

NON-CROSSING PARTITION LATTICES IN FINITE REAL REFLECTION GROUPS

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ABSTRACT. For a finite real reflection group W with Coxeter element γ we give a case-free proof that the closed interval, $[I, \gamma]$, forms a lattice in the partial order on W induced by reflection length. Key to this is the construction of an isomorphic lattice of spherical simplicial complexes. We also prove that the greatest element in this latter lattice embeds in the type W simplicial generalised associahedron, and we use this fact to give a new proof that the geometric realisation of this associahedron is a sphere.

1. INTRODUCTION

Let W be a finite real reflection group. Associated to W is a finite type Artin group or generalised braid group, $A(W)$. Much of the work on finite type Artin groups takes Garside's paper, [14], as its starting point, using the set of fundamental reflections as a generating set for W and a corresponding standard generating set for $A(W)$. Recently it has been shown that when the set of *all* reflections is used as a generating set for W and a corresponding generating set is used for $A(W)$, a parallel theory can be constructed. In particular, the new positive monoid embeds in $A(W)$ and new $K(\pi, 1)$'s for $A(W)$ have been constructed. The larger generating set gives $A(W)$ a second structure as a Garside group. The larger generating set is proposed in [7], but Daan Krammer has used it independently in unpublished work. In the case of the braid group B_n , where W is the symmetric group Σ_n , the larger generating set coincides with the band generators from [5]. The second Garside structure for B_n is described in [2] and the structure for general finite W in [3]. The general construction of $K(A(W), 1)$'s is described in [11], using ideas from [4].

A central result needed in the development of this parallel theory is that the closed interval, $[I, \gamma]$, bounded by the identity I and a Coxeter element γ , forms a lattice in the partial order on W induced by reflection length. The lattice property is used to prove the embedding of the positive monoid and the asphericity of the new $K(\pi, 1)$'s. Existing proofs of the lattice property use the classification of finite real reflection groups with different methods applied to the different groups. The symmetric group is handled in [8], and the C_n and D_n groups in [10]. Bessis treats all cases in [3].

Since the lattice in the symmetric group case coincides with the lattice of non-crossing partitions introduced by Kreweras in [16], it has become customary to call

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this lattice, in the case of general W , the type W non-crossing partition lattice. The type C_n and type D_n lattices are studied in [17] and [1], respectively.

In this paper we give a new proof of the lattice property that is independent of the classification of finite real reflection groups. For this purpose we introduce a new simplicial complex which is a geometric model for the partially ordered set $[I, \gamma]$. If n is the rank of W , this simplicial complex will lie in the $(n-1)$ -sphere, S^{n-1} , in \mathbf{R}^n , and its vertex set will consist of a set of positive roots for W .

The layout of the paper is as follows. In Section 2 we motivate the construction we use. In Section 3, we recall definitions and results about the order on the set of roots. We define the simplicial complex $X(\gamma)$ and give an example. We also collect results about the partial order on the orthogonal group. In Section 4 we extend some of the material from [18] and analyse the dot products of roots of W with elements in the γ orbits of vectors in the dual basis. In Section 5 we find simple systems for certain subgroups and describe $X(\sigma)$ when σ has length two. After some preparatory work in Section 6, we characterise the geometric realisation of $X(\sigma)$ and we prove the lattice property in Section 7. We conclude in Section 8 by explaining the connection between our construction and the generalised associahedra of [13]. In the process, we give a new proof that the latter is a spherical simplicial complex.

2. MOTIVATION

The idea behind the proof of the lattice property can be described without many of the technicalities that appear later. Let W be a finite, irreducible, real reflection group with reflection set \mathcal{R} . We recall from [10] the partial order on W given by the reflection length function l . For $u, w \in W$ we say

$$u \leq w \quad \Leftrightarrow \quad l(w) = l(u) + l(u^{-1}w),$$

where $l(v)$ is the smallest positive integer k such that v can be written as a product of k reflections from \mathcal{R} . Thus $u \leq w$ if and only if there is a shortest factorisation of u as a product of reflections which is a prefix of a shortest factorisation of w . Note that l is *not* the usual length function associated to a simple system for W .

We note that the partial order on W is the restriction to W of a partial order on the orthogonal group $O(n)$ which is introduced in [9] and investigated further in [10]. Some of the notation and results from those papers will be used here. In particular, if $A \in O(n)$, we associate to A two subspaces of \mathbf{R}^n , namely

$$M(A) = \text{im}(A - I) \quad \text{and} \quad F(A) = \ker(A - I),$$

which we call the moved space of A and the fixed space of A , respectively. We recall that $M(A) = F(A)^\perp$ and note that $F(A) = F(A^{-1})$. The main result of [9] implies that if V is a subspace of $M(A)$, then there is a unique $B \in O(n)$ satisfying $M(B) = V$ and $B \leq A$. It follows that the map M , which associates to an orthogonal transformation α its moved space $M(\alpha)$, restricts to give a poset isomorphism of the interval $[I, \gamma]$ in W onto its image (in the set of subspaces of \mathbf{R}^n). Here the set of subspaces of \mathbf{R}^n is partially ordered by inclusion and is, in fact, a lattice whose meet operation is given by subspace intersection. Unfortunately, the set of subspaces of the form $M(\sigma)$ for $\sigma \leq \gamma$ is not closed under intersection. There are many examples of group elements $\alpha, \beta \leq \gamma$ for which there is no element $\delta \leq \gamma$ in W satisfying $M(\delta) = M(\alpha) \cap M(\beta)$. (For example, if $W = A_3 = \Sigma_4$, then the reflections are transpositions and every 4-cycle is a Coxeter element. Consider

$\gamma = (1, 2, 3, 4)$ and the elements α and β given by

$$\alpha = (1, 3)\gamma = (1, 2)(3, 4) \text{ and } \beta = (2, 4)\gamma = (1, 4)(2, 3).$$

Each of α and β has a two-dimensional moved space, these moved spaces intersect inside the three-dimensional space $M(\gamma)$ in a one-dimensional subspace, but no transposition precedes both α and β .)

It was noted in [10] that a group element $\alpha \leq \gamma$ is characterised by its reflection set S_α , where

$$S_\alpha = \{R \in \mathcal{R} \mid R \leq \alpha\}.$$

It was also noted there that whenever α, β and δ are elements of the interval $[I, \gamma]$ in W with the properties that $\delta \leq \alpha, \beta$ and $S_\delta = S_\alpha \cap S_\beta$, then δ must be the greatest lower bound of α and β in W ; that is, if $\tau \in W$ satisfies $\tau \leq \alpha, \beta$, then $\tau \leq \delta$. This suggests that reflection sets might have the correct intersection properties. We will see later (Theorem 7.8) that this is in fact the case. To prove this we fix a set of simple roots for W and replace $M(\alpha)$ by the positive cone on the positive roots associated to those reflections which precede α . Two surprising things happen: (i) the intersection of the positive cones associated to two elements α and β of $[I, \gamma]$ is equal to the positive cone associated to some other element δ of $[I, \gamma]$, and (ii) this collection of positive cones intersects the unit sphere in a spherical simplicial complex which we denote $X(\gamma)$.

3. NOTATION AND MAIN DEFINITIONS

Several of the (now standard) case-free proofs of results about irreducible, finite real reflection groups are due to Steinberg [18]. We will use his numbering of the roots of W in the construction of our simplicial complex. Fix a fundamental chamber C with inward unit normals $\alpha_1, \dots, \alpha_n$ and let R_1, \dots, R_n be the corresponding reflections. Assume that $S_1 = \{\alpha_1, \dots, \alpha_s\}$ and $S_2 = \{\alpha_{s+1}, \dots, \alpha_n\}$ are orthonormal sets (from Lemma 2.2 of [18]). Let γ denote the Coxeter element given by $\gamma = R_1 R_2 \dots R_n$ and let h denote the order of γ in W . We note that $M(\gamma)$ is all of \mathbf{R}^n . As in [18], we set $\rho_i = R_1 R_2 \dots R_{i-1} \alpha_i$, where the α 's and the R 's are indexed cyclically modulo n . We will use the following explicit formulae which are easily verified:

$$\rho_i = \begin{cases} \alpha_i & \text{for } i = 1, \dots, s, \\ -\gamma(\alpha_i) & \text{for } i = s+1, \dots, n, \\ \gamma(\rho_{i-n}) & \text{for } i > n. \end{cases}$$

It is proved in [18] that the positive roots relative to C are $\rho_1, \dots, \rho_{nh/2}$ while the negative roots are $\rho_{(nh/2)+1}, \dots, \rho_{nh}$. Furthermore the last $n-s$ positive roots are a permutation (possibly trivial) of S_2 . We use \preceq to denote the total order on the set of positive roots $\{\rho_1, \dots, \rho_{nh/2}\}$ determined by the subscripts. For each i , let $R(\rho_i)$ denote the reflection in W with fixed subspace ρ_i^\perp which is given by

$$R(\rho_i)(x) = x - 2(\rho_i \cdot x)\rho_i.$$

Note 3.1. If $\rho_i, \rho_{i+1}, \dots, \rho_{i+n-1}$ are n consecutive roots in this ordering, then $R(\rho_i)R(\rho_{i+1}) \dots R(\rho_{i+n-1}) = \gamma^{-1}$.

In what follows, we often consider group elements which precede γ . For such an element σ (i.e. $\sigma \leq \gamma$) we let $P_\sigma = \{\tau_1, \tau_2, \dots, \tau_t\}$ be the set of positive roots whose reflections precede σ , and we assume that the subscripts are chosen so that $\tau_i \preceq \tau_j$ whenever $i \leq j$.

We will use (often without mention) the following facts about spherical simplices. Any linearly independent set of vectors in \mathbf{R}^n determines a spherical simplex provided the angle between each pair of vectors is less than π . For such a set the spherical simplex is obtained by intersecting the unit sphere with the positive cone on those vectors. The spherical simplex determined by the linearly independent set $\{v_1, v_2, \dots, v_k\}$ will be denoted by $\langle v_1, v_2, \dots, v_k \rangle$. Since any fixed set of positive roots for W lies in a single open halfspace, each linearly independent subset of such a set of positive roots determines a spherical simplex.

Definition 3.2. We define a set, $X = X(\gamma)$, of spherical simplices by declaring that

- the vertex set is $\{\rho_1, \rho_2, \dots, \rho_{nh/2}\}$,
- an edge joins ρ_i to ρ_j whenever $i < j$ and $R(\rho_i)R(\rho_j) \leq \gamma^{-1}$ and
- $\langle \rho_{i_1}, \rho_{i_2}, \dots, \rho_{i_k} \rangle$ forms a $(k-1)$ -simplex if and only if the vertices are distinct and pairwise joined by edges.

For $\sigma \leq \gamma$, let $X(\sigma)$ be the collection of simplices of $X(\gamma)$ whose vertices lie in P_σ .

Note 3.3. It will follow from Lemma 4.8 that a set of $k \leq n$ distinct vertices, $\{\rho_{i_1}, \rho_{i_2}, \dots, \rho_{i_k}\}$, with $1 \leq i_1 < i_2 < \dots < i_k \leq nh/2$, determines a $k-1$ simplex of X if and only if

$$l[R(\rho_{i_1})R(\rho_{i_2}) \dots R(\rho_{i_k})\gamma] = n - k.$$

The largest possible dimension for a simplex of X is $n-1$, and at least $nh/2 - n + 1$ of these occur by part (c) of Proposition 4.6 and by Lemma 4.8.

