{E}^x[h(Y_{\tau_{B(x,r)}});\tau_{B(x,r)}<T_C,Y_{\tau_{B(x,r)}}\notin B(x,2r)]\\&\le\zeta K\mathbb{P}^x(T_C<\tau_{B(x,r)})+N\mathbb{P}^x(\tau_{B(x,r)}<T_C)+\eta K\\&=\zeta K\mathbb{P}^x(T_C<\tau_{B(x,r)})+N(1-\mathbb{P}^x(T_C<\tau_{B(x,r)}))+\eta K,\end{aligned}$$

or

$$\frac{N}{K}\ge\frac{1-\eta-\zeta\mathbb{P}^x(T_C<\tau_{B(x,r)})}{1-\mathbb{P}^x(T_C<\tau_{B(x,r)})}.$$

Using (5.4) there exists $\beta>0$ such that $N\ge K(1+\beta)$. Therefore there exists $x'\in B(x,2r)$ with $h(x')\ge K(1+\beta)$.

Now suppose there exists $x_1\in B(x_0,\theta R/2)$ with $h(x_1)=K_1$. Define r_1 and A_1 in terms of K_1 analogously to (5.7) and (5.9). Using the above argument (with x_1 replacing x and x_2 replacing x'), there exists $x_2\in B(x_1,2r_1)$ with $h(x_2)=K_2\ge(1+\beta)K_1$. We continue and obtain r_2 and A_2 and then x_3,K_3,r_3,A_3, etc. Note $x_{i+1}\in B(x_i,2r_i)$ and $K_i\ge(1+\beta)^{i-1}K_1$. In view of (5.8), $\sum_i|x_{i+1}-x_i|\le c_{11}K_1^{-1/d}R$. If K_1 is big enough, we have a sequence $x_1,x_2,\dots$ contained in $B(x_0,3\theta R/4)$. Since $K_i\ge(1+\beta)^{i-1}K_1$ and $r_i\le c_{12}K_i^{-1/d}R$, there will be a first integer i for which $r_i<2r_0$. But for all $y\in B(x_i,\theta r_i)$ we have $h(y)\ge c_8h(x_i)$, so then $A_i=B(x_i,\theta r_i)$, a contradiction to (5.10). $\square$

Corollary 5.3. *Let ξ_i be an i.i.d. sequence of symmetric random vectors taking values in $\mathbb{Z}^d$ with finite second moments. Let $X_n=\sum_{i=1}^n\xi_i$ and suppose X_n is aperiodic. Suppose there exists c_1 such that*

$$\mathbb{P}(\xi_1=y)\le c_1\mathbb{P}(\xi_1=y')$$

whenever $|y-y'|\le|y|/3$. Then there exists c_2 and R_0 such that for all R larger than R_0 and any $w\notin B(x_0,R)$,

$$\mathbb{P}^x(X_{\tau_{B(x_0,R)}}=w)\le c_2\mathbb{P}^y(X_{\tau_{B(x_0,R)}}=w),\qquad x,y\in B(x_0,R/2).$$

Proof. We let $C_{xy} = \mathbb{P}(\xi_1 = y - x)$. Since the ξ_i are symmetric, then the X_n form a symmetric Markov chain, and it is easy to see that (A1)–(A4) are satisfied. We then apply Theorem 5.1 to $h(x) = \mathbb{P}^x(Y_{\tau_{B(x_0,R)}} = w)$. □

6. Central limit theorem

Suppose we have a sequence C^n_{xy} of conductances satisfying (A1), (A2), and (A3) with constants and φ independent of n. Let $Y^{(n)}_t$ be the corresponding continuous time Markov chains on $\mathbb{Z}^d$ and set

$$Z^{(n)}_t = Y^{(n)}_{n^2 t}/n.$$

As noted previously, the Dirichlet form corresponding to the process $Z^{(n)}$ is

$$\mathcal{E}_n(f,f) = n^{2-d} \sum_{x,y \in n^{-1}\mathbb{Z}^d} (f(y) - f(x))^2 C^n_{nx,ny}. \tag{6.1}$$

We will also need to discuss the form

$$\mathcal{E}^R_n(f,f) = n^{2-d} \sum_{x,y \in n^{-1}\mathbb{Z}^d} (f(y) - f(x))^2 C^{n,R}_{nx,ny}, \tag{6.2}$$

where $C^{n,R}_{k,l}, k,l \in \mathbb{Z}^d$ is equal to $C^n_{k,l}$ if $|k-l| \le nR$ and 0 otherwise.

Since the state space of $Z^{(n)}$ is $n^{-1}\mathbb{Z}^d$ while the limit process will have $\mathbb{R}^d$ as its state space, we need to exercise some care with the domains of the functions we deal with. First, if g is defined on $\mathbb{R}^d$, we define $R_n(g)$ to be the restriction of g to $n^{-1}\mathbb{Z}^d$:

$$R_n(g)(x) = g(x), \qquad x \in n^{-1}\mathbb{Z}^d.$$

If g is defined on $n^{-1}\mathbb{Z}^d$, we next define an extension of g to $\mathbb{R}^d$. The one we use is defined as follows. For $k \in \mathbb{Z}^d$, let

$$Q_n(k) = \prod_{j=1}^d [n^{-1}k_j, n^{-1}(k_j+1)].$$

When $d = 1$, we define the extension, $E_n(g)$, to be linear in each $Q_n(k)$ and to agree with g on the endpoints of each interval $Q_n(k)$. For $d > 1$ we define $E_n(g)$ inductively. We use the definition in the $(d-1)$-dimensional case to define $E_n(g)$ on each face of each $Q_n(k)$. We define $E_n(g)$ in the interior of a $Q_n(k)$ so that if L is any line segment contained in the $Q_n(k)$ that is parallel to one of the coordinate axes, then $E_n(g)$ is linear on L. For example, when $d = 2$, $n = 1$, and $k = (0,0)$, then

$$\begin{aligned} E_n(g)(s,t) = g(0,0)&(1-s)(1-t) + g(0,1)(1-s)t + g(1,0)s(1-t) \\ &+ g(1,1)st, \qquad\qquad 0 \le s,t \le 1. \end{aligned}$$

Recall that e^j is the unit vector in the x_j direction and let (x,y) denote the inner product in $\mathbb{R}^d$. If $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$, let $\mathcal{P}(k)$ be the union of the line segment from 0 to $(k_1, 0, \dots, 0)$, the line segment from $(k_1, 0, \dots, 0)$ to $(k_1, k_2, 0, \dots, 0)$, ..., and the line segment from $(k_1, \dots, k_{d-1}, 0)$ to k. For $z \in n^{-1}\mathbb{Z}^d$ and $1 \le i \le d$, let

$$\begin{aligned} L^i_z = \{(y,k) \in (n^{-1}\mathbb{Z}^d)^2 &: y + n^{-1}\mathcal{P}(nk) \\ &\text{contains the line segment from } z \text{ to } z + n^{-1}e^i\}. \end{aligned}$$

We note that $(x,k) \in L_z^i$ for $z \in n^{-1}\mathbb{Z}^d$ if and only if $(x+k)_l = z_l$ for $l = 1, ..., i-1$, $x_l = z_l$ for $l = i+1, ..., d$ and $z_i \in [x_i \wedge (x+k)_i, x_i \vee (x+k)_i)$. So, for each k, the number of x that satisfies $(x,k) \in L_z^i$ is at most $n|k_i|$.

