ON CROSSED HOMOMORPHISMS ON SYMPLECTIC MAPPING CLASS GROUPS

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ABSTRACT. For a symplectic manifold (M, ω) with a relation Q between Chern classes of it and the cohomology class of the symplectic form ω , we construct a crossed homomorphism F_Q on the symplectomorphism group of (M, ω) with values in the cohomology group of M and show an application to the symplectic flux group. Moreover we see that F_Q descends to a crossed homomorphism on the symplectic mapping class group of (M, ω) and show a nontrivial example of it.

1. INTRODUCTION

There are a lot of works for the mapping class group $\mathcal{M}$ of an oriented surface Σ , which is the group of path components of the diffeomorphism group $Diff_+(\Sigma)$ of Σ with the C^∞ topology. If we take an area form ω_Σ on Σ , we also obtain the group $\mathcal{S}$ of path components of the group of area preserving diffeomorphisms. By Moser [6], $\mathcal{S}$ is isomorphic to $\mathcal{M}$. Since a symplectic manifold (M, ω) is a high-dimensional analogue to (Σ, ω_Σ) , the group $\mathcal{S}(M, \omega)$ of path components of the symplectomorphism group of (M, ω) is considered as a high-dimensional analogue to the mapping class group $\mathcal{M}$.

As works for $\mathcal{M}$, there are constructions of crossed homomorphisms from $\mathcal{M}$ to the first (co)homology group of Σ and investigations of them by Morita [7] and by Trapp [11]. In this paper, we carry out an analogy of them in terms of differential forms. For a symplectic manifold (M, ω) with a relation Q between Chern classes of (M, ω) and the cohomology class of the symplectic form ω , we construct a crossed homomorphism F_Q from the symplectomorphism group $Symp(M, \omega)$ of (M, ω) to the cohomology group of M using Chern-Simons forms or Bott homomorphisms. F_Q has a connection with the flux homomorphism which is a homomorphism from the identity component of $Symp(M, \omega)$ to $H^1(M; \mathbb{R})$. So we can show an application of F_Q to the flux group Γ_ω for (M, ω) , which gives an extension of a result for Γ_ω by Kędra-Kotschick-Morita [2]. Moreover we show that F_Q descends to a crossed homomorphism $\mathcal{F}_Q$ from the symplectic mapping group $\mathcal{S}(M, \omega)$ to a quotient of the cohomology group of M and that $\mathcal{F}_Q$ is generally nontrivial by giving a nontrivial example.

This paper is organized as follows. In section 2, we define our crossed homomorphisms and state the main results. In sections 3 and 4, we introduce some tools for

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the proof of Theorem A: Bott homomorphisms and a derivation formula for a curve of pushforward connections. In section 5, we prove Theorem A. In section 6, we give a topological definition of our crossed homomorphisms, which is needed for the computation of a nontrivial example. In section 7, we recall Thurston's theorem for a symplectic fibration and consider diffeomorphisms on it with some properties. These are also needed for the construction of our example. In sections 8 and 9, we show a nontrivial example of our crossed homomorphisms on the symplectic mapping class group.

2. DEFINITION OF OUR CROSSED HOMOMORPHISMS AND MAIN RESULTS

Let (M, ω) be a closed symplectic manifold of dimension $2n$. Let $p: \mathcal{J} \rightarrow M$ be the fiber bundle over M whose fiber at each $x \in M$ is the space of positive compatible complex structures on the symplectic vector space $(T_x M, \omega_x)$. Note that $\mathcal{J}$ is a smooth manifold. A section $J: M \rightarrow \mathcal{J}$ is an ω -positive compatible almost complex structure on M . Since the fibers of $\mathcal{J}$ are contractible, J is unique up to homotopy. Hence the Chern classes of (M, ω) are defined as those of the complex vector bundle (TM, J) . Similarly Chern classes of a symplectic vector bundle $(E, \omega_E) \rightarrow N$ of $\text{rank}_{\mathbb{R}} = 2n$ over a manifold N are defined. For each invariant symmetric multilinear function $q_j \in I_n^j := I^j(U(n))$ of degree j on the Lie algebra $\mathfrak{u}(n)$, the associated Chern class of (E, ω_E) in $H^{2j}(N; \mathbb{R})$ is denoted by $q_j(E, \omega_E)$. For a complex vector bundle ξ of $\text{rank}_{\mathbb{C}} = n$, its Chern class associated with q_j is denoted by $q_j(\xi)$. Let $I_n[w]$ be the polynomial algebra of one variable w with coefficients in the invariant Weil algebra $I_n = \sum_{j \geq 0} I_n^j$ of $U(n)$. Put $I_n[w]^k = \sum_{j=0}^k I_n^j \cdot w^{k-j}$. For each $Q = \sum_{j=0}^k q_j \cdot w^{k-j} \in I_n[w]^k$ and each $a \in H^2(N; \mathbb{R})$, we can consider the cohomology class $Q((E, \omega_E), a) = \sum_{j=0}^k q_j(E, \omega_E) \cup a^{k-j} \in H^{2k}(N; \mathbb{R})$. In particular, we put $Q(M, \omega) := Q((TM, \omega), [\omega]) \in H^{2k}(M; \mathbb{R})$ for the symplectic manifold (M, ω) .

We assume that there exists $Q = \sum_{j=0}^k q_j \cdot w^{k-j} \in I_n[w]^k$ such that the equality

$$Q(M, \omega) = \sum_{j=0}^k q_j(TM, \omega) \cup [\omega]^{k-j} = 0$$

holds in $H^{2k}(M; \mathbb{R})$. We fix such a $Q \in I_n[w]^k$.

The pullback bundle p^*TM over $\mathcal{J}$ has a canonical hermitian structure which is given by the hermitian form $h_{J_x}(\cdot, \cdot) = \omega_x(\cdot, J_x \cdot) - \sqrt{-1}\omega_x(\cdot, \cdot)$ on the fiber $p^*TM|_{J_x} = T_x M$ at each $J_x \in \mathcal{J}$ with $p(J_x) = x$. Let $\text{Symp}(M, \omega)$ be the group of symplectomorphisms on (M, ω) . The standard action of $\text{Symp}(M, \omega)$ on M lifts to ones on $\mathcal{J}$ and on p^*TM given, respectively, by

$$\text{Symp}(M, \omega) \times \mathcal{J} \ni (\varphi, J_x) \mapsto \varphi(J_x) := d\varphi_x \cdot J_x \cdot d\varphi_{\varphi(x)}^{-1} \in \mathcal{J}_{\varphi(x)} \subset \mathcal{J}$$

and

$$\text{Symp}(M, \omega) \times p^*TM \ni (\varphi, v) \mapsto d\varphi_x(v) \in T_{\varphi(x)}M = p^*TM|_{\varphi(J_x)} \subset p^*TM,$$

where $v \in p^*TM|_{J_x} = T_x M$. The action on p^*TM is one of $U(n)$ vector bundle isomorphisms.

The assumption $Q(M, \omega) = 0$ implies $\sum_{j=0}^k q_j(p^*TM) \cup [p^*\omega]^{k-j} = 0$ in $H^{2k}(\mathcal{J}; \mathbb{R})$.

This means that, if we take a $U(n)$ -connection A on p^*TM with curvature form F_A , there exists a $2k-1$ -form μ on $\mathcal{J}$ satisfying

$$\sum_{j=0}^k q_j(F_A^{(j)}) \wedge p^*\omega^{k-j} + d\mu = 0$$

since $q_j(p^*TM)$ is represented by the $2j$ -form $q_j(F_A^{(j)}) = q_j(F_A, F_A, \dots, F_A)$ by Chern-Weil theory.

We define the map

$$\tilde{F}_Q: \text{Symp}(M, \omega) \rightarrow \Omega^{2k-1}(M)$$

by

$$\tilde{F}_Q(\varphi) = J^* \left\{ \sum_{j=0}^k j \int_0^1 q_j(\alpha, F_{A_t}^{(j-1)}) dt \wedge p^*\omega^{k-j} + \varphi_*\mu - \mu \right\},$$

where $\alpha = \varphi_*A - A$ and $A_t = A + t\alpha$. The integrals are relative Chern-Simons forms.

Put

$$Q' = \sum_{j=0}^k (k-j)q_j \cdot \omega^{k-j-1} \in I_n[\omega]^{k-1};$$

then we have $Q'(M, \omega) = \sum_{j=0}^k (k-j)q_j(TM, \omega) \cup [\omega]^{k-j-1}$ in $H^{2k-2}(M; \mathbb{R})$.

Theorem A. *For a closed symplectic $2n$ -manifold (M, ω) with $Q(M, \omega) = 0$ for some $Q \in I_n[\omega]^k$, the map $F_Q: \text{Symp}(M, \omega) \ni \varphi \mapsto [\tilde{F}_Q(\varphi)] \in H^{2k-1}(M; \mathbb{R})$ is a well-defined crossed homomorphism. Its class $[F_Q]$ in the group cohomology $H^1(\text{Symp}(M, \omega), H^{2k-1}(M; \mathbb{R}))$ is independent of the choice of connection A , $2k-1$ form μ and section J . Moreover the equality $\frac{d}{ds}F_Q(\varphi_s) = Q'(M, \omega) \cup [\iota(X_s)\omega]$ holds for any smooth path $\{\varphi_s\}_{s \in [0,1]}$ in $\text{Symp}(M, \omega)$ with family of vector fields X_s defined by $\frac{d}{ds}\varphi_s = X_s \circ \varphi_s$.*

The map F_Q is a crossed homomorphism if and only if it satisfies $F_Q(\varphi\psi) = F_Q(\varphi) + \varphi_*F_Q(\psi)$ for any $\varphi, \psi \in \text{Symp}(M, \omega)$, where the action of $\text{Symp}(M, \omega)$ on $H^*(M; \mathbb{R})$ is given by $(\varphi, a) \mapsto \varphi_*a := (\varphi^{-1})^*a$ for each $(\varphi, a) \in \text{Symp}(M, \omega) \times H^*(M; \mathbb{R})$.

Let $\text{Symp}_0(M, \omega)$ be the identity component of $\text{Symp}(M, \omega)$. For any $\varphi \in \text{Symp}_0(M, \omega)$, choose a path $\varphi_s \in \text{Symp}_0(M, \omega)$ ($0 \leq s \leq 1$) from $\varphi_0 = id$ to $\varphi_1 = \varphi$. The flux homomorphism

$$\text{Flux}^\sim: \text{Symp}_0^\sim(M, \omega) \rightarrow H^1(M; \mathbb{R})$$

is defined by $\text{Flux}^\sim(\varphi) = [\int_0^1 \iota(X_s)\omega ds] \in H^1(M; \mathbb{R})$, where $\text{Symp}_0^\sim(M, \omega)$ is the universal covering of $\text{Symp}_0(M, \omega)$. Let Γ_ω be the subgroup of $H^1(M; \mathbb{R})$ which is the image by $\text{Flux}^\sim$ of the fiber of the universal covering at the identity. Γ_ω is called the flux group. $\text{Flux}^\sim$ descends to a homomorphism:

$$\text{Flux}: \text{Symp}_0(M, \omega) \rightarrow H^1(M; \mathbb{R})/\Gamma_\omega,$$

which is also called the flux homomorphism.

