MULTISCALE-BUMP STANDING WAVES WITH A CRITICAL FREQUENCY FOR NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. In this paper we study the existence and qualitative property of standing wave solutions $\psi(x, t) = e^{-\frac{iEt}{\hbar}}u(x)$ for the nonlinear Schrödinger equation $i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \Delta \psi - W(x)\psi + |\psi|^{p-1}\psi = 0$ with E being a critical frequency in the sense that $\inf_{x \in \mathbb{R}^N} W(x) = E$. We show that if the zero set of $V = W - E$ has k isolated connected components $Z_i (i = 1, \dots, k)$ such that the interior of Z_i is not empty and ∂Z_i is smooth, V has t isolated zero points, $b_i, i = 1, \dots, t$, and V has l critical points $a_i (i = 1, \dots, l)$ such that $V(a_i) > 0$, then for $\hbar > 0$ small, there exists a standing wave solution which is trapped in a neighborhood of $\bigcup_{i=1}^k Z_i \cup (\bigcup_{i=1}^t \{b_i\}) \cup (\bigcup_{i=1}^l \{a_i\})$. Moreover the amplitudes of the standing wave around $\bigcup_{i=1}^k Z_i, \bigcup_{i=1}^t \{b_i\}$ and $\bigcup_{i=1}^l \{a_i\}$ are of a different order of $\hbar$. This type of multi-scale solution has never before been obtained.

1. INTRODUCTION

The study of the Schrödinger equation is one of the main objects of quantum physics. The evolution of a group of identical particles interacting with each other in ultra-cold states, in particular, Bose-Einstein condensates, is described via Hartree approximation, to an excellent degree of accuracy by nonlinear Schrödinger equations. The equation also arises in many fields of physics. For instance, when we describe the propagation of light in some nonlinear optical materials, the nonlinear Schrödinger equations in nonlinear optics are reduced from Maxwell's equations. The nonlinear Schrödinger equation is typically the form

$$(1.1) \quad i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + W(x)\psi - |\psi|^{p-1}\psi, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

where $\hbar$ denotes the Plank constant, i is the imaginary unit, $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$ and $1 < p < +\infty$ for $N = 1, 2$. In physical problems, a cubic nonlinearity, $p = 3$, is common. In this case, the equation is called the Gross-Pitaevskii equation. In this paper, we consider the existence of standing waves of equation (1.1), namely of special solutions of the form $\psi(x, t) = e^{-\frac{iEt}{\hbar}}u(x)$, where $u(x) > 0$ for $x \in \mathbb{R}^N$,

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for small $\hbar > 0$. The transition from quantum mechanics to classical mechanics can be formally performed by letting $\hbar \rightarrow 0$, and thus the existence of solutions for $\hbar$ small has important physical interest. Standing wave solutions of (1.1) for small $\hbar$ are usually referred to as semi-classical bound states. It is easily checked that ψ of this form satisfies equation (1.1) if and only if the function u satisfies

$$(1.2) \quad \frac{\hbar^2}{2m} \Delta u - (W(x) - E)u + |u|^{p-1}u = 0, \quad x \in \mathbb{R}^N.$$

In this paper, we investigate problem (1.2) when

$$\inf_{x \in \mathbb{R}^N} W(x) = E.$$

Under the condition that

$$\inf_{x \in \mathbb{R}^N} W(x) > E,$$

Rabinowitz proved in [30] that problem (1.2) has a ground state solution (mountain-pass solution) for $\hbar > 0$ small when $\inf_{x \in \mathbb{R}^N} W(x) < \liminf_{|x| \rightarrow \infty} W(x)$; furthermore, the following well-understood problem in $\mathbb{R}^N$ plays a crucial role in the construction of solutions of (1.2) and is considered the limiting equation for (1.2) as $\hbar \rightarrow 0$:

$$(1.3) \quad \Delta u - u + |u|^{p-1}u = 0, \quad x \in \mathbb{R}^N.$$

On the other hand, for

$$\inf_{x \in \mathbb{R}^N} W(x) < E,$$

it is not difficult to see that problem (1.2) has no ground-state solutions (mountain-pass solutions) if $\hbar > 0$ is sufficiently small. In this sense, $E = \inf_{x \in \mathbb{R}^N} W(x)$ is called a critical frequency (or energy) for the nonlinear Schrödinger equation (1.1), or problem (1.2), in [4] by Byeon and Wang.

Under the condition $\inf_{x \in \mathbb{R}^N} W(x) > E$, there have been enormous investigations on problem (1.2). In [18], Floer and Weinstein considered the case $N = 1$ and $p = 3$. Assuming that W is globally bounded, for a given nondegenerate critical point of the potential W , they constructed a positive solution $u_\hbar$ to (1.2) for small $\hbar > 0$. The solutions concentrate near some critical points of $W(x)$ as $\hbar \rightarrow 0$. Their method, based on an interesting Lyapunov-Schmidt finite dimensional reduction, was extended by Oh in [28, 29] to include a similar result in higher dimensions, provided $1 < p < \frac{N+2}{N-2}$. Other existence results for positive solutions of problem (1.2) under the condition $\inf_{x \in \mathbb{R}^N} W(x) > E$ can be found in [1, 2, 3, 10, 12, 13, 14, 15, 16, 17, 20, 22, 24, 25, 27, 31, 32].

Uniqueness of solutions concentrating on several nondegenerate critical points of W is studied by Cao and Heinz in [8]. See also [11].

In all the above mentioned works, the authors use the ground state solution of the limiting equation (1.3) as a building block to construct single-bump or multi-bump solutions for (1.2) with each bump looking like a translated ground state solution.

It seems that Byeon and Wang [4] were the first to study energy level and the asymptotic behavior of positive solutions to problem (1.2) under the condition $\inf_{x \in \mathbb{R}^N} W(x) = E$. For an isolated component A of $\{x \in \mathbb{R}^N \mid W(x) = E\}$, satisfying

$\overline{\text{int}(A)} = A$, and $\text{int}(A) = \bigcup_{i \in I} A_i$, where $\{A_i\}_{i \in I}$ are connected components of

$\text{int}(A)$, they proved in [4] that there is a solution $u_{\hbar}$ such that $\left(\frac{\sqrt{2m}}{\hbar}\right)^{\frac{2}{p-1}} u_{\hbar}(x)$ converges to a function w as $\hbar \rightarrow 0$, where $w = 0$ for $x \in \mathbb{R}^N \setminus \Omega$, and $w|_{\Omega}$ is a ground state (least energy solution) of

$$(1.4) \quad \begin{cases} \Delta u + u^p = 0 & x \in \Omega, \\ u > 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Ω is one of A_i satisfying

$$\inf \left\{ \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} \mid u \in H_0^1(\Omega) \setminus \{0\}, \int_{\Omega} |\nabla u|^2 = \int_{\Omega} |u|^{p+1} \right\} \\ = \inf \{c_i \mid i \in I\}$$

with c_i defined by

$$(1.5) \quad c_i = \inf \left\{ \frac{1}{2} \int_{A_i} |\nabla u|^2 - \frac{1}{p+1} \int_{A_i} |u|^{p+1} \mid u \in H_0^1(A_i) \setminus \{0\}, \right. \\ \left. \int_{A_i} |\nabla u|^2 = \int_{A_i} |u|^{p+1} \right\}.$$

It is easy to see that the height of the bump for $u_{\hbar}$ tends to zero as $\hbar \rightarrow 0$. This is in striking contrast to the case $\inf_{x \in \mathbb{R}^N} W(x) > E$.

In [4], Byeon and Wang also proved the existence of solutions concentrating on an isolated zero point b of $V(x)$. Their result shows that the height of the bump of the solution concentrating on b and that of the solution concentrating on Ω are of different order.

Results on the existence of localized solutions of [4] are extended in [5] to more general nonlinearities.

The solutions obtained in [4] have one bump. In [9], Cao and Noussair obtained the existence of solutions concentrating simultaneously on several sets of $\{A_i \mid i \in I\}$, that is, solutions $u_{\hbar}$ such that $\hbar^{-\frac{2}{p-1}} u_{\hbar}(x) \rightarrow w$ as $\hbar \rightarrow 0$ and w satisfies $w = 0$ for $\mathbb{R}^N \setminus (\bigcup_{j=1}^k \Omega_j)$ with Ω_j being one set of $\{A_i \mid i \in I\}$ and $w|_{\Omega_j} (j = 1, \dots, k)$ a positive solution of (1.4) with Ω replaced by Ω_j provided $\hbar$ is small enough. The energy of $u_{\hbar}$ is approximately $\hbar^{\frac{2(p+1)}{p-1}} \sum_{i=1}^k c_i$, where c_i is defined by (1.5) with A_i replaced by Ω_i . Although the solution $u_{\hbar}$ has several bumps, the height of the bumps are of the same order.

The aim of this paper is to construct solutions such that they have several bumps in $\mathbb{R}^N$ and the heights of the bumps are of different order. In order to state the result precisely, we need to give some notation first.

Set $V(x) = W(x) - E$, $\varepsilon = \frac{\hbar}{\sqrt{2m}}$. Then (1.2) becomes

$$(1.6) \quad \begin{cases} -\varepsilon^2 \Delta u + V(x)u = |u|^{p-1}u = 0, & x \in \mathbb{R}^N, \\ \lim_{|x| \rightarrow \infty} u(x) = 0, \end{cases}$$

where $p \in \left(1, \frac{N+2}{N-2}\right)$ for $N \geq 3$, and $p \in (1, +\infty)$ for $N = 1, 2$.

For any given set $A \subset \mathbb{R}^N$, $r > 0$, define $A^r = \{x \in \mathbb{R}^N \mid \text{dist}(x, A) \leq r\}$ and $A_{\varepsilon}^r = \{x \in \mathbb{R}^N \mid \varepsilon x \in A^r\}$.

In this paper, we always assume that the potential V satisfies the following conditions:

$$(\mathbf{V1}) \quad V \in C^{2,\alpha}(\mathbb{R}^N), \inf_{x \in \mathbb{R}^N} V(x) = 0, 0 < V_0 = \lim_{|x| \rightarrow \infty} \inf V(x).$$

Let

$$\mathcal{Z} = \{x \in \mathbb{R}^N \mid V(x) = 0\}.$$

The potential $V(x)$ may have different behavior near different points in $\mathcal{Z}$. In this paper, we are particularly interested in the following two cases.

The first case is that $\mathcal{Z}$ contains some connected components with nonempty interior. More precisely, we assume that

(V2-1) $\mathcal{Z}$ has k isolated components Ω_i ($i = 1, \dots, k$) such that the interior of each Ω_i , denoted by $\text{int}(\Omega_i)$, is not empty, and $\Omega_i = \text{int}(\Omega_i)$. Furthermore, for some $r > 0$, $\Omega_i^{8r} \cap (\mathcal{Z} \setminus \Omega_i) = \emptyset$;

(V2-2) For each Ω_i ($i = 1, \dots, k$), the problem

$$(1.7) \quad \begin{cases} -\Delta u = |u|^{p-1}u, & x \in \Omega_i, \\ u(x) = 0, & \partial\Omega_i \end{cases}$$

has a unique positive solution W_i . Moreover W_i is nondegenerate, that is,

$$\begin{cases} -\Delta u = pW_i^{p-1}u, & x \in \Omega_i, \\ u(x) = 0, & x \in \partial\Omega_i, \end{cases}$$

has only a trivial solution $u \equiv 0$.

Remark 1.1. If Ω_i is a ball for each $i = 1, \dots, k$, then **(V2-2)** holds (see [21]). If p is close to $\frac{N+2}{N-2}$ for $N \geq 3$, then by a result of Grossi [20], **(V2-2)** holds if Ω_i is convex and symmetric with respect to the coordinate plane.

The second case is that $\mathcal{Z}$ contains some isolated zero points $\{b_1, \dots, b_t\}$ of $V(x)$. We assume

(V3) There are points $b_i \in \mathcal{Z}$, $i = 1, \dots, t$, such that

$$V(x) = |x - b_i|^{m_i} + Q(x - b_i) \quad \text{for } x \text{ close to } b_i,$$

where $m_i > 0$ is a real number, $\lim_{|x| \rightarrow 0} Q(x)|x|^{-m_i} = 0$.

By a result of [7], the problem

$$(1.8) \quad \begin{cases} -\Delta u + |x|^{m_i}u = u^p, & x \in \mathbb{R}^N, \\ \lim_{|x| \rightarrow \infty} u(x) = 0 \end{cases}$$

has a unique positive solution $\widehat{W}_{m_i}$. Furthermore, $\widehat{W}_{m_i}$ is nondegenerate, that is,

$$\begin{cases} -\Delta u + |x|^{m_i}u = p\widehat{W}_{m_i}^{p-1}u, & x \in \mathbb{R}^N, \\ \lim_{|x| \rightarrow \infty} u(x) = 0 \end{cases}$$

has only a trivial solution $u \equiv 0$.