Recall that $\operatorname{sgn} r$ is equal to 1 if $r > 0$, equal to 0 if $r = 0$, and equal to -1 if $r < 0$. We define a map a^n from $\mathbb{R}^d$ into $\mathcal{M}$, the collection of $d \times d$ matrices as follows: Fix R. If $x \in n^{-1}\mathbb{Z}^d$, let the (i,j)-th element of a^n be given by

$$\big(a^n(x)\big)_{ij} = \sum_{(y,k) \in L_x^i} C_{ny,n(y+k)}^{n,R} n k_j \operatorname{sgn} k_i. \tag{6.3}$$

For general $x = (x_i)_{i=1}^d \in \mathbb{R}^d$, we define $a^n(x) := a^n([x]_n)$, where we set $[x]_n = (n^{-1}[nx_i])_{i=1}^d$. $a^n(x)$ is not symmetric in general, but under (A5), we see that $(a^n(x))_{ij}$ is bounded for all i, j (which can be proved similarly to (6.21) below), and when n is large, we can use Cauchy-Schwarz, etc., as in the symmetric case. Note that if $C_{xy}^n = 0$ for $|x-y| > 1$ (i.e., the nearest neighbor case), then the expression in (6.3) is equal to $2C_{nx,nx+e^i}^n$ if $i = j$ and equal to 0 if $i \neq j$. (In particular, $a^n(x)$ is symmetric in this case.)

We make the following assumption.

(A5) There exist $R > 0$ and a Borel measurable $a : \mathbb{R}^d \to \mathcal{M}$ such that a is symmetric and uniformly elliptic, the map $x \to a(x)$ is continuous, and a^n converges to a uniformly on compacts sets.

We will see from the proofs below that if (A5) holds for one R, then it holds for every $R > 1$ and the limit a is independent of R.

Since a is uniformly elliptic, if we define

$$\mathcal{E}_a(f,f) = \int_{\mathbb{R}^d} (\nabla f(x), a(x)\nabla f(x))dx,$$

then $(\mathcal{E}_a, H^1(\mathbb{R}^d))$ is a regular Dirichlet form on $L^2(\mathbb{R}^d, dx)$ where $H^1(\mathbb{R}^d)$ is the Sobolev space of square integrable functions with one square integrable derivative. Further, it is well-known that the corresponding heat kernel $p^a(t,x,y)$ satisfies the following estimate:

$$c_1 t^{-d/2} \exp\Big(-c_2 \frac{|x-y|^2}{t}\Big) \le p^a(t,x,y) \le c_3 t^{-d/2} \exp\Big(-c_4 \frac{|x-y|^2}{t}\Big), \tag{6.4}$$

for all $t > 0$ and all $x, y \in \mathbb{R}^d$. As a consequence, the corresponding diffusion (which we denote by $\{Z_t\}$) can be defined without ambiguity from any starting point.

In this section we prove the following central limit theorem. Let $C([0,t_0];\mathbb{R}^d)$ be the collection of continuous paths from $[0,t_0]$ to $\mathbb{R}^d$.

Theorem 6.1. *Suppose (A1)-(A3) and (A5) hold.*

(a) Then for each x and each t_0 the $\mathbb{P}^{[x]_n}$-law of $\{Z_t^{(n)}; 0 \le t \le t_0\}$ converges weakly with respect to the topology of the space $D([0,t_0],\mathbb{R}^d)$. The limit probability gives full measure to $C([0,t_0],\mathbb{R}^d)$.

(b) If Z_t is the canonical process on $C([0,\infty),\mathbb{R}^d)$ and $\mathbb{P}^x$ is the weak limit of the $\mathbb{P}^{[x]_n}$-laws of $Z^{(n)}$, then the process $\{Z_t, \mathbb{P}^x\}$ has continuous paths and is the symmetric process corresponding to the Dirichlet form $\mathcal{E}_a$.

Before giving the proof, we discuss three examples. First, suppose each $X^{(n)}$ is the sum of i.i.d. random vectors. Then the C_{xy}^n will depend only on $y-x$, and so the $a^n(x)$ will be constant in the variable x. Therefore, if convergence holds, the limit $a(x)$ will be constant in x. This means that the limit is a linear transformation of d-dimensional Brownian motion, as one would expect.

For another example, suppose the $X^{(n)}$ are nearest neighbor Markov chains, i.e., $C^n_{xy} = 0$ if $|x-y| \neq 1$. Then in this case the result of [SZ] is included in our Corollary 6.5 and 6.7.

Third, suppose $C^n_{xy} = C_{xy}$ does not depend on n. Unless C_{xy} is a function only of $y-x$, then (2.6) of [SZ] (which is (6.29) below) will not be satisfied, and this situation is covered by Theorem 6.1 but not by the results of [SZ]. To be fair, the goal of [SZ] was not to obtain a general central limit theorem, but instead to come up with a way of approximating diffusions by Markov chains. Condition (A5) is restrictive. For this $C^n_{xy} = C_{xy}$ case, if we further assume that $C_{xy} = 0$ for $|x-y| > 1$, then $a(x)$ is always a constant matrix. Indeed, in this case the expression in (6.3) is equal to $2C_{nx,nx+e^i}\delta_{ij}$, which converges to $(a(x))_{ij}$ uniformly on compacts as $n \to \infty$ by (A5). So, for any $m \in \mathbb{N}$, the limit of $a^n(x/m)$ is equal to $a(x)$, i.e., $a(x/m) = a(x)$. Since a is continuous, we conclude $a(x) = a(0)$ for all $x \in \mathbb{R}^d$.

Before we prove Theorem 6.1, we prove a proposition showing tightness of the laws of $Z^{(n)}$.

Proposition 6.2. *Suppose $\{n_j\}$ is a subsequence. Then there exists a further subsequence $\{n_{j_k}\}$ such that*

(a) For each f that is C^∞ on $\mathbb{R}^d$ with compact support, $E_{n_{j_k}}(P^{n_{j_k}}_t R_{n_{j_k}}(f))$ converges uniformly on compact subsets; if we denote the limit by $P_t f$, then the operator P_t is linear and extends to all continuous functions on $\mathbb{R}^d$ with compact support and is the semigroup of a symmetric strong Markov process on $\mathbb{R}^d$ with continuous paths.

(b) For each x and each t_0 the $\mathbb{P}^{[x]_{n_{j_k}}}$ law of $\{Z^{(n_{j_k})}_t; 0 \le t \le t_0\}$ converges weakly to a probability $\mathbb{P}^x$ giving full measure to $C([0,t_0];\mathbb{R}^d)$.

Proof. Let $t_0 > 0$ and $\eta > 0$. Let τ_n be stopping times bounded by t_0 and let $\delta_n \to 0$. Then by Proposition 3.4 and the strong Markov property,

$$\limsup_{n\to\infty} \mathbb{P}(|Z^{(n)}_{\tau_n+\delta_n} - Z^{(n)}_{\tau_n}| > \eta) = 0.$$

This, Proposition 3.4, and [A] imply that the laws of the $\{Z^{(n)}\}$ are tight in $D[0,t_0]$ for each t_0.