A large part of this paper is devoted to proving that $X(\sigma)$ is a spherical simplicial complex for each $\sigma \leq \gamma$ and to characterising its geometric realisation. The case of H_3 is illustrated in Figure 1 and Figure 2. Figure 1 shows the moved spaces of the elements in $[I, \gamma]$ intersected with the 2-sphere and projected stereographically onto \mathbf{R}^2 , while the second figure shows the simplicial complex $X(\gamma)$. Note that the arrangement of the two-dimensional moved spaces in the first figure is the image of the Coxeter arrangement under the (non-orthogonal) linear transformation $(\gamma - I)^{-1}$. This follows because, whenever $\gamma = R(\sigma_1)R(\sigma_2)R(\sigma_3)$, then

$$\begin{aligned} M(R(\sigma_2)R(\sigma_3)) &= M(R(\sigma_1)\gamma) \\ &= [F(R(\sigma_1)\gamma)]^\perp \\ &= [(\gamma - I)^{-1}M(R(\sigma_1))]^\perp \\ &= [(\gamma - I)^{-1}(\sigma_1)]^\perp. \end{aligned}$$

The third equality follows from Lemma 3 of [9], since γ has a trivial fixed space. Note 3.1 implies that each pair of consecutive positive roots lies in the moved space of some length two element of $[I, \gamma]$. The second figure shows the restriction of the moved spaces to the span of the positive roots within them, producing the spherical simplicial complex $X(\gamma)$ with vertex set coinciding with the set of positive roots.

Let $\{\beta_1, \dots, \beta_n\}$ be the dual basis to $\{\alpha_1, \dots, \alpha_n\}$. Thus

$$\beta_i \cdot \alpha_j = \delta_{ij}, \text{ for } 1 \leq i, j \leq n, \text{ where } \delta_{ij} \text{ is the Kronecker delta.}$$

With a slight change of notation from [18] we define the vectors μ_i for $1 \leq i \leq nh$ by $\mu_i = R_1 R_2 \dots R_{i-1} \beta_i$, where the β 's and the R 's are again indexed cyclically

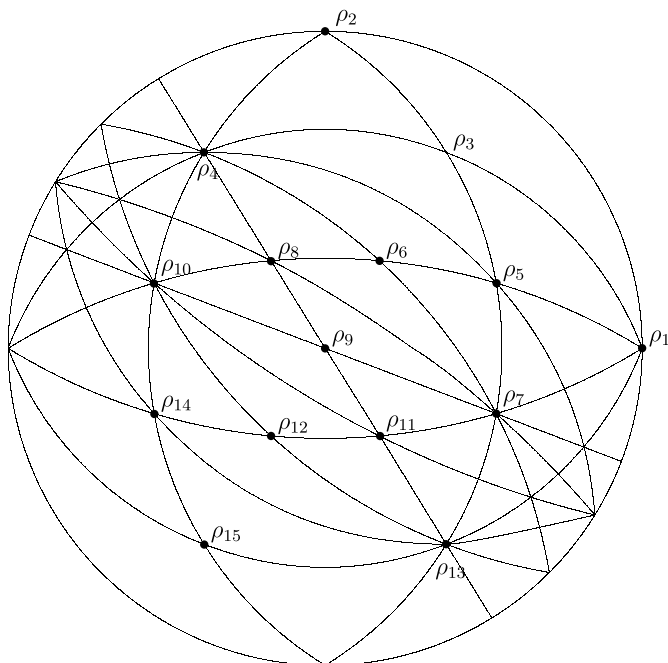


FIGURE 1

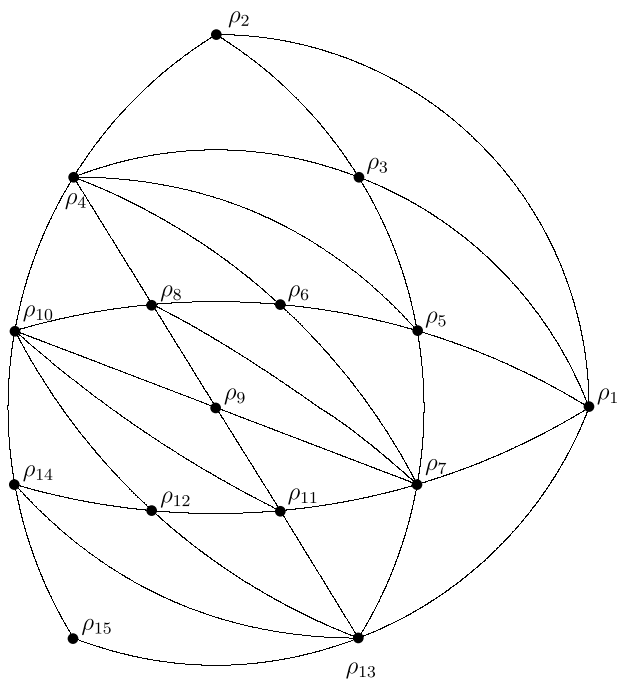


FIGURE 2

modulo n . As with the ρ 's the following explicit formulae are readily verified:

$$\mu_i = \begin{cases} \beta_i & \text{for } i = 1, \dots, n, \\ \gamma(\mu_{i-n}) & \text{for } i > n. \end{cases}$$

If σ is an arbitrary root, then we define

$$\mu(\sigma) = -2(\gamma - I)^{-1}\sigma.$$

We will see in part (a) of Corollary 4.2 that if $\sigma = \rho_i$ for some i , then $\mu(\rho_i) = \mu_i$.

For convenience, we record some results which will be used in later calculations.

$$(3.1) \quad \text{If } \alpha \leq \beta, \text{ then } M(\alpha) \subset M(\beta) \text{ and } F(\beta) \subset F(\alpha).$$

This equation follows from Corollary 1 of [9].

If $R(\vec{v})$ denotes the reflection in \vec{v}^\perp , then

$$(3.2) \quad \vec{v} \in M(\alpha) \Rightarrow R(\vec{v})\alpha = \alpha R[\alpha^{-1}(\vec{v})].$$

This equation follows from Property (A-4) of Chapter 1 of [15].

$$(3.3) \quad \text{If } \alpha \leq \beta \leq \delta, \text{ then } \alpha^{-1}\beta \leq \alpha^{-1}\delta \text{ and } \beta\alpha^{-1} \leq \delta\alpha^{-1}.$$

The first part of this equation follows from Proposition 3 of [9] while the second part follows from the fact that conjugation in W preserves reflection length.

$$(3.4) \quad \text{If } \alpha, \beta \leq \delta \text{ and } M(\alpha) \subset M(\beta), \text{ then } \alpha \leq \beta.$$

This equation is a consequence of Theorem 1 of [9].

If $l(\alpha) = k$ and R_i are reflections, then

$$(3.5) \quad \alpha = R_1 \dots R_k \Rightarrow M(R_j) \subset M(R_1\alpha) \text{ for } j = 2, \dots, k.$$

The preceding equation follows since $R_1\alpha$ has length $k - 1$ and $R_1\alpha = R_2 \dots R_k$ is a shortest expression for $R_1\alpha$ as a product of reflections.

$$(3.6) \quad \text{If } \alpha \leq \beta \leq \delta, \text{ then } \beta^{-1}\delta \leq \alpha^{-1}\delta \text{ and } \delta\beta^{-1} \leq \delta\alpha^{-1}.$$

The first inequality follows from

$$\begin{aligned} l(\alpha^{-1}\delta) &= l(\delta) - l(\alpha) \\ &= l(\delta) - l(\beta) + l(\beta) - l(\alpha) \\ &= l(\beta^{-1}\delta) + l(\alpha^{-1}\beta) \\ &= l(\beta^{-1}\delta) + l(\delta^{-1}\alpha[\alpha^{-1}\beta]\alpha^{-1}\delta) \\ &= l(\beta^{-1}\delta) + l(\delta^{-1}\beta\alpha^{-1}\delta) \\ &= l(\beta^{-1}\delta) + l([\beta^{-1}\delta]^{-1}\alpha^{-1}\delta), \end{aligned}$$

and the second is similar.

If R_1 and R_2 are distinct reflections with $R_1 \leq \alpha$ and $R_2 \leq \alpha$, then

$$(3.7) \quad R_1R_2 \leq \alpha \Leftrightarrow R_2 \leq R_1\alpha \Leftrightarrow R_1 \leq \alpha R_2.$$

To show (3.7) we note that $R_1R_2 \leq \alpha$ is equivalent to $\alpha = R_1R_2R_3 \dots R_k$ for some reflections R_3, \dots, R_k , where $k = l(\alpha)$. This, in turn, is equivalent to $\alpha = R_1(R_3 \dots R_k)^{R_2}R_2$.

4. SOME GEOMETRY OF THE ROOT SYSTEM

Our later work relies heavily on a detailed analysis of the geometry of the vectors ρ_i and μ_i . We present the necessary results in this section.

Proposition 4.1. $\gamma(\mu_i) = \mu_i - 2\rho_i$ for $1 \leq i \leq nh$.

Proof. Because of the recursions satisfied by both ρ_i and μ_i it is sufficient to establish this result for $1 \leq i \leq n$. In this case $\mu_i = \beta_i$ and, since $R_j(\beta_i) = \beta_i$ if $j \neq i$, we have

$$\begin{aligned} \gamma(\beta_i) &= (R_1 \dots R_n)\beta_i \\ &= (R_1 \dots R_i)\beta_i \\ &= (R_1 \dots R_{i-1})(\beta_i - 2\alpha_i) \\ &= \beta_i - 2(R_1 \dots R_{i-1})\alpha_i \\ &= \beta_i - 2\rho_i \quad \text{as required.} \quad \square \end{aligned}$$

Corollary 4.2. With ρ_i and μ_i defined as above we have

- (a) $(\gamma - I)\mu_i = -2\rho_i$,
- (b) $\mu_i \cdot \rho_i = 1$,
- (c) $\mu_i \in F(R(\rho_i)\gamma)$.

Proof. Part (a) is immediate from Proposition 4.1. Part (b) follows from Proposition 4.1 and the fact that $\mu_i \cdot \mu_i = \gamma(\mu_i) \cdot \gamma(\mu_i)$. From Proposition 4.1 and part (b) we see that $\gamma(\mu_i) = R(\rho_i)(\mu_i)$, which is equivalent to statement (c). \square

Note 4.3. We observe that parts (b) and (c) of Corollary 4.2 characterise μ_i as the unique vector in the one-dimensional subspace $F(R(\rho_i)\gamma)$ satisfying $\mu_i \cdot \rho_i = 1$.

In what follows we make extensive use of the properties of the matrix of dot products $[\mu_i \cdot \rho_j]$. Before establishing the general result we present some examples.