Let us recall some known results about (1.6). When **(V1)** **(V2-1)** are satisfied, then for each Ω_i , by a result of Byeon and Wang [4], (1.6) has a solution $W_{\varepsilon,i}$ for

ε small. $W_{\varepsilon,i}$ satisfies that as $\varepsilon \rightarrow 0$,

$$(1.9) \quad \varepsilon^{-\frac{2}{p-1}} W_{\varepsilon,i}(x) \rightarrow W_i(x) \text{ in } C^{1,\alpha}(\Omega_i) \text{ and } H^1(\mathbb{R}^N),$$

$$(1.10) \quad W_{\varepsilon,i}(x) \leq C e^{-\gamma_0 \text{dist}(x, \Omega_i)/\varepsilon} \text{ for } x \in \mathbb{R}^N,$$

$$(1.11) \quad \int_{\mathbb{R}^N} (|\nabla W_{\varepsilon,i}|^2 + W_{\varepsilon,i}^{p+1}) \leq C \varepsilon^{\frac{2(p+1)}{p-1}},$$

where $C > 0, \gamma_0 > 0$ are some constants. Here, we assume that $W_i = 0$ in $\mathbb{R}^N \setminus \Omega_i$.

When V satisfies **(V1)**–**(V3)**, following a result of Byeon and Wang [4], (1.6) has a positive solution $\widetilde{W}_{\varepsilon,i}(x)$. As $\varepsilon \rightarrow 0$, $\widetilde{W}_{\varepsilon,i}(x)$ satisfies

$$(1.12) \quad \widehat{W}_{\varepsilon,i}(x) =: \varepsilon^{-\frac{2m_i}{(p-1)(m_i+2)}} \widetilde{W}_{\varepsilon,i}(\varepsilon^{\frac{2}{m_i+2}}x + b_i) \rightarrow \widehat{W}_{m_i}(x), \text{ in } H^1(\mathbb{R}^N),$$

where $\widehat{W}_{m_i}$ is the positive solution of (1.8). Furthermore

$$(1.13) \quad \widetilde{W}_{\varepsilon,i}(x) \leq C e^{-\gamma_1|x-b_i|/\varepsilon} \text{ for } x \in \mathbb{R}^N,$$

$$(1.14) \quad \int_{\mathbb{R}^N} (|\nabla \widetilde{W}_{\varepsilon,i}|^2 + \widetilde{W}_{\varepsilon,i}^{p+1}) \leq C \varepsilon^{\frac{2m_i(p+1)}{(p-1)(m_i+2)} + \frac{2N}{m_i+2}},$$

where $C > 0$ and $\gamma_1 > 0$ are some constants.

Since $V(x) \geq 0$, every point in $\mathcal{Z}$ is a critical point of V . Of course, V may have critical points, where V is positive. We assume

(V4) There are $\{a_1, \dots, a_l\}$, such that $V(a_i) > 0$, $i = 1, \dots, l$, and a_i is a nondegenerate critical point of V , $i = 1, \dots, l$. That is, the matrix $(\nabla^2 V(a_i))$ is invertible.

It is well known that

$$(1.15) \quad \begin{cases} -\Delta u + V(a_i)u = u^p, & x \in \mathbb{R}^N, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty \end{cases}$$

has a unique positive solution which will be denoted by $U_i(x)$. $U_i(x)$ is a radial, decreasing function and satisfies

$$(1.16) \quad \begin{cases} U_i(x) e^{\sqrt{V(a_i)}|x|} |x|^{\frac{N-1}{2}} \rightarrow c_i > 0, \\ \nabla U_i(x) \cdot \frac{x}{|x|} e^{\sqrt{V(a_i)}|x|} |x|^{\frac{N-1}{2}} \rightarrow -c_i, \end{cases}$$

as $|x| \rightarrow +\infty$.

Define the inner product of $H^1(\mathbb{R}^N)$ by

$$\langle u, v \rangle_\varepsilon = \int_{\mathbb{R}^N} (\varepsilon^2 \nabla u \nabla v + V(x)uv)$$

and the norm induced by $\langle \cdot, \cdot \rangle_\varepsilon$ is denoted by $\|\cdot\|_\varepsilon$.

We will use the convention that if $l = 0$, then the summation from 0 to l is zero. Now we are ready to state our main results.

Theorem 1.1. *Suppose that **(V1)** holds. Let $\Omega_1, \dots, \Omega_k$ be the connected components of $\mathcal{Z}$, such that **(V2-1)** and **(V2-2)** hold. Let $b_1, \dots, b_t$ be the points such that **(V3)** holds. Let $a_1, \dots, a_l$ be the points such that **(V4)** holds. Then for ε small, (1.6) has a positive solution u_ε such that*

$$u_\varepsilon(x) = \sum_{i=1}^k W_{\varepsilon,i}(x) + \sum_{i=1}^t \widetilde{W}_{\varepsilon,i} + \sum_{i=1}^l U\left(\frac{x - y_{\varepsilon,i}}{\varepsilon}\right) + \omega_\varepsilon(x),$$

where $\omega_\varepsilon \in H^1(\mathbb{R}^N)$ satisfying

$$(1.17) \quad |\omega_\varepsilon(x)| \leq Ce^{-\sigma/\varepsilon}, \quad x \in \mathbb{R}^N \setminus \bigcup_{i=1}^l B_\delta(a_i),$$

for some small constants $\sigma > 0$ and $\delta > 0$,

$$(1.18) \quad \|\omega_\varepsilon\|_\varepsilon = O(\varepsilon^{\frac{N}{2}+2}), \quad \|\omega_\varepsilon\|_{L^\infty} = o(1),$$

$$(1.19) \quad y_{\varepsilon,i} \rightarrow a_i, \quad i = 1, \dots, l, \quad \text{as } \varepsilon \rightarrow 0.$$

By the same argument for the proof of Theorem 1.1, we can prove the following special case (corresponding to $k = t = 0$ in Theorem 1.1).

Theorem 1.2. *Suppose that (V1) holds. Let $a_1, \dots, a_l$ be the points such that (V4) holds. Then for ε small, (1.6) has a positive solution u_ε such that*

$$u_\varepsilon(x) = \sum_{i=1}^l U\left(\frac{x - y_{\varepsilon,i}}{\varepsilon}\right) + \omega_\varepsilon(x),$$

where $\omega_\varepsilon \in H^1(\mathbb{R}^N)$ satisfying (1.17), (1.18) and (1.19) in Theorem 1.1.

By (1.9), (1.12), (1.16) and (1.17), u_ε obtained in Theorem 1.1 is multi-scale, because u_ε is approximately of level $\varepsilon^{\frac{2}{p-1}}$ around Ω_i , is approximately of level $\varepsilon^{\frac{2m_i}{(p-1)(m_i+2)}}$ around b_i , and u_ε has a positive lower bound independent of ε around a_i for small ε . Therefore the solution obtained in Theorem 1.1 takes the profile of $W_{\varepsilon,i}$, $\tilde{W}_{\varepsilon,i}$ and $U\left(\frac{x-y_{\varepsilon,i}}{\varepsilon}\right)$ respectively near the corresponding domains or points.

Let us point out that the result in Theorem 1.2, which generalizes the corresponding results of Oh in [28] and [29] to nonnegative potentials (a uniform positive lower bound is needed in [28] and [29]), is new. The proof of this result does not follow the reduction procedure in a standard way, because $\|u\|_\varepsilon$ cannot control the L^p norm of u uniformly if $\mathcal{Z}$ is not empty.

The only known results on the existence of solutions with multi-scale were obtained by Byeon and Oshita in [6], where they require $N \leq 5$ and only consider the cases that $p = 2, 3$ or $p \geq 4$ for $N = 1, 2$, $p = 2, 3$ or $4 \leq p < 5$ for $N = 3$ and $p = 2$ for $N = 4, 5$. In the present paper, we consider the case $N \geq 1$ and allow $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$, $1 < p < +\infty$ for $N = 1, 2$. Although in principle the method used in [6] and that used in the present paper is similar, the way of constructing the solutions is different. Roughly speaking, taking $k = 1, l = 1$ and $i = 1$ for instance, in [6] the main terms of solution were taken as

$$w_{\varepsilon,y} = W_{\varepsilon,1} + \tilde{W}_{\varepsilon,1} + U\left(\frac{x-y}{\varepsilon}\right) + \sum_{i=1}^4 \varepsilon^i U_{i,y}(x),$$

where $U_{i,y}$ are deduced inductively. The correction term $v_{\varepsilon,y}$ is then obtained so that $w_{\varepsilon,y} + v_{\varepsilon,y}$ is a real solution. To analyze the interaction between bumps and to obtain estimates for $v_{\varepsilon,y}$, expansion of $f(s) = |s|^{p-1}s$ up to the fourth order is used. This requires that $f \in C^4(\mathbb{R})$ (see page 1894 in [6]) and therefore p is assumed to satisfy that either p is an integer or $4 \leq p < 5$. To make sure that the solution obtained is nonnegative, they need to show that the L^2 norm of the negative part of the solution is sufficiently small provided it is not identically zero. To obtain this, $p \geq 2$ is assumed which requires that $N \leq 5$. In our paper we take $f(s) = s_+^p$

and we use a different way of constructing solutions. To be more precise, we take the main terms of the solution in the following way:

$$w_{\varepsilon,y} = W_{\varepsilon,1} + \tilde{W}_{\varepsilon,1} + U\left(\frac{x-y}{\varepsilon}\right).$$

Then we try to obtain a corresponding correction term ω_ε in a subset with small H^1 and L^∞ norm. Furthermore ω_ε should be exponentially small in Ω_1 and near b_1 (see (1.17) and (1.18)), which is necessary to guarantee that $w_{\varepsilon,y_\varepsilon} + \omega_\varepsilon$ is really a solution with multi-scale bumps. To obtain existence of such a correction term is the key point, and we take a lot of efforts to accomplish this.

We end this section by giving the outline of our proof of the main results.

For the simplicity of the notation we only prove the case $k = t = l = 1$. In this case, we drop all the subscripts i in Theorem 1.1. Thus, Theorem 1.1 can be stated as

Theorem 1.3. *Suppose that $k = t = l = 1$. Assume (V1), (V2), (V3) and (V4) hold. Then for ε small, (1.6) has a solution u_ε such that*

$$u_\varepsilon(x) = W_\varepsilon(x) + \tilde{W}_\varepsilon(x) + U\left(\frac{x-y_\varepsilon}{\varepsilon}\right) + \omega_\varepsilon(x),$$

where $\omega_\varepsilon(x)$ and y_ε satisfy (1.17), (1.18) and (1.19) in Theorem 1.1.

Let us give some notation.

Define $U_{\varepsilon,y}(x) = U\left(\frac{x-y}{\varepsilon}\right)$,

$$\begin{aligned} E_{\varepsilon,y} &= \left\{ \omega \in H^1(\mathbb{R}^N) \mid \left\langle \omega, \frac{\partial U_{\varepsilon,y}}{\partial x_i} \right\rangle_\varepsilon = 0, \ i = 1, \dots, N \right\}, \\ \Sigma_{\varepsilon,\delta,\sigma} &= \left\{ (y, \omega) \mid y \in B_\delta(a_1), \omega \in E_{\varepsilon,y}, \|\omega\|_\varepsilon \leq \varepsilon^{\frac{N}{2}+\sigma} \right\}, \\ \Phi_\varepsilon(x) &= W_\varepsilon(x) + \tilde{W}_\varepsilon(x). \end{aligned}$$

To prove Theorem 1.3, we will construct a solution of the form $u_\varepsilon = \Phi_\varepsilon(x) + u$, with u satisfying

$$(1.20) \quad \begin{cases} -\varepsilon^2 \Delta u + V(x)u = u_+^p + g_\varepsilon(x, u), & x \in \mathbb{R}^N, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty, \end{cases}$$

where

$$(1.21) \quad g_\varepsilon(x, t) = (\Phi_\varepsilon(x) + t)_+^p - W_\varepsilon^p(x) - \tilde{W}_\varepsilon^p(x) - t_+^p.$$

Set

$$\begin{aligned} G_\varepsilon(x, t) &= \int_0^t g_\varepsilon(x, \tau) d\tau, \\ I(u) &= \frac{1}{2} \int_{\mathbb{R}^N} (\varepsilon^2 |\nabla u|^2 + V(x)u^2) - \frac{1}{p+1} \int_{\mathbb{R}^N} u_+^{p+1} - \int_{\mathbb{R}^N} G_\varepsilon(x, u), \end{aligned}$$

and

$$J(y, \omega) = I(U_{\varepsilon,y} + \omega) \quad \text{for } (y, \omega) \in \Sigma_{\varepsilon,\delta,\sigma}.$$

It is known that for δ small $u_\varepsilon = U_{\varepsilon,y} + \omega$ is a critical point of I in $H^1(\mathbb{R}^N)$ if and only if (y, ω) is a critical point of J . A point (y, ω) is a critical point of J if

and only if there exist $D_j \in \mathbb{R}$ ($j = 1, \dots, N$) such that

$$(1.22) \quad \left\langle \frac{\partial J}{\partial \omega}, \varphi \right\rangle = \sum_{j=1}^N D_j \left\langle \frac{\partial U_{\varepsilon, y}}{\partial x_j}, \varphi \right\rangle_{\varepsilon} \quad \forall \varphi \in H^1(\mathbb{R}^N),$$

$$(1.23) \quad \frac{\partial J}{\partial y_j} = \sum_{i=1}^N D_i \left\langle \frac{\partial^2 U_{\varepsilon, y}}{\partial x_i \partial x_j}, \omega \right\rangle_{\varepsilon}.$$

Our strategy in proving Theorem 1.3 is first to find, for given $y \in B_{\delta}(a_1)$, $\omega_{\varepsilon, y} \in E_{\varepsilon, y}$ such that (1.22) is satisfied provided δ and ε are small. Then we try to obtain a suitable point y_{ε} in $B_{\delta}(a_1)$ so that (1.23) is satisfied as well. The main difficulties in the proof of the result are the following. First, we try to glue together bumps of different scales. To make sure that ω_{ε} is a small term of the all the bumps, we need to prove that the infinite norm of ω_{ε} is of higher order term of the heights of all the bumps. Second, as we pointed out earlier, if $\mathcal{Z}$ is not empty, the norm $\|u\|_{\varepsilon}$ cannot control the L^p norm of u uniformly. So we cannot carry out the reduction procedure in a standard way. In this paper, we achieve the first step by carrying out the reduction argument in a carefully chosen subset of $E_{\varepsilon, y}$, and we finish the second step by using a degree argument.