Fix t_0 and $\eta > 0$. $Z^{(n)}$ will have a jump of size larger than η before time t_0 only if $|Y^{(n)}_t - Y^{(n)}_{t-}| \ge \eta n$ for some $t \le n^2 t_0$. By the Lévy system formula, the probability of this is bounded by

$$\begin{aligned}\mathbb{E}^x \sum_{s\le n^2 t_0} 1_{(|Y^{(n)}_s - Y^{(n)}_{s-}| \ge \eta n)} &= \mathbb{E}^x \int_0^{n^2 t_0} \sum_{|x - Y^{(n)}_s| \ge \eta n} C^n_{Y^{(n)}_s x}\, ds \\ &\le c_1 (n^2 t_0) \sum_{i \ge \eta n} \varphi(i) i^{d-1} \\ &\le c_1 t_0 \eta^{-2} \sum_{i\ge \eta n} \varphi(i) i^{d+1},\end{aligned}$$

which tends to 0 by dominated convergence as $n \to \infty$. Since this is true for each t_0 and $\eta > 0$ we conclude that any subsequential limit point of the sequence $Z^{(n)}$ will have continuous paths.

From this point on the argument is fairly standard. We give a sketch, leaving the details to the reader. Take a countable dense subset $\{t_i\}$ of $[0, \infty)$ and a countable dense subset $\{f_m\}$ of the C^∞ functions on $\mathbb{R}^d$ with compact support. Let P_t^n be the semigroup for $Z^{(n)}$. In view of Theorem 4.9, $E_{n_j}(P_{t_i}^{n_j}(R_{n_j}(f_m)))$ will be equicontinuous. By a diagonalization argument, we can find a subsequence $\{n_{j_k}\}$ of $\{n_j\}$ such that for each i and m, as $n_{j_k} \to \infty$, these functions converge uniformly on compact sets. Call the limit $P_{t_i} f_m$. Using the equicontinuity, we can define $P_t f_m$ by continuity for all t, and because the norm of each P_t is bounded by 1, we can also define $P_t f$ by continuity for f continuous with compact support. Using the equicontinuity yet again, it is easy to see that the P_t satisfy the semigroup property and that P_t maps continuous functions with compact support into continuous functions. One can thus construct a strong Markov process that has P_t as its semigroup. The symmetry of $P_t^{(n)}$ leads to the symmetry of P_t.

For each x, the $\mathbb{P}^{[x]_{n_j}}$ laws of $\{Z_t^{(n_j)}; 0 \le t \le t_0\}$ are tight. Fix x, let $\{n'\}$ be any subsequence of $\{n_{j_k}\}$ along which the $\mathbb{P}^{[x]_{n'}}$ converge weakly, and let $\mathbb{P}$ be the weak limit of the subsequence $\mathbb{P}^{[x]_{n'}}$. Suppose F is a continuous functional on $C([0, t_0]; \mathbb{R}^d)$ of the form $F(\omega) = \prod_{\ell=1}^L g_i(\omega(s_\ell))$, where the g_i are continuous with compact support and $0 \le s_1 < \cdots < s_L \le t_0$. When $L = 1$, then

$$\begin{aligned}
\mathbb{E}\, g_1(Z_{s_1}) &= \lim \mathbb{E}^{\,[x]_{n'}} R_{n'}(g_1)(Z_{s_1}^{(n')}) \\
&= \lim P_{s_1}^{n'} R_{n'}(g_1)([x]_{n'}) \\
&= P_{s_1} g_1(x).
\end{aligned}$$

Thus the one-dimensional distributions of a subsequential limit point of the $\mathbb{P}^{[x]_{n_{j_k}}}$ do not depend on the subsequence $\{n'\}$. Using the Markov property of $Z^{(n)}$ and the equicontinuity, a similar argument shows that the same is true of the L-dimensional distributions. Therefore there must be weak convergence along the subsequence $\{n_{j_k}\}$. As proved above, the weak limit is concentrated on the set of continuous paths. $\square$

Proof of Theorem 6.1. We denote the Dirichlet form for the process $Z^{(n)}$ by $\mathcal{E}_n$. Suppose f, g are C^∞ on $\mathbb{R}^d$ with compact support. Let U_λ^n be the λ-resolvent for $Z^{(n)}$; this means that

$$U_\lambda^n h(x) = \mathbb{E}^{\,x} \int_0^\infty e^{-\lambda t} h(Z_t^{(n)})\, dt$$

for $x \in n^{-1}\mathbb{Z}^d$ and h having domain $n^{-1}\mathbb{Z}^d$. We write P_t^n for the semigroup for $Z^{(n)}$.

Using Proposition 6.2, we need to show that if we have a subsequential limit point of the P_t^n in the sense of that proposition, then the limiting process corresponds to the Dirichlet form $\mathcal{E}_a$. Let $\{n'\}$ be a subsequence of $\{n\}$ for which the subsequence converges in the sense of Proposition 6.2, and let U_λ be the λ-resolvent of the limiting process.

Let $F_{n'} = U_\lambda^{n'}(R_{n'}(f))$. Then

$$\mathcal{E}_{n'}(F_{n'}, R_{n'}(g)) = (R_{n'}(f), R_{n'}(g)) - \lambda(F_{n'}, R_{n'}(g)), \tag{6.5}$$

where we let $(h_1, h_2) = \sum_{x \in n^{-1}\mathbb{Z}^d} h_1(x) h_2(x) \mu_x^D$ for functions defined on $n^{-1}\mathbb{Z}^d$. (Recall that our base measure is μ^D.) Let $H_n = E_n(F_n)$ and $H = U_\lambda f$. The equicontinuity result of Theorem 4.9 and Proposition 6.2 shows that the $H_{n'}$ converges uniformly on compacts to H. If we can show

$$\mathcal{E}_a(H,g) = (f,g) - \lambda(H,g), \tag{6.6}$$

this will show that the λ-resolvent for the limiting process is the same as the λ-resolvent for the process corresponding to $\mathcal{E}_a$, and the proof will be complete; we also use (h_1,h_2) to denote $\int h_1(x)h_2(x)\,dx$ when h_1,h_2 are functions defined on $\mathbb{R}^d$.

Next, since $f\in L^2(\mathbb{R}^d)$ and f is C^∞, then $R_n(f)\in L^2(d\mu_n)$. Standard Dirichlet form theory shows that

$$\|U^n_\lambda(R_n(f))\|_2 \le \frac{1}{\lambda}\|R_n(f)\|_2,$$

that is, the L^2 norm of F_n is bounded in n. We see that

$$\int |\nabla H_n(x)|^2\,dx \le c_1\mathcal{E}_n(F_n,F_n) = c_1((R_n(f),F_n) - \lambda(F_n,F_n)) \tag{6.7}$$

is bounded in n. Using the imbedding of $W^{1,2}$ into L^2, we conclude that $\{H_n\}$ is a compact sequence in $L^2(\mathbb{R}^d)$; here $W^{1,2}$ is the space of L^2 functions whose gradient is square integrable. Since $H_{n'}$ converges on compacts to H, it follows that $H_{n'}$ converges in L^2 to H. We also note that by (6.5)

$$\mathcal{E}_n(F_n,F_n) = (R_n(f),F_n) - \lambda(F_n,F_n) \tag{6.8}$$

is uniformly bounded in n.