Example 4.4. Consider the symmetry group A_2 of the equilateral triangle. A simple system for A_2 is

$$\alpha_1 = a(1, -1, 0), \alpha_2 = a(0, 1, -1),$$

where $a = \sqrt{2}/2$. The dual basis to $\{\alpha_1, \alpha_2\}$ is

$$\beta_1 = b(2, -1, -1), \beta_2 = b(1, 1, -2),$$

where $b = \sqrt{2}/3$ and the dual basis is computed in the 2-dimensional subspace of \mathbf{R}^3 given by $x + y + z = 0$. Using the formula for a reflection we can generate the following table:

i	ρ_i	μ_i
1	$a(1, -1, 0)$	$b(2, -1, -1)$
2	$a(1, 0, -1)$	$b(1, 1, -2)$
3	$a(0, 1, -1)$	$b(-1, 2, -1)$

The entry in the i^{th} row and j^{th} column of the following table is $\mu_i \cdot \rho_j$:

	ρ_1	ρ_2	ρ_3
μ_1	1	1	0
μ_2	0	1	1
μ_3	-1	0	1

Example 4.5. Consider the symmetry group C_3 of the cube. A simple system for C_3 is

$$\alpha_1 = (1, 0, 0), \alpha_2 = (\sqrt{2}/2)(0, 1, -1), \alpha_3 = (\sqrt{2}/2)(-1, 0, 1).$$

The dual basis to $\{\alpha_1, \alpha_2, \alpha_3\}$ is

$$\beta_1 = (1, 1, 1), \beta_2 = (0, \sqrt{2}, 0), \beta_3 = (0, \sqrt{2}, \sqrt{2}).$$

As in the preceding example we can generate the following table:

i	ρ_i	μ_i
1	$(1, 0, 0)$	$(1, 1, 1)$
2	$(\sqrt{2}/2)(0, 1, -1)$	$\sqrt{2}(0, 1, 0)$
3	$(\sqrt{2}/2)(1, 1, 0)$	$\sqrt{2}(0, 1, 1)$
4	$(0, 1, 0)$	$(-1, 1, 1)$
5	$(\sqrt{2}/2)(1, 0, 1)$	$\sqrt{2}(0, 0, 1)$
6	$(\sqrt{2}/2)(0, 1, 1)$	$\sqrt{2}(-1, 0, 1)$
7	$(0, 0, 1)$	$(-1, -1, 1)$
8	$(\sqrt{2}/2)(-1, 1, 0)$	$\sqrt{2}(-1, 0, 0)$
9	$(\sqrt{2}/2)(-1, 0, 1)$	$\sqrt{2}(-1, -1, 0)$

The entry in the i^{th} row and j^{th} column of the following table is $\mu_i \cdot \rho_j$:

1	0	$\sqrt{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1	0	0
0	1	1	$\sqrt{2}$	0	1	0	1	0
0	0	1	$\sqrt{2}$	1	2	$\sqrt{2}$	1	1
-1	0	0	1	0	$\sqrt{2}$	1	$\sqrt{2}$	$\sqrt{2}$
0	-1	0	0	1	1	$\sqrt{2}$	0	1
$-\sqrt{2}$	-1	-1	0	0	1	$\sqrt{2}$	1	2
-1	$-\sqrt{2}$	$-\sqrt{2}$	-1	0	0	1	0	$\sqrt{2}$
$-\sqrt{2}$	0	-1	0	-1	0	0	1	1
$-\sqrt{2}$	-1	$-\sqrt{2}$	$-\sqrt{2}$	-1	-1	0	0	1

The second table in each of these examples exhibits certain symmetry properties which are valid in the general case and which we now address.

Proposition 4.6. *The quantities $\mu_i \cdot \rho_j$ have the following properties:*

- (a) $\mu_i \cdot \rho_j = -\mu_{j+n} \cdot \rho_i$ for all i and j .
- (b) $\mu_i \cdot \rho_j \geq 0$, for $1 \leq i \leq j \leq nh/2$.
- (c) $\mu_{i+t} \cdot \rho_i = 0$, for $1 \leq t \leq n-1$ and for all i .
- (d) $\mu_j \cdot \rho_i \leq 0$ for $1 \leq i < j \leq nh/2$.

Proof. (a) Since $\gamma(\mu_i) = \mu_i - 2\rho_i$ by Proposition 4.1, we compute

$$\begin{aligned}
 -\mu_{j+n} \cdot \rho_i &= -\gamma(\mu_j) \cdot (-1/2)(\gamma(\mu_i) - \mu_i) \\
 &= 1/2(\gamma(\mu_j) \cdot \gamma(\mu_i) - \gamma(\mu_j) \cdot \mu_i) \\
 &= 1/2(\mu_j \cdot \mu_i - \gamma(\mu_j) \cdot \mu_i) \\
 &= 1/2(\mu_j - \gamma(\mu_j)) \cdot \mu_i \\
 &= \rho_j \cdot \mu_i.
 \end{aligned}$$

(b) Since $\rho_p \cdot \mu_q = \gamma(\rho_p) \cdot \gamma(\mu_q)$ we can assume that $1 \leq i \leq n$ and $i < j \leq nh/2$. The result follows from the fact that ρ_j is a positive root and μ_i is part of the dual basis to $\{\alpha_1, \dots, \alpha_n\}$.

(c) Since $\gamma = R(\rho_{i+t})R(\rho_{i+t-1}) \dots R(\rho_i) \dots R(\rho_{i+t-n+1})$ (by Note 3.1) equation (3.5) implies that $\rho_i \cdot \mu_{i+t} = 0$.

(d) If $j < i + n$, this follows from part (c). If $j \geq i + n$, this property follows from parts (a) and (b). \square

Corollary 4.7. *If $1 \leq i_1 < i_2 < \dots < i_m < k \leq nh/2$, then ρ_k does not lie in the positive cone on $\{\rho_{i_1}, \dots, \rho_{i_m}\}$.*

Proof. If the vector \vec{x} is expressible as a non-negative linear combination of the roots $\rho_{i_1}, \dots, \rho_{i_m}$, then $\mu_k \cdot \vec{x} \leq 0$, since $\mu_k \cdot \rho_{i_j} \leq 0$ for $j = 1, \dots, m$. As $\mu_k \cdot \rho_k = 1$, the vector ρ_k is not expressible in this manner. \square

The following result provides information about the simplices in $X(\gamma)$.

Lemma 4.8. *Suppose $\sigma_1 \prec \sigma_2 \prec \dots \prec \sigma_k$ are chosen from the set $\{\rho_1, \dots, \rho_{nh/2}\}$. Then the following are equivalent:*

- (a) $l(R(\sigma_1)R(\sigma_2) \dots R(\sigma_k)\gamma) = n - k$.
- (b) $\mu(\sigma_i) \cdot \sigma_j = 0$ whenever $i > j$.

Proof. (a) \Rightarrow (b): Let $\delta = R(\sigma_1)R(\sigma_2) \dots R(\sigma_k)\gamma$ and assume that (a) is true. If $i > j$, then repeated application of equation (3.2) gives

$$\begin{aligned} \gamma &= R(\sigma_k) \dots R(\sigma_i) \dots R(\sigma_j) \dots R(\sigma_1)\delta \\ &= R(\sigma_i) \dots R(\sigma_j) \dots R(\sigma_1)\delta R(\gamma^{-1}(\sigma_k)) \dots R(\gamma^{-1}(\sigma_{i+1})). \end{aligned}$$

Since $l(\delta) = n - k$, it follows from equation (3.5) that $R(\sigma_j) \leq R(\sigma_i)\gamma$. Thus $\sigma_j \in M(R(\sigma_i)\gamma)$ and hence $\mu(\sigma_i) \cdot \sigma_j = 0$ (since $\mu(\sigma_i) \in F(R(\sigma_i)\gamma)$).

(b) \Rightarrow (a): Assume that (b) is true. Then the $k \times k$ matrix $[\mu(\sigma_i) \cdot \sigma_j]$ is upper triangular with ones on the diagonal (Corollary 4.2 and part (c) of Proposition 4.6). Thus this matrix is non-singular and it follows that each of $\{\sigma_1, \dots, \sigma_k\}$ and $\{\mu(\sigma_1), \dots, \mu(\sigma_k)\}$ is a linearly independent set.

Next we show that $R(\sigma_k)R(\sigma_{k-1}) \dots R(\sigma_i) \leq \gamma$ by reverse induction on i . The case $i = k$ is immediate since $M(\gamma) = \mathbf{R}^n$. Now suppose that $i < k$ and that $R(\sigma_k)R(\sigma_{k-1}) \dots R(\sigma_{i+1}) \leq \gamma$. If $i + 1 \leq j \leq k$, then using equation (3.6) and equation (3.1) we get

$$\begin{aligned} R(\sigma_j) &\leq R(\sigma_k)R(\sigma_{k-1}) \dots R(\sigma_{i+1}) \leq \gamma \\ \Rightarrow R(\sigma_j)\gamma &\geq R(\sigma_{i+1}) \dots R(\sigma_{k-1})R(\sigma_k)\gamma \\ \Rightarrow F[R(\sigma_j)\gamma] &\subseteq F[R(\sigma_{i+1}) \dots R(\sigma_{k-1})R(\sigma_k)\gamma] \\ \Rightarrow \mu(\sigma_j) &\in F[R(\sigma_{i+1}) \dots R(\sigma_{k-1})R(\sigma_k)\gamma], \end{aligned}$$

because $\mu(\sigma_j) \in F[R(\sigma_j)\gamma]$ by part (c) of Corollary 4.2. Thus $\{\mu(\sigma_{i+1}), \dots, \mu(\sigma_k)\}$ is a basis for $F[R(\sigma_{i+1}) \dots R(\sigma_{k-1})R(\sigma_k)\gamma]$ (since this subspace has dimension $k - i$ by the inductive hypothesis). As σ_i is orthogonal to each vector in this basis it follows that

$$\sigma_i \in M[R(\sigma_{i+1}) \dots R(\sigma_{k-1})R(\sigma_k)\gamma]$$

and hence $R(\sigma_k) \dots R(\sigma_{i+1})R(\sigma_i) \leq \gamma$, which completes the inductive step.

Finally, since $l(R(\sigma_k) \dots R(\sigma_2)R(\sigma_1)) = k$ (by linear independence of $\sigma_1, \dots, \sigma_k$) and $R(\sigma_k) \dots R(\sigma_2)R(\sigma_1) \leq \gamma$, we can conclude that $R(\sigma_1)R(\sigma_2) \dots R(\sigma_k)\gamma$ has length $n - k$. \square

5. THE SIMPLE SYSTEM FOR $\sigma \leq \gamma$

In this and the following sections we will consider a fixed element $\sigma \leq \gamma$. As before, we let $P_\sigma = \{\tau_1, \tau_2, \dots, \tau_t\}$ be the set of positive roots whose reflections precede σ and we assume that $\tau_1 \prec \tau_2 \prec \dots \prec \tau_t$. In this section we will characterise the simple system associated to σ and we will analyse the case where $l(\sigma) = 2$.

Let $k = l(\sigma)$. Then P_σ is the set of positive roots for a (possibly reducible) rank- k , reflection group $W_\sigma = \{\alpha \in W \mid M(\alpha) \subseteq M(\sigma)\}$. (Note that the elements of W_σ need not necessarily precede σ .) Let $\Delta = \{\delta_1, \delta_2, \dots, \delta_k\}$ be the simple system contained in P_σ and choose the subscripts so that $\delta_1 \prec \delta_2 \prec \dots \prec \delta_k$.

Proposition 5.1. *The ordered elements $\delta_1, \delta_2, \dots, \delta_k$ of Δ are determined recursively by the fact that δ_i is the last positive root in $M(\sigma R(\delta_k) R(\delta_{k-1}) \dots R(\delta_{i+1}))$. In particular, $\delta_k = \tau_t$.*

Proof. Since τ_t is the last root in P_σ we can apply Corollary 4.7 to τ_t and the set $P_\sigma - \{\tau_t\}$ to deduce that τ_t cannot be a positive linear combination of the elements of $P_\sigma - \{\tau_t\}$. Thus $P_\sigma - \{\tau_t\}$ cannot contain a simple system and $\delta_k = \tau_t$.

Next we show that $U_k = M(\sigma R(\delta_k))$ contains $\{\delta_1, \dots, \delta_{k-1}\}$. To begin, we show that $U_k = M(\sigma) \cap \varphi^\perp$, where $\varphi = \gamma(\mu(\delta_k))$. Note that $U_k = M(\sigma R(\delta_k))$ has dimension $k - 1$. By parts (a) and (b) of Proposition 4.6, if $1 \leq i \leq t$, then

$$\varphi \cdot \tau_i = \gamma(\mu(\tau_t)) \cdot \tau_i = -\tau_t \cdot \mu(\tau_i) \leq 0.$$

For $i = t$, we obtain $\varphi \cdot \tau_t = -1$. Thus τ_t is not an element of $M(\sigma) \cap \varphi^\perp$, and this subspace must also have dimension $k - 1$. Since $M(\sigma R(\delta_k))$ is contained in $M(\sigma)$ and

$$M(\sigma R(\delta_k)) \subseteq M(\gamma R(\delta_k)) = M(R[\gamma(\delta_k)]\gamma) = \mu(\gamma(\delta_k))^\perp = \gamma(\mu(\delta_k))^\perp$$

(by equation (3.3) and equation (3.2)), it follows that $U_k \subseteq M(\sigma) \cap \varphi^\perp$, and hence these subspaces are equal.

