ABELIAN STRICT APPROXIMATION IN AW^* -ALGEBRAS AND WEYL-VON NEUMANN TYPE THEOREMS

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Dedicated to Professor E. Effros on his 70th birthday

ABSTRACT. In this paper, for a C^* -algebra A with $M = M(A)$ an AW^* -algebra, or equivalently, for an essential, norm-closed, two-sided ideal A of an AW^* -algebra M , we investigate the strict approximability of the elements of M from commutative C^* -subalgebras of A . In the relevant case of the norm-closed linear span A of all finite projections in a semi-finite AW^* -algebra M we shall give a complete description of the strict closure in M of any maximal abelian self-adjoint subalgebra (masa) of A . We shall see that the situation is completely different for discrete, respectively continuous, M :

In the discrete case, for any masa C of A , the strict closure of C is equal to the relative commutant $C' \cap M$, while in the continuous case, under certain conditions concerning the center valued quasitrace of the finite reduced algebras of M (satisfied by all von Neumann algebras), C is already strictly closed. Thus in the continuous case no elements of M which are not already belonging to A can be strictly approximated from commutative C^* -subalgebras of A .

In spite of this pathology of the strict topology in the case of the norm-closed linear span of all finite projections of a continuous semi-finite AW^* -algebra, we shall prove that in general situations also including this case, any normal $y \in M$ is equal modulo A to some $x \in M$ which belongs to an order theoretical closure of an appropriate commutative C^* -subalgebra of A . In other words, if we replace the strict topology with order theoretical approximation, Weyl-von Neumann-Berg-Sikonia type theorems will hold in substantially greater generality.

INTRODUCTION

Let A be a C^* -algebra. The *multiplier algebra* of A is the C^* -subalgebra

$$\{x \in A^{**}; \quad xa, \quad ax \in A \text{ for all } a \in A\}$$

of the second dual A^{**} (see [25], Section 3.12, or [30], Chapter 2). A natural locally convex vector space topology on $M(A)$, called the *strict topology* β , is defined by the seminorms

$$x \mapsto \|xa\| \text{ and } x \mapsto \|ax\|, \quad a \in A.$$

It is complete and compatible with the duality between $M(A)$ and A^* . Hence the strict topology is weaker than the norm-topology on $M(A)$, but stronger than the

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restriction to $M(A)$ of the weak $*$ topology of A^{**} . In particular, A is strictly dense in $M(A)$.

We notice that for A the C^* -algebra $K(H)$ of all compact linear operators on a complex Hilbert space H , $M(A)$ can be identified with the C^* -algebra $B(H)$ of all bounded linear operators on H , and on every bounded subset of $B(H)$ the strict topology coincides with the s^* -topology.

More generally, if M is an AW^* -algebra (see [17], or [6], §4, or [28], §9) and A is an essential, norm-closed, two-sided ideal of M , then, by a theorem of B. E. Johnson, M can be identified with $M(A)$ (see [13] or [26]). Thus the pairs $(A, M(A))$, where A is a C^* -algebra such that $M(A)$ is an AW^* -algebra, are exactly the pairs (A, M) , where M is an AW^* -algebra and A is an essential, norm-closed, two-sided ideal of M .

A relevant case of an essential, norm-closed, two-sided ideal of an AW^* -algebra is the norm-closed linear subspace A generated by all finite projections of a semi-finite AW^* -algebra M . Then there are central projections p_1, p_2, p_3 of M with $p_1 + p_2 + p_3 = 1_M$ such that Mp_1 is finite, Mp_2 is properly infinite and discrete, while Mp_3 is properly infinite and continuous (see [6], §15, Theorem 1). Since $Ap_1 = Mp_1$, the non-trivial cases are Ap_2 and Ap_3 , with $M(Ap_2) = Mp_2$ properly infinite and discrete and $M(Ap_3) = Mp_3$ properly infinite and continuous.

Throughout the C^* -algebra theory the possibility to reduce certain verifications to the case of commutative C^* -algebras is an important issue. Concerning the strict approximation of the multipliers of a C^* -algebra A with elements of A , such a reduction would be clearly possible if every normal $x \in M(A)$ would belong to the strict closure of some commutative C^* -subalgebra of A , hence to the strict closure of some maximal abelian self-adjoint subalgebra (*masa*) C_x of A , in which case we say that x belongs to the *abelian strict closure* of A . Unfortunately, this is not true even in the case of $A = K(H)$ with infinite-dimensional H .

However, by the classical Weyl-von Neumann-Berg-Sikonia (WNBS) Theorem, if $A = K(H)$, where H is a separable complex Hilbert space, then every normal element of $M(A) = B(H)$ is of the form $a + x$ with $a \in A$ and x in the strict closure of some masa of A . Extensions of this result to general C^* -algebras A , which are σ -unital (that is, the unit of $M(A)$ is the strict limit of a sequence in A , which of course can be chosen to belong to a commutative C^* -subalgebra of A) and with $M(A)$ of real rank zero (see [9]), were obtained in [23] and [31] (see also [12], [19], [20], [21]).

It is natural to ask to which extent generalizations of the WNBS Theorem hold if we renounce to one of the two assumptions above. For general σ -unital C^* -algebras the only result we know is contained in our previous paper [10]. There we proved a partial extension of the WNBS Theorem for an arbitrary σ -unital C^* -algebra A (see [10], Theorem 1), which implies that each element $y \in M(A)$ is of the form $a + x_1 + x_2$, where $a \in A$, $x_1 \in B_1$, $x_2 \in B_2$ with B_1, B_2 separable C^* -subalgebras of $M(A)$ such that every normal element of B_j , $j = 1, 2$, belongs to the abelian strict closure of A . Moreover, if y is self-adjoint, then x_1, x_2 can be chosen self-adjoint, so in this situation x_1, x_2 themselves belong to the abelian strict closure of A . Though unsatisfactory, this result still allows the reduction of the proof of an important result of L. G. Brown (concerning the non-existence of non-zero separable hereditary C^* -subalgebras of the corona algebra of a σ -unital C^* -algebra; see [8]) to the commutative case.

In the present paper we discuss abelian strict approximability for a C^* -algebra A which is the norm-closed linear subspace generated by all finite projections of some semi-finite AW^* -algebra M . We recall that any AW^* -algebra is of real rank zero, so in this case $M(A) = M$ is of real rank zero. Since the abelian strict closure of A is the union of all $\overline{C}^\beta$ with C a masa of A , we are interested in describing $\overline{C}^\beta$ for any masa C of A . We shall see that the situation is completely different for discrete, respectively continuous, M :

In the discrete case $\overline{C}^\beta$ is equal to the relative commutant $C' \cap M(A)$ (Theorem 1.2), while in the continuous case, under a certain condition on the centre valued quasitrace of the reduced AW^* -subalgebras of $M(A)$ by finite projections (always satisfied if $M(A)$ is a von Neumann algebra), C is already strictly closed (Theorem 2.6).

Consequently, if M is a properly infinite, continuous, semi-finite ($= \Pi_\infty$) AW^* -algebra satisfying the above mentioned condition and A is the norm-closed linear span of all finite projections of M , then the unit of $M(A) = M$ does not belong to the abelian strict closure of A , that is, there is no approximate unit for A contained in a commutative $*$ -subalgebra of A . In particular, in this case A is not σ -unital. We notice that it was already shown in [1], Proposition 4.5, that the norm-closed linear span of all finite projections of a type Π_∞ factor is a non- σ -unital C^* -algebra. Nevertheless, also in this case WNBS type theorems can be proved. Indeed, if A is the norm-closed linear subspace generated by all finite projections of some countably decomposable semi-finite W^* -factor M , then, according to [32], Theorem 3.1, every normal $y \in M(A) = M$ is of the form $a + x$ with $a \in A$ and x in the s^* -closure in M of some masa C of A . Since the s^* -closure of a commutative $*$ -subalgebra of a W^* -algebra is equal to its monotone order closure (cf. [14] and [24]), it is natural to expect that for extensions of the WNBS Theorem to non- σ -unital C^* -algebras the strict closure should be replaced by an order theoretical closure. Along this line we prove several WNBS type theorems in a general setting which includes the case of the norm-closed linear span of all finite projections of a countably decomposable semi-finite AW^* -algebra.

More precisely, we prove that if $\mathcal{J}$ is a norm-closed two-sided ideal of a (unital) Rickart C^* -algebra M (a Rickart C^* -algebra is a C^* -algebra in which every positive element has a support projection, in particular it is of real rank zero), which has a countable “order theoretical approximate unit”, then any normal $y \in M$ is of the form $y = a + x$, where $a \in A$ is of arbitrarily small norm and x belongs to the order theoretical closure of some masa of $\mathcal{J}$ (Theorem 3.2 and the subsequent remark). Moreover, the above x can be chosen as a particular infinite linear combination of a sequence of mutually orthogonal projections from $\mathcal{J}$ (Theorems 3.5 and 3.6).

Since only little of the specific properties of Rickart C^* -algebras is used, we are left with the question as to which extent the above mentioned WNBS type theorems hold if M is assumed to be only a C^* -algebra of real rank zero.

1. ABELIAN STRICT CLOSURE IN DISCRETE AW^* -ALGEBRAS

First we prove a general result concerning a masa C of a C^* -algebra A , whose multiplier algebra is an AW^* -algebra; that is, according to the theorem of B. E. Johnson quoted in the Introduction (see [13] or [26]), a masa C of an essential, norm-closed, two-sided ideal A of some AW^* -algebra. We notice that a part of this result holds for a masa of an essential, norm-closed, two-sided ideal of

any Rickart C^* -algebra. We shall restrict ourselves to unital Rickart C^* -algebras, because adjoining a unit to a non-unital Rickart C^* -algebra M , we obtain a unital Rickart C^* -algebra $\widetilde{M}$ (see [6], §5, Theorem 1, or [28], 9.11.(1)), and it is easy to see that every essential, norm-closed, two-sided ideal of M is an essential, norm-closed, two-sided ideal also of $\widetilde{M}$.

Any essential, two-sided ideal $\mathcal{J}$ of a C^* -algebra M induces a strict topology $\beta_{\mathcal{J}}$ on M , defined by the seminorms

$$M \ni x \mapsto \|xa\| \text{ and } x \mapsto \|ax\|, \quad a \in \mathcal{J}.$$

With this definition, the usual strict topology on the multiplier algebra of a C^* -algebra A is β_A .

For the basic facts concerning Rickart C^* -algebras and AW^* -algebras see [6], §§ 3, 4 and 5, or [28], §9.

Lemma 1.1. *Let M be a unital C^* -algebra, $\mathcal{J}$ an essential, norm-closed, two-sided ideal of M , and C a masa of $\mathcal{J}$. By the strict topology on M we shall understand $\beta_{\mathcal{J}}$, which of course is the usual strict topology when M is an AW^* -algebra and so can be identified with the multiplier algebra $M(\mathcal{J})$. Then*

- (i) *every $x \geq 0$ in the strict closure of C in M belongs to the strict closure of $\{b \in C; 0 \leq b \leq x\}$ in M .*

Let us next assume that M is a Rickart C^ -algebra. Then*

- (ii) *for every $0 \leq b \in C$ and every $\delta > 0$ there is a projection $f_{\delta} \in C$ such that*

$$bf_{\delta} \geq \delta f_{\delta}, \quad b(1_M - f_{\delta}) \leq \delta(1_M - f_{\delta}),$$

so C is the norm-closed linear span of its projections;

- (iii) *any projection e in the strict closure of C in M belongs to the strict closure of $\{f \in C; f \leq e \text{ projection}\}$ in M ;*
- (iv) *any projection e in the relative commutant $C' \cap M$ is the least upper bound of $\{f \in C; f \leq e \text{ projection}\}$ in the projection lattice of M , in particular $C' \cap M$ is a masa of M .*

Finally, assuming M to be an AW^ -algebra,*

- (v) *the relative commutant $C' \cap M$ is the AW^* -subalgebra of M generated by C , so $C' \cap M$ can be identified with $M(C)$;*
- (vi) *the strict closure of C in M coincides with $C' \cap M$ if and only if C contains a two-sided approximate unit for $\mathcal{J}$, in which case the strict topology of $M(C) = C' \cap M$ is the restriction of the strict topology of $M(\mathcal{J}) = M$.*

Proof. The strict closure $\overline{C}^{\beta_{\mathcal{J}}}$ of C being an abelian C^* -subalgebra of $M(A)$, we have for every $b \in C$

$$(x - b)^*(x - b) \geq (x - \operatorname{Re} b)^2 \geq (x - (\operatorname{Re} b)_+)^2 \geq (x - b_o)^2,$$

where

$$b_o = \frac{1}{2} \left(x + (\operatorname{Re} b)_+ - |x - (\operatorname{Re} b)_+| \right)$$

denotes the greatest lower bound of x and $(\operatorname{Re} b)_+$ in the Hermitian part of $\overline{C}^{\beta_{\mathcal{J}}}$. Since

$$0 \leq b_o \leq (\operatorname{Re} b)_+ \in C \subset \mathcal{J},$$

by [25], Proposition 1.4.5, we have $b_o \in \mathcal{J}$, so

$$b_o \in C' \cap \mathcal{J} = C.$$

Thus, for every $a \in \mathcal{J}$ and $b \in C$ we have $\|(x - b)a\| \geq \|(x - b_o)a\|$ for some $0 \leq b_o \leq x$ in C and (i) follows.

For (ii) put

$$f_\delta = \text{support of } (b - \delta 1_M)_+ \text{ in } M.$$

Then f_δ commutes with every element of C and

$$bf_\delta \geq \delta f_\delta, \quad b(1_M - f_\delta) \leq \delta(1_M - f_\delta).$$

In particular, $f_\delta \leq \frac{1}{\delta}b \in A$ and [25], Proposition 1.4.5, yields $f_\delta \in \mathcal{J}$. Consequently $f_\delta \in C' \cap A = C$.