We need to know that

$$|\mathcal{E}^R_n(F_n,R_n(g)) - \mathcal{E}_{a^n}(H_n,g)| \to 0 \tag{6.9}$$

as $n\to\infty$. The proof of this is a bit lengthy and we defer it to Lemma 6.3 below.

We also need to show that

$$|\mathcal{E}_n(F_n,R_n(g)) - \mathcal{E}^R_n(F_n,R_n(g))| \to 0 \tag{6.10}$$

as $n\to\infty$. This follows because by Cauchy-Schwarz, we have

$$\begin{aligned}
\Big| &\sum_{x,y\in n^{-1}\mathbb{Z}^d} (F_n(y)-F_n(x))n^{2-d}C^n_{nx,ny}(R_n(g)(y)-R_n(g)(x))\\
&\quad - \sum_{x,y\in n^{-1}\mathbb{Z}^d} (F_n(y)-F_n(x))n^{2-d}C^{n,R}_{nx,ny}(R_n(g)(y)-R_n(g)(x))\Big|\\
&\le c\mathcal{E}_n(F_n,F_n)^{1/2}\Big[\sum_{x,y\in n^{-1}\mathbb{Z}^d} n^{2-d}(C^n_{nx,ny}-C^{n,R}_{nx,ny})(R_n(g)(y)-R_n(g)(x))^2\Big]^{1/2}.
\end{aligned}$$

The term within the brackets on the last line is bounded by

$$c(\|\nabla g\|^2_\infty + \|g\|^2_\infty) \sup_{x\in n^{-1}\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d,|x-y|>nR} |x-y|^2C^n_{xy} \le c' \sum_{i>nR} i^{d-1}i^2\varphi(i),$$

which will be less than ε^2 if n is large.

Using (6.5), (6.6), (6.9), and (6.10), we see that it suffices to show

$$\mathcal{E}_{a^{n'}}(H_{n'},g) \to \mathcal{E}_a(H,g). \tag{6.11}$$

Now

$$|\mathcal{E}_{a^{n'}}(H_{n'},g) - \mathcal{E}_a(H_{n'},g)| = \Big|\int \nabla H_{n'}\cdot(a^{n'}-a)\nabla g\Big|. \tag{6.12}$$

Since ∇g is bounded with compact support and $|\nabla H_{n'}|$ is bounded in L^2, then (A5) and the Cauchy-Schwarz inequality tell us that the right hand side of (6.12) tends to 0 as $n \to \infty$. Therefore we need to show

$$\mathcal{E}_a(H_{n'}, g) \to \mathcal{E}_a(H, g). \tag{6.13}$$

But if ∇h is bounded with compact support, then

$$\int (\nabla H_{n'})\, h = -\int H_{n'} \nabla h \to -\int H\, \nabla h = \int (\nabla H)\, h. \tag{6.14}$$

If we take the supremum over such h that also have L^2 norm bounded by 1, then Fatou's lemma and the Cauchy-Schwarz inequality show that ∇H is in L^2. If h is bounded with compact support, let $\varepsilon > 0$ and approximate h by a C^1 function $\tilde h$ with compact support such that $\|h - \tilde h\|_2 \le \varepsilon$. Since $|\nabla H_n|$ is bounded in L^2, then $|\int \nabla H_{n'}(h - \tilde h)| \le c_1\varepsilon$ and $|\int \nabla H(h - \tilde h)| \le c_1 \varepsilon$. So by (6.14)

$$\limsup_{n'\to\infty} \Big| \int \nabla H_{n'} h - \int \nabla H\, h \Big| \le 2c_1\varepsilon.$$

Because ε is arbitrary, we have

$$\int \nabla H_{n'}\, h \to \int \nabla H\, h. \tag{6.15}$$

If we apply (6.15) with $h = a\nabla g$, we obtain (6.13). $\square$

To complete the proof we have

Lemma 6.3. *With the notation of the above proof,*

$$|\mathcal{E}^R_n(F_n, R_n(g)) - \mathcal{E}_{a^n}(H_n, g)| \to 0$$

as $n \to \infty$.

Proof. Step 1. Let $\varepsilon, \eta_1, \eta_2, \delta > 0$ and let $\{\mathcal{S}_m\}$ be a collection of cubes with disjoint interiors whose union contains the support of g and such that the oscillation of a on each $\mathcal{S}_m$ is less than η_1 and the oscillation of ∇g on each $\mathcal{S}_m$ is less than η_2. One way to construct such a collection is to take a cube large enough to contain the support of g, divide it into 2^d equal subcubes, and then divide each of the subcubes and so on until the oscillation restrictions are satisfied.

Step 2. Let $\mathcal{S}'_m$ be the cube with the same center as $\mathcal{S}_m$ but side length $(1-2\delta)$ times as long. Let $A = \bigcup_m (\mathcal{S}_m - \mathcal{S}'_m)$. We claim it suffices to show that

$$\begin{aligned}
\Big| &\int_{A^c} \nabla H_n(x) \cdot a^n(x) \nabla g(x)\, dx \\
&\quad - \sum_{x\notin A, x\in n^{-1}\mathbb{Z}^d} \sum_{y \in n^{-1}\mathbb{Z}^d} (F_n(y) - F_n(x)) n^{2-d} C^{n,R}_{nx,ny} (R_n(g)(y) - R_n(g)(x)) \Big| \\
&\to 0
\end{aligned} \tag{6.16}$$

as $n \to \infty$. To see this, note first that by Cauchy-Schwarz and (6.7)

$$\begin{aligned}
\Big| \int_A \nabla H_n(x) \cdot a^n(x) \nabla g(x)\, dx \Big| &\le \mathcal{E}_{a^n}(H_n, H_n)^{1/2} \Big(\int_A \nabla g(x) \cdot a^n(x) \nabla g(x)\, dx \Big)^{1/2} \\
&\le c\mathcal{E}_{a^n}(H_n, H_n)^{1/2} \|\nabla g\|_\infty |A|^{1/2}
\end{aligned}$$

will be less than ε if δ is taken sufficiently small. Next note that for any $x \in n^{-1}\mathbb{Z}^d$,

$$\sum_{y\in n^{-1}\mathbb{Z}^d} n^{2-d} C^{n,R}_{nx,ny}(R_n(g)(y) - R_n(g)(x))^2 \le n^{-d}\|\nabla g\|^2_\infty \sum_{y\in n^{-1}\mathbb{Z}^d} C^{n,R}_{nx,ny}|ny-nx|^2 \le cn^{-d}.$$

So by Cauchy-Schwarz and (6.8)

$$\begin{aligned}&\Big|\sum_{x\in A, x\in n^{-1}\mathbb{Z}^d}\ \sum_{y\in n^{-1}\mathbb{Z}^d} (F_n(y)-F_n(x))n^{2-d}C^{n,R}_{nx,ny}(R_n(g)(y)-R_n(g)(x))\Big| \\ &\quad\le \mathcal{E}_n(F_n,F_n)^{1/2}\Big(\sum_{x\in A, x\in n^{-1}\mathbb{Z}^d}\ \sum_{y\in n^{-1}\mathbb{Z}^d} n^{2-d}C^{n,R}_{nx,ny}(R_n(g)(y)-R_n(g)(x))^2\Big)^{1/2} \\ &\quad\le c\mathcal{E}_n(F_n,F_n)^{1/2}\Big(n^{-d}\,\mathrm{card}\,(A\cap n^{-1}\mathbb{Z}^d)\Big)^{1/2},\end{aligned} \tag{6.17}$$

which will be less than ε if δ is taken small enough and n is large.

