# SEQUENTIAL FOURIER-FEYNMAN TRANSFORM, CONVOLUTION AND FIRST VARIATION 

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#### Abstract

Cameron and Storvick introduced the concept of a sequential Fourier-Feynman transform and established the existence of this transform for functionals in a Banach algebra $\hat{\mathcal{S}}$ of bounded functionals on classical Wiener space. In this paper we investigate various relationships between the sequential Fourier-Feynman transform and the convolution product for functionals which need not be bounded or continuous. Also we study the relationships involving this transform and the first variation.


## 1. Introduction and preliminaries

The concept of an $L_{1}$ analytic Fourier-Feynman transform for functionals on classical Wiener space ( $C_{0}[0, T], m$ ) was introduced by Brue in [1]. In [2], Cameron and Storvick introduced an $L_{2}$ analytic Fourier-Feynman transform on classical Wiener space. In [15], Johnson and Skoug developed an $L_{p}$ analytic Fourier-Feynman transform theory for $1 \leq p \leq 2$ that extended the results in [2] and gave various relationships between the $L_{1}$ and $L_{2}$ theories. In [12], Huffman, Park and Skoug defined a convolution product for functionals on classical Wiener space and they obtained various results on the analytic Fourier-Feynman transform and the convolution product [13, 14]. In [16], Park, Skoug and Storvick investigated various relationships among the first variation, the convolution product and the analytic Fourier-Feynman transform for functionals on classical Wiener space which belong to the Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick in [3].

Recently, Chang, Kim, Song and Yoo studied analytic Fourier-Feynman transform, convolution and the first variation for functionals on abstract Wiener space, product abstract Wiener space and the space of abstract Wiener space valued continuous functions $[8,9,10,11]$ which extended the results in $[12,13,14,16]$. For a detailed survey of previous work, see [17].

On the other hand, Cameron and Storvick gave a simple definition of the sequential Feynman integral which is applicable to a rather large class of functionals [4]. In [5], they used the sequential Feynman integral to define a sequential FourierFeynman transform and established the existence of the sequential Fourier-Feynman transform for functionals in the Banach algebra $\hat{\mathcal{S}}$ introduced by Cameron and

[^0]Storvick in [4]. Moreover they showed that the set of sequential Fourier-Feynman transforms $\left\{\Gamma_{p}: p \in \mathbb{R}\right\}$ forms an abelian group of isometries of $\hat{\mathcal{S}}$.

In this paper we study the sequential Fourier-Feynman transform, convolution and first variation of functionals which need not be bounded or continuous. In Section 2, we prove the existence of the sequential Fourier-Feynman transform and a translation theorem for functionals of the form $F(x)=G(x) \Psi(x(T))$, where $G$ is in the Banach algebra $\hat{\mathcal{S}}$ and $\Psi=\Psi_{1}+\Psi_{2}$ where $\Psi_{1} \in L_{1}(\mathbb{R})$ and $\Psi_{2}$ is the Fourier transform of a complex Borel measure of bounded variation on $\mathbb{R}$.

In Section 3, we show that the sequential Fourier-Feynman transform of the convolution product is a product of the sequential Fourier-Feynman transforms for functionals studied in Section 2. We also obtain Parseval's relation for functionals in $\hat{\mathcal{S}}$.

Finally, in Section 4, we study relationships involving the sequential FourierFeynman transform and the first variation. The transform can be taken with respect to the first or the second argument of the variation.

Let $C_{0}[0, T]$ be the space of continuous functions $x(t)$ on $[0, T]$ such that $x(0)=0$. Let a subdivision $\sigma$ of $[0, T]$ be given:

$$
\sigma: 0=\tau_{0}<\tau_{1}<\cdots<\tau_{m}=T
$$

and let $X \equiv X(t)$ be a polygonal curve in $C_{0}[0, T]$ based on a subdivision $\sigma$ and the real numbers $\vec{\xi} \equiv\left\{\xi_{k}\right\}$, that is,

$$
X(t) \equiv X(t, \sigma, \vec{\xi})
$$

where

$$
X(t, \sigma, \vec{\xi})=\frac{\xi_{k-1}\left(\tau_{k}-t\right)+\xi_{k}\left(t-\tau_{k-1}\right)}{\tau_{k}-\tau_{k-1}}
$$

when $\tau_{k-1} \leq t \leq \tau_{k}, k=1,2, \cdots, m$ and $\xi_{0}=0$. If there is a sequence of subdivisions $\left\{\sigma_{n}\right\}$, then $\sigma, m$ and $\tau_{k}$ will be replaced by $\sigma_{n}, m_{n}$ and $\tau_{n, k}$.

Let $q \neq 0$ be a given real number and let $F(x)$ be a functional defined on a subset of $C_{0}[0, T]$ containing all the polygonal curves in $C_{0}[0, T]$. Let $\left\{\sigma_{n}\right\}$ be a sequence of subdivisions such that $\left\|\sigma_{n}\right\| \rightarrow 0$ and let $\left\{\lambda_{n}\right\}$ be a sequence of complex numbers with $\operatorname{Re} \lambda_{n}>0$ such that $\lambda_{n} \rightarrow-i q$. Then if the integral in the right hand side of (1.1) exists for all $n$ and if the following limit exists and is independent of the choice of the sequences $\left\{\sigma_{n}\right\}$ and $\left\{\lambda_{n}\right\}$, we say that the sequential Feynman integral with parameter $q$ exists and is denoted by
(1.1)

$$
\int^{\mathrm{sf} f_{q}} F(x) d x=\lim _{n \rightarrow \infty} \gamma_{\sigma_{n}, \lambda_{n}} \int_{\mathbb{R}^{m_{n}}} \exp \left\{-\frac{\lambda_{n}}{2} \int_{0}^{T}\left|\frac{d X}{d t}\left(t, \sigma_{n}, \vec{\xi}\right)\right|^{2} d t\right\} F\left(X\left(\cdot, \sigma_{n}, \vec{\xi}\right)\right) d \vec{\xi},
$$

where

$$
\gamma_{\sigma, \lambda}=\left(\frac{\lambda}{2 \pi}\right)^{m / 2} \prod_{k=1}^{m}\left(\tau_{k}-\tau_{k-1}\right)^{-1 / 2}
$$

Let

$$
\begin{aligned}
W_{\lambda}(\sigma, \vec{\xi}) & \equiv \gamma_{\sigma, \lambda} \exp \left\{-\frac{\lambda}{2} \int_{0}^{T}\left|\frac{d X}{d t}(t, \sigma, \vec{\xi})\right|^{2} d t\right\} \\
& =\left(\frac{\lambda}{2 \pi}\right)^{m / 2} \prod_{k=1}^{m}\left(\tau_{k}-\tau_{k-1}\right)^{-1 / 2} \exp \left\{-\frac{\lambda}{2} \sum_{k=1}^{m} \frac{\left(\xi_{k}-\xi_{k-1}\right)^{2}}{\tau_{k}-\tau_{k-1}}\right\} .
\end{aligned}
$$

Thus in terms of $W_{\lambda}(\sigma, \vec{\xi})$, the sequential Feynman integral can be written

$$
\int^{\mathrm{sf}_{q}} F(x) d x=\lim _{n \rightarrow \infty} \int_{\mathbb{R}^{m_{n}}} W_{\lambda_{n}}\left(\sigma_{n}, \vec{\xi}\right) F\left(X\left(\cdot, \sigma_{n}, \vec{\xi}\right)\right) d \vec{\xi} .
$$

Let $D[0, T]$ be the class of elements $x \in C_{0}[0, T]$ such that $x$ is absolutely continuous on $[0, T]$ and its derivative $x^{\prime} \in L_{2}[0, T]$.
Definition 1.1. Let $q$ be a nonzero real number. For $y \in D[0, T]$, we define the sequential Fourier-Feynman transform $\Gamma_{q}(F)$ of $F$ by the formula

$$
\begin{equation*}
\Gamma_{q}(F)(y)=\int^{\mathrm{sf}_{q}} F(x+y) d x \tag{1.2}
\end{equation*}
$$

if it exists.
Definition 1.2. Let $q$ be a nonzero real number. For $y \in D[0, T]$, we define the convolution $(F * G)_{q}$ of $F$ and $G$ by the formula

$$
\begin{equation*}
(F * G)_{q}(y)=\int^{\mathrm{sf}_{q}} F\left(\frac{y+x}{\sqrt{2}}\right) G\left(\frac{y-x}{\sqrt{2}}\right) d x \tag{1.3}
\end{equation*}
$$

if it exists.
Definition 1.3. Let $y \in C_{0}[0, T]$. The first variation of $F$ in the direction $y$ is defined by the formula

$$
\begin{equation*}
\delta F(x \mid y)=\left.\frac{\partial}{\partial h} F(x+h y)\right|_{h=0} \tag{1.4}
\end{equation*}
$$

if it exists.
Remark 1.4. Cameron and Storvick [5] defined the sequential Fourier-Feynman transform by the formula

$$
\begin{equation*}
\Gamma_{q}(F)(y)=\int^{\mathrm{sf}_{1 / q}} F(x+y) d x \tag{1.5}
\end{equation*}
$$

for $y \in D[0, T]$. The two definitions (1.2) and (1.5) are essentially the same. For more discussions, see Remark 3.5.

For $u, v \in L_{2}[0, T]$ and $x \in C_{0}[0, T]$, we let

$$
\langle u, v\rangle=\int_{0}^{T} u(t) v(t) d t
$$

and

$$
(u, x)=\int_{0}^{T} u(t) d x(t)
$$

the Paley-Wiener-Zygmund stochastic integral. For a subdivision $\sigma$ of $[0, T]$, we let

$$
\langle v, 1\rangle_{k}=\int_{\tau_{k-1}}^{\tau_{k}} v(t) d t
$$

for $k=1, \cdots, m$. If there is a sequence of subdivisions $\left\{\sigma_{n}\right\}$, then $\langle v, 1\rangle_{k}$ will be replaced by $\langle v, 1\rangle_{n, k}$.