For (iii) let $0 \neq a \in \mathcal{J}$ and $\varepsilon > 0$ be arbitrary. According to (i) there exists $0 \leq b \leq e$ in C such that

$$\|(e - b)a\| < \frac{\varepsilon}{2}.$$

Further, by (ii) there is a projection $f \in C$ with

$$bf \geq \frac{\varepsilon}{2\|a\|}f, \quad b(1_M - f) \leq \frac{\varepsilon}{2\|a\|} \cdot (1_M - f).$$

Then $f \leq e$ and $e - f \leq (e - bf)^2$, so

$$\begin{aligned} \|(e - f)a\| &= \|a^*(e - f)a\|^{1/2} \\ &\leq \|a^*(e - bf)^2a\|^{1/2} = \|(e - bf)e\| \\ &\leq \|(e - b)e\| + \|b(1_M - f)e\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2\|a\|}\|a\| = \varepsilon. \end{aligned}$$

For (iv) we have to show that if a projection $g \in M$ majorizes all projections $C \ni f \leq e$, then $g \geq e$, that is, e is equal to the greatest lower bound $e \wedge g$ of e and g in the projection lattice of M . Let us assume that

$$e_o = e - e \wedge g \neq 0.$$

Since $\mathcal{J}$ is essential ideal in M , there exists $a \in \mathcal{J}$ with $ae_o \neq 0$. Choosing some $0 < \delta < \|e_o a^* a e_o\|$ and putting

$$e_1 = \text{support of } (e_o a^* a e_o - \delta 1_M)_+ \text{ in } M,$$

we have

$$0 \neq e_1 \leq \frac{1}{\delta}e_o a^* a e_o \in \mathcal{J}.$$

Clearly, $e_1 \leq e_o$ and [25], Proposition 1.4.5, yields also $e_1 \in \mathcal{J}$. Furthermore, for every projection $f \in C$ we get successively

$$\begin{aligned} fe &\in C' \cap \mathcal{J} = C \text{ and } fe \leq e, \\ fe &\leq e \wedge g, \text{ hence } fe_o = (fe)e_o = 0, \\ fe_1 &= (fe_o)e_1 = 0. \end{aligned}$$

Taking into account (ii), it follows that

$$be_1 = 0 \text{ for all } b \in C,$$

in particular

$$e_1 \in C' \cap \mathcal{J} = C.$$

But then $e_1 \leq e_o \leq e$ implies $e_1 \leq e \wedge g$, which contradicts $0 \neq e_1 \leq e_o = e - e \wedge g$.

In particular, $C' \cap M$ is commutative. For the proof we notice that, since $C' \cap M$ is a Rickart C^* -subalgebra of M (see [6], §5, Proposition 5, or [28], 9.12.(1)), it is

the norm-closed linear span of its projections (see e.g. [28], 9.4) and therefore it is enough to show that any two projections $e_1, e_2 \in C' \cap M$ commute. But the *-automorphism $M \ni x \mapsto (2e_2 - 1_M)x(2e_2 - 1_M) \in M$ leaves fixed C , hence also the least upper bound of any projection family in C in the projection lattice of M . Therefore it leaves fixed e_1 , that is, $e_1e_2 = e_2e_1$.

Moreover, $C' \cap M$ is a masa of M . Indeed, if $C_o \supset C' \cap M$ is a commutative subalgebra of M , then $C_o \supset C$ and thus we also have $C_o \subset C_o' \cap M \subset C' \cap M$.

For (v) we first notice that $C' \cap M$ is an AW^* -subalgebra of M containing C (see [6], §4, Proposition 8, or [28], 9.24.(1)). Now let N be any AW^* -subalgebra of M containing C . By (iv) N contains all projections from $C' \cap M$, hence $N \supset C' \cap M$. Consequently $C' \cap M$ is the AW^* -subalgebra of M generated by C .

Further, C is a two-sided ideal of $C' \cap M$:

$$b \in C \text{ and } y \in C' \cap M \implies by \in C' \cap \mathcal{J} = C.$$

Moreover, it is essential, because a projection $e \in C' \cap M$ with $Ce = \{0\}$ belongs to the AW^* -subalgebra of $C' \cap M$ generated by C only if $e = 0$. Hence we can identify $C' \cap M$ with $M(C)$ (see [13] or [26]).

Finally we prove (vi). If the strict closure of C in M is $C' \cap M \ni 1_M$, then there exists a net $(u_\iota)_\iota$ in C with $u_\iota \rightarrow 1_M$ strictly in M , that is,

$$\|a - u_\iota a\| \rightarrow 0 \text{ and } \|a - u_\iota a\| \rightarrow 0 \text{ for all } a \in \mathcal{J}.$$

Conversely, let us assume that C contains a two-sided approximate unit $(u_\iota)_\iota$ for $\mathcal{J}$. Then the strict topology β_C of $M(C) = C' \cap M$ agrees with the strict topology $\beta_{\mathcal{J}}$ of $M(\mathcal{J}) = M$ on every norm bounded subset of $C' \cap M$. Indeed, if $(y_\lambda)_\lambda$ is a norm bounded net in $C' \cap M$, convergent to 0 with respect to β_C , and $0 \neq a \in \mathcal{J}$, $\varepsilon > 0$ are arbitrary, then there exists ι_o such that

$$\|y_\lambda\| \cdot \|a - u_{\iota_o} a\| < \frac{\varepsilon}{2} \text{ for all } \lambda,$$

and then there exists some λ_o with

$$\|y_\lambda u_{\iota_o}\| < \frac{\varepsilon}{2\|a\|} \text{ for every } \lambda \geq \lambda_o.$$

It follows for every $\lambda \geq \lambda_o$:

$$\|y_\lambda a\| \leq \|y_\lambda(a - u_{\iota_o} a)\| + \|y_\lambda u_{\iota_o} a\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2\|a\|} \|a\| = \varepsilon.$$

But β_C is the finest locally convex vector space topology on $C' \cap M$ that agrees with β_C on every norm bounded subset of $C' \cap M$ (see [29], Cor. 2.7). Thus the restriction of $\beta_{\mathcal{J}}$ to $C' \cap M$, which is plainly finer than β_C , is actually equal to β_C . In particular, the β_C -density of C in $M(C)$ implies the $\beta_{\mathcal{J}}$ -density of C in $C' \cap M$. $\square$

It is well known that every commutative AW^* -algebra Z is monotone complete (see e.g. [28], 9.26, Proposition 1). If M is an arbitrary AW^* -algebra, we call

$$\Phi : \{e \in M; e \text{ projection}\} \rightarrow Z^+$$

completely additive whenever, for every family $(e_\iota)_\iota$ of mutually orthogonal projections in M , we have

$$\Phi\left(\bigvee_\iota e_\iota\right) = \sum_\iota \Phi(e_\iota),$$

where the sum stands for the least upper bound in Z^+ of all finite sums of $\Phi(e_\iota)$.

Now we describe the strict closure of a masa of the norm-closed two-sided ideal generated by the finite projections of a discrete semi-finite AW^* -algebra :

Theorem 1.2 (on the abelian strict closure in discrete AW^* -algebras). *Let M be a discrete AW^* -algebra, A the norm-closed linear span of all finite projections of M , and C a masa of A . Then the strict closure of C in $M(A) = M$ is equal to $C' \cap M$.*

Proof. According to Lemma 1.1 (vi), we have to show that C contains a two-sided approximate unit for A . Without loss of generality we may assume that $A \neq \{0\}$, hence $C \neq \{0\}$.

Let $(e_\iota)_{\iota \in I}$ be a maximal family of mutually orthogonal non-zero projections in C . Then

$$\bigvee_{\iota} e_\iota = 1_M.$$

Indeed, $e_o = 1_M - \bigvee_{\iota} e_\iota$ belongs to $C' \cap M$, so Lemma 1.1 (iv) yields

$$e_o = \bigvee \{f \in C; f \leq e_o \text{ projection}\}.$$

Thus $e_o \neq 0$ would imply the existence of some projection $0 \neq f \leq e_o$ in C , contradicting the maximality of $(e_\iota)_{\iota \in I}$.

Denoting by Z the centre of M , we call central partition of 1_M any set of mutually orthogonal projections in Z with least upper bound 1_M . The projections

$$\bigvee_{p \in \mathcal{P}} \left(\sum_{\iota \in I_p} e_\iota \right) p, \quad \mathcal{P} \text{ a central partition of } 1_M, \quad I_p \subset I \text{ finite for any } p \in \mathcal{P}$$

belong to $C' \cap M$ and are finite (see [6], §15, Proposition 8), hence they belong to $C' \cap A = C$. We show that their family is an (increasing positive) approximate unit for A . For we have to prove that every finite projection e in M has the property

$$(P) \quad \begin{cases} \text{for every } \varepsilon > 0 \text{ there are } \mathcal{P} \text{ and } I_p, p \in \mathcal{P}, \text{ with} \\ \left\| \left(1_M - \bigvee_{p \in \mathcal{P}} \left(\sum_{\iota \in I_p} e_\iota \right) p \right) e \right\| \leq \varepsilon. \end{cases}$$

But standard arguments show that every finite projection e in M is of the form

$$e = \bigvee_{n \geq 1} (e_{n,1} + \cdots + e_{n,n}) p_n,$$

where $p_n, n \geq 1$ are mutually orthogonal projections in Z and, for every $n \geq 1$, $e_{n,1}, \dots, e_{n,n}$ are mutually orthogonal abelian projections of central support p_n (use [6], §18, Exercises 3, 4, and Proposition 1), so it is enough to prove (P) for every abelian projection e in M . Moreover, since every abelian projection is majorized by an abelian projection of central support 1_M , without loss of generality we can restrict ourselves to the case of an abelian projection e of central support 1_M .

For every $x \in M$ there exists a unique $\Phi_e(x) \in Z$ such that

$$exe = \Phi_e(x)e$$

(see [6], §15, Proposition 6, and §5). Clearly, $\Phi_e : M \rightarrow Z$ is a conditional expectation and, according to [18], Lemma 7, it is completely additive on the projection lattice of M . Furthermore, $Z \ni z \mapsto ze \in Ze$ being $*$ -isomorphism, we have

$$\|xe\|^2 = \|ex^*xe\| = \|\Phi_e(x^*x)e\| = \|\Phi_e(x^*x)\|, \quad x \in M.$$

Now, by the complete additivity of Φ_e ,

$$\sum_{\iota} \Phi_e(e_{\iota}) = \Phi_e(1_M) = 1_M.$$

Thus, according to [18], Lemma 5, for every $\varepsilon > 0$ there exist a central partition $\mathcal{P}$ of 1_M and finite sets $I_p \subset I, p \in \mathcal{P}$ such that

$$\left\| \left(1_M - \sum_{\iota \in I_p} \Phi_e(e_{\iota}) \right) p \right\| \leq \varepsilon^2 \text{ for all } p \in \mathcal{P}.$$

But then we have for every $p \in \mathcal{P}$

$$\begin{aligned} \left\| \left(1_M - \sum_{\iota \in I_p} e_{\iota} \right) p e \right\|^2 &= \left\| \Phi_e \left(1_M - \sum_{\iota \in I_p} e_{\iota} \right) p \right\|^2 \\ &= \left\| \left(1_M - \sum_{\iota \in I_p} \Phi_e(e_{\iota}) \right) p \right\|^2 \leq \varepsilon^2, \end{aligned}$$

so, taking into account [17], Lemma 2.5,

$$\left\| \left(1_M - \bigvee_{p \in \mathcal{P}} \left(\sum_{\iota \in I_p} e_{\iota} \right) p \right) e \right\| = \sup_{p \in \mathcal{P}} \left\| \left(1_M - \sum_{\iota \in I_p} e_{\iota} \right) p e \right\| \leq \varepsilon. \quad \square$$

2. ABELIAN STRICT CLOSURE IN CONTINUOUS AW^* -ALGEBRAS

For the treatment of the case of continuous M we need several lemmas on AW^* -algebras, which could be of interest for themselves. First we extend [32], Lemma 2.2, concerning a Darboux property of normal functionals on von Neumann algebras without minimal projections, to the case of centre valued completely additive maps on the projection lattice of a continuous AW^* -algebra (similar results can be found in [5] and, for tracial maps, in [15], Proposition 3.13, [16], Proposition 27).

Lemma 2.1. *Let M be a continuous AW^* -algebra, Z its centre, C a masa of M , and*

$$\Phi : \{e \in M; e \text{ projection}\} \rightarrow Z^+$$

a completely additive map such that

$$\Phi(ep) = \Phi(e)p, \quad e \in M \text{ and } p \in Z \text{ projections.}$$

Then, for every projection $e \in C$,

$$\{z \in Z; 0 \leq z \leq \Phi(e)\} = \{\Phi(f); e \geq f \in C \text{ projection}\}.$$

Proof. a) First we prove that for every projection $0 \neq g \in C$ there exists a projection $0 \neq h \leq g$ in C such that

$$\Phi(h) \leq \frac{1}{2}\Phi(g).$$

The case $\Phi(g) = 0$ being trivial, we can assume without loss of generality that $\Phi(g) \neq 0$.

Let $(g_{\iota})_{\iota}$ be a maximal family of mutually orthogonal projections in Cg such that $\Phi(g_{\iota}) = 0$ for every ι . Put $g_1 = g - \bigvee_{\iota} g_{\iota} \in C$. Then

$$\Phi(g_1) = \Phi(g) - \sum_{\iota} \Phi(g_{\iota}) = \Phi(g) \neq 0,$$

so $g_1 \neq 0$. By the maximality of $(g_i)_i$, for no projection $0 \neq g' \leq g_1$ in C can hold $\Phi(g') = 0$.

Now there exists a projection $g_2 \leq g_1$ in C such that $g_2 \notin Zg_1$. Let us assume the contrary, that is, that

$$C = Zg_1 + C(1_M - g_1).$$

There exist projections $h_1, h_2 \in M$ such that $g_1 = h_1 + h_2$ and $h_1 \sim h_2$ ([6], §19, Theorem 1), and then

$$C \subset Zh_1 + Zh_2 + C(1_M - g_1)$$

and the maximal abelianness of C imply that

$$C = Zh_1 + Zh_2 + C(1_M - g_1).$$

Thus

$$h_1, h_2 \in Cg_1 = Zg_1.$$

But, denoting by $z(g_1)$ the central support of g_1 ,

$$Zz(g_1) \ni z \mapsto zg_1 \in Zg_1$$

is a $*$ -isomorphism and it follows that h_1 and h_2 have orthogonal central supports, in contradiction to $h_1 \sim h_2 \neq 0$.