If we integrate the formula for a derivative of $F_Q(\varphi_s)$ in Theorem A in the case when φ_s is a path from the identity to φ , then we obtain $F_Q(\varphi) = Q'(M, \omega) \cup Flux \sim([\varphi_s])$, where $[\varphi_s]$ denotes the element of $Symp_0^\sim(M, \omega)$ defined by the path φ_s . This implies $\Gamma_\omega \subset \ker[Q'(M, \omega) \cup]$, where $Q'(M, \omega) \cup$ is the homomorphism from $H^1(M; \mathbb{R})$ to $H^{2k-1}(M; \mathbb{R})$ defined by the cup product with $Q'(M, \omega)$. This result has already been given by Kędra-Kotschick-Morita [2] in the case when $[\omega]^k$ is proportional to a product of fundamental Chern classes of (M, ω) . In the case of $k = 1$, namely in the case when $[\omega]$ is proportional to the first Chern class $c_1(M, \omega)$, Kotschick-Morita [3] proved $\Gamma_\omega = \{0\}$ and that $Flux$ extends to the whole group $Symp(M, \omega)$ as a crossed homomorphism. Our crossed homomorphism F_Q with $Q = w - \lambda c_1$ also gives an extension of $Flux$.

Corollary B. $\Gamma_\omega \subset \bigcap_Q \ker[Q'(M, \omega) \cup]$, where Q runs over $\bigcup_{k \geq 0} \{Q \in I_n[w]^k \mid Q(M, \omega) = 0\}$.

Since $Flux$ is surjective and the restriction of F_Q to $Symp_0(M, \omega)$ factors through $H^1(M; \mathbb{R})/\Gamma_\omega$,

$$F_Q|_{Symp_0(M, \omega)} : Symp_0(M, \omega) \xrightarrow{Flux} H^1(M; \mathbb{R})/\Gamma_\omega \xrightarrow{Q'(M, \omega) \cup} H^{2k-1}(M; \mathbb{R}),$$

we have the following proposition.

Proposition C. Under the assumption of Theorem A, if $Q'(M, \omega) \cup : H^1(M; \mathbb{R}) \rightarrow H^{2k-1}(M; \mathbb{R})$ is a nontrivial homomorphism, then the cohomology class $[F_Q]$ is nonzero in $H^1(Symp(M, \omega), H^{2k-1}(M; \mathbb{R}))$.

Put

$$D_Q(M, \omega) = Q'(M, \omega) \cup H^1(M; \mathbb{R});$$

then it is a subgroup of $H^{2k-1}(M; \mathbb{R})$ invariant under the standard action by $Symp(M, \omega)$. Let $\mathcal{S}(M, \omega)$ be the quotient $Symp(M, \omega)/Symp_0(M, \omega)$, which is called the symplectic mapping class group of (M, ω) .

Corollary D. F_Q descends to a crossed homomorphism

$$\mathcal{F}_Q : \mathcal{S}(M, \omega) \rightarrow H^{2k-1}(M; \mathbb{R})/D_Q(M, \omega),$$

whose cohomology class $[\mathcal{F}_Q] \in H^1(\mathcal{S}(M, \omega), H^{2k-1}(M; \mathbb{R})/D_Q(M, \omega))$ is independent of the choice of connection A , $2k - 1$ form μ and almost complex structure J .

Next we give a nontrivial example of the crossed homomorphism $\mathcal{F}_Q$. Let $(B = T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}, \beta)$ be the standard symplectic torus of dimension $2n$ with coordinates $(x_i, y_i)_{i=1}^n$ and symplectic form $\beta = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$. Its cohomology ring is given by $H^*(T^{2n}; \mathbb{Z}) = \bigwedge(dx_i, dy_i)$ with abuse of notation.

Let $L_i \rightarrow T^{2n}$ be a complex $U(1)$ -line bundle over T^{2n} for $1 \leq i \leq n$ with $\gamma_i := c_1(L_i) = dx_i \wedge dy_i \in H^2(T^{2n}; \mathbb{Z})$. Put $V = L_1 \oplus \cdots \oplus L_n$; then V is a complex $U(n)$ -vector bundle of $\text{rank}_{\mathbb{C}} V = n$. Let $\pi : M = P(V) \rightarrow T^{2n}$ be the projective bundle associated with V ; then it is a symplectic fibration with fiber $\mathbb{C}P^{n-1}$ with the standard symplectic (Fubini-Study) form. By Thurston's theorem in [10] (see also [5] or Theorem 7.1), it is a symplectic manifold with symplectic form ω_K for any sufficiently large $K > 0$.

Let $\mathcal{L}$ be the Hopf line bundle associated with the projective bundle M . Put $\eta = -c_1(\mathcal{L})$; then we have $H^*(M; \mathbb{R}) = H^*(T^{2n}; \mathbb{R})[\eta]/\sim$ with relation $\eta^n + c_1(V)\eta^{n-1} + \dots + c_n(V) = 0$. Put

$$Q_{K,2} = \bar{c}_2 - \frac{1}{2(1-nK)^2} [nw^2 - 2\bar{c}_1 w + \{n(n-1)K^2 - 2(n-1)K + 1\}\bar{c}_1^2]$$

in $I_{2n-1}[w]^2$, where $\bar{c}_i \in I_{2n-1}^i$ ($i = 1, 2$) is the form corresponding to the i -th Chern class. We can show $Q_{K,2}(M, \omega_K) = 0$, $Q'_{K,2} = -\frac{1}{(1-nK)^2}(nw - \bar{c}_1)$ and $D_{Q_{K,2}}(M, \omega_K) = (c_1 - n[\omega_K]) \cup H^1(M; \mathbb{R})$. By Corollary D, we have the crossed homomorphism

$$\mathcal{F}_{Q_{K,2}}: \mathcal{S}(M, \omega_K) \rightarrow H^3(M; \mathbb{R})/(c_1 - n[\omega_K]) \cup H^1(M; \mathbb{R}),$$

where $c_1 = \bar{c}_1(TM, \omega_K)$.

Let $\mathcal{S}_0(M, \omega_K)$ be the subgroup of $\mathcal{S}(M, \omega_K)$ consisting of all elements which act trivially on $H^*(M; \mathbb{R})$. The restriction of $\mathcal{F}_{Q_{K,2}}$ to $\mathcal{S}_0(M, \omega_K)$ is a homomorphism.

Theorem E. *The image of $[\mathcal{F}_{Q_{K,2}}]$ by the induced homomorphism $H^1(\mathcal{S}(M, \omega_K), \mathcal{H}) \rightarrow H^1(\mathcal{S}_0(M, \omega_K), \mathcal{H})$ by the restriction, where*

$$\mathcal{H} = H^3(M; \mathbb{R})/(c_1 - n[\omega_K]) \cup H^1(M; \mathbb{R}),$$

is nonzero, hence so is $[\mathcal{F}_{Q_{K,2}}]$.

3. BOTT HOMOMORPHISMS

In this section we shall recall Bott homomorphisms (see [12]) which are useful for our computation.

Let G be a Lie group, but we consider only $G = U(n)$ in this paper. Let $\pi: P \rightarrow M$ be a principal G -bundle over a manifold M . Let A_h ($h = 0, 1, \dots, r$) be $r+1$ connection forms on P and $\Delta^r := \{(t_0, t_1, \dots, t_r) \in \mathbb{R}^{r+1} | t_h \geq 0, \sum_{h=0}^r t_h = 1\}$ the standard r -simplex. Then we have the average connection $\tilde{A} = \sum_{h=0}^r t_h A_h$ on the product principal G -bundle $\pi \times id: P \times \Delta^r \rightarrow M \times \Delta^r$. Let $I^k(G)$ be the vector space of invariant, symmetric, multilinear functions on the k -th product $\mathfrak{g}^k$ of the Lie algebra $\mathfrak{g}$ of G with values in $\mathbb{R}$. For each $c \in I^k(G)$, put $\Delta(A_0, \dots, A_r)c = (-1)^{[\frac{r+1}{2}]} \int_{\Delta^r} c(F_{\tilde{A}}^{(k)})$, where $F_{\tilde{A}}$ is the curvature form of $\tilde{A}$, $c(F_{\tilde{A}}^{(k)})$ denotes $c(F_{\tilde{A}}, F_{\tilde{A}}, \dots, F_{\tilde{A}})$ and the orientation of Δ^r is given by $dt_1 \wedge \dots \wedge dt_r$ ($t_0 = 1 - \sum_{h=1}^r t_h$). Then we have the Bott homomorphism $\Delta(A_0, \dots, A_r): I^k(G) \rightarrow \Omega^{2k-r}(M)$. The Bott homomorphisms have the following properties:

$$(1) \quad d\Delta(A_0, \dots, A_r)c = \sum_{h=0}^r (-1)^h \Delta(A_0, \dots, A_{h-1}, A_{h+1}, \dots, A_r)c;$$

in particular, $d\Delta(A_0)c = 0$ for $r = 0$.

$$(2) \quad \Delta(A_0): I(G) \rightarrow \Omega^*(M) \text{ is the Chern-Weil homomorphism;}$$

that is, the equality $\Delta(A_0)c = c(F_{A_0}^{(k)})$ holds.

$$(3) \quad \Delta(A_0, A_1)c = k \int_0^1 c(\alpha, F_{A_t}^{(k-1)}) dt,$$

where $\alpha = A_1 - A_0$ and $A_t = A_0 + t\alpha$.

Let Q be another principal $U(n)$ -bundle over M and $\tilde{\varphi}: Q \rightarrow P$ a $U(n)$ -bundle isomorphism over a diffeomorphism φ of M . Then the pushforward connection $\tilde{\varphi}_*A = (\tilde{\varphi}^{-1})^*A$ can be considered.

$$(4) \quad \Delta(\tilde{\varphi}_*A_0, \tilde{\varphi}_*A_1, \dots, \tilde{\varphi}_*A_r)c = \varphi_*\Delta(A_0, A_1, \dots, A_r)c,$$

where $\varphi_* = (\varphi^{-1})^*$.

4. THE DERIVATIVE OF A CURVE OF PUSHFORWARD CONNECTIONS

In this section we shall derive a derivation formula for a curve of pushforward connections.

In the setting of Theorem A, let A be a $U(n)$ -connection on p^*TM and $\nabla_A: \Omega^0(p^*TM) \rightarrow \Omega^1(p^*TM)$ the covariant derivative with respect to it. For any $\varphi \in \text{Symp}(M, \omega)$, φ_*A denotes the pushforward connection of A by φ , which is the pullback connection $(\varphi^{-1})^*A$ of A by the inverse of φ .

Here and hereafter we use the same notation for the actions of $\varphi \in \text{Symp}(M, \omega)$ on M , on $\mathcal{J}$ and on p^*TM . In terms of a covariant derivative, the pushforward connection is given by $\nabla_{\varphi_*A} = \varphi_*\nabla_A = \varphi_*(\nabla_A((\varphi^{-1})^*(\cdot)))$.

Let J_x and K_y be two points of $\mathcal{J}$ with $p(J_x) = x$ and $p(K_y) = y$, respectively. Let (U, u^i) and (V, v^j) be local coordinate systems of $\mathcal{J}$ around J_x and K_y with orthonormal frames $e = (e_1, \dots, e_n)$ for $p^*TM|_U$ and $f = (f_1, \dots, f_n)$ for $p^*TM|_V$, respectively. We can put $\nabla_A e = e \otimes a$ for some $u(n)$ -valued 1-form a on U .

For $\varphi \in \text{Symp}(M, \omega)$, we assume $\varphi(J_x) = K_y$ and consider it near J_x and K_y . Put $\psi = \varphi^{-1}$. We can set $\varphi^*f = e \otimes \Phi^{-1}$, where Φ is a $U(n)$ -valued function on U near J_x . Similarly we can put $\psi^*e = f \otimes \Psi^{-1}$, where Ψ is a $U(n)$ -valued function on V near K_y . Then we have $\psi^*\Phi = \Psi^{-1}$ and $\varphi^*\Psi = \Phi^{-1}$. Direct computations show the following lemma.