Let $\mathcal{M} \equiv \mathcal{M}\left(L_{2}[0, T]\right)$ be the class of complex measures of finite variation defined on $\mathcal{B}\left(L_{2}[0, T]\right)$, the Borel measurable subsets of $L_{2}[0, T]$. A functional $F$ defined on
a subset of $C_{0}[0, T]$ that contains $D[0, T]$ is said to be an element of $\hat{\mathcal{S}} \equiv \hat{\mathcal{S}}\left(L_{2}[0, T]\right)$ if there exists a measure $f \in \mathcal{M}$ such that for $x \in D[0, T]$,

$$
\begin{equation*}
F(x)=\int_{L_{2}[0, T]} \exp \left\{i\left\langle v, x^{\prime}\right\rangle\right\} d f(v) \tag{1.6}
\end{equation*}
$$

Cameron and Storvick showed in [4] that each functional $F \in \hat{\mathcal{S}}$ is sequential Feynman integrable and

$$
\begin{equation*}
\int^{\mathrm{sf}_{q}} F(x) d x=\int_{L_{2}[0, T]} \exp \left\{-\frac{i}{2 q}\|v\|_{2}^{2}\right\} d f(v) \tag{1.7}
\end{equation*}
$$

Moreover they showed in [5] that for each functional $F \in \hat{\mathcal{S}}$, its sequential FourierFeynman transform exists and is given by

$$
\begin{equation*}
\Gamma_{q}(F)(y)=\int_{L_{2}[0, T]} \exp \left\{i\left\langle v, y^{\prime}\right\rangle-\frac{i}{2 q}\|v\|_{2}^{2}\right\} d f(v) \tag{1.8}
\end{equation*}
$$

We finish this section by introducing three lemmas which are useful in Section 2 and Section 3.

Lemma 1.5 (Lemma 3.2 in [4]). Let $v \in L_{2}[0, T]$. Let $\left\{\sigma_{n}\right\}$ be a sequence of subdivisions of $[0, T]$ such that $\left\|\sigma_{n}\right\| \rightarrow 0$. Define the averaged function $v_{\sigma_{n}}$ for $v$ on $\sigma_{n}$ by

$$
v_{\sigma_{n}}(t) \equiv \begin{cases}\frac{\langle v, 1\rangle_{n, k}}{\tau_{n, k}-\tau_{n, k-1}}, & \text { if } \tau_{n, k-1} \leq t<\tau_{n, k}, \quad k=1, \cdots, m_{n} \\ 0, & \text { if } t=T\end{cases}
$$

Then

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|v_{\sigma_{n}}\right\|_{2}^{2}=\|v\|_{2}^{2} \tag{1.9}
\end{equation*}
$$

The following lemma is essentially the same as Lemma 1 in [6]. But we rewrite it with our notation.

Lemma 1.6. Let $v \in L_{2}[0, T]$. Let $\sigma$ be a subdivision of $[0, T]$ and let $\operatorname{Re} \lambda>0$. Let

$$
\begin{equation*}
=\gamma_{\sigma, \lambda} \int_{\mathbb{R}^{m-1}} \exp \left\{-\frac{\lambda}{2} \sum_{k=1}^{m} \frac{\left(\xi_{k}-\xi_{k-1}\right)^{2}}{\tau_{k}-\tau_{k-1}}+i \sum_{k=1}^{m}\langle v, 1\rangle_{k} \frac{\xi_{k}-\xi_{k-1}}{\tau_{k}-\tau_{k-1}}\right\} d \xi_{1} \cdots d \xi_{m-1} \tag{1.10}
\end{equation*}
$$

Then we have
$J_{\sigma, \lambda}\left(\xi_{m}, v\right)=\left(\frac{\lambda}{2 \pi T}\right)^{1 / 2} \exp \left\{\frac{\langle v, 1\rangle^{2}}{2 \lambda T}-\frac{1}{2 \lambda} \sum_{k=1}^{m} \frac{\langle v, 1\rangle_{k}^{2}}{\tau_{k}-\tau_{k-1}}+\frac{1}{2 T}\left(-\lambda \xi_{m}^{2}+2 i \xi_{m}\langle v, 1\rangle\right)\right\}$.
Using Lemma 1.5, we easily obtain the following lemma.
Lemma 1.7. Let $v$ and $\left\{\sigma_{n}\right\}$ be given as in Lemma 1.5. Let $\left\{\lambda_{n}\right\}$ be a sequence of complex numbers such that $\operatorname{Re} \lambda_{n}>0$ and $\lambda_{n} \rightarrow-i q$ as $n \rightarrow \infty$. Let $J_{\sigma_{n}, \lambda_{n}}(\xi, v)$ be given by (1.11) with $\sigma, \lambda$ and $\xi_{m}$ replaced by $\sigma_{n}, \lambda_{n}$ and $\xi$. Then we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} J_{\sigma_{n}, \lambda_{n}}(\xi, v)=\left(\frac{-i q}{2 \pi T}\right)^{1 / 2} \exp \left\{K_{q}(\xi, v)\right\} \tag{1.12}
\end{equation*}
$$

where

$$
\begin{equation*}
K_{q}(\xi, v)=\frac{i}{2 q T}\langle v, 1\rangle^{2}-\frac{i}{2 q}\|v\|_{2}^{2}+\frac{i}{2 T}\left(q \xi^{2}+2 \xi\langle v, 1\rangle\right) . \tag{1.13}
\end{equation*}
$$

## 2. Sequential Fourier-Feynman transform

In [6], Cameron and Storvick established the sequential Feynman integrability of functionals of the form

$$
F(x)=G(x) \Psi(x(T)),
$$

where $G \in \hat{\mathcal{S}}$ and $\Psi$ need not be bounded or continuous.
In this section we prove the existence of the sequential Fourier-Feynman transform and a translation theorem for such functionals.

Theorem 2.1. For $x \in D[0, T]$, let

$$
\begin{equation*}
F(x)=G(x) \Psi(x(T)), \tag{2.1}
\end{equation*}
$$

where $G \in \hat{\mathcal{S}}$ is given by (1.6) with corresponding measure $g$ in $\mathcal{M}$ and $\Psi \in L_{1}(\mathbb{R})$. Then for each nonzero real $q$, the sequential Fourier-Feynman transform $\Gamma_{q}(F)$ of $F$ exists and is given by
(2.2) $\Gamma_{q}(F)(y)=\left(\frac{-i q}{2 \pi T}\right)^{1 / 2} \int_{L_{2}[0, T]} \int_{\mathbb{R}} \exp \left\{i\left\langle v, y^{\prime}\right\rangle+K_{q}(\xi, v)\right\} \Psi(\xi+y(T)) d \xi d g(v)$
for $y \in D[0, T]$, where $K_{q}(\xi, v)$ is given by (1.13).
Proof. Let $\sigma: 0=\tau_{0}<\tau_{1}<\cdots<\tau_{m}=T$ be a subdivision of $[0, T]$, let $\lambda$ be a complex number with $\operatorname{Re} \lambda>0$, and let

$$
\begin{aligned}
I_{\sigma, \lambda}(F) & =\int_{\mathbb{R}^{m}} W_{\lambda}(\sigma, \vec{\xi}) F(X(\cdot, \sigma, \vec{\xi})+y) d \vec{\xi} \\
& =\gamma_{\sigma, \lambda} \int_{\mathbb{R}^{m}} \exp \left\{-\frac{\lambda}{2} \sum_{k=1}^{m} \frac{\left(\xi_{k}-\xi_{k-1}\right)^{2}}{\tau_{k}-\tau_{k-1}}\right\} F(X(\cdot, \sigma, \vec{\xi})+y) d \vec{\xi} .
\end{aligned}
$$

By (2.1) and (1.6), we have for $y \in D[0, T]$,

$$
\begin{aligned}
F(X(\cdot, \sigma, \vec{\xi})+y)= & \int_{L_{2}[0, T]} \exp \left\{i \sum_{k=1}^{m}\langle v, 1\rangle_{k} \frac{\xi_{k}-\xi_{k-1}}{\tau_{k}-\tau_{k-1}}+i\left\langle v, y^{\prime}\right\rangle\right\} \\
& \Psi\left(\xi_{m}+y(T)\right) d g(v)
\end{aligned}
$$

Using the Fubini theorem and Lemma 1.6, we have

$$
I_{\sigma, \lambda}(F)=\int_{L_{2}[0, T]} \int_{\mathbb{R}} J_{\sigma, \lambda}\left(\xi_{m}, v\right) \exp \left\{i\left\langle v, y^{\prime}\right\rangle\right\} \Psi\left(\xi_{m}+y(T)\right) d \xi_{m} d g(v)
$$

where $J_{\sigma, \lambda}\left(\xi_{m}, v\right)$ is given by (1.11).
Now let $\left\{\sigma_{n}\right\}$ be a sequence of subdivisions of $[0, T]$ such that $\left\|\sigma_{n}\right\| \rightarrow 0$, and let $\left\{\lambda_{n}\right\}$ be a sequence of complex numbers such that $\operatorname{Re} \lambda_{n}>0$ and $\lambda_{n} \rightarrow-i q$ as $n \rightarrow \infty$. Then we have

$$
I_{\sigma_{n}, \lambda_{n}}(F)=\int_{L_{2}[0, T]} \int_{\mathbb{R}} J_{\sigma_{n}, \lambda_{n}}(\xi, v) \exp \left\{i\left\langle v, y^{\prime}\right\rangle\right\} \Psi(\xi+y(T)) d \xi d g(v)
$$

But by the Schwartz inequality,

$$
\frac{1}{T}\langle v, 1\rangle^{2}-\sum_{k=1}^{m_{n}} \frac{\langle v, 1\rangle_{n, k}^{2}}{\tau_{n, k}-\tau_{n, k-1}} \leq 0
$$

and so we apply the dominated convergence theorem and use Lemma 1.7 to conclude that
$\lim _{n \rightarrow \infty} I_{\sigma_{n}, \lambda_{n}}(F)=\left(\frac{-i q}{2 \pi T}\right)^{1 / 2} \int_{L_{2}[0, T]} \int_{\mathbb{R}} \exp \left\{i\left\langle v, y^{\prime}\right\rangle+K_{q}(\xi, v)\right\} \Psi(\xi+y(T)) d \xi d g(v)$, and this completes the proof.

In [4], Cameron and Storvick gave a translation theorem for the sequential Feynman integral for functionals in $\hat{\mathcal{S}}$. In our next corollary, we give a translation theorem for functionals we considered in Theorem 2.1.