Step 3. Let x_m be the center of $\mathcal{S}_m$. Define $\overline{g}$ by requiring $\overline{g}$ to be linear on each $\mathcal{S}_m$ and satisfying $\overline{g}(x_m) = g(x_m)$, $\nabla\overline{g}(x_m) = \nabla g(x_m)$. We claim it suffices to show that

$$\begin{aligned}&\Big|\int_{A^c} \nabla H_n(x)\cdot a^n(x)\nabla\overline{g}(x)\,dx \\ &\qquad - \sum_{x\notin A, x\in n^{-1}\mathbb{Z}^d}\ \sum_{y\in n^{-1}\mathbb{Z}^d} (F_n(y)-F_n(x))n^{2-d}C^{n,R}_{nx,ny}(R_n(\overline{g})(y)-R_n(\overline{g})(x))\Big| \\ &\qquad \to 0.\end{aligned} \tag{6.18}$$

To see this, note that

$$\begin{aligned}&\Big|\int_{A^c}\nabla H_n(x)\cdot a^n(x)\nabla\overline{g}(x)\,dx - \int_{A^c}\nabla H_n(x)\cdot a^n(x)\nabla g(x)\,dx\Big| \\ &\quad\le \mathcal{E}_{a^n}(H_n,H_n)^{1/2}\Big(\int_{A^c}\nabla(\overline{g}-g)(x)\cdot a^n(x)\nabla(\overline{g}-g)(x)\,dx\Big)^{1/2} \\ &\quad\le c\mathcal{E}_{a^n}(H_n,H_n)^{1/2}\eta_2,\end{aligned}$$

which will be less than ε if η_2 is chosen small enough. A similar argument shows that the difference between the second term in (6.18) and the corresponding term with $\overline{g}$ replaced by g is small; cf. Step 2.

Step 4. Let $\overline{C}^n_{xy} = C^{n,\delta/2}_{xy}$ and define

$$\overline{a}^n(x) \text{ by } (\overline{a}^n(x))_{ij} = \sum_{(y,k)\in L^i_x} \overline{C}^n_{ny,n(y+k)} nk_j \mathrm{sgn}\, k_i.$$

We claim it suffices to show that

$$\begin{aligned}&\Big|\int_{A^c} \nabla H_n(x)\cdot \overline{a}^n(x)\nabla\overline{g}(x)\,dx \\ &\qquad - \sum_{x\notin A, x\in n^{-1}\mathbb{Z}^d}\ \sum_{y\in n^{-1}\mathbb{Z}^d} (F_n(y)-F_n(x))n^{2-d}\overline{C}^n_{nx,ny}(R_n(\overline{g})(y)-R_n(\overline{g})(x))\Big| \\ &\qquad \to 0.\end{aligned} \tag{6.19}$$

To prove this, we first note that the following can be proved in the same way as (6.10):

$$\begin{aligned}&\Big|\sum_{x\notin A, x\in n^{-1}\mathbb{Z}^d}\sum_{y\in n^{-1}\mathbb{Z}^d}(F_n(y)-F_n(x))n^{2-d}\overline{C}^n_{nx,ny}(R_n(\overline{g})(y)-R_n(\overline{g})(x))\\ &-\sum_{x\notin A, x\in n^{-1}\mathbb{Z}^d}\sum_{y\in n^{-1}\mathbb{Z}^d}(F_n(y)-F_n(x))n^{2-d}C^{n,R}_{nx,ny}(R_n(\overline{g})(y)-R_n(\overline{g})(x))\Big|\to 0,\end{aligned}$$

as $n\to\infty$. Next,

$$\begin{aligned}&\Big|\int_{A^c}\nabla H_n(x)\cdot\overline{a}^n(x)\nabla\overline{g}(x)\,dx-\int_{A^c}\nabla H_n(x)\cdot a^n(x)\nabla\overline{g}(x)\,dx\Big|\\ &\quad\le c\Big(\int_{A^c}(\nabla H_n(x))^2dx\Big)^{1/2}\Big(\int_{A^c}(\overline{a}^n(x)-a^n(x))(\nabla\overline{g}(x))^2\Big)^{1/2}.\end{aligned}\tag{6.20}$$

We can estimate

$$\begin{aligned}\Big|(\overline{a}^n(x)-a^n(x))_{ij}\Big|&\le\sum_{(y,k)\in L^i_x}|\overline{C}^n_{ny,n(y+k)}-C^n_{ny,n(y+k)}|n|k_j|\\ &\le c_1\sup_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d,|x-y|>n\delta/2}|x-y|^2C^n_{xy}\le c_2\sum_{i>n\delta/2}i^{d-1}i^2\varphi(i),\end{aligned}\tag{6.21}$$

where in the second inequality, we used the fact that for each k, the number of y that satisfies $(y,k)\in L^i_z$ is at most $n|k_i|$ (as mentioned when we defined L^i_z). So the right hand side of (6.20) will be less than ε if n is large.

Step 5. We have chosen the $\mathcal{S}_m$ so that the oscillation of a on each $\mathcal{S}_m$ is at most η_1. Since we have that the a^n converge to the a uniformly on compacts and there are only finitely many $\mathcal{S}_m$'s, then for n large the oscillation of a^n on any $\mathcal{S}_m$ will be at most $2\eta_1$.

Step 6. We will now prove (6.19). By Step 3, $\overline{g}$ is linear on each $\mathcal{S}_m$, so it is enough to discuss the case where $\overline{g}(x)=x_{j_0}$ on $\mathcal{S}'_m$ for some j_0 and then use a linearity argument. Noting that $H_n=F_n$ on $n^{-1}\mathbb{Z}^d$, define

$$\begin{aligned}&\mathcal{E}^{\mathcal{S}'_m}_n(H_n,\overline{g})\\ &\quad:=\sum_{x\in\mathcal{S}'_m\cap n^{-1}\mathbb{Z}^d}\sum_{y\in n^{-1}\mathbb{Z}^d}(H_n(y)-H_n(x))n^{2-d}\overline{C}^n_{nx,ny}(R_n(\overline{g})(y)-R_n(\overline{g})(x)).\end{aligned}$$