We claim that $\Phi(g_2)\Phi(g_1 - g_2) \neq 0$. Indeed, otherwise there would exist a projection $p \in Z$ such that

$$\Phi(g_2) = \Phi(g_2)p \text{ and } \Phi(g_1 - g_2)p = 0$$

and it would follow successively that

$$\begin{aligned} \Phi(g_2(1_M - p)) &= 0 \text{ and } \Phi((g_1 - g_2)p) = 0, \\ (g_2(1_M - p)) &= 0 \text{ and } (g_1 - g_2)p = 0, \\ g_2 &= g_2p = g_1p \in Zg_1. \end{aligned}$$

Let $q \in Z$ denote the support projection of $(\Phi(g_1) - 2\Phi(g_2))_+$. Then

$$\begin{aligned} \Phi(g_1q) - 2\Phi(g_2q) &= (\Phi(g_1) - 2\Phi(g_2))_+ \geq 0, \\ \Phi(g_2q) &\leq \frac{1}{2}\Phi(g_1q) \leq \frac{1}{2}\Phi(g_1) \leq \frac{1}{2}\Phi(g). \end{aligned}$$

Similarly,

$$\Phi((g_1 - g_2)(1_M - q)) \leq \frac{1}{2}\Phi(g).$$

But we cannot simultaneously have

$$\Phi(g_2q) = 0 \text{ and } \Phi((g_1 - g_2)(1_M - q)) = 0,$$

because this would imply

$$\Phi(g_2)\Phi(g_1 - g_2) = \Phi(g_2q)\Phi(g_1 - g_2) + \Phi(g_2)\Phi((1_M - q)(g_1 - g_2)) = 0.$$

Therefore, putting $h = g_2q$ if $\Phi(g_2q) \neq 0$ and $h = (g_1 - g_2)(1_M - q)$ otherwise, h is a non-zero projection in C , majorized by g , such that $\Phi(h) \leq \frac{1}{2}\Phi(g)$.

b) Now let $e \in C$ be a projection and let $x \in Z$, $0 \leq z \leq \Phi(e)$ be arbitrary. Choose a maximal family $(f_i)_i$ of mutually orthogonal projections in Ce satisfying

$$\sum_i \Phi(f_i) \leq z.$$

Then the projection $f = \bigvee_{\iota} f_{\iota} \leq e$ belongs to C and

$$\Phi(f) = \sum_{\iota} \Phi(f_{\iota}) \leq z.$$

We claim that actually $\Phi(f) = z$.

For let us assume the contrary. Then there exist a projection $0 \neq p \in Z$ and $\varepsilon > 0$ such that

$$(z - \Phi(f))p \geq \varepsilon p.$$

The projection $g = (e - f)p \in C$ is not zero, because otherwise it would follow

$$0 = (\Phi(e) - \Phi(f))p \geq (z - \Phi(f))p \geq \varepsilon p,$$

contradicting $p \neq 0, \varepsilon > 0$. Choosing an integer $n \geq 1$ with $2^{-n}\|\Phi(e - f)\| \leq \varepsilon$, the n -fold application of a) yields the existence of a projection $0 \neq h \leq g$ in C such that

$$\Phi(h) \leq 2^{-n}\Phi((e - f)p) \leq \varepsilon p.$$

Since $0 \neq h \in Ce$ is orthogonal to every f_{ι} and

$$\Phi(h) + \sum_{\iota} \Phi(f_{\iota}) = \Phi(h) + \Phi(f) \leq \varepsilon p + \Phi(f) \leq z,$$

the maximality of $(f_{\iota})_{\iota}$ is contradicted. $\square$

It is well known that if the projection family $(e_{\iota})_{\iota}$ in a finite AW^* -algebra M is upward directed and, for some projection $f \in M$, $e_{\iota} \prec f$ for all ι , then $\bigvee_{\iota} e_{\iota} \prec f$ (see [6], §33, Exercise 1). The above statement actually holds in any AW^* -algebra M under the only assumption of the finiteness of f (see Appendix, Corollary 1). Here we give a proof for this, assuming additionally that the projections e_{ι} are the finite partial sums of a family of mutually orthogonal projections in M :

Lemma 2.2. *Let M be an AW^* -algebra, $f \in M$ a finite projection, and $(e_{\iota})_{\iota \in I}$ a family of mutually orthogonal projections in M such that*

$$\sum_{\iota \in F} e_{\iota} \prec f \text{ for every finite } F \subset I.$$

Then

$$\bigvee_{\iota \in I} e_{\iota} \prec f.$$

Proof. According to the theory of Murray-von Neumann equivalence for projections in AW^* -algebras, we can assume without loss of generality that either fMf is of type I_n for some natural number $n \geq 1$, or that it is continuous (see [6], §15, Theorem 1, §18, Theorem 2, and §6, Corollary 2 of Proposition 4).

Let us first assume that fMf is of type I_n . By the Zorn Lemma there exists a maximal set $\mathcal{P}$ of mutually orthogonal central projections in M such that

$$\text{card } \{\iota \in I; pe_{\iota} \neq 0\} \leq n \text{ for every } p \in \mathcal{P}.$$

We claim that $\bigvee \mathcal{P} = 1_M$. Let us assume that $p_o = \bigvee \mathcal{P} \neq 1_M$. Then we can recursively find $n + 1$ indices $\iota_1, \dots, \iota_{n+1} \in I$ such that

$$p_1 = (1_M - p_o)z(e_{\iota_1}) \dots z(e_{\iota_{n+1}}) \neq 0,$$

where $z(e_{\iota})$ denotes the central support of e_{ι} . By the assumption of the lemma there exist mutually orthogonal projections $f_{\iota_1}, \dots, f_{\iota_{n+1}} \leq f$ in M such that $e_{\iota_j} \sim f_{\iota_j}$ for every $1 \leq j \leq n + 1$. For every $1 \leq j \leq n + 1$, the central support of $p_1 f_{\iota_j}$

is p_1 , there exists an abelian projection $g_j \leq p_1 f_{\iota_j}$ of central support p_1 (see [6], §18, exercise 4). But then $g_1, \dots, g_{n+1}$ are mutually orthogonal, equivalent, non-zero projections in fMf (see [6], §18, Proposition 1), which contradicts [6], §18, Proposition 4.

By the very orthogonal additivity of equivalence in AW^* -algebras (see [6], §11, Proposition 2) we conclude that

$$\bigvee_{\iota \in I} e_\iota = \bigvee \left\{ \sum_{pe_\iota \neq 0} pe_{\iota_j} ; p \in \mathcal{P} \right\} \prec \bigvee \{pf ; p \in \mathcal{P}\} = f.$$

Let us next assume that fMf is continuous and let $x \mapsto x^\natural$ denote the centre valued dimension function of the finite AW^* -algebra fMf (see [6], Ch.6).

For every $\iota \in I$ there exists a projection $e'_\iota \leq f$ in M such that $e_\iota \sim e'_\iota$. Since $(e'_\iota)^\natural$ does not depend on the choice of e'_ι , we can put

$$e_\iota^\natural = (e'_\iota)^\natural.$$

By the assumption of the lemma, for every finite $F \subset I$ we can choose the projections e'_ι , $\iota \in F$, mutually orthogonal and then

$$\sum_{\iota \in F} e_\iota^\natural = \sum_{\iota \in F} (e'_\iota)^\natural = \left(\sum_{\iota \in F} e'_\iota \right)^\natural \leq f.$$

It follows that all sums

$$\sum_{\iota \in J} e_\iota^\natural \leq f, \quad J \subset I,$$

exist in the monotone complete centre of fMf .

Now let us consider the set of all families of mutually orthogonal projections in fMf

$$(f_\iota)_{\iota \in J} \text{ with } J \subset I,$$

for which $f_\iota \sim e_\iota$ for every $\iota \in J$. We can endow this set with the partial order

$$(f_\iota)_{\iota \in J} \leq (f'_\iota)_{\iota \in J'} \iff J \subset J' \text{ and } f_\iota = f'_\iota \text{ for all } \iota \in J.$$

By the Zorn Lemma there exists a maximal element $(f_\iota)_{\iota \in J}$ of the above partially ordered set. We claim that then $J = I$. Let us assume the existence of some $\iota_o \in I \setminus J$. Since

$$e_{\iota_o}^\natural + \left(\bigvee_{\iota \in J} f_\iota \right)^\natural = e_{\iota_o}^\natural + \sum_{\iota \in J} f_\iota^\natural \leq \sum_{\iota \in I} e_\iota^\natural \leq f,$$

that is,

$$e_{\iota_o}^\natural \leq \left(f - \bigvee_{\iota \in J} f_\iota \right)^\natural,$$

by [6], §33, Theorem 3 (a particular case of the above Lemma 2.1) there exists a projection $f_{\iota_o} \leq f - \bigvee_{\iota \in J} f_\iota$ in M such that $f_{\iota_o}^\natural = e_{\iota_o}^\natural = (e'_{\iota_o})^\natural$, hence $f_{\iota_o} \sim e'_{\iota_o} \sim e_{\iota_o}$. But this contradicts the maximality of $(f_\iota)_{\iota \in J}$.

By the general additivity of equivalence in AW^* -algebras (see [6], §20, Theorem 1) we can conclude also in this case that

$$\bigvee_{\iota \in I} e_\iota \sim \bigvee_{\iota \in I} f_\iota \leq f. \quad \square$$

Let M be a semi-finite AW^* -algebra, and A the norm-closed linear span of all finite projections of M . We then recall that $M = M(A)$.

Let us call a masa $\tilde{C}$ of M M -semi-finite if $\tilde{C} \cap A$ is an essential ideal of $\tilde{C}$ or, equivalently, if every non-zero projection in $\tilde{C}$ majorizes a non-zero projection in $\tilde{C} \cap A$ (cf. with [16], Definition 1). For $\tilde{C} \subset M$ are equivalent:

- 1) $\tilde{C}$ is an M -semi-finite masa of M ;
- 2) $\tilde{C} = C' \cap M$ for some masa C of A .

Indeed, 2) implies 1) by Lemma 1.1 (iv), while 1) $\Rightarrow$ 2) follows by noticing that, according to the M -semi-finiteness of $\tilde{C}$, every projection in $\tilde{C}$ is the least upper bound of a family of mutually orthogonal projections from $C = \tilde{C} \cap A$, and so $C' \cap M = \tilde{C} \cap M = \tilde{C}$, $C' \cap A = (C' \cap M) \cap A = \tilde{C} \cap A = C$.

The following result extends [15], Theorem 3.18, and [16], Corollary 31, in the case of an M -semi-finite masa :

Theorem 2.3 (on labeling Murray-von Neumann equivalence classes). *Let M be a semi-finite AW^* -algebra, A the norm closed linear span of all finite projections of M , and C a masa of A . Then*

- (i) *for any projections $M \ni f \leq e \in C' \cap M$ there exists a projection $f \sim g \leq e$ in $C' \cap M$;*
- (ii) *for any projections $M \ni f \leq e \in C' \cap M$ of equal central supports, f finite and e properly infinite, there is a set $\mathcal{P}$ of mutually orthogonal central projections in M with $\bigvee \mathcal{P} = 1_M$ such that, for every $p \in \mathcal{P}$, ep is the least upper bound in the projection lattice of M of some family of mutually orthogonal projections from C , each one of which is equivalent in M to fp .*

Proof. (a) First we prove (i) in the case $e \in C$. Similarly as in the proof of Lemma 2.2, we can assume without loss of generality that either $eMe = eAe$ is of type I_n for some natural number $n \geq 1$, or it is continuous.

If eMe is of type I_n , by [15], Lemma 3.7, there exist mutually orthogonal projections $e_1, \dots, e_n \in C$ with $\sum_{j=1}^n e_j = e$, such that each e_j is abelian in M and has the same central support in M as e (actually [15], Lemma 3.7, is proved only for von Neumann algebras, but an inspection of the proof shows that it also works without any change in the realm of the AW^* -algebras). On the other hand, using [6], §18, Exercise 4 and Proposition 4, it is easy to see that there exist mutually orthogonal abelian projections $f_1, \dots, f_n \in M$ with $\sum_{j=1}^n f_j = f$ and central supports $z(f) = z(f_1) \geq \dots \geq z(f_n)$. By [6], §18, Proposition 1, it follows that $f_j \sim e_j z(f_j)$ for all $1 \leq j \leq n$, so f is equivalent to $C \ni \sum_{j=1}^n e_j z(f_j) \leq e$.

Now let us assume that eMe is continuous and let $x \mapsto x^{\natural}$ denote the centre valued dimension function of the finite AW^* -algebra eMe . Then Lemma 2.1 yields the existence of a projection $C \ni g \leq e$ such that $g^{\natural} = f^{\natural}$, hence $g \sim f$.

(b) Next we prove (i) in the case $f \in A$.

By Lemma 1.1 (iv) there exists a family $(e_{\iota})_{\iota \in I}$ of mutually orthogonal projections in C such that

$$e = \bigvee_{\iota \in I} e_{\iota}.$$

Let $\mathcal{P}$ be a maximal set of mutually orthogonal central projections in M such that, for every $p \in \mathcal{P}$, there is a finite set $F_p \subset I$ with

$$fp \prec p \sum_{\iota \in F_p} e_\iota \in C.$$

By the above part (a) of the proof, for every $p \in \mathcal{P}$ there exists a projection

$$g(p) \in C \text{ with } fp \sim g(p) \leq p \sum_{\iota \in F_p} e_\iota.$$

If $\bigvee \mathcal{P} = 1_M$, then $f = \bigvee \{fp; p \in \mathcal{P}\}$ is equivalent to $C' \cap M \ni \bigvee \{g(p); p \in \mathcal{P}\} \leq e$, so let us assume in the sequel that $p_o = 1_M - \bigvee \mathcal{P} \neq 0$.