Corollary 2.2. Let $F$ be given as in Theorem 2.1 and let $y \in D[0, T]$. Then for each nonzero real $q$, each side of the following equation exists and

$$
\begin{equation*}
\int^{\mathrm{sf}_{q}} F(x+y) d x=\exp \left\{\frac{i q}{2}\left\|y^{\prime}\right\|_{2}^{2}\right\} \int^{\mathrm{sf}_{q}} F(x) \exp \left\{-i q\left(y^{\prime}, x\right)\right\} d x \tag{2.3}
\end{equation*}
$$

Proof. Let

$$
\tilde{F}(x)=F(x) \exp \left\{-i q\left(y^{\prime}, x\right)\right\} .
$$

Using (2.1) and (1.6), we have for $x \in D[0, T]$,

$$
\tilde{F}(x)=\int_{L_{2}[0, T]} \exp \left\{i\left\langle v-q y^{\prime}, x^{\prime}\right\rangle\right\} d g(v) \Psi(x(T))
$$

Let $\tilde{g}$ be a measure in $\mathcal{M}$ defined by

$$
\begin{equation*}
\tilde{g}(E)=g\left(E+q y^{\prime}\right) \tag{2.4}
\end{equation*}
$$

for $E \in \mathcal{B}\left(L_{2}[0, T]\right)$. Then

$$
\tilde{F}(x)=\int_{L_{2}[0, T]} \exp \left\{i\left\langle w, x^{\prime}\right\rangle\right\} d \tilde{g}(w) \Psi(x(T))
$$

which is of the form (2.1). Thus by Theorem 2.1,

$$
\Gamma_{q}(\tilde{F})(0)=\left(\frac{-i q}{2 \pi T}\right)^{1 / 2} \int_{L_{2}[0, T]} \int_{\mathbb{R}} \exp \left\{K_{q}(\xi, w)\right\} \Psi(\xi) d \xi d \tilde{g}(w)
$$

Using the expressions (1.13), (2.4) and letting $\xi=\eta+y(T)$, we have

$$
\begin{aligned}
\Gamma_{q}(\tilde{F})(0)= & \left(\frac{-i q}{2 \pi T}\right)^{1 / 2} \int_{L_{2}[0, T]} \int_{\mathbb{R}} \exp \left\{i\left\langle v, y^{\prime}\right\rangle+K_{q}(\eta, v)-\frac{i q}{2}\left\|y^{\prime}\right\|_{2}^{2}\right\} \\
& \Psi(\eta+y(T)) d \eta d g(v)
\end{aligned}
$$

Hence we have proven that

$$
\Gamma_{q}(\tilde{F})(0)=\exp \left\{-\frac{i q}{2}\left\|y^{\prime}\right\|_{2}^{2}\right\} \Gamma_{q}(F)(y)
$$

which completes the proof of Corollary 2.2 .
Let $\mathcal{T}$ be the set of functions $\Psi$ defined on $\mathbb{R}$ by

$$
\begin{equation*}
\Psi(r)=\int_{\mathbb{R}} \exp \{i r s\} d \rho(s) \tag{2.5}
\end{equation*}
$$

where $\rho$ is a complex Borel measure of bounded variation on $\mathbb{R}$.
For $s \in \mathbb{R}$, let $\phi(s)$ be the function $v$ such that $v(t)=s$ for $0 \leq t \leq T$; thus $\phi: \mathbb{R} \rightarrow L_{2}[0, T]$ is continuous. For $E \in \mathcal{B}\left(L_{2}[0, T]\right)$, let

$$
\begin{equation*}
\psi(E)=\rho\left(\phi^{-1}(E)\right) \tag{2.6}
\end{equation*}
$$

Thus $\psi \in \mathcal{M}$. Transforming the right hand member of (2.5), we have for $x \in$ $D[0, T]$,

$$
\begin{equation*}
\Psi(x(T))=\int_{L_{2}[0, T]} \exp \left\{i\left\langle u, x^{\prime}\right\rangle\right\} d \psi(u), \tag{2.7}
\end{equation*}
$$

and $\Psi(x(T))$, considered as a functional of $x$, is an element of $\hat{\mathcal{S}}$.
Theorem 2.3. For $x \in D[0, T]$, let $F(x)=G(x) \Psi(x(T))$ where $G \in \hat{\mathcal{S}}$ and $\Psi \in \mathcal{T}$ are given by (1.6) with corresponding measure $g$ in $\mathcal{M}$ and (2.5) respectively. Then for each nonzero real $q$, the sequential Fourier-Feynman transform $\Gamma_{q}(F)$ of $F$ exists, belongs to $\hat{\mathcal{S}}$ and is given by

$$
\begin{equation*}
\Gamma_{q}(F)(y)=\int_{L_{2}[0, T]} \int_{\mathbb{R}} \exp \left\{i\left\langle v+s, y^{\prime}\right\rangle-\frac{i}{2 q}\|v+s\|_{2}^{2}\right\} d \rho(s) d g(v) \tag{2.8}
\end{equation*}
$$

for $y \in D[0, T]$.
Proof. Because $\hat{\mathcal{S}}$ is a Banach algebra, and $G$ and $\Psi(x(T))$ are in $\hat{\mathcal{S}}$, we see that $F \in \hat{\mathcal{S}}$. By (1.6) and (2.7), we have for $x \in D[0, T]$,

$$
F(x)=\int_{L_{2}^{2}[0, T]} \exp \left\{i\left\langle v+u, x^{\prime}\right\rangle\right\} d g(v) d \psi(u)
$$

Making the substitution $w=v+u$ on the inner integral, we have

$$
F(x)=\int_{L_{2}^{2}[0, T]} \exp \left\{i\left\langle w, x^{\prime}\right\rangle\right\} d_{w} g(w-u) d \psi(u)
$$

Then by the unsymmetric Fubini theorem (Theorem 6.1 in [3]), we have

$$
F(x)=\int_{L_{2}[0, T]} \exp \left\{i\left\langle w, x^{\prime}\right\rangle\right\} d h(w)
$$

where $h$ is a complex measure on $\mathcal{B}\left(L_{2}[0, T]\right)$ defined by

$$
\begin{equation*}
h(E)=\int_{L_{2}[0, T]} g(E-u) d \psi(u) \tag{2.9}
\end{equation*}
$$

Now, by applying (1.8) to the expression for $F$ above, we have

$$
\begin{equation*}
\Gamma_{q}(F)(y)=\int_{L_{2}[0, T]} \exp \left\{i\left\langle w, y^{\prime}\right\rangle-\frac{i}{2 q}\|w\|_{2}^{2}\right\} d h(w) \tag{2.10}
\end{equation*}
$$

for $y \in D[0, T]$. By Theorem 6.1 in [3] and the transformation $v=w-u$, we have

$$
\Gamma_{q}(F)(y)=\int_{L_{2}^{2}[0, T]} \exp \left\{i\left\langle v+u, y^{\prime}\right\rangle-\frac{i}{2 q}\|v+u\|_{2}^{2}\right\} d g(v) d \psi(u)
$$

Using (2.6) and the Fubini theorem, we obtain (2.8).
Finally from (2.10), it is easy to see that

$$
\Gamma_{q}(F)(y)=\int_{L_{2}[0, T]} \exp \left\{i\left\langle w, y^{\prime}\right\rangle\right\} d \tilde{h}(w)
$$

where $\tilde{h}$ is a complex Borel measure on $\mathcal{B}\left(L_{2}[0, T]\right)$ defined by

$$
\tilde{h}(E)=\int_{E} \exp \left\{-\frac{i}{2 q}\|w\|_{2}^{2}\right\} d h(w)
$$

and so $\Gamma_{q}(F)$ belongs to $\hat{\mathcal{S}}$.

Corollary 2.4. Let $F$ be given as in Theorem 2.3 and let $y \in D[0, T]$. Then our translation theorem (2.3) holds.

Proof. Without loss of generality, we may assume that $F \in \hat{\mathcal{S}}$. Let $\tilde{F}(x)$ be given as in the proof of Corollary 2.2. By the same method as in the proof of Corollary 2.2 , we write for $x \in D[0, T]$

$$
\tilde{F}(x)=\int_{L_{2}[0, T]} \exp \left\{i\left\langle w, x^{\prime}\right\rangle\right\} d \tilde{f}(v)
$$

where $\tilde{f}$ is the measure on $\mathcal{B}\left(L_{2}[0, T]\right)$ defined by $\tilde{f}(E)=f\left(E+q y^{\prime}\right)$. Now by (1.8),

$$
\Gamma_{q}(\tilde{F})(0)=\int_{L_{2}[0, T]} \exp \left\{-\frac{i}{2 q}\left\|v-q y^{\prime}\right\|_{2}^{2}\right\} d f(v)=\exp \left\{-\frac{i q}{2}\left\|y^{\prime}\right\|_{2}^{2}\right\} \Gamma_{q}(F)(y)
$$

and this completes the proof.
From Theorems 2.1, 2.3, Corollaries 2.2, 2.4 and the linearity of the sequential Feynman integral, we have the following results.

Theorem 2.5. For $x \in D[0, T]$, let $F(x)=G(x) \Psi(x(T))$ where $G \in \hat{\mathcal{S}}$ and $\Psi=$ $\Psi_{1}+\Psi_{2} \in L_{1}(\mathbb{R})+\mathcal{T}$. Then for each nonzero real $q$, the sequential Fourier-Feynman transform $\Gamma_{q}(F)$ of $F$ exists, and $\Gamma_{q}(F)(y)$ for $y \in D[0, T]$ is equal to the sum of the right hand side of equations (2.2) and (2.8).

Corollary 2.6. Let $F$ be given as in Theorem 2.5 and let $y \in D[0, T]$. Then our translation theorem (2.3) holds.

## 3. Convolution product for the sequential Fourier-Feynman transform

In this section we show that the sequential Fourier-Feynman transform of the convolution product is a product of the sequential Fourier-Feynman transforms for the functionals studied in Section 2.

We begin with the existence theorem for the convolution product for functionals in $\hat{\mathcal{S}}$.