Now suppose τ is an arbitrary element of $P_\sigma \cap U_k$ and write

$$\tau = a_1 \delta_1 + \dots + a_k \delta_k, \text{ with each } a_i \geq 0.$$

Since $\tau \in \varphi^\perp$ and $\delta_i \cdot \varphi \leq 0$ for $i = 1, \dots, k$, we must have $a_i = 0$ or $\delta_i \cdot \varphi = 0$ for each i . Thus τ is a linear combination of those δ 's in $M(\sigma) \cap \varphi^\perp = U_k$. However, the span of $P_\sigma \cap U_k$ is all of U_k since $l(\sigma R(\tau_t)) = k - 1$. Hence the $k - 1$ simple roots $\delta_1, \dots, \delta_{k-1}$ must lie in U_k .

The theorem follows since we can now apply the same arguments as above to the shorter element $\sigma' = \sigma R(\delta_k)$. \square

Note 5.2. One can show that δ_i is the first root in $M(R(\delta_{i-1}) \dots R(\delta_2) R(\delta_1) \sigma)$ in an analogous way and, in particular, that $\delta_1 = \tau_1$.

Definition 5.3. We will refer to the intersection with S^{n-1} of the positive cone on $\{\delta_1, \delta_2, \dots, \delta_k\}$ as the fat simplex associated to σ .

Theorem 5.4. *If $\sigma \leq \gamma$ has length 2, then $X(\sigma)$ consists of the t roots from P_σ and the $(t - 1)$ 1-cells given by $\langle \tau_i, \tau_{i+1} \rangle$ for $i = 1, 2, \dots, t - 1$.*

Proof. Suppose $\sigma \leq \gamma$ has length two. Since both $R(\tau_i)\sigma$ and $\sigma R(\tau_i)$ have length one for each i , each subspace $M(\sigma) \cap \mu(\tau_i)^\perp$ contains precisely one positive root τ_j , and each τ_j lies in precisely one subspace of the form $M(\sigma) \cap \mu(\tau_i)^\perp$. It follows that the $t \times t$ matrix $A = [\mu(\tau_i) \cdot \tau_j]$ has precisely one zero in each row and column. Note

that the diagonal entries are all 1, the entries below the diagonal are non-positive, while the entries above the diagonal are non-negative (by Proposition 4.6). By Proposition 5.1 and Note 5.2, $\sigma = R(\tau_1)R(\tau_t)$ and hence $\tau_t \cdot \mu(\tau_1) = 0$. Let $i_1 > 1$ be the value for which $\sigma = R(\tau_{i_1})R(\tau_1)$. Thus $\mu(\tau_{i_1}) \cdot \tau_1 = 0$ and $X(\sigma)$ contains the edge $\langle \tau_1, \tau_{i_1} \rangle$. As row i_1 contains only one zero and $i_1 \leq t$, we must have $\mu(\tau_{i_1}) \cdot \tau_t > 0$ (by Proposition 4.6). Now, since $\{\tau_1, \tau_t\}$ is a simple system, we can write each τ_j as a non-negative linear combination of τ_1 and τ_t . Thus we obtain $\tau_j \cdot \mu(\tau_{i_1}) \geq 0$. It follows that each entry in row i_1 of A is non-negative. Since every row after the second has at least two entries below the diagonal and no more than one of these can be zero, we get $i_1 \leq 2$. Since we know that $i_1 \neq 1$ we deduce that i_1 must be equal to 2. If $t = 2$ the proof is complete.

The rest of the proof uses induction. Assume that $r < t$ and that

$$\sigma = R(\tau_1)R(\tau_t) = R(\tau_2)R(\tau_1) = \cdots = R(\tau_r)R(\tau_{r-1}).$$

We know that $\sigma = R(\tau_{i_r})R(\tau_r)$ for some $i_r > r$. Thus

$$\mu(\tau_{i_r}) \cdot \tau_r = 0 \text{ while } \mu(\tau_{i_r}) \cdot \tau_t > 0.$$

If $j \geq r$, then Corollary 4.7 and the fact that $M(\sigma)$ is two-dimensional imply that τ_j can be expressed as a non-negative linear combination of τ_r and τ_t . As in the case $r = 1$, we obtain $\tau_j \cdot \mu(\tau_{i_r}) \geq 0$ for $r \leq j \leq t$. It follows that each of the last $t - r + 1$ entries in row i_r of A is non-negative. Since row j has $j - 1$ entries below the diagonal and no more than one of these can be zero, we get $i_r \leq r + 1$. Since we know that $i_r > r$ we deduce that i_r must be equal to $r + 1$. \square

In the case $l(\sigma) = 2$, the action of the simple reflections on P_σ can now be deduced.

Lemma 5.5. *If $l(\sigma) = 2$ and $P_\sigma = \{\tau_1, \dots, \tau_t\}$ is the ordered set of positive roots for reflections preceding σ , then*

$$\begin{aligned} R(\tau_1)(\tau_i) &= \begin{cases} \tau_{t-i+2} & \text{for } 2 \leq i \leq t, \\ -\tau_1 & \text{for } i = 1, \end{cases} \\ R(\tau_t)(\tau_i) &= \begin{cases} \tau_{t-i} & \text{for } 1 \leq i \leq t-1, \\ -\tau_t & \text{for } i = t. \end{cases} \end{aligned}$$

Proof. We will consider the case of τ_1 . The case of τ_t is similar. From the proof of Theorem 5.4 it follows that the only expressions for σ as a product of two reflections are

$$\sigma = R(\tau_1)R(\tau_t) = R(\tau_2)R(\tau_1) = R(\tau_3)R(\tau_2) = \cdots = R(\tau_t)R(\tau_{t-1}).$$

Since $R(\tau_1)$ permutes $P_\sigma - \{\tau_1\}$ (Lemma A in I.4.3 of [15]) and

$$\sigma = R(\tau_1)R(\tau_t) = (R(\tau_1)R(\tau_t)R(\tau_1))R(\tau_1) = R(R(\tau_1)\tau_t)R(\tau_1)$$

we deduce that $R(\tau_1)\tau_t = \tau_2$. In general, suppose that $1 < i + 1 \leq t$ and that $R(\tau_1)\tau_{i+1} = \tau_{t-(i+1)+2}$. Conjugate $R(\tau_t)R(\tau_1) = \sigma^{-1} = R(\tau_i)R(\tau_{i+1})$ by $R(\tau_1)$ to obtain

$$R(\tau_1)R(\tau_t) = \sigma = R(R(\tau_1)\tau_i)R(R(\tau_1)\tau_{i+1}) = R(R(\tau_1)\tau_i)R(\tau_{t-i+1}).$$

We deduce that $R(\tau_1)\tau_i = \tau_{(t-i+1)+1} = \tau_{t-i+2}$ (by comparing with the list of factorisations for σ). Now use reverse induction on i , starting at $i = t$. \square

Note 5.6. The proofs of Lemma 5.5 and Theorem 5.4 can be shortened considerably by using the notion of a reflection ordering as introduced in [6] and developed in [12]. It follows from the definition of ρ_i together with Exercise 6.2 of Chapter 5 of [6] and Proposition 2.13 of [12] that

$$R(\rho_1), R(\rho_2), \dots, R(\rho_{nh/2})$$

is a reflection ordering on W , in the sense that the induced order on the set of reflections in any dihedral subgroup is

$$a < aba < ababa < \dots < bab < b,$$

where a, b is a set of simple reflections for the dihedral group. From this it follows that when $l(\sigma) = 2$,

$$\tau_{2k+1} = [R(\tau_1)R(\tau_t)]^k \tau_1 \text{ and } \tau_{2k+2} = [R(\tau_1)R(\tau_t)]^k R(\tau_1)\tau_t.$$

Lemma 5.5 follows immediately from this. The identities

$$\sigma = R(\tau_1)R(\tau_t) = R(\tau_2)R(\tau_1) = R(\tau_3)R(\tau_2) = \dots = R(\tau_t)R(\tau_{t-1})$$

also follow from the expressions of τ_i in terms τ_1 and τ_t , and Theorem 5.4 can be reduced to the argument that the subspaces $M(R(\tau_i)\sigma)$ and $M(\sigma R(\tau_i))$ are one-dimensional. However, we prefer to use the current proofs which, although longer, do not require the rather technical facts about reflection orderings.

Lemma 5.7. Assume that $R(\rho_i)R(\rho_j) \leq \gamma$, where ρ_i and ρ_j are distinct positive roots. Then

- (a) $\rho_i \cdot \rho_j \leq 0$ if $i < j$.
- (b) $\rho_i \cdot \rho_j \geq 0$ if $i > j$.

Proof. Let $P_\sigma = \{\tau_1, \dots, \tau_t\}$ be the ordered set of positive roots whose reflections precede $\sigma = R(\rho_i)R(\rho_j)$. By Theorem 5.4, $\{\tau_1, \tau_t\}$ is the corresponding simple system and the only expressions for σ as a product of two reflections are

$$\sigma = R(\tau_1)R(\tau_t) = R(\tau_2)R(\tau_1) = R(\tau_3)R(\tau_2) = \dots = R(\tau_t)R(\tau_{t-1}).$$

Since $\sigma = R(\tau_1)R(\tau_t)$ is the only case in which the roots appear in increasing order, if $i < j$, then we must have $\rho_i = \tau_1$ and $\rho_j = \tau_t$. Part (a) now follows from $\rho_i \cdot \rho_j = \tau_1 \cdot \tau_t \leq 0$ (since $\{\tau_1, \tau_t\}$ is a simple system).

If $i > j$, then $\rho_i = \tau_{r+1}$ and $\rho_j = \tau_r$ for some $r \geq 1$. Thus we need to verify that $\tau_i \cdot \tau_{i+1} \geq 0$ for $i = 1, \dots, t-1$. Using Lemma 5.5 and the facts that reflection is an isometry and $\{\tau_1, \tau_t\}$ is a simple system, we obtain

$$\tau_1 \cdot \tau_2 = R(\tau_1)\tau_1 \cdot R(\tau_1)\tau_2 = -\tau_1 \cdot \tau_t \geq 0.$$

For the same reasons,

$$\tau_2 \cdot \tau_3 = R(\tau_1)R(\tau_t)(-\tau_t) \cdot R(\tau_1)R(\tau_t)\tau_1 = -\tau_t \cdot \tau_1 \geq 0.$$

Finally, if $i \geq 2$, then

$$\begin{aligned} \tau_i \cdot \tau_{i+1} &= R(\tau_t)R(\tau_1)\tau_i \cdot R(\tau_t)R(\tau_1)\tau_{i+1} \\ &= R(\tau_t)\tau_{t-i+2} \cdot R(\tau_t)\tau_{t-i+1} = \tau_{i-2} \cdot \tau_{i-1}, \end{aligned}$$

and the result follows by induction on i . □

Note 5.8. It is immediate from Lemma 5.7 that the edges of X can now be characterised by the following geometric criterion. There is an edge joining the distinct vertices ρ_i and ρ_j if and only if the vectors subtend a non-obtuse angle and one of $R(\rho_i)R(\rho_j)$ or $R(\rho_j)R(\rho_i)$ precedes γ .

6. WALLS OF FAT SIMPLICES

In this section we continue to investigate a fixed element σ of length k which precedes γ in W . Using the same notation as in the preceding section, we find a dual basis to $\{\delta_1, \dots, \delta_k\}$ (Corollary 6.8). We also determine the first top-dimensional simplex of $X(\sigma)$ in the lexicographic order.

