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FREE BOUNDARY REGULARITY CLOSE TO INITIAL STATE FOR PARABOLIC OBSTACLE PROBLEM

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ABSTRACT. In this paper we study the behavior of the free boundary $\partial\{u > \psi\}$, arising in the following complementary problem:

$$(Hu)(u-\psi) = 0, \qquad u \geq \psi(x,t) \quad \text{in } Q^+,$$
$$Hu \leq 0,$$
$$u(x,t) \geq \psi(x,t) \quad \text{on } \partial_p Q^+.$$

Here ∂_p denotes the parabolic boundary, H is a parabolic operator with certain properties, Q^+ is the upper half of the unit cylinder in $\mathbf{R}^{n+1}$, and the equation is satisfied in the viscosity sense. The obstacle ψ is assumed to be continuous (with a certain smoothness at $\{x_1 = 0,\, t = 0\}$), and coincides with the boundary data $u(x,0) = \psi(x,0)$ at time zero. We also discuss applications in financial markets.

1. Introduction and main results

1.1. **Backgrounds.** Recent years have seen a new trend of analyzing free boundaries close to fixed boundaries. These types of problems seem to have a variety of applications, especially when certain experiments are done in a small and confined container, so that the interface between the reaction and nonreaction zone come in touch with the "wall" of the container. A similar type of question, within application, is the behavior of the interface close to initial state, when an evolutionary problem is considered.

In this paper we study properties of free boundaries that appear in nonlinear parabolic problems of obstacle type, close to initial state. To the author's knowledge the only known results are those for the 1-space dimensional case, and with a very specific obstacle/initial state. This, however, is very well studied, due to its applications to mathematical finance.

To set the problem, denote by Q^+ the upper half of the unit cylinder in $\mathbf{R}^{n+1}$, and let $\psi(x,t)$ be a continuous function, with a fixed given modulus of continuity.

Let us for convenience set

$$H(u) = F(D^2u, Du, u, x, t) - D_t u \tag{1.1}$$

and

$$H_0(u) = F(D^2u, 0, 0, 0, 0) - D_t u \tag{1.2}$$

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where F is a uniformly elliptic operator, which will be carefully defined below. Now consider a solution to the parabolic obstacle problem: u is a continuous function satisfying

$$(u-\psi)H(u)=0, \quad u\geq\psi \quad \text{in } Q^+, \tag{1.3}$$

$$H(u)\leq 0, \tag{1.4}$$

with boundary datum

$$u(x,t)=g(x,t)\geq\psi(x,t) \qquad \partial_p Q^+. \tag{1.5}$$

Here the notation ∂_p stands for parabolic boundary; see [W1]. In other words u is the *smallest* supersolution (to $H(v)=0$) over ψ, with given boundary values.

1.2. **Main results.** To formulate our main theorems we need some definitions. We set

$$\mathcal{E}(u)=\{(x,t)\in Q_1^+ :\ u(x,t)=\psi(x,t)\},$$
$$\mathcal{C}(u)=\{(x,t)\in Q_1^+ : u(x,t)>\psi(x,t)\},$$

where ψ will satisfy the following conditions. For the obstacle $\psi=\psi_\alpha$ we will assume

$$\psi(x,t)=(x_1^+)^\alpha\psi_1(x,t)+\psi_2(x,t), \tag{1.6}$$

with ψ_1,ψ_2 in $C_x^0\cap C_t^0$, and

$$\psi_1(0,0)=1, \qquad |\psi_2(x,t)|\leq(|x|^2+t)^{\alpha/2}\tau_0(|x|^2+t),$$

where $\tau_0(r)\to 0$ as r tends to zero. We also denote by τ the modulus of continuity for ψ in $\overline{Q^+}$. This assumption makes it possible to get rid of ψ_2 in a scaled version of the equation, and in a global setting. For simplicity the reader may consider the situation where $\psi_1=1$ and $\psi_2=0$. After all, in a blow-up (global) version this is the case.

Actually, there are a variety of possible obstacles that one may consider. We hope to be able to treat such problems in a forthcoming paper. One example is $\psi=(E-\min(x_1^+,x_2^+))^+$, in two space dimensions, that appears in finance. It relates to the so-called max-option for two assets, with exercise price E.

It is also noteworthy that many of possible examples that one can give have direct applications in mathematical finance (see [BD]).