Since there is no term involving different $\mathcal{S}'_m$'s, we will consider each $\mathcal{S}'_m$ separately. We will fix an $x_0\in\mathcal{S}'_m$ and look at the terms involving $H_n(x_0+n^{-1}e_i)-H_n(x_0)$. First, by an elementary computation using the definition of the linear extension map E_n, we have

$$\int_{Q_n(x_0)}\frac{\partial H_n}{\partial x_i}dx=\frac{1}{2^{d-1}n^{d-1}}\sum_{z\in V_i(x_0)}(H_n(z+n^{-1}e_i)-H_n(z))\tag{6.22}$$

where $V_i(x_0)$ is the collection of vertices of the face of $Q_n(x_0)$ perpendicular to e_i and with the smaller e_i component. (E.g., for a square, $V_1(x_0)$ is the two leftmost corners, and $V_2(x_0)$ is the two lower corners.) So

$$\begin{aligned}\int_{Q_n(x_0)} &(\nabla H_n, \overline{a}^n \nabla \overline{g}) dx \\ &= \sum_{i,j=1}^d \int_{Q_n(x_0)} \frac{\partial}{\partial x_i} H_n \overline{a}^n_{ij} \frac{\partial}{\partial x_j} \overline{g}\, dx = \sum_i \overline{a}^n_{ij_0}(x_0) \int_{Q_n(x_0)} \frac{\partial}{\partial x_i} H_n dx \\ &= \sum_{i=1}^d \overline{a}^n_{ij_0}(x_0) \frac{1}{2^{d-1} n^{d-1}} \sum_{z \in V_i(x_0)} (H_n(z + n^{-1} e_i) - H_n(z)).\end{aligned}$$

Summing over all cubes that contain $H_n(x_0 + n^{-1} e^i) - H_n(x_0)$, the coefficient in front of $H_n(x_0 + n^{-1} e^i) - H_n(x_0)$ will be

$$\frac{n^{1-d}}{2^{d-1}} \sum_{z \in V_i(x_0 + n^{-1} e^i - e_*)} \overline{a}^n_{ij_0}(z), \tag{6.23}$$

where $e_* = (1/n, ..., 1/n)$.

We next look at $\mathcal{E}_n^{\mathcal{S}'_m}(H_n, \overline{g})$. Since $\overline{g}(x+k) - \overline{g}(x) = k_{j_0}$ where $k = (k_1, ..., k_d)$, we have

$$\mathcal{E}_n^{\mathcal{S}'_m}(H_n, \overline{g}) = n^{2-d} \sum_{\substack{x \in \mathcal{S}'_m \cap n^{-1} Z^d, \\ k \in n^{-1} Z^d}} (H_n(x+k) - H_n(x)) \overline{C}^n_{nx, n(x+k)} k_{j_0}.$$

Let us fix x and k and replace $(H_n(x+k) - H_n(x))$ by $\sum_{m=1}^{|k|} (H_n(z_{m+1}) - H_n(z_m))$ (here $|k| := |k_1| + ... + |k_d|$ and $|z_{m+1} - z_m| = 1/n$) so that the union of the line segments belongs to $x + n^{-1}\mathcal{P}(k)$. We will get a term of the form $H_n(x_0 + n^{-1} e_i) - H_n(x_0)$ if $z_m = x_0$ and $z_{m+1} = x_0 + n^{-1} e_i$ (we get $H_n(x_0) - H_n(x_0 + n^{-1} e_i)$ if $z_{m+1} = x_0$ and $z_m = x_0 + n^{-1} e_i$), so the contribution will be

$$n^{2-d} \overline{C}^n_{nx, n(x+k)} k_{j_0} (\mathrm{sgn}\ k_i).$$

Summing over $x \in \mathcal{S}'_m \cap n^{-1} Z^d, k \in n^{-1} \mathbb{Z}^d$, we have that the coefficient in front of $H_n(x_0 + n^{-1} e^i) - H_n(x_0)$ for $\mathcal{E}_n^{\mathcal{S}'_m}(H_n, \overline{g})$ is

$$n^{2-d} \sum_{\substack{x \in \mathcal{S}'_m \cap n^{-1} Z^d, \\ (x,k) \in L^i_{x_0}}} \overline{C}^n_{nx, n(x+k)} k_{j_0} (\mathrm{sgn}\ k_i). \tag{6.24}$$

On the other hand, by the definition of $\overline{a}^n$, we have

$$n^{2-d} \sum_{(x,k) \in L^i_{x_0}} \overline{C}^n_{nx, n(x+k)} k_{j_0} (\mathrm{sgn}\ k_i) = n^{1-d} \overline{a}^n_{ij_0}(x_0). \tag{6.25}$$

Let $\mathcal{S}''_m$ be the cube with the same center as $\mathcal{S}'_m$ but side length $(1 - 2\delta)$ times as long. If $x_0 \in \mathcal{S}''_m \cap n^{-1} \mathbb{Z}^d$, then the expressions in (6.24) and (6.25) are equal, because $\overline{C}^n_{nx, n(x+k)} = 0$ for $x \notin \mathcal{S}'_m \cap n^{-1} \mathbb{Z}^d, (x,k) \in L^i_{x_0}$. Since the oscillation of a^n on each $\mathcal{S}'_m$ is less than $2\eta_1$ as in Step 5, by (6.21) the oscillation of $\overline{a}^n$ on each $\mathcal{S}'_m$ is less that $3\eta_1$. Thus, when $x_0 \in \mathcal{S}''_m \cap n^{-1} \mathbb{Z}^d$, we see that the absolute value of the difference between (6.23) and (6.24) is bounded by $3\eta_1 n^{1-d}$. (Note that $\mathrm{card}\, V_i(x_0 + n^{-1} e^i - e_*) = 2^{d-1}$ is used here.) Now, if $x_0 \in (\mathcal{S}'_m - \mathcal{S}''_m) \cap n^{-1} \mathbb{Z}^d$, then the difference between (6.23) and (6.24) is bounded by $c_3 n^{1-d}$, because similar to (6.21) we have

$$\sum_{(x,k)\in L^i_{x_0}} \overline{C}^n_{nx,n(x+k)} n k_{j_0}(\text{sgn } k_i) \le c_1 \sup_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} |x-y|^2 C^n_{xy} \le c_2 \sum_i i^{d-1} i^2 \varphi(i) =: c_3.$$

Set $H_{x_0,i} := H_n(x_0+n^{-1}e^i) - H_n(x_0)$, $A' := (\bigcup_m (\mathcal{S}'_m - \mathcal{S}''_m)) \cap n^{-1}\mathbb{Z}^d$ and $B := (\bigcup_m \mathcal{S}''_m) \cap n^{-1}\mathbb{Z}^d$. Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\Big| \textstyle\int_{\cup_m \mathcal{S}'_m} (\nabla H_n, \overline{a}^n \nabla \overline{g}) dx - \sum_m \mathcal{E}^{\mathcal{S}'_m}_n (H_n, \overline{g}) \Big| \qquad (6.26)\\
\le\ & \eta_1 n^{1-d} \textstyle\sum_{x_0\in B, i=1,\cdots,d} |H_{x_0,i}| + c_3 n^{1-d} \sum_{x_0\in A', i=1,\cdots,d} |H_{x_0,i}| \\
\le\ & \eta_1 \Big(d n^{-d} \text{card}\, B \Big)^{1/2} \Big(n^{2-d} \textstyle\sum_{x_0\in n^{-1}\mathbb{Z}^d, i} (H_{x_0,i})^2 \Big)^{1/2} \qquad (6.27)\\
& + c_3 \Big(d n^{-d} \text{card}\, A' \Big)^{1/2} \Big(n^{2-d} \textstyle\sum_{x_0\in n^{-1}\mathbb{Z}^d, i} (H_{x_0,i})^2 \Big)^{1/2} \\
\le\ & c_4(\eta_1+\varepsilon) \Big(n^{2-d} \textstyle\sum_{x_0\in n^{-1}\mathbb{Z}^d, i} (H_{x_0,i})^2 \Big)^{1/2} \le c_5(\eta_1+\varepsilon),
\end{aligned}$$

if δ is taken small enough and n is large. We thus complete the proof of (6.19). □

When $d=1$, Lemma 6.3 can be proved under much milder conditions.