Theorem 3.1. Let $F_{j} \in \hat{\mathcal{S}}$ be given by (1.6) with corresponding measures $f_{j}$ in $\mathcal{M}$ for $j=1,2$. Then for each nonzero real number $q$, the convolution $\left(F_{1} * F_{2}\right)_{q}$ exists, belongs to $\hat{\mathcal{S}}$ and is given by

$$
\begin{equation*}
\left(F_{1} * F_{2}\right)_{q}(y)=\int_{L_{2}^{2}[0, T]} \exp \left\{\frac{i}{\sqrt{2}}\left\langle v_{1}+v_{2}, y^{\prime}\right\rangle-\frac{i}{4 q}\left\|v_{1}-v_{2}\right\|_{2}^{2}\right\} d f_{1}\left(v_{1}\right) d f_{2}\left(v_{2}\right) \tag{3.1}
\end{equation*}
$$

for $y \in D[0, T]$.
Proof. Let $\sigma$ and $\lambda$ be given as in the proof of Theorem 2.1 and let

$$
I_{\sigma, \lambda}\left(F_{1}, F_{2}\right)=\int_{\mathbb{R}^{m}} W_{\lambda}(\sigma, \vec{\xi}) F_{1}\left(\frac{y+X(\cdot, \sigma, \vec{\xi})}{\sqrt{2}}\right) F_{2}\left(\frac{y-X(\cdot, \sigma, \vec{\xi})}{\sqrt{2}}\right) d \vec{\xi} .
$$

Using (1.6), the Fubini theorem, we have for $y \in D[0, T]$

$$
I_{\sigma, \lambda}\left(F_{1}, F_{2}\right)=\int_{L_{2}^{2}[0, T]} J_{\sigma, \lambda}\left(\frac{v_{1}-v_{2}}{\sqrt{2}}\right) \exp \left\{\frac{i}{\sqrt{2}}\left\langle v_{1}+v_{2}, y^{\prime}\right\rangle\right\} d f_{1}\left(v_{1}\right) d f_{2}\left(v_{2}\right)
$$

where

$$
\begin{aligned}
J_{\sigma, \lambda}\left(\frac{v_{1}-v_{2}}{\sqrt{2}}\right)= & \gamma_{\sigma, \lambda} \int_{\mathbb{R}^{m}} \exp \left\{-\frac{\lambda}{2} \sum_{k=1}^{m} \frac{\left(\xi_{k}-\xi_{k-1}\right)^{2}}{\tau_{k}-\tau_{k-1}}\right. \\
& \left.+\frac{i}{\sqrt{2}} \sum_{k=1}^{m}\left\langle v_{1}-v_{2}, 1\right\rangle_{k} \frac{\xi_{k}-\xi_{k-1}}{\tau_{k}-\tau_{k-1}}\right\} d \vec{\xi} .
\end{aligned}
$$

Using the integration formula

$$
\int_{\mathbb{R}} \exp \left\{-a \xi^{2}+i b \xi\right\} d \xi=\left(\frac{\pi}{a}\right)^{1 / 2} \exp \left\{-\frac{b^{2}}{4 a}\right\}
$$

when $\operatorname{Re} a>0$ and $b \in \mathbb{R}$, we have

$$
J_{\sigma, \lambda}\left(\frac{v_{1}-v_{2}}{\sqrt{2}}\right)=\exp \left\{-\frac{1}{4 \lambda} \sum_{k=1}^{m} \frac{\left\langle v_{1}-v_{2}, 1\right\rangle_{k}^{2}}{\tau_{k}-\tau_{k-1}}\right\} .
$$

Let $\left\{\sigma_{n}\right\}$ and $\left\{\lambda_{n}\right\}$ be given as in the proof of Theorem 2.1. Then we have (3.2)

$$
I_{\sigma_{n}, \lambda_{n}}\left(F_{1}, F_{2}\right)=\int_{L_{2}^{2}[0, T]} J_{\sigma_{n}, \lambda_{n}}\left(\frac{v_{1}-v_{2}}{\sqrt{2}}\right) \exp \left\{\frac{i}{\sqrt{2}}\left\langle v_{1}+v_{2}, y^{\prime}\right\rangle\right\} d f_{1}\left(v_{1}\right) d f_{2}\left(v_{2}\right)
$$

where

$$
J_{\sigma_{n}, \lambda_{n}}\left(\frac{v_{1}-v_{2}}{\sqrt{2}}\right)=\exp \left\{-\frac{1}{4 \lambda_{n}} \sum_{k=1}^{m_{n}} \frac{\left\langle v_{1}-v_{2}, 1\right\rangle_{n, k}^{2}}{\tau_{n, k}-\tau_{n, k-1}}\right\} .
$$

By Lemma 1.5,

$$
\lim _{n \rightarrow \infty} J_{\sigma_{n}, \lambda_{n}}\left(\frac{v_{1}-v_{2}}{\sqrt{2}}\right)=\exp \left\{-\frac{i}{4 q}\left\|v_{1}-v_{2}\right\|_{2}^{2}\right\}
$$

Now applying the bounded convergence theorem to (3.2), we obtain (3.1).
Let $\mu$ be a set function on $\mathcal{B}\left(L_{2}^{2}[0, T]\right)$, defined by

$$
\mu(E)=\int_{E} \exp \left\{-\frac{i}{4 q}\left\|v_{1}-v_{2}\right\|_{2}^{2}\right\} d f_{1}\left(v_{1}\right) d f_{2}\left(v_{2}\right)
$$

Then $\mu \in \mathcal{M}\left(L_{2}^{2}[0, T]\right)$. Define a function $\phi: L_{2}^{2}[0, T] \rightarrow L_{2}[0, T]$ by $\phi\left(v_{1}, v_{2}\right)=$ $\frac{1}{\sqrt{2}}\left(v_{1}+v_{2}\right)$. Then $\phi$ is a Borel measurable function and so $k \equiv \mu \circ \phi^{-1}$ is in $\mathcal{M}$. Using the change of variable theorem, we have

$$
\begin{equation*}
\left(F_{1} * F_{2}\right)_{q}(y)=\int_{L_{2}[0, T]} \exp \left\{i\left\langle u, y^{\prime}\right\rangle\right\} d k(u) \tag{3.3}
\end{equation*}
$$

and this completes the proof.
Theorem 3.2. Let $F_{j}, j=1,2$, be given as in Theorem 3.1. Then for each nonzero real number $q$ and for $y \in D[0, T], \Gamma_{q}\left(\left(F_{1} * F_{2}\right)_{q}\right)(y)$ exists and

$$
\begin{equation*}
\Gamma_{q}\left(\left(F_{1} * F_{2}\right)_{q}\right)(y)=\Gamma_{q}\left(F_{1}\right)\left(\frac{y}{\sqrt{2}}\right) \Gamma_{q}\left(F_{2}\right)\left(\frac{y}{\sqrt{2}}\right) \tag{3.4}
\end{equation*}
$$

Proof. From the expression (3.3) and (1.8), we have

$$
\begin{equation*}
\Gamma_{q}\left(F_{1} * F_{2}\right)_{q}(y)=\int_{L_{2}[0, T]} \exp \left\{i\left\langle u, y^{\prime}\right\rangle-\frac{i}{2 q}\|u\|_{2}^{2}\right\} d k(u) \tag{3.5}
\end{equation*}
$$

for $y \in D[0, T]$. Using the definition of $k$, we rewrite $\Gamma_{q}\left(F_{1} * F_{2}\right)_{q}(y)$ as
$\Gamma_{q}\left(F_{1} * F_{2}\right)_{q}(y)=\int_{L_{2}^{2}[0, T]} \exp \left\{\frac{i}{\sqrt{2}}\left\langle v_{1}+v_{2}, y^{\prime}\right\rangle-\frac{i}{2 q}\left(\left\|v_{1}\right\|_{2}^{2}+\left\|v_{2}\right\|_{2}^{2}\right)\right\} d f_{1}\left(v_{1}\right) d f_{2}\left(v_{2}\right)$
which is equal to the right hand side of (3.4).
Theorem 3.3. Let $F_{j}, j=1,2$, be given as in Theorem 3.1. Then for each nonzero real number $q$, Parseval's relation

$$
\begin{equation*}
\int^{\mathrm{sf}-q} \Gamma_{q}\left(F_{1}\right)\left(\frac{y}{\sqrt{2}}\right) \Gamma_{q}\left(F_{2}\right)\left(\frac{y}{\sqrt{2}}\right) d y=\int^{\mathrm{sf}_{q}} F_{1}\left(\frac{y}{\sqrt{2}}\right) F_{2}\left(-\frac{y}{\sqrt{2}}\right) d y \tag{3.6}
\end{equation*}
$$

holds.
Proof. From (3.4), we have

$$
\int^{\mathrm{sf}-q} \Gamma_{q}\left(F_{1}\right)\left(\frac{y}{\sqrt{2}}\right) \Gamma_{q}\left(F_{2}\right)\left(\frac{y}{\sqrt{2}}\right) d y=\int^{\text {sf }-q} \Gamma_{q}\left(\left(F_{1} * F_{2}\right)_{q}\right)(y) d y
$$

Moreover from (3.5), we have

$$
\begin{equation*}
\Gamma_{q}\left(F_{1} * F_{2}\right)_{q}(y)=\int_{L_{2}[0, T]} \exp \left\{i\left\langle u, y^{\prime}\right\rangle\right\} d \tilde{k}(u) \tag{3.7}
\end{equation*}
$$

where $\tilde{k} \in M$ is a measure on $\mathcal{B}\left(L_{2}[0, T]\right)$ defined by

$$
\tilde{k}(E)=\int_{E} \exp \left\{-\frac{i}{2 q}\|u\|_{2}^{2}\right\} d k(u)
$$

and $k$ is the measure given in the proof of Theorem 3.1. Applying (1.7) to the expression (3.7), we obtain

$$
\int^{\mathrm{sf}_{-q}} \Gamma_{q}\left(F_{1} * F_{2}\right)_{q}(y) d y=\int_{L_{2}^{2}[0, T]} \exp \left\{-\frac{i}{4 q}\left\|v_{1}-v_{2}\right\|_{2}^{2}\right\} d f_{1}\left(v_{1}\right) d f_{2}\left(v_{2}\right)
$$

On the other hand, using (1.6) and (1.7), it is easy to see that the right hand side of (3.6) is equal to the right hand side of the last expression above, and this completes the proof.