2. THE REDUCTION

As indicated in section 2, we will consider the positive solution of

$$(2.1) \quad \begin{cases} -\varepsilon^2 \Delta u + V(x)u = u_+^p + g_{\varepsilon}(x, u), & x \in \mathbb{R}^N, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty. \end{cases}$$

We will reduce the problem of finding a solution for (2.1) concentrating in near a_1 to a finite dimensional problem this section.

In what follows, we assume that $b_1 = 0$.

Expanding $J(y, \omega)$ at $\omega \equiv 0$, we get

$$(2.2) \quad J(y, \omega) = J(y, 0) + \ell_{\varepsilon}(\omega) + \frac{1}{2}Q_{\varepsilon}(\omega, \omega) + R_{\varepsilon}(\omega)$$

with

$$(2.3) \quad \ell_{\varepsilon}(\omega) = \langle U_{\varepsilon, y}, \omega \rangle_{\varepsilon} - \int_{\mathbb{R}^N} U_{\varepsilon, y}^p \omega - \int_{\mathbb{R}^N} g_{\varepsilon}(x, U_{\varepsilon, y}) \omega,$$

$$(2.4) \quad Q_{\varepsilon}(\omega, \eta) = \langle \omega, \eta \rangle_{\varepsilon} - p \int_{\mathbb{R}^N} U_{\varepsilon, y}^{p-1} \omega \eta - \int_{\mathbb{R}^N} g'_{\varepsilon}(x, U_{\varepsilon, y}) \omega \eta$$

for $\omega, \eta \in E_{\varepsilon, y}$,

$$(2.5) \quad \begin{aligned} R_{\varepsilon}(\omega) = & \frac{1}{p+1} \int_{\mathbb{R}^N} \left[(U_{\varepsilon, y} + \omega)_{+}^{p+1} - U_{\varepsilon, y}^{p+1} - (p+1)U_{\varepsilon, y}^p \omega - \frac{1}{2}(p+1)pU_{\varepsilon, y}^{p-1} \omega^2 \right] \\ & + \int_{\mathbb{R}^N} \left[G_{\varepsilon}(x, U_{\varepsilon, y} + \omega) - G_{\varepsilon}(x, U_{\varepsilon, y}) - g_{\varepsilon}(x, U_{\varepsilon, y}) \omega - \frac{1}{2}g'_{\varepsilon}(x, U_{\varepsilon, y}) \omega^2 \right]. \end{aligned}$$

Here and in what follows we always denote $g'_{\varepsilon}(x, t) = \frac{\partial g_{\varepsilon}(x, t)}{\partial t}$.

Our first preliminary result is the following

Lemma 2.1. *$\ell_{\varepsilon}(\omega)$ is a bounded linear functional and there exists a constant $C > 0$ such that*

$$(2.6) \quad |\ell_{\varepsilon}(\omega)| \leq C (|V(y) - V(x_0)| + \varepsilon |\nabla V(y)| + \varepsilon^2) \varepsilon^{\frac{N}{2}} \|\omega\|_{\varepsilon}.$$

Proof. From the equation satisfied by $U_{\varepsilon,y}$, we have

$$(2.7) \quad \ell_\varepsilon(\omega) = \int_{\mathbb{R}^N} (V(x) - V(x_0))U_{\varepsilon,y}\omega - \int_{\mathbb{R}^N} g_\varepsilon(x, U_{\varepsilon,y})\omega.$$

For the first term in (2.7), using the estimate

$$\int_{\mathbb{R}^N} \omega^2 \leq C \int_{\mathbb{R}^N} (|\nabla \omega|^2 + V(x)\omega^2) \leq C\varepsilon^{-2} \|\omega\|_\varepsilon^2,$$

we have

$$(2.8) \quad \begin{aligned} & \left| \int_{\mathbb{R}^N} (V(x) - V(x_0))U_{\varepsilon,y}\omega \right| \\ &= \left| \left(\int_{B_\delta(y)} + \int_{\mathbb{R}^N \setminus B_\delta(y)} \right) (V(x) - V(x_0))U_{\varepsilon,y}\omega \right| \\ &\leq |V(y) - V(x_0)| \left(\int_{B_\delta(y)} U_{\varepsilon,y}^2 \right)^{\frac{1}{2}} \left(\int_{B_\delta(y)} \omega^2 \right)^{\frac{1}{2}} \\ &\quad + \left| \int_{B_\delta(y)} (V(x) - V(y))U_{\varepsilon,y}\omega \right| + Ce^{-\gamma\delta/\varepsilon} \left(\int_{\mathbb{R}^N} \omega^2 \right)^{\frac{1}{2}} \\ &\leq C\varepsilon^{\frac{N}{2}} (|V(y) - V(x_0)| + \varepsilon|\nabla V(y)| + \varepsilon^2) \|\omega\|_\varepsilon, \end{aligned}$$

where $\gamma > 0$ is some constant.

On the other hand,

$$(2.9) \quad \begin{aligned} & \left| \int_{\mathbb{R}^N} g_\varepsilon(x, U_{\varepsilon,y})\omega \right| \\ &\leq C \int_{\mathbb{R}^N} \left[W_\varepsilon^{p-1}(U_{\varepsilon,y} + \widetilde{W}_\varepsilon) + U_{\varepsilon,y}^{p-1}(W_\varepsilon + \widetilde{W}_\varepsilon) + \widetilde{W}_\varepsilon^{p-1}(U_{\varepsilon,y} + W_\varepsilon) \right] |\omega| \\ &\leq Ce^{-\gamma/\varepsilon} \|\omega\|_\varepsilon \end{aligned}$$

for some $\gamma > 0$. It is easy to see that (2.6) follows from (2.8) and (2.9). $\square$

Lemma 2.2. Q_ε is a bounded bilinear operator on $E_{\varepsilon,y}$ and there is a positive constant C such that

$$(2.10) \quad |Q_\varepsilon(\omega, \eta)| \leq C\|\omega\|_\varepsilon\|\eta\|_\varepsilon.$$

Proof. With direct computations and using (1.16), we have

$$(2.11) \quad \begin{aligned} & \left| \int_{\mathbb{R}^N} U_{\varepsilon,y}^{p-1}\omega\eta \right| \\ &\leq \left| \left(\int_{B_\delta(y)} + \int_{\mathbb{R}^N \setminus B_\delta(y)} \right) U_{\varepsilon,y}^{p-1}\omega\eta \right| \\ &\leq \left(\int_{B_\delta(y)} V(x)\omega^2 \right)^{\frac{1}{2}} \left(\int_{B_\delta(y)} V(x)\eta^2 \right)^{\frac{1}{2}} + Ce^{-2\gamma/\varepsilon} \int_{\mathbb{R}^N \setminus B_\delta(y)} |\omega||\eta| \\ &\leq (C + Ce^{-\gamma/\varepsilon}) \|\omega\|_\varepsilon \|\eta\|_\varepsilon \end{aligned}$$

and

$$(2.12) \quad \left| \int_{\mathbb{R}^N} g'_\varepsilon(x, U_{\varepsilon, y}) \omega \eta \right| \leq C \left(\int_{\mathbb{R}^N} W_\varepsilon^{p-1} |\omega| |\eta| + \int_{\mathbb{R}^N} \widetilde{W}_\varepsilon^{p-1} |\omega| |\eta| + \int_{\mathbb{R}^N} U_{\varepsilon, y}^{\frac{p-1}{2}} (W_\varepsilon + \widetilde{W}_\varepsilon)^{\frac{p-1}{2}} |\omega| |\eta| \right).$$

It is easy to see

$$(2.13) \quad \begin{aligned} \int_{\mathbb{R}^N} W_\varepsilon^{p-1} |\omega| |\eta| &= \varepsilon^2 \int_{\mathbb{R}^N} (\varepsilon^{-\frac{2}{p-1}} W_\varepsilon)^{p-1} |\omega| |\eta| \\ &\leq C \varepsilon^2 (\int_{\mathbb{R}^N} |\omega|^2)^{\frac{1}{2}} (\int_{\mathbb{R}^N} |\eta|^2)^{\frac{1}{2}} \\ &\leq C \|\omega\|_\varepsilon \|\eta\|_\varepsilon. \end{aligned}$$

Denote $\tilde{\omega}(x) = \omega(\varepsilon^{\frac{2}{m+2}} x)$, $\delta_\varepsilon = \varepsilon^{-\frac{2}{m+2}} \delta$. Then for $\varepsilon > 0$ small,

$$(2.14) \quad \begin{aligned} \varepsilon^{\frac{m}{m+2}} \left(\int_{B_\delta(0)} \omega^2(x) dx \right)^{\frac{1}{2}} &= \varepsilon^{\frac{m+N}{m+2}} \left(\int_{B_{\delta_\varepsilon}(0)} \tilde{\omega}^2(\xi) d\xi \right)^{\frac{1}{2}} \\ &\leq C \varepsilon^{\frac{m+N}{m+2}} \left(\int_{B_{\delta_\varepsilon}(0)} (|\nabla \tilde{\omega}|^2 + |\xi|^m \tilde{\omega}^2(\xi) d\xi) \right)^{\frac{1}{2}} \\ &\leq C \left(\int_{B_\delta(0)} (\varepsilon^2 |\nabla \omega|^2 + V(x) \omega^2) dx \right)^{\frac{1}{2}} \\ &\leq C \|\omega\|_\varepsilon. \end{aligned}$$

Hence by (2.14) and (1.12),

$$(2.15) \quad \begin{aligned} \int_{\mathbb{R}^N} \widetilde{W}_\varepsilon^{p-1} |\omega| |\eta| &= \varepsilon^{\frac{2m}{m+2}} \int_{B_\delta(0)} \widehat{W}_\varepsilon^{p-1} (\varepsilon^{-\frac{2}{m+2}} x) |\omega(x)| |\eta(x)| dx \\ &\quad + C e^{-2\gamma/\varepsilon} (\int_{\mathbb{R}^N \setminus B_\delta(0)} |\omega|^2)^{\frac{1}{2}} (\int_{\mathbb{R}^N \setminus B_\delta(0)} |\eta|^2)^{\frac{1}{2}} \\ &\leq C \varepsilon^{\frac{2m}{m+2}} (\int_{B_\delta(0)} |\omega|^2)^{\frac{1}{2}} (\int_{B_\delta(0)} |\eta|^2)^{\frac{1}{2}} \\ &\quad + C e^{-2\gamma/\varepsilon} \|\omega\|_\varepsilon \|\eta\|_\varepsilon \\ &\leq C \|\omega\|_\varepsilon \|\eta\|_\varepsilon. \end{aligned}$$

Finally,

$$(2.16) \quad \begin{aligned} \int_{\mathbb{R}^N} U_{\varepsilon, y}^{\frac{p-1}{2}} (W_\varepsilon + \widetilde{W}_\varepsilon)^{\frac{p-1}{2}} |\omega| |\eta| &\leq C e^{-2\gamma/\varepsilon} (\int_{\mathbb{R}^N} |\omega|^2)^{\frac{1}{2}} (\int_{\mathbb{R}^N} |\eta|^2)^{\frac{1}{2}} \\ &\leq C \|\omega\|_\varepsilon \|\eta\|_\varepsilon. \end{aligned}$$

So, (2.10) follows easily from (2.11)-(2.13), (2.15) and (2.16). $\square$

From Lemma 2.2 we obtain the existence of a linear operator $L_\varepsilon : E_{\varepsilon,y} \rightarrow E_{\varepsilon,y}$ such that for any $\omega, \eta \in E_{\varepsilon,y}$

$$\langle L_\varepsilon \omega, \eta \rangle_\varepsilon = Q_\varepsilon(\omega, \eta).$$

For the L_ε defined as above we have

Proposition 2.3. *There are $\varepsilon_1 > 0, \delta_1 > 0$ and $\rho > 0$ such that for $\varepsilon \in (0, \varepsilon_1), \delta \in (0, \delta_1]$ and for all $y \in B_\delta(a_1), \omega \in E_{\varepsilon,y}$*

$$\|L_\varepsilon \omega\|_\varepsilon \geq \rho \|\omega\|_\varepsilon.$$

Proof. Suppose our conclusion is false. Then there exist a sequence $\{\varepsilon_n\}, \varepsilon_n \rightarrow 0, \{y_{\varepsilon_n}\} \subset \mathbb{R}^N, y_{\varepsilon_n} \rightarrow a_1$ and $\omega_{\varepsilon_n} \in E_{\varepsilon_n, y_{\varepsilon_n}}$ such that

$$\|L_{\varepsilon_n} \omega_{\varepsilon_n}\|_{\varepsilon_n} = o(1) \|\omega_{\varepsilon_n}\|_{\varepsilon_n}.$$

To simplify the notation, we write ε_n and ω_{ε_n} as ε and ω_ε , respectively, and write y_{ε_n} as y_ε . By our assumption we have

$$(2.17) \quad \int_{\mathbb{R}^N} (\varepsilon^2 \nabla \omega_\varepsilon \nabla \eta + V(x) \omega_\varepsilon \eta) - p \int_{\mathbb{R}^N} (\Phi_\varepsilon + U_{\varepsilon, y_\varepsilon})^{p-1} \omega_\varepsilon \eta = o(1) \|\omega\|_\varepsilon \|\eta\|_\varepsilon$$

for $\omega, \eta \in E_{\varepsilon, y_\varepsilon}$, that is, $\phi = \omega, \eta$ satisfies

$$(2.18) \quad \int_{\mathbb{R}^N} \left(\varepsilon^2 \nabla \frac{\partial U_{\varepsilon, y_\varepsilon}}{\partial x_i} \nabla \phi + V(x) \frac{\partial U_{\varepsilon, y_\varepsilon}}{\partial x_i} \phi \right) = 0, \quad i = 1, \dots, N.$$

We claim that the following estimates hold.

