AFFINE GEOMETRIC CRYSTALS AND LIMIT OF PERFECT CRYSTALS

MASAKI KASHIWARA, TOSHIKI NAKASHIMA, AND MASATO OKADO

ABSTRACT. For every non-exceptional affine Lie algebra, we explicitly construct a positive geometric crystal associated with a fundamental representation. We also show that its ultra-discretization is isomorphic to the limit of certain perfect crystals of the Langlands dual affine Lie algebra.

1. INTRODUCTION

The theory of perfect crystals was introduced in [7], [8] in order to study certain physical models, called vertex models, which are associated with quantum R -matrices. A perfect crystal is a (pseudo-)crystal base of a finite-dimensional module of a quantum affine algebra $U_q(\mathfrak{g})$ (for more details, see 3.3), and it is labeled by a positive integer l , called the level (Definition 3.6). One of the most important properties of perfect crystals is as follows: for a perfect crystal B of level l and the crystal $B(\lambda)$ of the irreducible $U_q(\mathfrak{g})$ -module with a dominant integral weight λ of level l as a highest weight, there exist a unique dominant integral weight μ of level l and an isomorphism of crystals:

$$B(\lambda) \cong B(\mu) \otimes B.$$

By iterating this isomorphism, we obtain the path realization of $B(\lambda)$, which plays a crucial role in solving the vertex models. In [8], for each $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, D_{n+1}^{(2)}, A_{2n}^{(2)}$, perfect crystals are constructed explicitly by patching two classical crystals. It is conjectured that certain “Kirillov-Reshetikhin modules” have perfect crystal bases and all perfect crystals are obtained as a tensor product of those perfect crystal bases ([2], [3]), but, it is still far from the complete classification of perfect crystals.

Let $\{B_l\}_{l \geq 1}$ be a family of perfect crystals. If it satisfies certain conditions (Definition 3.7), there exists a *limit* B_∞ of $\{B_l\}_{l \geq 1}$. In such a case the family $\{B_l\}_{l \geq 1}$ is called a *coherent family* of perfect crystals ([6]). Let $B(\infty)$ be the crystal of the lower nilpotent subalgebras $U_q^-(\mathfrak{g})$ of the quantum algebra $U_q(\mathfrak{g})$. Then, similarly to perfect crystals, we have an isomorphism of crystals:

$$B(\infty) \otimes B_\infty \cong B(\infty).$$

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By iterating this isomorphism, the path realization of $B(\infty)$ is obtained. In [6], for $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, D_{n+1}^{(2)}, A_{2n}^{(2)}$, explicit forms of the limit of perfect crystals are described, which are also reviewed in §3 below.

A. Berenstein and D. Kazhdan introduced the theory of geometric crystals ([1]), which is a structure on geometric objects analogous to crystals. Let I be the index set of simple roots of $\mathfrak{g}$. A geometric crystal consists of a variety X , a rational $\mathbb{C}^\times$ -action $e_i: \mathbb{C}^\times \times X \rightarrow X$ and rational functions $\gamma_i, \varepsilon_i: X \rightarrow \mathbb{C}$ which satisfy certain conditions (see Definition 2.1). This structure resembles the one of crystals; for instance, we have the tensor product of a pair of geometric crystals similarly to the tensor product of crystals. There is a more direct and remarkable relation between crystals and geometric crystals, that is, the *ultra-discretization* functor UD . This is a functor from the category of *positive* geometric crystals to that of crystals, where a positive geometric crystal is a geometric crystal equipped with a positive structure (see §2.4). Applying this functor, positive rational functions are transferred to piecewise linear functions by the following simple correspondence:

$$x \times y \mapsto x + y, \quad \frac{x}{y} \mapsto x - y, \quad x + y \mapsto \max(x, y).$$

The purpose of this paper is to construct a positive affine geometric crystal $\mathcal{V}(\mathfrak{g})$, whose ultra-discretization is isomorphic to a limit of perfect crystals $B_\infty(\mathfrak{g}^L)$, where $\mathfrak{g}^L$ is the Langlands dual of $\mathfrak{g}$.

Let G (resp. $\mathfrak{g}$) be the affine Kac-Moody group (resp. algebra) associated with a generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$. Let $B^\pm$ be the Borel subgroup and T the maximal torus. Set $y_i(c) := \exp(cf_i)$, and let $\alpha_i^\vee(c) \in T$ be the image of $c \in \mathbb{C}^\times$ by the group morphism $\mathbb{C}^\times \rightarrow T$ induced by the simple coroot $\alpha_i^\vee$ as in §2.1. We set $Y_i(c) := y_i(c^{-1})\alpha_i^\vee(c) = \alpha_i^\vee(c)y_i(c)$. Let W (resp. $\widetilde{W}$) be the Weyl group (resp. the extended Weyl group) associated with $\mathfrak{g}$. The Schubert cell $X_w := BwB/B$ ($w = s_{i_1} \cdots s_{i_k} \in W$) is isomorphic to the variety

$$B_w^- := \{Y_{i_1}(x_1) \cdots Y_{i_k}(x_k) \mid x_1, \dots, x_k \in \mathbb{C}^\times\} \subset B^-,$$

and X_w has a natural geometric crystal structure ([1], [14]).

We choose $0 \in I$ as in [9], and let $\{\varpi_i\}_{i \in I \setminus \{0\}}$ be the set of level 0 fundamental weights. Let $W(\varpi_i)$ be the fundamental representation of $U_q(\mathfrak{g})$ with ϖ_i as an extremal weight ([9]). Let us denote its reduction at $q = 1$ by the same notation $W(\varpi_i)$. It is a finite-dimensional $\mathfrak{g}$ -module. The module $W(\varpi_i)$ is irreducible over $U_q(\mathfrak{g})$, but its reduction at $q = 1$ is not necessarily an irreducible $\mathfrak{g}$ -module. We set $\mathbb{P}(\varpi_i) := (W(\varpi_i) \setminus \{0\})/\mathbb{C}^\times$.

For any $i \in I$, define (see §3.2 for the definition of the inner product),

$$(1.1) \quad c_i^\vee := \max(1, \frac{2}{(\alpha_i, \alpha_i)}).$$

Then the translation $t(c_i^\vee \varpi_i)$ belongs to $\widetilde{W}$ (see §5.1). For a subset J of I , let us denote by $\mathfrak{g}_J$ the subalgebra of $\mathfrak{g}$ generated by $\{e_i, f_i\}_{i \in J}$. For an integral weight μ , define $I(\mu) := \{j \in I \mid \langle h_j, \mu \rangle \geq 0\}$.

Then we have the following conjecture ([2, 3, 9]).

- Conjecture 1.1.** (1) For any $i \in I \setminus \{0\}$ and any positive integer n , there exists a finite-dimensional irreducible $U_q(\mathfrak{g})$ -module $W(n\varpi_i)$ with extremal weight $n\varpi_i$, called the “Kirillov-Reshetikhin module”.
- (2) The module $W(n\varpi_i)$ has a global crystal basis.
- (3) Its crystal $B(n\varpi_i)$ is perfect if and only if $n \in c_i^\vee \mathbb{Z}$, and its level is equal to $n/c_i^\vee$.
- (4) $\{B(nc_i^\vee \varpi_i)\}_n$ is a coherent family of perfect crystals. We denote by $B_\infty(\varpi_i)$ its limit crystal.

Now, let us state our conjecture.

Conjecture 1.2. For any $i \in I \setminus \{0\}$, there exist a unique variety X endowed with a positive $\mathfrak{g}$ -geometric crystal structure and a rational mapping $\pi: X \rightarrow \mathbb{P}(\varpi_i)$ satisfying the following properties:

- (1) for an arbitrary extremal vector $u \in W(\varpi_i)_\mu$, writing the translation $t(c_i^\vee \mu)$ as $\iota w \in \widetilde{W}$ with a Dynkin diagram automorphism ι and $w = s_{i_1} \cdots s_{i_k}$ (see §5.1), there exists a birational mapping $\xi: B_w^- \rightarrow X$ such that ξ is a morphism of $\mathfrak{g}_{I(\mu)}$ -geometric crystals and such that the composition $\pi \circ \xi: B_w^- \rightarrow \mathbb{P}(\varpi_i)$ coincides with $Y_{i_1}(x_1) \cdots Y_{i_k}(x_k) \mapsto Y_{i_1}(x_1) \cdots Y_{i_k}(x_k) \overline{u}$, where $s_{i_1} \cdots s_{i_k}$ is a reduced expression of w and $\overline{u}$ is the line including u ,
- (2) the ultra-discretization of X is isomorphic to the crystal $B_\infty(\varpi_i)$ of the Langlands dual $\mathfrak{g}^L$.

In this paper, we construct a positive geometric crystal for some $i = 1$ and for $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, D_{n+1}^{(2)}, A_{2n}^{(2)}$, with this conjecture as a guide.

Let $W(\varpi_1)$ be the fundamental $\mathfrak{g}$ -module as above ([9]) and $u_1 := u_{\varpi_1}$ an extremal weight vector of $W(\varpi_1)$ with the weight ϖ_1 .

Then $t(c_1^\vee \varpi_1)$ belongs to $\widetilde{W}$ and (see §5.1) there exist $w_1 \in W$ and a Dynkin diagram automorphism ι such that $t(c_1^\vee \varpi_1) = \iota \cdot w_1$. Associated with this w_1 , we define

$$X_1 := B_{w_1}^-.$$

Then $I(\varpi_1) = I \setminus \{0\}$ and X_1 has a structure of a $\mathfrak{g}_{I \setminus \{0\}}$ -geometric crystal.

Let us choose another extremal weight vector u_2 with extremal weight η such that $I(\eta) = I \setminus \{i_2\}$ for some $i_2 \neq 0$. Then there exists $w_2 \in W$ such that $t(c_1^\vee \eta) = \iota \cdot w_2$. We define similarly

$$X_2 := B_{w_2}^-,$$

which has a structure of a $\mathfrak{g}_{I \setminus \{i_2\}}$ -geometric crystal. We have the rational mapping $X_\nu \rightarrow \mathbb{P}(\varpi_1)$ ($\nu = 1, 2$).

Then we see (by a case-by-case calculation) that there exists a unique positive birational mapping $\xi: X_1 \rightarrow X_2$ such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\xi} & X_2 \\ & \searrow & \downarrow \\ & & \mathbb{P}(\varpi_1) \end{array}$$

commutes and ξ commutes with e_i for $i \neq 0, i_2$. Moreover, ξ is an isomorphism of $\mathfrak{g}_{I \setminus \{0, i_2\}}$ -geometric crystals.

Now the $\mathfrak{g}$ -geometric crystal $\mathcal{V}(\mathfrak{g})$ is obtained by patching X_1 and X_2 by ξ . The relations (1)-(4) in Definition 2.1 are obvious for $(i, j) \neq (0, i_2), (i_2, 0)$ and we check these relations for $(i, j) = (0, i_2), (i_2, 0)$ by hand. In this paper we choose $\eta = \sigma\varpi_1$ for some Dynkin diagram automorphism σ except the case $\mathfrak{g} = A_{2n}^{(2)}$.

We then show that the ultra-discretization of $\mathcal{V}(\mathfrak{g})$ is isomorphic to the crystal $B_\infty(\varpi_1)$ of the Langlands dual $\mathfrak{g}^L$.

The organization of the paper is as follows. In §2, we review basic facts about geometric crystals following [1], [14], [15]. In §3, we shall recall the notion of the limit of perfect crystals and give examples of the limit $B_\infty(\mathfrak{g}^L)$ of perfect crystals following [6]. In §4, we present the explicit form of the fundamental representation $W(\varpi_1)$ for each affine Lie algebra $\mathfrak{g}$. In §5, we shall construct the affine geometric crystal $\mathcal{V}(\mathfrak{g})$ associated with the fundamental representation $W(\varpi_1)$. In §6, we prove that the ultra-discretization $\mathcal{UD}(\mathcal{V}(\mathfrak{g}))$ is isomorphic to $B_\infty(\mathfrak{g}^L)$.

2. GEOMETRIC CRYSTALS

In this section, we review Kac-Moody groups and geometric crystals following [1], [4], [5], [11], [14], [16].

2.1. Kac-Moody algebras and Kac-Moody groups. Fix a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ with a finite index set I . Let $(\mathfrak{t}, \{\alpha_i\}_{i \in I}, \{\alpha_i^\vee\}_{i \in I})$ be the associated root data, where $\mathfrak{t}$ is a vector space over $\mathbb{C}$ and $\{\alpha_i\}_{i \in I} \subset \mathfrak{t}^*$ and $\{\alpha_i^\vee\}_{i \in I} \subset \mathfrak{t}$ are linearly independent satisfying $\alpha_j(\alpha_i^\vee) = a_{ij}$.

The Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ associated with A is the Lie algebra over $\mathbb{C}$ generated by $\mathfrak{t}$ and the Chevalley generators e_i and f_i ($i \in I$) with the usual defining relations ([5], [16]). There exists a root space decomposition $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{t}^*} \mathfrak{g}_\alpha$. Denote the set of roots by $\Delta := \{\alpha \in \mathfrak{t}^* \mid \alpha \neq 0, \mathfrak{g}_\alpha \neq (0)\}$. Set $Q = \sum_i \mathbb{Z}\alpha_i$, $Q_+ = \sum_i \mathbb{Z}_{\geq 0}\alpha_i$, $Q^\vee = \sum_i \mathbb{Z}\alpha_i^\vee$ and $\Delta_+ = \Delta \cap Q_+$. An element of Δ_+ is called a *positive root*. Let $P \subset \mathfrak{t}^*$ be a weight lattice such that $\mathbb{C} \otimes P = \mathfrak{t}^*$, whose element is called a weight.

Define the simple reflections $s_i \in \text{Aut}(\mathfrak{t})$ ($i \in I$) by $s_i(h) := h - \alpha_i(h)\alpha_i^\vee$, which generate the Weyl group W . They induce the action of W on $\mathfrak{t}^*$ by $s_i(\lambda) := \lambda - \lambda(\alpha_i^\vee)\alpha_i$. Set $\Delta^{\text{re}} := \{w(\alpha_i) \mid w \in W, i \in I\}$, whose elements are called real roots.

Let G be the Kac-Moody group associated with $\mathfrak{g}$ ([16]). Let $U_\alpha := \exp \mathfrak{g}_\alpha$ ($\alpha \in \Delta^{\text{re}}$) be the one-parameter subgroup of G . The group G is generated by U_α ($\alpha \in \Delta^{\text{re}}$). Let $U^\pm$ be the subgroup generated by $U_{\pm\alpha}$ ($\alpha \in \Delta_+^{\text{re}} = \Delta^{\text{re}} \cap Q_+$), i.e., $U^\pm := \langle U_{\pm\alpha} \mid \alpha \in \Delta_+^{\text{re}} \rangle$.

For any $i \in I$, there exists a unique group homomorphism $\phi_i: SL_2(\mathbb{C}) \rightarrow G$ such that

$$\phi_i \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) = \exp(te_i), \quad \phi_i \left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right) = \exp(tf_i) \quad (t \in \mathbb{C}).$$

Set $\alpha_i^\vee(c) := \phi_i \left(\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \right)$, $x_i(t) := \exp(te_i)$, $y_i(t) := \exp(tf_i)$, $G_i := \phi_i(SL_2(\mathbb{C}))$, $T_i := \alpha_i^\vee(\mathbb{C}^\times)$ and $N_i := N_{G_i}(T_i)$. Let T be the subgroup of G with the Lie algebra $\mathfrak{t}$, which is called a *maximal torus* in G , and let $B^\pm = U^\pm T$ be the Borel subgroup of G . Let N be the subgroup of G generated by the N_i 's. Then we have the isomorphism $\phi: W \xrightarrow{\sim} N/T$ defined by $\phi(s_i) = N_i T/T$. An element $\bar{s}_i := x_i(-1)y_i(1)x_i(-1) = \phi_i \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)$ in $N_G(T)$ is a representative of $s_i \in W = N_G(T)/T$.

2.2. Geometric crystals. Let W be the Weyl group associated with $\mathfrak{g}$. Define $R(w)$ for $w \in W$ by

$$R(w) := \{(i_1, i_2, \dots, i_l) \in I^l \mid w = s_{i_1} s_{i_2} \cdots s_{i_l}\},$$

where l is the length of w , i.e., $R(w)$ is the set of reduced expressions of w .

Let X be a variety, $\gamma_i: X \rightarrow \mathbb{C}$ and $\varepsilon_i: X \rightarrow \mathbb{C}$ ($i \in I$) rational functions on X , and $e_i: \mathbb{C}^\times \times X \rightarrow X$ ($(c, x) \mapsto e_i^c(x)$) a rational $\mathbb{C}^\times$ -action. For $w \in W$ and $\mathbf{i} = (i_1, \dots, i_l) \in R(w)$, set $\alpha^{(j)} := s_{i_l} \cdots s_{i_{j+1}}(\alpha_{i_j})$ ($1 \leq j \leq l$) and

$$\begin{aligned} e_{\mathbf{i}}: T \times X &\rightarrow X \\ (t, x) &\mapsto e_{\mathbf{i}}^t(x) := e_{i_1}^{\alpha^{(1)}(t)} e_{i_2}^{\alpha^{(2)}(t)} \cdots e_{i_l}^{\alpha^{(l)}(t)}(x). \end{aligned}$$

Definition 2.1. A quadruple $(X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ is a G (or $\mathfrak{g}$)-geometric crystal if

- (1) $\{1\} \times X \subset \text{dom}(e_i)$ for any $i \in I$.
- (2) $\gamma_j(e_i^c(x)) = c^{a_{ij}} \gamma_j(x)$.
- (3) $e_{\mathbf{i}} = e_{\mathbf{i}'}$ for any $w \in W$ and $\mathbf{i}, \mathbf{i}' \in R(w)$.
- (4) $\varepsilon_i(e_i^c(x)) = c^{-1} \varepsilon_i(x)$.

Note that the condition (3) is equivalent to the following so-called *Verma relations*:

$$\begin{aligned} e_i^{c_1} e_j^{c_2} &= e_j^{c_2} e_i^{c_1} & \text{if } a_{ij} = a_{ji} = 0, \\ e_i^{c_1} e_j^{c_1 c_2} e_i^{c_2} &= e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1} & \text{if } a_{ij} = a_{ji} = -1, \\ e_i^{c_1} e_j^{c_1^2 c_2} e_i^{c_1 c_2} e_j^{c_2} &= e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1^2 c_2} e_i^{c_1} & \text{if } a_{ij} = -2, a_{ji} = -1, \\ e_i^{c_1} e_j^{c_1^3 c_2} e_i^{c_1^2 c_2} e_j^{c_1 c_2} e_i^{c_2} &= e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1^3 c_2} e_i^{c_1^2 c_2} e_j^{c_1 c_2} e_i^{c_1} & \text{if } a_{ij} = -3, a_{ji} = -1. \end{aligned}$$

Note that the last formula is different from the one in [1], [14], [15], which seems to be incorrect.

2.3. Geometric crystal on Schubert cell. Let $X := G/B$ be the flag variety, which is the inductive limit of finite-dimensional projective varieties. For $w \in W$, let $X_w := BwB/B \subset X$ be the Schubert cell associated with w , which has a natural geometric crystal structure ([1], [14]). For $\mathbf{i} = (i_1, \dots, i_k) \in R(w)$, set

$$(2.1) \quad B_{\mathbf{i}}^- := \{Y_{\mathbf{i}}(c_1, \dots, c_k) := Y_{i_1}(c_1) \cdots Y_{i_k}(c_k) \mid c_1, \dots, c_k \in \mathbb{C}^\times\} \subset B^-$$

where $Y_i(c) := y_i(c^{-1}) \alpha_i^\vee(c)$. Then $B_{\mathbf{i}}^-$ is birationally isomorphic to X_w and endowed with the induced geometric crystal structure. The explicit form of the action e_i^c on $B_{\mathbf{i}}^-$ is given by

$$e_i^c(Y_{i_1}(c_1) \cdots Y_{i_k}(c_k)) = Y_{i_1}(C_1) \cdots Y_{i_k}(C_k)$$

where

$$(2.2) \quad C_j := c_j \cdot \frac{\sum_{1 \leq m \leq j, i_m = i} \frac{c}{c_1^{a_{i_1, i}} \cdots c_{m-1}^{a_{i_{m-1}, i}} c_m} + \sum_{j < m \leq k, i_m = i} \frac{1}{c_1^{a_{i_1, i}} \cdots c_{m-1}^{a_{i_{m-1}, i}} c_m}}{\sum_{1 \leq m < j, i_m = i} \frac{c}{c_1^{a_{i_1, i}} \cdots c_{m-1}^{a_{i_{m-1}, i}} c_m} + \sum_{j \leq m \leq k, i_m = i} \frac{1}{c_1^{a_{i_1, i}} \cdots c_{m-1}^{a_{i_{m-1}, i}} c_m}}.$$

We also have the explicit forms of the rational functions ε_i and γ_i :

$$\varepsilon_i(Y_{i_1}(c_1) \cdots Y_{i_l}(c_k)) = \sum_{1 \leq m \leq k, i_m = i} \frac{1}{c_1^{a_{i_1, i}} \cdots c_{m-1}^{a_{i_{m-1}, i}} c_m},$$

$$\gamma_i(Y_{i_1}(c_1) \cdots Y_{i_l}(c_k)) = c_1^{a_{i_1, i}} \cdots c_k^{a_{i_k, i}}.$$

2.4. Positive structure, ultra-discretization and tropicalization. Let us recall the notions of positive structure and ultra-discretization/tropicalization.

The setting below is the same as in [15]. Set $R := \mathbb{C}(c)$ and define

$$\begin{aligned} v: R \setminus \{0\} &\longrightarrow \mathbb{Z} \\ f(c) &\longmapsto \deg(f(c)). \end{aligned}$$

Here $\deg$ is the degree of the pole at $c = \infty$. Note that for $f_1, f_2 \in R \setminus \{0\}$, we have

$$(2.3) \quad v(f_1 f_2) = v(f_1) + v(f_2), \quad v\left(\frac{f_1}{f_2}\right) = v(f_1) - v(f_2).$$

We say that a non-zero rational function $f(c) \in \mathbb{C}(c)$ is *positive* if f can be expressed as a ratio of polynomials with positive coefficients. If $f_1, f_2 \in R$ are positive, then we have

$$(2.4) \quad v(f_1 + f_2) = \max(v(f_1), v(f_2)).$$

Let $T \simeq (\mathbb{C}^\times)^l$ be an algebraic torus over $\mathbb{C}$ and $X^*(T) := \text{Hom}(T, \mathbb{C}^\times)$ (resp. $X_*(T) := \text{Hom}(\mathbb{C}^\times, T)$) be the lattice of characters (resp. co-characters) of T . A non-zero rational function on an algebraic torus T is called *positive* if it is written as g/h where g and h are positive linear combinations of characters of T .