Definition 1.1. We say a continuous function u belongs to the class $\mathcal{G}_1(n,M,H,\psi_\alpha)$ if u satisfies equations (1.3)–(1.5), and $\|u\|_{\infty,Q_1^+}\leq M$ (supremum norm).

We denote by $\mathcal{G}_\infty(n,M,H_0,\alpha)$ all "global solutions" in the entire parabolic upper half-space $\mathbf{R}_+^{n+1}$, with α-growth , i.e., solutions in the entire space $\mathbf{R}_+^{n+1}$, w.r.t. the operator H_0, and with growth

$$(x_1^+)^\alpha\leq u(x,t)\leq M(|x|^2+t)^{\alpha/2}, \qquad u(x,0)=(x_1^+)^\alpha,$$

and u solves (1.3)-(1.4) in $\mathbf{R}_+^{n+1}$.

Finally, we stress once again that the operator H should have the properties mentioned in Section 1.3 (entitled Conditions on H) below.

Our first result asserts that local solutions have growth of order α, at the origin. This will be used in a scaling argument in our main theorems.

Theorem 1.2. *There is a universal constant $C_0 = C_0(n, H, \psi_\alpha)$ such that if $u \in \mathcal{G}_1(n, M, H, \psi_\alpha)$, then*

$$\sup_{Q_r^+} |u| \le C_0 M r^\alpha \qquad \text{for } r < 1.$$

Another tool needed in the main theorem is a compactness argument. For this we need (at least) a uniform continuity for the class $\mathcal{G}$.

Let us first introduce the notation $X = (x, t)$ and $|X| = \sqrt{x^2 + |t|}$. In the next theorem we assume only that ψ is τ-continuous and it does not necessarily need to have the form (1.6).

Theorem 1.3. *There is a universal constant C, such that if $u \in \mathcal{G}_1(n, M, H, \psi)$ (with ψ τ-continuous), then*

$$|u(X) - u(Y)| \le C\tau_2(|X - Y|),$$

where $\tau_2(r) := \max(r^{1/4}, \tau(r^{1/4}))$, and τ is the modulus of continuity for ψ.

Next we formulate a qualitative result for the behavior of solutions close to the initial state. We consider the two cases $\alpha \ge 1$, and $\alpha < 1$ separately.

Theorem 1.4. *For $\alpha \ge 1$ there exists $r_0 > 0$, and a modulus of continuity $\sigma(r)$ such that if $u \in \mathcal{G}_1(n, M, H, \psi_\alpha)$, then*

$$\mathcal{E}(u) \cap Q_{r_0}^+ \subset \{(x, t) :\ t \le |x|^2 \sigma(|x|)\} \cap Q_{r_0}^+. \tag{1.7}$$

Here r_0 and σ depend on the class $\mathcal{G}$ only.

Corollary 1.5. *In Theorem 1.4 assume $\psi_2 = 0$. Suppose also $1/2 \le \psi_1(x^0, 0) \le 1$ for all points $x^0 \in B_{1/2}$ on the x_1-axis, and close to the origin. Under these assumptions we have*

$$\mathcal{E}(u) \cap Q_{r_0}^+ \subset \{(x, t) :\ t \le x_1^2 \sigma(|x_1|)\} \cap Q_{r_0}^+. \tag{1.8}$$

The case $\alpha < 1$ offers a different geometric behavior for the coincident set. Here is our result in this case.

Theorem 1.6. *For $\alpha < 1$, there exist positive constants r_0, c_α, and a modulus of continuity σ, such that if $u \in \mathcal{G}_1(n, M, H, \psi_\alpha)$, then*

$$\mathcal{E}(u) \cap Q_{r_0}^+ \subset P_\sigma \cup T_\sigma \tag{1.9}$$

where

$$P_\sigma := \{(x, t) :\ x_1 > 0,\ t \le (c_\alpha + \sigma(|x|))x_1^2)\}$$

and

$$T_\sigma := \{(x, t) :\ t \le \sigma((|x|)|x|^2)\}.$$

Before closing this section we give some explanation/definition of the viscosity solutions. We also introduce the exact condition imposed on the nonlinear operator F or H.

1.3. **Conditions on H.** In this paper the following conditions are imposed on the operator H, i.e. on F, as $H(D^2u, Du, u, x, t) = F(D^2u, Du, u, x, t) - D_tu$. When there is no ambiguity we use also the notation $H(u)$, $F(u)$.