First, for any $R > 0$,

$$(2.19) \quad \int_{B_{\varepsilon R}(y_\varepsilon)} \omega_\varepsilon^2 = o(1) \|\omega_\varepsilon\|_\varepsilon^2.$$

Second, for any given $R > 0$, let $R_\varepsilon = \varepsilon^{\frac{2}{m+2}} R$. Then

$$(2.20) \quad \varepsilon^{\frac{2m}{m+2}} \int_{B_{R_\varepsilon}(0)} \omega_\varepsilon^2 = o(1) \|\omega_\varepsilon\|_\varepsilon^2.$$

Finally,

$$(2.21) \quad \varepsilon^2 \int_{\Omega} \omega_\varepsilon^2 = o(1) \|\omega_\varepsilon\|_\varepsilon^2.$$

Suppose that (2.19), (2.20) and (2.21) are true. We will show from (2.19)-(2.21) that

$$(2.22) \quad \int_{\mathbb{R}^N} (\Phi_\varepsilon + U_{\varepsilon, y_\varepsilon})^{p-1} \omega_\varepsilon^2 = o(1) \|\omega_\varepsilon\|_\varepsilon^2 + o_R(1) \|\omega_\varepsilon\|_\varepsilon^2,$$

where $o_R(1)$ denotes quantities that go to 0 as $R \rightarrow \infty$.

To show (2.22), let us first give some estimates.

Similar to the proof of (2.14), we have

$$\begin{aligned} & \varepsilon^{\frac{2m}{m+2}} \int_{B_\delta(0) \setminus B_{R_\varepsilon}(0)} \widehat{W}_\varepsilon^{p-1}(\varepsilon^{-\frac{2}{m+2}} x) \omega_\varepsilon^2(x) \\ & \leq o_R(1) \varepsilon^{\frac{2m}{m+2}} \int_{B_\delta(0) \setminus B_{R_\varepsilon}(0)} \omega_\varepsilon^2 \leq o_R(1) \|\omega_\varepsilon\|_\varepsilon^2. \end{aligned}$$

On the other hand, from $U_{\varepsilon, y_\varepsilon}(x) = o_R(1)$ for $x \in \mathbb{R}^N \setminus B_{\varepsilon R}(y_\varepsilon)$, we obtain

$$\begin{aligned} & \int_{B_\delta(y_\varepsilon) \setminus B_{\varepsilon R}(y_\varepsilon)} U_\varepsilon^{p-1} \left(\frac{x - y_\varepsilon}{\varepsilon} \right) \omega_\varepsilon^2(x) \\ & \leq o_R(1) \int_{B_\delta(y_\varepsilon) \setminus B_{\varepsilon R}(y_\varepsilon)} V(x) \omega_\varepsilon^2 \leq o_R(1) \|\omega_\varepsilon\|_\varepsilon^2. \end{aligned}$$

By the exponential decay of $\widehat{W}_\varepsilon$ and U , we have

$$\varepsilon^{\frac{2m}{m+2}} \int_{\mathbb{R}^N \setminus B_\delta(0)} \widehat{W}_\varepsilon^{p-1} \left(\varepsilon^{-\frac{2}{m+2}} x \right) \omega_\varepsilon^2(x) \leq O(e^{-\gamma/\varepsilon}) \|\omega_\varepsilon\|_\varepsilon^2$$

and

$$\int_{\mathbb{R}^N \setminus B_\delta(y_\varepsilon)} U_\varepsilon^{p-1} \left(\frac{x - y_\varepsilon}{\varepsilon} \right) \omega_\varepsilon^2 \leq O(e^{-\gamma/\varepsilon}) \|\omega_\varepsilon\|_\varepsilon^2,$$

where $\gamma > 0$ is a small constant.

Combining all the above estimates, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^N} (\Phi_\varepsilon + U_{\varepsilon, y_\varepsilon})^{p-1} \omega_\varepsilon^2 \\ & \leq C \int_{\mathbb{R}^N} W_{\varepsilon, y_\varepsilon}^{p-1} \omega_\varepsilon^2 + C \int_{\mathbb{R}^N} \widehat{W}_\varepsilon^{p-1} \omega_\varepsilon^2 + C \int_{\mathbb{R}^N} U_{\varepsilon, y_\varepsilon}^{p-1} \omega_\varepsilon^2 + O(e^{-\gamma/\varepsilon}) \|\omega_\varepsilon\|_\varepsilon^2 \\ & \leq C \varepsilon^2 \int_{\mathbb{R}^N} (\varepsilon^{-\frac{2}{p-1}} W_\varepsilon)^{p-1} |\omega|_\varepsilon^2 + C \sup_{|x| \leq R_\varepsilon} (\varepsilon^{-\frac{2}{p-1} - \frac{m}{m+2}} \widehat{W}_\varepsilon(x))^{p-1} \varepsilon^{\frac{2m}{m+2}} \int_{B_{R_\varepsilon}(0)} \omega_\varepsilon^2 \\ & \quad + C \varepsilon^{\frac{2m}{m+2}} \int_{B_\delta(0) \setminus B_{R_\varepsilon}(0)} \widehat{W}_\varepsilon^{p-1} \left(\varepsilon^{-\frac{2}{m+2}} x \right) \omega_\varepsilon^2 + C \varepsilon^{\frac{2m}{m+2}} \int_{\mathbb{R}^N \setminus B_\delta(0)} \widehat{W}_\varepsilon^{p-1} \left(\varepsilon^{-\frac{2}{m+2}} x \right) \omega_\varepsilon^2 \\ & \quad + C \int_{B_{\varepsilon R}(y_\varepsilon)} U^{p-1} \left(\frac{x - y_\varepsilon}{\varepsilon} \right) \omega_\varepsilon^2(x) dx \\ & \quad + C \int_{\mathbb{R}^N \setminus B_\delta(y_\varepsilon)} U^{p-1} \left(\frac{x - y_\varepsilon}{\varepsilon} \right) \omega_\varepsilon^2(x) dx + C \int_{B_\delta(y_\varepsilon) \setminus B_{\varepsilon R}(y_\varepsilon)} U^{p-1} \left(\frac{x - y_\varepsilon}{\varepsilon} \right) \omega_\varepsilon^2(x) dx \\ & \leq C \varepsilon^2 \int_{\Omega} \omega_\varepsilon^2(x) dx + o(1) \|\omega\|_\varepsilon^2 + o_R(1) \|\omega\|_\varepsilon^2 \\ & = o(1) \|\omega\|_\varepsilon^2 + o_R(1) \|\omega\|_\varepsilon^2, \end{aligned}$$

which implies (2.22).

Taking $\eta = \omega_\varepsilon$ in (2.17), we have from (2.22)

$$\begin{aligned} o(1) \|\omega_\varepsilon\|_\varepsilon^2 &= \int_{\mathbb{R}^N} (\varepsilon^2 |\nabla \omega_\varepsilon|^2 + V(x) \omega_\varepsilon^2) - p \int_{\mathbb{R}^N} (\Phi_\varepsilon + U_{\varepsilon, y_\varepsilon})^{p-1} \omega_\varepsilon^2 \\ &= (1 + o(1) + o_R(1)) \|\omega_\varepsilon\|_\varepsilon^2, \end{aligned}$$

which is a contradiction.

It remains to prove (2.19), (2.20) and (2.21).

First, we prove (2.19). Let $\|\phi\|_\varepsilon^* = (\int_{\mathbb{R}^N} (|\nabla \phi|^2 + V(\varepsilon x + y_\varepsilon) \phi^2) dx)^{\frac{1}{2}}$. Rescaling $\tilde{\omega}_\varepsilon(x) = \omega_\varepsilon(\varepsilon x + y_\varepsilon)$, then by (2.17) we have

$$(2.23) \quad \int_{\mathbb{R}^N} (\nabla \tilde{\omega}_\varepsilon \nabla \eta + V(\varepsilon x + y_\varepsilon) \tilde{\omega}_\varepsilon \eta) - p \int_{\mathbb{R}^N} (\Phi_\varepsilon(\varepsilon x + y_\varepsilon) + U(x))^{p-1} \tilde{\omega}_\varepsilon \eta = o(1) \|\tilde{\omega}_\varepsilon\|_\varepsilon^* \|\eta\|_\varepsilon^*$$

provided η satisfies

$$(2.24) \quad \int_{\mathbb{R}^N} \left(\nabla \frac{\partial U(x)}{\partial x_i} \nabla \eta + V(\varepsilon x + y_\varepsilon) \frac{\partial U(x)}{\partial x_i} \eta \right) = 0, \quad i = 1, \dots, N.$$

Without loss of generality, we assume $\|\tilde{\omega}_\varepsilon\|_\varepsilon^* = 1$. Since $V(y_\varepsilon) \geq c'' > 0$, we see that there is a $\tilde{\omega} \in H^1(\mathbb{R}^N)$ such that

$$\begin{aligned}\tilde{\omega}_\varepsilon &\rightharpoonup \tilde{\omega} \quad \text{weakly in } H_{\text{loc}}^1(\mathbb{R}^N), \\ \tilde{\omega}_\varepsilon &\rightarrow \tilde{\omega} \quad \text{a.e. in } \mathbb{R}^N.\end{aligned}$$

We claim that $\tilde{\omega}$ satisfies for all $\eta \in C_0^\infty(\mathbb{R}^N)$,

$$(2.25) \quad \int_{\mathbb{R}^N} (\nabla \tilde{\omega} \nabla \eta + V(a_1) \tilde{\omega} \eta) = p \int_{\mathbb{R}^N} U^{p-1}(x) \tilde{\omega} \eta.$$

In fact, for any given $\eta \in C_0^\infty(\mathbb{R}^N)$, by projecting η into $\tilde{E}_{\varepsilon, y_\varepsilon}$, the set of functions such that (2.24) is satisfied, we then have $c_{j,\varepsilon} \in \mathbb{R}$ ($j = 1, \dots, N$) such that $\eta_\varepsilon = \eta - \sum_{j=1}^N c_{j,\varepsilon} \frac{\partial U(x)}{\partial x_j} \in \tilde{E}_{\varepsilon, y_\varepsilon}$. It is not hard to show by (2.24) that $c_{j,\varepsilon}$ is uniformly bounded. Replacing η in (2.23) by η_ε we have

$$\begin{aligned}(2.26) \quad & \int_{\mathbb{R}^N} \left(\nabla \tilde{\omega}_\varepsilon \nabla (\eta - \sum_{j=1}^N c_{j,\varepsilon} \frac{\partial U(x)}{\partial x_j}) + V(\varepsilon x + y_\varepsilon) \tilde{\omega}_\varepsilon (\eta - \sum_{j=1}^N c_{j,\varepsilon} \frac{\partial U(x)}{\partial x_j}) \right) \\ & - p \int_{\mathbb{R}^N} (\Phi_\varepsilon(\varepsilon x + y_\varepsilon) + U(x))^{p-1} \tilde{\omega}_\varepsilon (\eta - \sum_{j=1}^N c_{j,\varepsilon} \frac{\partial U(x)}{\partial x_j}) \\ & = o(1) \|\tilde{\omega}_\varepsilon\|_\varepsilon^* \|\eta\|_\varepsilon^*.\end{aligned}$$

Taking $\varepsilon \rightarrow 0$ in (2.26), we have from $y_\varepsilon \rightarrow a_1$,

$$\begin{aligned}(2.27) \quad & \int_{\mathbb{R}^N} (\nabla \tilde{\omega} \nabla \eta + V(a_1) \tilde{\omega} \eta) - p \int_{\mathbb{R}^N} U^{p-1}(x) \tilde{\omega} \eta \\ & - \sum_{j=1}^N c_j \left(\int_{\mathbb{R}^N} \left(\nabla \tilde{\omega} \nabla \frac{\partial U(x)}{\partial x_j} + V(a_1) \tilde{\omega} \frac{\partial U(x)}{\partial x_j} \right) - p \int_{\mathbb{R}^N} U^{p-1}(x) \tilde{\omega} \frac{\partial U(x)}{\partial x_j} \right) = 0,\end{aligned}$$

where $c_{j,\varepsilon} \rightarrow c_j$ (up to a subsequence). But

$$\int_{\mathbb{R}^N} \left(\nabla \tilde{\omega} \nabla \frac{\partial U(x)}{\partial x_j} + V(a_1) \tilde{\omega} \frac{\partial U(x)}{\partial x_j} \right) - p \int_{\mathbb{R}^N} U^{p-1}(x) \tilde{\omega} \frac{\partial U(x)}{\partial x_j} = 0.$$

Thus, (2.25) follows.