Definition 2.2. Let $f: T \rightarrow T'$ be a rational mapping between two algebraic tori T and T' . We say that f is *positive*, if $\chi \circ f$ is positive for any character $\chi: T' \rightarrow \mathbb{C}$.

Denote by $\text{Mor}^+(T, T')$ the set of positive rational mappings from T to T' .

Lemma 2.3 ([1]). *For any $f \in \text{Mor}^+(T_1, T_2)$ and $g \in \text{Mor}^+(T_2, T_3)$, the composition $g \circ f$ is well-defined and belongs to $\text{Mor}^+(T_1, T_3)$.*

By Lemma 2.3, we can define a category $\mathcal{T}_+$ whose objects are algebraic tori over $\mathbb{C}$ and whose arrows are positive rational mappings.

Let $f: T \rightarrow T'$ be a positive rational mapping of algebraic tori T and T' . We define a map $\widehat{f}: X_*(T) \rightarrow X_*(T')$ by

$$\langle \chi, \widehat{f}(\xi) \rangle = v(\chi \circ f \circ \xi),$$

where $\chi \in X^*(T')$ and $\xi \in X_*(T)$.

Lemma 2.4 ([1]). *For any algebraic tori T_1, T_2, T_3 , and positive rational mappings $f \in \text{Mor}^+(T_1, T_2)$, $g \in \text{Mor}^+(T_2, T_3)$, we have $\widehat{g \circ f} = \widehat{g} \circ \widehat{f}$.*

By this lemma, we obtain a functor

$$\begin{aligned} \mathcal{UD}: \quad \mathcal{T}_+ &\longrightarrow \text{Set} \\ T &\longmapsto X_*(T) \\ (f: T \rightarrow T') &\longmapsto (\widehat{f}: X_*(T) \rightarrow X_*(T')). \end{aligned}$$

Let us come back to the situation in §2.2. Hence G is a Kac-Moody group and T is its Cartan subgroup.

Definition 2.5 ([1]). Let $\chi = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ be a G -geometric crystal, T' an algebraic torus and $\theta: T' \rightarrow X$ a birational mapping. The mapping θ is called a *positive structure* on χ if it satisfies

- (1) for any $i \in I$ the rational functions $\gamma_i \circ \theta: T' \rightarrow \mathbb{C}$ and $\varepsilon_i \circ \theta: T' \rightarrow \mathbb{C}$ are positive,
- (2) for any $i \in I$, the rational mapping $e_{i,\theta}: \mathbb{C}^\times \times T' \rightarrow T'$ defined by $e_{i,\theta}(c, t) := \theta^{-1} \circ e_i^c \circ \theta(t)$ is positive.

Let $\theta: T' \rightarrow X$ be a positive structure on a geometric crystal $\chi = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$. Applying the functor $\mathcal{UD}$ to the positive rational mappings $e_{i,\theta}: \mathbb{C}^\times \times T' \rightarrow T'$ and $\gamma_i \circ \theta, \varepsilon_i \circ \theta: T' \rightarrow \mathbb{C}^\times$, we obtain

$$\begin{aligned} \tilde{e}_i &:= \mathcal{UD}(e_{i,\theta}): \mathbb{Z} \times X_*(T') \rightarrow X_*(T'), \\ \text{wt}_i &:= \mathcal{UD}(\gamma_i \circ \theta), \varepsilon_i := \mathcal{UD}(\varepsilon_i \circ \theta): X_*(T') \rightarrow \mathbb{Z}. \end{aligned}$$

Hence the quadruple $(X_*(T'), \{\tilde{e}_i\}_{i \in I}, \{\text{wt}_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ is a free pre-crystal structure (see [1, 2.2]) and we denote it by $\mathcal{UD}_{\theta, T'}(\chi)$. We thus have the following theorem:

Theorem 2.6 ([1], [14]). *For any geometric crystal $\chi = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ and a positive structure $\theta: T' \rightarrow X$, the associated pre-crystal $\mathcal{UD}_{\theta, T'}(\chi)$ is a crystal (see [1, 2.2]).*

Now, let $\mathcal{GC}^+$ be the category whose object is a triplet (χ, T', θ) , where $\chi = (X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ is a geometric crystal and $\theta: T' \rightarrow X$ is a positive structure on χ , and the morphism $f: (\chi_1, T'_1, \theta_1) \rightarrow (\chi_2, T'_2, \theta_2)$ is given by a morphism $\varphi: X_1 \rightarrow X_2$ of geometric crystals such that

$$f := \theta_2^{-1} \circ \varphi \circ \theta_1: T'_1 \rightarrow T'_2$$

is a positive rational mapping. Let $\mathcal{CR}$ be the category of crystals. Then by the theorem above, we have

Corollary 2.7. *$\mathcal{UD}$ defines a functor*

$$\begin{aligned} \mathcal{UD}: \mathcal{GC}^+ &\longrightarrow \mathcal{CR}, \\ (\chi, T', \theta) &\mapsto X_*(T'), \\ (f: (\chi_1, T'_1, \theta_1) \rightarrow (\chi_2, T'_2, \theta_2)) &\mapsto (\hat{f}: X_*(T'_1) \rightarrow X_*(T'_2)). \end{aligned}$$

We call the functor $\mathcal{UD}$ “*ultra-discretization*” as in [14], [15], while it is called “*tropicalization*” in [1]. For a crystal B , if there exists a geometric crystal χ and a positive structure $\theta: T' \rightarrow X$ on χ such that $\mathcal{UD}(\chi, T', \theta) \simeq B$ as crystals, we call an object (χ, T', θ) in $\mathcal{GC}^+$ a *tropicalization* of B .

3. LIMITS OF PERFECT CRYSTALS

We review the limits of perfect crystals following [6]. (See also [7], [8].)

3.1. Crystals. First we recall the notion of crystals, which is obtained by abstracting the combinatorial properties of crystal bases.

Definition 3.1. A *crystal* B is a set endowed with the following maps:

$$\begin{aligned} \text{wt}: B &\longrightarrow P, \\ \varepsilon_i: B &\longrightarrow \mathbb{Z} \sqcup \{-\infty\}, \quad \varphi_i: B \longrightarrow \mathbb{Z} \sqcup \{-\infty\} \quad \text{for } i \in I, \\ \tilde{e}_i: B \sqcup \{0\} &\longrightarrow B \sqcup \{0\}, \quad \tilde{f}_i: B \sqcup \{0\} \longrightarrow B \sqcup \{0\} \quad \text{for } i \in I, \\ \tilde{e}_i(0) &= \tilde{f}_i(0) = 0. \end{aligned}$$

Those maps satisfy the following axioms: for all $b, b_1, b_2 \in B$, we have

$$\begin{aligned} \varphi_i(b) &= \varepsilon_i(b) + \langle \alpha_i^\vee, \text{wt}(b) \rangle, \\ \text{wt}(\tilde{e}_i b) &= \text{wt}(b) + \alpha_i \text{ if } \tilde{e}_i b \in B, \\ \text{wt}(\tilde{f}_i b) &= \text{wt}(b) - \alpha_i \text{ if } \tilde{f}_i b \in B, \\ \tilde{e}_i b_2 = b_1 &\iff \tilde{f}_i b_1 = b_2, \\ \varepsilon_i(b) = -\infty &\implies \tilde{e}_i b = \tilde{f}_i b = 0. \end{aligned}$$

The following tensor product structure is one of the most crucial properties of crystals.

Theorem 3.2. Let B_1 and B_2 be crystals. Set $B_1 \otimes B_2 := \{b_1 \otimes b_2 \mid b_j \in B_j \ (j=1, 2)\}$. Then we have

- (1) $B_1 \otimes B_2$ is a crystal,
- (2) for $b_1 \in B_1$ and $b_2 \in B_2$, we have

$$\begin{aligned} \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2); \end{cases} \\ \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \\ \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2). \end{cases} \end{aligned}$$

Definition 3.3. Let B_1 and B_2 be crystals. A *morphism* of crystals $\psi: B_1 \longrightarrow B_2$ is a map $\psi: B_1 \sqcup \{0\} \longrightarrow B_2 \sqcup \{0\}$ satisfying

- (1) $\psi(0) = 0$, $\psi(B_1) \subset B_2$,
- (2) for $b, b' \in B_1$, $\tilde{f}_i b = b'$ implies $\tilde{f}_i \psi(b) = \psi(b')$,
- (3) $\text{wt}(\psi(b)) = \text{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, $\varphi_i(\psi(b)) = \varphi_i(b)$ for any $b \in B_1$.

In particular, a bijective morphism is called an *isomorphism of crystals*.

Example 3.4. (1) If (L, B) is a crystal base, then B is a crystal.
 (2) For the crystal base $(L(\infty), B(\infty))$ of the subalgebra $U_q^-(\mathfrak{g})$ of the quantum algebra $U_q(\mathfrak{g})$, $B(\infty)$ is a crystal.
 (3) For $\lambda \in P$, set $T_\lambda := \{t_\lambda\}$. We define a crystal structure on T_λ by

$$\tilde{e}_i(t_\lambda) = \tilde{f}_i(t_\lambda) = 0, \quad \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty, \quad \text{wt}(t_\lambda) = \lambda.$$

Definition 3.5. To a crystal B , a colored oriented graph is associated by

$$b_1 \xrightarrow{i} b_2 \iff \tilde{f}_i b_1 = b_2.$$

We call this graph the *crystal graph* of B .

3.2. Affine weights. Let $\mathfrak{g}$ be an affine Lie algebra, and let the sets $\mathfrak{t}$, $\{\alpha_i\}_{i \in I}$ and $\{\alpha_i^\vee\}_{i \in I}$ be as in §2.1. We take $\mathfrak{t}$ so that $\dim \mathfrak{t} = \sharp I + 1$. Let $\delta \in Q_+$ be a unique element satisfying

$$\{\lambda \in Q \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \text{ for any } i \in I\} = \mathbb{Z}\delta,$$

and let $\mathbf{c} \in \sum_i \mathbb{Z}_{\geq 0} \alpha_i^\vee \subset \mathfrak{g}$ be a unique central element satisfying

$$\{h \in Q^\vee \mid \langle h, \alpha_i \rangle = 0 \text{ for any } i \in I\} = \mathbb{Z}\mathbf{c}.$$

We write ([4, 6.1])

$$(3.1) \quad \mathbf{c} = \sum_i a_i^\vee \alpha_i^\vee, \quad \delta = \sum_i a_i \alpha_i.$$

Let $(\cdot, \cdot)$ be the non-degenerate W -invariant symmetric bilinear form on $\mathfrak{t}^*$ normalized by $(\delta, \lambda) = \langle \mathbf{c}, \lambda \rangle$ for $\lambda \in \mathfrak{t}^*$. Let us set $\mathfrak{t}_{\text{cl}}^* := \mathfrak{t}^*/\mathbb{C}\delta$ and let $\text{cl}: \mathfrak{t}^* \rightarrow \mathfrak{t}_{\text{cl}}^*$ be the canonical projection. Then we have $\mathfrak{t}_{\text{cl}}^* \cong \bigoplus_i (\mathbb{C}\alpha_i^\vee)^*$. Set $\mathfrak{t}_0^* := \{\lambda \in \mathfrak{t}^* \mid \langle \mathbf{c}, \lambda \rangle = 0\}$, $(\mathfrak{t}_{\text{cl}}^*)_0 := \text{cl}(\mathfrak{t}_0^*)$. Then we have a positive-definite symmetric form on $(\mathfrak{t}_{\text{cl}}^*)_0$ induced by the one on $\mathfrak{t}^*$. Let $\Lambda_i \in \mathfrak{t}_{\text{cl}}^*$ ($i \in I$) be a weight such that $\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{i,j}$, which is called a *fundamental weight*. We choose P so that $P_{\text{cl}} := \text{cl}(P)$ coincides with $\bigoplus_{i \in I} \mathbb{Z}\Lambda_i$ and we call P_{cl} the *classical weight lattice*.

3.3. Definitions of a perfect crystal and its limit. Let $\mathfrak{g}$ be an affine Lie algebra, P_{cl} the classical weight lattice as above and set $(P_{\text{cl}})_l^+ := \{\lambda \in P_{\text{cl}} \mid \langle \mathbf{c}, \lambda \rangle = l, \langle \alpha_i^\vee, \lambda \rangle \geq 0\}$ for $l \in \mathbb{Z}_{>0}$.

Definition 3.6. We say that a crystal B is *perfect* of level l if

- (1) $B \otimes B$ is connected as a crystal graph.
- (2) There exists $\lambda_0 \in P_{\text{cl}}$ such that

$$\text{wt}(B) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0} \text{cl}(\alpha_i), \quad \sharp B_{\lambda_0} = 1.$$

- (3) There exists a finite-dimensional $U'_q(\mathfrak{g})$ -module V with a crystal pseudo-base B_{ps} such that $B \cong B_{ps}/\pm 1$.
- (4) The maps $\varepsilon, \varphi: B^{\min} := \{b \in B \mid \langle c, \varepsilon(b) \rangle = l\} \rightarrow (P_{\text{cl}}^+)_l$ are bijective, where $\varepsilon(b) := \sum_i \varepsilon_i(b) \Lambda_i$ and $\varphi(b) := \sum_i \varphi_i(b) \Lambda_i$.

Let $\{B_l\}_{l \geq 1}$ be a family of perfect crystals of level l and set $J := \{(l, b) \mid l > 0, b \in B_l^{\min}\}$.

Definition 3.7. A crystal B_∞ with an element b_∞ is called the *limit* of $\{B_l\}_{l \geq 1}$ if

- (1) $\text{wt}(b_\infty) = \varepsilon(b_\infty) = \varphi(b_\infty) = 0$.
- (2) For any $(l, b) \in J$, there exists an embedding of crystals:

$$\begin{aligned} f_{(l,b)}: \quad T_{\varepsilon(b)} \otimes B_l \otimes T_{-\varphi(b)} &\hookrightarrow B_\infty \\ t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)} &\mapsto b_\infty. \end{aligned}$$

- (3) $B_\infty = \bigcup_{(l,b) \in J} \text{Im} f_{(l,b)}$.

As for the crystal T_λ , see Example 3.4 (3). If the limit of a family $\{B_l\}$ exists, we say that $\{B_l\}$ is a *coherent family* of perfect crystals.

The following is one of the most important properties of the limits of perfect crystals.

Proposition 3.8 ([6]). *Let $B(\infty)$ be the crystal as in Example 3.4 (2). Then we have the following isomorphism of crystals:*

$$B(\infty) \otimes B_\infty \xrightarrow{\sim} B(\infty).$$

In the rest of this section, for each affine Lie algebra $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, D_{n+1}^{(2)}, A_{2n}^{(2)}$, we explicitly describe an example of $B_\infty = B_\infty(\mathfrak{g})$ following [6].

3.4. $A_n^{(1)}$ ($n \geq 2$). The Cartan matrix $(a_{ij})_{i,j \in I}$ ($I := \{0, 1, \dots, n\}$) of type $A_n^{(1)}$ is

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i \equiv j \pm 1 \pmod{n+1}, \\ 0 & \text{otherwise.} \end{cases}$$

We have $\mathbf{c} = \sum_{i \in I} \alpha_i^\vee$ and $\delta = \sum_{i \in I} \alpha_i$. A limit of perfect crystals is given as follows:

$$B_\infty(A_n^{(1)}) = \{(b_1, \dots, b_{n+1}) \in \mathbb{Z}^{n+1} \mid \sum_{i=1}^{n+1} b_i = 0\},$$

for $b = (b_1, \dots, b_{n+1}) \in B_\infty(A_n^{(1)})$. We have

$$\begin{cases} \tilde{e}_0 b = (b_1 - 1, b_2, \dots, b_n, b_{n+1} + 1), \\ \tilde{e}_i b = (b_1, \dots, b_i + 1, b_{i+1} - 1, \dots, b_{n+1}) \quad (i = 1, \dots, n), \\ \tilde{f}_i = \tilde{e}_i^{-1}, \end{cases}$$

and

$$\begin{cases} \text{wt}(b) = (b_{n+1} - b_1)\Lambda_0 + \sum_{i=1}^n (b_i - b_{i+1})\Lambda_i, \\ \varepsilon_0(b) = b_1, \quad \varepsilon_i(b) = b_{i+1} \quad (i = 1, \dots, n), \\ \varphi_0(b) = b_{n+1}, \quad \varphi_i(b) = b_i \quad (i = 1, \dots, n). \end{cases}$$

3.5. $B_n^{(1)}$ ($n \geq 3$). The Cartan matrix $(a_{ij})_{i,j \in I}$ ($I := \{0, 1, \dots, n\}$) of type $B_n^{(1)}$ is

$$a_{ij} = \begin{cases} 2 & i = j, \\ -1 & |i - j| = 1 \text{ and } (i, j) \neq (0, 1), (1, 0), (n, n-1) \text{ or } (i, j) = (0, 2), (2, 0), \\ -2 & (i, j) = (n, n-1), \\ 0 & \text{otherwise.} \end{cases}$$

The Dynkin diagram is

where σ is a Dynkin diagram automorphism which we use later. We have

$$\mathbf{c} = \alpha_0^\vee + \alpha_1^\vee + 2 \sum_{i=2}^{n-1} \alpha_i^\vee + \alpha_n^\vee, \quad \delta = \alpha_0 + \alpha_1 + 2 \sum_{i=2}^n \alpha_i.$$

A limit of perfect crystals is given as follows:

$$B_\infty(B_n^{(1)}) = \{(b_1, \dots, b_n, \bar{b}_n, \dots, \bar{b}_1) \in \mathbb{Z}^{n-1} \times \left(\frac{1}{2}\mathbb{Z}\right)^2 \times \mathbb{Z}^{n-1} \mid \sum_{i=1}^n (b_i + \bar{b}_i) = 0, b_n + \bar{b}_n \in \mathbb{Z}\},$$

for $b = (b_1, \dots, b_n, \bar{b}_n, \dots, \bar{b}_1) \in B_\infty(B_n^{(1)})$. We have

$$\begin{aligned} \tilde{e}_0 b &= \begin{cases} (b_1, b_2 - 1, \dots, \bar{b}_2, \bar{b}_1 + 1) & \text{if } b_2 > \bar{b}_2, \\ (b_1 - 1, b_2, \dots, \bar{b}_2 + 1, \bar{b}_1) & \text{if } b_2 \leq \bar{b}_2, \end{cases} \\ \tilde{e}_i b &= \begin{cases} (b_1, \dots, b_i + 1, b_{i+1} - 1, \dots, \bar{b}_1) & \text{if } b_{i+1} > \bar{b}_{i+1}, \\ (b_1, \dots, \bar{b}_{i+1} + 1, \bar{b}_i - 1, \dots, \bar{b}_1) & \text{if } b_{i+1} \leq \bar{b}_{i+1}, \end{cases} \quad (i = 1, \dots, n-1), \\ \tilde{e}_n b &= (b_1, \dots, b_n + \frac{1}{2}, \bar{b}_n - \frac{1}{2}, \dots, \bar{b}_1), \\ \tilde{f}_i &= \tilde{e}_i^{-1}, \end{aligned}$$

and

$$\begin{cases} \text{wt}(b) = (\bar{b}_1 - b_1 + \bar{b}_2 - b_2)\Lambda_0 + \sum_{i=1}^{n-1} (b_i - \bar{b}_i + \bar{b}_{i+1} - b_{i+1})\Lambda_i + 2(b_n - \bar{b}_n)\Lambda_n, \\ \varepsilon_0(b) = b_1 + (b_2 - \bar{b}_2)_+, \quad \varepsilon_i(b) = \bar{b}_i + (b_{i+1} - \bar{b}_{i+1})_+ \quad (i = 1, \dots, n-1), \\ \varepsilon_n(b) = 2\bar{b}_n, \quad \varphi_0(b) = \bar{b}_1 + (\bar{b}_2 - b_2)_+, \quad \varphi_i(b) = b_i + (\bar{b}_{i+1} - b_{i+1})_+ \quad (i = 1, \dots, n-1), \\ \varphi_n(b) = 2b_n. \end{cases}$$

Note that the presentation above is slightly different from the one in [6]. But we see that they are equivalent by the correspondence $\nu_n + \frac{1}{2}\nu_0 \leftrightarrow b_n$ and $\bar{\nu}_n + \frac{1}{2}\nu_0 \leftrightarrow \bar{b}_n$.

3.6. $C_n^{(1)}$ ($n \geq 2$). The Cartan matrix $(a_{ij})_{i,j \in I}$ ($I := \{0, 1, \dots, n\}$) of type $C_n^{(1)}$ is

$$a_{ij} = \begin{cases} 2 & i = j, \\ -1 & |i - j| = 1 \text{ and } (i, j) \neq (1, 0), (n-1, n), \\ -2 & (i, j) = (1, 0), (n-1, n), \\ 0 & \text{otherwise.} \end{cases}$$

Then the Dynkin diagram is

where σ is the Dynkin diagram automorphism $\sigma: \alpha_i \mapsto \alpha_{n-i}$. We have

$$\mathbf{c} = \sum_{i \in I} \alpha_i^\vee, \quad \delta = \alpha_0 + 2 \sum_{i=1}^{n-1} \alpha_i + \alpha_n.$$

A limit of perfect crystals is given as follows:

$$B_\infty(C_n^{(1)}) = \{(b_1, \dots, b_n, \bar{b}_n, \dots, \bar{b}_1) \in \mathbb{Z}^{2n} \mid \sum_{i=1}^n (b_i + \bar{b}_i) \in 2\mathbb{Z}\},$$

for $b = (b_1, \dots, b_n, \bar{b}_n, \dots, \bar{b}_1) \in B_\infty(C_n^{(1)})$. We have

$$\begin{aligned} \tilde{e}_0 b &= \begin{cases} (b_1 - 2, b_2, \dots, \bar{b}_2, \bar{b}_1) & \text{if } b_1 > \bar{b}_1 + 1, \\ (b_1 - 1, b_2, \dots, \bar{b}_2, \bar{b}_1 + 1) & \text{if } b_1 = \bar{b}_1 + 1, \\ (b_1, b_2, \dots, \bar{b}_2, \bar{b}_1 + 2) & \text{if } b_1 \leq \bar{b}_1, \end{cases} \\ \tilde{e}_i b &= \begin{cases} (b_1, \dots, b_i + 1, b_{i+1} - 1, \dots, \bar{b}_1) & \text{if } b_{i+1} > \bar{b}_{i+1}, \\ (b_1, \dots, \bar{b}_{i+1} + 1, \bar{b}_i - 1, \dots, \bar{b}_1) & \text{if } b_{i+1} \leq \bar{b}_{i+1}, \end{cases} \quad (1 \leq i < n), \\ \tilde{e}_n b &= (b_1, \dots, b_n + 1, \bar{b}_n - 1, \dots, \bar{b}_1), \\ \tilde{f}_i &= \tilde{e}_i^{-1}, \end{aligned}$$

and

$$\begin{cases} \text{wt}(b) = (\bar{b}_1 - b_1)\Lambda_0 + \sum_{i=1}^{n-1} (b_i - \bar{b}_i + \bar{b}_{i+1} - b_{i+1})\Lambda_i + (b_n - \bar{b}_n)\Lambda_n, \\ \varepsilon_0(b) = -\frac{1}{2}l(b) + (b_1 - \bar{b}_1)_+, \varepsilon_i(b) = \bar{b}_i + (b_{i+1} - \bar{b}_{i+1})_+ \quad (1 \leq i < n), \varepsilon_n(b) = \bar{b}_n, \\ \varphi_0(b) = -\frac{1}{2}l(b) + (\bar{b}_1 - b_1)_+, \varphi_i(b) = b_i + (\bar{b}_{i+1} - b_{i+1})_+ \quad (1 \leq i < n), \varphi_n(b) = b_n, \end{cases}$$

where $l(b) := \sum_{i=1}^n (b_i + \bar{b}_i)$.