Lemma 4.1. *In the trivialization $p^*TM|_V \cong V \times \mathbb{C}^n$ near K_y with respect to the orthonormal frame f , the covariant derivative with respect to φ_*A is given by*

$$\nabla_{\varphi_*A} = d + \psi^*(-d\Phi\Phi^{-1} + \Phi a\Phi^{-1}) = d + \Psi^{-1}d\Psi + \Psi^{-1}(\psi^*a)\Psi$$

and its curvature F_{φ_*A} is given by

$$F_{\varphi_*A} = \psi^*(\Phi F_a \Phi^{-1}) = \Psi^{-1}\psi^*F_a\Psi,$$

where $F_a = da + a \wedge a$ is a local expression of the curvature of A near J_x .

Next we compute the derivative $\frac{d}{ds}(\varphi_{s*}A - A)$ of a smooth path φ_s in $\text{Symp}(M, \omega)$. Let $\tilde{X}_s$ be the smooth family of vector fields on $\mathcal{J}$ given by $\frac{d}{ds}\varphi_s = \tilde{X}_s \circ \varphi_s$. Hereafter we omit the parameter s if it is not particularly needed. In the trivialization $p^*TM|_V \cong V \times \mathbb{C}^n$ of Lemma 4.1, $\alpha = \varphi_*A - A$ is given by

$$\alpha = \Psi^{-1}d\Psi + \Psi^{-1}(\psi^*a)\Psi - b,$$

where b is the column vector valued 1-form defined by $\nabla_A f := (\nabla_A f_1, \dots, \nabla_A f_n) = f \otimes b$. A direct computation shows that

$$\dot{\alpha} = \frac{d}{ds}\alpha = \Psi^{-1}\{d(\dot{\Psi}\Psi^{-1}) + [\psi^*a, \dot{\Psi}\Psi^{-1}] + \frac{d}{ds}(\psi^*a)\}\Psi.$$

To begin with, we take $\varphi_0 = id$ and consider the derivative at $s = 0$. We can assume $e = f$, $b = a$ and $\Phi_0 = \Psi_0 = I$. Hence we have

$$\dot{\alpha}_0 = d\dot{\Psi} + [a, \dot{\Psi}] + \frac{d}{ds}(\psi^*a)|_{s=0} = d_a\{\dot{\Psi} + \iota(\tilde{Y}_0)a\} + \iota(\tilde{Y}_0)F_a,$$

where $d_a = d + [a, \cdot]$ and $\tilde{Y}_s$ is defined by $\frac{d}{ds}\psi_s = \tilde{Y}_s \circ \psi_s$ on $\mathcal{J}$ with $\tilde{Y}_0 = \tilde{Y}_s|_{s=0}$. We have $\tilde{Y}_s = -\psi_{s*}\tilde{X}_s$.

We note that d_a and F_a define the global objects d_A , which are the covariant exterior differential and F_A , respectively. It is easy to see that $\beta := \dot{\Psi} + \iota(\tilde{Y}_0)a$ also

defines a global section of the bundle $End(p^*TM)$ over $\mathcal{J}$, which can be denoted by $\beta_{\dot{\varphi}_0, A}$. Therefore we get

$$\dot{\alpha}_0 = d_A \beta_{\dot{\varphi}_0, A} + \iota(\tilde{Y}_0)F_A.$$

Next we consider the general case. For any smooth curve φ_s , put $\xi_u = \varphi_{u+s}\varphi_s^{-1}$; then we have $\xi_0 = id$. Since we have

$$\frac{d}{ds}(\varphi_{s*}A - A) = \frac{d}{du}(\varphi_{u+s}\varphi_s^{-1}\varphi_s)_*A|_{u=0} = \frac{d}{du}\xi_{u*}\varphi_{s*}A|_{u=0},$$

we obtain

$$\frac{d}{ds}(\varphi_{s*}A - A) = d_{\varphi_{s*}A}\beta_{\xi_0, \varphi_{s*}A} + \iota(\tilde{V}_0)F_{\varphi_{s*}A},$$

where $\frac{d}{du}\xi_u^{-1} = \tilde{V}_u \circ \xi_u^{-1}$. The equality $\tilde{V}_0 = \frac{d}{du}\xi_u^{-1}|_{u=0} = -\tilde{X}_s$ implies the following lemma.

Lemma 4.2. *Let A be a $U(n)$ -connection on p^*TM and φ_s a smooth curve in $Symp(M, \omega)$ with a smooth family of vector fields $\tilde{X}_s$ on $\mathcal{J}$ defined by $\frac{d}{ds}\varphi_s = \tilde{X}_s \circ \varphi_s$. Then the equality*

$$\frac{d}{ds}(\varphi_{s*}A - A) = d_{\varphi_{s*}A}\beta_{\xi_0, \varphi_{s*}A} - \iota(\tilde{X}_s)F_{\varphi_{s*}A}$$

holds for some $\beta_{\xi_0, \varphi_{s}A} \in \Gamma(End(p^*TM))$, where $\xi_u = \varphi_{u+s} \circ \varphi_s^{-1}$ and $\dot{\xi}_0 = \frac{d\xi_u}{du}|_{u=0}$.*

5. PROOF OF THEOREM A

In this section we shall prove Theorem A.

Let (M, ω) be a closed symplectic manifold of dimension $2n$ and fix $Q = \sum_j q_j \cdot w^{k-j} \in I_n[w]^k$ satisfying $Q(M, \omega) = 0$ in $H^{2k}(M; \mathbb{R})$. As in section 2, we have the map $\tilde{F}_Q: Symp(M, \omega) \rightarrow \Omega^{2k-1}(M)$, which can be rewritten as

$$\tilde{F}_Q(\varphi) = J^* \left\{ \sum_{j=0}^k \Delta(A, \varphi_*A)q_j \wedge p^*\omega^{k-j} + \varphi_*\mu - \mu \right\}$$

in terms of Bott homomorphisms. We note $\Delta(A, \varphi_*A)q_0 = 0$.

Put

$$\tilde{f}_Q(\varphi) = \sum_{j=0}^k \Delta(A, \varphi_*A)q_j \wedge p^*\omega^{k-j} + \varphi_*\mu - \mu;$$

then we have $\tilde{F}_Q = J^* \circ \tilde{f}_Q$.

Lemma 5.1. *The map $\tilde{f}_Q: Symp(M, \omega) \rightarrow \Omega^{2k-1}(\mathcal{J})$ induces a crossed homomorphism $f_Q: Symp(M, \omega) \rightarrow H^{2k-1}(\mathcal{J}; \mathbb{R})$. Moreover the cohomology class $[f_Q]$ in $H^1(Symp(M, \omega), H^{2k-1}(\mathcal{J}))$ is independent of the choice of connection A and $2k-1$ form μ .*

Proof. Using the properties of Bott homomorphisms in section 3 and the definition of μ , we can easily show $d\tilde{f}_Q(\varphi) = 0$ and

$$\tilde{f}_Q(\varphi\psi) - \tilde{f}_Q(\varphi) - \varphi_*\tilde{f}_Q(\psi) = d \left\{ - \sum_{j=0}^k \Delta(A, \varphi_*A, (\varphi\psi)_*A)q_j \wedge p^*\omega^{k-j} \right\}$$

for any $\varphi, \psi \in \text{Symp}(M, \omega)$. The map

$$f_Q: \text{Symp}(M, \omega) \ni \varphi \mapsto [\tilde{f}_Q(\varphi)] \in H^{2k-1}(\mathcal{J}; \mathbb{R})$$

turns out to be a well-defined crossed homomorphism.

Let B and ν be another connection and $2k-1$ form satisfying

$$\sum_{j=0}^k \Delta(B)q_j \wedge p^* \omega^{k-j} + d\nu = 0.$$

Put

$$\alpha = \nu - \mu + \sum_{j=0}^k \Delta(A, B)q_j \wedge p^* \omega^{k-j};$$

then it is easy to see that α is a closed $2k-1$ form on $\mathcal{J}$.

Put

$$\tilde{g}_Q(\varphi) = \sum_{j=0}^k \Delta(B, \varphi_* B)q_j \wedge p^* \omega^{k-j} + \varphi_* \nu - \nu$$

for any $\varphi \in \text{Symp}(M, \omega)$; then it also induces a crossed homomorphism g_Q . A computation shows

$$\begin{aligned} & \tilde{g}_Q(\varphi) - \tilde{f}_Q(\varphi) \\ &= \varphi_* \alpha - \alpha + d \left[\sum_{j=1}^k \{ -\Delta(B, \varphi_* A, \varphi_* B)q_j + \Delta(A, B, \varphi_* A)q_j \} \wedge p^* \omega^{k-j} \right]. \end{aligned}$$

Therefore $[g_Q(\varphi)] - [f_Q(\varphi)] = \varphi_*[\alpha] - [\alpha]$ holds in $H^{2k-1}(\mathcal{J}; \mathbb{R})$. This shows $g_Q - f_Q = \delta\alpha$ as a 1-cochain of the group $\text{Symp}(M, \omega)$ with values in $H^{2k-1}(\mathcal{J}; \mathbb{R})$; hence $[g_Q] = [f_Q]$ in $H^1(\text{Symp}(M, \omega), H^{2k-1}(\mathcal{J}; \mathbb{R}))$. $\square$

Since any ω -positive compatible almost complex structure $J: M \rightarrow \mathcal{J}$ induces the same isomorphism $H^*(\mathcal{J}; \mathbb{R}) \xrightarrow{\cong} H^*(M; \mathbb{R})$, we have the first part of Theorem A.

Next we prove the second part of Theorem A. We compute the derivative $\frac{d}{ds} f_Q(\varphi_s)$ of a smooth curve φ_s in $\text{Symp}(M, \omega)$. There is the following derivation formula for Bott homomorphisms or Chern-Simons forms in the general setting.

Let $P \rightarrow M$ be a G -bundle and $q \in I^j(G)$ an invariant polynomial of degree j . For a connection $\bar{\theta}$ on the product G -bundle $P \times I \rightarrow M \times I$ with coordinate $s \in I = [0, 1]$, we consider the corresponding family $\theta_s = \bar{\theta}|_{s=\text{const}}$ of connections on P . Let $\bar{\theta}'$ and θ'_s be another pair. Then we have

$$(5.1) \quad \frac{\partial}{\partial s} \Delta(\theta_s, \theta'_s)q = j \left\{ q \left(\frac{\partial \theta'_s}{\partial s}, F_{\theta'_s}^{(j-1)} \right) - q \left(\frac{\partial \theta_s}{\partial s}, F_{\theta_s}^{(j-1)} \right) \right\} + \text{exact form}$$

(see [12]). We apply this formula to our situation. Take $\bar{\theta} = A$ and $\bar{\theta}' = \varphi_{s*} A$, which are connections on $p^* TM \times I$, where A is considered as a constant along I . Then we have $\theta_s = A$, which is independent of s , and $\theta'_s = \varphi_{s*} A$ with abuse of

notation. Take $q = q_j$. In this case, the formula (5.1) and Lemma 4.2 imply

$$\begin{aligned} \frac{d}{ds} \Delta(A, \varphi_{s*} A) q_j &= j q_j (d_{\varphi_{s*} A} \beta_{\xi_0, \varphi_{s*} A} - \iota(\tilde{X}_s) F_{\varphi_{s*} A}, F_{\varphi_{s*} A}^{(j-1)}) + \text{exact form} \\ &= j d q_j (\beta_{\xi_0, \varphi_{s*} A}, F_{\varphi_{s*} A}^{(j-1)}) - \iota(\tilde{X}_s) q_j (F_{\varphi_{s*} A}^{(j)}) + \text{exact form} \\ &= -\iota(\tilde{X}_s) \Delta(\varphi_{s*} A) q_j + d R_j, \end{aligned}$$

where R_j is some $2j - 2$ -form on $\mathcal{J}$ for each j . On the other hand, we have

$$\frac{d}{ds} \psi_s^* \mu = \frac{d}{du} (\eta_u^* \psi_s^* \mu)|_{u=0} = -d\{\iota(\tilde{X}_s) \psi_s^* \mu\} - \iota(\tilde{X}_s) d(\psi_s^* \mu),$$

where $\eta_u = \xi_u^{-1} = \varphi_s \varphi_{u+s}^{-1} = \psi_s^{-1} \psi_{s+u}$ with $\frac{d}{du} \eta_u|_{u=0} = -\tilde{X}_s$.