(1) $F(A, p, v, X)$ is defined on $\mathcal{S}^{n-1} \times \mathbf{R}^n \times \mathbf{R} \times Q_1^+$.
(2) F is uniformly elliptic with ellipticity constants λ, Λ, i.e.,

$$\lambda\|N\| \leq F(A+N, ...) - F(A, ...) \leq \Lambda\|N\|,$$

where A and N are arbitrary $n \times n$ symmetric matrices with $N \geq 0$.
(3) F has the homogeneity property

$$F(sA, sp, sv, x, t) \leq sF(A, p, v, x, t),$$

for all positive numbers s.
(4) F is C^0 in all its variables, and bounded on compact sets.
(5) $H_r(v) := F(D^2v, rDv, r^2v, rx, r^2t) - D_tv$ has the maximum/comparison principle for small enough r, in compact sets.
(6) The obstacle problem for the operator $H_r(v)$ admits a solution in compact sets with appropriate boundary conditions.
(7) The Dirichlet problem for $H(v)$ is stable under boundary-value perturbations, on bounded domains.

It should be remarked that most of the standard operators do have all of the properties above. However, the reason we have taken such a general formulation for H rather than specifying it, is for future references, and to make it easier for the reader to adapt the results of this paper to their situations.

Definition 1.7. Viscosity solutions to (1.3)-(1.5) are continuous functions u with the property that if at $(x^0, t^0, u(x^0))$ the graph of u can be touched, locally from above and below, by polynomials

$$P(x,t) = \frac{1}{2}x^TAx + bt + cx + d,$$

then P should satisfy equation (1.3)-(1.5), pointwise at (x^0, t^0).

For viscosity solutions in the elliptic case we refer to the excellent book of L. Caffarelli and X. Cabre [CC] for further details. For the parabolic case we refer to [W1], [W2], [W3]. It is known that viscosity solutions to operators defined above have the usual maximum/minimum, and comparison principle as well as compactness properties.

1.4. **Applications to finance.** The obstacle problem defined above, at least when H is the heat operator with lower order terms, appears naturally in the valuation of American type options in financial markets. Such options give its owner the right (but not obligation) of selling the option at any time during its life time, if the owner is better for doing so. We refer the reader to the paper by M. Broadie and J. Detemple [BD] for background and more details.

Now to fix ideas we denote $\mathbf{S}_t = (S_t^1, S_t^2)$ to be the price vector of underlying assets at time t. The price S_t^i follows the standard stochastic model

$$dS_t^i = (r - \delta_i)S_t^i dt + \sigma_i dW_t^i, \qquad i = 1, 2,$$

where r is the constant interest rate, δ_i is the dividend rate of the i-th stock, and σ_i is the volatility of the price of the corresponding asset. The notation W_t^i also stands for the standard Brownian motion, over a probability filtered space $(\Omega, \mathcal{F}, \mathcal{P})$, with P as the risk-neutral measure.

Now the value V of the American option is given by

$$V(S,t) = \sup_{\tau} E\left(e^{-r(T-t)}\psi(\mathbf{S}_\tau)\right) \tag{1.10}$$

with the stopping time τ varying over all $\mathcal{F}_t$-adapted random variables, and $\psi(S)$ the option payoff. Here $(\mathcal{F}_t)_{t\geq 0}$ denotes the P completion of the natural filtration associated to $(W_t^i)_{t\geq 0}$. This completion comes from the so-called completeness of markets, so that one has a unique solution to the problem.

Standard theory of stochastic control can now be used to show that the value function V satisfies a variational inequality, here written in the complementary form,

$$\mathcal{L}V + \partial_t V \leq 0, \qquad (\mathcal{L}V + \partial_t V)\,(\psi - V) = 0, \qquad V \geq \psi\ ,$$

a.e. on $\mathbb{R}^n \times [0,T)$, and with condition

$$V(x,T) = \psi(x,T).$$

Here the elliptic operator $\mathcal{L}$ is given by

$$\mathcal{L}V = (r-\delta_1)S^1\frac{\partial V}{\partial S^1} + (r-\delta_2)S^2\frac{\partial V}{\partial S^2}$$
$$+\frac{1}{2}\left((\sigma_1 S^1)^2\frac{\partial^2 V}{\partial (S^1)^2} + \sigma_1\sigma_2 S^1 S^2\frac{\partial^2 V}{\partial S^1 \partial S^2} + (\sigma_2 S^2)^2\frac{\partial^2 V}{\partial (S^2)^2}\right) - rV\ .$$

The backward parabolic equation can be turned into a forward equation by a change of variable, and hence we arrive at the case of the parabolic obstacle problem in this paper.