3.7. $D_n^{(1)}$ ($n \geq 4$). The Cartan matrix $(a_{ij})_{i,j \in I}$ ($I := \{0, 1, \dots, n\}$) of type $D_n^{(1)}$ is

$$a_{ij} = \begin{cases} 2 & i = j, \\ -1 & |i - j| = 1 \text{ and } 1 \leq i, j \leq n-1 \\ & \text{or } (i, j) = (0, 2), (2, 0), (n-2, n), (n, n-2), \\ 0 & \text{otherwise.} \end{cases}$$

The Dynkin diagram is

where σ is the Dynkin diagram automorphism $\sigma: \alpha_0 \leftrightarrow \alpha_1$ and $\sigma\alpha_i = \alpha_i$ for $i \neq 0, 1$. We have

$$\mathbf{c} = \alpha_0^\vee + \alpha_1^\vee + 2 \sum_{i=2}^{n-2} \alpha_i^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee, \quad \delta = \alpha_0 + \alpha_1 + 2 \sum_{i=2}^{n-2} \alpha_i + \alpha_{n-1} + \alpha_n.$$

A limit of perfect crystals is given as follows:

$$B_{\infty}(D_n^{(1)}) = \{(b_1, \dots, b_n, \bar{b}_{n-1}, \dots, \bar{b}_1) \in \mathbb{Z}^{2n-1} \mid \sum_{i=1}^n b_i + \sum_{i=1}^{n-1} \bar{b}_i = 0\},$$

for $b = (b_1, \dots, b_n, \bar{b}_{n-1}, \dots, \bar{b}_1) \in B_{\infty}(D_n^{(1)})$. We have

$$\begin{aligned} \tilde{e}_0 b &= \begin{cases} (b_1, b_2 - 1, \dots, \bar{b}_2, \bar{b}_1 + 1) & \text{if } b_2 > \bar{b}_2, \\ (b_1 - 1, b_2, \dots, \bar{b}_2 + 1, \bar{b}_1) & \text{if } b_2 \leq \bar{b}_2, \end{cases} \\ \tilde{e}_i b &= \begin{cases} (b_1 \dots, b_i + 1, b_{i+1} - 1, \dots, \bar{b}_1) & \text{if } b_{i+1} > \bar{b}_{i+1}, \\ (b_1 \dots, \bar{b}_{i+1} + 1, \bar{b}_i - 1, \dots, \bar{b}_1) & \text{if } b_{i+1} \leq \bar{b}_{i+1}, \end{cases} \quad (i = 1, \dots, n-2), \\ \tilde{e}_{n-1} b &= (b_1, \dots, b_{n-1} + 1, b_n - 1, \bar{b}_{n-1}, \dots, \bar{b}_1), \\ \tilde{e}_n b &= (b_1, \dots, b_{n-1}, b_n + 1, \bar{b}_{n-1} - 1, \dots, \bar{b}_1), \\ \tilde{f}_i &= \tilde{e}_i^{-1}, \end{aligned}$$

and

$$\begin{cases} \text{wt}(b) = (\bar{b}_1 - b_1 + \bar{b}_2 - b_2)\Lambda_0 + \sum_{i=1}^{n-2} (b_i - \bar{b}_i + \bar{b}_{i+1} - b_{i+1})\Lambda_i \\ \quad + (b_{n-1} - \bar{b}_{n-1} - b_n)\Lambda_{n-1} + (b_{n-1} - \bar{b}_{n-1} + b_n)\Lambda_n, \\ \varepsilon_0(b) = b_1 + (b_2 - \bar{b}_2)_+, \quad \varepsilon_i(b) = \bar{b}_i + (b_{i+1} - \bar{b}_{i+1})_+ \quad (i = 1, \dots, n-2), \\ \varepsilon_{n-1}(b) = b_n + \bar{b}_{n-1}, \quad \varepsilon_n(b) = \bar{b}_{n-1}, \\ \varphi_0(b) = \bar{b}_1 + (\bar{b}_2 - b_2)_+, \quad \varphi_i(b) = b_i + (\bar{b}_{i+1} - b_{i+1})_+ \quad (i = 1, \dots, n-2), \\ \varphi_{n-1}(b) = b_{n-1}, \quad \varphi_n(b) = b_{n-1} + b_n. \end{cases}$$

3.8. $A_{2n-1}^{(2)}$ ($n \geq 3$). The Cartan matrix $(a_{ij})_{i,j \in I}$ ($I := \{0, 1, \dots, n\}$) of type $A_{2n-1}^{(2)}$ is

$$a_{ij} = \begin{cases} 2 & i = j, \\ -1 & |i - j| = 1 \text{ and } 1 \leq i, j \leq n-1 \text{ or } (i, j) = (0, 2), (2, 0), (n, n-1), \\ -2 & (i, j) = (n-1, n), \\ 0 & \text{otherwise.} \end{cases}$$

The Dynkin diagram is

where σ is the Dynkin diagram automorphism $\sigma: \alpha_0 \leftrightarrow \alpha_1$. We have

$$\mathbf{c} = \alpha_0^\vee + \alpha_1^\vee + 2 \sum_{i=2}^n \alpha_i^\vee, \quad \delta = \alpha_0 + \alpha_1 + 2 \sum_{i=2}^{n-1} \alpha_i + \alpha_n.$$

A limit of perfect crystals is given as follows:

$$B_{\infty}(A_{2n-1}^{(2)}) = \{(b_1, \dots, b_n, \bar{b}_n, \dots, \bar{b}_1) \in \mathbb{Z}^{2n} \mid \sum_{i=1}^n (b_i + \bar{b}_i) = 0\},$$

for $b = (b_1, \dots, b_n, \bar{b}_n, \dots, \bar{b}_1) \in B_\infty(A_{2n-1}^{(2)})$. We have

$$\begin{aligned}\tilde{e}_0 b &= \begin{cases} (b_1, b_2 - 1, \dots, \bar{b}_2, \bar{b}_1 + 1) & \text{if } b_2 > \bar{b}_2, \\ (b_1 - 1, b_2, \dots, \bar{b}_2 + 1, \bar{b}_1) & \text{if } b_2 \leq \bar{b}_2, \end{cases} \\ \tilde{e}_i b &= \begin{cases} (b_1, \dots, b_i + 1, b_{i+1} - 1, \dots, \bar{b}_1) & \text{if } b_{i+1} > \bar{b}_{i+1}, \\ (b_1, \dots, \bar{b}_{i+1} + 1, \bar{b}_i - 1, \dots, \bar{b}_1) & \text{if } b_{i+1} \leq \bar{b}_{i+1}, \end{cases} \quad (i = 1, \dots, n-1), \\ \tilde{e}_n b &= (b_1, \dots, b_{n-1}, b_n + 1, \bar{b}_n - 1, \dots, \bar{b}_1), \\ \tilde{f}_i &= \tilde{e}_i^{-1},\end{aligned}$$

and

$$\begin{cases} \text{wt}(b) = (\bar{b}_1 - b_1 + \bar{b}_2 - b_2)\Lambda_0 + \sum_{i=1}^{n-1} (b_i - \bar{b}_i + \bar{b}_{i+1} - b_{i+1})\Lambda_i + (b_n - \bar{b}_n)\Lambda_n, \\ \varepsilon_0(b) = b_1 + (b_2 - \bar{b}_2)_+, \varepsilon_i(b) = \bar{b}_i + (b_{i+1} - \bar{b}_{i+1})_+ \quad (i = 1, \dots, n-1), \\ \varepsilon_n(b) = \bar{b}_n, \varphi_0(b) = \bar{b}_1 + (\bar{b}_2 - b_2)_+, \varphi_i(b) = b_i + (\bar{b}_{i+1} - b_{i+1})_+ \quad (i = 1, \dots, n-1) \\ \varphi_n(b) = b_n. \end{cases}$$

3.9. $D_{n+1}^{(2)}$ ($n \geq 2$). The Cartan matrix $(a_{ij})_{i,j \in I}$ ($I := \{0, 1, \dots, n\}$) of type $D_{n+1}^{(2)}$ is

$$a_{ij} = \begin{cases} 2 & i = j, \\ -1 & |i - j| = 1 \text{ and } (i, j) \neq (0, 1), (n, n-1), \\ -2 & (i, j) = (0, 1), (n, n-1), \\ 0 & \text{otherwise.} \end{cases}$$

The Dynkin diagram is

where σ is the Dynkin diagram automorphism $\sigma: \alpha_i \leftrightarrow \alpha_{n-i}$ ($i = 0, 1, \dots, n$). We have

$$\mathbf{c} = \alpha_0^\vee + \alpha_1^\vee + 2 \sum_{i=2}^{n-1} \alpha_i^\vee + \alpha_n^\vee, \quad \delta = \sum_{i=0}^n \alpha_i.$$

A limit of perfect crystals is given as follows:

$$B_\infty(D_{n+1}^{(2)}) = \{(b_1, \dots, b_n, \bar{b}_n, \dots, \bar{b}_1) \in \mathbb{Z}^{n-1} \times \left(\frac{1}{2}\mathbb{Z}\right)^2 \times \mathbb{Z}^{n-1} \mid b_n + \bar{b}_n \in \mathbb{Z}\},$$

for $b = (b_1, \dots, b_n, \bar{b}_n, \dots, \bar{b}_1) \in B_\infty(D_{n+1}^{(2)})$. We have

$$\begin{aligned}\tilde{e}_0 b &= \begin{cases} (b_1 - 1, b_2, \dots, \bar{b}_2, \bar{b}_1) & \text{if } b_1 > \bar{b}_1, \\ (b_1, b_2, \dots, \bar{b}_2, \bar{b}_1 + 1) & \text{if } b_1 \leq \bar{b}_1, \end{cases} \\ \tilde{e}_i b &= \begin{cases} (b_1, \dots, b_i + 1, b_{i+1} - 1, \dots, \bar{b}_1) & \text{if } b_{i+1} > \bar{b}_{i+1}, \\ (b_1, \dots, \bar{b}_{i+1} + 1, \bar{b}_i - 1, \dots, \bar{b}_1) & \text{if } b_{i+1} \leq \bar{b}_{i+1}, \end{cases} \quad (1 \leq i < n), \\ \tilde{e}_n b &= (b_1, \dots, b_n + \frac{1}{2}, \bar{b}_n - \frac{1}{2}, \dots, \bar{b}_1), \\ \tilde{f}_i &= \tilde{e}_i^{-1},\end{aligned}$$

and

$$\begin{cases} \text{wt}(b) = 2(\bar{b}_1 - b_1)\Lambda_0 + \sum_{i=1}^{n-1} (b_i - \bar{b}_i + \bar{b}_{i+1} - b_{i+1})\Lambda_i + 2(b_n - \bar{b}_n)\Lambda_n, \\ \varepsilon_0(b) = -l(b) + 2(b_1 - \bar{b}_1)_+, \varepsilon_i(b) = \bar{b}_i + (b_{i+1} - \bar{b}_{i+1})_+ (1 \leq i < n), \varepsilon_n(b) = 2\bar{b}_n, \\ \varphi_0(b) = -l(b) + 2(\bar{b}_1 - b_1)_+, \varphi_i(b) = b_i + (\bar{b}_{i+1} - b_{i+1})_+ (1 \leq i < n), \varphi_n(b) = 2b_n, \end{cases}$$

where $l(b) := \sum_{i=1}^n (b_i + \bar{b}_i)$. Note that the presentation above is slightly different from the one in [6]. As in §3.5, they are equivalent by the correspondence $\nu_n + \frac{1}{2}\nu_0 \leftrightarrow b_n$ and $\bar{\nu}_n + \frac{1}{2}\nu_0 \leftrightarrow \bar{b}_n$.

3.10. $A_{2n}^{(2)}$ ($n \geq 2$). The Cartan matrix $(a_{ij})_{i,j \in I}$ ($I := \{0, 1, \dots, n\}$) of type $A_{2n}^{(2)}$ is

$$a_{ij} = \begin{cases} 2 & i = j, \\ -1 & |i - j| = 1 \text{ and } (i, j) \neq (0, 1), (n-1, n), \\ -2 & (i, j) = (0, 1), (n-1, n), \\ 0 & \text{otherwise.} \end{cases}$$

The Dynkin diagram is

Note that there exists no Dynkin diagram automorphism in this case. We have

$$\mathbf{c} = \alpha_0^\vee + 2 \sum_{i=1}^n \alpha_i^\vee, \quad \delta = 2 \sum_{i=0}^{n-1} \alpha_i + \alpha_n.$$

A limit of perfect crystals is given as follows:

$$B_\infty(A_{2n}^{(2)}) = \{(b_1, \dots, b_n, \bar{b}_n, \dots, \bar{b}_1) \in \mathbb{Z}^{2n}\},$$

for $(b_1, \dots, b_n, \bar{b}_n, \dots, \bar{b}_1) \in B_\infty(A_{2n}^{(2)})$. We have

$$\begin{aligned} \tilde{e}_0 b &= \begin{cases} (b_1 - 1, b_2, \dots, \bar{b}_2, \bar{b}_1) & \text{if } b_1 > \bar{b}_1, \\ (b_1, b_2, \dots, \bar{b}_2, \bar{b}_1 + 1) & \text{if } b_1 \leq \bar{b}_1, \end{cases} \\ \tilde{e}_i b &= \begin{cases} (b_1, \dots, b_i + 1, b_{i+1} - 1, \dots, \bar{b}_1) & \text{if } b_{i+1} > \bar{b}_{i+1}, \\ (b_1, \dots, \bar{b}_{i+1} + 1, \bar{b}_i - 1, \dots, \bar{b}_1) & \text{if } b_{i+1} \leq \bar{b}_{i+1}, \end{cases} \quad (1 \leq i < n), \\ \tilde{e}_n b &= (b_1, \dots, b_{n-1}, b_n + 1, \bar{b}_n - 1, \dots, \bar{b}_1), \\ \tilde{f}_i &= \tilde{e}_i^{-1}, \end{aligned}$$

and

$$\begin{cases} \text{wt}(b) = 2(\bar{b}_1 - b_1)\Lambda_0 + \sum_{i=1}^{n-1} (b_i - \bar{b}_i + \bar{b}_{i+1} - b_{i+1})\Lambda_i + (b_n - \bar{b}_n)\Lambda_n, \\ \varepsilon_0(b) = -l(b) + 2(b_1 - \bar{b}_1)_+, \varepsilon_i(b) = \bar{b}_i + (b_{i+1} - \bar{b}_{i+1})_+ (1 \leq i < n), \varepsilon_n(b) = \bar{b}_n, \\ \varphi_0(b) = -l(b) + 2(\bar{b}_1 - b_1)_+, \varphi_i(b) = b_i + (\bar{b}_{i+1} - b_{i+1})_+ (1 \leq i < n), \varphi_n(b) = b_n, \end{cases}$$

where $l(b) := \sum_{i=1}^n (b_i + \bar{b}_i)$.

4. FUNDAMENTAL REPRESENTATIONS

4.1. Fundamental representation $W(\varpi_1)$. Let $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, D_{n+1}^{(2)}, A_{2n}^{(2)}$, and let $\{\Lambda_i \mid i \in I\}$ be the set of fundamental weights as in §3. Let $\varpi_1 := \Lambda_1 - a_1^\vee \Lambda_0$ be the (level 0) fundamental weight, where $i = 1$ is the node of the Dynkin diagram as in §3 and $a_i^\vee$ is given in (3.1).

Let $V(\varpi_1)$ be the extremal weight module over $U_q(\mathfrak{g})$ associated with ϖ_1 ([9]) and let $W(\varpi_1) \cong V(\varpi_1)/(z_1 - 1)V(\varpi_1)$ be the fundamental representation of $U'_q(\mathfrak{g})$ where z_1 is a $U'_q(\mathfrak{g})$ -linear automorphism on $V(\varpi_1)$ (see [9, Sect. 5]).

By [9, Theorem 5.17], $W(\varpi_1)$ is a finite-dimensional irreducible integrable $U'_q(\mathfrak{g})$ -module and has a global basis with a simple crystal. Thus, we can consider its specialization $q = 1$ and obtain a finite-dimensional $\mathfrak{g}$ -module, which will be denoted by the same notation $W(\varpi_1)$.

Now we present the explicit form of $W(\varpi_1)$ for $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, D_{n+1}^{(2)}, A_{2n}^{(2)}$.

4.2. $A_n^{(1)}$ ($n \geq 2$). The global basis of $W(\varpi_1)$ is

$$\{v_1, v_2, \dots, v_{n+1}\},$$

and we have

$$\text{wt}(v_i) = \Lambda_i - \Lambda_{i-1} \quad (i = 1 \cdots, n+1),$$

where we understand $\Lambda_{n+1} = \Lambda_0$. The explicit actions of the f_i 's are

$$\begin{aligned} f_i v_i &= v_{i+1} \quad (1 \leq i \leq n), & f_0 v_{n+1} &= v_1, \\ f_i v_j &= 0 & \text{otherwise.} \end{aligned}$$

Its crystal graph is:

4.3. $B_n^{(1)}$ ($n \geq 3$). The global basis of $W(\varpi_1)$ is

$$\{v_1, v_2, \dots, v_n, v_0, v_{\overline{n}}, \dots, v_{\overline{2}}, v_{\overline{1}}\},$$

and we have

$$\begin{aligned} \text{wt}(v_i) &= \Lambda_i - \Lambda_{i-1}, & \text{wt}(v_{\overline{i}}) &= \Lambda_{i-1} - \Lambda_i & (i \neq 0, 2, n), \\ \text{wt}(v_2) &= -\Lambda_0 - \Lambda_1 + \Lambda_2, & \text{wt}(v_{\overline{2}}) &= \Lambda_0 + \Lambda_1 - \Lambda_2, \\ \text{wt}(v_n) &= 2\Lambda_n - \Lambda_{n-1}, & \text{wt}(v_{\overline{n}}) &= \Lambda_{n-1} - 2\Lambda_n, & \text{wt}(v_0) &= 0. \end{aligned}$$

The explicit forms of the actions by the f_i 's are

$$\begin{aligned} f_i v_i &= v_{i+1}, & f_i v_{\overline{i+1}} &= v_{\overline{i}} & (i = 1, \dots, n-1), \\ f_n v_n &= v_0, & f_n v_0 &= 2v_{\overline{n}}, \\ f_0 v_{\overline{2}} &= v_1, & f_0 v_{\overline{1}} &= v_2, \\ f_i v_j &= 0 & \text{otherwise.} \end{aligned}$$

Its crystal graph is

4.4. $C_n^{(1)}$ ($n \geq 2$). The global basis of $W(\varpi_1)$ is

$$\{v_1, v_2, \dots, v_n, v_{\overline{n}}, \dots, v_{\overline{2}}, v_{\overline{1}}\},$$

and we have

$$\text{wt}(v_i) = \Lambda_i - \Lambda_{i-1}, \quad \text{wt}(v_{\overline{i}}) = \Lambda_{i-1} - \Lambda_i \quad (i = 1 \cdots, n).$$

The explicit forms of the actions by the f_i 's are

$$\begin{aligned} f_i v_i &= v_{i+1}, & f_i v_{\overline{i+1}} &= v_{\overline{i}} & (i = 1, \dots, n-1), \\ f_n v_n &= v_{\overline{n}}, & f_0 v_{\overline{1}} &= v_1, \\ f_i v_j &= 0 & \text{otherwise.} \end{aligned}$$

Its crystal graph is

4.5. $D_n^{(1)}$ ($n \geq 4$). The global basis of $W(\varpi_1)$ is

$$\{v_1, v_2, \dots, v_n, v_{\overline{n}}, \dots, v_{\overline{2}}, v_{\overline{1}}\},$$

and we have

$$\begin{aligned} \text{wt}(v_i) &= \Lambda_i - \Lambda_{i-1}, & \text{wt}(v_{\overline{i}}) &= \Lambda_{i-1} - \Lambda_i & (i \neq 2, n-1), \\ \text{wt}(v_2) &= -\Lambda_0 - \Lambda_1 + \Lambda_2, & \text{wt}(v_{\overline{2}}) &= \Lambda_0 + \Lambda_1 - \Lambda_2, \\ \text{wt}(v_{n-1}) &= \Lambda_{n-1} + \Lambda_n - \Lambda_{n-1}, & \text{wt}(v_{\overline{n-1}}) &= \Lambda_{n-2} - \Lambda_{n-1} - \Lambda_n. \end{aligned}$$

The explicit forms of the actions by the f_i 's are

$$\begin{aligned} f_i v_i &= v_{i+1}, & f_i v_{\overline{i+1}} &= v_{\overline{i}} & (i = 1, \dots, n-1), \\ f_n v_n &= v_{\overline{n-1}}, & f_n v_{n-1} &= v_{\overline{n}}, \\ f_0 v_{\overline{2}} &= v_1, & f_0 v_{\overline{1}} &= v_2, \\ f_i v_j &= 0 & \text{otherwise.} \end{aligned}$$