Definition 6.1. For each $i = 1, \dots, k$, we define ϵ_i to be the root given by

$$\epsilon_i = [R(\delta_1) \dots R(\delta_{i-1})](\delta_i).$$

Proposition 6.2. For each $i = 1, \dots, k$, the vector ϵ_i is a positive root. Moreover

$$\sigma = R(\epsilon_k)R(\epsilon_{k-1}) \dots R(\epsilon_1)$$

and for each $i = 1, \dots, k$

$$\sigma = R(\epsilon_i)R(\delta_1) \dots R(\delta_{i-1})R(\delta_{i+1}) \dots R(\delta_k).$$

Proof. The first statement is a special case of Lemma D in Section I.4.3 of [15]. The remaining statements follow from

$$\begin{aligned} R(\epsilon_i) &= R([R(\delta_1) \dots R(\delta_{i-1})](\delta_i)) \\ &= [R(\delta_1)R(\delta_2) \dots R(\delta_{i-1})]R(\delta_i)[R(\delta_{i-1}) \dots R(\delta_2)R(\delta_1)] \end{aligned}$$

since each reflection has order 2. \square

Corollary 6.3. The walls of the spherical simplex on the vertices $\{\delta_1, \dots, \delta_k\}$ in the subspace $M(\sigma)$ are given by intersecting the planes $\mu^\perp(\epsilon_i)$ with the unit sphere in $M(\sigma)$.

Proof. The second identity in Proposition 6.2 implies that the subspace $\mu^\perp(\epsilon_i)$ contains the set $\{\delta_1, \dots, \delta_{i-1}, \delta_{i+1}, \dots, \delta_k\}$, for $i = 1, \dots, k$. \square

Note 6.4. We will see in the next section that this simplex is the geometric realisation of $X(\sigma)$.

Definition 6.5. For each $i = 1, \dots, t$, let $\mu'(\tau_i)$ be the orthogonal projection of $\mu(\tau_i)$ into $M(\sigma)$.

Proposition 6.6. For each $i, j \in \{1, \dots, t\}$, we have $\mu'(\tau_i) \cdot \tau_j = \mu(\tau_i) \cdot \tau_j$ and $\mu'(\tau_i) \in F(R(\tau_i)\sigma)$. In particular,

$$\sigma(\mu'(\tau_i)) = [R(\tau_i)](\mu'(\tau_i)) = \mu'(\tau_i) - 2\tau_i.$$

Proof. Write $\mu(\tau_i) = \mu'(\tau_i) + \vec{y}$, where $\mu'(\tau_i) \in M(\sigma)$ and $\vec{y} \in [M(\sigma)]^\perp = F(\sigma)$. Then $\mu(\tau_i) \cdot \tau_j = \mu'(\tau_i) \cdot \tau_j$ for any $1 \leq j \leq t$. Now $\mu(\tau_i) \in F(R(\tau_i)\gamma) \subset F(R(\tau_i)\sigma)$, because $R(\tau_i)\sigma \leq R(\tau_i)\gamma$ by equation (3.3). But \vec{y} is also an element of $F(R(\tau_i)\sigma)$, since $F(R(\tau_i)\sigma)$ contains $F(\sigma)$ by equation (3.1). It follows that $\mu'(\tau_i)$ is an element of $F(R(\tau_i)\sigma)$. The final claim of the proposition follows from this. \square

Proposition 6.7. For $1 \leq i \leq k$, we have $\mu(\epsilon_i) \cdot \delta_i = 1$.

Proof. Fix i and let $\sigma' = R(\delta_1) \dots R(\delta_i)$ (which precedes σ and hence γ). Write $\mu(\epsilon_i) = \mu''(\epsilon_i) + \bar{z}$, where $\mu''(\epsilon_i) \in M(\sigma')$ and $\bar{z} \in M(\sigma')^\perp = F(\sigma')$. Applying Proposition 6.6 to σ' yields

$$\begin{aligned}
 \mu(\epsilon_i) \cdot \delta_i &= \sigma'[\mu(\epsilon_i)] \cdot \sigma'[\delta_i] \\
 &= \sigma'[\mu''(\epsilon_i) + \bar{z}] \cdot \sigma'[\delta_i] \\
 &= [\mu''(\epsilon_i) - 2\epsilon_i + \bar{z}] \cdot R(\delta_1) \dots R(\delta_{i-1})R(\delta_i)[\delta_i] \\
 &= [\mu(\epsilon_i) - 2\epsilon_i] \cdot R(\delta_1) \dots R(\delta_{i-1})[-\delta_i] \\
 &= [\mu(\epsilon_i) - 2\epsilon_i] \cdot [-\epsilon_i] \\
 &= 1 \quad \text{as required.} \quad \square
 \end{aligned}$$

The following corollary is immediate from Corollary 6.3 and Proposition 6.7

Corollary 6.8. *The dual basis to $\{\delta_1, \dots, \delta_k\}$ is $\{\mu'(\epsilon_1), \dots, \mu'(\epsilon_k)\}$.*

It need not be the case that $\epsilon_1 \prec \dots \prec \epsilon_k$. However the induced order on this set still determines a factorisation of σ (Proposition 6.11 below). First we make a definition.

Definition 6.9. Let $\theta_1, \dots, \theta_k$ be the reordering of $\epsilon_1, \dots, \epsilon_k$ for which $\theta_i \prec \theta_j$ whenever $i < j$.

Lemma 6.10. *If $i < j$ but $\epsilon_i > \epsilon_j$, then $R(\epsilon_j)R(\epsilon_i) = R(\epsilon_i)R(\epsilon_j)$.*

Proof. Assume that $i < j$ and that $\epsilon_i > \epsilon_j$. We need to show that $\epsilon_i \cdot \epsilon_j = 0$. Since $\{\delta_1, \dots, \delta_k\}$ is a simple system, each of the dot products $\delta_i \cdot \delta_j$ is non-positive for $i \neq j$. It follows that $[R(\delta_{j-1})](\delta_j)$ is a non-negative linear combination of δ_{j-1} and δ_j and, by induction, that $[R(\delta_{i+1}) \dots R(\delta_{j-1})](\delta_j)$ is a non-negative linear combination of $\delta_{i+1}, \delta_{i+2}, \dots, \delta_j$. Hence, $\delta_i \cdot [R(\delta_{i+1}) \dots R(\delta_{j-1})](\delta_j) \leq 0$. Now, since $i < j$, we can compute that

$$\begin{aligned}
 \epsilon_i \cdot \epsilon_j &= [R(\delta_1) \dots R(\delta_{i-1})](\delta_i) \cdot [R(\delta_1) \dots R(\delta_{j-1})](\delta_j) \\
 &= \delta_i \cdot [R(\delta_i) \dots R(\delta_{j-1})](\delta_j) \\
 &= [R(\delta_i)](\delta_i) \cdot [R(\delta_{i+1}) \dots R(\delta_{j-1})](\delta_j) \\
 &= -\delta_i \cdot [R(\delta_{i+1}) \dots R(\delta_{j-1})](\delta_j) \\
 &\geq 0.
 \end{aligned}$$

However, since $i < j$, the identity $\sigma = R(\epsilon_k) \dots R(\epsilon_j) \dots R(\epsilon_i) \dots R(\epsilon_1)$ implies that $R(\epsilon_j)R(\epsilon_i)$ precedes σ and hence γ . Now part (a) of Lemma 5.7 implies that $\epsilon_i \cdot \epsilon_j \leq 0$ (because $\epsilon_i > \epsilon_j$). We conclude that $\epsilon_i \cdot \epsilon_j = 0$, as required. \square

Proposition 6.11. *The elements $\theta_1, \dots, \theta_k$ satisfy the identity*

$$\sigma = R(\theta_k) \dots R(\theta_1).$$

Proof. We know that $\sigma = R(\epsilon_k) \dots R(\epsilon_1)$. If $\epsilon_i = \theta_i$ for each i , there is nothing to prove. Otherwise, repeated application of Lemma 6.10 yields the required result. \square

Corollary 6.12. *The simplex $\langle \theta_1, \dots, \theta_k \rangle$ is the first top dimensional simplex of $X(\sigma)$ in the lexicographic order.*

Proof. Since $\theta_1 < \theta_2 < \cdots < \theta_k$ and $\sigma = R(\theta_k) \cdots R(\theta_1)$, the $(k-1)$ -simplex $\langle \theta_1, \dots, \theta_k \rangle$ is in $X(\sigma)$. As $\{\delta_1, \dots, \delta_k\}$ is the simple system corresponding to P_σ , $\{\theta_1, \dots, \theta_k\}$ is a rearrangement of $\{\epsilon_1, \dots, \epsilon_k\}$ and

$$\mu(\epsilon_a) \cdot \delta_b = \begin{cases} 0 & \text{for } a \neq b, \\ 1 & \text{for } a = b, \end{cases}$$

it follows that $\mu(\theta_i) \cdot \tau_j \geq 0$ for $1 \leq i \leq k$ and $1 \leq j \leq t$. However, Proposition 4.6 implies that whenever $\tau_j < \theta_i$, the dot product $\mu(\theta_i) \cdot \tau_j$ is non-positive, and hence it must be zero. Therefore

$$\{\tau_j \mid \tau_j < \theta_i\} \subseteq M(\sigma) \cap \mu(\theta_i)^\perp \cap \mu(\theta_{i+1})^\perp \cap \cdots \cap \mu(\theta_k)^\perp$$

which is an $(i-1)$ -dimensional subspace of \mathbf{R}^n since the θ 's are linearly independent. Now if $\langle \tau_{i_1}, \dots, \tau_{i_k} \rangle$ is a $(k-1)$ -simplex of $X(\sigma)$ with $\tau_{i_1} < \tau_{i_2} < \cdots < \tau_{i_k}$, then $\langle \tau_{i_1}, \dots, \tau_{i_j} \rangle$ is a $(j-1)$ -simplex for each $j \leq k$, and this forces $\tau_{i_j} \geq \theta_j$. \square

We finish this section with two examples. The first shows that the first k roots of P_σ may fail to span a top dimensional simplex. The second example illustrates that even in the case $\sigma = \gamma$, it is possible that there is some $i < j$ for which $\epsilon_j \prec \epsilon_i$.

Example 6.13. Consider the symmetry group of the 4-dimensional cube. One simple system of unit vectors for this group is

$$\alpha_1 = (1, 0, 0, 0), \alpha_2 = (\sqrt{2}/2)(0, 1, 0, -1),$$

$$\alpha_3 = (\sqrt{2}/2)(-1, 0, 0, 1), \alpha_4 = (\sqrt{2}/2)(0, -1, 1, 0).$$

The element $\gamma = [1, 2, 3, 4]$ (in the notation of [10]) is one of the Coxeter elements determined by this simple system, where

$$[1, 2, 3, 4](x, y, z, w) = (-w, x, y, z).$$

The element $\sigma = [1, 2, 3]$ precedes γ , has length three and its first three positive roots are

$$(1, 0, 0, 0), \quad (\sqrt{2}/2)(1, 1, 0, 0), \quad (0, 1, 0, 0).$$

These cannot span a 2-simplex since they are linearly dependent.

Example 6.14. Consider the group $A_3 = \Sigma_4$. We have seen that the four-cycle $\gamma = (1, 2, 3, 4)$ is a Coxeter element. The simple system corresponding to the factorisation $\gamma = (1, 3)(1, 2)(3, 4)$ is given by normalising

$$(1, 0, -1, 0), \quad (-1, 1, 0, 0), \quad (0, 0, 1, -1).$$

For notational convenience we will identify roots with the transpositions they determine. The global order is the following:

$$(1, 3), (2, 3), (1, 4), (2, 4), (3, 4), (1, 2).$$

We use Proposition 5.1 to find the simple system. First $\delta_3 = (1, 2)$ since this is the last root. Since $(1, 2, 3, 4)(1, 2) = (1, 3, 4)$, $\delta_2 = (3, 4)$. Finally, $\delta_1 = (1, 3)$. Thus $\epsilon_1 = (1, 3) = \tau_1$, $\epsilon_2 = (1, 4) = \tau_3$ and $\epsilon_3 = (2, 3) = \tau_2$ so that $\epsilon_1 \prec \epsilon_3 \prec \epsilon_2$.