In general, the obstacle ψ is nonsmooth at some point x^0, at time of maturity $t = T$. Examples of such obstacles can be found in [BD].

A direct application of our results tells us that the option value is Lipschitz in space and half-Lipschitz in time up to maturity, e.g. in the case of the maximum of two options. The results also describe the behavior of the exercise region (for put option) close to maturity.

2. Proof of Theorems 1.2-1.3

Define

$$S_j(u) = \|u\|_{\infty, Q^+_{2^{-j}}}\,.$$

To prove Theorem 1.2, it suffices, in view of an iteration argument, to show the following lemma.

Lemma 2.1. *For $u \in \mathcal{G}_1(n, M, H, \psi, \alpha)$, there exists a constant C_1 such that*

$$S_{j+1}(u) \leq \max\left(C_1 M 2^{-j\alpha}, \frac{S_j(u)}{2^\alpha}, \frac{S_{j-1}(u)}{2^{2\alpha}}, \cdots, \frac{S_0(u)}{2^{(j+1)\alpha}}\right) \qquad \text{for } j \in \mathbb{N}.$$

The constant C_1 depends on the class $\mathcal{G}$.

Proof. We use a contradictory argument. Suppose the conclusion fails. Then, for $j \in \mathbb{N}$, there are $\{u_j\} \in \mathcal{G}_1$, and positive integers $\{k_j\}$ such that

$$S_{k_j+1}(u_j) \geq \max\left(jM2^{-k_j\alpha}, \frac{S_{k_j}(u_j)}{2^\alpha}, \frac{S_{k_j-1}(u_j)}{2^{2\alpha}}, \cdots, \frac{S_0(u_j)}{2^{(k_j+1)\alpha}}\right). \tag{2.1}$$

A crucial point is that the maximum value $S_{k_j}(u_j)$ cannot be taken at $t = 0$, due to the assumptions on ψ. Hence it is realized at (x^j, t^j) with $t^j > 0$,

$$S_{k_j+1}(u_j) = u_j(x^j, t^j). \tag{2.2}$$

Since $|u_j| \le M$, by (2.1), $j2^{-2k_j}$ is bounded. Hence $k_j \to \infty$. Now set

$$\tilde u_j(x,t) = \frac{u_j(2^{-k_j}x, 2^{-2k_j}t)}{S_{k_{j+1}}(u_j)} \qquad \text{in } Q^+_{2^j}.$$

Then, for $(\tilde x^j, \tilde t^j) = (x^j 2^{k_j}, t^j 2^{2k_j})$,

$$\tilde u_j(\tilde x^j, \tilde t^j) = 1, \tag{2.3}$$

and for $m < k_j$

$$\|\tilde u_j\|_{\infty, Q^+_{2^m}} = \frac{S_{k_j - m}}{S_{k_j+1}} \le 2^{(m+1)\alpha}. \tag{2.4}$$

Moreover $\tilde u_j$ solves the scaled version of the obstacle problem, i.e., when we replace ψ and H with

$$\psi_j(x,t) = \psi(2^{-k_j}x, 2^{-2k_j}t)/S_{k_j+1} \quad \to \quad 0 \tag{2.5}$$

and

$$H_j(D^2v, Dv, v, x, t) = H(D^2v, 2^{-k_j}Dv, 2^{-2k_j}v, 2^{-k_j}x, 2^{-2k_j}t),$$

respectively. Since

$$c_j := \|\psi_j\|_{\infty,K} \le \frac{C_K}{Mj} \quad \to \quad 0 \qquad (K \text{ compact})$$

uniformly on compact sets, one expects that for a point (x,t), the solution-sequence either tends to zero at (x,t) or stays away from the obstacle. The latter case implies, in particular, that we have a solution at this point, rather than a super-solution, and hence one expects uniform convergence at such points. We need to give this argument some mathematical rigor.