Remark 3.4. From a careful look at the expressions (3.1), (3.3) and (3.7), and the proof of Theorem 3.3, we easily obtain the following alternative form of Parseval's relation:

$$
\int^{\text {sf }_{-q}} \Gamma_{q / 2}\left(F_{1}\right)(y) \Gamma_{q / 2}\left(F_{2}\right)(y) d y=\int^{\text {sf }_{q}} F_{1}(y) F_{2}(-y) d y
$$

for every nonzero real number $q$.
Remark 3.5. If we adopt Cameron and Storvick's definition of the sequential FourierFeynman transform ((1.5) in Section 1), then the equality (3.6) still holds true, but relationship (3.4) should be

$$
\Gamma_{q}\left(\left(F_{1} * F_{2}\right)_{1 / q}\right)(y)=\Gamma_{q}\left(F_{1}\right)\left(\frac{y}{\sqrt{2}}\right) \Gamma_{q}\left(F_{2}\right)\left(\frac{y}{\sqrt{2}}\right)
$$

On the other hand, if we adopt Cameron and Storvick's definition of the sequential Fourier-Feynman transform and if we define the convolution product by the formula

$$
(F * G)_{q}(y)=\int^{\mathrm{sf}_{1 / q}} F\left(\frac{y+x}{\sqrt{2}}\right) G\left(\frac{y-x}{\sqrt{2}}\right) d x
$$

then the equality (3.4) still holds true, but (3.6) should be

$$
\int^{\mathrm{sf}-1 / q} \Gamma_{q}\left(F_{1}\right)\left(\frac{y}{\sqrt{2}}\right) \Gamma_{q}\left(F_{2}\right)\left(\frac{y}{\sqrt{2}}\right) d y=\int^{\mathrm{sf}_{1 / q}} F_{1}\left(\frac{y}{\sqrt{2}}\right) F_{2}\left(-\frac{y}{\sqrt{2}}\right) d y
$$

for every nonzero real number $q$.
From now on, we investigate the convolution product for functionals of the form $F(x)=G(x) \Psi(x(T))$, where $G \in \hat{\mathcal{S}}$ and $\Psi$ need not be bounded or continuous.

Theorem 3.6. For $j=1,2$ and $x \in D[0, T]$, let $F_{j}(x)=G_{j}(x) \Psi_{j}(x(T))$ where $G_{j} \in \hat{\mathcal{S}}$ is given by (1.6) with corresponding measures $g_{j}$ in $\mathcal{M}$ and $\Psi_{j} \in L_{1}(\mathbb{R}) \cap$ $L_{2}(\mathbb{R})$. Then for each nonzero real number $q$ and for $y \in D[0, T]$, the convolution product $\left(F_{1} * F_{2}\right)_{q}(y)$ exists and is given by

$$
\begin{align*}
\left(F_{1} * F_{2}\right)_{q}(y)= & \left(\frac{-i q}{2 \pi T}\right)^{1 / 2} \int_{L_{2}^{2}[0, T]} \int_{\mathbb{R}} \exp \left\{\frac{i}{\sqrt{2}}\left\langle v_{1}+v_{2}, y^{\prime}\right\rangle+K_{q}\left(\xi, \frac{v_{1}-v_{2}}{\sqrt{2}}\right)\right\}  \tag{3.8}\\
& \Psi_{1}\left(\frac{y(T)+\xi}{\sqrt{2}}\right) \Psi_{2}\left(\frac{y(T)-\xi}{\sqrt{2}}\right) d \xi d g_{1}\left(v_{1}\right) d g_{2}\left(v_{2}\right)
\end{align*}
$$

where $K_{q}(\xi, v)$ is given by (1.13).
Proof. Let $\left\{\sigma_{n}\right\}$ and $\left\{\lambda_{n}\right\}$ be given as in the proof of Theorem 2.1 and let

$$
I_{\sigma_{n}, \lambda_{n}}\left(F_{1}, F_{2}\right)=\int_{\mathbb{R}^{m_{n}}} W_{\lambda_{n}}\left(\sigma_{n}, \vec{\xi}\right) F_{1}\left(\frac{y+X\left(\cdot, \sigma_{n}, \vec{\xi}\right)}{\sqrt{2}}\right) F_{2}\left(\frac{y-X\left(\cdot, \sigma_{n}, \vec{\xi}\right)}{\sqrt{2}}\right) d \vec{\xi} .
$$

Using the expression (1.6), the Fubini theorem and Lemma 1.6, we have

$$
\begin{aligned}
I_{\sigma_{n}, \lambda_{n}}\left(F_{1}, F_{2}\right)= & \int_{L_{2}^{2}[0, T]} \int_{\mathbb{R}} J_{\sigma_{n}, \lambda_{n}}\left(\xi, \frac{v_{1}-v_{2}}{\sqrt{2}}\right) \exp \left\{\frac{i}{\sqrt{2}}\left\langle v_{1}+v_{2}, y^{\prime}\right\rangle\right\} \\
& \Psi_{1}\left(\frac{y(T)+\xi}{\sqrt{2}}\right) \Psi_{2}\left(\frac{y(T)-\xi}{\sqrt{2}}\right) d \xi d g_{1}\left(v_{1}\right) d g_{2}\left(v_{2}\right)
\end{aligned}
$$

where $J_{\sigma_{n}, \lambda_{n}}(\xi, v)$ is given by (1.11). Now, the fact that $\Psi_{1}, \Psi_{2} \in L_{2}(\mathbb{R})$ and the Schwartz inequality justify the application of the dominated convergence theorem to the last expression above; so by Lemma 1.7, we have the desired result.

Theorem 3.7. Let $F_{j}, j=1,2$, be given as in Theorem 3.6. Then for each nonzero real number $q$ and for $y \in D[0, T]$, the sequential Fourier-Feynman transform $\Gamma_{q}\left(\left(F_{1} * F_{2}\right)_{q}\right)(y)$ exists and

$$
\begin{equation*}
\Gamma_{q}\left(\left(F_{1} * F_{2}\right)_{q}\right)(y)=\Gamma_{q}\left(F_{1}\right)\left(\frac{y}{\sqrt{2}}\right) \Gamma_{q}\left(F_{2}\right)\left(\frac{y}{\sqrt{2}}\right) \tag{3.9}
\end{equation*}
$$

Proof. Let $\left\{\sigma_{n}\right\}$ and $\left\{\lambda_{n}\right\}$ be given as in the proof of Theorem 2.1 and let

$$
I_{\sigma_{n}, \lambda_{n}}\left(\left(F_{1} * F_{2}\right)_{q}\right)=\int_{\mathbb{R}^{m_{n}}} W_{\lambda_{n}}\left(\sigma_{n}, \vec{\eta}\right)\left(F_{1} * F_{2}\right)_{q}\left(X\left(\cdot, \sigma_{n}, \vec{\eta}\right)+y\right) d \vec{\eta} .
$$

Using the expression (3.8), the Fubini theorem and Lemma 1.6, we have

$$
\begin{aligned}
I_{\sigma_{n}, \lambda_{n}}\left(\left(F_{1} * F_{2}\right)_{q}\right)= & \left(\frac{-i q}{2 \pi T}\right)^{1 / 2} \int_{L_{2}^{2}[0, T]} \int_{\mathbb{R}^{2}} J_{\sigma_{n}, \lambda_{n}}\left(\eta, \frac{v_{1}+v_{2}}{\sqrt{2}}\right) \\
& \exp \left\{\frac{i}{\sqrt{2}}\left\langle v_{1}+v_{2}, y^{\prime}\right\rangle+K_{q}\left(\xi, \frac{v_{1}-v_{2}}{\sqrt{2}}\right)\right\} \\
& \Psi_{1}\left(\frac{y(T)+\eta+\xi}{\sqrt{2}}\right) \Psi_{2}\left(\frac{y(T)+\eta-\xi}{\sqrt{2}}\right) d \eta d \xi d g_{1}\left(v_{1}\right) d g_{2}\left(v_{2}\right)
\end{aligned}
$$

where $J_{\sigma_{n}, \lambda_{n}}(\eta, v)$ and $K_{q}(\xi, v)$ are given by (1.11) and (1.13) respectively. By the same reason as in the proof of Theorem 3.6, we apply dominated convergence theorem to the last expression above and so by Lemma 1.7, we have that

$$
\begin{aligned}
\Gamma_{q}\left(\left(F_{1} * F_{2}\right)_{q}\right)(y)= & \left(\frac{-i q}{2 \pi T}\right) \int_{L_{2}^{2}[0, T]} \int_{\mathbb{R}^{2}} \exp \left\{\frac{i}{\sqrt{2}}\left\langle v_{1}+v_{2}, y^{\prime}\right\rangle\right. \\
& \left.+K_{q}\left(\xi, \frac{v_{1}-v_{2}}{\sqrt{2}}\right)+K_{q}\left(\eta, \frac{v_{1}+v_{2}}{\sqrt{2}}\right)\right\} \Psi_{1}\left(\frac{y(T)+\eta+\xi}{\sqrt{2}}\right) \\
& \Psi_{2}\left(\frac{y(T)+\eta-\xi}{\sqrt{2}}\right) d \eta d \xi d g_{1}\left(v_{1}\right) d g_{2}\left(v_{2}\right) .
\end{aligned}
$$

A direct calculation shows that

$$
K_{q}\left(\xi, \frac{v_{1}-v_{2}}{\sqrt{2}}\right)+K_{q}\left(\eta, \frac{v_{1}+v_{2}}{\sqrt{2}}\right)=K_{q}\left(\frac{\eta+\xi}{\sqrt{2}}, v_{1}\right)+K_{q}\left(\frac{\eta-\xi}{\sqrt{2}}, v_{1}\right),
$$

and finally using (2.2) we obtain (3.9).
Theorem 3.8. For $j=1,2$ and $x \in D[0, T]$, let $F_{j}(x)=G_{j}(x) \Psi_{j}(x(T))$ where $G_{j} \in \hat{\mathcal{S}}$ is given by (1.6) with corresponding measures $g_{j}$ in $\mathcal{M}$ and $\Psi_{j} \in \mathcal{T}$ is given by (2.5) with corresponding complex Borel measures $\rho_{j}$. Then for each nonzero real number $q$ and for $y \in D[0, T]$, the convolution product $\left(F_{1} * F_{2}\right)_{q}(y)$ exists and is given by

$$
\begin{align*}
\left(F_{1} * F_{2}\right)_{q}(y)= & \int_{L_{2}^{2}[0, T]} \int_{\mathbb{R}^{2}} \exp \left\{\frac{i}{\sqrt{2}}\left\langle v_{1}+v_{2}+s_{1}+s_{2}, y^{\prime}\right\rangle\right.  \tag{3.10}\\
& \left.+\frac{i}{4 q}\left\|v_{1}-v_{2}+s_{1}-s_{2}\right\|_{2}^{2}\right\} d \rho\left(s_{1}\right) d \rho\left(s_{2}\right) d g_{1}\left(v_{1}\right) d g_{2}\left(v_{2}\right)
\end{align*}
$$