Its crystal graph is

4.6. $A_{2n-1}^{(2)}$ ($n \geq 3$). The global basis of $W(\varpi_1)$ is

$$\{v_1, v_2, \dots, v_n, v_{\overline{n}}, \dots, v_{\overline{2}}, v_{\overline{1}}\},$$

and we have

$$\begin{aligned} \text{wt}(v_i) &= \Lambda_i - \Lambda_{i-1}, & \text{wt}(v_{\overline{i}}) &= \Lambda_{i-1} - \Lambda_i & (i \neq 2), \\ \text{wt}(v_2) &= -\Lambda_0 - \Lambda_1 + \Lambda_2, & \text{wt}(v_{\overline{2}}) &= \Lambda_0 + \Lambda_1 - \Lambda_2. \end{aligned}$$

The explicit forms of the actions by the f_i 's are

$$\begin{aligned} f_i v_i &= v_{i+1}, & f_i v_{\overline{i+1}} &= v_{\overline{i}} & (i = 1, \dots, n-1), \\ f_n v_n &= v_{\overline{n}}, \\ f_0 v_{\overline{2}} &= v_1, & f_0 v_{\overline{1}} &= v_2, \\ f_i v_j &= 0 & \text{otherwise.} \end{aligned}$$

Its crystal graph is

4.7. $D_{n+1}^{(2)}$ ($n \geq 2$). The global basis of $W(\varpi_1)$ is

$$\{v_1, v_2, \dots, v_n, v_0, v_{\overline{n}}, \dots, v_{\overline{2}}, v_{\overline{1}}, \phi\},$$

and we have

$$\begin{aligned} \text{wt}(v_i) &= \Lambda_i - \Lambda_{i-1}, & \text{wt}(v_{\overline{i}}) &= \Lambda_{i-1} - \Lambda_i & (i \neq 0, 1, n), \\ \text{wt}(v_1) &= \Lambda_1 - 2\Lambda_0, & \text{wt}(v_{\overline{1}}) &= 2\Lambda_0 - \Lambda_1, \\ \text{wt}(v_n) &= 2\Lambda_n - \Lambda_{n-1}, & \text{wt}(v_{\overline{n}}) &= \Lambda_{n-1} - 2\Lambda_n, \\ \text{wt}(v_0) &= 0, & \text{wt}(\phi) &= 0. \end{aligned}$$

The explicit forms of the actions by the f_i 's are

$$\begin{aligned} f_i v_i &= v_{i+1}, & f_i v_{\overline{i+1}} &= v_{\overline{i}} & (i = 1, \dots, n-1), \\ f_n v_n &= v_0, & f_n v_0 &= 2v_{\overline{n}}, \\ f_0 v_{\overline{1}} &= \phi, & f_0 \phi &= 2v_1, \\ f_i v_j &= 0, & f_i \phi &= 0 & \text{otherwise.} \end{aligned}$$

Its crystal graph is

4.8. $A_{2n}^{(2)\dagger}$ ($n \geq 2$). We take the Cartan data transposed from that in §3.10. Then the Dynkin diagram is

In this case, we denote this type by $A_{2n}^{(2)\dagger}$ in order to distinguish it from the one in §3.10. Then the global basis of $W(\varpi_1)$ is

$$\{v_1, v_2, \dots, v_n, v_0, v_{\overline{n}}, \dots, v_{\overline{2}}, v_{\overline{1}}\},$$

and we have

$$\begin{aligned} \text{wt}(v_i) &= \Lambda_i - \Lambda_{i-1}, & \text{wt}(v_{\overline{i}}) &= \Lambda_{i-1} - \Lambda_i & (i = 1, \dots, n-1), \\ \text{wt}(v_n) &= 2\Lambda_n - \Lambda_{n-1}, & \text{wt}(v_0) &= 0, & \text{wt}(v_{\overline{n}}) &= \Lambda_{n-1} - 2\Lambda_n. \end{aligned}$$

The explicit forms of the actions by the f_i 's are

$$\begin{aligned} f_i v_i &= v_{i+1}, & f_i v_{\overline{i+1}} &= v_{\overline{i}} & (i = 1, \dots, n-1), \\ f_n v_n &= v_{\overline{0}}, & f_n v_0 &= 2v_{\overline{n}}, \\ f_0 v_{\overline{1}} &= v_1, \\ f_i v_j &= 0, & \text{otherwise.} \end{aligned}$$

Its crystal graph is

5. AFFINE GEOMETRIC CRYSTALS

In this section, we shall construct the affine geometric crystal $\mathcal{V}(\mathfrak{g})$, which is realized in the fundamental representation $W(\varpi_1)$.

5.1. **Translation** $t(\tilde{\varpi}_1)$. For $\xi_0 \in (\mathfrak{t}_{\text{cl}}^*)_0$, let $t(\xi_0)$ be as in [9, Sect.4]:

$$t(\xi_0)(\lambda) := \lambda + (\delta, \lambda)\xi - (\xi, \lambda)\delta - \frac{(\xi, \xi)}{2}(\delta, \lambda)\delta$$

for $\xi \in \mathfrak{t}^*$ such that $\text{cl}(\xi) = \xi_0$. Then $t(\xi_0)$ does not depend on the choice of ξ , and it is well-defined.

Let $c_i^\vee$ be as in (1.1). Then $t(m\varpi_i)$ belongs to $\widetilde{W}$ if and only if $m \in c_i^\vee \mathbb{Z}$. Setting $\widetilde{\varpi}_i := c_i^\vee \varpi_i$ ($i \in I$) ([10]), $t(\widetilde{\varpi}_1)$ is expressed as follows (see *e.g.* [12]):

$$t(\widetilde{\varpi}_1) = \begin{cases} \iota(s_n s_{n-1} \cdots s_2 s_1) & A_n^{(1)} \text{ case,} \\ \iota(s_1 \cdots s_n)(s_{n-1} \cdots s_2 s_1) & B_n^{(1)}, A_{2n-1}^{(2)} \text{ cases,} \\ (s_0 s_1 \cdots s_n)(s_{n-1} \cdots s_2 s_1) & C_n^{(1)}, D_{n+1}^{(2)} \text{ cases,} \\ \iota(s_1 \cdots s_n)(s_{n-2} \cdots s_2 s_1) & D_n^{(1)} \text{ case,} \\ (s_0 s_1 \cdots s_n)(s_{n-1} \cdots s_2 s_1) & A_{2n}^{(2)\dagger} \text{ case,} \end{cases}$$

where ι is the Dynkin diagram automorphism

$$\iota = \begin{cases} \sigma & \mathfrak{g} = A_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}, \\ \alpha_0 \leftrightarrow \alpha_1 \text{ and } \alpha_{n-1} \leftrightarrow \alpha_n & \mathfrak{g} = D_n^{(1)}. \end{cases}$$

Now, we know that each $t(\widetilde{\varpi}_1)$ is in the form w_1 or $\iota \cdot w_1$ for some $w_1 \in W$, *e.g.*, $w_1 = s_n \cdots s_1$ for $A_n^{(1)}$, $w_1 = (s_1 \cdots s_n)(s_{n-1} \cdots s_1)$ for $B_n^{(1)}$, *etc.*, *...*

In the case $\mathfrak{g} = A_{2n}^{(2)\dagger}$, $\eta := \text{wt}(v_{\overline{n}}) = \Lambda_{n-1} - 2\Lambda_n$ is a unique weight of $W(\varpi_1)$ which satisfies $\langle \alpha_i^\vee, \eta \rangle \geq 0$ for $i \neq n$. For this η we have

$$(5.1) \quad t(\eta) = (s_n s_{n-1} \cdots s_1)(s_0 s_1 \cdots s_{n-1}) =: w_2,$$

which will be used later.

5.2. Affine geometric crystals. Let σ be the Dynkin diagram automorphism as in §3.4–3.9 and $w_1 = s_{i_1} \cdots s_{i_k}$ be as in the previous subsection. Set

$$(5.2) \quad \mathcal{V}(\mathfrak{g}) := \{v(x_1, \dots, x_k) := Y_{i_1}(x_1) \cdots Y_{i_k}(x_k)v_1 \mid x_1, \dots, x_k \in \mathbb{C}^\times\} \subset W(\varpi_1).$$

Since the vector v_1 is a highest weight vector in $W(\varpi_1)$ as a $\mathfrak{g}_0$ -module, $\mathcal{V}(\mathfrak{g})$ has a G_0 -geometric crystal structure. Moreover $(\mathbb{C}^\times)^k \rightarrow \mathcal{V}(\mathfrak{g})$ is a birational morphism. We shall define a G -geometric crystal structure on $\mathcal{V}(\mathfrak{g})$ by using the Dynkin diagram automorphism σ except for $A_{2n}^{(2)}$. This σ induces an automorphism of $W(\varpi_1)$, which is also denoted by $\sigma: W(\varpi_1) \rightarrow W(\varpi_1)$. In the subsequent subsections, we shall show the following theorems by case-by-case arguments.

Theorem 5.1. (1) *Case $\mathfrak{g} \neq A_{2n}^{(2)\dagger}$. For $x = (x_1, \dots, x_k) \in (\mathbb{C}^\times)^k$, there exist a unique $y = (y_1, \dots, y_k) \in (\mathbb{C}^\times)^k$ and a positive rational function $a(x)$ such that*

$$(5.3) \quad v(y) = a(x)\sigma(v(x)), \quad \varepsilon_{\sigma(i)}(v(y)) = \varepsilon_i(v(x)) \quad \text{if } i, \sigma(i) \neq 0.$$

(2) *Case $\mathfrak{g} = A_{2n}^{(2)\dagger}$. Associated with w_1 and w_2 as in the previous section, we define*

$$\mathcal{V}(\mathfrak{g}) := \{v_1(x) = Y_0(x_0)Y_1(x_1) \cdots Y_n(x_n)Y_{n-1}(\bar{x}_{n-1}) \cdots Y_1(\bar{x}_1)v_1 \mid x_i, \bar{x}_i \in \mathbb{C}^\times\},$$

$$\mathcal{V}_2(\mathfrak{g}) := \{v_2(y) = Y_n(y_n) \cdots Y_1(y_1)Y_0(y_0)Y_1(\bar{y}_1) \cdots Y_{n-1}(\bar{y}_{n-1})v_{\overline{n}} \mid y_i, \bar{y}_i \in \mathbb{C}^\times\}.$$

For any $x \in (\mathbb{C}^\times)^{2n}$ there exist a unique $y \in (\mathbb{C}^\times)^{2n}$ and a rational function $a(x)$ such that $v_2(y) = a(x)v_1(x)$.

Now, using this theorem, we define the rational mapping

$$(5.4) \quad \begin{aligned} \bar{\sigma}: \mathcal{V}(\mathfrak{g}) &\longrightarrow \mathcal{V}(\mathfrak{g}), & \bar{\sigma}: \mathcal{V}(\mathfrak{g}) &\longrightarrow \mathcal{V}_2(\mathfrak{g}), \\ v(x) &\mapsto v(y) \quad (\mathfrak{g} \neq A_{2n}^{(2)\dagger}), & v_1(x) &\mapsto v_2(y) \quad (\mathfrak{g} = A_{2n}^{(2)\dagger}). \end{aligned}$$

Theorem 5.2. *The rational mapping $\bar{\sigma}$ is birational. If we define*

$$(5.5) \quad \begin{cases} e_0^c := \bar{\sigma}^{-1} \circ e_{\sigma(0)}^c \circ \bar{\sigma}, & \varepsilon_0 := \varepsilon_{\sigma(0)} \circ \bar{\sigma}, & \gamma_0 := \gamma_{\sigma(0)} \circ \bar{\sigma}, & \text{for } \mathfrak{g} \neq A_{2n}^{(2)\dagger}, \\ e_0^c := \bar{\sigma}^{-1} \circ e_0^c \circ \bar{\sigma}, & \varepsilon_0 := \varepsilon_0 \circ \bar{\sigma}, & \gamma_0 := \gamma_0 \circ \bar{\sigma}, & \text{for } \mathfrak{g} = A_{2n}^{(2)\dagger}, \end{cases}$$

then $(\mathcal{V}(\mathfrak{g}), \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ is an affine $\mathfrak{g}$ -geometric crystal.

Remark. In the case $\mathfrak{g} = A_{2n}^{(2)\dagger}$, $\mathcal{V}_2(\mathfrak{g})$ has a $\mathfrak{g}_{I \setminus \{n\}}$ -geometric crystal structure. Thus, $e_0, \gamma_0, \varepsilon_0$ are well defined on $\mathcal{V}_2(\mathfrak{g})$.

The following lemma is obvious and it shows Theorem 5.2 partially.

Lemma 5.3. *Suppose that $\mathfrak{g} \neq A_{2n}^{(2)\dagger}$. If there exists $\bar{\sigma}$ as above and*

$$(5.6) \quad e_{\sigma(i)}^c = \bar{\sigma} \circ e_i^c \circ \bar{\sigma}^{-1}, \quad \gamma_i = \gamma_{\sigma(i)} \circ \bar{\sigma}, \quad \varepsilon_i = \varepsilon_{\sigma(i)} \circ \bar{\sigma},$$

for $i \neq \sigma^{-1}(0)$, 0, then we obtain

(1)

$$\begin{aligned} e_0^{c_1} e_i^{c_2} &= e_i^{c_2} e_0^{c_1} \quad \text{if } a_{0i} = a_{i0} = 0, \\ e_0^{c_1} e_i^{c_1 c_2} e_0^{c_2} &= e_i^{c_2} e_0^{c_1 c_2} e_i^{c_2} \quad \text{if } a_{0i} = a_{i0} = -1, \\ e_0^{c_1} e_i^{c_1^2 c_2} e_0^{c_1 c_2} e_i^{c_2} &= e_i^{c_2} e_0^{c_1 c_2} e_i^{c_1^2 c_2} e_0^{c_1} \quad \text{if } a_{0i} = -2, a_{i0} = -1, \end{aligned}$$

$$(2) \quad \gamma_0(e_i^c(v(x))) = c^{a_{i0}} \gamma_0(v(x)) \quad \text{and} \quad \gamma_i(e_0^c(v(x))) = c^{a_{0i}} \gamma_i(v(x)),$$

$$(3) \quad \varepsilon_0(e_0^c(v(x))) = c^{-1} \varepsilon_0(v(x)).$$

Proof. For example, we have

$$\begin{aligned} \gamma_0(e_i^c(v(x))) &= \gamma_{\sigma(0)}(\bar{\sigma} e_i^c \bar{\sigma}^{-1}(\bar{\sigma}(v(x)))) \\ &= \gamma_{\sigma(0)}(e_{\sigma(i)}^c(\bar{\sigma}(v(x)))) = c^{a_{\sigma(0), \sigma(i)}} \gamma_{\sigma(0)}(\bar{\sigma}(v(x))) \\ &= c^{a_{i0}} \gamma_0(v(x)), \end{aligned}$$

where we use $a_{\sigma(0), \sigma(i)} = a_{0i}$ in the last equality. The other assertions are obtained similarly. $\square$

In the rest of this section, we shall prove Theorems 5.1 and 5.2 in each case.

5.3. $A_n^{(1)}$ -case. We have $w_1 := s_n s_{n-1} \cdots s_2 s_1$, and

$$\mathcal{V}(A_n^{(1)}) := \{Y_n(x_n) \cdots Y_2(x_2) Y_1(x_1) v_1 \mid x_i \in \mathbb{C}^\times\} \subset W(\varpi_1).$$

Since $\exp(c^{-1} f_i) = 1 + c^{-1} f_i$ on $W(\varpi_1)$, $v(x) = Y_n(x_n) \cdots Y_2(x_2) Y_1(x_1) v_1$ is explicitly written as

$$v(x) = v(x_1, \dots, x_n) = \left(\sum_{i=1}^n x_i v_i \right) + v_{n+1}.$$

Let $\sigma: \alpha_k \mapsto \alpha_{k+1}$ ($k \in I$) be the Dynkin diagram automorphism for $A_n^{(1)}$, which gives rise to the automorphism $\sigma: W(\varpi_1) \rightarrow W(\varpi_1)$. We have $\sigma v_i = v_{i+1}$. Then, we obtain

$$\sigma(v(x)) = v_1 + \sum_{i=1}^n x_i v_{i+1}.$$

Then the equation $v(y) = a(x)\sigma(v(x))$, *i.e.*

$$\sum_{i=1}^n y_i v_i + v_{n+1} = a(x)(v_1 + \sum_{i=1}^n x_i v_{i+1}),$$

is solved by

$$(5.7) \quad a(x) = \frac{1}{x_n}, \quad y_1 = \frac{1}{x_n}, \quad y_i = \frac{x_{i-1}}{x_n} \quad (i = 2, \dots, n),$$

that is,

$$(5.8) \quad \overline{\sigma}(v(x_1, \dots, x_n)) = v\left(\frac{1}{x_n}, \frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right).$$

The A_n -geometric crystal structure on $\mathcal{V}(A_n^{(1)})$ induced from the one on $B_{w_1}^-$ is given by:

$$(5.9) \quad e_i^c(v(x_1, \dots, x_n)) = v(x_1, \dots, cx_i, \dots, x_n) \quad (i = 1, \dots, n),$$

$$(5.10)$$

$$\gamma_1(v(x)) = \frac{x_1^2}{x_2}, \quad \gamma_i(v(x)) = \frac{x_i^2}{x_{i-1}x_{i+1}} \quad (i = 2, \dots, n-1), \quad \gamma_n(v(x)) = \frac{x_n^2}{x_{n-1}},$$

$$(5.11) \quad \varepsilon_i(v(x)) = \frac{x_{i+1}}{x_i} \quad (i = 1, \dots, n-1), \quad \varepsilon_n(v(x)) = \frac{1}{x_n}.$$

Then we have

$$\varepsilon_{i+1}(\overline{\sigma}(v(x))) = \begin{cases} \frac{x_{i+1}}{x_i} & \text{if } i = 1, \dots, n-2, \\ \frac{x_n}{x_{n-1}} & \text{if } i = n-1, \end{cases}$$

which implies $\varepsilon_{\sigma(i)}(\overline{\sigma}(v(x))) = \varepsilon_i(v(x))$, and then we have completed the proof of Theorem 5.1 in this case.

Now, we define e_0^c, γ_0 and ε_0 by

$$(5.12) \quad e_0^c := \overline{\sigma}^{-1} \circ e_1^c \circ \overline{\sigma}, \quad \gamma_0 := \gamma_1 \circ \overline{\sigma}, \quad \varepsilon_0 := \varepsilon_1 \circ \overline{\sigma}.$$

Their explicit forms are

$$(5.13) \quad e_0^c(v(x)) = v\left(\frac{x_1}{c}, \frac{x_2}{c}, \dots, \frac{x_n}{c}\right),$$

$$(5.14) \quad \gamma_0(v(x)) = \frac{1}{x_1 x_n}, \quad \varepsilon_0(v(x)) = x_1.$$

Thus, we can check (5.6) easily, and then Lemma 5.3 reduces the proof of Theorem 5.2 to the statements:

$$(5.15) \quad e_0^{c_1} e_n^{c_1 c_2} e_0^{c_2} = e_n^{c_2} e_0^{c_1 c_2} e_n^{c_1},$$

$$(5.16) \quad \gamma_0(e_n^c(v(x))) = c^{-1} \gamma_0(v(x)), \quad \gamma_n(e_0^c(v(x))) = c^{-1} \gamma_n(v(x)).$$

These are immediate from (5.9)–(5.14). Thus, we obtain Theorem 5.2.

5.4. $B_n^{(1)}$ -**case**. We have $w_1 = s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_1$, and

$$\mathcal{V}(B_n^{(1)}) := \{v(x) = Y_1(x_1) \cdots Y_n(x_n) Y_{n-1}(\bar{x}_{n-1}) \cdots Y_1(\bar{x}_1) v_1 \mid x_i, \bar{x}_i \in \mathbb{C}^\times\}.$$

It follows from the explicit description of $W(\varpi_1)$ as in §4.3 that $y_i(c^{-1}) = \exp(c^{-1}f_i)$ on $W(\varpi_1)$ can be written as:

$$\exp(c^{-1}f_i) = \begin{cases} 1 + c^{-1}f_i & i \neq n, \\ 1 + c^{-1}f_n + \frac{1}{2c^2}f_n^2 & i = n. \end{cases}$$

Therefore, we have

$$v(x_1, \dots, x_n, \bar{x}_{n-1}, \dots, \bar{x}_1) = \left(\sum_{i=1}^n \xi_i(x) v_i \right) + x_n v_0 + \left(\sum_{i=2}^n x_{i-1} v_i^- \right) + v_{\overline{1}},$$

$$\text{where } \xi_i(x) := \begin{cases} x_1 \bar{x}_1 & i = 1, \\ \frac{x_{i-1} \bar{x}_{i-1} + x_i \bar{x}_i}{x_{i-1}} & i \neq 1, n, \\ \frac{x_{n-1} \bar{x}_{n-1} + x_n^2}{x_{n-1}} & i = n. \end{cases}$$

Since $\sigma v_1 = v_{\overline{1}}$, $\sigma v_{\overline{1}} = v_1$ and $\sigma v_k = v_k$ otherwise, we have

$$\sigma(v(x)) = v_1 + \left(\sum_{i=2}^n \xi_i(x) v_i \right) + x_n v_0 + \left(\sum_{i=2}^n x_{i-1} v_i^- \right) + x_1 \bar{x}_1 v_{\overline{1}}.$$

The equation $v(y) = a(x)\sigma(v(x))$ ($x, y \in (\mathbb{C}^\times)^{2n-1}$) has a unique solution:

$$(5.17) \quad \begin{aligned} a(x) &= \frac{1}{x_1 \bar{x}_1}, & y_i &= a(x) x_i = \frac{x_i}{x_1 \bar{x}_1} \quad (1 \leq i \leq n), \\ \bar{y}_i &= a(x) \bar{x}_i = \frac{\bar{x}_i}{x_1 \bar{x}_1} \quad (1 \leq i < n). \end{aligned}$$

Hence we have the rational mapping:

$$(5.18) \quad \bar{\sigma}(v(x)) := v(y) = v\left(\frac{x_1}{x_1 \bar{x}_1}, \frac{x_2}{x_1 \bar{x}_1}, \dots, \frac{x_n}{x_1 \bar{x}_1}, \frac{\bar{x}_{n-1}}{x_1 \bar{x}_1}, \dots, \frac{\bar{x}_1}{x_1 \bar{x}_1}\right).$$

By the explicit form of $\bar{\sigma}$ in (5.18), we have $\bar{\sigma}^2 = \text{id}$, which means that the morphism $\bar{\sigma}$ is birational. In this case, the second condition in Theorem 5.1 is trivial since $\sigma(i) = i$ if i , $\sigma(i) \neq 0$. Thus, the proof of Theorem 5.1 in this case is completed.