Since $\tilde u_j$ is locally bounded, we may extract a subsequence (with the same label) converging locally, and weakly in $L^q(\mathbf{R}^{n+1}_+)$ $(1 < q < +\infty)$ to a limit function u_0. We need to show that the convergence is uniform, i.e.,

$$\tilde u_j \ \to u_0 \qquad \text{uniformly} \tag{2.6}$$

on any compact set $K \subset \mathbf{R}^{n+1}_+$, and (provided j is large)

$$\tilde u_j \qquad \text{is a solution to } H_j, \text{ on } K, \tag{2.7}$$

rather than a (strict) super-solution.

Suppose, for the moment, that (2.6)-(2.7) hold. Then, as $H_j \to H_0$, and by uniform convergence we have that $\tilde u_j \to u_0$ uniformly in any compact set $K \subset \mathbf{R}^{n+1}_+$. The limit function u_0 is itself a solution to H_0 and it satisfies the following

$$0 \le u_0 \le C(|x|^2 + t)^{\alpha/2}, \qquad \text{by } (2.4) - (2.5).$$

Moreover, by (2.5) the obstacle and the initial value become as small as we wish within any compact set. Hence by comparison principle $\tilde u_j$ must be smaller than a solution to the obstacle problem for H_j with the constant c_j as the obstacle and as the initial data. However, due to the ellipticity and boundedness of F, the function $v_j = c(a|x - x^0|^2 + bt) + c_j$ is a super-solution and definitely above $c_j \ge \psi_j$ in $Q^+_1(x^0, 0)$ if the constants c, a, b are chosen appropriately. Similarly one can establish that $v_j \geq \tilde{u}_j$ on the parabolic boundary of $Q_1^+(x^0, 0)$. Observe that due to (2.4) we have uniform boundedness on compact sets, and hence we may choose c, a, b independent of j, but depending on x^0. In particular v_j is larger than any solution to the obstacle problem (in $Q_1^+(x^0, 0)$, with an obstacle smaller than c_j. Hence $v_j \geq \tilde{u}_j$ in $Q_1^+(x^0, 0)$, and in the limit $u_0 \leq c(a|x - x^0|^2 + bt)$ in $Q_1^+(x^0, 0)$, and in particular $u_0(x^0, 0) = 0$. As x^0 is arbitrary we conclude

$$u_0(x, 0) = 0. \tag{2.8}$$

Next we claim that (after scaling) the point (x^j, t^j) (corresponding to the maximum value in the cylinder $Q^+_{2^{-k_j-1}}(0, 0)$; see (2.2)) cannot come too close to the initial boundary $\{t = 0\}$. Indeed, we already know from discussions preceding (2.2) that $t^j > 0$.

Since we are using uniform convergence in compact sets of $\mathbf{R}^{n+1}_+$ we need to show that this point does not come too close to $\{t = 0\}$. Actually this follows from the barrier

$$v_j = c(a|x - \tilde{x}^j|^2 + bt) + c_j \geq \tilde{u}_j$$

we constructed above. Here the constants a, b, c can be taken so that $ca \geq 4^\alpha$, and $b > 0$ large enough, so that v_j is a super-solution. The conclusion here is that

$$\tilde{t}^j \geq \frac{1 - c_j}{bc}, \tag{2.9}$$

and that (in the limit with $(\tilde{x}^0, \tilde{t}^0) := \lim(x^j, t^j)$, maybe for another subsequence)

$$u_0(\tilde{x}^0, \tilde{t}^0) = 1, \qquad \frac{1}{bc} \leq \tilde{t}^0 \leq 1. \tag{2.10}$$

We want to show that such a function u_0 cannot exist, and hence the contradiction. So let

$$v(x, t) = \epsilon(|x^2| + Nt)^\beta, \qquad \beta > \alpha.$$

Then one easily computes that for large enough N, v is a super-solution to the equation H_0 and has nonnegative initial values; observe that by (2.8) the function u_0 is zero at $t = 0$. For large enough (x, t) it also holds that $v \geq u_0$, and hence by comparison principle $v \geq u_0$ in Q_R^+ provided $R = R_\epsilon$ is large enough. Letting ϵ be very small and R very large, we obtain $u_0 < 1/2$ on Q_1^+, contradicting (2.10).

Final step: To complete the proof we need to verify (2.6)-(2.7).