On the other hand, by $\tilde{\omega}_\varepsilon \in \tilde{E}_{\varepsilon, y_\varepsilon}$ we have

$$\int_{\mathbb{R}^N} \left(\nabla \tilde{\omega}_\varepsilon \nabla \frac{\partial U(x)}{\partial x_j} + V(a_1) \tilde{\omega}_\varepsilon \frac{\partial U(x)}{\partial x_j} \right) = 0, \quad i = 1, \dots, N.$$

From the fact that U is non-degenerate, we have $\tilde{\omega} \equiv 0$ in $\mathbb{R}^N$ (see [18], for example). Therefore for any $R > 0$,

$$\int_{B_R(0)} \tilde{\omega}_\varepsilon^2 \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

which, together with $\|\omega_\varepsilon\|_\varepsilon^2 = \varepsilon^N (\|\tilde{\omega}_\varepsilon\|_\varepsilon^*)^2 = \varepsilon^N$, yields (2.19).

Now we turn to the proof of (2.20).

Set $\hat{\omega}(x) = \varepsilon^{-\frac{2m}{(p-1)(m+2)}} \omega(\varepsilon^{\frac{2}{m+2}} x)$. Define

$$\|[\hat{\omega}]\|_\varepsilon = \left(\int_{\mathbb{R}^N} (|\nabla \hat{\omega}|^2 + \varepsilon^{-\frac{2m}{m+2}} V(\varepsilon^{\frac{2}{m+2}} x) \hat{\omega}^2(x)) dx \right)^{\frac{1}{2}}.$$

Assume $||\hat{\omega}_\varepsilon||_\varepsilon = 1$ (otherwise take $\frac{\hat{\omega}_\varepsilon}{||\hat{\omega}_\varepsilon||_\varepsilon}$). Then (2.17) is equivalent to

$$(2.28) \quad \begin{aligned} & \int_{\mathbb{R}^N} \left(\nabla \hat{\omega}_\varepsilon \nabla \hat{\eta} + \varepsilon^{-\frac{2m}{m+2}} V(\varepsilon^{\frac{2}{m+2}} x) \hat{\omega}_\varepsilon \hat{\eta} \right) \\ & - p \varepsilon^{-\frac{2m}{m+2}} \int_{\mathbb{R}^N} \left(\Phi_\varepsilon(\varepsilon^{\frac{2}{m+2}} x) + U\left(\frac{\varepsilon^{\frac{2}{m+2}} x - y_\varepsilon}{\varepsilon}\right) \right)^{p-1} \hat{\omega}_\varepsilon \hat{\eta} \\ & = o(1) ||\hat{\omega}_\varepsilon||_\varepsilon ||\hat{\eta}||_\varepsilon, \end{aligned}$$

for any $\hat{\eta} \in H^1(\mathbb{R}^N)$, satisfying

$$(2.29) \quad \int_{\mathbb{R}^N} (\nabla \hat{U}_{\varepsilon,j} \nabla \hat{\eta} + \varepsilon^{-\frac{2m}{m+2}} V(\varepsilon^{\frac{2}{m+2}} x) \hat{U}_{\varepsilon,j} \hat{\eta}) = 0,$$

where $\hat{U}_{\varepsilon,j}(x) = \varepsilon^{-\frac{2m}{(p-1)(m+2)}} \frac{\partial U_{\varepsilon,y_\varepsilon}(z)}{\partial x_j} \Big|_{z=\varepsilon^{\frac{2}{m+2}} x}$.

Since $\varepsilon^{-\frac{2m}{m+2}} V(\varepsilon^{\frac{2}{m+2}} x) \geq c'|x|^m$ if $|x| \leq R$, it is not difficult to prove that there is an $\hat{\omega} \in H^1(\mathbb{R}^N)$, such that

$$\begin{aligned} \hat{\omega}_\varepsilon &\rightharpoonup \hat{\omega} \quad \text{weakly in } H_{\text{loc}}^1(\mathbb{R}^N), \\ \hat{\omega}_\varepsilon &\rightarrow \hat{\omega} \quad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

For any $\eta \in C_0^\infty(\mathbb{R}^N)$, choose $f_{\varepsilon,j}$, such that $\eta_\varepsilon = \eta - \sum_{j=1}^N f_{\varepsilon,j} \hat{U}_{\varepsilon,j}$ satisfies (2.29). Since η has compact support, and $\hat{U}_{\varepsilon,j}$ is exponentially small in $B_R(0)$ for any $R > 0$, we see that $|f_{\varepsilon,j}| \leq e^{-\gamma/\varepsilon}$ for some $\gamma > 0$.

Inserting η_ε into (2.28), using

$$\begin{aligned} & \varepsilon^{-\frac{2m}{m+2}} \int_{\mathbb{R}^N} \left(\Phi_\varepsilon(\varepsilon^{\frac{2}{m+2}} x) + U\left(\frac{\varepsilon^{\frac{2}{m+2}} x - y_\varepsilon}{\varepsilon}\right) \right)^{p-1} \hat{\omega}_\varepsilon \eta \\ & = \int_{\mathbb{R}^N} \left(\varepsilon^{-\frac{2m}{(p-1)(m+2)}} W_\varepsilon(\varepsilon^{\frac{2}{m+2}} x) + \varepsilon^{-\frac{2m}{(p-1)(m+2)}} \widetilde{W}_\varepsilon(\varepsilon^{\frac{2}{m+2}} x) + U\left(\frac{\varepsilon^{\frac{2}{m+2}} x - y_\varepsilon}{\varepsilon}\right) \right)^{p-1} \hat{\omega}_\varepsilon \eta \\ & \longrightarrow \int_{\mathbb{R}^N} \widehat{W}^{p-1}(x) \hat{\omega} \eta, \end{aligned}$$

and

$$\int_{\mathbb{R}^N} \varepsilon^{-\frac{2m}{m+2}} V(\varepsilon^{\frac{2}{m+2}} x) \hat{\omega}_\varepsilon \eta \longrightarrow \int_{\mathbb{R}^N} |x|^m \hat{\omega} \eta,$$

we obtain that for any $\eta \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (\nabla \hat{\omega} \nabla \eta + |x|^m \hat{\omega} \eta) = p \int_{\mathbb{R}^N} \widehat{W}^{p-1} \hat{\omega} \eta,$$

which, by the fact that $\widehat{W}$ is non-degenerate, implies that $\hat{\omega} \equiv 0$. Thus for any given $R > 0$,

$$\int_{B_R(0)} \hat{\omega}_\varepsilon^2 \rightarrow 0,$$

which gives

$$\int_{B_{R_\varepsilon}(0)} \omega_\varepsilon^2 = o(1) \varepsilon^{\frac{2N}{m+2} + \frac{4m}{(p-1)(m+2)}}.$$

From $||\hat{\omega}||_\varepsilon^2 = 1$, we deduce

$$||\omega_\varepsilon||_\varepsilon^2 = \varepsilon^{\frac{2N}{m+2} + \frac{4m}{(p-1)(m+2)} + \frac{2m}{m+2}}.$$

Thus,

$$\varepsilon^{\frac{2m}{m+2}} \int_{B_{R_\varepsilon}(0)} \omega_\varepsilon^2 = o(1) \varepsilon^{\frac{2N}{m+2} + \frac{4m}{(p-1)(m+2)} + \frac{2m}{m+2}} = o(1) \|\omega_\varepsilon\|_\varepsilon^2,$$

and (2.20) follows.

To show (2.21), we define

$$||[\omega]|_\varepsilon^* = \left(\int_{\mathbb{R}^N} (|\nabla \omega|^2 + \varepsilon^{-2} V(x) \omega^2(x)) dx \right)^{\frac{1}{2}}.$$

Then, (2.17) is equivalent to

$$(2.30) \quad \int_{\mathbb{R}^N} \nabla \omega_\varepsilon \nabla \eta + \varepsilon^{-2} V(x) \omega_\varepsilon \eta - p \varepsilon^{-2} \int_{\mathbb{R}^N} (\Phi_\varepsilon + U_{\varepsilon,y})^{p-1} \omega_\varepsilon \eta = o(1) ||[\omega_\varepsilon]|_\varepsilon^* ||[\eta]|_\varepsilon^*,$$

for any $\eta \in E_{\varepsilon, y_\varepsilon}$.

Assume $||[\omega_\varepsilon]|_\varepsilon^* = 1$ (otherwise replace ω_ε by $\frac{\omega_\varepsilon}{||[\omega_\varepsilon]|_\varepsilon^*}$). Then ω_ε is bounded in $H^1(\mathbb{R}^N)$. Consequently there exists an $\omega \in H^1(\mathbb{R}^N)$ such that

$$\begin{aligned} \omega_\varepsilon &\rightharpoonup \omega \quad \text{weakly in } H^1(\mathbb{R}^N), \\ \omega_\varepsilon &\rightarrow \omega \quad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

Moreover from $\int_{\mathbb{R}^N} V(x) \omega_\varepsilon^2 \leq \varepsilon^2$, we obtain $\omega = 0$ for $x \in \mathbb{R}^N \setminus \mathcal{Z}$. For any $\eta \in H_0^1(\Omega)$, let $\eta_\varepsilon = \eta - \sum_{j=1}^N d_{j,\varepsilon} \frac{\partial U_{\varepsilon, y_\varepsilon}(x)}{\partial x_j}$, where $d_{j,\varepsilon} \in \mathbb{R}$ is chosen so that $\eta_\varepsilon \in E_{\varepsilon, y_\varepsilon}$.

It is not difficult to check that $d_{j,\varepsilon} = O(e^{-\gamma/\varepsilon})$ for some $\gamma > 0$ from the fact that $U_{\varepsilon, y_\varepsilon}(x) \leq C e^{-\sqrt{V(x_0)}|x-y_\varepsilon|/\varepsilon}$ and $y_\varepsilon \rightarrow x_0$.

Taking $\eta = \eta_\varepsilon$ in (2.30) and letting $\varepsilon \rightarrow 0$ we get

$$\int_{\Omega} \nabla \omega \nabla \eta = p \int_{\Omega} W^{p-1} \omega \eta$$

since both $W_\varepsilon(x)$ and $U_{\varepsilon, y_\varepsilon}$ are bounded by $e^{-\gamma/\varepsilon}$ for $x \in \Omega$, where $\gamma > 0$ is a constant. Therefore $\omega \equiv 0$ in $\mathbb{R}^N$. Thus

$$\int_{\Omega} \omega_\varepsilon^2 = o(1),$$

which, in view of $\|\omega\|_\varepsilon^2 = \varepsilon^2 (||[\omega]|_\varepsilon^*)^2 = \varepsilon^2$, implies (2.21). We therefore complete our proof of Proposition 2.3. $\square$

Denote $\widehat{\Omega} = \Omega \cup \{0\}$ and define for $\omega, \eta \in H_0^1(\widehat{\Omega}^{2\delta})$ the inner product of ω and η :

$$\begin{aligned} (\omega, \eta)_\varepsilon &= \int_{\widehat{\Omega}^{2\delta}} (\nabla \omega \nabla \eta + \varepsilon^{-2} V(x) \omega \eta) dx, \\ \widetilde{L}_\varepsilon \omega &= -\Delta \omega + \varepsilon^{-2} V(x) \omega - \varepsilon^{-2} p (\Phi_\varepsilon(x) + U_{\varepsilon, y}(x))^{p-1} \omega. \end{aligned}$$

Proposition 2.4. *Suppose (V2-2) and (V3-2) hold. Then there exists $\rho_1 > 0$ independent of ε such that*

$$||(\widetilde{L}_\varepsilon \omega)|_\varepsilon \geq \rho_1 ||(\omega)|_\varepsilon \text{ for any } \omega \in H_0^1(\widehat{\Omega}^{2\delta})$$

where $||(\cdot)|_\varepsilon$ is the norm defined by the inner product $(\cdot, \cdot)_\varepsilon$.