Using these formulas, properties of Bott homomorphisms and the fact that the interior product is an anti-derivative of degree -1 , we have

$$\begin{aligned} \frac{d}{ds} \tilde{f}_Q(\varphi_s) &= - \sum_{j=0}^k \{ \iota(\tilde{X}_s) \Delta(\varphi_{s*} A) q_j - d R_j \} \wedge p^* \omega^{k-j} - d\{\iota(\tilde{X}_s) \psi_s^* \mu\} - \iota(\tilde{X}_s) d\psi_s^* \mu \\ &= - \sum_{j=0}^k \left\{ \iota(\tilde{X}_s) \Delta(\varphi_{s*} A) q_j \right\} \wedge p^* \omega^{k-j} \\ &\quad + \iota(\tilde{X}_s) \psi_s^* \left\{ \sum_{j=0}^k \Delta(A) q_j \wedge p^* \omega^{k-j} \right\} + \text{exact form} \\ &= \sum_{j=0}^k \Delta(\varphi_{s*} A) q_j \wedge \iota(\tilde{X}_s) (p^* \omega^{k-j}) + \text{exact form} \\ &= \sum_{j=0}^k (k-j) \Delta(\varphi_{s*} A) q_j \wedge p^* \omega^{k-j-1} \wedge \iota(\tilde{X}_s) p^* \omega + \text{exact form}. \end{aligned}$$

Since $\varphi_{s*} A$ is also a $U(n)$ -connection on $p^* TM$, we obtain

$$\frac{d}{ds} f_Q(\varphi_s) = \sum_{j=0}^k (k-j) q_j (p^* TM) \cup p^* [\omega]^{k-j-1} \cup [\iota(\tilde{X}_s) p^* \omega].$$

Let X_s be the family of vector fields on M defined by $\frac{d}{ds} \varphi_s = X_s \circ \varphi_s$; then we have $J^*(\iota(\tilde{X}_s) p^* \omega) = \iota(X_s) \omega$. Therefore $\frac{d}{ds} F_Q(\varphi_s) = J^* \frac{d}{ds} f_Q(\varphi_s)$ implies the following proposition.

Proposition 5.2. *The equality $\frac{d}{ds} F_Q(\varphi_s) = \sum_{j=0}^k (k-j) q_j (M, \omega) \cup [\omega]^{k-j-1} \cup [\iota(X_s) \omega]$ holds in $H^{2k-1}(M; \mathbb{R})$.*

This proposition is the second part of Theorem A, which concludes the proof of it.

6. ANOTHER DEFINITION OF THE CROSSED HOMOMORPHISM

In this section we shall define another map $h_Q: \text{Symp}(M, \omega) \rightarrow H^{2k-1}(M; \mathbb{R})$ for the fixed A and μ in section 2 and show $h_Q = -F_Q$.

Let (M, ω) be a closed symplectic manifold of dimension $2n$. Fix $Q \in I_n[w]^k$ with $Q(M, \omega) = 0$ in $H^{2k}(M; \mathbb{R})$ as before. For any $\varphi \in \text{Symp}(M, \omega)$, we consider

the mapping tori $M_\varphi := [0, 1] \times M / (1, x) \sim (0, \varphi^{-1}(x))$ of φ and $\mathcal{J}_\varphi$ of $\varphi: \mathcal{J} \rightarrow \mathcal{J}$. Then we have the fiber bundle $p_\varphi: \mathcal{J}_\varphi \rightarrow M_\varphi$. Note that $\mathcal{J}_\varphi$ and M_φ are homotopy equivalent.

Let $pr: (-\varepsilon, 1 + \varepsilon) \times M \rightarrow M_\varphi$ and $pr: (-\varepsilon, 1 + \varepsilon) \times \mathcal{J} \rightarrow \mathcal{J}_\varphi$ be the obvious projections for small $\varepsilon > 0$. By considering ω as a closed 2-form on $(-\varepsilon, 1 + \varepsilon) \times M$ which is constant along $(-\varepsilon, 1 + \varepsilon)$, it descends to a closed 2-form ω_φ on M_φ . The pullback bundle $p_\varphi^* T^M M_\varphi$ of the tangent bundle $T^M M_\varphi$ along the fibers also has the canonical hermitian structure. In particular it is a complex vector bundle.

We consider the class

$$\bar{Q}_\varphi := Q((p_\varphi^* T^M M_\varphi, p_\varphi^* \omega_\varphi), [p_\varphi^* \omega_\varphi]) = \sum_{j=0}^k q_j(p_\varphi^* T^M M_\varphi, p_\varphi^* \omega_\varphi) \cup [p_\varphi^* \omega_\varphi]^{k-j}$$

in $H^{2k}(\mathcal{J}_\varphi; \mathbb{R})$. We identify $\mathcal{J}$ with the fiber at 0 : $\mathcal{J} = \mathcal{J} \times 0 \hookrightarrow \mathcal{J}_\varphi$. By the assumption $Q(M, \omega) = 0$, we have $\bar{Q}_\varphi|_{\mathcal{J}} = 0$.

Recall the notation in section 2: $p: \mathcal{J} \rightarrow M$ is the projection and A is a $U(n)$ -connection on $p^* TM$. Put $Q(A, p^* \omega) := \sum_{j=0}^k q_j(F_A^{(j)}) \wedge p^* \omega^{k-j}$; then it is an exact $2k$ -form on $\mathcal{J}$. There exists $\mu \in \Omega^{2k-1}(\mathcal{J})$ such that $d\mu + Q(A, p^* \omega) = 0$.

Take a $U(n)$ -connection A_φ on $p_\varphi^* T^M M_\varphi$ and a $2k-1$ form μ_φ on $\mathcal{J}_\varphi$ satisfying $\iota_{(-\varepsilon, \varepsilon)}^* A_\varphi = A$ and $\iota_{(-\varepsilon, \varepsilon)}^* \mu_\varphi = \mu$, where $\iota_{(-\varepsilon, \varepsilon)}: (-\varepsilon, \varepsilon) \times \mathcal{J} \hookrightarrow \mathcal{J}_\varphi$ is the inclusion and A and μ are considered as a connection and as a $2k-1$ form on $(\varepsilon, \varepsilon) \times \mathcal{J}$ which are constant along $[0, 1]$. Then we have $d\mu_\varphi + Q(A_\varphi, p_\varphi^* \omega_\varphi) = 0$ near $\mathcal{J}$. Therefore we have $[Q(A_\varphi, p_\varphi^* \omega_\varphi) + d\mu_\varphi] \in H^{2k}(\mathcal{J}_\varphi, \mathcal{J}; \mathbb{R})$ whose image in $H^{2k}(\mathcal{J}_\varphi; \mathbb{R})$ is $\bar{Q}_\varphi$.

We define the class $h_Q(\varphi)$ in $H^{2k-1}(\mathcal{J}; \mathbb{R})$ as the image of $[Q(A_\varphi, p_\varphi^* \omega_\varphi) + d\mu_\varphi]$ by the composition of isomorphisms:

$$\begin{aligned} H^{2k}(\mathcal{J}_\varphi, \mathcal{J}; \mathbb{R}) &\xrightarrow{\text{excision}} H^{2k}([0, 1] \times \mathcal{J}, \{0, 1\} \times \mathcal{J}; \mathbb{R}) \\ &\xrightarrow{\text{K\"unneth}} H^1([0, 1], \{0, 1\}; \mathbb{R}) \otimes H^{2k-1}(\mathcal{J}; \mathbb{R}) \\ &\cong H^{2k-1}(\mathcal{J}; \mathbb{R}). \end{aligned}$$

Lemma 6.1. *The map*

$$h_Q = h_{Q, A, \mu}: \text{Symp}(M, \omega) \ni \varphi \mapsto h_Q(\varphi) \in H^{2k-1}(\mathcal{J}; \mathbb{R})$$

is well-defined, that is, it is independent of the choice of A_φ and μ_φ .

Proof. Let B_φ and ν_φ be another connection on $p_\varphi^* T^M M_\varphi$ and the $2k-1$ form on $\mathcal{J}_\varphi$ satisfying $\iota_{(-\varepsilon, \varepsilon)}^* B_\varphi = A$ and $\iota_{(-\varepsilon, \varepsilon)}^* \nu_\varphi = \mu$. Then we have $d\nu_\varphi + Q(B_\varphi, p_\varphi^* \omega_\varphi) = 0$ near $\mathcal{J}$. We have only to show $[Q(B_\varphi, p_\varphi^* \omega_\varphi) + d\nu_\varphi] = [Q(A_\varphi, p_\varphi^* \omega_\varphi) + d\mu_\varphi]$ in $H^{2k}(\mathcal{J}_\varphi, \mathcal{J}; \mathbb{R})$. But it follows from

$$\begin{aligned} Q(B_\varphi, p_\varphi^* \omega_\varphi) + d\nu_\varphi &= Q(A_\varphi, p_\varphi^* \omega_\varphi) + d\mu_\varphi \\ &\quad + d \left\{ \sum_{j=0}^k \Delta(A_\varphi, B_\varphi) q_j \wedge p_\varphi^* \omega_\varphi^{k-j} + \nu_\varphi - \mu_\varphi \right\} \end{aligned}$$

on $\mathcal{J}_\varphi$ and $\sum_{j=0}^k \Delta(A_\varphi, B_\varphi) q_j \wedge p_\varphi^* \omega_\varphi^{k-j} + \nu_\varphi - \mu_\varphi \equiv 0$ near $\mathcal{J}$. $\square$

Proposition 6.2. $f_Q = -h_Q: \text{Symp}(M, \omega) \rightarrow H^{2k-1}(\mathcal{J}; \mathbb{R})$.

Proof. For any sufficiently small $\varepsilon > 0$, let $\lambda_\varepsilon: (-\varepsilon, 1+\varepsilon) \rightarrow \mathbb{R}$ be a smooth function satisfying $|\lambda_\varepsilon(t)| < 2\varepsilon$, $|\lambda'_\varepsilon(t)| \leq \text{const.}$ independent of ε and

$$\lambda_\varepsilon(t) = \begin{cases} -t, & t \in (-\varepsilon, \varepsilon], \\ 0, & t \in [2\varepsilon, 1-2\varepsilon], \\ 1-t, & t \in [1-\varepsilon, 1+\varepsilon). \end{cases}$$

Clearly there exists a family $\{\lambda_\varepsilon\}_\varepsilon$ of such functions.

Put $b = \varphi_* A - A$ on $\mathcal{J}$. The connection $\bar{A}_\varphi = A + (t + \nu_\varepsilon(t))b$ on $(-\varepsilon, 1+\varepsilon) \times p^*TM$ descends to a connection A_φ on $p^*T^M M_\varphi \cong (-\varepsilon, 1+\varepsilon) \times p^*TM / \sim$. Similarly the 2-form $p^*\omega$ on $(-\varepsilon, 1+\varepsilon) \times \mathcal{J}$ descends to the closed 2-form $p^*\omega_\varphi$. Therefore we can calculate forms on $\mathcal{J}_\varphi$ as those on $(-\varepsilon, 1+\varepsilon) \times \mathcal{J}$.