Let v be a solution to the obstacle problem in $Q_r(X^0)$ ($X^0 = (x^0, t^0)$) with boundary values g, and an obstacle ψ' with $|\psi'| < \epsilon$. Further, let v_0 and v_ϵ be solutions to the equation

$$H_j(D^2 w, Dw, w, x, t) = 0 \qquad \text{in } Q_r(X^0) \tag{2.11}$$

with (parabolic) boundary values g, and $g_\epsilon := \max(g, \epsilon)$, respectively. Then

$$v_0 \leq v \leq v_\epsilon,$$

where in the first inequality we have used the comparison principle, and in the second inequality we have used the fact that $v_\epsilon \geq \min g_\epsilon \geq \epsilon > \psi'$, that v_ϵ is a solution to H, and that v is the smallest super-solution above the obstacle.

Moreover if $v(X^0) \geq A > 0$, then $v_\epsilon(X^0) \geq A$. Now by compactness,

$$\sup_{Q_{r/2}(X^0)} |v_\epsilon - v_0| \leq C_\epsilon,$$

with C_ϵ tending to zero if ϵ does so. From here it follows that $v_0(X^0) \geq A/2$ if ϵ is small enough.

Next by C^α regularity, for solutions to (2.11), $v_0 \geq \gamma A$ in $Q_{r/2}(X^0)$, for some $\gamma > 0$, and hence by the above comparison $v \geq \gamma A$ on $Q_{r/2}(X^0)$.

Now using this argument we may conclude that for any $Z \in Q^+_{2^j}$ we may use $N = N_Z$ chains of cubes (of appropriate sizes) to reach this point from the point $\tilde{X}^j = (\tilde{x}^j, \tilde{t}^j) \in Q^+_{1/2}$, for which the maximum value (2.3) is realized for the function $\tilde{u}_j$. Observe that by (2.9) $\tilde{X}^j \in \{t > 1/2bc\}$, if j is large enough. Hence $\tilde{u}_j(Z) \geq \gamma^N A > 2c_j$ for all large enough j. In particular $\tilde{u}_j$ is a solution (and not a strict super-solution) in any compact set $K \in \mathbf{R}^{n+1}_+$, provided $j \geq j_K$ for j_K large. In other words the graph of $\tilde{u}_j$ does not touch the graph of the obstacle ψ_j. So by the uniform bound (2.4) we can conclude that $\tilde{u}_j$ converges uniformly on any compact set of $\mathbf{R}^{n+1}_+$ to a limit function. Obviously we may choose the subsequence which gave the weak-L^q limit function u_0 above. □

The proof of uniform continuity, Theorem 1.3, is given in the same way as that of the α-growth. Here however, one considers

$$S_j(u, Z) = \|u(\cdot) - u(Z)\|_{\infty, Q_{2^{-j}}(Z)\cap\mathbf{R}^{n+1}_+},$$

for any point $Z \in \partial\mathcal{E}$, and $j \geq 0$. Then as above (with slight simplification) one tries to prove that for $u \in \mathcal{G}_1(n, M, H, \psi, \alpha)$, there exists a constant C_1 such that

$$S_{j+1} \leq \max\left(C_1 M\tau(2^{-j}), \frac{S_j}{2}, \frac{S_{j-1}}{2^2}, \cdots, \frac{S_0}{2^{j+1}}\right) \tag{2.12}$$

($S_j = S_j(u, Z)$) for all $j \in \mathbb{N}$, and $Z \in \mathcal{E}$. This defines a modulus of continuity

$$\tau_1(r) = \max(C\sqrt{r}, \tau(\sqrt{r})) \tag{2.13}$$

for how the solution leaves the obstacle. The latter is a simple exercise and is left to the reader. Now if this fails we should have (compare with the proof of Theorem 1.2) the sequences

$$Z^j = (z^j, s^j) \in \partial\mathcal{E}(u_j) \cap Q^+_{1/2}, \qquad u_j, \qquad k_j$$

for which a reverse inequality of (2.12) holds:

$$S_{k_j+1} \geq \max\left(jM\tau(2^{-j}), \frac{S_{k_j}}{2}, \frac{S_{k_j-1}}{2^2}, \cdots, \frac{S_0}{2^{k_j+1}}\right).$$

Once again defining

$$\tilde{u}_j(x,t) = \frac{u_j(2^{-k_j}x + z^j, 2^{-2k_j}t + s^j) - u_j(Z^j)}{S_{k_j+1}(u_j, Z^j)} \qquad \text{in } B_{2^{k_j}} \times (-s^j, 2^{2k_j}),$$

we have the following sequences:

$$\tilde{u}_j, \qquad \tilde{X}^j \in B_{1/2} \times (-s^j, -s^j + 1/4), \qquad \psi_j, \qquad H_j$$

with $\tilde{u}_j$ solving a new obstacle problem for H_j in the cylinder $B_{2^{k_j-1}} \times (-s^j, 2^{2k_j-2})$, with

$$\tilde{u}_j(\tilde{X}^j) = 1, \qquad \tilde{u}_j(0) = 0, \qquad |\psi_j| \leq c_j \ \to \ 0.$$

Observe that, due to the τ-continuity of ψ, $\tilde{X}^j \notin \mathcal{E}_j$ (the coincidence set for $\tilde{u}_j$).