Proposition 2.4 can be proved by the similar arguments to those used in the proof of Proposition 2.3 and therefore we omit it here.

Define

$$\mathcal{S}_\varepsilon = \left\{ \omega \mid \omega \in E_{\varepsilon,y}, \begin{array}{ll} \|\omega\|_\varepsilon \leq \varepsilon^{\frac{N}{2}+2-\tau}, & |\omega(x)| \leq \varepsilon^\sigma e^{-\sigma|x-y|/(\varepsilon\delta)} \quad \text{for } x \in B_\delta(y), \\ |\omega(x)| \leq e^{-\sigma/\varepsilon} & \text{for } x \in \mathbb{R}^N \setminus B_\delta(y) \end{array} \right\}$$

where $\sigma > 0$ and $\tau \in (0, \sigma)$ are small constants to be determined.

Proposition 2.5. *There exists $\varepsilon_2 > 0$ such that for $\varepsilon \in (0, \varepsilon_2)$ there is a unique C^1 map: $y \in B_\varepsilon(a_1) \rightarrow \omega_\varepsilon \in H^1(\mathbb{R}^N)$ satisfying $\omega_\varepsilon \in \mathcal{S}_\varepsilon$ and for some $D_{j,\varepsilon} \in \mathbb{R}$*

$$(2.31) \quad I'(U_{\varepsilon,y} + \omega_\varepsilon) = \sum_{j=1}^N D_{j,\varepsilon} \frac{\partial U_{\varepsilon,y}}{\partial x_j}.$$

Moreover

$$(2.32) \quad \|\omega_\varepsilon\|_\varepsilon = O(\varepsilon^{\frac{N}{2}+2})$$

and

$$(2.33) \quad D_{j,\varepsilon} = O(\varepsilon^3).$$

Proof. By Lemma 2.1, there is $\omega_{\ell_\varepsilon} \in E_{\varepsilon,y}$ such that

$$\langle \omega_{\ell_\varepsilon}, \omega \rangle_\varepsilon = \ell_\varepsilon(\omega) \quad \text{for any } \omega \in E_{\varepsilon,y}$$

and

$$\begin{aligned} \|\omega_{\ell_\varepsilon}\|_\varepsilon &\leq C(|V(y) - V(a_1)| + \varepsilon|\nabla V(y)| + \varepsilon^2)\varepsilon^{\frac{N}{2}} \\ &= O(|y - a_1|^2 + \varepsilon|y - a_1| + \varepsilon^2)\varepsilon^{\frac{N}{2}} = O(\varepsilon^{\frac{N}{2}+2}). \end{aligned}$$

It follows from Lemma 2.2 and Proposition 2.3 that there exists an invertible linear operator L_ε such that

$$Q_\varepsilon(\omega, \omega) = \langle L_\varepsilon \omega, \omega \rangle_\varepsilon \quad \text{for all } \omega \in E_{\varepsilon,y}.$$

Hence to find a critical point of I is equivalent to solving

$$(2.34) \quad \omega_{\ell_\varepsilon} + L_\varepsilon \omega + R'_\varepsilon(\omega) = 0.$$

Define $G_\varepsilon(\omega) = -L_\varepsilon^{-1}\omega_{\ell_\varepsilon} - L_\varepsilon^{-1}R'_\varepsilon(\omega)$; then to solve (2.34) is equivalent to finding a fixed point of G_ε in $E_{\varepsilon,y}$. We will establish the existence of a fixed point $\omega_{\varepsilon,y}$ for $y \in B_\varepsilon(a_1)$ by the following four steps.

Step 1. There are $C > 0, \vartheta \in (0, 1)$ such that

$$(2.35) \quad \|G_\varepsilon(\omega)\|_\varepsilon \leq C\varepsilon^{\frac{N}{2}+2} \leq \varepsilon^{\frac{N}{2}+2-\tau} \quad \text{for any } \omega \in S_\varepsilon,$$

$$(2.36) \quad \|G_\varepsilon(\omega_1) - G_\varepsilon(\omega_2)\|_\varepsilon \leq \vartheta\|\omega_1 - \omega_2\|_\varepsilon \quad \text{for any } \omega_1, \omega_2 \in S_\varepsilon.$$

By direct calculations, we have

$$\langle R'_\varepsilon(\omega), \eta \rangle = \int_{\mathbb{R}^N} [(\Phi_\varepsilon + U_{\varepsilon,y} + \omega)_+^p - (\Phi_\varepsilon + U_{\varepsilon,y})^p - p(\Phi_\varepsilon + U_{\varepsilon,y})^{p-1}\omega] \eta.$$

Let $\bar{p} = \min\{p, 2\}$. Then

$$\begin{aligned}
 (2.37) \quad |\langle R'_\varepsilon(\omega), \eta \rangle| &\leq C \int_{\mathbb{R}^N} (|\omega|^p + |\omega|^{\bar{p}}) |\eta| \\
 &= C \left(\int_{B_\delta(y)} (|\omega|^p + |\omega|^{\bar{p}}) |\eta| + \int_{\mathbb{R}^N \setminus B_\delta(y)} (|\omega|^p + |\omega|^{\bar{p}}) |\eta| \right) \\
 &\leq C \varepsilon^{\sigma(\bar{p}-1)} \int_{B_\delta(y)} V(x) |\omega| |\eta| + C e^{-\sigma(\bar{p}-1)/\varepsilon} \int_{\mathbb{R}^N \setminus B_\delta(y)} |\omega| |\eta| \\
 &\leq C \varepsilon^{\sigma(\bar{p}-1)} \|\omega\|_\varepsilon \|\eta\|_\varepsilon \leq C \varepsilon^{\sigma(\bar{p}-1) + \frac{N}{2} + 2 - \tau} \|\eta\|_\varepsilon \\
 &\leq \varepsilon^{\frac{N}{2} + 2} \|\eta\|_\varepsilon,
 \end{aligned}$$

provided $\tau < (\bar{p} - 1)\sigma$, and ε_2 is small enough. Thus,

$$\|G_\varepsilon(\omega)\|_\varepsilon \leq C \|\omega_{\ell_\varepsilon}\|_\varepsilon + C \|R'_\varepsilon(\omega)\|_\varepsilon \leq C \varepsilon^{\frac{N}{2} + 2}.$$

So (2.35) is proved.

By direct calculation, we have

$$\begin{aligned}
 &\langle R'_\varepsilon(\omega_1) - R'_\varepsilon(\omega_2), \eta \rangle \\
 &= \int_{\mathbb{R}^N} [(\Phi_\varepsilon + U_{\varepsilon,y} + \omega_1)_+^p - (\Phi_\varepsilon + U_{\varepsilon,y} + \omega_2)_+^p - p(\Phi_\varepsilon + U_{\varepsilon,y})^{p-1}(\omega_1 - \omega_2)] \eta,
 \end{aligned}$$

from which we deduce by using argument as in (2.37),

$$\begin{aligned}
 (2.38) \quad &|\langle R'_\varepsilon(\omega_1) - R'_\varepsilon(\omega_2), \eta \rangle| \\
 &\leq C \int_{\mathbb{R}^N} \left(|\omega_1|^{p-1} + |\omega_2|^{p-1} + (\Phi_\varepsilon + U_{\varepsilon,y})^{p-2} (|\omega_1| + |\omega_2|) \right) |\omega_1 - \omega_2| |\eta| \\
 &\leq C \varepsilon^{\frac{p}{4}(\bar{p}-1)} \|\omega_1 - \omega_2\|_\varepsilon \|\eta\|_\varepsilon.
 \end{aligned}$$

So, (2.36) follows directly from (2.38).

Step 2. Let $\omega_1 = G_\varepsilon(\omega)$. Then there is a small $\mu > 0$, such that

$$(2.39) \quad |\omega_1(x)| \leq C \varepsilon^{(1+\mu)\sigma} \quad \text{for all } x \in \mathbb{R}^N \setminus \mathcal{Z}^\delta.$$

Indeed, ω_1 satisfies $L_\varepsilon \omega_1 = -\omega_{\ell_\varepsilon} - R'_\varepsilon(\omega)$ and there are $D_{j,\varepsilon} \in \mathbb{R}$ for $j = 1, \dots, N$ such that for any $\eta \in H^1(\mathbb{R}^N)$

$$(2.40) \quad \langle L_\varepsilon \omega_1, \eta \rangle_\varepsilon = -\langle \omega_{\ell_\varepsilon}, \eta \rangle_\varepsilon - \langle R'_\varepsilon(\omega), \eta \rangle + \sum_{j=1}^N D_{j,\varepsilon} \left\langle \frac{\partial U_{\varepsilon,y}}{\partial x_j}, \eta \right\rangle_\varepsilon.$$

We claim that $D_{j,\varepsilon} = O(\varepsilon^3)$ for $j = 1, \dots, N$. To this end, taking $\eta = \frac{\partial U_{\varepsilon,y}}{\partial x_j}$ respectively in (2.40) we obtain a linear system for $D_{j,\varepsilon}$

$$\begin{aligned}
 (2.41) \quad &\sum_{j=1}^N \left\langle \frac{\partial U_{\varepsilon,y}}{\partial x_i}, \frac{\partial U_{\varepsilon,y}}{\partial x_j} \right\rangle_\varepsilon D_{j,\varepsilon} = \left\langle L_\varepsilon \omega_1, \frac{\partial U_{\varepsilon,y}}{\partial x_j} \right\rangle_\varepsilon + \left\langle \omega_{\ell_\varepsilon}, \frac{\partial U_{\varepsilon,y}}{\partial x_j} \right\rangle_\varepsilon + \left\langle R'_\varepsilon(\omega), \frac{\partial U_{\varepsilon,y}}{\partial x_j} \right\rangle_\varepsilon.
 \end{aligned}$$

With direct calculations, we have

$$\begin{aligned} \left| \left\langle L_\varepsilon \omega_1, \frac{\partial U_{\varepsilon,y}}{\partial x_j} \right\rangle_\varepsilon \right| &\leq C \|\omega_1\|_\varepsilon \left\| \frac{\partial U_{\varepsilon,y}}{\partial x_j} \right\|_\varepsilon \leq C \varepsilon^{N+1}, \\ \left| \left\langle \omega_{\ell_\varepsilon}, \frac{\partial U_{\varepsilon,y}}{\partial x_j} \right\rangle_\varepsilon \right| &\leq C \|\omega_{\ell_\varepsilon}\|_\varepsilon \left\| \frac{\partial U_{\varepsilon,y}}{\partial x_j} \right\|_\varepsilon \leq C \varepsilon^{N+1}, \\ \left| \left\langle R'_\varepsilon(\omega), \frac{\partial U_{\varepsilon,y}}{\partial x_j} \right\rangle \right| &\leq C \|R'_\varepsilon(\omega)\|_\varepsilon \left\| \frac{\partial U_{\varepsilon,y}}{\partial x_j} \right\|_\varepsilon \leq C \varepsilon^{N+1}, \end{aligned}$$

and

$$\left\langle \frac{\partial U_{\varepsilon,y}}{\partial x_i}, \frac{\partial U_{\varepsilon,y}}{\partial x_j} \right\rangle_\varepsilon = \begin{cases} c_0 \varepsilon^{N-2}, & \text{if } i = j, \ i, j = 1, \dots, N, \\ O(\varepsilon^N), & \text{if } i \neq j, \ i, j = 1, \dots, N, \end{cases}$$

for some positive constant c_0 , depending on $V(a_1)$, p , and N only. We obtain our conclusion easily from the estimates in (2.41).