By the maximality of $\mathcal{P}$ and by the comparison theorem (see [6], §14, Corollary 1 of Proposition 7) we have

$$p_o \sum_{\iota \in F} e_\iota \prec f \text{ for every finite } F \subset I.$$

According to Lemma 2.2 it follows that

$$p_o e = \bigvee_{\iota \in I} p_o e_\iota \prec f,$$

so by the Schröder-Bernstein theorem (see [6], §12) we have

$$fp_o \sim ep_o.$$

Consequently $f = fp_o + \bigvee \{fp; p \in \mathcal{P}\}$ is equivalent to

$$C' \cap M \ni ep_o + \bigvee \{g(p); p \in \mathcal{P}\} \leq e.$$

(c) Now we prove (ii).

Let $\mathcal{P}$ be a maximal set of mutually orthogonal central projections in M such that, for every $p \in \mathcal{P}$, ep is the least upper bound in the projection lattice of M of some family of mutually orthogonal projections from C , each one of which is equivalent in M to fp . We claim that then $\bigvee \mathcal{P} = 1_M$.

Let us assume that $p_o = 1_M - \bigvee \mathcal{P} \neq 0$. We notice that $fp \neq 0$ for any central projection $0 \neq p \leq p_o$ in M : indeed, otherwise p would be orthogonal to the common central support of f and e , so $ep = 0$ would be equal to $fp = 0 \in C$, in contradiction with the maximality of $\mathcal{P}$.

Let $(e_\iota)_{\iota \in I}$ be a maximal family of mutually orthogonal projections in C such that $fp_o \sim e_\iota \leq ep_o$ for all $\iota \in I$. By the comparison theorem there exists a central projection $p_1 \leq p_o$ in M such that

$$\begin{aligned} \left(ep_o - \bigvee_{\iota \in I} e_\iota \right) p_1 &\prec fp_1, \\ \left(ep_o - \bigvee_{\iota \in I} e_\iota \right) (p_o - p_1) &\succ f(p_o - p_1). \end{aligned}$$

Then $p_1 \neq 0$: indeed, $p_1 = 0$ would imply

$$A \ni fp_o \prec ep_o - \bigvee_{\iota \in I} e_\iota \in C' \cap M$$

and, by the above proved (b), there would exist a projection $fp_o \sim e' \leq ep_o - \bigvee_{\iota \in I} e_\iota$ in $(C' \cap M) \cap A = C$, contradicting the maximality of $(e_\iota)_{\iota \in I}$. Put

$$e_o = ep_1 - \bigvee_{\iota \in I} e_\iota p_1 \prec fp_1.$$

Then e_o is finite and belongs to $C' \cap M$, so it belongs to $C' \cap A = C$. On the other hand, the proper infiniteness of e and $ep_1 \neq 0$ imply that $ep_1 = e_o + \bigvee_{\iota \in I} e_\iota p_1$ is properly infinite. It follows that the set I is necessarily infinite, hence containing an infinite sequence $\iota_1, \iota_2, \dots$.

For every $j \geq 1$, $e_o \prec fp_1 \sim e_{\iota_j} p_1 \in C$ and the above proved a) yield the existence of some projection $e_o \sim e_{\iota_j}^{(1)} \leq e_{\iota_j} p_1$ in C . In particular, all projections $e_{\iota_j}^{(1)}$ are equivalent, hence, the projections $e_\iota p_1$ being finite, the projections $e_{\iota_j}^{(2)} = e_{\iota_j} p_1 - e_{\iota_j}^{(1)}$ are also all equivalent (see [6], §17, Exercise 3). Consequently, the projections from C

$$e'_{\iota_1} = e_o + e_{\iota_1}^{(2)} \text{ and } e'_{\iota_j} = e_{\iota_{j-1}}^{(1)} + e_{\iota_j}^{(2)}, \quad j \geq 2,$$

are all equivalent in M to $e_{\iota_1}^{(1)} + e_{\iota_1}^{(2)} = e_{\iota_1} p_1 \sim fp_1$. Clearly, they are mutually orthogonal and

$$\bigvee_{j \geq 1} e'_{\iota_j} = e_o \vee \bigvee_{j \geq 1} e_{\iota_j}^{(1)} \vee \bigvee_{j \geq 1} e_{\iota_j}^{(2)} = e_o \vee \bigvee_{j \geq 1} e_{\iota_j} p_1.$$

Letting

$$e'_\iota = e_\iota p_1 \text{ for } \iota \in I \setminus \{\iota_1, \iota_2, \dots\},$$

we conclude that all projections e'_ι , $\iota \in I$, belong to C and are equivalent in M to fp_1 . Moreover, they are mutually orthogonal and

$$\bigvee_{\iota \in I} e'_\iota = \bigvee_{j \geq 1} e'_{\iota_j} \vee \bigvee_{\iota \neq \iota_j} e'_\iota = e_o \vee \bigvee_{j \geq 1} e_{\iota_j} p_1 \vee \bigvee_{\iota \neq \iota_j} e_\iota p_1 = e_o \vee \bigvee_{\iota \in I} e_\iota p_1 = ep_1.$$

But this contradicts the maximality of $\mathcal{P}$.

(d) Finally we prove (i) in full generality.

We can assume without loss of generality that either f is finite, or it is properly infinite. The case of finite f was already settled in (b), so it remains to consider only the case of properly infinite f .

Choose some finite projection $M \ni f_o \leq f$ of the same central support as f (see [6], §17, Exercise 19 iii)). According to the above proved (c), we can assume without loss of generality that there are families $(e_\iota)_{\iota \in I}$ and $(f_\kappa)_{\kappa \in K}$ of mutually orthogonal projections in M such that

$$e_\iota \sim f_o \sim f_\kappa \text{ for all } \iota \in I \text{ and } \kappa \in K,$$

$$\bigvee_{\iota \in I} e_\iota = e, \quad \bigvee_{\kappa \in K} f_\kappa = f.$$

If $\text{card } K \leq \text{card } I$, that is, if there exists an injective map $K \ni \kappa \mapsto \iota(\kappa) \in I$, then the projection $g = \bigvee_{\kappa \in K} e_{\iota(\kappa)} \leq e$ belongs to $C' \cap M$ and is equivalent to $\bigvee_{\kappa \in K} f_\kappa = f$. On the other hand, if $\text{card } I \leq \text{card } K$, then $e = \bigvee_{\iota \in I} e_\iota \prec \bigvee_{\kappa \in K} f_\kappa = f \leq e$ and the Schröder-Bernstein theorem imply that $e \sim f$. $\square$

Let us now prove the statement of [15], Theorem 3.18, and [16], Corollary 31, in the case of an M -semi-finite masa of an arbitrary semi-finite AW^* -algebra M :

Corollary 2.4. *Let M be a semi-finite AW^* -algebra, A the norm-closed linear span of all finite projections of M , and C a masa of A . If $e \in C' \cap M$ is a projection and $1 \leq n \leq \aleph_0$ is a cardinal number such that e is the least upper bound of n mutually orthogonal, equivalent projections from M , then there exist n mutually orthogonal projections in $C' \cap M$, all equivalent in M , whose least upper bound is e .*

Proof. It is enough to separately treat the case of finite, resp. properly infinite, e .

If e is finite, n can be only a natural number. Let $f_1, \dots, f_n$ be mutually orthogonal, equivalent projections in M with $\sum_{j=1}^n f_j = e$. By (i) in the above theorem there

exists a projection $f_1 \sim e_1 \leq e$ in C . Since e is finite, it follows that $\sum_{j=2}^n f_j \sim e - e_1$, so we can again apply (i) in the above theorem to get a projection $f_2 \sim e_2 \leq e - e_1$ in C . By induction we obtain n mutually orthogonal projections $e_1, \dots, e_n \in C$ such that $f_j \sim e_j$ for all j and $\sum_{j=1}^n e_j = e$.

Now let us assume that e is properly infinite and consider a set I of cardinality n . Choosing a finite projection $M \ni f \leq e$ of the same central support as e (see [6], §17, Exercise 19 iii)), (ii) in the above theorem entails the existence of a set $\mathcal{P}$ of mutually orthogonal central projections in M with $\bigvee \mathcal{P} = 1_M$ such that, for every $p \in \mathcal{P}$, ep is the least upper bound of some set $\mathcal{E}_p$ of mutually orthogonal projections from C , each one of which is equivalent in M to fp . If $ep \neq 0$, then $\mathcal{E}_p$ must be infinite, so there exists a partition $(\mathcal{E}_{p,\iota})_{\iota \in I}$ of $\mathcal{E}_p$ in n sets of equal cardinality. Then the projections $e_\iota = \bigvee_{p \neq 0} \bigvee \mathcal{E}_{p,\iota}$, $\iota \in I$, belong to $C' \cap M$, are mutually orthogonal and equivalent in M , and $\bigvee_{\iota \in I} e_\iota = e$. $\square$

Let M be a finite AW^* -algebra with centre Z and let $x \mapsto x^\natural$ denote its centre valued dimension function (see [6], Ch. 6). It is known (see [7], II, 1) that $\natural$ can be uniquely extended to a centre valued quasitrace on M , that is, to a map $\Phi : M \rightarrow Z$ such that

- Φ is linear on commutative $*$ -subalgebras of M ,
- $\Phi(a + ib) = \Phi(a) + i\Phi(b)$ for all selfadjoint $a, b \in M$,
- Φ acts identically on Z ,
- $0 \leq \Phi(x^*x) = \Phi(xx^*)$ for all $x \in M$,

and then

- $\Phi(a) \leq \Phi(b)$ whenever $a \leq b$ are selfadjoint elements of M ,
- Φ is norm continuous, more precisely, $\|\Phi(a) - \Phi(b)\| \leq \|a - b\|$ for all selfadjoint $a, b \in M$.

We shall also use the symbol $\natural$ to denote the above Φ .

According to classical results of F.J. Murray and J. von Neumann, the centre valued quasitrace of every finite W^* -algebra is additive, hence linear.

It is an open question, raised by I. Kaplansky, whether the centre valued quasitrace of every finite AW^* -algebra is additive. Recently U. Haagerup has proven that the answer to Kaplansky's question is positive for any finite AW^* -algebra which is generated (as an AW^* -algebra) by an exact C^* -subalgebra (see [11], Theorem 5.11, Proposition 3.12 and Lemma 3.7 (4)).

We notice that if M is a finite AW^* -algebra and $n \geq 1$ is an integer, then the $*$ -algebra $\text{Mat}_n(M)$ of all $n \times n$ matrices over M is again a finite AW^* -algebra (see [6], §62). Denoting by $\natural$ and $\natural_n$ the respective centre valued quasitraces, it is easily seen that

$$n \cdot \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & \\ & \ddots & \\ 0 & & 0 \end{pmatrix}^{\natural_n} = \begin{pmatrix} x^{\natural} & 0 & 0 \\ 0 & x^{\natural} & \\ & \ddots & \\ 0 & & x^{\natural} \end{pmatrix}, \quad x \in M.$$

Moreover the additivity of $\natural$ is equivalent to the validity of

$$2 \cdot \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}^{\natural_2} = \begin{pmatrix} x_{11}^{\natural} + x_{22}^{\natural} & 0 \\ 0 & x_{11}^{\natural} + x_{22}^{\natural} \end{pmatrix}.$$

Indeed, using the above equality, we get for all $0 \leq a, b \in M$

$$\begin{aligned} \begin{pmatrix} (a+b)^{\natural} & 0 \\ 0 & (a+b)^{\natural} \end{pmatrix} &= 2 \cdot \begin{pmatrix} a+b & 0 \\ 0 & 0 \end{pmatrix}^{\natural_2} \\ &= 2 \cdot \left[\begin{pmatrix} a^{1/2} & b^{1/2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^{1/2} & 0 \\ b^{1/2} & 0 \end{pmatrix} \right]^{\natural_2} \\ &= 2 \cdot \left[\begin{pmatrix} a^{1/2} & 0 \\ b^{1/2} & 0 \end{pmatrix} \begin{pmatrix} a^{1/2} & b^{1/2} \\ 0 & 0 \end{pmatrix} \right]^{\natural_2} \\ &= 2 \cdot \begin{pmatrix} a & a^{1/2}b^{1/2} \\ b^{1/2}a^{1/2} & b \end{pmatrix}^{\natural_2} \\ &= \begin{pmatrix} a^{\natural} + b^{\natural} & 0 \\ 0 & a^{\natural} + b^{\natural} \end{pmatrix}. \end{aligned}$$

Conversely, assuming that $\natural$ is additive, it is easy to verify that

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \longmapsto \frac{1}{2} \begin{pmatrix} x_{11}^{\natural} + x_{22}^{\natural} & 0 \\ 0 & x_{11}^{\natural} + x_{22}^{\natural} \end{pmatrix}$$

is a centre valued quasitrace on $\text{Mat}_2(M)$.

For a given $\delta > 0$, we say that the centre valued quasitrace $\natural$ of a finite AW^* -algebra M is δ -subadditive (resp. δ -superadditive) if the map $M_+ \ni a \mapsto (a^{\natural})^{\delta}$ is subadditive (resp. superadditive). Clearly, δ -subadditivity (δ -superadditivity) of $\natural$ implies its δ' -subadditivity (δ' -superadditivity) whenever $\delta' < \delta$ ($\delta' > \delta$). It was proven by U. Haagerup that $\natural$ is always $\frac{1}{2}$ -subadditive (see [11], Lemma 3.5 (1)) and it seems reasonable to conjecture that it is also always 2-superadditive (or, at least, k -superadditive for some $k \geq 1$).