Let $\nabla_A \stackrel{\text{loc}}{=} d + a$ be a local expression of the covariant derivative associated with the connection A on $\mathcal{J}$ in some local trivialization of p^*TM ; then we have

$$\nabla_{\bar{A}_\varphi} \stackrel{\text{loc}}{=} d^{\mathcal{J}_\varphi} + a + (t + \lambda_\varepsilon)b,$$

where $d^{\mathcal{J}_\varphi}$ denotes the exterior differential on $\mathcal{J}_\varphi$. The curvature $F_{\bar{A}_\varphi}$ of $\bar{A}_\varphi$ is given by

$$\begin{aligned} F_{\bar{A}_\varphi} &\stackrel{\text{loc}}{=} d^{\mathcal{J}_\varphi} \{a + (t + \lambda_\varepsilon)b\} + \{a + (t + \lambda_\varepsilon)b\} \wedge \{a + (t + \lambda_\varepsilon)b\} \\ &= F_t + dt \wedge (1 + \lambda'_\varepsilon)b + \lambda_\varepsilon B_t, \end{aligned}$$

where

$$F_t = F_{A+tb}^{\mathcal{J}} \stackrel{\text{loc}}{=} d^{\mathcal{J}}(a + tb) + (a + tb) \wedge (a + tb)$$

and

$$B_{t,\varepsilon} = d^{\mathcal{J}}b + (a + tb) \wedge b + b \wedge (a + tb) + \lambda_\varepsilon b \wedge b.$$

Here $d^{\mathcal{J}}$ denotes the exterior differential on $\mathcal{J}$. For any $q_j \in I_n^j$, we have

$$\begin{aligned} q_j(F_{\bar{A}_\varphi}^{(j)}) &= q_j((dt \wedge (1 + \lambda'_\varepsilon)b + F_t + \lambda_\varepsilon B_{t,\varepsilon})^{(j)}) \\ &= j(1 + \lambda'_\varepsilon)dt \wedge q_j(b, (F_t + \lambda_\varepsilon B_{t,\varepsilon})^{(j-1)}) + q((F_t + \lambda_\varepsilon B_{t,\varepsilon})^{(j)}). \end{aligned}$$

Put $\mu_\varphi = \xi_\varphi + dt \wedge \eta$; then we have $\xi_\varphi \equiv \mu$ near $0 \times \mathcal{J}$, $\xi_\varphi \equiv (\varphi^{-1})^*\mu$ near $1 \times \mathcal{J}$, $\eta \equiv 0$ near $\{0, 1\} \times \mathcal{J}$ and $d^{\mathcal{J}_\varphi}\mu_\varphi = d^{\mathcal{J}}\xi_\varphi + dt \wedge \xi'_\varphi - dt \wedge d^{\mathcal{J}}\eta$. Put $Q(\bar{A}_\varphi, p^*\omega_\varphi) = Q^{\mathcal{J}} + dt \wedge Q^I$. Since the equality $Q(\bar{A}_\varphi, p^*\omega_\varphi) + d\mu_\varphi = 0$ holds on $(-\varepsilon, \varepsilon) \times \mathcal{J}$, we have $Q^{\mathcal{J}} + d^{\mathcal{J}}\xi_\varphi = 0$ and $Q^I + \xi'_\varphi - d^{\mathcal{J}}\eta = 0$ on $(-\varepsilon, \varepsilon) \times \mathcal{J}$.

Note that the integration is defined by $\int_I \alpha = \int_I \beta dt$ for $I = [0, 1]$ and $\alpha = \beta \wedge dt$ by our convention.

A representative of $h_Q(\varphi)$ is given by

$$\begin{aligned} \int_I (Q(\bar{A}_\varphi, p^*\omega_\varphi) + d\mu_\varphi) &= \int_I dt \wedge (Q^I + \xi'_\varphi - d^{\mathcal{J}}\eta) \\ &= \left(\int_I dt \wedge Q^I \right) - [\xi_\varphi]_0^1 - d^{\mathcal{J}} \int_I dt \wedge \eta \\ &= \int_I dt \wedge \left\{ \sum_{j=0}^k j(1 + \lambda'_\varepsilon)q_j(b, (F_t + \lambda_\varepsilon B_{t,\varepsilon})^{(j-1)}) \wedge p^*\omega^{k-j} \right\} \\ &\quad - (\varphi^{-1})^*\mu + \mu - d^{\mathcal{J}} \int_I dt \wedge \eta, \end{aligned}$$

which implies

$$h_Q(\varphi) = \left[\int_I dt \wedge \left\{ \sum_{j=0}^k j(1 + \lambda'_\varepsilon) q_j(b, (F_t + \lambda_\varepsilon B_{t,\varepsilon})^{(j-1)}) \wedge p^* \omega^{k-j} \right\} - \varphi_* \mu + \mu \right].$$

For any sufficiently small $\varepsilon > 0$, we have $|\lambda'_\varepsilon| \leq \text{const}$, $|\text{supp}(\lambda_\varepsilon)| \leq 4\varepsilon$ and $|\lambda_\varepsilon| \leq \varepsilon$, hence $|B_{t,\varepsilon}| \leq \text{const}$. Take the limit as $\varepsilon \rightarrow 0$; then we have

$$\begin{aligned} h_Q(\varphi) &= \left[\int_I dt \wedge \sum_{j=0}^k j q_j(b, F_t^{(j-1)}) \wedge p^* \omega^{k-j} - \varphi_* \mu + \mu \right] \\ &= - \left[\sum_{j=0}^k \Delta(A, \varphi_* A) q_j \wedge p^* \omega^{k-j} + \varphi_* \mu - \mu \right] \\ &= -f_Q(\varphi). \end{aligned}$$

This finishes the proof of the proposition. $\square$

Put $Q_\varphi := \sum_{j=0}^k q_j(T^M M_\varphi, \omega_\varphi) \cup [\omega_\varphi]^{k-j} \in H^{2k}(M_\varphi; \mathbb{R})$; then the image of it by the isomorphism $p_\varphi^*: H^*(M_\varphi; \mathbb{R}) \rightarrow H^*(\mathcal{J}_\varphi; \mathbb{R})$ is $\bar{Q}_\varphi$.

Here we recall the Leray-Hirsch theorem [9] over $\mathbb{R}$. Let $\pi: Y \rightarrow X$ be a fibration over an arcwise-connected CW-complex X with fiber F . Assume that the cohomology group $H^*(F; \mathbb{R})$ is finitely generated and that the induced homomorphism $\iota^*: H^*(Y; \mathbb{R}) \rightarrow H^*(F; \mathbb{R})$ by the inclusion $\iota: F \hookrightarrow Y$ is surjective. Then we can take a homomorphism $s: H^*(F; \mathbb{R}) \rightarrow H^*(Y; \mathbb{R})$ of graded modules of degree 0 such that $\iota^* \circ s = \text{id}$. We can define an $\mathbb{R}$ -module isomorphism $\kappa_s: H^*(X; \mathbb{R}) \otimes H^*(F; \mathbb{R}) \rightarrow H^*(Y; \mathbb{R})$ by $\kappa_s(u \otimes v) = \pi^* u \cup s(v)$. We have the formula $\kappa_s(u \otimes v) = \pi^* u \cup \kappa_s(1 \otimes v)$.

Back to our situation, we consider the mapping torus M_φ of $\varphi \in \text{Symp}(M, \omega)$, which is a fiber bundle over S^1 with fiber M . In the case of $\varphi^* = \text{id}$ on $H^*(M; \mathbb{R})$, we can easily check that the induced homomorphism $\iota^*: H^*(M_\varphi; \mathbb{R}) \rightarrow H^*(M; \mathbb{R})$ of the inclusion $\iota: M \hookrightarrow M_\varphi$ is surjective. Hence we have an $\mathbb{R}$ -module isomorphism

$$(6.1) \quad \kappa_\varphi = \kappa_{\varphi, s}: H^*(S^1; \mathbb{R}) \otimes H^*(M; \mathbb{R}) \xrightarrow{\cong} H^*(M_\varphi; \mathbb{R}).$$

The following lemma, which is a corollary of the proof of Proposition 6.2, is needed later.

Lemma 6.3. *If $\varphi_* = \text{id}$ on $H^*(M; \mathbb{R})$, then the equalities*

$$Q_\varphi = \kappa_\varphi(t^* \otimes H_Q(\varphi)) = \kappa_\varphi(t^* \otimes (-F_Q(\varphi)))$$

hold. Here $H_Q := J^ \circ h_Q: \text{Symp}(M, \omega) \rightarrow H^{2k-1}(M; \mathbb{R})$, and t^* is the generator of $H^1(S^1; \mathbb{Z}) \subset H^1(S^1; \mathbb{R})$.*

Proof. By the assumption, it is easy to see that the induced homomorphism

$$\iota^*: H^*(\mathcal{J}_\varphi; \mathbb{R}) \rightarrow H^*(\mathcal{J}; \mathbb{R})$$

of the inclusion $\iota: \mathcal{J} \hookrightarrow \mathcal{J}_\varphi$ as a fiber is surjective. Therefore we have the short exact sequences

$$0 \rightarrow H^*(\mathcal{J}_\varphi, \mathcal{J}; \mathbb{R}) \rightarrow H^*(\mathcal{J}_\varphi; \mathbb{R}) \rightarrow H^*(\mathcal{J}; \mathbb{R}) \rightarrow 0.$$

On the other hand, by the Leray-Hirsch theorem, we have the $\mathbb{R}$ -module isomorphism

$$\bar{\kappa}_\varphi = \bar{\kappa}_{\varphi,s}: H^*(S^1; \mathbb{R}) \otimes H^*(\mathcal{J}; \mathbb{R}) \rightarrow H^*(\mathcal{J}_\varphi; \mathbb{R}).$$

These imply isomorphisms

$$H^{2k}(\mathcal{J}_\varphi, \mathcal{J}; \mathbb{R}) \cong H^1(S^1; \mathbb{R}) \otimes H^{2k-1}(\mathcal{J}; \mathbb{R}) \xrightarrow{\bar{\kappa}_\varphi} \ker[\iota^*: H^{2k}(\mathcal{J}_\varphi; \mathbb{R}) \rightarrow H^{2k}(\mathcal{J}; \mathbb{R})].$$

Under these isomorphisms, we have $\bar{Q}_\varphi = \bar{\kappa}_\varphi(t^* \otimes h_Q(\varphi)) = \bar{\kappa}_\varphi(t^* \otimes (-f_Q(\varphi)))$ by the proof of Proposition 6.2. Since any section of the bundle $p_\varphi: \mathcal{J}_\varphi \rightarrow M_\varphi$ is homotopy equivalent, we obtain the lemma. $\square$

7. THURSTON'S THEOREM

In this section we shall recall Thurston's theorem [10] (see also [5]) for a symplectic fibration and consider some related diffeomorphisms.

Let $\pi: M \rightarrow B$ be a symplectic fibration whose typical fiber is a symplectic manifold (F, σ) . By definition, there is an open covering $\{U_\alpha\}$ of B with trivialization

$$\begin{array}{ccc} \phi_\alpha: \pi^{-1}(U_\alpha) & \xrightarrow{\quad} & U_\alpha \times F \\ \downarrow \pi & & \downarrow \\ U_\alpha & \xlongequal{\quad} & U_\alpha \end{array}$$

for each U_α . Moreover put $\phi_\alpha(b) = \text{proj}_2 \circ \phi_\alpha|_{F_b = \pi^{-1}(b)}: F_b \rightarrow F$ and $\phi_{\beta\alpha}(b) = \phi_\beta(b) \circ \phi_\alpha(b)^{-1}$; then they define smooth maps $\phi_{\beta\alpha}: U_\alpha \cap U_\beta \rightarrow \text{Symp}(F, \sigma)$. For each $b \in B$, the canonical symplectic form σ_b is defined by $\sigma_b = \phi_\alpha(b)^*\sigma$. This form is independent of the choice of α .