Now, we set $e_0^c := \bar{\sigma} \circ e_1^c \circ \bar{\sigma}$, $\gamma_0 := \gamma_1 \circ \bar{\sigma}$ and $\varepsilon_0 := \varepsilon_1 \circ \bar{\sigma}$.

The explicit forms of the e_i 's, ε_i 's and γ_i 's are:

$$\begin{aligned}
e_0^c: \quad x_1 &\mapsto x_1 \frac{cx_1\bar{x}_1 + x_2\bar{x}_2}{c(x_1\bar{x}_1 + x_2\bar{x}_2)}, & x_i &\mapsto \frac{x_i}{c} \quad (2 \leq i \leq n), \\
\bar{x}_1 &\mapsto \bar{x}_1 \frac{x_1\bar{x}_1 + x_2\bar{x}_2}{cx_1\bar{x}_1 + x_2\bar{x}_2}, & \bar{x}_i &\mapsto \frac{\bar{x}_i}{c} \quad (2 \leq i \leq n-1), \\
e_i^c: \quad x_i &\mapsto x_i \frac{cx_i\bar{x}_i + x_{i+1}\bar{x}_{i+1}}{x_i\bar{x}_i + x_{i+1}\bar{x}_{i+1}}, & \bar{x}_i &\mapsto \bar{x}_i \frac{c(x_i\bar{x}_i + x_{i+1}\bar{x}_{i+1})}{cx_i\bar{x}_i + x_{i+1}\bar{x}_{i+1}}, \\
x_j &\mapsto x_j, & \bar{x}_j &\mapsto \bar{x}_j \quad (j \neq i) \quad (1 \leq i < n-1), \\
e_{n-1}^c: \quad x_{n-1} &\mapsto x_{n-1} \frac{cx_{n-1}\bar{x}_{n-1} + x_n^2}{x_{n-1}\bar{x}_{n-1} + x_n^2}, & \bar{x}_{n-1} &\mapsto \bar{x}_{n-1} \frac{c(x_{n-1}\bar{x}_{n-1} + x_n^2)}{cx_{n-1}\bar{x}_{n-1} + x_n^2}, \\
x_j &\mapsto x_j, & \bar{x}_j &\mapsto \bar{x}_j \quad (j \neq n-1), \\
e_n^c: \quad x_n &\mapsto cx_n, & x_j &\mapsto x_j \quad \bar{x}_j \mapsto \bar{x}_j \quad (j \neq n),
\end{aligned}$$

$$\begin{aligned}
\varepsilon_0(v(x)) &= \frac{x_1\bar{x}_1 + x_2\bar{x}_2}{x_1}, & \varepsilon_1(v(x)) &= \frac{1}{x_1} \left(1 + \frac{x_2\bar{x}_2}{x_1\bar{x}_1} \right), \\
\varepsilon_i(v(x)) &= \frac{x_{i-1}}{x_i} \left(1 + \frac{x_{i+1}\bar{x}_{i+1}}{x_i\bar{x}_i} \right) \quad (2 \leq i \leq n-2), \\
\varepsilon_{n-1}(v(x)) &= \frac{x_{n-2}}{x_{n-1}} \left(1 + \frac{x_n^2}{x_{n-1}\bar{x}_{n-1}} \right), & \varepsilon_n(v(x)) &= \frac{x_{n-1}}{x_n},
\end{aligned}$$

$$\begin{aligned}
\gamma_0(v(x)) &= \frac{1}{x_2\bar{x}_2}, & \gamma_1(v(x)) &= \frac{(x_1\bar{x}_1)^2}{x_2\bar{x}_2}, \\
\gamma_i(v(x)) &= \frac{(x_i\bar{x}_i)^2}{x_{i-1}\bar{x}_{i-1}x_{i+1}\bar{x}_{i+1}} \quad (2 \leq i \leq n-2), \\
\gamma_{n-1}(v(x)) &= \frac{(x_{n-1}\bar{x}_{n-1})^2}{x_{n-2}\bar{x}_{n-2}x_n^2}, & \gamma_n(v(x)) &= \frac{x_n^2}{x_{n-1}\bar{x}_{n-1}}.
\end{aligned}$$

Since $\sigma(i) = i$ for $i \neq 0, 1$, the condition (5.6) in Lemma 5.3 can be easily seen by (5.18) and by the explicit form of e_i , γ_i and ε_i ($i \in I$). Thus, in order to prove Theorem 5.2, it suffices to show that

$$(5.19) \quad e_0^{c_1} e_1^{c_2} = e_1^{c_2} e_0^{c_1},$$

$$(5.20) \quad \gamma_0(e_1^c(v(x))) = \gamma_0(v(x)), \quad \gamma_1(e_0^c(v(x))) = \gamma_1(v(x)).$$

It follows from the explicit formula above that

$$\begin{aligned}
e_0^{c_1} e_1^{c_2}(v(x)) &= e_1^{c_2} e_0^{c_1}(v(x)) \\
&= v \left(x_1 \frac{c_1 c_2 x_1 \bar{x}_1 + x_2 \bar{x}_2}{c_1(x_1 \bar{x}_1 + x_2 \bar{x}_2)}, \frac{x_2}{c_1}, \dots, \frac{\bar{x}_2}{c_1}, \bar{x}_1 \frac{c_2(x_1 \bar{x}_1 + x_2 \bar{x}_2)}{c_1 c_2 x_1 \bar{x}_1 + x_2 \bar{x}_2} \right),
\end{aligned}$$

which implies (5.19). We get (5.20) immediately from the formula above and we complete the proof of Theorem 5.2 for $B_n^{(1)}$.

5.5. $C_n^{(1)}$ -**case**. We have $w_1 = s_0 s_1 \cdots s_n s_{n-1} \cdots s_1$ and

$$\mathcal{V}(C_n^{(1)}) := \{v(x) = Y_0(x_0)Y_1(x_1) \cdots Y_n(x_n)Y_{n-1}(\bar{x}_{n-1}) \cdots Y_1(\bar{x}_1)v_1 \mid x_i, \bar{x}_i \in \mathbb{C}^\times\}.$$

Due to the explicit description of $W(\varpi_1)$ in §4.4, we obtain $y_i(c^{-1}) = \exp(c^{-1}f_i) = 1 + c^{-1}f_i$ on $W(\varpi_1)$. Hence we have

$$v(x_0, x_1, \dots, x_n, \bar{x}_{n-1}, \dots, \bar{x}_1) = \left(\sum_{i=1}^n \xi_i v_i \right) + \left(\sum_{i=1}^n x_{i-1} v_{\bar{i}} \right) \text{ where } \xi_i := \begin{cases} \frac{x_0 + x_1 \bar{x}_1}{x_0} & i = 1, \\ \frac{x_{i-1} \bar{x}_{i-1} + x_i \bar{x}_i}{x_{i-1}} & i \neq 1, n, \\ \frac{x_{n-1} \bar{x}_{n-1} + x_n}{x_{n-1}} & i = n. \end{cases}$$

The explicit forms of $\varepsilon_i(x)$ ($1 \leq i \leq n$) are:

$$\begin{aligned} \varepsilon_i(v(x)) &= \frac{x_{i-1}}{x_i} \left(1 + \frac{x_{i+1} \bar{x}_{i+1}}{x_i \bar{x}_i} \right) \quad (1 \leq i \leq n-2), \\ \varepsilon_{n-1}(v(x)) &= \frac{x_{n-2}}{x_{n-1}} \left(1 + \frac{x_n}{x_{n-1} \bar{x}_{n-1}} \right), \quad \varepsilon_n(v(x)) = \frac{x_{n-1}^2}{x_n}. \end{aligned}$$

Since $\sigma v_{i+1} = v_{\overline{n-i}}$ and $\sigma v_{\overline{i+1}} = v_{n-i}$ ($0 \leq i < n$), we have

$$\sigma(v(x)) = \left(\sum_{i=1}^n x_{n-i} v_i \right) + \left(\sum_{i=1}^n \xi_{n-i+1} v_{\bar{i}} \right).$$

We obtain a unique solution of the equations $v(y) = a(x)\sigma(v(x))$, $\varepsilon_{n-1}(v(y)) = \varepsilon_1(v(x))$ ($x, y \in (\mathbb{C}^\times)^{2n}$):

$$\begin{aligned} a(x) &= \frac{\bar{x}_{n-1}}{x_n} + \frac{1}{x_{n-1}}, \\ y_0 &= x_n \left(\frac{\bar{x}_{n-1}}{x_n} + \frac{1}{x_{n-1}} \right)^2, \\ y_i &= \left(\bar{x}_{n-i-1} + \frac{x_{n-i} \bar{x}_{n-i}}{x_{n-i-1}} \right) \left(\frac{\bar{x}_{n-1}}{x_n} + \frac{1}{x_{n-1}} \right) \quad (1 \leq i < n), \\ y_{n-1} &= \left(1 + \frac{x_1 \bar{x}_1}{x_0} \right) \left(\frac{\bar{x}_{n-1}}{x_n} + \frac{1}{x_{n-1}} \right), \\ y_n &= x_0 \left(\frac{\bar{x}_{n-1}}{x_n} + \frac{1}{x_{n-1}} \right)^2, \\ \bar{y}_{n-1} &= \frac{(x_{n-1} \bar{x}_{n-1} + x_n) x_0 x_1 \bar{x}_1}{(x_0 + x_1 \bar{x}_1) x_{n-1} x_n}, \\ \bar{y}_i &= \frac{(x_{n-1} \bar{x}_{n-1} + x_n) x_{n-i-1} x_{n-i} \bar{x}_{n-i}}{(x_{n-i-1} \bar{x}_{n-i-1} + x_{n-i} \bar{x}_{n-i}) x_{n-1} x_n} \quad (1 \leq i < n-1). \end{aligned}$$

Now we have the rational mapping $\bar{\sigma}: \mathcal{V}(C_n^{(1)}) \rightarrow \mathcal{V}(C_n^{(1)})$ defined by $v(x) \mapsto v(y)$. By the above explicit form of y , we have $\bar{\sigma}^2 = \text{id}$, which means that $\bar{\sigma}$ is birational.

By direct calculations, we have $\varepsilon_{n-i}(v(y)) = \varepsilon_i(v(x))$ ($1 \leq i \leq n-1$), and we complete the proof of Theorem 5.1 for $C_n^{(1)}$.

Let us define $e_0^c := \bar{\sigma} \circ e_n^c \circ \bar{\sigma}$ ($\bar{\sigma}^2 = \text{id}$), $\gamma_0 := \gamma_n \circ \bar{\sigma}$ and $\varepsilon_0 := \varepsilon_n \circ \bar{\sigma}$. The explicit forms of e_i , γ_i and ε_0 are

$$\begin{aligned} e_0^c: \quad x_0 &\mapsto x_0 \frac{(cx_0 + x_1\bar{x}_1)^2}{c(x_0 + x_1\bar{x}_1)^2}, & x_i &\mapsto x_i \frac{cx_0 + x_1\bar{x}_1}{c(x_0 + x_1\bar{x}_1)} \quad (1 \leq i \leq n-1), \\ x_n &\mapsto x_n \frac{(cx_0 + x_1\bar{x}_1)^2}{c^2(x_0 + x_1\bar{x}_1)^2}, & \bar{x}_i &\mapsto \bar{x}_i \frac{cx_0 + x_1\bar{x}_1}{c(x_0 + x_1\bar{x}_1)} \quad (1 \leq i \leq n-1), \\ e_i^c: \quad x_i &\mapsto x_i \frac{cx_i\bar{x}_i + x_{i+1}\bar{x}_{i+1}}{x_i\bar{x}_i + x_{i+1}\bar{x}_{i+1}}, & \bar{x}_i &\mapsto \bar{x}_i \frac{c(x_i\bar{x}_i + x_{i+1}\bar{x}_{i+1})}{cx_i\bar{x}_i + x_{i+1}\bar{x}_{i+1}}, \\ x_j &\mapsto x_j, & \bar{x}_j &\mapsto \bar{x}_j \quad (j \neq i), & (1 \leq i < n-1), \\ e_{n-1}^c: \quad x_{n-1} &\mapsto x_{n-1} \frac{cx_{n-1}\bar{x}_{n-1} + x_n}{x_{n-1}\bar{x}_{n-1} + x_n}, & \bar{x}_{n-1} &\mapsto \bar{x}_{n-1} \frac{c(x_{n-1}\bar{x}_{n-1} + x_n)}{cx_{n-1}\bar{x}_{n-1} + x_n}, \\ x_j &\mapsto x_j, & \bar{x}_j &\mapsto \bar{x}_j \quad (j \neq n-1), \\ e_n^c: \quad x_n &\mapsto cx_n, & x_j &\mapsto x_j, & \bar{x}_j &\mapsto \bar{x}_j \quad (j \neq n), \end{aligned}$$

$$\begin{aligned} \gamma_0(v(x)) &= \frac{x_0^2}{(x_1\bar{x}_1)^2}, & \gamma_1(v(x)) &= \frac{(x_1\bar{x}_1)^2}{x_0x_2\bar{x}_2}, \\ \gamma_i &= \frac{(x_i\bar{x}_i)^2}{x_{i-1}\bar{x}_{i-1}x_{i+1}\bar{x}_{i+1}} \quad (2 \leq i \leq n-2), \\ \gamma_{n-1}(v(x)) &= \frac{(x_{n-1}\bar{x}_{n-1})^2}{x_{n-2}\bar{x}_{n-2}x_n}, & \gamma_n(v(x)) &= \frac{x_n^2}{(x_{n-1}\bar{x}_{n-1})^2}, \\ \varepsilon_0(v(x)) &= \frac{1}{x_0} \left(1 + \frac{x_1\bar{x}_1}{x_0} \right)^2. \end{aligned}$$

Let us check the condition (5.6) in Lemma 5.3. The following are useful for this purpose:

$$(5.21) \quad y_i \bar{y}_i = a(x)^2 x_{n-i} \bar{x}_{n-i}, \quad y_0 = a(x)^2 x_n,$$

$$(5.22) \quad a(v(y)) = a(\bar{\sigma}(v(x))) = \frac{1}{a(v(x))}.$$

Using these we can easily check the two conditions $\gamma_i = \gamma_{\sigma(i)} \circ \bar{\sigma}$ and $\varepsilon_i = \varepsilon_{\sigma(i)} \circ \bar{\sigma}$. The condition $e_{\sigma(i)}^c = \bar{\sigma} \circ e_i^c \circ \bar{\sigma}^{-1}$ for $i = 2, \dots, n-2$ is also immediate from (5.21) and (5.22). Next let us see the case $i = 1, n-1$. We have

$$a(e_{n-1}^c(v(y))) = \frac{y_n + cy_{n-1}\bar{y}_{n-1}}{y_{n-1}y_n} \cdot \frac{x_1\bar{x}_1 + x_0}{cx_1\bar{x}_1 + x_0} = \frac{1}{a(v(x))}.$$

Using this, we can get $e_{n-i}^c = \bar{\sigma} \circ e_1^c \circ \bar{\sigma}^{-1}$ and then $e_1^c = \bar{\sigma} \circ e_{n-1}^c \circ \bar{\sigma}^{-1}$ since $\bar{\sigma}^2 = \text{id}$. Now, it remains to show that

$$e_0^{c_1} e_n^{c_2} = e_n^{c_2} e_0^{c_1}, \quad \gamma_0(e_n^c(v(x))) = \gamma_0(v(x)), \quad \gamma_n(e_0^c(v(x))) = \gamma_n(v(x)).$$

They easily follow from the explicit form of e_0^c . Thus, the proof of Theorem 5.2 in this case is completed.

5.6. $D_n^{(1)}$ -**case.** We have $w_1 = s_1 s_2 \cdots s_{n-1} s_n s_{n-2} s_{n-3} \cdots s_2 s_1$, and

$$\mathcal{V}(D_n^{(1)}) := \{v(x) = Y_1(x_1) \cdots Y_{n-1}(x_{n-1}) Y_n(x_n) Y_{n-2}(\bar{x}_{n-2}) \cdots Y_1(\bar{x}_1) v_1 \mid x_i, \bar{x}_i \in \mathbb{C}^\times\}.$$

It follows from the explicit form of $W(\varpi_1)$ in §4.5 that $y_i(c^{-1}) = \exp(c^{-1}f_i) = 1 + c^{-1}f_i$ on $W(\varpi_1)$. Thus, we have

$$v(x) = \left(\sum_{i=1}^{n-1} \xi_i(x) v_i \right) + x_n v_n + \left(\sum_{i=2}^n x_{i-1} v_i \right) + v_{\overline{1}},$$

$$\text{where } \xi_i(x) := \begin{cases} x_1 \bar{x}_1 & i = 1, \\ \frac{x_{i-1} \bar{x}_{i-1} + x_i \bar{x}_i}{x_{i-1}} & i \neq 1, n-1, \\ \frac{x_{n-2} \bar{x}_{n-2} + x_{n-1} x_n}{x_{n-2}} & i = n-1. \end{cases}$$

Since $\sigma v_1 = v_{\overline{1}}$, $\sigma v_{\overline{1}} = v_1$ and $\sigma v_k = v_k$ otherwise, we have

$$\sigma(v(x)) = v_1 + \left(\sum_{i=2}^n \xi_i(x) v_i \right) + x_n v_n + \left(\sum_{i=1}^{n-1} x_{i-1} v_i \right) + \xi_1 v_{\overline{1}}.$$

Then the equation $v(y) = a(x)\sigma(v(x))$ ($x, y \in (\mathbb{C}^\times)^{2n-2}$) has the following unique solution:

$$(5.23) \quad \begin{aligned} a(x) &= \frac{1}{x_1 \bar{x}_1}, & y_i &= a(x) x_i = \frac{x_i}{x_1 \bar{x}_1} \quad (1 \leq i \leq n), \\ \bar{y}_i &= a(x) \bar{x}_i = \frac{\bar{x}_i}{x_1 \bar{x}_1} \quad (1 \leq i \leq n-2). \end{aligned}$$

We define the rational mapping $\bar{\sigma}: \mathcal{V}(D_n^{(1)}) \longrightarrow \mathcal{V}(D_n^{(1)})$ by

$$(5.24) \quad \bar{\sigma}(v(x)) = v \left(\frac{x_1}{x_1 \bar{x}_1}, \frac{x_2}{x_1 \bar{x}_1}, \dots, \frac{x_n}{x_1 \bar{x}_1}, \frac{\bar{x}_{n-2}}{x_1 \bar{x}_1}, \dots, \frac{\bar{x}_1}{x_1 \bar{x}_1} \right).$$

It is immediate from (5.24) that $\bar{\sigma}^2 = \text{id}$, which implies the birationality of the morphism $\bar{\sigma}$. In this case, the second condition in Theorem 5.1 is trivial since $\sigma(i) = i$ if $i, \sigma(i) \neq 0$. Thus, the proof of Theorem 5.1 for $D_n^{(1)}$ is completed.

Now, we set $e_0^c := \bar{\sigma} \circ e_1^c \circ \bar{\sigma}$, $\gamma_0 := \gamma_1 \circ \bar{\sigma}$ and $\varepsilon_0 := \varepsilon_1 \circ \bar{\sigma}$. The explicit forms of e_i , ε_i and γ_i are:

$$\begin{aligned}
e_0^c: \quad x_1 &\mapsto x_1 \frac{cx_1\bar{x}_1 + x_2\bar{x}_2}{c(x_1\bar{x}_1 + x_2\bar{x}_2)}, & x_i &\mapsto \frac{x_i}{c} \quad (2 \leq i \leq n), \\
\bar{x}_1 &\mapsto \bar{x}_1 \frac{x_1\bar{x}_1 + x_2\bar{x}_2}{cx_1\bar{x}_1 + x_2\bar{x}_2}, & \bar{x}_i &\mapsto \frac{\bar{x}_i}{c} \quad (2 \leq i \leq n-2), \\
e_i^c: \quad x_i &\mapsto x_i \frac{cx_i\bar{x}_i + x_{i+1}\bar{x}_{i+1}}{x_i\bar{x}_i + x_{i+1}\bar{x}_{i+1}}, & \bar{x}_i &\mapsto \bar{x}_i \frac{c(x_i\bar{x}_i + x_{i+1}\bar{x}_{i+1})}{cx_i\bar{x}_i + x_{i+1}\bar{x}_{i+1}}, \\
x_j &\mapsto x_j, & \bar{x}_j &\mapsto \bar{x}_j \quad (j \neq i), & (1 \leq i \leq n-3), \\
e_{n-2}^c: \quad x_{n-2} &\mapsto x_{n-2} \frac{cx_{n-2}\bar{x}_{n-2} + x_{n-1}x_n}{x_{n-2}\bar{x}_{n-2} + x_{n-1}x_n}, \\
\bar{x}_{n-2} &\mapsto \bar{x}_{n-2} \frac{c(x_{n-2}\bar{x}_{n-2} + x_{n-1}x_n)}{cx_{n-2}\bar{x}_{n-2} + x_{n-1}x_n}, \\
x_j &\mapsto x_j, & \bar{x}_j &\mapsto \bar{x}_j \quad (j \neq n-2), \\
e_{n-1}^c: \quad x_{n-1} &\mapsto cx_{n-1}, & x_j &\mapsto x_j, & \bar{x}_j &\mapsto \bar{x}_j \quad (j \neq n-1), \\
e_n^c: \quad x_n &\mapsto cx_n, & x_j &\mapsto x_j, & \bar{x}_j &\mapsto \bar{x}_j \quad (j \neq n),
\end{aligned}$$

$$\begin{aligned}
\varepsilon_0(v(x)) &= \frac{x_1\bar{x}_1 + x_2\bar{x}_2}{x_1}, & \varepsilon_1(v(x)) &= \frac{1}{x_1} \left(1 + \frac{x_2\bar{x}_2}{x_1\bar{x}_1} \right), \\
\varepsilon_i(v(x)) &= \frac{x_{i-1}}{x_i} \left(1 + \frac{x_{i+1}\bar{x}_{i+1}}{x_i\bar{x}_i} \right) & (2 \leq i \leq n-3), \\
\varepsilon_{n-2}(v(x)) &= \frac{x_{n-3}}{x_{n-2}} \left(1 + \frac{x_{n-1}x_n}{x_{n-2}\bar{x}_{n-2}} \right), & \varepsilon_{n-1}(v(x)) &= \frac{x_{n-2}}{x_{n-1}}, \\
\varepsilon_n(v(x)) &= \frac{x_{n-2}}{x_n},
\end{aligned}$$

$$\begin{aligned}
\gamma_0(v(x)) &= \frac{1}{x_2\bar{x}_2}, & \gamma_1(v(x)) &= \frac{(x_1\bar{x}_1)^2}{x_2\bar{x}_2}, \\
\gamma_i(v(x)) &= \frac{(x_i\bar{x}_i)^2}{x_{i-1}\bar{x}_{i-1}x_{i+1}\bar{x}_{i+1}} & (2 \leq i \leq n-3), \\
\gamma_{n-2}(v(x)) &= \frac{(x_{n-2}\bar{x}_{n-2})^2}{x_{n-3}\bar{x}_{n-3}x_{n-1}x_n}, & \gamma_{n-1}(v(x)) &= \frac{x_{n-1}^2}{x_{n-2}\bar{x}_{n-2}}, \\
\gamma_n(v(x)) &= \frac{x_n^2}{x_{n-2}\bar{x}_{n-2}}.
\end{aligned}$$

By these formulas, we can show Theorem 5.2 for $D_n^{(1)}$ similarly to the one for $B_n^{(1)}$.