We notice as a curiosity that, for any two projections p, q in a finite AW^* -algebra M with centre valued quasitrace $\natural$,

$$(p+q)^{\natural} = p^{\natural} + q^{\natural}.$$

Indeed, since

$$\begin{aligned} \begin{pmatrix} p+q & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} p & \pm q \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p & 0 \\ \pm q & 0 \end{pmatrix}, \\ \begin{pmatrix} p & \pm pq \\ \pm qp & q \end{pmatrix} &= \begin{pmatrix} p & 0 \\ \pm q & 0 \end{pmatrix} \begin{pmatrix} p & \pm q \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

and $\begin{pmatrix} p & pq \\ qp & q \end{pmatrix}, \begin{pmatrix} p & -pq \\ -qp & q \end{pmatrix}$ commute, we have

$$\begin{aligned} \begin{pmatrix} (p+q)^{\natural} & 0 \\ 0 & (p+q)^{\natural} \end{pmatrix} &= 2 \begin{pmatrix} p+q & 0 \\ 0 & 0 \end{pmatrix}^{\natural_2} \\ &= \begin{pmatrix} p & pq \\ qp & q \end{pmatrix}^{\natural_2} + \begin{pmatrix} p & -pq \\ -qp & q \end{pmatrix}^{\natural_2} \\ &= 2 \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}^{\natural_2} \\ &= 2 \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}^{\natural_2} + 2 \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}^{\natural_2} \\ &= \begin{pmatrix} p^{\natural} & 0 \\ 0 & p^{\natural} \end{pmatrix} + \begin{pmatrix} q^{\natural} & 0 \\ 0 & q^{\natural} \end{pmatrix} \\ &= \begin{pmatrix} p^{\natural} + q^{\natural} & 0 \\ 0 & p^{\natural} + q^{\natural} \end{pmatrix}. \end{aligned}$$

This can also be deduced from Haagerup's result, taking into account that the C^* -algebra generated by two projections is of type I , hence nuclear, hence exact.

Lemma 2.5. *Let M be a finite AW^* -algebra, whose centre valued quasitrace $\natural$ is k -superadditive for some $k \geq 1$. Further, let $e_1, \dots, e_n \in M$ be mutually equivalent projections with $\sum_{j=1}^n e_j = 1_M$. Then there exists a projection $e_1 \sim p \in M$ such that, for every projection $f \in \{e_1, \dots, e_n\}' \cap M$,*

$$f^{\natural} \geq (1 - \|(1_M - f)p\|^2)n^{\frac{1}{k}-1}1_M.$$

Proof. Let $v_1, \dots, v_n \in M$ be partial isometries such that

$$v_j^* v_j = e_1, \quad v_j v_j^* = e_j, \quad 1 \leq j \leq n.$$

Since

$$\left(\frac{1}{\sqrt{n}} \sum_{j=1}^n v_j \right)^* \frac{1}{\sqrt{n}} \sum_{j=1}^n v_j = \frac{1}{n} \sum_{j_1, j_2=1}^n v_{j_1}^* v_{j_2} = \frac{1}{n} \sum_{j=1}^n v_j^* v_j = e_1,$$

$$p = \frac{1}{n} \sum_{j_1, j_2=1}^n v_{j_1} v_{j_2}^* = \frac{1}{\sqrt{n}} \sum_{j=1}^n v_j \left(\frac{1}{\sqrt{n}} \sum_{j=1}^n v_j \right)^*$$

is a projection in M equivalent to e_1 .
Now let the projection

$$f \in \{e_1, \dots, e_n\}' \cap M$$

be arbitrary and set $\delta = \|(1_M - f)p\|$. Since the case $\delta = 1$ is trivial, we can assume without loss of generality that $\delta < 1$. Then

$$\|p - pfp\| = \|(1_M - f)p\|^2 = \delta^2 < 1,$$

so $pfp \geq (1 - \delta^2)p$ is invertible in pMp . Thus the polar decomposition $fp = w \cdot |fp|$ exists in the C^* -algebra generated by p and f , and we have

$$w^* w = p, \quad f p f = w(p f p) w^* \geq (1 - \delta^2) w w^*.$$

Let us denote $\zeta = e^{i\frac{n}{\pi}}$. Then

$$u = \sum_{j=1}^n \zeta^j e_j \in \{e_1, \dots, e_n, f\}' \cap M$$

is unitary. Since

$$\begin{aligned} u^m p u^{-m} &= \frac{1}{n} \sum_{j, j_1, j_2, j'=1}^n \zeta^{mj} e_j v_{j_1} v_{j_2}^* \zeta^{-mj'} e_{j'} \\ &= \frac{1}{n} \sum_{j_1, j_2=1}^n \zeta^{m(j_1-j_2)} v_{j_1} v_{j_2}^* \end{aligned}$$

and

$$\sum_{m=1}^n \zeta^{mj} = 0 \text{ for every } 1 \leq j \leq n-1,$$

we have

$$\begin{aligned} \sum_{m=1}^n u^m p u^{-m} &= \frac{1}{n} \sum_{j_1, j_2=1}^n \left(\sum_{m=1}^n \zeta^{m(j_1-j_2)} \right) v_{j_1} v_{j_2}^* \\ &= \frac{1}{n} \sum_{j_1=1}^n n v_{j_1} v_{j_1}^* = 1_M. \end{aligned}$$

Therefore

$$f = f \sum_{m=1}^n u^m p u^{-m} f = \sum_{m=1}^n u^m (f p f) u^{-m} \geq (1 - \delta^2) \sum_{m=1}^n u^m w w^* u^{-m}$$

and, using the superadditivity of $\natural$, we get

$$\begin{aligned} f^\natural &\geq (1 - \delta^2) \left(\sum_{m=1}^n u^m w w^* u^{-m} \right)^\natural \\ &\geq (1 - \delta^2) \left(\sum_{m=1}^n ((u^m w w^* u^{-m})^\natural)^k \right)^{\frac{1}{k}} \\ &= (1 - \delta^2) \left(n ((w^* w)^\natural)^k \right)^{\frac{1}{k}} = (1 - \delta^2) n^{\frac{1}{k}} p^\natural. \end{aligned}$$

But $p^\natural = e_j^\natural$ for all $1 \leq j \leq n$, so

$$n p^\natural = \sum_{j=1}^n e_j^\natural = \left(\sum_{j=1}^n e_j \right)^\natural = 1_M,$$

and we conclude that $f^\natural \geq (1 - \delta^2) n^{\frac{1}{k}-1} 1_M$. $\square$

Now we are ready to prove the following

Theorem 2.6 (the abelian strict closure in continuous semi-finite AW^* -algebras). *Let M be a continuous semi-finite AW^* -algebra such that, for some finite projection $e_o \in M$ of central support 1_M and some $k \geq 1$, the centre valued quasitrace of $e_o M e_o$ is k -superadditive. Further, let A denote the norm-closed linear span of all finite projections of M , and C a masa of A . Then the strict closure of C in $M = M(A)$ is C .*

Proof. Let us assume that the strict closure $\overline{C}^\beta \subset C' \cap M$ of C contains some $0 \leq x \notin C$.

(a) First we prove that $\overline{C}^\beta$ contains some projection $e \notin C$.

Let e_δ denote, for every $\delta > 0$, the support of $(x - \delta 1_M)_+$ in the AW^* -subalgebra $C' \cap M$ of M . Then

$$xe_\delta \geq \delta e_\delta, \quad x(1_M - e_\delta) \leq \delta(1_M - e_\delta).$$

In particular, there exists $0 \leq y \in C' \cap M$ with $yx = e_\delta$. Moreover, $e_\delta \in \overline{C}^\beta$. Indeed, by Lemma 1.1 (i) there is a net $(b_\iota)_\iota$ in C with

$$0 \leq b_\iota \leq x \text{ for all } \iota, \quad b_\iota \rightarrow x \text{ strictly.}$$

Then $0 \leq yb_\iota \in C' \cap A = C$ for all ι and

$$\|(e_\delta - yb_\iota)a\| = \|y(x - b_\iota)a\| \leq \|y\| \cdot \|(x - b_\iota)a\| \rightarrow 0$$

for every $a \in A$.

(b) Next we prove the existence of an infinite sequence of mutually orthogonal projections $0 \neq e_1, e_2, \dots \in C$, all equivalent in M to $e_o q_o$ for some projection q_o in the centre Z of M , such that $\bigvee_{n \geq 1} e_n \in \overline{C}^\beta$.

Let e be a projection as in (a). Then e is not finite, so there exists a projection $q \in Z$ such that eq is properly infinite. But then, by the comparison theorem, there exists a projection $0 \neq q_o \in Z$ such that $e_o q_o \prec eq$. Since the central support of e_o is 1_M , we have $q_o \leq q$.

Now, according to (ii) in Theorem 2.3 (on labeling Murray-von Neumann equivalence classes), there exists a family $(e_\iota)_{\iota \in I}$ of mutually orthogonal projections in C , all equivalent in M to $e_o q_o \neq 0$, such that $\bigvee_{\iota \in I} e_\iota = eq_o$. I must be infinite, so it contains an infinite sequence $\iota_1, \iota_2, \dots$. Put

$$e_n = e_{\iota_n}, \quad n \geq 1.$$

Then $\bigvee_{n \geq 1} e_n$ belongs to $\overline{C}^\beta$. Indeed, since $\bigvee_{n \geq 1} e_n \in C' \cap M$, if $(b_\kappa)_\kappa$ is a net in C which converges strictly to e , then the net $(b_\kappa \bigvee_{n \geq 1} e_n)_\kappa$ is contained in C and converges clearly to $e \bigvee_{n \geq 1} e_n = \bigvee_{n \geq 1} e_n$ in the strict topology of M .

(c) Finally we prove that the statement in (b) leads to a contradiction.

Let us denote by $\natural$ the map $\bigcup_{n \geq 1} e_n M e_n \rightarrow Z q_o$ such that, for every $n \geq 1$, $e_n M e_n \ni x \mapsto x^\natural e_n$ is the centre valued quasitrace of $e_n M e_n$. It is easy to see that $\natural$ takes the same value in two projections from $\bigcup_{n \geq 1} e_n M e_n$ if and only if they are equivalent in M .

Let $n \geq 1$ be arbitrary and let $j_n = \lfloor n^{\frac{k+1}{k}} \rfloor \geq 1$ denote the integer part of $n^{\frac{k+1}{k}}$. According to Corollary 2.4, there exist projections

$$e_{n,1}, \dots, e_{n,j_n} \in C, \quad \sum_{j=1}^{j_n} e_{n,j} = e_n,$$

such that

$$e_{n,j}^\natural = \frac{1}{j_n} q_o \text{ for all } 1 \leq j \leq j_n.$$

Since $e_n \sim e_o q_o$, the centre valued quasitrace of $e_n M e_n$ is k -superadditive and Lemma 2.5 yields the existence of a projection $p_n \in e_n M e_n$ with $p_n^\natural = \frac{1}{j_n} q_o$ such

that, for every projection $g \in \{e_{n,1}, \dots, e_{n,j_n}\}' \cap e_n M e_n$,

$$g^\natural \geq (1 - \|(e_n - g)p_n\|^2) \frac{1}{j_n^{\frac{k-1}{k}}} q_o \geq (1 - \|(e_n - g)p_n\|^2) \frac{1}{n} q_o.$$

Now put $p = \bigvee_{n \geq 1} p_n$. Since $p_n^\natural = \frac{1}{j_n} q_o$ and $\sum_{n \geq 1} \frac{1}{j_n} < +\infty$, using Lemma 2.1 it is easy to verify that p is equivalent to a subprojection of the sum of finitely many e_n 's. In particular, p is finite, that is, $p \in A$. Therefore, $\bigvee_{n \geq 1} e_n$ being in $\overline{C}^\beta$, Lemma 1.1 (iii) yields the existence of a projection $\bigvee_{n \geq 1} e_n \geq f \in C$ with

$$\left\| \left(\bigvee_{n \geq 1} e_n - f \right) p \right\| \leq \frac{1}{\sqrt{2}}.$$

But then, for every $n \geq 1$, $f e_n$ is a projection in $C \cap e_n M e_n \subset \{e_{n,1}, \dots, e_{n,j_n}\}' \cap e_n M e_n$ and the above yield

$$(f e_n)^\natural \geq (1 - \|(e_n - f e_n)p_n\|^2) \frac{1}{n} q_o \geq \frac{1}{2n} q_o.$$

Since $\sum_{n \geq 1} \frac{1}{2n} = +\infty$, again using Lemma 2.1, it is easily seen that $f = \bigvee_{n \geq 1} (f e_n)$ is equivalent to $\bigvee_{n \geq 1} e_n$. In particular, f is properly infinite, in contradiction with $f \in C \subset A$. $\square$

3. WEYL-VON NEUMANN-BERG-SIKONIA TYPE THEOREMS

We recall that any Rickart C^* -algebra M is σ -normal, which means that, for every increasing sequence $(e_k)_{k \geq 1}$ of projections in M , the least upper bound of $(e_k)_{k \geq 1}$ in the projection lattice of M is actually its least upper bound in the ordered space M_h of all self-adjoint elements of M (see [4] or [27]). Therefore we shall speak in the sequel simply about the least upper bound of increasing sequences of projections in M .

Let us first prove a lemma about the sequential approximability of a projection in a Rickart C^* -algebra from a two-sided ideal :

Lemma 3.1. *Let M be a unital Rickart C^* -algebra, $\mathcal{J}$ a two-sided ideal of M , and $f \in M$ a projection. Then the following statements are equivalent :*

- (a) *there exists a sequence $(b_k)_{k \geq 1}$ of positive elements in $\mathcal{J}$ such that $b_k \leq f$ for all $k \geq 1$ and every projection $e \in M$ with $b_k \leq e$, $k \geq 1$ satisfies $f \leq e$;*
- (b) *there exists an increasing sequence $(f_k)_{k \geq 1}$ of projections in $\mathcal{J}$, whose least upper bound in M is f .*

Proof. Let us assume that (a) holds and put

$$f_{k,l} = \text{support of } \left(b_k - \frac{1}{l} 1_M \right)_+ \leq f, \quad k, l \geq 1,$$

$$f_n = \bigvee_{1 \leq k, l \leq n} f_{k,l} \text{ in the projection lattice of } M \leq f, \quad n \geq 1.$$

Since $b_k f_{k,l} \leq \frac{1}{l} f_{k,l}$, and so $f_{k,l}$ can be factorized by $b_k \leq f$, we have $f \geq f_{k,l} \in \mathcal{J}$ for all k and l . Further, using the validity of the Parallelogramm Law in all Rickart C^* -algebras (see [6], §13, Th. 1), we also obtain $f \geq f_n \in \mathcal{J}$, $n \geq 1$.