(A6) There exists $R>0$ and a Borel measurable $a : \mathbb{R}^d \to \mathcal{M}$ such that for each $r>0$

$$\lim_{n\to\infty} \int_{|x|\le r} |a^n(x) - a(x)| dx = 0. \qquad (6.28)$$

Corollary 6.4. *Let $d=1$ and suppose (A1)-(A3) and (A6) hold. Then the conclusions of Theorem* 6.1 *hold.*

Proof. The proof is similar to the proof of Theorem 6.1. Let us point out the places where we need modifications. First, we can prove that there exist $c_1, c_2 > 0$ such that $c_1 \le a^n(x) \le c_2$ for all $x\in\mathbb{R}^d$ and $n\in\mathbb{N}$. Indeed, by (A2) the lower bound is guaranteed and the upper bound can be proved similarly to (6.21). So, we know $\mathcal{E}_{a^n}(f,f)$ is bounded whenever $f, \nabla f \in L^2$. For the proof that the right hand side of (6.12) goes to 0 as $n\to\infty$, we use (6.28). (To be more precise, the convergence of a^n to a locally in L^2 is used there, which is guaranteed by (6.28) and the fact that the a^n are uniformly bounded.) Noting these facts, the proofs of Theorem 6.1 and Proposition 6.2 go the same way as above. For the proof of Lemma 6.3, in Step 1, we do not need to control the oscillation of a on each $\mathcal{S}_m$. Step 5 is not needed. We have that the expression (6.23) is equal to $\overline{a}^n(x_0)$, and this is equal to the expression in (6.25). (This is a key point; because of this we do not have to worry about the oscillation of a and a^n.) Finally, in the computation of (6.26), the difference on the set B is 0 due to the fact just mentioned, and we can prove that (6.26) is small directly. □

We now give an extension of the result in [SZ] to the case of unbounded range. Assume

(A7) There exists $R>0$ such that for each $r>1$

$$\lim_{n\to\infty} \sum_{k\in\mathbb{Z}^d} \sup_{|y|\le nr} \sup_{|x-y|\le nR} \Big| C^{n,R}_{x,x+k} - C^{n,R}_{y,y+k} \Big| = 0. \qquad (6.29)$$

Let the (i,j)-th element of b^n be given by

$$(6.30)\qquad \big(b^n(x)\big)_{ij} = \sum_{k\in n^{-1}\mathbb{Z}^d} C^{n,R}_{nx,n(x+k)} n^2 k_i k_j, \qquad x\in n^{-1}\mathbb{Z}^d.$$

For general $x=(x_i)_{i=1}^d\in\mathbb{R}^d$, define $b^n(x) := b^n([x]_n)$. Assume the b^n version of (A6);

(A8) There exists $R>0$ and a Borel measurable $a:\mathbb{R}^d\to\mathcal{M}$ such that for each $r>0$

$$(6.31)\qquad \lim_{n\to\infty}\int_{|x|\le r} |b^n(x)-a(x)|dx = 0.$$

We can recover and generalize the convergence theorem given in [SZ] as follows.

Corollary 6.5. *Suppose that (A1)-(A3), (A7), and (A8) hold. Then the conclusions of Theorem* 6.1 *hold.*

Proof. For each $\varepsilon>0$, let $R'=R'(\varepsilon)>0$ be an integer that satisfies $\sum_{s\ge R'}\varphi(s)s^{d+1} < \varepsilon$. Note that $C^{n,R}_{x,y} = C^{n,R'/n}_{x,y} + 1_{\{|x-y|>R'\}}C^{n,R}_{x,y}$. Then, for any $r\ge 1$, any $x\in n^{-1}\mathbb{Z}^d$ such that $|x|\le r$, and any $n\ge R'/R$, we have

$$\begin{aligned}
&\Big|\big(a^n(x)\big)_{ij} - \big(b^n(x)\big)_{ij}\Big|\\
&\quad\le \sum_{k'\in\mathbb{Z}^d}\Big|\sum_{y:(y,k')\in L^{i,*}_x} C^{n,R}_{ny,ny+k'}\operatorname{sgn} k'_i - C^{n,R}_{nx,nx+k'}k'_i\Big|\,|k'_j|\\
&\quad\le R'^2\Big(\sum_{k'\in\mathbb{Z}^d}\sup_{|y'|\le nr}\sup_{|x'-y'|\le R'}\Big|C^{n,R'/n}_{x',x'+k'} - C^{n,R'/n}_{y',y'+k'}\Big|\Big) + 2\sum_{s\ge R'}\varphi(s)s^{d+1}\\
&\quad\le R'^2\Big(\sum_{k'\in\mathbb{Z}^d}\sup_{|y'|\le nr}\sup_{|x'-y'|\le nR}\Big|C^{n,R}_{x',x'+k'} - C^{n,R}_{y',y'+k'}\Big|\Big) + 2\varepsilon,
\end{aligned}$$

where $L^{i,*}_z = \{(y,k')\in(n^{-1}\mathbb{Z}^d)\times\mathbb{Z}^d : y+n^{-1}\mathcal{P}(k'$ contains the line segment from z to $z+n^{-1}e^i\}$. In the second inequality, we used the fact that if $|k'|\le n\cdot R'/n$, $(y,k')\in L^{i,*}_x$ and $x'=nx, y'=ny$, then $|x'-y'| = n|x-y|\le n|k'/n| = k'\le n\cdot R'/n = R'$. Using (6.29) in (A7), the right hand side converges to 0 as $n\to\infty$. In other words,

$$(6.32)\qquad |(a^n(x))_{ij} - (b^n(x))_{ij}|\to 0 \text{ uniformly on compacts as } n\to\infty.$$

Similarly, for any $r\ge 1$, we can prove

$$(6.33)\qquad |(b^n(x))_{ij} - (b^n(y))_{ij}|\to 0 \;\text{ as }\; n\to\infty,\; |x-y|\le n^{-1}R,\; |x|\le r.$$

Now the proof of this corollary goes similarly to the proofs above. As before we point out places where we need modifications. First, as in Corollary 6.4, we can prove that there exist $c_1,c_2>0$ such that $c_1 I\le b^n(x)\le c_2 I$ for all $x\in\mathbb{R}^d$ and $n\in\mathbb{N}$. So we know $\mathcal{E}_{b^n}(f,f)$ is bounded whenever $f,\nabla f\in L^2$. As in Corollary 6.4, we use (6.31) to show that the right hand side of (6.12) goes to 0 as $n\to\infty$. Noting these facts, the proofs of Theorem 6.1 and Proposition 6.2 go in the same way as before. For the proof of Lemma 6.3, in Step 1, we do not need to control the oscillation of a on each $\mathcal{S}_m$. Step 4 with respect to b^n works due to (6.32). Step 5 is not needed. Thanks to (6.32) and (6.33), the difference between the expression in (6.23) (with a replaced by b) and the expression in (6.25) is small. (This is again the key point; because of this we do not have to worry about the oscillation of a

and b^n.) Finally, in the computation of (6.26), the difference on the set B is small due to the fact just mentioned. $\square$