Moreover $\Gamma_{q}\left(\left(F_{1} * F_{2}\right)_{q}\right)(y)$ exists and

$$
\begin{equation*}
\Gamma_{q}\left(\left(F_{1} * F_{2}\right)_{q}\right)(y)=\Gamma_{q}\left(F_{1}\right)\left(\frac{y}{\sqrt{2}}\right) \Gamma_{q}\left(F_{2}\right)\left(\frac{y}{\sqrt{2}}\right) \tag{3.11}
\end{equation*}
$$

Proof. From the first part of the proof of Theorem 2.3, we know that $F_{j}$ is in $\hat{\mathcal{S}}$. Hence we apply Theorem 3.1 to conclude that

$$
\left(F_{1} * F_{2}\right)_{q}(y)=\int_{L_{2}^{2}[0, T]} \exp \left\{\frac{i}{\sqrt{2}}\left\langle w_{1}+w_{2}, y^{\prime}\right\rangle-\frac{i}{4 q}\left\|w_{1}-w_{2}\right\|_{2}^{2}\right\} d h_{1}\left(w_{1}\right) d h_{2}\left(w_{2}\right)
$$

where $h_{j}$ is the complex measure defined by (2.9) with corresponding measures $g_{j}$ and $\psi_{j}$. Using Theorem 6.1 in [3] and the transformation $w_{j}=v_{j}+u_{j}$, we have

$$
\begin{aligned}
\left(F_{1} * F_{2}\right)_{q}(y)= & \int_{L_{2}^{4}[0, T]} \exp \left\{\frac{i}{\sqrt{2}}\left\langle v_{1}+u_{1}+v_{2}+u_{2}, y^{\prime}\right\rangle\right. \\
& \left.-\frac{i}{4 q}\left\|v_{1}+u_{1}-v_{2}-u_{2}\right\|_{2}^{2}\right\} d g_{1}\left(v_{1}\right) d \psi_{1}\left(u_{1}\right) d g_{2}\left(v_{2}\right) d \psi_{2}\left(u_{2}\right)
\end{aligned}
$$

Now, the Fubini theorem and the change of variable theorem give (3.10). Since $F_{j} \in \hat{\mathcal{S}}$, equation (3.11) follows immediately from Theorem 3.2.

Theorem 3.9. For $j=1,2$ and $x \in D[0, T]$, let $F_{j}(x)=G_{j}(x) \Psi_{j}(x(T))$ where $G_{j} \in \hat{\mathcal{S}}$ is given by (1.6) with corresponding measures $g_{j}$ in $\mathcal{M}, \Psi_{1} \in L_{1}(\mathbb{R})$ and $\Psi_{2} \in \mathcal{T}$ is given by (2.5) with corresponding complex Borel measure $\rho_{2}$. Then for each nonzero real number $q$ and for $y \in D[0, T]$, the convolution product $\left(F_{1} * F_{2}\right)_{q}(y)$ exists and is given by

$$
\begin{align*}
\left(F_{1} * F_{2}\right)_{q}(y)= & \left(\frac{-i q}{2 \pi T}\right)^{1 / 2} \int_{L_{2}^{2}[0, T]} \int_{\mathbb{R}^{2}} \exp \left\{\frac{i}{\sqrt{2}}\left\langle v_{1}+v_{2}+s, y^{\prime}\right\rangle\right.  \tag{3.12}\\
& \left.+K_{q}\left(\xi, \frac{v_{1}-v_{2}-s}{\sqrt{2}}\right)\right\} \Psi_{1}\left(\frac{y(T)+\xi}{\sqrt{2}}\right) d \xi d \rho_{2}(s) d g_{1}\left(v_{1}\right) d g_{2}\left(v_{2}\right),
\end{align*}
$$

where $K_{q}(\xi, v)$ is given by (1.13). Moreover $\Gamma_{q}\left(\left(F_{1} * F_{2}\right)_{q}\right)(y)$ exists and

$$
\begin{equation*}
\Gamma_{q}\left(\left(F_{1} * F_{2}\right)_{q}\right)(y)=\Gamma_{q}\left(F_{1}\right)\left(\frac{y}{\sqrt{2}}\right) \Gamma_{q}\left(F_{2}\right)\left(\frac{y}{\sqrt{2}}\right) \tag{3.13}
\end{equation*}
$$

Proof. As we know from the proof of Theorem 2.3, $F_{2}$ is expressed by

$$
F_{2}(x)=\int_{L_{2}[0, T]} \exp \left\{i\left\langle w_{2}, x^{\prime}\right\rangle\right\} d h_{2}\left(w_{2}\right)
$$

where $h_{2}$ is the complex measure defined by (2.9) with corresponding measures $g_{2}$ and $\psi_{2}$. Replacing $\Psi_{2}$ by the constant function $1, g_{2}$ by $h_{2}$ and $v_{2}$ by $w_{2}$ whenever they occur in the proof of Theorem 3.6, we have

$$
\begin{aligned}
\left(F_{1} * F_{2}\right)_{q}(y)= & \left(\frac{-i q}{2 \pi T}\right)^{1 / 2} \int_{L_{2}^{2}[0, T]} \int_{\mathbb{R}} \exp \left\{\frac{i}{\sqrt{2}}\left\langle v_{1}+w_{2}, y^{\prime}\right\rangle\right. \\
& \left.+K_{q}\left(\xi, \frac{v_{1}-w_{2}}{\sqrt{2}}\right)\right\} \Psi_{1}\left(\frac{y(T)+\xi}{\sqrt{2}}\right) d \xi d g_{1}\left(v_{1}\right) d h_{2}\left(w_{2}\right)
\end{aligned}
$$

Using (2.9), Theorem 6.1 in [3], the transformation $v_{2}=w_{2}-u_{2}$ and (2.6), we obtain (3.12).

The existence of $\Gamma_{q}\left(\left(F_{1} * F_{2}\right)_{q}\right)$ can also be obtained from the proof of Theorem 3.7 by replacing $\Psi_{2}$ by $1, g_{2}$ by $h_{2}$ and $v_{2}$ by $w_{2}$. Hence we have

$$
\begin{aligned}
I_{\sigma_{n}, \lambda_{n}}\left(\left(F_{1} * F_{2}\right)_{q}\right)= & \left(\frac{-i q}{2 \pi T}\right)^{1 / 2} \int_{L_{2}^{2}[0, T]} \int_{\mathbb{R}^{2}} J_{\sigma_{n}, \lambda_{n}}\left(\eta, \frac{v_{1}+w_{2}}{\sqrt{2}}\right) \\
& \exp \left\{\frac{i}{\sqrt{2}}\left\langle v_{1}+w_{2}, y^{\prime}\right\rangle+K_{q}\left(\xi, \frac{v_{1}-w_{2}}{\sqrt{2}}\right)\right\} \\
& \Psi_{1}\left(\frac{y(T)+\eta+\xi}{\sqrt{2}}\right) d \eta d \xi d g_{1}\left(v_{1}\right) d h_{2}\left(w_{2}\right) .
\end{aligned}
$$

Using (1.11) and (1.13), we rewrite the above expression as

$$
\begin{aligned}
I_{\sigma_{n}, \lambda_{n}}\left(\left(F_{1} * F_{2}\right)_{q}\right)= & \left(\frac{-i q}{2 \pi T}\right)^{1 / 2} \int_{L_{2}^{2}[0, T]} \int_{\mathbb{R}^{2}} P_{\sigma_{n}, \lambda_{n}}\left(v_{1}, w_{2}\right) \exp \left\{\frac{i}{\sqrt{2}}\left\langle v_{1}+w_{2}, y^{\prime}\right\rangle\right\} \\
& R_{\lambda_{n}}\left(\eta, \xi, v_{1}, w_{2}\right) \Psi_{1}\left(\frac{y(T)+\eta+\xi}{\sqrt{2}}\right) d \eta d \xi d g_{1}\left(v_{1}\right) d h_{2}\left(w_{2}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
P_{\sigma_{n}, \lambda_{n}}\left(v_{1}, w_{2}\right)= & \left(\frac{\lambda_{n}}{2 \pi T}\right)^{1 / 2} \exp \left\{\frac{\left\langle v_{1}+w_{2}, 1\right\rangle^{2}}{4 \lambda_{n} T}-\frac{1}{4 \lambda_{n}} \sum_{k=1}^{m_{n}} \frac{\left\langle v_{1}+w_{2}, 1\right\rangle_{k}^{2}}{\tau_{n, k}-\tau_{n, k-1}}\right. \\
& \left.+\frac{i}{4 q T}\left\langle v_{1}-w_{2}, 1\right\rangle^{2}-\frac{i}{4 q}\left\|v_{1}-w_{2}\right\|_{2}^{2}\right\}
\end{aligned}
$$

and
$R_{\lambda_{n}}\left(\eta, \xi, v_{1}, w_{2}\right)=\exp \left\{\frac{1}{2 T}\left[-\lambda_{n} \eta^{2}+\sqrt{2} i \eta\left\langle v_{1}+w_{2}, 1\right\rangle+i q \xi^{2}+\sqrt{2} i \xi\left\langle v_{1}-w_{2}, 1\right\rangle\right]\right\}$.
Letting $\alpha=\frac{\eta+\xi}{\sqrt{2}}$ and $\beta=\frac{\eta-\xi}{\sqrt{2}}$, we have

$$
\begin{aligned}
R_{\lambda_{n}}\left(\eta, \xi, v_{1}, w_{2}\right)= & \exp \left\{-\frac{\lambda_{n}-i q}{4 T} \beta^{2}+\frac{2 i\left\langle w_{2}, 1\right\rangle-\left(\lambda_{n}+i q\right) \alpha}{2 T} \beta\right. \\
& \left.-\frac{\lambda_{n}-i q}{4 T} \alpha^{2}+\frac{i\left\langle v_{1}, 1\right\rangle}{T} \alpha\right\}
\end{aligned}
$$