5.7. $A_{2n-1}^{(2)}$ -case. We have $w_1 = s_1s_2 \cdots s_ns_{n-1} \cdots s_2s_1$, and

$$\mathcal{V}(A_{2n-1}^{(2)}) := \{v(x) := Y_1(x_1) \cdots Y_n(x_n)Y_{n-1}(\bar{x}_{n-1}) \cdots Y_1(\bar{x}_1)v_1 \mid x_i, \bar{x}_i \in \mathbb{C}^\times\}.$$

In this case, $y_i(c^{-1}) = \exp(c^{-1}f_i) = 1 + c^{-1}f_i$ on $W(\varpi_1)$, and we have

$$v(x_1, \dots, x_n, \bar{x}_{n-1}, \dots, \bar{x}_1) = \left(\sum_{i=1}^n \xi_i v_i \right) + \left(\sum_{i=2}^n x_{i-1} v_i \right) + v_{\bar{1}},$$

$$\text{where } \xi_i := \begin{cases} x_1 \bar{x}_1 & i = 1, \\ \frac{x_{i-1} \bar{x}_{i-1} + x_i \bar{x}_i}{x_{i-1}} & i \neq 1, n, \\ \frac{x_{n-1} \bar{x}_{n-1} + x_n}{x_{n-1}} & i = n. \end{cases}$$

Since $\sigma v_1 = v_{\bar{1}}$, $\sigma v_{\bar{1}} = v_1$ and $\sigma v_k = v_k$ otherwise, we have

$$\sigma(v(x)) = v_1 + \left(\sum_{i=2}^n \xi_i v_i \right) + \left(\sum_{i=2}^n x_{i-1} v_i \right) + x_1 \bar{x}_1 v_{\bar{1}}.$$

Solving $v(y) = a(x)\sigma(v(x))$ ($x, y \in (\mathbb{C}^\times)^{2n-1}$), we obtain a unique solution:

(5.25)

$$a(x) = \frac{1}{x_1 \bar{x}_1}, \quad y_i = a(x)x_i = \frac{x_i}{x_1 \bar{x}_1}, \quad \bar{y}_i = a(x)\bar{x}_i = \frac{\bar{x}_i}{x_1 \bar{x}_1} \quad (1 \leq i \leq n-1),$$

$$y_n = \frac{x_n}{(x_1 \bar{x}_1)^2}.$$

Then we have

$$(5.26) \quad \bar{\sigma}(v(x)) = v \left(\frac{x_1}{x_1 \bar{x}_1}, \frac{x_2}{x_1 \bar{x}_1}, \dots, \frac{x_n}{(x_1 \bar{x}_1)^2}, \frac{\bar{x}_{n-1}}{x_1 \bar{x}_1}, \dots, \frac{\bar{x}_1}{x_1 \bar{x}_1} \right).$$

By the explicit form of $\bar{\sigma}$ in (5.26), we have $\bar{\sigma}^2 = \text{id}$, which means that the morphism $\bar{\sigma}$ is birational. In this case, the second condition in Theorem 5.1 is trivial since $\sigma(i) = i$ if i , $\sigma(i) \neq 0$. Thus, the proof of Theorem 5.1 for $A_{2n-1}^{(2)}$ is completed.

Now, we set $e_0^c := \bar{\sigma} \circ e_1^c \circ \bar{\sigma}$, $\gamma_0 := \gamma_1 \circ \bar{\sigma}$ and $\varepsilon_0 := \varepsilon_1 \circ \bar{\sigma}$. The explicit forms of e_i , ε_i and γ_i are:

$$\begin{aligned} e_0^c: \quad & x_1 \mapsto x_1 \frac{cx_1 \bar{x}_1 + x_2 \bar{x}_2}{c(x_1 \bar{x}_1 + x_2 \bar{x}_2)}, \quad x_i \mapsto \frac{x_i}{c} \quad (2 \leq i \leq n-1), \quad x_n \mapsto \frac{x_n}{c^2}, \\ & \bar{x}_1 \mapsto \bar{x}_1 \frac{x_1 \bar{x}_1 + x_2 \bar{x}_2}{cx_1 \bar{x}_1 + x_2 \bar{x}_2}, \quad \bar{x}_i \mapsto \frac{\bar{x}_i}{c} \quad (2 \leq i \leq n-1), \\ e_i^c: \quad & x_i \mapsto x_i \frac{cx_i \bar{x}_i + x_{i+1} \bar{x}_{i+1}}{x_i \bar{x}_i + x_{i+1} \bar{x}_{i+1}}, \quad \bar{x}_i \mapsto \bar{x}_i \frac{c(x_i \bar{x}_i + x_{i+1} \bar{x}_{i+1})}{cx_i \bar{x}_i + x_{i+1} \bar{x}_{i+1}}, \\ & x_j \mapsto x_j, \quad \bar{x}_j \mapsto \bar{x}_j \quad (j \neq i) \quad (1 \leq i < n-1), \\ e_{n-1}^c: \quad & x_{n-1} \mapsto x_{n-1} \frac{cx_{n-1} \bar{x}_{n-1} + x_n}{x_{n-1} \bar{x}_{n-1} + x_n}, \quad \bar{x}_{n-1} \mapsto \bar{x}_{n-1} \frac{c(x_{n-1} \bar{x}_{n-1} + x_n)}{cx_{n-1} \bar{x}_{n-1} + x_n}, \\ & x_j \mapsto x_j, \quad \bar{x}_j \mapsto \bar{x}_j \quad (j \neq n-1), \\ e_n^c: \quad & x_n \mapsto cx_n, \quad x_j \mapsto x_j, \quad \bar{x}_j \mapsto \bar{x}_j \quad (j \neq n). \end{aligned}$$

$$\begin{aligned} \varepsilon_0(v(x)) &= \frac{x_1 \bar{x}_1 + x_2 \bar{x}_2}{x_1}, \quad \varepsilon_1(v(x)) = \frac{1}{x_1} \left(1 + \frac{x_2 \bar{x}_2}{x_1 \bar{x}_1} \right), \\ \varepsilon_i(v(x)) &= \frac{x_{i-1}}{x_i} \left(1 + \frac{x_{i+1} \bar{x}_{i+1}}{x_i \bar{x}_i} \right) \quad (2 \leq i \leq n-2), \\ \varepsilon_{n-1}(v(x)) &= \frac{x_{n-2}}{x_{n-1}} \left(1 + \frac{x_n}{x_{n-1} \bar{x}_{n-1}} \right), \quad \varepsilon_n(v(x)) = \frac{x_{n-1}^2}{x_n}, \end{aligned}$$

$$\begin{aligned}\gamma_0(v(x)) &= \frac{1}{x_2\bar{x}_2}, \quad \gamma_1(v(x)) = \frac{(x_1\bar{x}_1)^2}{x_2\bar{x}_2}, \\ \gamma_i &= \frac{(x_i\bar{x}_i)^2}{x_{i-1}\bar{x}_{i-1}x_{i+1}\bar{x}_{i+1}} \quad (2 \leq i \leq n-2), \\ \gamma_{n-1}(v(x)) &= \frac{(x_{n-1}\bar{x}_{n-1})^2}{x_{n-2}\bar{x}_{n-2}x_n}, \quad \gamma_n(v(x)) = \frac{x_n^2}{(x_{n-1}\bar{x}_{n-1})^2}.\end{aligned}$$

We can show Theorem 5.2 for $A_{2n-1}^{(2)}$ similarly to the one for $B_n^{(1)}$.

5.8. $D_{n+1}^{(2)}$ -**case.** We have $w_1 = s_0s_1 \cdots s_ns_{n-1} \cdots s_2s_1$ and

$$\mathcal{V}(D_{n+1}^{(2)}) := \{v(x) := Y_0(x_0)Y_1(x_1) \cdots Y_n(x_n)Y_{n-1}(\bar{x}_{n-1}) \cdots Y_1(\bar{x}_1)v_1 \mid x_i, \bar{x}_i \in \mathbb{C}^\times\}.$$

It follows from the explicit description of $W(\varpi_1)$ as in §4.7 that on $W(\varpi_1)$:

$$y_i(c^{-1}) = \exp(c^{-1}f_i) = \begin{cases} 1 + c^{-1}f_i & i \neq 0, n, \\ 1 + c^{-1}f_i + \frac{1}{2c^2}f_i^2 & i = 0, n. \end{cases}$$

Then we have

$$v(x) = \left(\sum_{i=1}^n \xi_i(x)v_i \right) + x_nv_0 + x_0\phi + \left(\sum_{i=2}^n x_{i-1}v_{\bar{i}} \right) + x_0^2v_{\bar{1}}$$

where

$$\xi_i(x) := \begin{cases} \frac{x_0^2 + x_1\bar{x}_1}{x_0^2} & i = 1, \\ \frac{x_{i-1}\bar{x}_{i-1} + x_i\bar{x}_i}{x_{i-1}} & i \neq 1, n, \\ \frac{x_{n-1}\bar{x}_{n-1} + x_n^2}{x_{n-1}} & i = n. \end{cases}$$

Since $\sigma v_{i+1} = v_{\overline{n-i}}$ and $\sigma v_{\overline{i+1}} = v_{n-i}$ ($0 \leq i < n$) and $\sigma: v_0 \leftrightarrow \phi$, we also have

$$\sigma(v(x)) = \left(\sum_{i=1}^{n-1} x_{n-i}v_i \right) + x_0^2v_n + \left(\sum_{i=1}^n \xi_{n-i+1}v_{\bar{i}} \right) + x_n\phi + x_0v_0.$$

Solving $v(y) = a(x)\sigma(v(x))$ ($x, y \in (\mathbb{C}^\times)^{2n}$), we get a unique solution:

$$\begin{aligned}a(x) &= \frac{x_{n-1}\bar{x}_{n-1} + x_n^2}{x_{n-1}x_n^2}, \\ y_0 &= \frac{x_{n-1}\bar{x}_{n-1} + x_n^2}{x_{n-1}x_n}, \\ y_i &= \frac{(x_{n-i-1}\bar{x}_{n-i-1} + x_{n-i}\bar{x}_{n-i})(x_{n-1}\bar{x}_{n-1} + x_n^2)}{x_{n-i-1}x_{n-1}x_n^2} \quad (1 \leq i < n), \\ y_{n-1} &= \frac{(x_0^2 + x_1\bar{x}_1)(x_{n-1}\bar{x}_{n-1} + x_n^2)}{x_0^2x_{n-1}x_n^2}, \\ y_n &= \frac{x_0(x_{n-1}\bar{x}_{n-1} + x_n^2)}{x_{n-1}x_n^2}, \\ \bar{y}_i &= \frac{(x_{n-1}\bar{x}_{n-1} + x_n^2)x_{n-i-1}x_{n-i}\bar{x}_{n-i}}{(x_{n-i-1}\bar{x}_{n-i-1} + x_{n-i}\bar{x}_{n-i})x_{n-1}x_n^2} \quad (1 \leq i \leq n-2), \\ \bar{y}_{n-1} &= \frac{(x_{n-1}\bar{x}_{n-1} + x_n^2)x_0^2x_1\bar{x}_1}{(x_0^2 + x_1\bar{x}_1)x_{n-1}x_n^2}.\end{aligned}$$

Then we have the rational mapping $\bar{\sigma}: \mathcal{V}(D_{n+1}^{(2)}) \longrightarrow \mathcal{V}(D_{n+1}^{(2)})$ defined by $v(x) \mapsto v(y)$. The explicit forms of ε_i ($1 \leq i \leq n$) are as follows:

$$\begin{aligned}\varepsilon_1(v(x)) &= \frac{x_0^2}{x_1} \left(1 + \frac{x_2 \bar{x}_2}{x_1 \bar{x}_1} \right), \quad \varepsilon_n(v(x)) = \frac{x_{n-1}}{x_n}, \\ \varepsilon_i(v(x)) &= \frac{x_{i-1}}{x_i} \left(1 + \frac{x_{i+1} \bar{x}_{i+1}}{x_i \bar{x}_i} \right) \quad (2 \leq i \leq n-2), \\ \varepsilon_{n-1}(v(x)) &= \frac{x_{n-2}}{x_{n-1}} \left(1 + \frac{x_n^2}{x_{n-1} \bar{x}_{n-1}} \right).\end{aligned}$$

Then we get easily that $\varepsilon_{n-i}(v(y)) = \varepsilon_i(v(x))$ ($1 \leq i \leq n-1$), which finishes the proof of Theorem 5.1 for $D_{n+1}^{(2)}$.

Let us define $e_0^c := \bar{\sigma} \circ e_n^c \circ \bar{\sigma}$ ($\bar{\sigma}^2 = \text{id}$), $\gamma_0 := \gamma_n \circ \bar{\sigma}$ and $\varepsilon_0 := \varepsilon_n \circ \bar{\sigma}$. The explicit forms of e_i , γ_i and ε_0 are

$$\begin{aligned}e_0^c: \quad x_0 &\mapsto x_0 \frac{c^2 x_0^2 + x_1 \bar{x}_1}{c(x_0^2 + x_1 \bar{x}_1)}, \quad x_i \mapsto x_i \frac{c^2 x_0^2 + x_1 \bar{x}_1}{c^2(x_0^2 + x_1 \bar{x}_1)} \quad (1 \leq i \leq n), \\ \bar{x}_i &\mapsto \bar{x}_i \frac{c^2 x_0^2 + x_1 \bar{x}_1}{c^2(x_0^2 + x_1 \bar{x}_1)} \quad (1 \leq i \leq n-1), \\ e_i^c: \quad x_i &\mapsto x_i \frac{cx_i \bar{x}_i + x_{i+1} \bar{x}_{i+1}}{x_i \bar{x}_i + x_{i+1} \bar{x}_{i+1}}, \quad \bar{x}_i \mapsto \bar{x}_i \frac{c(x_i \bar{x}_i + x_{i+1} \bar{x}_{i+1})}{cx_i \bar{x}_i + x_{i+1} \bar{x}_{i+1}}, \\ x_j &\mapsto x_j, \quad \bar{x}_j \mapsto \bar{x}_j \quad (j \neq i), \quad (1 \leq i < n-1), \\ e_{n-1}^c: \quad x_{n-1} &\mapsto x_{n-1} \frac{cx_{n-1} \bar{x}_{n-1} + x_n^2}{x_{n-1} \bar{x}_{n-1} + x_n^2}, \quad \bar{x}_{n-1} \mapsto \bar{x}_{n-1} \frac{c(x_{n-1} \bar{x}_{n-1} + x_n^2)}{cx_{n-1} \bar{x}_{n-1} + x_n^2}, \\ x_j &\mapsto x_j, \quad \bar{x}_j \mapsto \bar{x}_j \quad (j \neq n-1), \\ e_n^c: \quad x_n &\mapsto cx_n, \quad x_j \mapsto x_j, \quad \bar{x}_j \mapsto \bar{x}_j \quad (j \neq n), \\ \gamma_0(v(x)) &= \frac{x_0^2}{x_1 \bar{x}_1}, \quad \gamma_1(v(x)) = \frac{(x_1 \bar{x}_1)^2}{x_0^2 x_2 \bar{x}_2}, \\ \gamma_i(v(x)) &= \frac{(x_i \bar{x}_i)^2}{x_{i-1} \bar{x}_{i-1} x_{i+1} \bar{x}_{i+1}} \quad (2 \leq i \leq n-2), \\ \gamma_{n-1}(v(x)) &= \frac{(x_{n-1} \bar{x}_{n-1})^2}{x_{n-2} \bar{x}_{n-2} x_n^2}, \quad \gamma_n(v(x)) = \frac{x_n^2}{x_{n-1} \bar{x}_{n-1}}, \\ \varepsilon_0(v(x)) &= \frac{x_0^2 + x_1 \bar{x}_1}{x_0^3}.\end{aligned}$$

Then similarly to the case $C_n^{(1)}$, we can show Theorem 5.2 for $D_{n+1}^{(2)}$.

5.9. $A_{2n}^{(2)\dagger}$ -case. As in the beginning of this section, we have $w_1 = s_0 s_1 \cdots s_n s_{n-1} \cdots s_2 s_1$ and

$$\mathcal{V}(A_{2n}^{(2)\dagger}) := \{v_1(x) = Y_0(x_0)Y_1(x_1) \cdots Y_n(x_n)Y_{n-1}(\bar{x}_{n-1}) \cdots Y_1(\bar{x}_1)v_1 \mid x_i, \bar{x}_i \in \mathbb{C}^\times\}.$$

By the explicit description of $W(\varpi_1)$ as in §4.8 on $W(\varpi_1)$ we have:

$$y_i(c^{-1}) = \exp(c^{-1}f_i) = \begin{cases} 1 + c^{-1}f_i & i \neq n, \\ 1 + c^{-1}f_n + \frac{1}{2c^2}f_n^2 & i = n. \end{cases}$$

Then we have

$$v_1(x) = \left(\sum_{i=1}^n \xi_i(x) v_i \right) + x_n v_0 + \left(\sum_{i=1}^n x_{i-1} v_i \right)$$

$$\text{where } \xi_i(x) := \begin{cases} \frac{x_0 + x_1 \bar{x}_1}{x_0} & i = 1, \\ \frac{x_{i-1} \bar{x}_{i-1} + x_i \bar{x}_i}{x_{i-1}} & i \neq 1, n, \\ \frac{x_{n-1} \bar{x}_{n-1} + x_n^2}{x_{n-1}} & i = n. \end{cases}$$

Next, for $w_2 = s_n s_{n-1} \cdots s_1 s_0 s_1 \cdots s_{n-1}$ we set

$$\mathcal{V}_2(A_{2n}^{(2)\dagger}) := \{v_2(y) = Y_n(y_n) \cdots Y_1(y_1) Y_0(y_0) Y_1(\bar{y}_1) \cdots Y_{n-1}(\bar{y}_{n-1}) v_{\bar{n}} \mid y_i, \bar{y}_i \in \mathbb{C}^\times\}.$$

Then we have

$$v_2(y) = \left(\sum_{i=1}^{n-1} y_i v_i \right) + y_n^2 v_n + y_n v_0 + \left(\sum_{i=1}^n \eta_i(y) v_i \right)$$

$$\text{where } \eta_i(y) := \begin{cases} \frac{y_0 + y_1 \bar{y}_1}{y_1} & i = 1, \\ \frac{y_{i-1} \bar{y}_{i-1} + y_i \bar{y}_i}{y_i} & i \neq 1, n, \\ \frac{y_{n-1} \bar{y}_{n-1} + y_n^2}{y_n^2} & i = n. \end{cases}$$

For $x \in (\mathbb{C}^\times)^{2n}$ there exist a unique $y = (y_0, \dots, \bar{y}_1) \in (\mathbb{C}^\times)^{2n}$ and $a(x)$ such that $v_2(y) = a(x) v_1(x)$. They are given by

$$\begin{aligned} a(x) &= \frac{x_{n-1} \bar{x}_{n-1} + x_n^2}{x_{n-1} x_n^2}, \\ y_0 &= a(x)^2 x_0 = \frac{x_0 (x_{n-1} \bar{x}_{n-1} + x_n^2)^2}{(x_{n-1} x_n^2)^2}, \\ y_1 &= a(x) \xi_1(x) = \frac{(x_{n-1} \bar{x}_{n-1} + x_n^2)(x_0 + x_1 \bar{x}_1)}{x_0 x_{n-1} x_n^2}, \\ y_i &= a(x) \xi_i(x) = \frac{(x_{i-1} \bar{x}_{i-1} + x_i \bar{x}_i)(x_{n-1} \bar{x}_{n-1} + x_n^2)}{x_{i-1} x_{n-1} x_n^2} \quad (1 < i < n), \\ y_n &= a(x) x_n = \frac{x_{n-1} \bar{x}_{n-1} + x_n^2}{x_{n-1} x_n}, \\ \bar{y}_1 &= a(x) \frac{x_0 x_1 \bar{x}_1}{x_0 + x_1 \bar{x}_1} = \frac{(x_{n-1} \bar{x}_{n-1} + x_n^2) x_0 x_1 \bar{x}_1}{(x_0 + x_1 \bar{x}_1) x_{n-1} x_n^2}, \\ \bar{y}_i &= \frac{(x_{n-1} \bar{x}_{n-1} + x_n^2) x_{i-1} \bar{x}_i}{(x_{i-1} \bar{x}_{i-1} + x_i \bar{x}_i) x_{n-1} x_n^2} \quad (1 < i \leq n-1). \end{aligned}$$

They define a rational mapping $\bar{\sigma}: \mathcal{V}(A_{2n}^{(2)\dagger}) \longrightarrow \mathcal{V}_2(A_{2n}^{(2)\dagger})$ ($v_1(x) \mapsto v_2(y)$). The inverse $\bar{\sigma}^{-1}: \mathcal{V}_2(A_{2n}^{(2)\dagger}) \longrightarrow \mathcal{V}(A_{2n}^{(2)\dagger})$ ($v_2(y) \mapsto v_1(x)$) is given by

$$\begin{aligned} a(y) &:= \frac{y_0 y_1}{y_0 + y_1 \bar{y}_1} \quad (= a(x)), \\ x_0 &= a(y)^{-1} \frac{y_0 + y_1 \bar{y}_1}{y_1}, \\ x_i &= a(y)^{-1} \frac{y_i \bar{y}_i + y_{i+1} \bar{y}_{i+1}}{y_{i+1}} \quad (1 \leq i \leq n-2), \\ x_{n-1} &= a(y)^{-1} \frac{y_{n-1} \bar{y}_{n-1} + y_n^2}{y_n^2}, \\ x_n &= a(y)^{-1} y_n, \\ \bar{x}_i &= a(y)^{-1} \frac{y_i \bar{y}_i y_{i+1}}{y_i \bar{y}_i + y_{i+1} \bar{y}_{i+1}} \quad (1 \leq i \leq n-2), \\ \bar{x}_{n-1} &= a(y)^{-1} \frac{y_{n-1} \bar{y}_{n-1} y_n^2}{y_{n-1} \bar{y}_{n-1} + y_n^2}, \end{aligned}$$

which means that the morphism $\bar{\sigma}: \mathcal{V}(A_{2n}^{(2)\dagger}) \longrightarrow \mathcal{V}_2(A_{2n}^{(2)\dagger})$ is birational. Thus, we obtain Theorem 5.1(2).