A symplectic form ω on M is called compatible with the fibration π if each fiber (F_b, σ_b) is a symplectic submanifold of (M, ω) , that is, the equality $\sigma_b = \iota_b^*\omega$ holds for each inclusion $\iota_b: F_b \hookrightarrow M$.

Theorem 7.1 (Thurston [10]). *Let $\pi: M \rightarrow B$ be a compact symplectic fibration with symplectic fiber (F, σ) and a connected symplectic base (B, β) , and let $\sigma_b \in \Omega^2(F_b)$ be the canonical symplectic form on F_b . If there exists $a \in H^2(M)$ satisfying $\iota_b^*a = [\sigma_b]$ for some (and hence any) $b \in B$, then for all sufficiently large $K > 0$, there exists a symplectic form $\omega_K \in \Omega^2(M)$ such that it is compatible with π and represents the class $a + K[\pi^*\beta]$.*

The proof of his theorem constructs a closed 2-form τ on M with $[\tau] = a$ satisfying $\iota_b^*\tau = \sigma_b$ for all $b \in B$. The required symplectic form is given by $\omega_K = \tau + K\pi^*\beta$ for any sufficiently large $K > 0$.

We fix such a τ . Let $\hat{\phi}: M \rightarrow M$ be a diffeomorphism of M over a symplectomorphism $\phi: (B, \beta) \rightarrow (B, \beta)$ such that its restriction $\hat{\phi}_b: (F_b, \sigma_b) \rightarrow (F_{\phi(b)}, \sigma_{\phi(b)})$ to the fiber F_b at each $b \in B$ is a symplectomorphism:

$$\begin{array}{ccc} \hat{\phi}: M & \xrightarrow{\quad} & M \\ \downarrow \pi & & \downarrow \pi \\ \phi: B & \xrightarrow{\quad} & B \end{array}$$

Moreover suppose that the induced homomorphism $\hat{\phi}^*: H^2(M) \rightarrow H^2(M)$ preserves the class $a = [\tau] \in H^2(M)$.

Let $Vert$ and Hor be the subbundles of TM whose fibers at $x \in M$ are given by $Vert_x := \ker d\pi_x$ and $Hor_x := (\ker d\pi_x)^\tau := \{\xi \in T_x M \mid \tau(\xi, \eta) = 0 \ \forall \eta \in Vert_x\}$, respectively. Then we have the splitting $TM = Vert \oplus Hor$ and the isomorphism $d\pi_x : Hor_x \rightarrow T_{\pi(x)}B$ for each $x \in M$.

Put

$$\chi_{\hat{\phi}} := \hat{\phi}^* \tau - \tau;$$

then we have

$$\hat{\phi}^* \omega_K = \omega_K + \chi_{\hat{\phi}}$$

and

$$\chi_{\hat{\phi}}|_{Vert} \equiv 0.$$

With respect to the splitting $TM = Vert \oplus Hor$, the 2-forms τ , $\pi^* \beta$ and $\chi_{\hat{\phi}}$ can be represented by

$$\tau = \begin{pmatrix} A & O \\ O & B \end{pmatrix}, \quad \pi^* \beta = \begin{pmatrix} O & O \\ O & C \end{pmatrix} \quad \text{and} \quad \chi_{\hat{\phi}} = \begin{pmatrix} O & E \\ F & G \end{pmatrix},$$

where A and C are nondegenerate. For each $t \in [0, 1]$, put

$$\Omega_t := \omega_K + t\chi_{\hat{\phi}} = \begin{pmatrix} A & tE \\ tF & B + KC + tG \end{pmatrix};$$

then we have $\Omega_0 = \omega_K$ and $\Omega_1 = \hat{\phi}^* \omega_K$. Moreover for any sufficiently large $K > 0$, Ω_t is nondegenerate and has the same cohomology class $[\Omega_t] = [\omega_K]$ for all $t \in [0, 1]$. Therefore we can apply Moser's method to them. There is a family of diffeomorphisms η_t on M satisfying $\eta_t^* \Omega_t = \Omega_0 = \omega_K$ and $\eta_0 = id_M$.

In particular we have

$$\omega_K = \eta_1^* \Omega_1 = \eta_1^* \hat{\phi}^* \omega_K = (\hat{\phi} \circ \eta_1)^* \omega_K,$$

which implies that the composition

$$\Phi := \hat{\phi} \circ \eta_1 : (M, \omega_K) \rightarrow (M, \omega_K)$$

is a symplectomorphism diffeotopic to $\hat{\phi}$. We have shown the following proposition.

Proposition 7.2. *Under the same assumption as Thurston's theorem, let $\hat{\phi} : M \rightarrow M$ be a fiberwise symplectomorphism over a symplectomorphism ϕ on (B, β) with $\hat{\phi}^* a = a$. Then for any sufficiently large $K > 0$, $\hat{\phi}$ is diffeotopic to a symplectomorphism $\Phi : (M, \omega_K) \rightarrow (M, \omega_K)$, where ω_K is the symplectic form on M in Thurston's theorem.*

For the symplectomorphism Φ on (M, ω_K) in this proposition, we consider the mapping torus $M_\Phi = [0, 1] \times M / (0, m) \sim (1, \Phi(m))$ of Φ . The closed 2-form ω_K defines a fiberwise symplectic form $\omega_{K, \Phi}$ on M_Φ . Let $T^M M_\Phi$ be the tangent bundle along the fibers M ; then the symplectic vector bundle $(T^M M_\Phi, \omega_{K, \Phi})$ over M_Φ is obtained. In order to calculate the Chern classes of it, we consider a family of the mapping tori $\{M_{\hat{\phi} \circ \eta_s}\}_{s \in [0, 1]}$ of $\{\hat{\phi} \circ \eta_s\}_{s \in [0, 1]}$. These mapping tori are diffeomorphic to each other as fiber bundles over S^1 . In particular we have the family of diffeomorphisms

$$\xi_s : M_{\hat{\phi}} \rightarrow M_{\hat{\phi} \circ \eta_s}$$

induced by

$$[0, 1] \times M \ni (t, m) \mapsto (t, \hat{\phi} \circ \eta_{s\nu(t)} \circ \hat{\phi}^{-1}(m)) \in [0, 1] \times M,$$

where $\nu(t)$ is a monotonically increasing function on $[0, 1]$ with $\nu(t) \equiv 0$ near $t = 0$ and $\nu(t) \equiv 1$ near $t = 1$. Put $\xi = \xi_1: M_{\hat{\phi}} \rightarrow M_{\hat{\phi} \circ \eta_1} = M_{\Phi}$.

For each $s \in [0, 1]$, the family of 2-forms

$$\bar{\Omega}_s := \{(\hat{\phi}^{-1})^* \Omega_{1+t(s-1)}\}_{t \in [0,1]}$$

on M parameterized by $t \in [0, 1]$ defines a fiberwise symplectic form $\bar{\Omega}_s$ (we use the same symbol) on $M_{\hat{\phi} \circ \eta_s}$ because of

$$(\hat{\phi} \circ \eta_s)^*(\hat{\phi}^{-1})^* \Omega_{1+t(s-1)} = \eta_s^* \Omega_s = \omega_K = (\hat{\phi}^{-1})^* \Omega_1.$$

Let $T^M M_{\hat{\phi} \circ \eta_s}$ be the tangent bundle of $M_{\hat{\phi} \circ \eta_s}$ along the fibers M as a fiber bundle over S^1 ; then the family $\{(T^M M_{\hat{\phi} \circ \eta_s}, \bar{\Omega}_s)\}_{s \in [0,1]}$ of symplectic vector bundles is obtained. The Chern classes of the symplectic vector bundles $(T^M M_{\hat{\phi}}, \bar{\Omega}_0) = (T^M M_{\hat{\phi} \circ \eta_0}, \bar{\Omega}_0)$ and $(T^M M_{\Phi}, \omega_{K,\Phi}) = (T^M M_{\hat{\phi} \circ \eta_1}, \bar{\Omega}_1)$ agree under the isomorphism $\xi^*: H^*(M_{\Phi}; \mathbb{R}) \cong H^*(M_{\hat{\phi}}; \mathbb{R})$.

Next we consider $(T^M M_{\hat{\phi}}, \bar{\Omega}_0)$. For each $t \in [0, 1]$, we have

$$\bar{\Omega}_0|_t = (\hat{\phi}^{-1})^* \Omega_{1-t} = \tau - t(\hat{\phi}^{-1})^* \chi_{\hat{\phi}} + K\pi^* \beta.$$

For each $\hat{m} = (t, m) \in [0, 1] \times M / \sim = M_{\hat{\phi}}$, put

$$T_{\hat{m}}^{MV} M_{\hat{\phi}} := \text{Vert}_m$$

and

$$T_{\hat{m}}^{MH} M_{\hat{\phi}} = \{\xi \in T_{\hat{m}}^M M_{\hat{\phi}} | (\tau - t(\hat{\phi}^{-1})^* \chi_{\hat{\phi}})(\xi, \eta) = 0 \text{ for } \forall \eta \in T_{\hat{m}}^{MV} M_{\hat{\phi}}\}.$$

Since $\hat{\phi}$ is a fiber mapping, we have $(\hat{\phi}^{-1})^* \text{Vert} \subset \text{Vert}$. The 2-form $\tau - t(\hat{\phi}^{-1})^* \chi_{\hat{\phi}}$ is nondegenerate on Vert for each t because of $\chi_{\hat{\phi}}|_{\text{Vert}} \equiv 0$. Therefore we have the splitting

$$T^M M_{\hat{\phi}} = T^{MV} M_{\hat{\phi}} \oplus T^{MH} M_{\hat{\phi}},$$

where $T^{MV} M_{\hat{\phi}} = \coprod_{\hat{m} \in M_{\hat{\phi}}} T_{\hat{m}}^{MV} M_{\hat{\phi}}$ and $T^{MH} M_{\hat{\phi}} = \coprod_{\hat{m} \in M_{\hat{\phi}}} T_{\hat{m}}^{MH} M_{\hat{\phi}}$ are subbundles of $T^M M_{\hat{\phi}}$.

With respect to the above splitting, we can take matrix expressions

$$\tau - t(\hat{\phi}^{-1})^* \chi_{\hat{\phi}} = \begin{pmatrix} A' & O \\ O & B' \end{pmatrix} \text{ and } \pi^* \beta = \begin{pmatrix} O & O \\ O & C' \end{pmatrix},$$

where A' and C' are nondegenerate; we then have

$$\bar{\Omega}_0 = \tau - t(\hat{\phi}^{-1})^* \chi_{\hat{\phi}} + K\pi^* \beta = \begin{pmatrix} A' & O \\ O & B' + KC' \end{pmatrix}.$$

Note that $T^{MV} M_{\hat{\phi}}$ is a symplectic vector bundle over $M_{\hat{\phi}}$ with symplectic structure $(\tau - t(\hat{\phi}^{-1})^* \chi_{\hat{\phi}})|_{T^{MV} M_{\hat{\phi}}} = \tau|_{T^{MV} M_{\hat{\phi}}}$, which is nothing but the canonical symplectic structures $\{\sigma_b\}_{b \in B}$ and is denoted by $\sigma_{\hat{\phi}}$. This symplectic vector bundle is denoted by $(T^{MV} M_{\hat{\phi}}, \sigma_{\hat{\phi}})$. The mapping torus B_{ϕ} of ϕ is also a symplectic fiber bundle over S^1 with fiber (B, β) . Its tangent bundle along the fibers is also a symplectic vector bundle $(T^B B_{\phi}, \beta_{\phi})$. Let $\sigma_{\hat{\phi}} \oplus \pi^* \beta_{\phi}$ be the symplectic structure on the vector bundle $T^M M_{\hat{\phi}}$ induced from $\sigma_{\hat{\phi}}$ and β_{ϕ} by the isomorphism $T^M M_{\hat{\phi}} \cong T^{MV} M_{\hat{\phi}} \oplus \pi^* T^B B_{\phi}$; then we have a symplectic vector bundle $(T^M M_{\hat{\phi}}, \sigma_{\hat{\phi}} \oplus \pi^* \beta_{\phi})$. Let J^{MV} be a positive compatible complex structure of $(T^{MV} M_{\hat{\phi}}, \sigma_{\hat{\phi}})$ and let $J_{B_{\phi}}^B$ be a fiberwise positive

compatible complex structure of (B_ϕ, β_ϕ) ; then a complex structure J of $T^M M_{\hat{\phi}}$ is induced from $J^{MV} \oplus \pi^* J_{B_\phi}^B$ by $T^M M_{\hat{\phi}} \cong T^{MV} M_{\hat{\phi}} \oplus \pi^* T^B B_\phi$. J is compatible to the symplectic form $\sigma_{\hat{\phi}} \oplus \pi^* \beta_\phi$.