To obtain an L^∞ estimate for ω_1 in $\mathbb{R}^N \setminus \mathcal{Z}^\delta$ we need to make use of the following equation satisfied by ω_1 :

$$(2.42) \quad -\varepsilon^2 \Delta \omega_1 + V(x) \omega_1 - p(\Phi_\varepsilon(x) + U_{\varepsilon,y})^{p-1} \omega_1 = f_\varepsilon(x)$$

with $f_\varepsilon(x) = f_\varepsilon^1(x) + f_\varepsilon^2(x) + f_\varepsilon^3(x)$, where f_ε^i ($i = 1, 2, 3$) is given by

$$(2.43) \quad f_\varepsilon^1(x) = (V(a_1) - V(x))U_{\varepsilon,y}(x) - \left[(\Phi_\varepsilon + U_{\varepsilon,y})^p - W_\varepsilon^p - \widehat{W}_\varepsilon^p - U_{\varepsilon,y}^p \right],$$

$$(2.44) \quad f_\varepsilon^2(x) = - \left[(\Phi_\varepsilon + U_{\varepsilon,y} + \omega)_+^p - (\Phi_\varepsilon + U_{\varepsilon,y})^p - p(\Phi_\varepsilon + U_{\varepsilon,y})^{p-1} \omega \right],$$

and

$$(2.45) \quad f_\varepsilon^3(x) = \sum_{j=1}^N \left[(V(x) - V(a_1)) \frac{\partial U_{\varepsilon,y}(x)}{\partial x_j} + p U_{\varepsilon,y}^{p-1} \frac{\partial U_{\varepsilon,y}(x)}{\partial x_j} \right] D_{j,\varepsilon}.$$

By the definition, we can find a constant $\gamma > 0$, such that

$$\begin{aligned} (2.46) \quad |f_\varepsilon^1(x)| &\leq |V(x_0) - V(y)|U_{\varepsilon,y}(x) + |V(x) - V(y)|U_{\varepsilon,y}(x) \\ &\quad + C \left[W_\varepsilon^{p-1}(W_\varepsilon + U_{\varepsilon,y}) + \widehat{W}_\varepsilon^{p-1}(W_\varepsilon + U_{\varepsilon,y}) + U_{\varepsilon,y}^{p-1}(W_\varepsilon + \widehat{W}_\varepsilon) \right] \\ &\leq |V(x_0) - V(y)|U_{\varepsilon,y}(x) + C|x - y|U_{\varepsilon,y}(x) + O(e^{-\gamma/\varepsilon}) \\ &\leq C\varepsilon, \end{aligned}$$

and

$$(2.47) \quad |f_\varepsilon^2(x)| \leq C(|\omega|^p + |\omega|^{\bar{p}}) \leq C\varepsilon^{\sigma \bar{p}} \leq \varepsilon^{(1+\mu)\sigma}.$$

Since $D_{j,\varepsilon} = O(\varepsilon^3)$, we have

$$(2.48) \quad |f_\varepsilon^3(x)| \leq \varepsilon.$$

Set $\tilde{\omega}_1(x) = \omega_1(\varepsilon x)$; then

$$(2.49) \quad -\Delta \tilde{\omega}_1 + V(\varepsilon x) \tilde{\omega}_1 = f_\varepsilon(\varepsilon x) + p(\Phi_\varepsilon(\varepsilon x) + U(x - \varepsilon^{-1}y))^{p-1} \tilde{\omega}_1.$$

From $\|\omega_1\|_\varepsilon \leq C\varepsilon^{\frac{N}{2}+2}$, we have for $z \in \mathbb{R}^N \setminus \mathcal{Z}_\varepsilon^{\frac{\delta}{2}}$

$$\left(\int_{B_2(z)} \tilde{\omega}_1^2 \right)^{\frac{1}{2}} \leq C\varepsilon^2.$$

By the Moser iteration technique (see Gilbarg and Trudinger [19]), using the fact that $(\Phi_\varepsilon(\varepsilon x) + U(x - \varepsilon^{-1}y))^{p-1}$ is bounded, we have

$$\begin{aligned} \|\tilde{\omega}_1\|_{L^\infty(B_1(z))} &\leq C (\|\tilde{\omega}_1\|_{L^2(B_2(z))} + \|f_\varepsilon\|_{L^\infty(B_2(z))}) \\ &\leq C\varepsilon^{(1+\mu)\sigma}, \end{aligned}$$

which implies (2.39). So, we complete step 2.

Step 3. There exists $\mu > 0$ such that

$$(2.50) \quad |\omega_1(x)| \leq e^{-(1+\mu)\sigma/\varepsilon} \text{ for } x \in \mathbb{R}^N \setminus B_\delta(y).$$

Fix $\theta \in (0, 1)$. By (2.43), we have from (1.9) and (1.10)

$$(2.51) \quad |f_\varepsilon^1(x)| \leq e^{-(1+10\mu)\sigma/\varepsilon} \text{ for } x \in \mathbb{R}^N \setminus B_{\delta(1-\theta)}(y),$$

provided σ is small such that $(1+10\mu)\sigma < \sqrt{V(x_0)}\delta(1-\theta)$, and

$$(2.52) \quad \begin{aligned} |f_\varepsilon^2(x)| &\leq C(|\omega|^p + |\omega|^{\bar{p}}) \\ &\leq \begin{cases} Ce^{-\bar{p}\sigma/\varepsilon}, & x \in \mathbb{R}^N \setminus B_\delta(y), \\ Ce^{-\bar{p}\sigma(1-\theta)/\varepsilon}, & x \in B_\delta(y) \setminus B_{\delta(1-\theta)}(y) \\ & x \in \mathbb{R}^N \setminus B_{\delta(1-\theta)}(y), \end{cases} \\ &\leq e^{-(1+10\mu)\sigma/\varepsilon}, \end{aligned}$$

provided μ is chosen so that $\bar{p}(1-\theta) > 1+10\mu$.

Similarly

$$(2.53) \quad |f_\varepsilon^3(x)| \leq e^{-(1+10\mu)\sigma/\varepsilon}, \quad x \in \mathbb{R}^N \setminus B_{\delta(1-\theta)}(y).$$

Suppose $\delta > 0, \gamma_0 > 0$ are small so that $V(x) \geq 6\gamma_0$ for $x \in \mathbb{R}^N \setminus \mathcal{Z}^\delta$. Let Ψ_1 be the solution of

$$\begin{cases} -\varepsilon^2 \Delta \Psi + 4\gamma_0 \Psi = 0, & x \in \mathbb{R}^N \setminus \mathcal{Z}^\delta, \\ \Psi(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty, \\ \Psi = 1 & \text{for } x \in \partial \mathcal{Z}^\delta. \end{cases}$$

By maximum principle, it is not difficult to see that

$$0 < \Psi_1(x) \leq Ce^{-3\gamma_0 \text{dist}(x, \mathcal{Z}^\delta)/\varepsilon}.$$

Let Ψ_2 be the solution of

$$\begin{cases} -\varepsilon^2 \Delta \Psi + 4\gamma_0 \Psi = 0, & x \in \mathbb{R}^N \setminus B_{\delta(1-\theta)}(y), \\ \Psi(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty, \\ \Psi = 1 & \text{for } x \in \partial B_{\delta(1-\theta)}(y). \end{cases}$$

We have

$$0 < \Psi_2(x) \leq Ce^{-3\gamma_0 \text{dist}(x, B_{\delta(1-\theta)}(y))/\varepsilon}.$$

Denote $a_\varepsilon(x) = V(x) - p(\Phi_\varepsilon(x) + U_{\varepsilon,y})^{p-1}$. Then

$$a_\varepsilon(x) \geq 6\gamma_0 - o(1) > 5\gamma_0 \quad \text{for } x \in \mathbb{R}^N \setminus (\mathcal{Z}^\delta \cup B_{\delta(1-\theta)}(y)).$$

Setting $\Psi = \Psi_1 + \Psi_2 + e^{-(1+5\mu)\sigma/\varepsilon}$ we have from (2.51)-(2.53) that for $x \in \mathbb{R}^N \setminus (\mathcal{Z}^\delta \cup B_{\delta(1-\theta)}(y))$

$$-\varepsilon^2 \Delta \Psi + a_\varepsilon(x) \Psi \geq -\varepsilon^2 \Delta \Psi + 5\gamma_0 \Psi \geq \gamma_0 e^{-(1+5\mu)\sigma/\varepsilon} \geq |f_\varepsilon|.$$

From step 2 we have $|\omega_1(x)| \leq C\varepsilon^{\frac{\sigma}{2}} < \Psi(x)$ for $x \in \partial(\mathcal{Z}^\delta \cup B_{\delta(1-\theta)}(y))$. Thus by maximum principle for $x \in \mathbb{R}^N \setminus (\mathcal{Z}^\delta \cup B_{\delta(1-\theta)}(y))$ we have

$$|\omega_1(x)| \leq \Psi(x).$$

Therefore

$$(2.54) \quad |\omega_1(x)| \leq Ce^{-(1+5\mu)\sigma/\varepsilon} \leq e^{-(1+4\mu)\sigma/\varepsilon}, \quad \text{for } x \in \mathbb{R}^N \setminus (\mathcal{Z}^\delta \cup B_{\delta(1-\theta)}(y)).$$

In particular we have

$$(2.55) \quad |\omega_1(x)| \leq e^{-(1+4\mu)\sigma/\varepsilon} \quad \text{for } x \in \mathcal{Z}^{3\delta} \setminus \mathcal{Z}^{\frac{5}{4}\delta}.$$

Next we show that (2.54) still holds even if $x \in \mathcal{Z}^\delta$. To this end, let Ψ_3 be the solution of

$$\begin{cases} -\Delta \Psi_3 = 0 & x \in \mathcal{Z}^{2\delta}, \\ \Psi_3(x) = \omega_1(x) & \text{for } x \in \partial \mathcal{Z}^{2\delta}. \end{cases}$$

Then it follows from maximum principle and (2.55) that

$$\Psi_3(x) \leq e^{-(1+4\mu)\sigma/\varepsilon} \quad \text{for } x \in \mathcal{Z}^{2\delta}.$$

Let $\bar{\omega}_1(x) = \omega_1(x) - \Psi_3(x)$. Then

$$(2.56) \quad \begin{aligned} & -\Delta \bar{\omega}_1 + \varepsilon^{-2}V(x)\bar{\omega}_1 - p\varepsilon^{-2}(\Phi_\varepsilon + U_{\varepsilon,y})^{p-1}\bar{\omega}_1 \\ & = \varepsilon^{-2}f_\varepsilon(x) - \varepsilon^{-2}V(x)\Psi_3(x) + p\varepsilon^{-2}(\Phi_\varepsilon + U_{\varepsilon,y})^{p-1}\Psi_3(x) \\ & = \varepsilon^{-2}f_\varepsilon(x) + O(e^{-(1+3\mu)\sigma/\varepsilon}). \end{aligned}$$

We claim that

$$(2.57) \quad \int_{\mathcal{Z}^{2\delta}} (|\nabla \bar{\omega}_1|^2 + \bar{\omega}_1^2) \leq Ce^{-(1+3\mu)\sigma/\varepsilon}.$$

Indeed, since for $\bar{\omega}_1 \in H_0^1(\widehat{\Omega}^{2\delta})$, it follows from Proposition 2.4 and the estimate of f_ε and (2.56) that

$$\int_{\widehat{\Omega}^{2\delta}} |\nabla \bar{\omega}_1|^2 \leq \int_{\widehat{\Omega}^{2\delta}} (|\nabla \bar{\omega}_1|^2 + \varepsilon^{-2}V(x)\bar{\omega}_1^2) \leq Ce^{-(1+3\mu)\sigma/\varepsilon},$$

which implies

$$\int_{\widehat{\Omega}^{2\delta}} (|\nabla \bar{\omega}_1|^2 + \bar{\omega}_1^2) \leq Ce^{-(1+3\mu)\sigma/\varepsilon},$$

since $\bar{\omega}_1 \in H_0^1(\widehat{\Omega}^{2\delta})$ and $\widehat{\Omega}^{2\delta}$ is bounded.

On the other hand, since the coefficient of the last term in the left hand side of (2.56), $p\varepsilon^{-2}(\Phi_\varepsilon + U_{\varepsilon,y})$, is exponentially small for $x \in \mathcal{Z}^{2\delta} \setminus \widehat{\Omega}^{2\delta}$, from the estimate of f_ε and (2.56) we have

$$\int_{\mathcal{Z}^{2\delta} \setminus \widehat{\Omega}^{2\delta}} (|\nabla \bar{\omega}_1|^2 + \bar{\omega}_1^2) \leq Ce^{-(1+3\mu)\sigma/\varepsilon},$$

and (2.57) therefore follows.

Rewrite (2.56) as

$$-\Delta \bar{\omega}_1 = \bar{f}_\varepsilon(x),$$

where

$$\bar{f}_\varepsilon(x) = -\varepsilon^{-2}V(x)\bar{\omega}_1 + p\varepsilon^{-2}(\Phi_\varepsilon + U_{\varepsilon,y})^{p-1}\bar{\omega}_1 + \varepsilon^{-2}f_\varepsilon(x) + O(e^{-(1+3\mu)\sigma/\varepsilon}).$$

Using the Moser iterative technique (see Gilbarg and Trudinger [19]), we have

$$(2.58) \quad \begin{aligned} \|\bar{\omega}_1\|_{L^\infty(\mathcal{Z}^{\frac{3}{2}\delta})} &\leq C(\|\bar{\omega}_1\|_{L^2(\mathcal{Z}^{2\delta})} + \|\bar{f}_\varepsilon\|_{L^2(\mathcal{Z}^{2\delta})}) \leq C\varepsilon^{-2}e^{-(1+3\mu)\sigma/\varepsilon} \\ &\leq Ce^{-(1+2\mu)\sigma/\varepsilon}, \end{aligned}$$

which, combined with (2.55), gives

$$\|\omega_1\|_{L^\infty(\mathcal{Z}^{2\delta})} \leq Ce^{-(1+2\mu)\sigma/\varepsilon}.$$

Thus, we complete step 3.

Step 4. $\omega_1 \in S_\varepsilon$.

By step 1, step 2 and step 3, we only need to obtain the estimate for $\omega_1(x)$ when $x \in B_\delta(y)$.