The actions of e_i ($i = 0, 1, \dots, n-1$) on $v_2(y)$ are induced from the ones on $Y_{\mathbf{i}_2}(y)$ ($\mathbf{i}_2 = (n, \dots, 1, 0, 1, \dots, n-1)$) since $e_i v_{\bar{n}} = 0$ for $i = 0, 1, \dots, n-1$. We also get $\gamma_i(v_2(y))$ and $\varepsilon_i(v_2(y))$ from the ones for $Y_{\mathbf{i}_2}(y)$ where $v_2(y) = \bar{\sigma}(v_1(x))$:

$$\begin{aligned} e_0^c: y_0 &\mapsto c y_0, \\ e_1^c: y_1 &\mapsto y_1 \frac{c y_1 \bar{y}_1 + y_0}{y_1 \bar{y}_1 + y_0}, \quad \bar{y}_1 \mapsto \bar{y}_1 \frac{c(y_1 \bar{y}_1 + y_0)}{c y_1 \bar{y}_1 + y_0}, \\ e_i^c: y_i &\mapsto y_i \frac{c y_i \bar{y}_i + y_{i-1} \bar{y}_{i-1}}{y_i \bar{y}_i + y_{i-1} \bar{y}_{i-1}}, \quad \bar{y}_i \mapsto \bar{y}_i \frac{c(y_i \bar{y}_i + y_{i-1} \bar{y}_{i-1})}{c y_i \bar{y}_i + y_{i-1} \bar{y}_{i-1}} \quad (i = 2, \dots, n-1), \\ \gamma_0(v_2(y)) &= \frac{y_0^2}{(y_1 \bar{y}_1)^2}, \quad \gamma_1(v_2(y)) = \frac{(y_1 \bar{y}_1)^2}{y_0 y_2 \bar{y}_2}, \\ \gamma_i(v_2(y)) &= \frac{(y_i \bar{y}_i)^2}{y_{i-1} \bar{y}_{i-1} y_{i+1} \bar{y}_{i+1}} \quad (i = 2, \dots, n-2), \quad \gamma_{n-1}(v_2(y)) = \frac{(y_{n-1} \bar{y}_{n-1})^2}{y_{n-2} \bar{y}_{n-2} y_n^2}, \\ \varepsilon_0(v_2(y)) &= \frac{y_1^2}{y_0}, \quad \varepsilon_1(v_2(y)) = \frac{y_2}{y_1} \left(1 + \frac{y_0}{y_1 \bar{y}_1} \right), \\ \varepsilon_{n-1}(v_2(y)) &= \frac{y_n^2}{y_{n-1}} \left(1 + \frac{y_{n-2} \bar{y}_{n-2}}{y_{n-1} \bar{y}_{n-1}} \right), \\ \varepsilon_i(v_2(y)) &= \frac{y_{i+1}}{y_i} \left(1 + \frac{y_{i-1} \bar{y}_{i-1}}{y_i \bar{y}_i} \right) \quad (i = 2, \dots, n-1). \end{aligned}$$

The explicit forms of $\varepsilon_i(v_1(x))$ and $\gamma_i(v_1(x))$ ($1 \leq i \leq n$) are also induced from the ones for $Y_{\mathbf{i}_1}(x) := Y_0(x_0) \cdots Y_1(\bar{x}_1)$, and we define $\varepsilon_0(v_1(x)) := \varepsilon_0(v_2(y))$ and

$$\gamma_0(v_1(x)) := \gamma_0(v_2(y)) \quad (v_2(y) := \bar{\sigma}(v_1(x))):$$

$$\varepsilon_0(v_1(x)) = \frac{1}{x_0} \left(1 + \frac{x_1 \bar{x}_1}{x_0} \right)^2,$$

$$\varepsilon_i(v_1(x)) = \frac{x_{i-1}}{x_i} \left(1 + \frac{x_{i+1} \bar{x}_{i+1}}{x_i \bar{x}_i} \right) \quad (1 \leq i \leq n-2),$$

$$\varepsilon_{n-1}(v_1(x)) = \frac{x_{n-2}}{x_{n-1}} \left(1 + \frac{x_n^2}{x_{n-1} \bar{x}_{n-1}} \right), \quad \varepsilon_n(v_1(x)) = \frac{x_{n-1}}{x_n},$$

$$\gamma_0(v_1(x)) = \frac{x_0^2}{(x_1 \bar{x}_1)^2}, \quad \gamma_1(v_1(x)) = \frac{(x_1 \bar{x}_1)^2}{x_0 x_2 \bar{x}_2},$$

$$\gamma_i(v_1(x)) = \frac{(x_i \bar{x}_i)^2}{x_{i-1} \bar{x}_{i-1} x_{i+1} \bar{x}_{i+1}} \quad (2 \leq i \leq n-2),$$

$$\gamma_{n-1}(v_1(x)) = \frac{(x_{n-1} \bar{x}_{n-1})^2}{x_{n-2} \bar{x}_{n-2} x_n^2}, \quad \gamma_n(v_1(x)) = \frac{x_n^2}{x_{n-1} \bar{x}_{n-1}}.$$

For $i = 0$, we define $e_0^c(v_1(x)) = \bar{\sigma}^{-1} \circ e_0^c \circ \bar{\sigma}(v_1(x)) = \bar{\sigma}^{-1} \circ e_0^c(v_2(y))$. Then we get

$$e_0^c: x_0 \mapsto x_0 \frac{(cx_0 + x_1 \bar{x}_1)^2}{c(x_0 + x_1 \bar{x}_1)^2}, \quad x_i \mapsto x_i \frac{cx_0 + x_1 \bar{x}_1}{c(x_0 + x_1 \bar{x}_1)} \quad (1 \leq i \leq n),$$

$$\bar{x}_i \mapsto \bar{x}_i \frac{cx_0 + x_1 \bar{x}_1}{c(x_0 + x_1 \bar{x}_1)} \quad (1 \leq i \leq n-1),$$

$$e_i^c: x_i \mapsto x_i \frac{cx_i \bar{x}_i + x_{i+1} \bar{x}_{i+1}}{x_i \bar{x}_i + x_{i+1} \bar{x}_{i+1}}, \quad \bar{x}_i \mapsto \bar{x}_i \frac{c(x_i \bar{x}_i + x_{i+1} \bar{x}_{i+1})}{cx_i \bar{x}_i + x_{i+1} \bar{x}_{i+1}} \quad (1 \leq i < n-1),$$

$$e_{n-1}^c: x_{n-1} \mapsto x_{n-1} \frac{cx_{n-1} \bar{x}_{n-1} + x_n^2}{x_{n-1} \bar{x}_{n-1} + x_n^2}, \quad \bar{x}_{n-1} \mapsto \bar{x}_{n-1} \frac{c(x_{n-1} \bar{x}_{n-1} + x_n^2)}{cx_{n-1} \bar{x}_{n-1} + x_n^2},$$

$$e_n^c: x_n \mapsto cx_n,$$

where we omit the unchanged parts. In order to prove Theorem 5.2, it suffices to show the following:

$$(5.27) \quad e_i^c = \bar{\sigma}^{-1} \circ e_i^c \circ \bar{\sigma}, \quad \gamma_i = \gamma_i \circ \bar{\sigma}, \quad \varepsilon_i = \varepsilon_i \circ \bar{\sigma} \quad (i \neq 0, n),$$

$$(5.28) \quad e_0^{c_1} e_n^{c_2} = e_n^{c_2} e_0^{c_1},$$

$$(5.29) \quad \gamma_n(e_0^c(v_1(x))) = \gamma_n(v_1(x)), \quad \gamma_0(e_n^c(v_1(x))) = \gamma_0(v_1(x)),$$

$$(5.30) \quad \varepsilon_0(e_0^c(v_1(x))) = c^{-1} \varepsilon_0(v_1(x)),$$

which are immediate from the above formulae. Let us show (5.27). Set $v_2(y) := \bar{\sigma}(v_1(x))$ and $v_1(x') := \bar{\sigma}^{-1}(e_i^c(v_2(y)))$ for $i = 2, \dots, n-2$. Then $x'_j = x_j$ and $\bar{x}'_j = \bar{x}_j$ for $j \neq i-1, i$, and we have

$$a(v_2(y)) = a(v_1(x)),$$

$$x'_i = \frac{1}{a(v_2(y))} \left(\bar{y}_{i+1} + \frac{cy_i \bar{y}_i}{y_{i+1}} \right) = x_i \frac{cx_i \bar{x}_i + x_{i+1} \bar{x}_{i+1}}{x_i \bar{x}_i + x_{i+1} \bar{x}_{i+1}},$$

$$\bar{x}'_i = \frac{1}{a(v_2(y))} \frac{cy_i \bar{y}_i y_{i+1}}{cy_i \bar{y}_i + y_{i+1} \bar{y}_{i+1}} = \bar{x}_i \frac{c(x_i \bar{x}_i + x_{i+1} \bar{x}_{i+1})}{cx_i \bar{x}_i + x_{i+1} \bar{x}_{i+1}},$$

$$x'_{i-1} = \frac{1}{a(v_2(y))} \left(\bar{y}_i + \frac{cy_{i-1} \bar{y}_{i-1}}{y_i} \right) = x_{i-1},$$

$$\bar{x}'_{i-1} = \frac{1}{a(v_2(y))} \frac{cy_{i-1} \bar{y}_{i-1} y_i}{cy_i \bar{y}_i + y_{i-1} \bar{y}_{i-1}} = \bar{x}_{i-1},$$

where the formula $y_i \bar{y}_i = a(v_1(x))^2 x_i \bar{x}_i$ is useful to obtain these results. Therefore we have $e_i^c = \bar{\sigma}^{-1} \circ e_i^c \circ \bar{\sigma}$ for $i = 2, \dots, n-2$. The others are obtained similarly.

6. ULTRA-DISCRETIZATION OF AFFINE GEOMETRIC CRYSTALS

In this section, we shall see that the limit B_∞ of perfect crystals as in §3 is isomorphic to the ultra-discretization of the geometric crystal obtained in the previous section.

Let $\mathcal{V}(\mathfrak{g})$ be the affine geometric crystal for $\mathfrak{g}$ as in the previous section. Let $m = \dim \mathcal{V}(\mathfrak{g})$. Then due to their explicit forms in the previous section, we have the following simple positive structure on $\mathcal{V}(\mathfrak{g})$:

$$(6.1) \quad \begin{aligned} \Theta: (\mathbb{C}^\times)^m &\longrightarrow \mathcal{V}(\mathfrak{g}), \\ x &\longmapsto v(x). \end{aligned}$$

Therefore, we obtain an affine crystal $\mathcal{UD}(\mathcal{V}(\mathfrak{g})) = (\mathcal{B}(\mathfrak{g}), \{\tilde{e}_i\}_{i \in I}, \{\text{wt}_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$.

Theorem 6.1. For $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, D_{n+1}^{(2)}$ and $A_{2n}^{(2)\dagger}$, let us denote its Langlands dual by $\mathfrak{g}^L$. Then we have the following isomorphism of crystals:

$$(6.2) \quad \mathcal{UD}(\mathcal{V}(\mathfrak{g})) \cong B_\infty(\mathfrak{g}^L).$$

Here $L: A_n^{(1)} \leftrightarrow A_n^{(1)}$, $L: B_n^{(1)} \leftrightarrow A_{2n-1}^{(2)}$, $L: C_n^{(1)} \leftrightarrow D_{n+1}^{(2)}$, $L: D_n^{(1)} \leftrightarrow D_n^{(1)}$ and $L: A_{2n}^{(2)} \leftrightarrow A_{2n}^{(2)\dagger}$.

In the following subsections, we shall show Theorem 6.1 in each case.

The following formula is useful for this purpose:

$$(6.3) \quad \mathcal{UD}\left(\frac{cx+y}{x+y}\right) = \max(c+x, y) - \max(x, y).$$

Note that, if $c = 1$,

$$\max(c+x, y) - \max(x, y) = \begin{cases} 1 & x \geq y, \\ 0 & x < y. \end{cases}$$

6.1. $A_n^{(1)}$ ($n \geq 2$). The crystal structure of $\mathcal{UD}(\mathcal{V}(A_n^{(1)}))$ is given as follows: $\mathcal{B}(A_n^{(1)}) = \mathbb{Z}^n$ and for $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$, we have

$$\begin{aligned} \tilde{e}_0(x) &= (x_1 - 1, x_2 - 1, \dots, x_n - 1), \\ \tilde{e}_i(x) &= (\dots, x_i + 1, \dots) \quad (1 \leq i \leq n), \\ \text{wt}_0(x) &= -x_1 - x_n, \quad \text{wt}_1(x) = 2x_1 - x_2, \\ \text{wt}_i(x) &= -x_{i-1} + 2x_i - x_{i+1} \quad (i = 2, \dots, n-1), \quad \text{wt}_n(x) = 2x_n - x_{n-1}, \\ \varepsilon_0(x) &= x_1, \quad \varepsilon_i(x) = x_{i+1} - x_i \quad (i = 1, \dots, n-1), \quad \varepsilon_n(x) = -x_n. \end{aligned}$$

Comparing with the result in §3.4 and the above formulae, we easily see that the following map gives an isomorphism of $A_n^{(1)}$ -crystals:

$$\begin{aligned} \mu: \mathcal{B}(A_n^{(1)}) = \mathbb{Z}^n &\longrightarrow B_\infty(A_n^{(1)}), \\ (x_1, \dots, x_n) &\longmapsto (b_1, \dots, b_n, b_{n+1}) = (x_1, x_2 - x_1, \dots, x_n - x_{n-1}, -x_n). \end{aligned}$$

Thus we have proved Theorem 6.1 for $A_n^{(1)}$.

6.2. $B_n^{(1)}$ ($n \geq 3$). Let us see the crystal structure of $\mathcal{UD}(\mathcal{V}(B_n^{(1)})) = (\mathcal{B}(B_n^{(1)}), \{\tilde{e}_i\}, \{\text{wt}_i\}, \{\varepsilon_i\})$. Due to the formula in §5.4, we have

$$\mathcal{B}(B_n^{(1)}) = \mathbb{Z}^{2n-1}$$

and for

$$x = (x_1, \dots, x_n, \bar{x}_{n-1}, \dots, \bar{x}_1) \in \mathcal{B}(B_n^{(1)}),$$

we have

$$\begin{aligned} \tilde{e}_0(x) &= \begin{cases} (x_1, x_2 - 1, \dots, x_n - 1, \dots, \bar{x}_2 - 1, \bar{x}_1 - 1) & \text{if } x_1 + \bar{x}_1 \geq x_2 + \bar{x}_2, \\ (x_1 - 1, x_2 - 1, \dots, x_n - 1, \dots, \bar{x}_2 - 1, \bar{x}_1) & \text{if } x_1 + \bar{x}_1 < x_2 + \bar{x}_2, \end{cases} \\ \tilde{e}_i(x) &= \begin{cases} (x_1, \dots, x_i + 1, \dots, \bar{x}_i, \dots, \bar{x}_1) & \text{if } x_i + \bar{x}_i \geq x_{i+1} + \bar{x}_{i+1}, \\ (x_1, \dots, x_i, \dots, \bar{x}_i + 1, \dots, \bar{x}_1) & \text{if } x_i + \bar{x}_i < x_{i+1} + \bar{x}_{i+1}, \end{cases} \\ &\quad (i = 1, \dots, n-2), \\ \tilde{e}_{n-1}(x) &= \begin{cases} (x_1, \dots, x_{n-1} + 1, x_n, \bar{x}_{n-1}, \dots, \bar{x}_1) & \text{if } x_{n-1} + \bar{x}_{n-1} \geq 2x_n, \\ (x_1, \dots, x_{n-1}, x_n, \bar{x}_{n-1} + 1, \dots, \bar{x}_1) & \text{if } x_{n-1} + \bar{x}_{n-1} < 2x_n, \end{cases} \\ \tilde{e}_n(x) &= (x_1, \dots, x_{n-1}, x_n + 1, \bar{x}_{n-1}, \dots, \bar{x}_1), \\ \text{wt}_0(x) &= -(x_2 + \bar{x}_2), \quad \text{wt}_1(x) = 2(x_1 + \bar{x}_1) - (x_2 + \bar{x}_2), \\ \text{wt}_i(x) &= -(x_{i-1} + \bar{x}_{i-1}) + 2(x_i + \bar{x}_i) - (x_{i+1} + \bar{x}_{i+1}) \quad (i = 2, \dots, n-2), \\ \text{wt}_{n-1}(x) &= -(x_{n-2} + \bar{x}_{n-2}) + 2(x_{n-1} + \bar{x}_{n-1}) - 2x_n, \\ \text{wt}_n(x) &= 2x_n - (x_{n-1} + \bar{x}_{n-1}), \\ \varepsilon_0(x) &= \max(x_1 + \bar{x}_1, x_2 + \bar{x}_2) - x_1 - \bar{x}_1 + (x_2 + \bar{x}_2 - x_1 - \bar{x}_1)_+, \\ \varepsilon_1(x) &= (x_2 + \bar{x}_2 - x_1 - \bar{x}_1)_+ - x_1, \\ \varepsilon_i(x) &= x_{i-1} - x_i + (x_{i+1} + \bar{x}_{i+1} - x_i - \bar{x}_i)_+ \quad (i = 2, \dots, n-2), \\ \varepsilon_{n-1}(x) &= x_{n-2} - x_{n-1} + (2x_n - x_{n-1} - \bar{x}_{n-1})_+, \quad \varepsilon_n(x) = x_{n-1} - x_n. \end{aligned}$$

Comparing the result in §3.8 and the above formulae, it is easy to see that the following map gives an isomorphism of $A_{2n-1}^{(2)}$ -crystals:

$$\begin{aligned} \mu: \mathcal{B}(B_n^{(1)}) &\longrightarrow B_\infty(A_{2n-1}^{(2)}) \\ (x_1, \dots, x_n, \dots, \bar{x}_1) &\mapsto (b_1, \dots, b_n, \bar{b}_n, \dots, \bar{b}_1), \end{aligned}$$

where

$$\begin{aligned} b_1 &= \bar{x}_1, \quad b_i = \bar{x}_i - \bar{x}_{i-1} \quad (i = 2, \dots, n-1), \quad b_n = x_n - \bar{x}_{n-1}, \\ \bar{b}_i &= x_{i-1} - x_i \quad (i = 2, \dots, n), \quad \bar{b}_1 = -x_1. \end{aligned}$$

We have proved Theorem 6.1 for $B_n^{(1)}$.

6.3. $C_n^{(1)}$ ($n \geq 2$). Let us see the crystal structure of $\mathcal{UD}(\mathcal{V}(C_n^{(1)})) = (\mathcal{B}(C_n^{(1)}), \{\tilde{e}_i\}, \{\text{wt}_i\}, \{\varepsilon_i\})$. We have $\mathcal{B}(C_n^{(1)}) = \mathbb{Z}^{2n}$ and for $x = (x_0, x_1, \dots, x_n, \bar{x}_{n-1}, \dots, \bar{x}_1) \in \mathcal{B}(C_n^{(1)})$, we have

$$\begin{aligned} \tilde{e}_0(x) &= \begin{cases} (x_0 + 1, x_1, \dots, x_n, \bar{x}_{n-1}, \dots, \bar{x}_2, \bar{x}_1) & \text{if } x_0 \geq x_1 + \bar{x}_1, \\ (x_0 - 1, \dots, x_{n-1} - 1, x_n - 2, \bar{x}_{n-1} - 1, \dots, \bar{x}_2 - 1, \bar{x}_1 - 1) & \text{if } x_0 < x_1 + \bar{x}_1, \end{cases} \\ \tilde{e}_i(x) &= \begin{cases} (x_0, \dots, x_i + 1, \dots, \bar{x}_i, \dots, \bar{x}_1) & \text{if } x_i + \bar{x}_i \geq x_{i+1} + \bar{x}_{i+1}, \\ (x_0, \dots, x_i, \dots, \bar{x}_i + 1, \dots, \bar{x}_1) & \text{if } x_i + \bar{x}_i < x_{i+1} + \bar{x}_{i+1}, \end{cases} \\ &\quad (i = 1, \dots, n-2), \\ \tilde{e}_{n-1}(x) &= \begin{cases} (x_0, \dots, x_{n-1} + 1, x_n, \bar{x}_{n-1}, \dots, \bar{x}_1) & \text{if } x_{n-1} + \bar{x}_{n-1} \geq x_n, \\ (x_0, \dots, x_{n-1}, x_n, \bar{x}_{n-1} + 1, \dots, \bar{x}_1) & \text{if } x_{n-1} + \bar{x}_{n-1} < x_n, \end{cases} \\ \tilde{e}_n(x) &= (x_0, \dots, x_{n-1}, x_n + 1, \bar{x}_{n-1}, \dots, \bar{x}_1), \end{aligned}$$

$$\begin{aligned} \text{wt}_0(x) &= 2x_0 - 2(x_1 + \bar{x}_1), \quad \text{wt}_1(x) = 2(x_1 + \bar{x}_1) - (x_0 + x_2 + \bar{x}_2), \\ \text{wt}_i(x) &= -(x_{i-1} + \bar{x}_{i-1}) + 2(x_i + \bar{x}_i) - (x_{i+1} + \bar{x}_{i+1}) \quad (i = 2, \dots, n-2) \\ \text{wt}_{n-1}(x) &= -(x_{n-2} + \bar{x}_{n-2}) + 2(x_{n-1} + \bar{x}_{n-1}) - x_n, \\ \text{wt}_n(x) &= 2x_n - 2(x_{n-1} + \bar{x}_{n-1}), \\ \varepsilon_0(x) &= -x_0 + 2(x_1 + \bar{x}_1 - x_0)_+, \\ \varepsilon_i(x) &= x_{i-1} - x_i + (x_{i+1} + \bar{x}_{i+1} - x_i - \bar{x}_i)_+ \quad (i = 1, \dots, n-2), \\ \varepsilon_{n-1}(x) &= x_{n-2} - x_{n-1} + (x_n - x_{n-1} - \bar{x}_{n-1})_+, \quad \varepsilon_n(x) = 2x_{n-1} - x_n. \end{aligned}$$

Comparing the result in §3.9 and the above formulae, we see that the following map gives an isomorphism of $D_{n+1}^{(2)}$ -crystals:

$$\begin{aligned} \mu: \mathcal{B}(C_n^{(1)}) &\longrightarrow B_\infty(D_{n+1}^{(2)}) \\ (x_0, x_1, \dots, x_n, \dots, \bar{x}_1) &\mapsto (b_1, \dots, b_n, \bar{b}_n, \dots, \bar{b}_1), \end{aligned}$$

where

$$\begin{aligned} b_1 &= \bar{x}_1, \quad b_i = \bar{x}_i - \bar{x}_{i-1} \quad (i = 2, \dots, n-1), \quad b_n = \frac{1}{2}x_n - \bar{x}_{n-1}, \\ \bar{b}_n &= x_{n-1} - \frac{1}{2}x_n, \quad \bar{b}_i = x_{i-1} - x_i \quad (i = 1, \dots, n-1). \end{aligned}$$

We have proved Theorem 6.1 for $C_n^{(1)}$.