Lemma 7.3. *For any sufficiently large $K > 0$, the complex structure J on $T^M M_{\hat{\phi}}$ is $\bar{\Omega}_0$ -tame, i.e. $\bar{\Omega}_0(\cdot, J(\cdot)) > 0$.*

Proof. From the matrix representation of $\bar{\Omega}_0$ above, we have

$$K > 0 \Rightarrow (B' + KC')(\cdot, \pi^* J_{B_\phi}^B(\cdot)) > 0 \Leftrightarrow \bar{\Omega}_0(\cdot, J(\cdot)) > 0.$$

This implies the lemma. $\square$

By this lemma, the Chern classes of $(T^M M_{\hat{\phi}}, \bar{\Omega}_0)$ agree with those of $(T^M M_{\hat{\phi}}, \sigma_{\hat{\phi}} \oplus \pi^* \beta_\phi)$. Since the Chern classes of $(T^M M_\Phi, \omega_{K,\Phi})$ and $(T^M M_{\hat{\phi}}, \bar{\Omega}_0)$ also agree under the isomorphism ξ^* , we have the following corollary.

Corollary 7.4. *The Chern classes of $(T^M M_\Phi, \omega_{K,\Phi})$ agree with those of $(T^M M_{\hat{\phi}}, \sigma_{\hat{\phi}} \oplus \pi^* \beta_\phi)$ under the isomorphism $\xi^*: H^*(M_\Phi; \mathbb{R}) \cong H^*(M_{\hat{\phi}}; \mathbb{R})$ for any sufficiently large $K > 0$.*

Next we assume $\hat{\phi} = id$ on $H^*(M; \mathbb{R})$. In this case, we have isomorphisms (6.1) $\kappa_\varphi: H^*(S^1; \mathbb{R}) \otimes H^*(M; \mathbb{R}) \xrightarrow{\cong} H^*(M_\varphi; \mathbb{R})$ ($\varphi = \hat{\phi}, \Phi$). Put $\Xi = \kappa_{\hat{\phi}}^{-1} \circ \xi^* \circ \kappa_\Phi$; then we have the following commutative diagram:

$$\begin{array}{ccc} H^*(M_\Phi; \mathbb{R}) & \xrightarrow{\xi^*} & H^*(M_{\hat{\phi}}; \mathbb{R}) \\ \uparrow \kappa_\Phi \cong & & \uparrow \kappa_{\hat{\phi}} \cong \\ H^*(S^1; \mathbb{R}) \otimes H^*(M; \mathbb{R}) & \xrightarrow{\Xi} & H^*(S^1; \mathbb{R}) \otimes H^*(M; \mathbb{R}). \end{array}$$

Since the restriction of ξ to the fiber $M = 0 \times M$ at $0 \in S^1$ is the identity, the matrix representation of Ξ of j -dimension with respect to the splitting $H^j(M; \mathbb{R}) \oplus H^1(S^1; \mathbb{R}) \otimes H^{j-1}(M; \mathbb{R})$ (whose elements are considered as column vectors) is given by

$$\Xi_j = \begin{pmatrix} id & 0 \\ * & id \end{pmatrix}.$$

Therefore, by Lemma 6.3, we have

$$\xi^* Q_\Phi = \xi^* \kappa_\Phi(t^* \otimes (-F_Q(\Phi))) = \kappa_{\hat{\phi}} \Xi(t^* \otimes (-F_Q(\Phi))) = \kappa_{\hat{\phi}}(t^* \otimes (-F_Q(\Phi))).$$

On the other hand, it is easy to see that we can put $\kappa_{\hat{\phi}}^{-1} \xi^*[\omega_{K,\Phi}] = [\omega_K] + t^* \otimes a$ for some $a \in H^1(M; \mathbb{R})$, where $t^* \in H^1(S^1; \mathbb{Z}) \subset H^1(S^1; \mathbb{R})$ is the generator. We have

$$\xi^*[\omega_{K,\Phi}] = \kappa_{\hat{\phi}}([\omega_K] + t^* \otimes a) = \kappa_{\hat{\phi}}[\omega_K] + \pi^* t^* \cup \kappa_{\hat{\phi}} a,$$

hence

$$(\xi^*[\omega_{K,\Phi}])^{k-j} = (\kappa_{\hat{\phi}}[\omega_K])^{k-j} + (k-j)(\kappa_{\hat{\phi}}[\omega_K])^{k-j-1} \cup \pi^* t^* \cup \kappa_{\hat{\phi}} a.$$

By Corollary 7.4, we have

$$\begin{aligned}
\xi^* Q_\Phi &= \xi^* \left(\sum_{j=0}^k q_j(T^M M_\Phi, \omega_{K,\Phi}) \cup [\omega_{K,\Phi}]^{k-j} \right) \\
&= \sum_{j=0}^k q_j(T^M M_{\hat{\phi}}, \sigma_{\hat{\phi}} \oplus \pi^* \beta_\phi) \cup (\xi^*[\omega_{K,\Phi}])^{k-j} \\
&= \sum_{j=0}^k q_j(T^M M_{\hat{\phi}}, \sigma_{\hat{\phi}} \oplus \pi^* \beta_\phi) \cup (\kappa_{\hat{\phi}}[\omega_K])^{k-j} \\
&\quad + \pi^* t^* \cup \sum_{j=0}^k (k-j) q_j(T^M M_{\hat{\phi}}, \sigma_{\hat{\phi}} \oplus \pi^* \beta_\phi) \cup (\kappa_{\hat{\phi}}[\omega_K])^{k-j-1} \cup \kappa_{\hat{\phi}} a.
\end{aligned}$$

In general, the equality $\pi^* t^* \cup x = \kappa_{\hat{\phi}}(t^* \otimes \iota^* x)$ holds for any $x \in H^j(M_{\hat{\phi}}; \mathbb{R})$, where $\iota: M = M \times 0 \hookrightarrow M_{\hat{\phi}}$ is the inclusion.

Using this fact, we have

$$\begin{aligned}
&\pi^* t^* \cup \sum_{j=0}^k (k-j) q_j(T^M M_{\hat{\phi}}, \sigma_{\hat{\phi}} \oplus \pi^* \beta_\phi) \cup (\kappa_{\hat{\phi}}[\omega_K])^{k-j-1} \cup \kappa_{\hat{\phi}} a \\
&= \kappa_{\hat{\phi}} \left(t^* \otimes \iota^* \left(\sum_{j=0}^k (k-j) q_j(T^M M_{\hat{\phi}}, \sigma_{\hat{\phi}} \oplus \pi^* \beta_\phi) \cup (\kappa_{\hat{\phi}}[\omega_K])^{k-j-1} \cup \kappa_{\hat{\phi}} a \right) \right) \\
&= \kappa_{\hat{\phi}} \left(t^* \otimes \left(\sum_{j=0}^k (k-j) q_j(TM, \omega_K) \cup [\omega_K]^{k-j-1} \cup a \right) \right).
\end{aligned}$$

Here we note $\iota^* q_j(T^M M_{\hat{\phi}}, \sigma_{\hat{\phi}} \oplus \pi^* \beta_\phi) = q_j(TM, \omega_1) = q_j(TM, \omega_K)$ for all $K > 0$ since we can take the same almost complex structure compatible to ω_K for all $K > 0$. Recall our notation:

$$Q((T^M M_{\hat{\phi}}, \sigma_{\hat{\phi}} \oplus \pi^* \beta_\phi), \kappa_{\hat{\phi}}[\omega_K]) = \sum_{j=0}^k q_j(T^M M_{\hat{\phi}}, \sigma_{\hat{\phi}} \oplus \pi^* \beta_\phi) \cup (\kappa_{\hat{\phi}}[\omega_K])^{k-j}.$$

Putting these equalities together, we have

$$\begin{aligned}
\kappa_{\hat{\phi}}(t^* \otimes (-F_Q(\Phi))) &= \xi^* Q_\Phi \\
&= Q((T^M M_{\hat{\phi}}, \sigma_{\hat{\phi}} \oplus \pi^* \beta_\phi), \kappa_{\hat{\phi}}[\omega_K]) \\
&\quad + \kappa_{\hat{\phi}} \left(t^* \otimes \left(\sum_{j=0}^k (k-j) q_j(TM, \omega_K) \cup [\omega_K]^{k-j-1} \cup a \right) \right),
\end{aligned}$$

which implies the following proposition.

Proposition 7.5. $\kappa_{\hat{\phi}}(t^* \otimes (-F_Q(\Phi))) \equiv Q((T^M M_{\hat{\phi}}, \sigma_{\hat{\phi}} \oplus \pi^* \beta_\phi), \kappa_{\hat{\phi}}[\omega_K])$ in $H^1(S^1; \mathbb{R}) \otimes H^{2k-1}(M; \mathbb{R})$ modulo $\kappa_{\hat{\phi}}(t^* \otimes D_Q(M, \omega_K))$.

8. THE FIRST CHERN CLASS OF A LINE BUNDLE OVER A TORUS

In this section we shall consider the first Chern class of a complex $U(1)$ -line bundle over a torus in preparation for our example.

Let L be a complex $U(1)$ -line bundle over a torus T^b of dimension b . Let $h^U: T^b \rightarrow U(1)$ be a smooth map; then it defines the $U(1)$ -automorphism

$$h: L \supset L_x \ni u \mapsto h^U(x)u \in L_x \subset L$$

of L over the identity. Let L_h be the mapping torus of h ; then it is a complex $U(1)$ -bundle over $S^1 \times T^b = T^{b+1}$. Let $t^* \in H^1(S^1; \mathbb{Z})$ be the generator. Let

$$\kappa: H^*(S^1; \mathbb{R}) \otimes H^*(T^b; \mathbb{R}) \rightarrow H^*(S^1 \times T^b; \mathbb{R})$$

be the isomorphism in Künneth's theorem, which can also be considered as the isomorphism (6.1) with $\varphi = id$ and $s = p^*: H^*(T^b; \mathbb{R}) \rightarrow H^*(S^1 \times T^b; \mathbb{R})$, where $p: S^1 \times T^b \rightarrow T^b$ is the projection.

For simplicity, we consider elements of $H^*(S^1; \mathbb{R})$ and $H^*(T^b; \mathbb{R})$ as sitting in $H^*(S^1 \times T^b; \mathbb{R})$ in the obvious way. In the case of $b = 1$, we have

$$c_1(L_h) = \kappa(t^* \otimes (h^U)^*u) = t^* \cup (h^U)^*u,$$

where $u \in H^1(U(1); \mathbb{Z})$ is the generator. Using this fact, it is easy to see the following lemma for any $b \geq 1$.