For $x \in B_\delta(y)$, we have

$$(2.59) \quad |\omega(x)|^{\bar{p}} \leq Ce^{-\bar{p}\sigma|x-y|/(\varepsilon\delta)} \varepsilon^{\sigma\bar{p}}$$

and

$$(2.60) \quad \begin{aligned} |V(x) - V(a_1)|U(\frac{x-y}{\varepsilon}) &\leq |V(x) - V(y)|U(\frac{x-y}{\varepsilon}) + |V(y) - V(a_1)|U(\frac{x-y}{\varepsilon}) \\ &\leq C|x-y|U(\frac{x-y}{\varepsilon}) + C\varepsilon^2U(\frac{x-y}{\varepsilon}) \\ &\leq C\varepsilon^{(1+\mu)\sigma}e^{-\sigma|x-y|/(\varepsilon\delta)}, \end{aligned}$$

provided σ is small so that $\sigma < \frac{1}{2}\delta\sqrt{V(x_0)}$. By step 2, for $x \in B_\delta(y)$,

$$(2.61) \quad \begin{aligned} (\Phi_\varepsilon + U_{\varepsilon,y})^{p-1}|\omega_1| &\leq C(W_\varepsilon^{p-1}(x) + \widetilde{W}_\varepsilon^{p-1}(x) + U^{p-1}(\frac{x-y}{\varepsilon}))|\omega_1(x)| \\ &\leq C\varepsilon^{(1+\mu)\sigma}e^{-\sigma|x-y|/(\varepsilon\delta)} \end{aligned}$$

if σ is small by (1.9), (1.10).

Let $\tilde{f}_\varepsilon(x) = f_\varepsilon(x) + p(\Phi_\varepsilon(x) + U_{\varepsilon,y}(x))^{p-1}\omega_1(x)$. Then

$$(2.62) \quad -\varepsilon^2\Delta\omega_1 + V(x)\omega_1 = \tilde{f}_\varepsilon(x).$$

By (2.59)-(2.61) we have

$$(2.63) \quad |\tilde{f}_\varepsilon(x)| \leq C\varepsilon^{(1+\mu)\sigma}e^{-\sigma|x-y|/(\varepsilon\delta)} \quad \forall x \in B_\delta(y).$$

On the other hand, let $\Psi_4(x) = \varepsilon^\sigma e^{-\sigma|x-y|/(\varepsilon\delta)}$. Then we have

$$(2.64) \quad -\varepsilon^2\Delta\Psi_4 + V(x)\Psi_4 \geq \frac{1}{2}V(x_0)\varepsilon^\sigma e^{-\sigma|x-y|/(\varepsilon\delta)} \geq |\tilde{f}_\varepsilon(x)|.$$

Notice that by step 3,

$$(2.65) \quad |\omega_1(x)| \leq Ce^{-(1+\mu)\sigma/\varepsilon} \leq \Psi_4(x) \quad \text{for } x \in \partial B_\delta(y).$$

By maximum principle we obtain from (2.62), (2.64) and (2.65)

$$|\omega_1(x)| \leq \Psi_4(x) = \varepsilon^\sigma e^{-\sigma|x-y|/(\varepsilon\delta)} \quad \text{for } x \in B_\delta(y),$$

which is exactly what we want. Hence step 4 is completed.

Thus, $G_\varepsilon(\omega)$ is a contraction mapping from $\mathcal{S}_\varepsilon$ to itself. Therefore, by the contraction mapping theorem, $G_\varepsilon(\omega)$ has a fixed point $\omega_\varepsilon \in \mathcal{S}_\varepsilon$. As a result,

$$(2.66) \quad \omega_{\ell_\varepsilon} + L_\varepsilon\omega_\varepsilon + R'_\varepsilon(\omega_\varepsilon) = 0.$$

Consequently, there exist $D_{j,\varepsilon} \in \mathbb{R}$ for $j = 1, \dots, N$ such that

$$I'(U_{\varepsilon,y} + \omega_\varepsilon) = \sum_{j=1}^N D_{j,\varepsilon} \frac{\partial U_{\varepsilon,y}}{\partial x_j}.$$

Moreover, by step 1 and step 2, we have

$$\begin{cases} \|\omega_\varepsilon\|_\varepsilon = O(\varepsilon^{\frac{N}{2}+2}), \\ D_{i,\varepsilon} = O(\varepsilon^3). \end{cases}$$

This completes our proof of Proposition 2.5. $\square$

3. PROOF OF THE MAIN RESULT

In this section we will give the proof of Theorem 1.3. The proof of Theorem 1.1 is similar. Let us point out that in the case $l = 0$, the result follows directly from Proposition 2.3 and the implicit function theorem.

Proof of Theorem 1.3. We need to prove that there is $y_\varepsilon \in B_\delta(a_1)$ such that (1.23) holds.

Direct calculations show that

$$\begin{aligned} & \frac{\partial J}{\partial y_j} \\ &= \langle U_{\varepsilon,y} + \omega_\varepsilon, \frac{\partial U_{\varepsilon,y}}{\partial y_j} \rangle - \int_{\mathbb{R}^N} (U_{\varepsilon,y} + \omega_\varepsilon)_+^p \frac{\partial U_{\varepsilon,y}}{\partial y_j} - \int_{\mathbb{R}^N} g_\varepsilon(x, U_{\varepsilon,y} + \omega_\varepsilon) \frac{\partial U_{\varepsilon,y}}{\partial y_j} \\ &= \int_{\mathbb{R}^N} V(x) U_{\varepsilon,y} \frac{\partial U_{\varepsilon,y}}{\partial y_j} - \int_{\mathbb{R}^N} ((U_{\varepsilon,y} + \omega_\varepsilon)_+^p - U_{\varepsilon,y}^p - p U_{\varepsilon,y}^{p-1} \omega_\varepsilon) \frac{\partial U_{\varepsilon,y}}{\partial y_j} \\ (3.1) \quad & - \int_{\mathbb{R}^N} g_\varepsilon(x, U_{\varepsilon,y} + \omega_\varepsilon) \frac{\partial U_{\varepsilon,y}}{\partial y_j} \\ &= \int_{\mathbb{R}^N} V(x) U_{\varepsilon,y} \frac{\partial U_{\varepsilon,y}}{\partial y_j} + O(e^{-\gamma/\varepsilon} + \varepsilon^{-1} \|\omega_\varepsilon\|_\varepsilon^2) \\ &= \frac{1}{2} \int_{\mathbb{R}^N} U_{\varepsilon,y}^2 \frac{\partial V}{\partial x_j} + O(\varepsilon^{N+3}) \\ &= \varepsilon^N \frac{\partial V(y)}{\partial y_j} \frac{1}{2} \int_{\mathbb{R}^N} U^2 + O(\varepsilon^{N+1+\alpha}), \end{aligned}$$

since

$$\int_{\mathbb{R}^N} U_{\varepsilon,y}^2 (x_i - y_i) dx = 0.$$

By step 2 in Proposition 2.5, we have

$$D_{j,\varepsilon} = O(\varepsilon^3).$$

So, it is easy to see that (1.23) is equivalent to

$$\nabla V(y) = O(\varepsilon^{1+\alpha}).$$

Since a_1 is a nondegenerate critical point of $V(x)$, we see that the above relation is equivalent to

$$y - a_1 = h_\varepsilon(y),$$

where $h_\varepsilon(y)$ satisfies $|h_\varepsilon(y)| \leq C\varepsilon^{1+\alpha}$ for all $y \in B_\varepsilon(a_1)$. Thus, h_ε maps $B_\varepsilon(a_1)$ into itself. By the fixed point theorem, we deduce that there is a $y_\varepsilon \in B_\varepsilon(a_1)$ such that (1.23) holds.

Therefore $U\left(\frac{x-y_\varepsilon}{\varepsilon}\right) + \omega_\varepsilon$ is a critical point of I in $H^1(\mathbb{R}^N)$, which yields that $u_\varepsilon(x) = W_\varepsilon(x) + \widetilde{W}_\varepsilon(x) + U\left(\frac{x-y_\varepsilon}{\varepsilon}\right) + \omega_\varepsilon$ is a solution of

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = u_+^p, & x \in \mathbb{R}^N, \\ u(x) \rightarrow 0, & \text{as } |x| \rightarrow \infty. \end{cases}$$

By the maximum principle, we see that $u_\varepsilon > 0$. $\square$

We can also use the above techniques to deal with some degenerate case. We have

Theorem 3.1. *Suppose that $k = t = l = 1$. Assume (V1), (V2), and (V3) hold. Suppose that a_1 is an isolated strict local maximum point or minimum point of $V(x)$ with $V(a_1) > 0$. Then for $\varepsilon > 0$ small, (1.6) has a solution u_ε such that*

$$u_\varepsilon(x) = W_\varepsilon(x) + \widetilde{W}_\varepsilon(x) + U\left(\frac{x-y_\varepsilon}{\varepsilon}\right) + \omega_\varepsilon(x)$$

where $\omega_\varepsilon(x)$ and y_ε satisfy (1.17), (1.18) and (1.19) in Theorem 1.1.

Proof. First, we need to modify the proof of Proposition 2.5. Define

$$\mathcal{M}_\varepsilon = \{y \in B_\delta(a_1) \mid |V(y) - V(a_1)| \leq \varepsilon^{(1+\tau)\sigma}\},$$

where $\tau > 0$ and $\sigma > 0$ are some small constants. Then, similar to the proof of Proposition 2.5, we can prove that for each $y \in \mathcal{M}_\varepsilon$, there is an $\omega_\varepsilon \in \mathcal{S}_\varepsilon$ such that (2.31) holds and

$$\|\omega_\varepsilon\|_\varepsilon = O(\varepsilon^{(1+\tau)\sigma}).$$

Define

$$K(y) = J(y, \omega_\varepsilon),$$

where ω_ε is the function obtained above.

It follows from (2.2), Lemmas 2.1 and 2.2 that

$$\begin{aligned} K(y) &= J(y, 0) + O(\|\ell_\varepsilon\|_\varepsilon \|\omega_\varepsilon\|_\varepsilon + \|\omega_\varepsilon\|_\varepsilon^2) \\ &= I(U_{\varepsilon, y}) + O(\varepsilon^{N+2(1+\tau)\sigma}). \end{aligned} \quad (3.2)$$

On the other hand, by the definition of $g_\varepsilon(x, t)$, it is easy to see

$$|G_\varepsilon(x, t)| \leq C(\widetilde{W}_\varepsilon + W_\varepsilon)(|t| + |t|^p).$$

Since $\widetilde{W}_\varepsilon$ and W_ε are exponentially small near y and $U_{\varepsilon, y}$ is exponentially small outside a small neighborhood of y , we obtain

$$\left| \int_{\mathbb{R}^N} G(x, U_{\varepsilon, y}) \right| \leq C e^{-\sigma'/\varepsilon}, \quad (3.3)$$

for some $\sigma' > 0$.

Combining (3.2) and (3.3), we obtain

$$K(y) = I^*(U_{\varepsilon, y}) + O(\varepsilon^{N+2(1+\tau)\sigma}), \quad (3.4)$$

where

$$I^*(u) = \frac{1}{2} \int_{\mathbb{R}^N} (\varepsilon^2 |\nabla u|^2 + V(x)u^2) - \frac{1}{p+1} \int_{\mathbb{R}^N} u_+^{p+1}.$$

Standard calculations show that

$$I^*(U_{\varepsilon, y}) = A\varepsilon^N + B(V(y) - V(a_1))\varepsilon^N + O(\varepsilon^{N+2}), \quad (3.5)$$

where $A = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^N} U^{p+1}$, $B = \frac{1}{2} \int_{\mathbb{R}^N} U^2$.

Combining (3.4) and (3.5), we obtain

$$(3.6) \quad K(y) = A\varepsilon^N + B(V(y) - V(a_1))\varepsilon^N + O(\varepsilon^{N+2(1+\tau)\sigma}).$$

Let us first consider the case that a_1 is a strict local maximum point. Consider

$$(3.7) \quad K_{\max} = \max\{K(y) \mid y \in \mathcal{M}_\varepsilon\}.$$

Suppose that $y_\varepsilon \in \mathcal{M}_\varepsilon$ achieves $K_{\max}$. We will prove that y_ε is an interior point of $\mathcal{M}_\varepsilon$. So y_ε is a critical point of $K(y)$.

We have

$$(3.8) \quad K(y_\varepsilon) \geq K(a_1) = A\varepsilon^N + O(\varepsilon^{N+2(1+\tau)\sigma}),$$

which, together with (3.6), gives

$$A\varepsilon^N + B(V(y_\varepsilon) - V(a_1))\varepsilon^N + O(\varepsilon^{N+2(1+\tau)\sigma}) \geq A\varepsilon^N + O(\varepsilon^{N+2(1+\tau)\sigma}).$$

As a result,

$$0 \leq V(a_1) - V(y_\varepsilon) \leq C\varepsilon^{2(1+\tau)\sigma} < \varepsilon^{(1+\tau)\sigma}.$$

This shows that y_ε is an interior point of $\mathcal{M}_\varepsilon$. Standard argument shows that (1.23) holds.

Suppose that a_1 is a strict local minimum point. Consider

$$(3.9) \quad K_{\min} = \min\{K(y) \mid y \in \mathcal{M}_\varepsilon\}.$$

Suppose that $\tilde{y}_\varepsilon$ achieves $K_{\min}$. Then, similar to the above argument, we can prove that $\tilde{y}_\varepsilon$ is an interior point of $\mathcal{M}_\varepsilon$, and thus a critical point of $K(y)$. $\square$

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