Remark 6.6. If (A7) does not hold, b^n need not be the right approximation in general. Indeed, here is an example where a^n converges to a, but b^n does not as $n \to \infty$. Suppose $d = 1$ and let $C^n_{k,k+i}$ equal r_i if k is odd, s_i if k is even, $i = 1, 2$. Then, we have

$$b^n(k/n) = \begin{cases} r_1 + s_1 + 8r_2, & \text{if } k \text{ is odd,} \\ r_1 + s_1 + 8s_2, & \text{if } k \text{ is even,} \end{cases}$$

$$a^n(k/n) = \begin{cases} 2r_1 + 4(r_2 + s_2), & \text{if } k \text{ is odd,} \\ 2s_1 + 4(r_2 + s_2), & \text{if } k \text{ is even.} \end{cases}$$

Suppose $r_1 = s_1$ and $r_2 \neq s_2$. Then, the value of $b^n(k/n)$ depends on whether k is odd or even, so b^n does not converge locally in L^2 as $n \to \infty$, whereas $a^n(k/n) = 2r_1 + 4(r_2 + s_2)$ is constant. In this case, the assumption of Theorem 6.1 (and Corollary 6.4) holds and $a(x) = 2r_1 + 4(r_2 + s_2)$.

Theorem 6.1 gives a central limit theorem for the processes $Y^{(n)}$. Note that the base measure for $Y^{(n)}$ is the uniform measure, which converges with respect to Lebesgue measure on $\mathbb{R}^d$. We finally discuss the convergence of the discrete time Markov chains $X^{(n)}$. Let Y^ν_t be the continuous time ν-symmetric Markov chain on $\mathbb{Z}^d$ which corresponds to $(\mathcal{E}, \mathcal{F})$. It is a time change of Y_t and it can be defined from X_n as follows. Let $\{U_i : i \in \mathbb{N}, x \in \mathbb{Z}^d\}$ be an independent collection of exponential random variables with parameter 1 that are independent of X_n. Define $T_0 = 0, T_n = \sum_{k=1}^n U_k$. Set $\widetilde{Y}^\nu = X_n$ if $T_n \le t < T_{n+1}$; then the laws of $\widetilde{Y}^\nu$ and Y^ν are the same. Let ν^D be a measure on $\mathcal{S}$ defined by $\nu^D(A) = D^{-d}\nu(DA)$ for $A \subset \mathcal{S}$. Since $\mathcal{S} \subset \mathbb{R}^d$, we will regard ν^D as a measure on $\mathbb{R}^d$ from time to time. By (A1), we see that $c_1\mu^D(A) \le \nu^D(A) \le c_2\mu^D(A)$ for all $A \subset \mathcal{S}$ and all d. So $\{\nu^D\}_D$ is tight and there is a convergent subsequence. We assume the following.

(A9) There exists a Borel measure $\bar{\nu}$ on $\mathbb{R}^d$ such that ν^D converges weakly to $\bar{\nu}$ as $D \to \infty$.

Let $Z^{\bar{\nu}}_t$ be the diffusion process corresponding to the Dirichlet form $\mathcal{E}_a$ considered on $L^2(\mathbb{R}^d, \bar{\nu})$. It is a time changed process of Z_t in Theorem 6.1. Note that by (A1), $\bar{\nu}$ is mutually absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$, so it charges no set of zero capacity. Further, the heat kernel for $Z^{\bar{\nu}}_t$ still enjoys the estimates (6.4).

Now we have a corresponding theorem for the discrete time Markov chains $X^{(n)}$. Define

$$W^{(n)}_t = X^{(n)}_{[n^2 t]}/n.$$

Corollary 6.7. *Suppose (A1)-(A3), (A5), and (A9) hold.*

(a) Then for each x and each t_0 the $\mathbb{P}^{[x]_n}$-law of $\{W^{(n)}_t; 0 \le t \le t_0\}$ converges weakly with respect to the topology of the space $D([0, t_0], \mathbb{R}^d)$. The limit probability gives full measure to $C([0, t_0], \mathbb{R}^d)$.

(b) If $Z^{\bar{\nu}}_t$ is the canonical process on $C([0, \infty), \mathbb{R}^d)$ and $\mathbb{P}^x$ is the weak limit of the $\mathbb{P}^{[x]_n}$-laws of $W^{(n)}$, then the process $\{Z^{\bar{\nu}}_t, \mathbb{P}^x\}$ has continuous paths and is the symmetric process corresponding to the Dirichlet form $\mathcal{E}_a$ considered on $L^2(\mathbb{R}^d, \bar{\nu})$.

Proof. Let $Y^{(n),\nu}_t$ be the continuous time Markov chains on $\mathbb{Z}^d$ corresponding to $\mathcal{E}_n$ considered on $L^2(\mathbb{Z}^d, \nu)$, and set $Z^{(n),\nu}_t = Y^{(n),\nu}_{n^2 t}/n$. Then, by changing the measure

μ^D to ν^D in the proof, we have the results corresponding to Theorem 6.1 for $Z_t^{(n),\nu}$ and $Z_t^{\bar{\nu}}$. So it suffices to show that there is a metric for $D([0,t_0],\mathbb{R}^d)$ with respect to which the distance between $W^{(n)}$ and $Z^{(n),\nu}$ goes to 0 in probability, where in the definition of $Z^{(n),\nu}$ we use the realization of $Y^{(n),\nu}$ given in terms of the $X^{(n)}$ by means of independent exponential random variables of parameter 1.

We use the J_1 topology of Skorokhod; see [Bi]. The paths of $Y^{(n),\nu}$ agree with those of $X^{(n)}$ except that the times of the jumps do not agree. Note that $X^{(n)}$ jumps at times k/n^2, while $Y^{(n),\nu}$ jumps at times T_k/n^2. So it suffices to show that if T_k is the sum of i.i.d. exponentials with parameter 1, then for each $\eta > 0$ and each t_0

$$\mathbb{P}(\sup_{k\le[n^2t_0]} |T_k - k| \ge n^2\eta) \to 0$$

as $n\to\infty$. But by Doob's inequality, the above probability is bounded by

$$\frac{4\,\mathrm{Var}\;T_{[n^2t_0]}}{(n^2\eta)^2} = \frac{4[n^2t_0]}{(n^2\eta)^2} \to 0$$

as desired. □

Remark 6.8. We remark that the definition of a^n, and hence the statement of (A5), depends on the definition of $\mathcal{P}(k)$ and of the extension operator E_n. It would be nice to have a central limit theorem with a more robust statement.

Remark 6.9. We make a few comments comparing the central limit theorem in our paper and the convergence theorem in [SZ] in the case of bounded range. The result in [SZ] requires a smoothness condition on the conductances C^n_{xy}, while we require smoothness instead on the a^n. Thus our theorem has weaker hypotheses, and as Remark 6.6 shows, there are examples where one set of hypotheses holds and the other set does not. On the other hand, if (A1)-(A3) hold, then the $\{b^n\}$ will automatically be symmetric, equi-bounded and equi-uniformly elliptic; if in addition $b^n\to a$, then a will be bounded and uniformly elliptic and this does not need to be assumed.

Acknowledgements

The authors thank an anonymous referee for a careful reading of the manuscript and helpful comments.

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