Now $(f_n)_{n \geq 1}$ is an increasing sequence, whose least upper bound in the projection lattice of M is f . Indeed, if $e \in M$ is a projection which majorizes every f_n , hence every $f_{k,l}$, then we have for all k and l

$$b_k^{\frac{1}{2}}(1_M - e)b_k^{\frac{1}{2}} \leq b_k^{\frac{1}{2}}(1_M - f_{k,l})b_k^{\frac{1}{2}} \leq \frac{1}{l}(1_M - f_{k,l}),$$

$$\|(1_M - e)b_k^{\frac{1}{2}}\|^2 \leq \frac{1}{l}.$$

Thus

$$b_k = e b_k e \leq e \text{ for all } k \geq 1$$

and it follows that $f \leq e$.

Conversely, (b) obviously implies (a) with $b_k = f_k$. $\square$

For unital Rickart C^* -algebras we have the following Weyl-von Neumann-Berg-Sikonia type result (cf. with [32], Theorem 3.1, and [1], §4) :

Theorem 3.2. *Let M be a unital Rickart C^* -algebra, and $\mathcal{J}$ a norm-closed two-sided ideal of M , which contains a sequence of positive elements such that 1_M is the only projection in M majorizing the sequence. Then, for any normal $y \in M$ and every $\varepsilon > 0$, there exist a masa C of $\mathcal{J}$ and an element x of the masa $C' \cap M$ of M , such that*

- 1) C contains an increasing sequence of projections, whose least upper bound in M is 1_M ,
- 2) $y - x \in \mathcal{J}$ and $\|y - x\| \leq \varepsilon$.

Remark 3.3. We notice that in Theorem 3.2 $C' \cap M$ is the sequentially monotone closure of C in the following sense : every $0 \leq a \in C' \cap M$ is the least upper bound in M_h of some increasing sequence of positive elements from $\mathcal{J}$.

Indeed, if $(e_k)_{k \geq 1}$ is an increasing sequence of projections in C , whose least upper bound in M is 1_M , then $(a^{1/2}e_k a^{1/2})_{k \geq 1}$ is an increasing sequence of positive elements from $\mathcal{J}$, whose least upper bound in A_h is $a^{1/2}1_M a^{1/2} = a$ (see [28], 9.14, the remark after Proposition 3). $\square$

For the proof of Theorem 3.2 we need the next result on quasi-central approximate units, implicitly contained in [32], Proposition 1.2 :

Lemma 3.4. *Let M be a unital Rickart C^* -algebra, $\mathcal{J}$ an essential, norm-closed, two-sided ideal of M , and $B \subset M$ a commutative C^* -subalgebra. Then the upward directed set of all projections of $\mathcal{J}$ contains a subnet $(e_i)_{i \in I}$ which, besides being automatically approximate unit for $\mathcal{J}$, is quasi-central for B , that is,*

$$\lim_i \|e_i b - b e_i\| = 0 \text{ for all } b \in B.$$

Proof. Passing to the Rickart C^* -subalgebra of M generated by B and 1_M (see e.g. [28], 9.11 (3)), we can assume without loss of generality that B is a Rickart C^* -subalgebra of M containing 1_M .

Let $\mathcal{P}$ denote the set of all finite sets P of projections from B satisfying the equality $\sum_{p \in P} p = 1_M$ and set

$$I = \{f \in \mathcal{J}; f \text{ projection}\} \times \mathcal{P}.$$

We endow I with a partial order by putting $(f_1, P_1) \leq (f_2, P_2)$ whenever $f_1 \leq f_2$ and the C^* -algebra $C^*(P_1)$ generated by P_1 is contained in $C^*(P_2)$ (that is, the

partition P_2 is a refinement of P_1). Clearly, in this way I becomes an upward directed ordered set.

Let $\iota = (f, P) \in I$ be arbitrary. For every $p \in P$, the right support $\mathbf{r}(fp)$ of fp is equivalent in M to the left support $\mathbf{l}(fp) \leq f \in \mathcal{J}$ (see [2] or [3]), so it belongs to $\mathcal{J}$. Thus

$$e_\iota = \sum_{p \in P} \mathbf{r}(fp)$$

is a projection in $\mathcal{J}$. Since every $\mathbf{r}(fp) \leq p$ belongs to the commutant P' , then also $e_\iota \in P'$. Furthermore,

$$f \leq e_\iota.$$

Indeed, for every $q \in P$,

$$fq = fq\mathbf{r}(fq) = \sum_{p \in P} fq\mathbf{r}(fp) = fq e_\iota,$$

so

$$f = f \sum_{q \in P} q = \sum_{q \in P} fqe_\iota = fe_\iota \leq e_\iota.$$

It is easily seen that

$$\iota_1 \leq \iota_2 \Rightarrow e_{\iota_1} \leq e_{\iota_2},$$

so $(e_\iota)_{\iota \in I}$ is a subnet of the upward directed set of all projections of $\mathcal{J}$.

Now, the upward directed set of all projections f of $\mathcal{J}$ is an increasing approximate unit for $\mathcal{J}$. Indeed, $\{x \in \mathcal{J}; \lim_f \|x(1_M - f)\| = 0\}$ is a norm-closed linear subspace of $\mathcal{J}$ containing all projections from $\mathcal{J}$, hence it is equal to $\mathcal{J}$. Thus also the subnet $(e_\iota)_{\iota \in I}$ is an approximate unit for $\mathcal{J}$.

On the other hand, the norm-closed linear subspace $\{b \in B; \lim_{\iota} \|e_\iota b - be_\iota\| = 0\}$ contains every projection from B : for any projection $p \in B$ and every $\iota = (f, P)$ with $p \in C^*(P)$ we have $e_\iota \in P' \cap A = C^*(P)' \cap \mathcal{J}$, so $e_\iota p - pe_\iota = 0$. Consequently the above subspace of B is actually equal to B . $\square$

Proof of Theorem 3.2. Put $y_1 = \frac{1}{2}(y + y^*)$, $y_2 = \frac{1}{2i}(y - y^*)$ and

$$p_j(\lambda) = \text{support of } (y_j - \lambda 1_M) \text{ in } M, \quad \lambda \in \mathbb{R}.$$

Further, let $\{\lambda_1, \lambda_2, \dots\}$ be the countable set of all rational numbers. Then

$$a = \sum_{k=1}^{\infty} 3^{-(2k-1)}(2p_1(\lambda_k) - 1_{A^{**}}) + \sum_{k=1}^{\infty} 3^{-2k}(2p_2(\lambda_k) - 1_{A^{**}}) + \frac{1}{2}1_{A^{**}} \in M,$$

$$0 \leq a \leq 1_M$$

and it is easy to see that the C^* -subalgebra of M generated by a and 1_M contains all projections $p_j(\lambda)$, $j = 1, 2$, $\lambda \in \mathbb{Q}$, hence also $y = y_1 + iy_2$. Therefore there exists a continuous function $f : [0, +\infty) \rightarrow \mathbb{C}$ such that $y = f(a)$. Furthermore, by a well known continuity property of the functional calculus (see e.g. [28], 1.18 (5)), there exists some $\delta > 0$ such that

$$0 \leq b \in M, \|a - b\| \leq \delta \implies \|f(a) - f(b)\| \leq \varepsilon.$$

Now, by Lemma 3.1, there exists an increasing sequence $(f_k)_{k \geq 1}$ of projections in $\mathcal{J}$, whose least upper bound in M is 1_M . Using Lemma 3.4, we can then construct by induction a sequence $0 = e_0 \leq e_1 \leq e_2 \leq \dots$ of projections in $\mathcal{J}$ such that

$$f_k \leq e_k, \quad \|e_k a - a e_k\| \leq 2^{-k-1} \delta.$$

Since the elements e_k and $(e_k - e_{k-1})a(e_k - e_{k-1})$ of $\mathcal{J}$ are mutually commuting, there exists a masa C of $\mathcal{J}$ containing all of them. Then C contains the increasing projection sequence $(e_k)_{k \geq 1}$, whose least upper bound in M is 1_M .

Let us denote

$$\begin{aligned} b_0 &= a, \\ b_n &= \sum_{k=1}^n (e_k - e_{k-1})a(e_k - e_{k-1}) + (1_M - e_n)a(1_M - e_n), \quad n \geq 1. \end{aligned}$$

Then, for every $n \geq 1$,

$$\begin{aligned} &b_{n-1} - b_n \\ &= (1_M - e_{n-1})(a - (1_M - e_n)a(1_M - e_n) - e_n a e_n)(1_M - e_{n-1}) \\ &= (1_M - e_{n-1}) \cdot [e_n, e_n a - a e_n] \cdot (1_M - e_{n-1}), \\ &\|b_{n-1} - b_n\| \leq 2\|e_n a - a e_n\| \leq 2^{-n} \delta. \end{aligned}$$

It follows that $\sum_{n=1}^{\infty} \|b_{n-1} - b_n\| \leq \delta$, so the sequence $(b_n)_{n \geq 1}$ is norm convergent to some $b \in M(A)^+$ and

$$\|a - b\| = \lim_{n \rightarrow \infty} \|b_0 - b_n\| \leq \delta.$$

Put $x = f(b)$.

We claim that $b \in C' \cap M$, hence also $x \in C' \cap M$. Since $C' \cap M$ is a masa of M (see Lemma 1.1 (iv)), it is enough to prove that b is commuting with all elements $a' \in C' \cap M \subset \{e_k, (e_k - e_{k-1})a(e_k - e_{k-1}); k \geq 1\}' \cap M$. We notice that, for every $n \geq 1$,

$$b_n a' - a' b_n = (1_M - e_n)(a a' - a' a)(1_M - e_n),$$

hence

$$\begin{aligned} |b_n a' - a' b_n|^2 &\leq (1_M - e_n) |a a' - a' a|^2 (1_M - e_n) \\ &\leq \|a a' - a' a\|^2 (1_M - e_n). \end{aligned}$$

Therefore

$$|b_n a' - a' b_n|^2 \leq \|a a' - a' a\|^2 (1_M - e_k), \quad n \geq k \geq 1,$$

and, passing to the limit for $n \rightarrow \infty$, we get for every $k \geq 1$

$$\begin{aligned} |b a' - a' b|^2 &\leq \|a a' - a' a\|^2 (1_M - e_k), \\ \text{support of } |b a' - a' b|^2 \text{ in } M &\text{ is } \leq 1_M - e_k. \end{aligned}$$

Since the least upper bound of $(e_k)_{k \geq 1}$ in M is 1_M , it follows that $b a' - a' b = 0$.

Finally, according to the choice of δ , $\|a - b\| \leq \delta$ implies that

$$\|y - x\| = \|f(a) - f(b)\| \leq \varepsilon.$$

On the other hand,

$$\begin{aligned} a - b_n &= \sum_{k=1}^n (b_{k-1} - b_k) \\ &= \sum_{k=1}^n (1_M - e_{k-1}) \cdot [e_k, e_k a - a e_k] \cdot (1_M - e_{k-1}) \in \mathcal{J} \end{aligned}$$

implies by passing to the limit for $n \rightarrow \infty$ that $a - b \in \mathcal{J}$. Using the Weierstrass Approximation Theorem, we infer that $y - x = f(a) - f(b) \in \mathcal{J}$. $\square$

We shall prove that in Theorem 3.2 the element x can be found under the form of an “infinite linear combination” of a sequence of mutually orthogonal projections from $\mathcal{J}$. To this aim we need an appropriate understanding of the summation of series in Rickart C^* -algebras.

We recall that every commutative Rickart C^* -algebra C is sequentially monotone complete (see e.g. [28], 9.16, Proposition 1). Thus, if $(a_k)_{k \geq 1}$ is a sequence in C^+ such that the partial sums $\sum_{k=1}^n a_k$, $n \geq 1$, are bounded, then there exists the least upper bound in C_h ,

$$\sum_{k=1}^{\infty} a_k = \sup \left\{ \sum_{k=1}^n a_k; n \geq 1 \right\} \in C^+.$$

Next let M be an arbitrary Rickart C^* -algebra, $(a_k)_{k \geq 1}$ a bounded sequence in M^+ such that the supports $\mathbf{s}(a_k)$, $k \geq 1$, are mutually orthogonal, and $(e_k)_{k \geq 1}$ a sequence of mutually orthogonal projections in M , for which $\mathbf{s}(a_k) \leq e_k$, $k \geq 1$ (we can take, for example, $e_k = \mathbf{s}(a_k)$). Then $\{a_k; k \geq 1\} \cup \{e_k; k \geq 1\}$ generates a commutative Rickart C^* -subalgebra C of M , so there exists $a = \sum_{k=1}^{\infty} a_k \in C^+$. Moreover, a is the least upper bound of the partial sums $\{\sum_{k=1}^n a_k; n \geq 1\}$ even in M_h . Indeed, by the σ -normality of the Rickart C^* -algebras, $\bigvee_{k=1}^{\infty} e_k$ is the least upper bound in M_h of the sequence $(\bigvee_{k=1}^n e_k)_{n \geq 1}$, and it follows that

$$\begin{aligned} a &= a^{1/2} \left(\bigvee_{k=1}^{\infty} e_k \right) a^{1/2} \text{ is the least upper bound in } M_h \text{ of} \\ \text{the increasing sequence } &a^{1/2} \left(\bigvee_{k=1}^n e_k \right) a^{1/2} = \sum_{k=1}^n a_k, n \geq 1 \end{aligned}$$

(see [28], 9.14, the remark after Proposition 3). In particular, a is the only element of M_h satisfying the conditions

$$a e_k = a_k, k \geq 1, \quad \mathbf{s}(a) \leq \bigvee_{k=1}^{\infty} e_k.$$

For the sake of completeness we notice that, by the above characterization, if $(e_k)_{k \geq 1}$ is a sequence of mutually orthogonal projections in M , then $\sum_{k=1}^{\infty} e_k = \bigvee_{k=1}^{\infty} e_k$.