7. CHARACTERISATION OF $|X(\sigma)|$ AND PROOF OF THE LATTICE PROPERTY

In this section we show that if $\sigma \leq \gamma$, then $X(\sigma)$ is a simplicial complex. We characterise the geometric realisation $|X(\sigma)|$ of the subcomplex $X(\sigma)$ and use this characterisation to prove that the interval $[I, \gamma]$ in W is a lattice. Throughout this section we continue to use the notation of the earlier sections.

We begin with a technical result (Proposition 7.2) about the separation properties of the hyperplanes $\{\mu(\tau_i)^\perp\}$ which is used in the proof of Proposition 7.6. It depends on Proposition 7.1 which is stated for convenient reference and which concerns the action of σ^{-1} on P_σ . Using the factorisation $\sigma^{-1} = R(\delta_k) \dots R(\delta_1)$, this is a special case of Theorem B in Section I.4.3 of [15].

Proposition 7.1. *If $\tau_s \in P_\sigma$, then $\sigma^{-1}(\tau_s) \in -P_\sigma$ if and only if $\tau_s = \epsilon_i$ for some i with $1 \leq i \leq k$.*

Proposition 7.2. *Let τ_a and τ_s be elements of P_σ such that $\tau_a \succeq \theta_k$, $\tau_s \prec \tau_a$, $\tau_s \notin \{\theta_1, \theta_2, \dots, \theta_{k-1}, \theta_k\}$ and $\tau_a \in \mu(\tau_s)^\perp$. Then we can find two roots τ_b and τ_c in P_σ with $\tau_b, \tau_c \prec \tau_a$, which both lie on the hyperplane $\mu(\tau_a)^\perp$, but which are separated by the hyperplane $\mu(\tau_s)^\perp$.*

Proof. Since $\tau_a \cdot \mu(\tau_s) = 0$, $\tau_a \in M(R(\tau_s)\gamma)$ and the element $R(\tau_s)R(\tau_a)$ precedes γ by equation (3.7).

Construction of τ_b : Let $Q = \{\tau_i \in P_\sigma \mid R(\tau_i) \leq R(\tau_s)R(\tau_a)\}$. Since $\tau_s \prec \tau_a$, the proof of Lemma 5.7 implies that (i) the set $\{\tau_s, \tau_a\}$ is a simple system which spans Q and (ii) τ_s is the first root and τ_a the last root of Q . By Lemma 5.5, the element τ_b given by $\tau_b = [R(\tau_a)](\tau_s)$ is in Q and $s < b < a$. Since

$$R(\tau_a)R(\tau_b) = R(\tau_a)(R(\tau_a)R(\tau_s)R(\tau_a)) = R(\tau_s)R(\tau_a) \leq \gamma,$$

we deduce that $\tau_b \cdot \mu(\tau_a) = 0$. Finally $\tau_b \cdot \mu(\tau_s) > 0$ because

$$\mu(\tau_s) \cdot \tau_b = \mu(\tau_s) \cdot [R(\tau_a)](\tau_s) = \mu(\tau_s) \cdot \{\tau_s - 2(\tau_a \cdot \tau_s)\tau_a\} = 1$$

since $\mu(\tau_s) \cdot \tau_s = 1$ and $\mu(\tau_s) \cdot \tau_a = 0$.

Construction of τ_c : As τ_s is not an element of $\{\theta_1, \dots, \theta_k\} = \{\epsilon_1, \dots, \epsilon_k\}$, the root τ_c defined by $\tau_c = \sigma^{-1}(\tau_s)$ is an element of P_σ (by Proposition 7.1). Since $R(\tau_s)R(\tau_a) \leq \gamma$ and $M[R(\tau_s)R(\tau_a)] \subset M(\sigma)$ we deduce by equation (3.4) that $R(\tau_s)R(\tau_a) \leq \sigma$. This gives $R(\tau_a) \leq R(\tau_s)\sigma = \sigma R(\tau_c)$ by equation (3.2) and hence $R(\tau_a)R(\tau_c) \leq \sigma \leq \gamma$ by equation (3.7). It follows that $\tau_c \cdot \mu(\tau_a) = 0$. Now $\tau_c \cdot \mu(\tau_s) < 0$ since

$$\begin{aligned} \mu(\tau_s) \cdot \tau_c &= \mu'(\tau_s) \cdot \tau_c \\ &= \mu'(\tau_s) \cdot \sigma^{-1}(\tau_s) \\ &= \sigma(\mu'(\tau_s)) \cdot \tau_s \\ &= \{\mu'(\tau_s) - 2\tau_s\} \cdot \tau_s \text{ by Proposition 6.6} \\ &= 1 - 2 = -1. \end{aligned}$$

As $\mu(\tau_s) \cdot \tau_c < 0$, Proposition 4.6 implies that $\tau_c \prec \tau_s$ and hence $\tau_c \prec \tau_a$. □

Definition 7.3. If τ_i is a root in P_σ and ρ is any positive root, we define

- $\mu(\tau_i)^+ = \{x \in \mathbf{R}^n \mid x \cdot \mu(\tau_i) \geq 0\}$ (a positive halfspace),
- $\mu(\tau_i)^- = \{x \in \mathbf{R}^n \mid x \cdot \mu(\tau_i) \leq 0\}$ (a negative halfspace),

- $X(\sigma, \rho)$ = the set of simplices of X whose vertices both lie in $M(\sigma)$ and precede ρ in the total order,
- $c[X(\sigma, \rho)]$ = the positive cone on the set $|X(\sigma, \rho)|$,
- $Y(\sigma, \rho)$ = the positive cone on those roots which both lie in $M(\sigma)$ and precede ρ in the total order,
- $Z(\sigma, \tau_i) = M(\sigma) \cap \mu(\theta_1)^+ \cap \cdots \cap \mu(\theta_k)^+ \cap \mu(\tau_{i+1})^- \cap \cdots \cap \mu(\tau_t)^-$, for $\tau_i \geq \theta_k$.

Proposition 7.4. *For each $i = 1, \dots, t$, the set $X(\sigma, \tau_i)$ is a simplicial complex.*

Proof. We use induction on i . First note that $X(\sigma, \tau_1) = \{\langle \tau_1 \rangle\}$, a zero-dimensional simplicial complex.

Assume now that $i \geq 1$ and that $X(\sigma, \tau_i)$ is a simplicial complex. By definition, if $\tau_j \in X(\sigma, \tau_i)$, then $\langle \tau_j, \tau_{i+1} \rangle \in X(\sigma, \tau_{i+1})$ if and only if $R(\tau_{i+1})R(\tau_j) \leq \gamma$ since $\tau_j \prec \tau_{i+1}$. However, $R(\tau_{i+1})R(\tau_j) \leq \gamma$ if and only if $\tau_j \cdot \mu(\tau_{i+1}) = 0$, because $\mu(\tau_{i+1}) \in F(R(\tau_j)\gamma)$. It follows from Proposition 4.6 that the only vertices of $X(\sigma, \tau_i)$ that are not joined to τ_{i+1} by an edge in $X(\sigma, \tau_{i+1})$, lie in the interior of the halfspace $\mu(\tau_{i+1})^+$. Hence, $\mu(\tau_{i+1})^\perp \cap |X(\sigma, \tau_i)|$ is a simplicial complex. Now each simplex in $X(\sigma, \tau_{i+1}) \setminus X(\sigma, \tau_i)$ is of the form $\langle \tau_{a_1}, \dots, \tau_{a_b}, \tau_{i+1} \rangle$ where $\tau_{a_1} \prec \cdots \prec \tau_{a_b} \prec \tau_{i+1}$ and where $\tau_{a_c} \in \mu(\tau_{i+1})^\perp$ for $c = 1, \dots, b$. Thus the simplex $\langle \tau_{a_1}, \dots, \tau_{a_b} \rangle$ of $X(\sigma, \tau_i)$ is contained in $\mu(\tau_{i+1})^\perp$. Using this, it is straightforward to verify that the intersection of any two simplices in $X(\sigma, \tau_{i+1})$ is itself a simplex in $X(\sigma, \tau_{i+1})$. \square

Corollary 7.5. *For each $\sigma \leq \gamma$, $X(\sigma)$ is a simplicial complex of dimension $l(\sigma) - 1$. In particular, $X(\gamma)$ is a simplicial complex of dimension $n - 1$.*

Proposition 7.6. *For $\theta_k \preceq \tau_i \preceq \tau_t$, we have $c[X(\sigma, \tau_i)] = Y(\sigma, \tau_i) = Z(\sigma, \tau_i)$.*

Proof. It suffices to show that $Z(\sigma, \tau_i)$ is contained in $c[X(\sigma, \tau_i)]$ because $c[X(\sigma, \rho)]$ is contained in $Y(\sigma, \rho)$ (by definition) and $Y(\sigma, \tau_i)$ is contained in $Z(\sigma, \tau_i)$ when $\tau_i \geq \theta_k$ (by Proposition 4.6). The proof is by induction on i , starting at the value i_0 for which $\tau_{i_0} = \theta_k$.

Base step: Let $F_0 = X(\sigma, \tau_{i_0-1})$ be the subcomplex of $X(\gamma)$ whose vertex set is $\{\tau_1, \tau_2, \dots, \tau_{i_0-1}\}$. Since $\langle \theta_1, \theta_2, \dots, \theta_k \rangle \in X(\sigma)$ (by Corollary 6.12) and $\theta_1, \theta_2, \dots, \theta_{k-1} \in F_0$, it follows that the simplex $\langle \theta_1, \theta_2, \dots, \theta_{k-1} \rangle$ is in F_0 . Furthermore, if $\sigma_0 = [R(\theta_k)](\sigma)$, then $F_0 = X(\sigma_0, \tau_{i_0-1})$. By induction on $k = l(\sigma)$, we can assume that the assertion of the theorem is valid if σ is replaced by the length $k - 1$ element σ_0 and hence $c[X(\sigma_0, \tau_{i_0-1})] = Z(\sigma_0, \tau_{i_0-1})$. (The base case of this inner induction is trivial since $k = 1$ corresponds to a rank 1 group.) Thus $|F_0|$ is convex and $(k - 2)$ -dimensional.

Now let $V_0 = X(\sigma, \tau_{i_0})$, which has vertex set $\{\tau_1, \tau_2, \dots, \tau_{i_0}\}$. Since $\mu(\theta_k) \cdot \tau_j = 0$ whenever $\tau_j \prec \theta_k$ (as in the proof of Corollary 6.12), it follows that $|V_0|$ is the cone with base $|F_0|$ and apex τ_{i_0} . Thus $|V_0|$ is convex and $(k - 1)$ -dimensional.

The containment of $Z(\sigma, \tau_{i_0})$ in $c[X(\sigma, \tau_{i_0})]$ is demonstrated by examining the supports of the facets of the positive cone, $c[V_0]$, on $|V_0|$. Each support is of the form $M(\sigma) \cap \mu(\tau_j)^\perp$ for some τ_j .