Hence evaluating the integral with respect to $\beta$, we obtain

$$
\begin{align*}
I_{\sigma_{n}, \lambda_{n}}\left(\left(F_{1} * F_{2}\right)_{q}\right)= & \left(\frac{-i q}{2 \pi T}\right)^{1 / 2} \int_{L_{2}^{2}[0, T]} \int_{\mathbb{R}}\left(\frac{4 \pi T}{\lambda_{n}-i q}\right)^{1 / 2} P_{\sigma_{n}, \lambda_{n}}\left(v_{1}, w_{2}\right)  \tag{3.14}\\
& \exp \left\{\frac{i}{\sqrt{2}}\left\langle v_{1}+w_{2}, y^{\prime}\right\rangle+\frac{1}{4\left(\lambda_{n}-i q\right) T}\left[2 i\left\langle w_{2}, 1\right\rangle-\left(\lambda_{n}+i q\right) \alpha\right]^{2}\right. \\
& \left.-\frac{\lambda_{n}-i q}{4 T} \alpha^{2}+\frac{i}{T} \alpha\left\langle v_{1}, 1\right\rangle\right\} \Psi_{1}\left(\frac{y(T)}{\sqrt{2}}+\alpha\right) d \alpha d g_{1}\left(v_{1}\right) d h_{2}\left(w_{2}\right)
\end{align*}
$$

Now by the Schwartz inequality

$$
\frac{1}{T}\left\langle v_{1}+w_{2}, 1\right\rangle^{2}-\sum_{k=1}^{m_{n}} \frac{\left\langle v_{1}+w_{2}, 1\right\rangle_{n, k}^{2}}{\tau_{n, k}-\tau_{n, k-1}} \leq 0
$$

and a direct calculation shows that

$$
\operatorname{Re}\left\{\frac{1}{4\left(\lambda_{n}-i q\right) T}\left[2 i\left\langle w_{2}, 1\right\rangle-\left(\lambda_{n}+i q\right) \alpha\right]^{2}-\frac{\lambda_{n}-i q}{4 T} \alpha^{2}\right\} \leq 0
$$

Now we apply the dominated convergence theorem to the expression (3.14) to conclude that

$$
\begin{aligned}
\Gamma_{q}\left(\left(F_{1} * F_{2}\right)_{q}\right)(y)= & \left(\frac{-i q}{2 \pi T}\right)^{1 / 2} \int_{L_{2}^{2}[0, T]} \int_{\mathbb{R}} \exp \left\{\frac{i}{\sqrt{2}}\left\langle v_{1}+w_{2}, y^{\prime}\right\rangle-\frac{i}{2 q}\left\|w_{2}\right\|_{2}^{2}\right. \\
& \left.+K_{q}\left(\alpha, v_{1}\right)\right\} \Psi_{1}\left(\frac{y(T)}{\sqrt{2}}+\alpha\right) d \alpha d g_{1}\left(v_{1}\right) d h_{2}\left(w_{2}\right)
\end{aligned}
$$

Finally, from (2.2) and (1.8) we obtain (3.13).
From Theorems 3.6, 3.7, 3.8, 3.9 and the linearity of the sequential Feynman integral, we have the following result.

Corollary 3.10. For $j=1,2$ and $x \in D[0, T]$, let $F_{j}(x)=G_{j}(x) \Psi_{j}(x(T))$ where $G_{j} \in \hat{\mathcal{S}}$ is given by (1.6) with corresponding measures $g_{j}$ in $\mathcal{M}$ and $\Psi_{j}=\Psi_{j_{1}}+\Psi_{j_{2}} \in$
$L_{1}(\mathbb{R}) \cap L_{2}(\mathbb{R})+\mathcal{T}$. Then for each nonzero real number $q$ and for $y \in D[0, T]$, $\left(F_{1} * F_{2}\right)_{q}(y)$ and $\Gamma_{q}\left(\left(F_{1} * F_{2}\right)_{q}\right)(y)$ exist and

$$
\begin{equation*}
\Gamma_{q}\left(\left(F_{1} * F_{2}\right)_{q}\right)(y)=\Gamma_{q}\left(F_{1}\right)\left(\frac{y}{\sqrt{2}}\right) \Gamma_{q}\left(F_{2}\right)\left(\frac{y}{\sqrt{2}}\right) \tag{3.15}
\end{equation*}
$$

## 4. Sequential Fourier-Feynman transform and the first variation

In this section we study the sequential Fourier-Feynman transform and the first variation of functionals studied in Sections 2 and 3.

We begin with the formula for the first variation of functionals in $F \in \hat{\mathcal{S}}$.
Theorem 4.1. Let $F \in \hat{\mathcal{S}}$ be given by (1.6) with $\int_{L_{2}[0, T]}\|v\|_{2}|d f(v)|<\infty$ and let $y \in D[0, T]$. Then for each $x \in D[0, T]$, the first variation $\delta F(x \mid y)$ exists, belongs to $\hat{\mathcal{S}}$ as a function of $x \in D[0, T]$ and is given by

$$
\begin{equation*}
\delta F(x \mid y)=\int_{L_{2}[0, T]} \exp \left\{i\left\langle v, x^{\prime}\right\rangle\right\} d f_{y}(v) \tag{4.1}
\end{equation*}
$$

for $x \in D[0, T]$, where $f_{y}$ is given by

$$
\begin{equation*}
f_{y}(E)=i \int_{E}\left\langle v, y^{\prime}\right\rangle d f(v) \tag{4.2}
\end{equation*}
$$

for $E \in \mathcal{B}\left(L_{2}[0, T]\right)$.
Proof. For $x, y \in D[0, T]$, we have

$$
\delta F(x \mid y)=\left.\frac{\partial}{\partial h} \int_{L_{2}[0, T]} \exp \left\{i\left\langle v, x^{\prime}+h y^{\prime}\right\rangle\right\} d f(v)\right|_{h=0}
$$

Since

$$
\int_{L_{2}[0, T]}\left|\left\langle v, y^{\prime}\right\rangle\right||d f(v)| \leq\left\|y^{\prime}\right\|_{2} \int_{L_{2}[0, T]}\|v\|_{2}|d f(v)|<\infty
$$

we can take the partial derivative under the integral sign, so

$$
\delta F(x \mid y)=\int_{L_{2}[0, T]} i\left\langle v, y^{\prime}\right\rangle \exp \left\{i\left\langle v, x^{\prime}\right\rangle\right\} d f(v)
$$

and we obtain (4.1). Since

$$
\left\|f_{y}\right\| \leq\left\|y^{\prime}\right\|_{2} \int_{L_{2}[0, T]}\|v\|_{2}|d f(v)|<\infty
$$

$f_{y} \in \mathcal{M}$, and we complete the proof.
Theorem 4.2. Let $F$ be given as in Theorem 4.1. Let $y \in D[0, T]$ and let $q$ be a nonzero real number. Then

$$
\begin{equation*}
\Gamma_{q}(\delta F(\cdot \mid y))(x)=\delta \Gamma_{q} F(x \mid y) \tag{4.3}
\end{equation*}
$$

for $x \in D[0, T]$.
Proof. Using the expressions (1.8) and (4.1) we have

$$
\Gamma_{q}(\delta F(\cdot \mid y))(x)=\int_{L_{2}[0, T]} \exp \left\{i\left\langle v, x^{\prime}\right\rangle-\frac{i}{2 q}\|v\|_{2}^{2}\right\} d f_{y}(v)
$$

On the other hand, using the expression (1.8) and taking the partial derivative under the integral sign, we have

$$
\begin{aligned}
\delta \Gamma_{q} F(x \mid y) & =\left.\frac{\partial}{\partial h}\left(\int_{L_{2}[0, T]} \exp \left\{i\left\langle v, x^{\prime}+h y^{\prime}\right\rangle-\frac{i}{2 q}\|v\|_{2}^{2}\right\} d f(v)\right)\right|_{h=0} \\
& =\int_{L_{2}[0, T]} i\left(v, y^{\prime}\right) \exp \left\{i\left\langle v, x^{\prime}\right\rangle-\frac{i}{2 q}\|v\|_{2}^{2}\right\} d f(v) .
\end{aligned}
$$

By (4.2) we obtain (4.3).
Theorem 4.3. Let $F$ be given as in Theorem 4.1. Let $x \in D[0, T]$ and let $q$ be $a$ nonzero real number. Then

$$
\begin{equation*}
\Gamma_{q}(\delta F(x \mid \cdot))(y)=\delta F(x \mid y) \tag{4.4}
\end{equation*}
$$

for $y \in D[0, T]$.
Proof. Let $\left\{\sigma_{n}\right\}$ and $\left\{\lambda_{n}\right\}$ be given as in the proof of Theorem 2.1 and let

$$
I_{\sigma_{n}, \lambda_{n}}(\delta F(x \mid \cdot))=\int_{\mathbb{R}^{m_{n}}} W_{\lambda_{n}}\left(\sigma_{n}, \vec{\xi}\right) \delta F\left(x \mid X\left(\cdot, \sigma_{n}, \vec{\xi}\right)+y\right) d \vec{\xi} .
$$

By Theorem 4.1 we have for $y \in D[0, T]$

$$
\begin{aligned}
\delta F\left(x \mid X\left(\cdot, \sigma_{n}, \vec{\xi}\right)+y\right)= & \int_{L_{2}[0, T]}\left(i\left\langle v, y^{\prime}\right\rangle+i \sum_{k=1}^{m_{n}}\langle v, 1\rangle_{n, k} \frac{\xi_{k}-\xi_{k-1}}{\tau_{n, k}-\tau_{n, k-1}}\right) \\
& \exp \left\{i\left\langle v, x^{\prime}\right\rangle\right\} d f(v) .
\end{aligned}
$$

The Fubini theorem and a direct calculation show that

$$
I_{\sigma_{n}, \lambda_{n}}(\delta F(x \mid \cdot))=\int_{L_{2}[0, T]} i\left\langle v, y^{\prime}\right\rangle \exp \left\{i\left\langle v, x^{\prime}\right\rangle\right\} d f(v)
$$

Hence

$$
\lim _{n \rightarrow \infty} I_{\sigma_{n}, \lambda_{n}}(\delta F(x \mid \cdot))=\int_{L_{2}[0, T]} i\left\langle v, y^{\prime}\right\rangle \exp \left\{i\left\langle v, x^{\prime}\right\rangle\right\} d f(v),
$$

and by Theorem 4.1, the proof is completed.
From now on, we investigate the first variation of functionals of the form $F(x)=$ $G(x) \Psi(x(T))$.