Now let $(x_k)_{k \geq 1}$ be a bounded sequence in M such that, denoting by $\mathbf{l}(x_k)$ the left support of x_k and by $\mathbf{r}(x_k)$ the right one, the projections $\mathbf{l}(x_k) \vee \mathbf{r}(x_k)$, $k \geq 1$, are mutually orthogonal. Then we can define

$$\sum_{k=1}^{\infty} x_k = \left(\sum_{k=1}^{\infty} (\operatorname{Re} x_k)_+ - \sum_{k=1}^{\infty} (\operatorname{Re} x_k)_- \right) + i \left(\sum_{k=1}^{\infty} (\operatorname{Im} x_k)_+ - \sum_{k=1}^{\infty} (\operatorname{Im} x_k)_- \right).$$

It is easy to see that, if $(e_k)_{k \geq 1}$ is any sequence of mutually orthogonal projections in M such that $\mathbf{l}(x_k) \vee \mathbf{r}(x_k) \leq e_k$, $k \geq 1$, then $\sum_{k=1}^{\infty} x_k$ is the only element $x \in M$ for which

$$(3.1) \quad x e_k = e_k x = x_k, \quad k \geq 1, \quad \mathbf{l}(x) \vee \mathbf{r}(x) \leq \bigvee_{k=1}^{\infty} e_k.$$

By the above, considering the direct product C^* -algebra

$$\bigoplus_{k=1}^{\infty} e_k M e_k = \left\{ (y_k)_{k \geq 1} \in \prod_{k=1}^{\infty} e_k M e_k ; \sup_{k \geq 1} \|y_k\| < +\infty \right\},$$

the mapping

$$\bigoplus_{k=1}^{\infty} e_k M e_k \ni (y_k)_{k \geq 1} \longmapsto \sum_{k=1}^{\infty} y_k \in M$$

is well defined and it is an injective $*$ -homomorphism. Consequently

$$(3.2) \quad \left\| \sum_{k=1}^{\infty} x_k \right\| = \sup_{k \geq 1} \|x_k\|.$$

Finally, let $(e_k)_{k \geq 1}$ be a sequence of mutually orthogonal projections in M , and

$$(x_k)_{k \geq 1}, (y_k)_{k \geq 1} \in \bigoplus_{k=1}^{\infty} e_k M e_k.$$

Denoting by $\overline{\operatorname{lin}} \{x_k - y_k ; k \geq 1\}$ the norm-closed linear subspace of M generated by $\{x_k - y_k ; k \geq 1\}$, we have

$$(3.3) \quad \sum_{k=1}^{\infty} x_k - \sum_{k=1}^{\infty} y_k \in \overline{\operatorname{lin}} \{x_k - y_k ; k \geq 1\} \text{ if } \|x_k - y_k\| \longrightarrow 0.$$

Indeed, according to (3.2), we have :

$$\left\| \sum_{k=1}^{\infty} x_k - \sum_{k=1}^{\infty} y_k - \underbrace{\sum_{k=1}^n (x_k - y_k)}_{\in \overline{\operatorname{lin}} \{x_k - y_k ; k \geq 1\}} \right\| = \sup_{k \geq n+1} \|x_k - y_k\| \xrightarrow{n \rightarrow \infty} 0.$$

A slight modification of the proof of Theorem 3.2 yields the following Weyl-von Neumann-Berg-Sikonia type result, which is much closer to [32], Theorem 3.1, than Theorem 3.2.

Theorem 3.5. *Let M be a unital Rickart C^* -algebra, and $\mathcal{J}$ a norm-closed two-sided ideal of M which contains a sequence of positive elements such that 1_M is the only projection in M majorizing the sequence. Then, for any normal $y \in M$ and every $\varepsilon > 0$, there are*

- a sequence $(p_k)_{k \geq 1}$ of mutually orthogonal projections in $\mathcal{J}$,
- a sequence $(\lambda_k)_{k \geq 1}$ in the spectrum $\sigma(y)$ of y ,

such that

- 1) the least upper bound of $(p_n)_{n \geq 1}$ in M is 1_M ,
- 2) $y - \sum_{k=1}^{\infty} \lambda_k p_k \in \mathcal{J}$ and $\left\| y - \sum_{k=1}^{\infty} \lambda_k p_k \right\| \leq \varepsilon$.

Proof. Repeating word for word the arguments from the first paragraph of the proof of Theorem 3.2, we get $a \in M$ with $0 \leq a \leq 1_M$, a continuous function $f : [0, +\infty) \rightarrow \mathbb{C}$ and $\delta > 0$, such that $y = f(a)$ and

$$(3.4) \quad 0 \leq b \in M, \|a - b\| \leq \delta \implies \|f(a) - f(b)\| \leq \varepsilon.$$

Subtracting from a an appropriate positive multiple of 1_M and modifying f correspondingly, if necessary, we can assume that $0 \in \sigma(a)$.

Choose a sequence $\delta/3 = \delta_1 > \delta_2 > \dots > 0$ which converges to 0. According to the upper semicontinuity of the spectrum, there exist

$$\begin{array}{ccccccc} \eta_1 & > & \eta_2 & > & \dots & > & 0 \\ \wedge & & \wedge & & & & \\ \delta/3 & = & \delta_1 & > & \delta_2 & > & \dots \end{array}$$

such that the spectrum of every $b \in M$ with $\|a - b\| \leq \eta_k$ is contained in

$$U_{\delta_k}(\sigma(a)) = \{\mu \in \mathbb{C}; |\mu - \lambda(\mu)| < \delta_k \text{ for some } \lambda(\mu) \in \sigma(a)\}.$$

Arguing again as in the proof of Theorem 3.2, we can construct a sequence $0 = e_0 \leq e_1 \leq e_2 \leq \dots$ of projections in $\mathcal{J}$, whose least upper bound in M is 1_M , such that

$$\|e_k a - a e_k\| \leq 2^{-k-1} \eta_{k+1} \text{ for all } k \geq 1.$$

Then setting

$$\begin{aligned} b_o &= a, \\ b_n &= \sum_{k=1}^n (e_k - e_{k-1}) a (e_k - e_{k-1}) + (1_M - e_n) a (1_M - e_n), \quad n \geq 1, \end{aligned}$$

we have

$$b_{n-1} - b_n = (1_M - e_{n-1}) \cdot [e_n, e_n a - a e_n] \cdot (1_M - e_{n-1}), \quad n \geq 1,$$

so $\|b_{n-1} - b_n\| \leq 2^{-n} \eta_{n+1} \leq 2^{-n} \delta/3$ and $b_{n-1} - b_n \in \mathcal{J}$. Therefore the sequence $(b_n)_{n \geq 1}$ is norm convergent to some $b_\infty \in M^+$, for which $\|a - b_\infty\| \leq \delta/3$ and $a - b_\infty \in \mathcal{J}$.

We claim that

$$b_\infty = \sum_{k=1}^{\infty} (e_k - e_{k-1}) a (e_k - e_{k-1}).$$

Indeed, since

$$b_n (e_k - e_{k-1}) = (e_k - e_{k-1}) b_n = (e_k - e_{k-1}) a (e_k - e_{k-1}), \quad n \geq k \geq 1,$$

by passing to the limit for $n \rightarrow \infty$ we get

$$b_\infty(e_k - e_{k-1}) = (e_k - e_{k-1})b_\infty = (e_k - e_{k-1})a(e_k - e_{k-1}), \quad k \geq 1.$$

Thus, taking into account that $\bigvee_{k=1}^\infty (e_k - e_{k-1}) = \bigvee_{k=1}^\infty e_k = 1_M$, the description (3.1) yields the desired equality.

We notice that, for every $k \geq 1$,

$$(3.5) \quad \sigma((e_k - e_{k-1})a(e_k - e_{k-1})) \subset U_{\delta_k}(\sigma(a)).$$

Indeed, since the norm of

$$\begin{aligned} & a - \left((e_k - e_{k-1})a(e_k - e_{k-1}) + (1_{A^{**}} - (e_k - e_{k-1}))a(1_{A^{**}} - (e_k - e_{k-1})) \right) \\ &= \left[[e_k - e_{k-1}, a], 1_{A^{**}} - (e_k - e_{k-1}) \right] \end{aligned}$$

is majorized by $2(\|e_k a - a e_k\| + \|e_{k-1} a - a e_{k-1}\|) \leq 2(2^{-k-2}\eta_{k+1} + 2^{-k-1}\eta_k) < \eta_k$, by the choice of η_k we have

$$\begin{aligned} & \sigma((e_k - e_{k-1})a(e_k - e_{k-1})) \\ & \subset \sigma\left((e_k - e_{k-1})a(e_k - e_{k-1}) + (1_{A^{**}} - (e_k - e_{k-1}))a(1_{A^{**}} - (e_k - e_{k-1}))\right) \cup \{0\} \\ & \subset U_{\delta_k}(\sigma(a)). \end{aligned}$$

For any $k \geq 1$, let $[r_1^{(k)}, r_2^{(k)}]$ denote the smallest compact interval in $\mathbb{R}$ containing the spectrum $\sigma((e_k - e_{k-1})a(e_k - e_{k-1}))$. Choose

$$r_1^{(k)} = \mu_1^{(k)} < \dots < \mu_j^{(k)} < \dots < \mu_{j_k}^{(k)} = r_2^{(k)}$$

in $\sigma((e_k - e_{k-1})a(e_k - e_{k-1}))$ such that $|\mu_j^{(k)} - \mu_{j-1}^{(k)}| \leq \eta_k$ for all $2 \leq j \leq j_k$. Then there exist mutually orthogonal projections $(p_j^{(k)})_{1 \leq j \leq j_k}$ in $\mathcal{J}$ such that

$$\sum_{j=1}^{j_k} p_j^{(k)} = e_k - e_{k-1} \quad \text{and} \quad \left\| (e_k - e_{k-1})a(e_k - e_{k-1}) - \sum_{j=1}^{j_k} \mu_j^{(k)} p_j^{(k)} \right\| \leq \eta_k.$$

For example, we can set $p_j^{(k)} = e_j^{(k)} - e_{j+1}^{(k)}$, $1 \leq j \leq j_k$, where

$$e_j^{(k)} = \mathbf{s}\left(\left((e_k - e_{k-1})a(e_k - e_{k-1}) - \mu_j^{(k)}(e_k - e_{k-1})\right)_+\right), \quad 1 \leq j \leq j_k,$$

and $e_{j_k+1}^{(k)} = 0$ (see e.g. [28], 9.9, Proposition 1). Using (3.5), we can find for every $\mu_j^{(k)}$ some $\lambda_j^{(k)} \in \sigma(a)$ with $|\lambda_j^{(k)} - \mu_j^{(k)}| < \delta_k$ and then

$$\left\| (e_k - e_{k-1})a(e_k - e_{k-1}) - \sum_{j=1}^{j_k} \lambda_j^{(k)} p_j^{(k)} \right\| \leq \eta_k + \delta_k < 2\delta_k \leq 2\delta/3.$$

Now $\bigcup_{k=1}^\infty \{p_j^{(k)}; 1 \leq j \leq j_k\}$ consists of mutually orthogonal projections in M , whose least upper bound in M is 1_M , while $\bigcup_{k=1}^\infty \{\lambda_j^{(k)}; 1 \leq j \leq j_k\} \subset \sigma(a)$. Set

$b = \sum_{k=1}^{\infty} \sum_{j=1}^{j_k} \lambda_j^{(k)} p_j^{(k)} \in M^+$. Then (3.2) yields

$$\begin{aligned} \|b_{\infty} - b\| &= \left\| \sum_{k=1}^{\infty} (e_k - e_{k-1}) a (e_k - e_{k-1}) - \sum_{k=1}^{\infty} \sum_{j=1}^{j_k} \lambda_j^{(k)} p_j^{(k)} \right\| \\ &= \sup_{k \geq 1} \left\| (e_k - e_{k-1}) a (e_k - e_{k-1}) - \sum_{j=1}^{j_k} \lambda_j^{(k)} p_j^{(k)} \right\| \leq 2\delta/3, \end{aligned}$$

so $\|a - b\| \leq \|a - b_{\infty}\| + \|b_{\infty} - b\| \leq \delta/3 + 2\delta/3 = \delta$. On the other hand, since

$$\underbrace{\left\| (e_k - e_{k-1}) a (e_k - e_{k-1}) - \sum_{j=1}^{j_k} \lambda_j^{(k)} p_j^{(k)} \right\|}_{\in \mathcal{J}} < 2\delta_k \longrightarrow 0,$$

(3.3) implies that $b_{\infty} - b \in \mathcal{J}$, hence $a - b = (a - b_{\infty}) + (b_{\infty} - b) \in \mathcal{J}$.