One of the facets of $c[V_0]$ contains $|F_0|$ and hence it has support $M(\sigma) \cap \mu(\tau_{i_0})^\perp$. Each of the other facets of $c[V_0]$ contains the vertex $\tau_{i_0} = \theta_k$ and hence its support is of the form $M(\sigma) \cap \mu(\tau_j)^\perp$ for some $j \neq i_0$. In fact, τ_j must belong to either $\{\theta_1, \dots, \theta_{k-1}\}$ or $\{\tau_j \mid \tau_j \succ \theta_k\}$ in this case. For $\mu(\tau_j)^\perp$ cannot separate $|F_0|$ (because $|F_0| \subset |V_0|$ which is contained in one of the half-spaces bounded by $\mu(\tau_j)^\perp$)

and hence if $\tau_j \notin \{\theta_1, \dots, \theta_{k-1}\}$, then Proposition 7.2 implies that we must have $\tau_j \succ \theta_k$. Thus the set of facets of $c[V_0]$ is of the form

$$\{M(\sigma) \cap \mu(\tau_j)^\perp \mid \tau_j \in \{\theta_{i_1}, \dots, \theta_{i_a}, \theta_k\} \cup \{\tau_{j_1}, \dots, \tau_{j_b}\}\}$$

where $j_l > i_0$ for $l = 1, 2, \dots, b$. It follows that

$$\begin{aligned} Z(\sigma, \tau_{i_0}) &= M(\sigma) \cap \mu(\theta_1)^+ \cap \dots \cap \mu(\theta_k)^+ \cap \mu(\tau_{i_0+1})^- \cap \dots \cap \mu(\tau_t)^- \\ &\subseteq M(\sigma) \cap \mu(\theta_{i_1})^+ \cap \dots \cap \mu(\theta_{i_a})^+ \cap \mu(\theta_k)^+ \\ &\quad \cap \mu(\tau_{j_1})^- \cap \dots \cap \mu(\tau_{j_b})^- \\ &= c[X(\sigma, \tau_{i_0})]. \end{aligned}$$

Inductive step: Assume now that $i \geq i_0$ and that $c[X(\sigma, \tau_i)] = Z(\sigma, \tau_i)$. Let F and V be the subcomplexes of $X(\gamma)$ whose sets of vertices are

$$\{\tau_j \mid 1 \leq j \leq i \text{ and } \mu(\tau_{i+1}) \cdot \tau_j = 0\}$$

and

$$\{\tau_{i+1}\} \cup \{\tau_j \mid 1 \leq j \leq i \text{ and } \mu(\tau_{i+1}) \cdot \tau_j = 0\},$$

respectively. Then $|V|$ is a cone with base $|F|$ and apex τ_{i+1} . We prove that the closure, Z , of $Z(\sigma, \tau_{i+1}) \setminus Z(\sigma, \tau_i)$ is contained in the positive cone, $c[V]$, on $|V|$. Since $|V|$ is contained in $|X(\sigma, \tau_{i+1})|$ and $Z(\sigma, \tau_{i+1}) = Z \cup Z(\sigma, \tau_i)$, it will then follow that $Z(\sigma, \tau_{i+1})$ is contained in $c[X(\sigma, \tau_{i+1})]$, as required.

First we show that F is $(k-2)$ -dimensional. Denote the length $(k-1)$ element $[R(\tau_{i+1})](\sigma)$ by σ' . Note that $\sigma' \leq R(\tau_{i+1})\gamma$ so that $M(\sigma') \subset \mu(\tau_{i+1})^\perp$. Apply the procedures of Sections 5 and 6 to σ' (i) to obtain a simple system $\{\delta'_1, \delta'_2, \dots, \delta'_{k-1}\}$ for the set $P_{\sigma'}$ of positive roots in $M(\sigma')$ and (ii) to calculate the reordering $\{\theta'_1, \theta'_2, \dots, \theta'_{k-1}\}$ of the set $\{\epsilon'_1, \epsilon'_2, \dots, \epsilon'_{k-1}\}$ for which

$$\sigma' = R(\theta'_{k-1}) \dots R(\theta'_2) R(\theta'_1)$$

where $\theta'_1 \prec \theta'_2 \prec \dots \prec \theta'_{k-1}$ and where ϵ'_j is given by

$$\epsilon'_j = [R(\delta'_1)R(\delta'_2) \dots R(\delta'_{j-1})](\delta'_j) \text{ for } j = 1, \dots, k-1.$$

Since $\{\theta'_1, \theta'_2, \dots, \theta'_{k-1}\} \subset \mu(\tau_{i+1})^\perp$, if we show that $\tau_{i+1} > \theta'_j$ for $j = 1, \dots, k-1$, then it will follow that $\langle \theta'_1, \theta'_2, \dots, \theta'_{k-1} \rangle \in F$ and hence F is $(k-2)$ -dimensional.

Fix $j \in \{1, \dots, k-1\}$. From $\epsilon'_j \cdot \mu(\tau_{i+1}) = 0$ we deduce that $\epsilon'_j \neq \tau_{i+1}$ and that $R(\epsilon'_j) \leq R(\tau_{i+1})\gamma$. Therefore the length two element $\sigma'' = R(\tau_{i+1})R(\epsilon'_j)$ precedes γ and hence σ by equation (3.4).

Assume now that $\epsilon'_j \succ \tau_{i+1}$ and let $Q = P_\sigma \cap M(\sigma'')$. Then, as in the proof of Lemma 5.5, $\{\tau_{i+1}, \epsilon'_j\}$ is the simple system for Q , with τ_{i+1} the first root and ϵ'_j the last root. Thus $R(\tau_{i+1})\epsilon'_j$ is a positive root and $R(\tau_{i+1})\epsilon'_j \succ \tau_{i+1}$ (by Lemma 5.5).

Now consider $(\sigma')^{-1}(\epsilon'_j)$, which is negative, by Proposition 7.1. However

$$(\sigma')^{-1}\epsilon'_j = (R(\tau_{i+1})\sigma)^{-1}\epsilon'_j = \sigma^{-1}([R(\tau_{i+1})](\epsilon'_j)).$$

Since $[R(\tau_{i+1})](\epsilon'_j)$ is positive, Proposition 7.1 implies that $R(\tau_{i+1})\epsilon'_j = \epsilon_a$ for some a . Thus $[R(\tau_{i+1})](\epsilon'_j)$ precedes τ_{i+1} (since each ϵ_b does), which contradicts the earlier conclusion that $[R(\tau_{i+1})](\epsilon'_j) \succ \tau_{i+1}$. Thus the assumption that $\epsilon'_j \succ \tau_{i+1}$ must have been false.

Next we show that $|F|$ is spherically convex. Since $c[X(\sigma, \tau_i)] = Z(\sigma, \tau_i)$ by the inductive hypothesis, we obtain

$$\begin{aligned} |F| &= M(R(\tau_{i+1})\sigma) \cap c[X(\sigma, \tau_i)] \cap S^{n-1} \\ &= M(R(\tau_{i+1})\sigma) \cap Z(\sigma, \tau_i) \cap S^{n-1} \end{aligned}$$

which is convex.

Now $|V|$, being a cone with a convex $(k-2)$ -dimensional base, must itself be convex and $(k-1)$ -dimensional. The proof that Z is contained in $c[V]$ involves a close examination of the facets of $c[V]$. First one argues that each facet of $c[V]$ is of the form $M(\sigma) \cap \mu(\tau_j)^\perp$ for some

$$\tau_j \in \{\theta_{i_1}, \dots, \theta_{i_a}\} \cup \{\tau_{i+1}, \tau_{j_1}, \dots, \tau_{j_b}\},$$

where $j_l > i+1$ for $l = 1, \dots, b$. (This step is similar to the corresponding argument for $c[V_0]$, with τ_{i+1} taking the place of θ_k .) It then follows that

$$\begin{aligned} Z &= M(\sigma) \cap \mu(\theta_1)^+ \cap \dots \cap \mu(\theta_k)^+ \cap \mu(\tau_{i+1})^+ \cap \mu(\tau_{i+2})^- \cap \dots \cap \mu(\tau_t)^- \\ &\subseteq M(\sigma) \cap \mu(\theta_{i_1})^+ \cap \dots \cap \mu(\theta_{i_a})^+ \cap \mu(\tau_{i+1})^+ \cap \mu(\tau_{j_i})^- \cap \dots \cap \mu(\tau_{j_b})^- \\ &= c[V] \quad \text{as required.} \quad \square \end{aligned}$$

Corollary 7.7. *For each $\sigma \leq \gamma$ the set $|X(\sigma)|$ is spherically convex. Furthermore, $|X(\sigma)|$ is the intersection with S^{n-1} of the positive cone on the set P_σ and we have*

$$|X(\sigma)| = S^{n-1} \cap M(\sigma) \cap \mu(\theta_1)^+ \cap \dots \cap \mu(\theta_k)^+.$$

Proof. Apply Proposition 7.6 with $i = t$. \square

We are now in a position to prove that $[I, \gamma]$ is a lattice.

Theorem 7.8. *If W is a finite real reflection group equipped with the partial order \leq defined by reflection length and γ is a Coxeter element for W , then the interval $[I, \gamma]$ is a lattice.*

Proof. Choose a simple system and Coxeter element γ for W as in Section 3. For $\sigma \leq \gamma$ and $\sigma \neq I$, construct the simplicial complex $X(\sigma)$ and define $X(I)$ to be the empty set \emptyset . We have seen that $X(\sigma)$ has dimension $l(\sigma) - 1$ and its vertex set is

$$P_\sigma = \{\rho_i \mid 1 \leq i \leq nh/2 \text{ and } R(\rho_i) \leq \sigma\}.$$

Furthermore, $X(\sigma)$ is a subcomplex of $X(\gamma)$ and, by Corollary 7.7, $|X(\sigma)|$ is spherically convex.

Assume now that α and β both precede γ . Then $X(\alpha) \cap X(\beta)$ is a sub-complex of $X(\gamma)$ and, since $|X(\alpha) \cap X(\beta)| = |X(\alpha)| \cap |X(\beta)|$, the set $|X(\alpha) \cap X(\beta)|$ is convex by Corollary 7.7. If d denotes the dimension of $X(\alpha) \cap X(\beta)$, then $|X(\alpha) \cap X(\beta)|$ is a union of d -dimensional simplices and, for each d -simplex $\langle v_0, \dots, v_d \rangle$ in $X(\alpha) \cap X(\beta)$, the linear subspaces $\text{span}(\{v_0, \dots, v_d\})$ and $\text{span}(X)$ coincide. We associate an element $\sigma \leq \gamma$ to $X(\alpha) \cap X(\beta)$ as follows. Choose a d -simplex $\langle v_0, \dots, v_d \rangle \in X$ and assume (without loss of generality) that $v_0 \prec v_1 \prec \dots \prec v_d$. Then

$$R(v_0)R(v_1)\dots R(v_d) \leq \gamma^{-1} \quad \text{by definition of } X(\gamma),$$

and we define σ to be $R(v_d)\dots R(v_1)R(v_0)$. Since $\text{span}(v_0, \dots, v_d) = M(\sigma) = \text{span}(X(\alpha) \cap X(\beta))$, it follows that $M(\sigma)$ contains and is spanned by the vertex set of $X(\alpha) \cap X(\beta)$. However this vertex set is $P_\alpha \cap P_\beta$, and the theorem now follows by our remarks in Section 2. \square

8. RELATIONSHIP WITH GENERALISED ASSOCIAHEDRA

In this section we embed the complex $X(\gamma)$ in a larger simplicial complex $EX(\gamma)$ whose vertex set consists of all positive roots and the negatives of the simple roots, and we show that the geometric realisation of $EX(\gamma)$ is a sphere. If W is crystallographic, then we show that $EX(\gamma)$ coincides with the simplicial generalised associahedron for W (defined in [13]).

Recall that $S_1 = \{\alpha_1, \dots, \alpha_s\}$ and $S_2 = \{\alpha_{s+1}, \dots, \alpha_n\}$ is a partition of the set of simple roots into two orthonormal sets. We will use the notation

$$-S_1 = \{-\alpha_1, \dots, -\alpha_s\} \quad \text{and} \quad -S_2 = \{-\alpha_{s+1}, \dots, -\alpha_n\}.$$

We note that the subscripting on the ρ 's can be applied to negative indices with the convention that $\rho_{-k} = \rho_{nh-k}$.