From here we can argue as in the proof of the previous theorem to reach a contradiction. We leave the details out. It is, however, noteworthy that before using the barrier function v_j in the previous case, we need to apply the minimum principle to conclude that the limit function u_0 is identically zero below $t = 0$.

Next, using this we can define

$$u_X(Y) = \frac{u(x + d_X y, t + (d_X)^2 s) - u(X)}{\tau_1(d_X)}, \qquad X = (x,t), \quad Y = (y,s),$$

for any given X in $\mathbf{R}^{n+1}_+$, with d_X the parabolic distance to the free boundary. Hence u_X is a solution to the re-scaled version of the parabolic equation $H(D^2v, Dv, v, x, t) = 0$ in $Q_1 \cap \{t > -d_X^2\}$, and by (2.12)-(2.13) uniformly bounded. Standard parabolic estimates [W1] imply u_X is uniformly continuous in the cylindrical domain $Q_{1/2} \cap \{t > -d_X^2\}$, so that

$$|u_X(Y)| = |u_X(Y) - u_X(0)| \leq \tau_1(|Y|),$$

i.e. for any point $\tilde{Y} = X + (d_X y, d_X^2 s) \in Q_{d_X}(X)$, with $Y = (y,s) \in Q_1(0)$, $s > -d_X^2$, we have

$$|u(\tilde{Y}) - u(X)| \leq \tau_1(|Y|)\tau_1(d_X) = \tau_1(d_X)\tau_1\left(\frac{|\tilde{Y} - X|}{d_X}\right) \leq \tau_1\left(\sqrt{|\tilde{Y} - X|}\right),$$

and we have a modulus of continuity $\tau_2(r) := \tau_1(\sqrt{r}) = \max(r^{1/4}, \tau(r^{1/4}))$.

3. Proof of main theorems

3.1. **Proof of Theorem 1.4.** For the proof of Theorem 1.4, we will show that given $\epsilon > 0$ there exists $r_\epsilon > 0$ such that for $u \in \mathcal{G}_1$ we have

$$\mathcal{E}(u) \cap Q^+_{r_\epsilon} \subset \{(x,t) : \ t < \epsilon|x|^2\} \cap Q^+_{r_\epsilon}. \tag{3.1}$$

Once we have this we may define the reverse relation $\epsilon(r)$ as the modulus of continuity.

Suppose (3.1) fails. Then there exist a sequence $u_j \in \mathcal{G}_1$, and $X^j = (x^j, t^j) \in \mathcal{E}(u_j)$ with $t^j \geq \epsilon|x^j|^2$, and $r_j = |X^j| \searrow 0$. Scaling u_j,

$$\tilde{u}_j(x,t) = \frac{u_j(r_j x, r_j^2 t)}{r_j^\alpha}.$$

We'll have a new sequence of solutions to the obstacle problem in Q^+_{1/r_j} with obstacle $\psi_j(x,t) = \psi(r_j x, r_j^2 t)/r^\alpha$, initial data $\psi_j(x,0)$, and the operator $H_j(v) = F_j(v) - D_t v$, where

$$H_j(D^2v, Dv, v, r_j x, r_j^2 t) = H(D^2 v, r_j Dv, r_j^2 v, r_j x, r_j^2 t), \qquad r_j = 2^{k_j}.$$

Since the ingredients are uniformly continuous, there is a limit function u_0 (after passing to a subsequence) which solves the limiting obstacle problem, with $\psi_0 = (x_1^+)^\alpha$ in $\mathbf{R}^{n+1}_+$, and H_0 as the operator. Observe that ψ_0 is a sub-solution to the operator H_0 when $\alpha \geq 1$.