Theorem 4.4. Let $F(x)=G(x) \Psi(x(T)), x \in D[0, T]$, where $G \in \hat{\mathcal{S}}$ is given by (1.6) with $\int_{L_{2}[0, T]}\|v\|_{2}|d g(v)|<\infty$, and $\Psi \in L_{1}(\mathbb{R})$ with $\Psi^{\prime}$ exists. Let $y \in D[0, T]$. Then

$$
\begin{equation*}
\delta F(x \mid y)=\delta G(x \mid y) \Psi(x(T))+G(x) \Psi^{\prime}(x(T)) y(T) \tag{4.5}
\end{equation*}
$$

for $x \in D[0, T]$. Also the right hand side of (4.5) can be expressed explicitly using (4.1) and (1.6).

Proof. For $x, y \in D[0, T]$, we have

$$
\delta F(x \mid y)=\left.\frac{\partial}{\partial h}(G(x+h y) \Psi(x(T)+h y(T)))\right|_{h=0},
$$

and this is equal to the right hand side of (4.5).

Theorem 4.5. Let $F$ be given as in Theorem 4.4 and suppose that $\Psi^{\prime} \in L_{1}(\mathbb{R})$. Let $y \in D[0, T]$ and let $q$ be a nonzero real number. Then

$$
\begin{equation*}
\Gamma_{q}(\delta F(\cdot \mid y))(x)=\delta \Gamma_{q} F(x \mid y) \tag{4.6}
\end{equation*}
$$

for $x \in D[0, T]$.
Proof. First note that $\delta G(\cdot \mid y) \in \hat{\mathcal{S}}$ with corresponding measure $g_{y}$, where $g_{y}$ is the measure induced by $g$ as $f_{y}$ was induced by $f$ in (4.2). Using Theorem 2.1 we have

$$
\begin{aligned}
& \Gamma_{q}(\delta G(\cdot \mid y) \Psi(\cdot(T)))(x) \\
= & \left(\frac{-i q}{2 \pi T}\right)^{1 / 2} \int_{L_{2}[0, T]} \int_{\mathbb{R}} \exp \left\{i\left\langle v, x^{\prime}\right\rangle+K_{q}(\xi, v)\right\} \Psi(\xi+x(T)) d \xi d g_{y}(v)
\end{aligned}
$$

for $x \in D[0, T]$. Similarly, since $\Psi^{\prime} \in L_{1}(\mathbb{R})$, we have

$$
\begin{aligned}
& \Gamma_{q}\left(G(\cdot) \Psi^{\prime}(\cdot(T))\right)(x) \\
= & \left(\frac{-i q}{2 \pi T}\right)^{1 / 2} \int_{L_{2}[0, T]} \int_{\mathbb{R}} \exp \left\{i\left\langle v, x^{\prime}\right\rangle+K_{q}(\xi, v)\right\} \Psi^{\prime}(\xi+x(T)) d \xi d g(v)
\end{aligned}
$$

for $x \in D[0, T]$. Hence by (4.5) and (4.2),

$$
\begin{aligned}
\Gamma_{q}(\delta F(\cdot \mid y))(x)= & \left(\frac{-i q}{2 \pi T}\right)^{1 / 2} \int_{L_{2}[0, T]} \int_{\mathbb{R}} \exp \left\{i\left\langle v, x^{\prime}\right\rangle+K_{q}(\xi, v)\right\} \\
& {\left[i\left\langle v, y^{\prime}\right\rangle \Psi(\xi+x(T))+\Psi^{\prime}(\xi+x(T)) y(T)\right] d \xi d g(v) }
\end{aligned}
$$

On the other hand, using (2.2) and taking the partial derivative under the integral sign, we see that $\delta \Gamma_{q}(F)(x \mid y)$ can also be expressed as the right hand side of the last expression above, and this completes the proof.

Theorem 4.6. Let $F$ be given as in Theorem 4.4. Let $x \in D[0, T]$ and let $q$ be $a$ nonzero real number. Then

$$
\begin{equation*}
\Gamma_{q}(\delta F(x \mid \cdot))(y)=\delta F(x \mid y) \tag{4.7}
\end{equation*}
$$

for $y \in D[0, T]$.
Proof. Obviously the sequential Fourier-Feynman transform of a constant function is the constant itself. Moreover it is easy to see that if we let $H(y)=y(T)$, $y \in D[0, T]$, then

$$
\Gamma_{q}(H)(y)=H(y)=y(T) .
$$

Hence using the relationship (4.5) for $\delta F(x \mid y)$ and applying Theorem 4.3, we have

$$
\begin{aligned}
\Gamma_{q}(\delta F(x \mid \cdot))(y) & =\Gamma_{q}(\delta G(x \mid \cdot))(y) \Psi(x(T))+G(x) \Psi^{\prime}(x(T)) y(T) \\
& =\delta G(x \mid y) \Psi(x(T))+G(x) \Psi^{\prime}(x(T)) y(T) .
\end{aligned}
$$

Equation (4.7) now follows from (4.5).
Theorem 4.7. Let $F(x)=G(x) \Psi(x(T)), x \in D[0, T]$, where $G \in \hat{\mathcal{S}}$ is given as in Theorem 4.4 and $\Psi \in \mathcal{T}$ is given by (2.5) with $\int_{\mathbb{R}}|s||d \rho(s)|<\infty$. Let $y \in D[0, T]$. Then

$$
\begin{equation*}
\delta F(x \mid y)=i \int_{L_{2}[0, T]} \int_{\mathbb{R}}\left\langle v+s, y^{\prime}\right\rangle \exp \left\{i\left\langle v+s, x^{\prime}\right\rangle\right\} d \rho(s) d g(v) \tag{4.8}
\end{equation*}
$$

for $x \in D[0, T]$.

Proof. As we see in the proof of Theorem 2.3, $F$ belongs to $\hat{\mathcal{S}}$. Moreover by Theorem 6.1 in [3] and the transformation $v=w-u$, it is easy to see that

$$
\begin{aligned}
\int_{L_{2}[0, T]}\|w\|_{2}|d h(w)| & \leq \int_{L_{2}[0, T]} \int_{\mathbb{R}}\left(\|v\|_{2}+|s|\right)|d \rho(s) \| d g(v)| \\
& =\|\rho\| \int_{L_{2}[0, T]}\|v\|_{2}|d g(v)|+\|g\| \int_{\mathbb{R}}|s||d \rho(s)|<\infty
\end{aligned}
$$

where $\|g\|$ and $\|\rho\|$ denote the total variation of $g$ and $\rho$, respectively. Applying Theorem 4.1 to the expression for $F$, we have

$$
\delta F(x \mid y)=i \int_{L_{2}[0, T]}\left\langle w, y^{\prime}\right\rangle \exp \left\{i\left\langle w, x^{\prime}\right\rangle\right\} d h(w)
$$

where $h$ is given by (2.9). Using Theorem 6.1 in [3] and transforming $v=w-u$ once more, we have

$$
\delta F(x \mid y)=i \int_{L_{2}^{2}[0, T]}\left\langle v+u, y^{\prime}\right\rangle \exp \left\{i\left\langle v+u, x^{\prime}\right\rangle\right\} d g(v) d \psi(u)
$$

where $\psi$ is given by (2.6). Finally (2.6) and the Fubini theorem give (4.8).
Now we apply Theorems 4.2 and 4.3 to the functional $F$ in Theorem 4.7 to obtain the following results.

Theorem 4.8. Let $F$ be given as in Theorem 4.7. Let $q$ be a nonzero real number. Then

$$
\begin{equation*}
\Gamma_{q}(\delta F(\cdot \mid y))(x)=\delta \Gamma_{q} F(x \mid y) \tag{4.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\Gamma_{q}(\delta F(x \mid \cdot))(y)=\delta F(x \mid y) \tag{4.10}
\end{equation*}
$$

for $x, y \in D[0, T]$.
From Theorems 4.4, 4.7 and the linearity of the sequential Feynman integral, we have

Corollary 4.9. Let $F(x)=G(x) \Psi(x(T)), x \in D[0, T]$, where $G \in \hat{\mathcal{S}}$ is given as in Theorem 4.4 and $\Psi=\Psi_{1}+\Psi_{2} \in L_{1}(\mathbb{R})+\mathcal{T}$ with $\Psi_{1}^{\prime}$ exists and $\Psi_{2}$ satisfies $\int_{\mathbb{R}}|s||d \rho(s)|<\infty$. For any $x, y \in D[0, T], \delta F(x \mid y)$ exists and is equal to the sum of the right hand sides of equations (4.5) and (4.8).

From Theorems 4.5, 4.6, 4.8 and the linearity of the sequential Feynman integral, we have

Corollary 4.10. Let $F$ be given as in Corollary 4.9. Let $q$ be a nonzero real number. Then

$$
\begin{equation*}
\Gamma_{q}(\delta F(x \mid \cdot))(y)=\delta F(x \mid y) \tag{4.11}
\end{equation*}
$$

In addition, if $\Psi_{1}^{\prime} \in L_{1}(\mathbb{R})$, then

$$
\begin{equation*}
\Gamma_{q}(\delta F(\cdot \mid y))(x)=\delta \Gamma_{q} F(x \mid y) \tag{4.12}
\end{equation*}
$$

for all $x, y \in D[0, T]$.

Remark 4.11. We thank the referee for calling our attention to the paper of Cameron and Storvick [7], where, for functionals $F$ we considered in Theorem 2.5, they showed that for each nonzero real $q$ the sequential Feynman integral of $F$ with parameter $q$ and the analytic Feynman integral of $F$ with parameter $q$ both exist and are equal. This implies that for each $F(x)=G(x) \Psi(x(T))$ where $G \in \hat{\mathcal{S}}$ and $\Psi=\Psi_{1}+\Psi_{2} \in L_{1}(\mathbb{R})+\mathcal{T}$, if either the sequential Fourier-Feynman transform of $F$ exists or the analytic Fourier-Feynman transform of $F$ exists, then both exists and equality holds. The same conclusions follow for the convolution products of functionals in $\hat{\mathcal{S}}$.

Hence all of the results established for the sequential Feynman integral setting in Section 2 also hold for the analytic Feynman integral setting. Moreover, using results in [7], Theorems 3.1, 3.2, 3.3 and Remark 3.4 in Section 3 can also be obtained from Theorems 3.2, 3.3, 3.4 and equation (3.9a) in [13], respectively. In the same way, Theorems 4.1, 4.2 and 4.3 in Section 4 can also be obtained from Lemma 3.1, Theorems 3.2 and 3.3 in [16], respectively.

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