Definition 8.1. We define a set, $EX = EX(\gamma)$, of simplices by declaring that

- the vertex set is the ordered set $\{\rho_i \mid -n + s + 1 \leq i \leq nh/2 + s\}$,
- an edge joins ρ_i to ρ_j whenever $R(\rho_i)R(\rho_j) \leq \gamma^{-1}$, $i < j$ and $\rho_i \neq -\rho_j$, and
- $\langle \rho_{i_1}, \rho_{i_2}, \dots, \rho_{i_k} \rangle$ forms a $(k-1)$ -simplex if the vertices are distinct and pairwise joined by edges.

We note that it follows from the definitions that the simplices of $X(\gamma)$ are simplices of $EX(\gamma)$. The extra vertices are precisely the negatives of the simple roots $\{\alpha_1, \dots, \alpha_n\}$. First, for $k = 1, \dots, n-s$, $\rho_{s+k} = -\gamma(\alpha_{s+k})$ implies that $\rho_{-n+s+k} = -\alpha_{s+k}$. Secondly the set $\{\rho_{nh/2+1}, \dots, \rho_{nh/2+s}\}$ is a permutation of $\{-\alpha_1, \dots, -\alpha_s\}$.

Theorem 8.2. $EX(\gamma)$ is a simplicial complex and $|EX(\gamma)|$ is a sphere of dimension $n-1$.

Proof. Let C be the spherical cross-polytope whose set of vertices is $\{\pm\alpha_1, \dots, \pm\alpha_n\}$. Then C is a simplicial complex whose geometric realisation $|C|$ is the unit sphere in \mathbf{R}^n . We will show that $EX(\gamma)$ is a simplicial subdivision of C .

Consider the simplicial subdivision C' of C which is defined as follows. Let K be the sub-complex of C which consists of the simplex $A = \langle \alpha_1, \dots, \alpha_n \rangle$ and all its faces. We proved earlier that $|X(\gamma)| = |K|$. Thus $X(\gamma)$ is a simplicial subdivision of K .

Extend this subdivision to the rest of C as follows. Any simplex of C whose vertices are contained in $\{-\alpha_1, \dots, -\alpha_n\}$ is not subdivided. If B is a simplex of C which contains both positive and negative roots, then we can write

$$B = \langle \alpha_{i_1}, \dots, \alpha_{i_p} \rangle * \langle -\alpha_{j_1}, \dots, -\alpha_{j_q} \rangle$$

where $i_1 < \dots < i_p$, $j_1 < \dots < j_q$, $*$ denotes the spherical join, and $\{i_1, \dots, i_p\}$ and $\{j_1, \dots, j_p\}$ are disjoint. We extend the subdivision of the simplex $\langle \alpha_{i_1}, \dots, \alpha_{i_p} \rangle$ (in $X(\gamma)$) by taking the join of this subdivision with the simplex $\langle -\alpha_{j_1}, \dots, -\alpha_{j_q} \rangle$.

Suppose $J \subset \{1, \dots, s\}$, $K \subset \{s+1, \dots, n\}$ and that $\{\rho_{i_1}, \dots, \rho_{i_a}\}$ is an ordered set of positive roots. Then $\{-\alpha_j \mid j \in J \cup K\} \cup \{\rho_{i_1}, \dots, \rho_{i_a}\}$ is the vertex set for a simplex in C' or in $EX(\gamma)$ if and only if

$$R(\rho_{i_1}) \dots R(\rho_{i_a}) \leq \left(\prod_{l \in K} R(\rho_l) \right) \gamma^{-1} \left(\prod_{m \in J} R(\rho_m) \right).$$

It now follows that $C' = EX(\gamma)$ as required. \square

For convenience we recall some facts from [13] and express them in a manner consistent with our earlier notation. If W is a crystallographic finite reflection group, then the simplicial generalised associahedron, $GA(W)$, for the simple system $S_1 \cup S_2$ is a simplicial complex whose set of vertices, denoted $\Omega_{\geq -1}$, consists of all the positive roots and the negative simple roots. Two piecewise-linear involutions τ_+ and τ_- are introduced in [13]. It can be shown that they are determined by

$$\tau_+(\beta) = \begin{cases} [R_1 R_2 \dots R_s](\beta) & \text{if } \beta \notin -S_2, \\ \beta & \text{if } \beta \in -S_2, \end{cases}$$

$$\tau_-(\beta) = \begin{cases} [R_{s+1} R_{s+2} \dots R_n](\beta) & \text{if } \beta \notin -S_1, \\ \beta & \text{if } \beta \in -S_1. \end{cases}$$

From this we deduce that the action of $\tau_+ \tau_-$ on $\Omega_{\geq -1}$ is given by

$$\tau_+ \tau_-(\beta) = \begin{cases} \gamma(\beta) & \text{if } \beta \notin (-S_1) \cup S_2, \\ -\beta & \text{if } \beta \in (-S_1) \cup S_2, \end{cases}$$

and the action of the inverse $\tau_- \tau_+$ is given by

$$\tau_- \tau_+(\beta) = \begin{cases} \gamma^{-1}(\beta) & \text{if } \beta \notin (-S_2) \cup S_1, \\ -\beta & \text{if } \beta \in (-S_2) \cup S_1. \end{cases}$$

The compatibility degree $(\alpha \parallel \beta)$ of any two elements $\alpha, \beta \in \Omega_{\geq -1}$ is characterised in [13] by the conditions

- (i) $(-\alpha_i \parallel \alpha) = \max\{[\alpha : \alpha_i], 0\}$, where $[\alpha : \alpha_i] = \alpha \cdot \mu_i$ for $i = 1, \dots, n$, and
- (ii) $(\alpha \parallel \beta) = (\tau_{\pm} \alpha \parallel \tau_{\pm} \beta)$ for all α and β .

We recall that two vertices α and β in $GA(W)$ are connected by an edge if and only if $(\alpha \parallel \beta) = 0$. Finally, we note that $GA(W)$ is completely determined by its one-skeleton.

Theorem 8.3. *If W is crystallographic, then $EX(\gamma)$ coincides with the simplicial generalised associahedron $GA(W)$.*

Proof. We continue to use the earlier notation. First note that the vertex set $\Omega_{\geq -1}$ of the associahedron is the same as that of $EX(\gamma)$. Since each of $EX(\gamma)$ and $GA(W)$ is determined by its one-skeleton, it suffices to show that two vertices α and β are joined by an edge in $EX(\gamma)$ if and only if they are joined by an edge in $GA(W)$. To simplify notation we will write $\rho_i \xrightarrow{GA} \rho_j$ or $\rho_i \xrightarrow{EX} \rho_j$ if ρ_i and ρ_j are connected by an edge in the simplicial complex $GA(W)$ or $EX(\gamma)$ respectively. Note that the ordering on the vertices of $EX(\gamma)$ is such that they are arranged into the following sets in the given order:

$$-S_2, S_1, \gamma(-S_2), \gamma^2(-S_2), \dots, \gamma^{-1}(-S_1), S_2, -S_1.$$

Suppose $\rho_i \prec \rho_j$. Then $\rho_i \xrightarrow{EX} \rho_j$ is equivalent to $R(\rho_i)R(\rho_j) \leq \gamma^{-1}$. First we can assume $\rho_j \neq -\rho_i$ since neither complex contains an edge from ρ_i to $-\rho_i$.

Case 1. If $\rho_i \in -S_2$, then $\rho_i = -\alpha_p$ for some p satisfying $s+1 \leq p \leq n$. Then $\rho_i \xrightarrow{GA} \rho_j$ if and only if $(-\alpha_p \parallel \rho_j) = 0$. However, by equation (3.4), this is equivalent to $R(\rho_j) \leq R(\alpha_1) \dots R(\alpha_{p-1})R(\alpha_{p+1}) \dots R(\alpha_n)$. But, since $p > s$, we have $R(\alpha_1) \dots R(\alpha_{p-1})R(\alpha_{p+1}) \dots R(\alpha_n) = \gamma R(\alpha_p)$. Thus $\rho_i \xrightarrow{GA} \rho_j$ is equivalent by equation (3.7) to $R(\rho_j)R(\rho_i) \leq \gamma$ which in turn is the criterion for $\rho_i \xrightarrow{EX} \rho_j$.

Case 2. If $\rho_i \in S_1$, then $\rho_i = \alpha_p$, for some p satisfying $1 \leq p \leq s$ and $\tau_- \tau_+(\rho_i) = -\rho_i = -\alpha_p \in -S_1$.

Case 2(a). If ρ_j is also in $\in S_1$, then $\tau_- \tau_+(\rho_j) = -\rho_j \in -S_1$, and applying $\tau_- \tau_+$ to both roots gives

$$(\rho_i \parallel \rho_j) = (-\rho_i \parallel -\rho_j) = 0$$

since ρ_j and ρ_i are distinct simple roots. By definition of edges in the generalised associahedron we must have $\rho_i \xrightarrow{GA} \rho_j$. However, in this subcase, $\rho_j = \alpha_q$ with $1 \leq p < q \leq s$, so that

$$R(\rho_j)R(\rho_i) = R(\rho_i)R(\rho_j) \leq \gamma$$

which is again equivalent to the criterion for $\rho_i \xrightarrow{EX} \rho_j$.

Case 2(b). If $\rho_j \notin S_1$, then $j > s$, and we know that $\tau_- \tau_+(\rho_j) = \gamma^{-1}(\rho_j)$. Now $\rho_i \xrightarrow{GA} \rho_j$ if and only if $(-\alpha_p \parallel \gamma^{-1}(\rho_j)) = 0$. However this is equivalent by equation (3.4) to

$$R[\gamma^{-1}(\rho_j)] \leq R(\alpha_1) \dots R(\alpha_{p-1})R(\alpha_{p+1}) \dots R(\alpha_n).$$

Since $1 \leq p \leq s$, we have $R(\alpha_1) \dots R(\alpha_{p-1})R(\alpha_{p+1}) \dots R(\alpha_n) = R(\alpha_p)\gamma$. Thus $\rho_i \xrightarrow{GA} \rho_j$ is equivalent to $R(\rho_i)R[\gamma^{-1}(\rho_j)] \leq \gamma$ by equation (3.7). But

$$R(\rho_i)R[\gamma^{-1}(\rho_j)] \leq \gamma \Leftrightarrow R(\rho_i) \leq \gamma R[\gamma^{-1}(\rho_j)] = R(\rho_j)\gamma,$$

using equations (3.7) and (3.2). The last condition in turn is the criterion for $\rho_i \xrightarrow{EX} \rho_j$.

Case 3. If $i > s$ let m be the smallest positive integer with the property that

$$(\tau_- \tau_+)^m(\rho_i) \in (-S_2) \cup S_1.$$

Now $\rho_i \xrightarrow{GA} \rho_j$ if and only if $(\tau_- \tau_+)^m(\rho_i) \xrightarrow{GA} (\tau_- \tau_+)^m(\rho_j)$. However, Case 1 or Case 2 now applies to this new pair of roots and $(\tau_- \tau_+)^m = \gamma^{-m}$ when applied to ρ_i and ρ_j . Thus $\rho_i \xrightarrow{GA} \rho_j$ if and only if

$$R(\gamma^{-m}(\rho_j))R(\gamma^{-m}(\rho_i)) \leq \gamma,$$

by the proof in cases 1 and 2. But this is equivalent to $R(\gamma^{-m}(\rho_i)) \leq R(\gamma^{-m}(\rho_j))\gamma$ or $\rho_{i-mn} \cdot \mu_{j-mn} = 0$. Since γ is an isometry this is equivalent to $\rho_i \cdot \mu_j = 0$ and hence to $\rho_i \xrightarrow{EX} \rho_j$. \square

Note 8.4. We observe that Theorem 8.3 provides a new proof that the simplicial generalised associahedron is a simplicial complex whose geometric realisation is a sphere. This proof is independent of the classification of finite real reflection groups and extends the work of Fomin and Zelevinsky [13] to include the non-crystallographic finite reflection groups. Furthermore Definition 8.1 and Theorem 8.3 characterise the notion of compatibility (in the sense of Section 3.2 of [13]) in terms of non-crossing partitions.

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