Using the characterization (3.1), it is easy to deduce that

$$f(b) = \sum_{k=1}^{\infty} \sum_{j=1}^{j_k} f(\lambda_j^{(k)}) p_j^{(k)},$$

where, by the Spectral Mapping Theorem, $\bigcup_{k=1}^{\infty} \{f(\lambda_j^{(k)}); 1 \leq j \leq j_k\}$ is contained in $f(\sigma(a)) = \sigma(f(a)) = \sigma(y)$. On the other hand, (3.4) yields the norm estimation $\|y - f(b)\| = \|f(a) - f(b)\| \leq \varepsilon$. Finally, using $a - b \in \mathcal{J}$ and the Weierstrass Approximation Theorem, we infer also that $y - f(b) = f(a) - f(b) \in \mathcal{J}$. $\square$

If in the above theorem we are not requiring the norm estimation in 2), then the coefficients λ_k can be chosen even in the essential spectrum of y modulo $\mathcal{J}$:

Theorem 3.6. *Let M be a unital Rickart C^* -algebra, and $\mathcal{J}$ a norm-closed two-sided ideal of M , which contains a sequence of positive elements such that 1_M is the only projection in M majorizing the sequence. For any normal $y \in M$ there are*

- *a sequence $(p_k)_{k \geq 1}$ of mutually orthogonal projections in $\mathcal{J}$,*
- *a sequence $(\lambda_k)_{k \geq 1}$ in the spectrum $\sigma_{\mathcal{J}}(y)$ of the canonical image of y in the quotient C^* -algebra $M/\mathcal{J}$*

such that

- 1) *the least upper bound of $(p_n)_{n \geq 1}$ in M is 1_M ,*
- 2) *$y - \sum_{k=1}^{\infty} \lambda_k p_k \in \mathcal{J}$.*

For the proof we need the next lifting result, which is essentially [32], Proposition 2.1 :

Lemma 3.7. *Let M be a unital Rickart C^* -algebra, and $\mathcal{J}$ a norm-closed two-sided ideal of M . For any self-adjoint $a \in M$ there exists a self-adjoint $b \in M$ such that $\sigma(b) = \sigma_{\mathcal{J}}(b)$ and $a - b \in \mathcal{J}$.*

Proof. A moment's reflection shows that the proof of [32], Proposition 2.1, works for M unital Rickart C^* -algebra instead of W^* -algebra. $\square$

Proof of Theorem 3.6. Again repeating the arguments from the first paragraph of the proof of Theorem 3.2, we get some $a \in M$ with $0 \leq a \leq 1_M$ and a continuous function $f : [0, +\infty) \rightarrow \mathbb{C}$ such that $y = f(a)$. Now, according to Lemma 3.7, there exists a self-adjoint $b \in M$ such that $\sigma(b) = \sigma_{\mathcal{J}}(b)$ and $a - b \in \mathcal{J}$. In particular, $\sigma(b) = \sigma_{\mathcal{J}}(a) \subset [0, 1]$, and so $0 \leq b \leq 1_M$.

Let x denote the normal element $f(b)$. Using the Weierstrass Approximation Theorem, we infer that $y - x \in \mathcal{J}$; hence, by the Spectral Mapping Theorem, $\sigma(x) = f(\sigma(b)) = f(\sigma_{\mathcal{J}}(a)) = \sigma_{\mathcal{J}}(y)$. Now Theorem 3.5 yields the existence of

- a sequence $(p_k)_{k \geq 1}$ of mutually orthogonal projections in $\mathcal{J}$,
- a sequence $(\lambda_k)_{k \geq 1}$ in $\sigma(x) = \sigma_{\mathcal{J}}(y)$,

such that the least upper bound of $(p_n)_{n \geq 1}$ in M is 1_M and $x - \sum_{k=1}^{\infty} \lambda_k p_k \in \mathcal{J}$. Then $y - \sum_{k=1}^{\infty} \lambda_k p_k = (y - x) + (x - \sum_{k=1}^{\infty} \lambda_k p_k) \in \mathcal{J}$. $\square$

Let us say that a C^* -algebra A is σ -subunital if there exists a sequence $(b_n)_{n \geq 1}$ in A^+ , whose least upper bound in $M(A)_h$ is $1_{A^{**}}$. Clearly, if A is σ -unital, then it is σ -subunital. For commutative A the two notions coincide. However, if M is a countably decomposable type II_{∞} -factor and A is the norm-closed linear span of all finite projections of M , then A is not σ -unital (see [1], Proposition 4.5), but it is easily seen that it is σ -subunital.

We remark that the sequence $(b_n)_{n \geq 1}$ in the definition of the σ -subunitalness can be considered a kind of “approximate unit with respect to the order structure”. Indeed, according to [28], 9.14, the remark after Proposition 3, if the least upper bound of $(b_n)_{n \geq 1}$ in $M(A)_h$ is $1_{A^{**}}$ and $x \in M(A)$, then the least upper bound of the sequence $(x^* b_n x)_{n \geq 1}$ in $M(A)_h$ is $x^* x$.

By Theorems 3.5 and 3.6 we have :

Corollary 3.8. *Let A be a σ -subunital C^* -algebra, whose multiplier algebra $M(A)$ is a Rickart C^* -algebra. For any normal $y \in M(A)$ and any $\varepsilon > 0$ there exist*

- a sequence $(p_k)_{k \geq 1}$ of mutually orthogonal projections in A ,
- a sequence $(\lambda_k)_{k \geq 1}$ in the spectrum $\sigma(y)$ of y ,

such that

- 1) the least upper bound of $(p_n)_{n \geq 1}$ in $M(A)_h$ is $1_{A^{**}}$,
- 2) $y - \sum_{k=1}^{\infty} \lambda_k p_k \in A$ and $\left\| y - \sum_{k=1}^{\infty} \lambda_k p_k \right\| \leq \varepsilon$.

Moreover, if we don't require the second inequality in 2), then the sequence $(\lambda_k)_{k \geq 1}$ can be chosen even in the spectrum of the canonical image of y in the corona algebra $C(A) = M(A)/A$. $\square$

In particular, the above corollary can be applied to $A = K(H)$, where H is a separable complex Hilbert space, in which case the series $\sum_{k=1}^{\infty} \lambda_k p_k$ converges even with respect to the strict topology of $M(A) = B(H)$. This is the statement of the classical Weyl-von Neumann-Berg-Sikonia Theorem, but convergence with respect to the strict topology is also used in its subsequent extensions to σ -unital C^* -algebras with real rank zero multiplier algebra (see e.g. [23], [31], [12], [19], [20], [21]).

On the other hand, in the early extension from [32] of the Weyl-von Neumann-Berg-Sikonia Theorem to the norm-closed linear span A of all finite projections of an arbitrary semi-finite W^* -factor M , which for M of type II_{∞} turns out not to be σ -unital, the series $\sum_{k=1}^{\infty} \lambda_k p_k$ is proved to converge only with respect to

the s^* -topology. The reason why here a weaker topology than the strict topology should be used, is given by Theorem 2.6: if M is a type Π_∞ W^* -factor and we assume that a sum $\sum_{k=1}^\infty \lambda_k p_k$ with $p_k \in A$ is strictly convergent, then, according to Theorem 2.6, we must have $\sum_{k=1}^\infty \lambda_k p_k \in A$.

APPENDIX

We give here, for the convenience of the reader, a treatment of a set-theoretical result of T. Iwamura (see [22], Appendix II) and two applications to the theory of AW^* -algebras.

Proposition. *Let $I, \leq$ be an upward directed partially ordered uncountable set. Then, there exist a well order $\preccurlyeq$ on I and a family $(I_\iota)_{\iota \in I}$ of subsets of I such that*

- I_ι is upward directed for every $\iota \in I$,
- $\text{card } I_\iota < \text{card } I$, $\iota \in I$,
- $I_{\iota_1} \subset I_{\iota_2}$ whenever $\iota_1 \prec \iota_2$,
- $\bigcup_{\iota \in I} I_\iota = I$.

Proof. By Zermelo's theorem there exists a well order $\preccurlyeq$ on I . We can choose it such that

$$(\star) \quad \text{card } \{\iota' \in I; \iota' \prec \iota\} < \text{card } I \text{ for every } \iota \in I.$$

Indeed, if there exists some $\iota \in I$ such that

$$\text{card } \{\iota' \in I; \iota' \prec \iota\} = \text{card } I,$$

then there exists a smallest ι with respect to $\preccurlyeq$, having the above property. Choose for this ι a bijection

$$\Phi : I \rightarrow \{\iota' \in I; \iota' \prec \iota\}$$

and replace $\preccurlyeq$ by the well order, according to which ι_1 less than or equal to ι_2 means $\Phi(\iota_1) \preccurlyeq \Phi(\iota_2)$.

We notice that, I being infinite, $(\star)$ implies that I does not contain a largest element with respect to $\preccurlyeq$.

Let us denote

$$J_\iota = \{\iota' \in I; \iota' \prec \iota\}, \quad \iota \in I.$$

Then

$$\text{card } J_\iota < \text{card } I, \quad \iota \in I,$$

$$J_{\iota_1} \subset J_{\iota_2} \text{ whenever } \iota_1 \prec \iota_2,$$

$$\bigcup_{\iota \in I} J_\iota = I.$$

On the other hand, $I, \leq$ being upward directed, we can choose for each finite $F \subset I$ some $\iota(F) \in I$ such that

$$\iota \leq \iota(F) \text{ for all } \iota \in F.$$

Denote for every $J \subset I$

$$D_1(J) = J \cup \{\iota(F); F \subset J \text{ finite}\}.$$

We notice that

$$D_1(J) \text{ is finite for } J \text{ finite,}$$

$$\text{card } D_1(J) = \text{card } J \text{ for } J \text{ infinite}$$

and

$$D_1(J_1) \subset D_1(J_2) \text{ whenever } J_1 \subset J_2.$$

Now we define by recursion

$$\begin{aligned} D_{n+1}(J) &= D_1(D_n(J)) \supset D_n(J), \quad n \geq 1 \text{ integer,} \\ D_\omega(J) &= \bigcup_{n \geq 1} D_n(J). \end{aligned}$$

Then

$$\begin{aligned} D_\omega(J) &\text{ is countable for } J \text{ finite,} \\ \text{card } D_\omega(J) &= \text{card } J \text{ for } J \text{ infinite} \end{aligned}$$

and

$$D_\omega(J_1) \subset D_\omega(J_2) \text{ whenever } J_1 \subset J_2.$$

Moreover, $D_\omega(J), \leq$ is upward directed for every $J \subset I$.

Now, putting

$$I_\iota = D_\omega(J_\iota), \quad \iota \in I,$$

it is easy to see that all conditions from the statement are satisfied. $\square$

The first corollary extends Lemma 2.2 (compare with [6], §33, Exercise 1):

Corollary 1. *Let M be an AW^* -algebra, $f \in M$ a finite projection, and $(e_\iota)_{\iota \in I}$ an upward directed family of projections in M such that*

$$e_\iota \prec f \text{ for all } \iota \in I.$$

Then

$$\bigvee_{\iota \in I} e_\iota \prec f.$$

Proof. The case of countable I can easily be reduced to Lemma 2.2. Indeed, choosing a cofinal sequence $\iota_1 \leq \iota_2 \leq \dots$ in I , we have

$$\bigvee_{\iota \in I} e_\iota = \bigvee_{n \geq 1} e_{\iota_n} = e_{\iota_1} \vee \bigvee_{n \geq 1} (e_{\iota_{n+1}} - e_{\iota_n})$$

and we can apply Lemma 2.2 to f and the family $e_{\iota_1}, e_{\iota_2} - e_{\iota_1}, e_{\iota_3} - e_{\iota_2}, \dots$

For the proof in the general case let $f \in M$ be a finite projection and let us assume the existence of some upward directed family $(e_\iota)_{\iota \in I}$ of projections in M such that

$$e_\iota \prec f \text{ for all } \iota \in I, \text{ but } \bigvee_{\iota \in I} e_\iota \not\prec f.$$

Choose among all such families one with I of the smallest cardinality. By the first part of the proof I is then uncountable.

Let the well order $\preceq$ on I and the family $(I_\iota)_{\iota \in I}$ of subsets of I be as in the above proposition.

According to the minimality property of $\text{card } I$, we have

$$p_\iota = \bigvee_{\iota' \in I_\iota} e_{\iota'} \prec f, \quad \iota \in I.$$

On the other hand,

$$p_{\iota_1} \leq p_{\iota_2} \text{ whenever } \iota_1 \prec \iota_2,$$

$$\bigvee_{\iota \in I} p_{\iota} = \bigvee_{\iota \in I} e_{\iota}.$$

Consequently, denoting

$$q_{\iota} = p_{\iota} - \bigvee_{\iota' \prec \iota} p_{\iota'} \leq p_{\iota}, \quad \iota \in I,$$

the projections $(q_{\iota})_{\iota \in I}$ are mutually orthogonal and

$$\sum_{\iota \in F} q_{\iota} \prec f \text{ for any finite } F \subset I.$$

By Lemma 2.2 it follows that

$$\bigvee_{\iota \in I} q_{\iota} \prec f.$$

But

$$\bigvee_{\iota \in I} q_{\iota} = \bigvee_{\iota \in I} p_{\iota} = \bigvee_{\iota \in I} e_{\iota}.$$

Indeed, otherwise there would exist a smallest $\iota \in I$ with respect to $\prec$ such that

$$(\star\star) \quad p_{\iota} \not\leq \bigvee_{\iota' \in I} q_{\iota'}.$$

But then we would have

$$\bigvee_{\iota'' \prec \iota} p_{\iota''} \leq \bigvee_{\iota' \in I} q_{\iota'},$$

which contradicts $(\star\star)$. □

For M an arbitrary AW^* -algebra and Z a commutative AW^* -algebra we call

$$\Phi : \{e \in M; e \text{ projection}\} \rightarrow Z^+$$

normal if, for every upward directed family $(e_{\iota})_{\iota}$ of projections in M , we have

$$\Phi\left(\bigvee_{\iota} e_{\iota}\right) = \sup \Phi(e_{\iota}),$$

where $\sup$ denotes the least upper bound in Z^+ . Clearly,

$$\Phi \text{ normal} \implies \Phi \text{ completely additive},$$

but, using the above proposition similarly as in the proof of Corollary 1, we also get the converse implication (which should be known, but for which we have no reference):

Corollary 2. *Let M, Z be AW^* -algebras, Z commutative, and $\Phi : \{e \in M; e \text{ projection}\} \rightarrow Z^+$. Then*

$$\Phi \text{ normal} \iff \Phi \text{ completely additive}.$$

In particular, the centre valued dimension function of a finite AW^* -algebra is normal (see [6], §33, Exercise 4). Also, if M is a discrete AW^* -algebra and $e \in M$ is an abelian projection of central support 1_M , then the map Φ_e considered in the proof of Theorem 1.2 (on the abelian strict closure in discrete AW^* -algebras) is normal on the projection lattice of M .

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