6.4. $D_n^{(1)}$ ($n \geq 4$). Let us see the crystal structure of $\mathcal{UD}(\mathcal{V}(D_n^{(1)})) = (\mathcal{B}(D_n^{(1)}), \{\tilde{e}_i\}, \{\text{wt}_i\}, \{\varepsilon_i\})$. We have $\mathcal{B}(D_n^{(1)}) = \mathbb{Z}^{2n-2}$ and for $x = (x_1, \dots, x_n, \bar{x}_{n-2}, \dots, \bar{x}_1) \in \mathcal{B}(D_n^{(1)})$, we have

$$\begin{aligned} \tilde{e}_0(x) &= \begin{cases} (x_1, x_2 - 1, \dots, x_n - 1, \dots, \bar{x}_2 - 1, \bar{x}_1 - 1) & \text{if } x_1 + \bar{x}_1 \geq x_2 + \bar{x}_2, \\ (x_1 - 1, x_2 - 1, \dots, x_n - 1, \dots, \bar{x}_2 - 1, \bar{x}_1) & \text{if } x_1 + \bar{x}_1 < x_2 + \bar{x}_2, \end{cases} \\ \tilde{e}_i(x) &= \begin{cases} (x_1, \dots, x_i + 1, \dots, \bar{x}_i, \dots, \bar{x}_1) & \text{if } x_i + \bar{x}_i \geq x_{i+1} + \bar{x}_{i+1}, \\ (x_1, \dots, x_i, \dots, \bar{x}_i + 1, \dots, \bar{x}_1) & \text{if } x_i + \bar{x}_i < x_{i+1} + \bar{x}_{i+1}, \end{cases} \\ &\quad (i = 1, \dots, n-3), \\ \tilde{e}_{n-2}(x) &= \begin{cases} (x_1, \dots, x_{n-2} + 1, \dots, \bar{x}_{n-2}, \dots, \bar{x}_1) & \text{if } x_{n-2} + \bar{x}_{n-2} \geq x_{n-1} + x_n, \\ (x_1, \dots, x_{n-2}, \dots, \bar{x}_{n-2} + 1, \dots, \bar{x}_1) & \text{if } x_{n-2} + \bar{x}_{n-2} < x_{n-1} + x_n, \end{cases} \\ \tilde{e}_{n-1}(x) &= (x_1, \dots, x_{n-1} + 1, x_n, \dots, \bar{x}_1), \\ \tilde{e}_n(x) &= (x_1, \dots, x_{n-1}, x_n + 1, \dots, \bar{x}_1), \\ \text{wt}_0(x) &= -(x_2 + \bar{x}_2), \quad \text{wt}_1(x) = 2(x_1 + \bar{x}_1) - (x_2 + \bar{x}_2), \\ \text{wt}_i(x) &= -(x_{i-1} + \bar{x}_{i-1}) + 2(x_i + \bar{x}_i) - (x_{i+1} + \bar{x}_{i+1}) \quad (i = 2, \dots, n-3), \\ \text{wt}_{n-2}(x) &= -(x_{n-3} + \bar{x}_{n-3}) + 2(x_{n-2} + \bar{x}_{n-2}) - (x_{n-1} + x_n), \\ \text{wt}_{n-1}(x) &= -(x_{n-2} + \bar{x}_{n-2}) + 2(x_{n-1} + \bar{x}_{n-1}), \quad \text{wt}_n(x) = 2x_n - (x_{n-2} + \bar{x}_{n-2}), \\ \varepsilon_0(x) &= \max(x_1 + \bar{x}_1, x_2 + \bar{x}_2) - x_1 = \bar{x}_1 + (x_2 + \bar{x}_2 - x_1 - \bar{x}_1)_+, \\ \varepsilon_1(x) &= (x_2 + \bar{x}_2 - x_1 - \bar{x}_1)_+ - x_1, \\ \varepsilon_i(x) &= x_{i-1} - x_i + (x_{i+1} + \bar{x}_{i+1} - x_i - \bar{x}_i)_+ \quad (i = 2, \dots, n-3), \\ \varepsilon_{n-2}(x) &= x_{n-3} - x_{n-2} + (x_{n-1} + x_n - x_{n-2} - \bar{x}_{n-2})_+, \\ \varepsilon_{n-1}(x) &= x_{n-2} - x_{n-1}, \quad \varepsilon_n(x) = x_{n-2} - x_n. \end{aligned}$$

Then the following map gives an isomorphism of $D_n^{(1)}$ -crystals:

$$\begin{aligned} \mu: \mathcal{B}(D_n^{(1)}) &\longrightarrow B_\infty(D_n^{(1)}) \\ (x_1, \dots, x_{n-1}, x_n, \bar{x}_{n-2}, \dots, \bar{x}_1) &\mapsto (b_1, \dots, b_n, \bar{b}_{n-1}, \dots, \bar{b}_1), \end{aligned}$$

where

$$\begin{aligned} b_1 &= \bar{x}_1, \quad b_i = \bar{x}_i - \bar{x}_{i-1} \quad (i = 2, \dots, n-2), \quad b_{n-1} = x_{n-1} - \bar{x}_{n-2}, \\ b_n &= x_n - x_{n-1}, \quad \bar{b}_{n-1} = x_{n-2} - x_n, \quad \bar{b}_i = x_{i-1} - x_i \quad (i = 2, \dots, n-2), \\ \bar{b}_1 &= -\sum_{i=1}^n b_i - \sum_{i=2}^{n-1} \bar{b}_i = -x_1. \end{aligned}$$

We have proved Theorem 6.1 for $D_n^{(1)}$.

6.5. $A_{2n-1}^{(2)}$ ($n \geq 3$). The crystal structure of $\mathcal{UD}(\mathcal{V}(A_{2n-1}^{(2)})) = (\mathcal{B}(A_{2n-1}^{(2)}), \{\tilde{e}_i\}, \{\text{wt}_i\}, \{\varepsilon_i\})$ is given by: $\mathcal{B}(A_{2n-1}^{(2)}) = \mathbb{Z}^{2n-1}$ and for $x = (x_1, \dots, x_n, \bar{x}_{n-1}, \dots, \bar{x}_1) \in \mathcal{B}(A_{2n-1}^{(2)})$, we have

$\mathcal{B}(A_{2n-1}^{(2)})$, we have

$$\begin{aligned} \tilde{e}_0(x) &= \begin{cases} (x_1, x_2 - 1, \dots, x_n - 2, \dots, \bar{x}_2 - 1, \bar{x}_1 - 1) & \text{if } x_1 + \bar{x}_1 \geq x_2 + \bar{x}_2, \\ (x_1 - 1, x_2 - 1, \dots, x_n - 2, \dots, \bar{x}_2 - 1, \bar{x}_1) & \text{if } x_1 + \bar{x}_1 < x_2 + \bar{x}_2, \end{cases} \\ \tilde{e}_i(x) &= \begin{cases} (x_1, \dots, x_i + 1, \dots, \bar{x}_i, \dots, \bar{x}_1) & \text{if } x_i + \bar{x}_i \geq x_{i+1} + \bar{x}_{i+1}, \\ (x_1, \dots, x_i, \dots, \bar{x}_i + 1, \dots, \bar{x}_1) & \text{if } x_i + \bar{x}_i < x_{i+1} + \bar{x}_{i+1}, \end{cases} \\ &\quad (i = 1, \dots, n-2), \\ \tilde{e}_{n-1}(x) &= \begin{cases} (x_1, \dots, x_{n-1} + 1, x_n, \bar{x}_{n-1}, \dots, \bar{x}_1) & \text{if } x_{n-1} + \bar{x}_{n-1} \geq x_n, \\ (x_1, \dots, x_{n-1}, x_n, \bar{x}_{n-1} + 1, \dots, \bar{x}_1) & \text{if } x_{n-1} + \bar{x}_{n-1} < x_n, \end{cases} \\ \tilde{e}_n(x) &= (x_1, \dots, x_{n-1}, x_n + 1, \bar{x}_{n-1}, \dots, \bar{x}_1), \\ \text{wt}_0(x) &= -(x_2 + \bar{x}_2), \quad \text{wt}_1(x) = 2(x_1 + \bar{x}_1) - (x_2 + \bar{x}_2), \\ \text{wt}_i(x) &= -(x_{i-1} + \bar{x}_{i-1}) + 2(x_i + \bar{x}_i) - (x_{i+1} + \bar{x}_{i+1}) \quad (i = 2, \dots, n-2), \\ \text{wt}_{n-1}(x) &= -(x_{n-2} + \bar{x}_{n-2}) + 2(x_{n-1} + \bar{x}_{n-1}) - x_n, \\ \text{wt}_n(x) &= 2x_n - 2(x_{n-1} + \bar{x}_{n-1}), \\ \varepsilon_0(x) &= \max(x_1 + \bar{x}_1, x_2 + \bar{x}_2) - x_1 = \bar{x}_1 + (x_2 + \bar{x}_2 - x_1 - \bar{x}_1)_+, \\ \varepsilon_1(x) &= (x_2 + \bar{x}_2 - x_1 - \bar{x}_1)_+ - x_1, \\ \varepsilon_i(x) &= x_{i-1} - x_i + (x_{i+1} + \bar{x}_{i+1} - x_i - \bar{x}_i)_+ \quad (i = 2, \dots, n-2), \\ \varepsilon_{n-1}(x) &= x_{n-2} - x_{n-1} + (x_n - x_{n-1} - \bar{x}_{n-1})_+, \quad \varepsilon_n(x) = 2x_{n-1} - x_n. \end{aligned}$$

Then the following map gives an isomorphism of $B_n^{(1)}$ -crystals:

$$\begin{aligned} \mu: \mathcal{B}(A_{2n-1}^{(2)}) &\longrightarrow B_\infty(B_n^{(1)}) \\ (x_1, \dots, x_n, \dots, \bar{x}_1) &\mapsto (b_1, \dots, b_n, \bar{b}_n, \dots, \bar{b}_1), \end{aligned}$$

where

$$\begin{aligned} b_1 &= \bar{x}_1, \quad b_i = \bar{x}_i - \bar{x}_{i-1} \quad (i = 2, \dots, n-1), \quad b_n = \frac{1}{2}x_n - \bar{x}_{n-1}, \\ \bar{b}_n &= x_{n-1} - \frac{1}{2}x_n, \quad \bar{b}_i = x_{i-1} - x_i \quad (i = 2, \dots, n-1), \quad \bar{b}_1 = -x_1. \end{aligned}$$

We have proved Theorem 6.1 for $A_{2n-1}^{(2)}$.

6.6. $D_{n+1}^{(2)}$ ($n \geq 2$). Let us see the crystal structure of $\mathcal{UD}(\mathcal{V}(D_{n+1}^{(2)})) = (\mathcal{B}(D_{n+1}^{(2)}), \{\tilde{e}_i\}, \{\text{wt}_i\}, \{\varepsilon_i\})$. We have $\mathcal{B}(D_{n+1}^{(2)}) = \mathbb{Z}^{2n}$ and for $x = (x_0, x_1, \dots, x_n, \bar{x}_{n-1}, \dots, \bar{x}_1) \in \mathcal{B}(D_{n+1}^{(2)})$, we have

$$\tilde{e}_0(x) = \begin{cases} (x_0 + 1, x_1, \dots, x_n, \bar{x}_{n-1}, \dots, \bar{x}_2, \bar{x}_1) & \text{if } 2x_0 \geq x_1 + \bar{x}_1, \\ (x_0, x_1 - 1, \dots, x_n - 1, \bar{x}_{n-1} - 1, \dots, \bar{x}_1 - 1) & \text{if } 2x_0 + 1 = x_1 + \bar{x}_1, \\ (x_0 - 1, \dots, x_{n-1} - 2, x_n - 2, \bar{x}_{n-1} - 2, \dots, \bar{x}_2 - 2, \bar{x}_1 - 2) & \text{if } 2x_0 + 1 < x_1 + \bar{x}_1, \end{cases}$$

where we use the formula

$$\max(2+x, y) - \max(x, y) = \begin{cases} 2 & x \geq y, \\ 1 & x+1 = y \quad (x, y \in \mathbb{Z}), \\ 0 & x+1 < y, \end{cases}$$

$$\begin{aligned} \tilde{e}_i(x) &= \begin{cases} (x_1, \dots, x_i+1, \dots, \bar{x}_i, \dots, \bar{x}_1) & \text{if } x_i + \bar{x}_i \geq x_{i+1} + \bar{x}_{i+1}, \\ (x_1, \dots, x_i, \dots, \bar{x}_i+1, \dots, \bar{x}_1) & \text{if } x_i + \bar{x}_i < x_{i+1} + \bar{x}_{i+1}, \end{cases} \\ &\quad (i = 1, \dots, n-2), \\ \tilde{e}_{n-1}(x) &= \begin{cases} (x_1, \dots, x_{n-1}+1, x_n, \bar{x}_{n-1}, \dots, \bar{x}_1) & \text{if } x_{n-1} + \bar{x}_{n-1} \geq 2x_n, \\ (x_1, \dots, x_{n-1}, x_n, \bar{x}_{n-1}+1, \dots, \bar{x}_1) & \text{if } x_{n-1} + \bar{x}_{n-1} < 2x_n, \end{cases} \\ \tilde{e}_n(x) &= (x_1, \dots, x_{n-1}, x_n+1, \bar{x}_{n-1}, \dots, \bar{x}_1), \\ \text{wt}_0(x) &= 2x_0 - (x_1 + \bar{x}_1), \quad \text{wt}_1(x) = 2(x_1 + \bar{x}_1) - (2x_0 + x_2 + \bar{x}_2), \\ \text{wt}_i(x) &= -(x_{i-1} + \bar{x}_{i-1}) + 2(x_i + \bar{x}_i) - (x_{i+1} + \bar{x}_{i+1}) \quad (i = 2, \dots, n-2), \\ \text{wt}_{n-1}(x) &= -(x_{n-2} + \bar{x}_{n-2}) + 2(x_{n-1} + \bar{x}_{n-1}) - 2x_n, \\ \text{wt}_n(x) &= 2x_n - (x_{n-1} + \bar{x}_{n-1}), \\ \varepsilon_0(x) &= -x_0 + (x_1 + \bar{x}_1 - 2x_0)_+, \quad \varepsilon_1(x) = 2x_0 - x_1 + (x_2 + \bar{x}_2 - x_1 - \bar{x}_1)_+, \\ \varepsilon_i(x) &= x_{i-1} - x_i + (x_{i+1} + \bar{x}_{i+1} - x_i - \bar{x}_i)_+ \quad (i = 2, \dots, n-2), \\ \varepsilon_{n-1}(x) &= x_{n-2} - x_{n-1} + (2x_n - x_{n-1} - \bar{x}_{n-1})_+, \quad \varepsilon_n(x) = x_{n-1} - x_n. \end{aligned}$$

Then the following map gives an isomorphism of $C_n^{(1)}$ -crystals:

$$\begin{aligned} \mu: \mathcal{B}(D_{n+1}^{(2)}) &\longrightarrow B_\infty(C_n^{(1)}) \\ (x_0, x_1, \dots, x_n, \dots, \bar{x}_1) &\mapsto (b_1, \dots, b_n, \bar{b}_n, \dots, \bar{b}_1), \end{aligned}$$

where

$$b_1 = \bar{x}_1, \quad b_i = \bar{x}_i - \bar{x}_{i-1} \quad (i = 2, \dots, n), \quad \bar{b}_i = x_{i-1} - x_i \quad (i = 2, \dots, n), \quad \bar{b}_1 = 2x_0 - x_1.$$

We have proved Theorem 6.1 for $D_{n+1}^{(2)}$.

6.7. $A_{2n}^{(2)\dagger}$ ($n \geq 2$). Let $\mathcal{V}(A_{2n}^{(2)\dagger})$ be the affine $A_{2n}^{(2)\dagger}$ -geometric crystal as in §5.9. We shall see that the crystal structure of $\mathcal{UD}(\mathcal{V}(A_{2n}^{(2)\dagger})) = (\mathcal{B}(A_{2n}^{(2)\dagger}), \{\tilde{e}_i\}, \{\text{wt}_i\}, \{\varepsilon_i\})$ is given by: $\mathcal{B}(A_{2n}^{(2)\dagger}) = \mathbb{Z}^{2n}$ and for $x = (x_0, x_1, \dots, x_n, \bar{x}_{n-1}, \dots, \bar{x}_1) \in \mathcal{B}(A_{2n}^{(2)\dagger})$, we have

$$\begin{aligned} \tilde{e}_0(x) &= \begin{cases} (x_0+1, x_1, \dots, x_n, \bar{x}_{n-1}, \dots, \bar{x}_2, \bar{x}_1) & \text{if } x_0 \geq x_1 + \bar{x}_1, \\ (x_0-1, x_1-1, \dots, x_{n-1}-1, x_n-1, \bar{x}_{n-1}-1, \dots, \bar{x}_1-1) & \text{if } x_0 < x_1 + \bar{x}_1, \end{cases} \\ \tilde{e}_i(x) &= \begin{cases} (x_1, \dots, x_i+1, \dots, \bar{x}_i, \dots, \bar{x}_1) & \text{if } x_i + \bar{x}_i \geq x_{i+1} + \bar{x}_{i+1}, \\ (x_1, \dots, x_i, \dots, \bar{x}_i+1, \dots, \bar{x}_1) & \text{if } x_i + \bar{x}_i < x_{i+1} + \bar{x}_{i+1}, \end{cases} \\ &\quad (i = 1, \dots, n-2), \\ \tilde{e}_{n-1}(x) &= \begin{cases} (x_1, \dots, x_{n-1}+1, x_n, \bar{x}_{n-1}, \dots, \bar{x}_1) & \text{if } x_{n-1} + \bar{x}_{n-1} \geq 2x_n, \\ (x_1, \dots, x_{n-1}, x_n, \bar{x}_{n-1}+1, \dots, \bar{x}_1) & \text{if } x_{n-1} + \bar{x}_{n-1} < 2x_n, \end{cases} \\ \tilde{e}_n(x) &= (x_1, \dots, x_{n-1}, x_n+1, \bar{x}_{n-1}, \dots, \bar{x}_1), \end{aligned}$$

$$\begin{aligned}
 \text{wt}_0(x) &= 2x_0 - 2(x_1 + \bar{x}_1), & \text{wt}_1(x) &= 2(x_1 + \bar{x}_1) - (x_0 + x_2 + \bar{x}_2), \\
 \text{wt}_i(x) &= -(x_{i-1} + \bar{x}_{i-1}) + 2(x_i + \bar{x}_i) - (x_{i+1} + \bar{x}_{i+1}) \quad (i = 2, \dots, n-2), \\
 \text{wt}_{n-1}(x) &= -(x_{n-2} + \bar{x}_{n-2}) + 2(x_{n-1} + \bar{x}_{n-1}) - 2x_n, \\
 \text{wt}_n(x) &= 2x_n - (x_{n-1} + \bar{x}_{n-1}), \\
 \varepsilon_0(x) &= -x_0 + 2(x_1 + \bar{x}_1 - x_0)_+, \\
 \varepsilon_i(x) &= x_{i-1} - x_i + (x_{i+1} + \bar{x}_{i+1} - x_i - \bar{x}_i)_+ \quad (i = 1, \dots, n-2), \\
 \varepsilon_{n-1}(x) &= x_{n-2} - x_{n-1} + (2x_n - x_{n-1} - \bar{x}_{n-1})_+, & \varepsilon_n(x) &= x_{n-1} - x_n.
 \end{aligned}$$

Then the following map gives an isomorphism of $A_{2n}^{(2)}$ -crystals:

$$\begin{aligned}
 \mu: \mathcal{B}(A_{2n}^{(2)\dagger}) &\longrightarrow B_\infty(A_{2n}^{(2)}) \\
 (x_0, x_1, \dots, x_n, \dots, \bar{x}_1) &\mapsto (b_1, \dots, b_n, \bar{b}_n, \dots, \bar{b}_1),
 \end{aligned}$$

where

$$\begin{aligned}
 b_1 &= \bar{x}_1, & b_i &= \bar{x}_i - \bar{x}_{i-1} \quad (i = 2, \dots, n-1), & b_n &= x_n - \bar{x}_{n-1}, \\
 \bar{b}_i &= x_{i-1} - x_i \quad (i = 1, \dots, n).
 \end{aligned}$$

We have proved Theorem 6.1 for $A_{2n}^{(2)\dagger}$.

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RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KITASHIWAKAWA,
SAKYO-KU, KYOTO 606, JAPAN

E-mail address: masaki@kurims.kyoto-u.ac.jp

DEPARTMENT OF MATHEMATICS, SOPHIA UNIVERSITY, KIOICHO 7-1, CHiyODA-KU, TOKYO 102-
8554, JAPAN

E-mail address: toshiki@mm.sophia.ac.jp

DEPARTMENT OF MATHEMATICAL SCIENCE, GRADUATE SCHOOL OF ENGINEERING SCIENCE,
OSAKA UNIVERSITY, TOYONAKA, OSAKA 560-8531, JAPAN

E-mail address: okado@sigmath.es.osaka-u.ac.jp