On the other hand the point $\tilde{X}^j := (x^j/r_j, t^j/r_j^2) \in \{|X| = 1\} \cap \{t \geq \epsilon|x|^2\}$, and $u_j(\tilde{X}^j) = \psi_j(\tilde{X}^j)$. Therefore the limit point X^0 (again after passing to a subsequence) will be in the set $\{|X| = 1\} \cap \{t \geq \epsilon|x|^2\}$, and $u_j(\tilde{X}^0) = \psi_j(\tilde{X}^0)$.

Now u_0 being a super-solution cannot touch $\psi_0 = (x_1^+)^\alpha$, a (strict) sub-solution, from above. Hence we have reached a contradiction and Theorem 1.4 is proved.

3.2. **Proof of Theorem 1.6.** To prove Theorem 1.6 we use a similar argument as above. We first need to classify global solutions.

Lemma 3.1. *Let u be a global solution to the obstacle problem in $\mathbf{R}^{n+1}_+$, for the operator $H_0 = F_0 - D_t$, and with both the initial value and obstacle $(x_1^+)^\alpha$, $0 < \alpha < 1$. Then there is a constant $0 < c_\alpha < \infty$ such that*

$$\mathcal{E} = \{(x,t):\ 0 < t \le c_\alpha (x_1^+)^2\}.$$

Proof. We first show that the set $\mathcal{E}$ is not empty. Indeed, if this was the case, then the solution $u > (x_1^+)^\alpha$ for $t > 0$. In particular $H_0(u) = 0$ in $t > 0$. Due to parabolic regularity [W1]-[W2], one then concludes that

$$\lim_{t\to 0} u_t(x,t) = \lim_{t\to 0} F_0(D^2u(x,t)), \qquad x \in \mathbf{R}^n, \quad x_1 \ne 0.$$

Since $D^2u(x,0) = D^2x_1^\alpha$ for $x_1 > 0$, then we must have $F_0(D^2u(x,0)) < 0$ $(x_1 > 0)$. In particular $u_t(x,0) < 0$ when $x_1 > 0$ and hence the graph of u would go below x_1^α for $x_1 > 0$. A contradiction. Hence the set $\mathcal{E}$ is nonvoid.

By uniqueness of solutions we conclude conclude that $u(rx, r^2t)/r^\alpha$ is also a solution (since the initial data is invariant under such a scaling). Hence if $(x,t) \in \mathcal{E}$, then so is $(rx, r^2t) \in \mathcal{E}$.

Another geometric property that we can derive is from the fact that $u(x,t) \ge u(x,0)$. Indeed, by shifting in the t-direction and comparing the two solutions, and using the comparison principle we have that u is nondecreasing in t.

It is also apparent that no point in $\{x_1 < 0,\ t > 0\}$ can be a free boundary point, due to strong maximum principle.

All the above implies that

$$\mathcal{E} = \{(x,t):\ 0 < t \le c_\alpha (x_1^+)^2\},$$

for some constant $0 < c_\alpha \le \infty$.

To complete the proof we need to make sure $c_\alpha < \infty$, or in other words that the set $\{x_1 > 0, t > 0\} \setminus \mathcal{E}$ is nonempty. This is however obvious since otherwise $\{x_1 = 0, t > 0\}$ would be included in the free boundary and consequently u takes its minimum value (zero here) at interior points. Hence it cannot be a supersolution. □

Next we prove Theorem 1.6.

As in the proof of Theorem 1.4, we claim that given ϵ there exists a $r_\epsilon > 0$ such that

$$\partial\{u > \psi\} \subset P_\epsilon \cup \{t < \epsilon|x|^2\},$$

where

$$P_\epsilon := \{(x,t):\ x_1 > 0,\ t \le (c_\alpha + \epsilon)x_1^2)\}.$$

If this fails, then there are $X^j = (x^j, t^j)$ (with $|X^j| =: r_j \searrow 0$), $u_j \in \mathcal{G}_1$ such that $u_j(X^j) = \psi(X^j)$ and

$$X^j \notin P_\epsilon \cup \{t < \epsilon|x|^2\}. \tag{3.2}$$

Once again upon scaling $\tilde u_j(x,t) = u_j(r_jx, r_j^2t)/r_j^\alpha$ and repeating the above argument we arrive at a global solution u_0 with a free boundary point X^0, such that

$$|X^0| = 1 \qquad \text{and} \qquad X^0 \notin P_\epsilon \cup \{t < \epsilon|x|^2\}. \tag{3.3}$$

This contradicts the classification of global solutions, Lemma 3.1. □

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