Lemma 8.1.

$$c_1(L_h) = \kappa(t^* \otimes (h^U)^*u + 1 \otimes c_1(L)) = t^* \cup (h^U)^*u + c_1(L) \in H^2(S^1 \times T^b; \mathbb{R}).$$

Next we consider $U(1)$ -line bundles $L_i \rightarrow T^b$ ($1 \leq i \leq n$) and their direct sum $V = L_1 \oplus \cdots \oplus L_n$ over T^b . Let $h_i: L_i \rightarrow L_i$ be the $U(1)$ -automorphism defined as above by a smooth map $h_i^U: T^b \rightarrow U(1)$ for $1 \leq i \leq n$. Then we have the $U(n)$ -bundle automorphism $h = h_1 \oplus \cdots \oplus h_n$ of $V = L_1 \oplus \cdots \oplus L_n$. The mapping torus V_h of h is identified with the direct sum $V_h = (L_1)_{h_1} \oplus \cdots \oplus (L_n)_{h_n}$ of the mapping tori $(L_i)_{h_i}$.

Let $\pi: P(V) \rightarrow T^b$ and $\pi_h: P(V_h) \rightarrow S^1 \times T^b$ be the projective bundles associated with V and V_h , respectively. Let $\mathcal{L} \rightarrow P(V)$ and $\tilde{\mathcal{L}}_h \rightarrow P(V_h)$ be the Hopf line bundles. The automorphism h on V induces a fiber map $\hat{h}$ on $P(V)$ and a $U(1)$ -automorphism $\tilde{h}$ on $\mathcal{L}$ over $\hat{h}$. The mapping torus $\mathcal{L}_{\tilde{h}}$ of $\tilde{h}$ is a line bundle over the mapping torus $P(V)_{\hat{h}}$ of $\hat{h}$.

$$\begin{array}{ccccc} \mathcal{L} & \xrightarrow{\tilde{h}} & \mathcal{L} & & \tilde{\mathcal{L}}_h & & \mathcal{L}_{\tilde{h}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P(V) & \xrightarrow{\hat{h}} & P(V) & & P(V_h) & & P(V)_{\hat{h}} \\ \downarrow & & \downarrow & & \downarrow \pi_h & & \downarrow \pi_{\tilde{h}} \\ T^b & \xrightarrow{id} & T^b & & S^1 \times T^b & & S^1 \times T^b. \end{array}$$

Clearly the following lemma is true.

Lemma 8.2. $P(V)_{\tilde{h}} \cong P(V_h)$ as $\mathbb{C}P^{n-1}$ -bundles and $\mathcal{L}_{\tilde{h}} \cong \tilde{\mathcal{L}}_h$ as line bundles.

9. AN EXAMPLE

In this section we shall give the nontriviality of our crossed homomorphisms on the symplectic mapping class group, which is the proof of Theorem E.

Let $B = T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ be the standard $2n$ -dimensional torus with the standard coordinates $(x_i, y_i)_{i=1}^n$ and the standard symplectic structure $\beta = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$ for $n \geq 3$. Its cohomology ring is given by $H^*(T^{2n}; \mathbb{Z}) = \bigwedge (dx_i, dy_i)$ with abuse of notation.

Let $L_i \rightarrow T^{2n}$ be a complex $U(1)$ -line bundle over T^{2n} with first Chern class $\gamma_i := c_1(L_i) = dx_i \wedge dy_i \in H^2(T^{2n}; \mathbb{Z})$ for $1 \leq i \leq n$. Then the direct sum $V = L_1 \oplus \cdots \oplus L_n \rightarrow T^{2n}$ is a complex vector bundle with total Chern class

$$c(V) = \prod_{i=1}^n c(L_i) = \prod_{i=1}^n (1 + \gamma_i) = e^\beta,$$

where β also denotes the cohomology class of β . Hereafter we use the same symbol for a cohomology class and its representative differential form and omit the symbol $\cup$ of the cup product in cohomological computation. Let $\pi: M = P(V) \rightarrow T^{2n}$ be the projective bundle associated with V ; then it is a symplectic fibration with fiber $\mathbb{C}P^{n-1}$ with the Fubini-Study form. Let $\mathcal{L} \rightarrow M$ be the Hopf line bundle associated with the projective bundle M . Put

$$\eta = c_1(\mathcal{L}^*) = -c_1(\mathcal{L}) \in H^2(M; \mathbb{Z});$$

then the cohomology group of M is the $H^*(T^{2n}; \mathbb{R})$ -module

$$H^*(M; \mathbb{R}) = H^*(T^{2n}; \mathbb{R})[\eta] / \sim$$

with relation

$$\eta^n + c_1(V)\eta^{n-1} + \cdots + c_n(V) = 0.$$

The tangent bundle $T^{\mathbb{C}P}M$ along the fibers is obtained by the exact sequence

$$0 \rightarrow \underline{\mathbb{C}} \rightarrow \mathcal{L}^* \otimes \pi^*V \rightarrow T^{\mathbb{C}P}M \rightarrow 0$$

of vector bundles, which implies

$$c_1(T^{\mathbb{C}P}M) = n\eta + \pi^*\beta;$$

hence

$$\iota_b^*\eta = \frac{1}{n}\iota_b^*c_1(T^{\mathbb{C}P}M)$$

for any $b \in B$. Therefore $\eta \in H^2(M; \mathbb{R})$ can be taken as the class a in Thurston's theorem (Theorem 7.1) for the symplectic fibration M (if necessary the symplectic forms on fibers are multiplied by a common constant). By the proof of Thurston's theorem, there exists a closed 2-form τ on M satisfying $[\tau] = a = \eta$ and $\iota_b^*\tau = \sigma_b$ for all $b \in B$. Moreover for any sufficiently large $K > 0$, $\omega_K = \tau + K\pi^*\beta$ is a symplectic form on M . Because of $c(T^{2n}, \beta) = 1$, the total Chern class of (M, ω_K)

is computed as

$$\begin{aligned} c(M, \omega_K) &= c(T^{\mathbb{C}P} M) = c\left(\bigoplus_{i=1}^n \mathcal{L} \otimes \pi^* L_i\right) \\ &= \prod_{i=1}^n (1 + \eta + \gamma_i) \\ &= (1 + \eta)^n + (1 + \eta)^{n-1} \sigma_1 + (1 + \eta)^{n-2} \sigma_2 + \cdots + \sigma_n, \end{aligned}$$

where σ_i is the i -th fundamental symmetric polynomial of $\gamma_1, \dots, \gamma_n$ for $1 \leq i \leq n$.

Put $s_j = \sum_{i=1}^n \gamma_i^j$; then we have

$$s_1 = \sigma_1 = \beta, \quad s_2 = s_3 = \cdots = 0$$

because of $\gamma_i^2 = 0$ for all i . By using Newton's formulas which are given by

$$\begin{cases} s_j - \sigma_1 s_{j-1} + \sigma_2 s_{j-2} - \cdots + (-1)^{j-1} \sigma_{j-1} s_1 + (-1)^j j \sigma_j = 0 & (j = 1, \dots, n) \\ s_j - \sigma_1 s_{j-1} + \cdots + (-1)^j \sigma_n s_{j-n} = 0 & (j = n+1, n+2, \dots), \end{cases}$$

we obtain

$$\sigma_{j-1} s_1 - j \sigma_j = 0 \quad (j = 1, \dots, n),$$

hence

$$\sigma_j = \frac{1}{j!} \sigma_1^j = \frac{1}{j!} \beta^j \quad (j = 1, \dots, n).$$

Since we have $\beta^j = 0$ for all $j \geq n+1$, we obtain

$$(9.1) \quad c(M, \omega_K) = \sum_{j=0}^n (1 + \eta)^{n-j} \frac{1}{j!} \beta^j = (1 + \eta)^n \exp \frac{\beta}{1 + \eta}.$$

Since $\pi^*: H^*(T^{2n}; \mathbb{R}) \rightarrow H^*(M; \mathbb{R})$ is injective, hereafter we shall omit π^* and write β instead of $\pi^* \beta$, etc., so we have

$$\omega_K = \eta + K\beta$$

and

$$c_1 := c_1(M, \omega_K) = n\eta + \sum_{i=1}^n \gamma_i = n\eta + \beta,$$

hence

$$\eta = \frac{\omega_K - Kc_1}{1 - nK} \quad \text{and} \quad \beta = \frac{-n\omega_K + c_1}{1 - nK}$$

in $H^*(M; \mathbb{R})$. We can eliminate η and β from the formula (9.1):

$$c(M, \omega_K) = \left(1 + \frac{\omega_K - Kc_1}{1 - nK}\right)^n \exp \frac{c_1 - n\omega_K}{1 - nK + \omega_K - Kc_1}.$$

Put

$$\begin{aligned} Q_K &:= \sum_{k \geq 0} Q_{K,k} := \sum_{k \geq 0} \bar{c}_k - (1 + \bar{\eta})^n \exp \frac{\bar{\beta}}{1 + \bar{\eta}} \\ &= \sum_{k \geq 0} \bar{c}_k - \left(1 + \frac{w - K\bar{c}_1}{1 - nK}\right)^n \exp \frac{\bar{c}_1 - nw}{1 - nK + w - K\bar{c}_1} \end{aligned}$$

in $\sum_{k \geq 0} I_{2n-1}[w]^k$, where $\bar{c}_i$ ($i \geq 0$) is the corresponding multilinear form to the i -th

Chern class, $\bar{\eta} = \frac{w - K\bar{c}_1}{1 - nK}$, $\bar{\beta} = \frac{-nw + \bar{c}_1}{1 - nK}$ and $Q_{K,k} \in I_{2n-1}[w]^k$ for all $k \geq 0$. Then we have $Q_{K,0} = Q_{K,1} = 0$ and

$$\begin{aligned} Q'_K &:= \sum_{k \geq 0} Q'_{K,k} := \frac{\bar{\beta}}{1 - nK} (1 + \bar{\eta})^{n-2} \exp \frac{\bar{\beta}}{1 + \bar{\eta}} \\ &= \frac{\bar{\beta}}{1 - nK} \left[1 + \{(n-2)\bar{\eta} + \bar{\beta}\} + \left\{ \frac{(n-2)(n-3)}{2!} \bar{\eta}^2 + (n-3)\bar{\eta}\bar{\beta} + \frac{1}{2!} \bar{\beta}^2 \right\} + \cdots \right]. \end{aligned}$$

Consequently, for any sufficiently large $K > 0$, we have the symplectic manifold (M, ω_K) with the relations $Q_{K,k}(M, \omega_K) = 0$ ($k = 2, \dots, 2n-1$) and the crossed homomorphisms

$$\mathcal{F}_{Q_{K,k}}: \mathcal{S}(M, \omega_K) \rightarrow H^{2k-1}(M; \mathbb{R})/D_{Q_{K,k}}(M, \omega_K)$$

by Theorem A.

Next we show that these crossed homomorphisms are nontrivial for large $K > 0$. As in section 8, let $h: V = L_1 \oplus \cdots \oplus L_n \rightarrow V$ be the $U(n)$ -bundle isomorphism over the identity of T^{2n} corresponding to a smooth map

$$h^U = \begin{pmatrix} h_1^U & & O \\ & \ddots & \\ O & & h_n^U \end{pmatrix}: T^{2n} \rightarrow U(n)$$

with $h_i^U: T^{2n} \rightarrow U(1)$ ($1 \leq i \leq n$). The induced fiber map on $M = P(V)$ is denoted by

$$\hat{h}: M \rightarrow M,$$

whose restriction to each fiber is a symplectomorphism.

Lemma 9.1. $\hat{h}^* = id$ on $H^*(M; \mathbb{R})$.

Proof. The cohomology group of M is given by $H^*(M; \mathbb{R}) = H^*(T^{2n}; \mathbb{R})[\eta]/\sim$. Since the bundle isomorphism h on V naturally induces a $U(1)$ -bundle isomorphism $\tilde{h}: \mathcal{L} \rightarrow \mathcal{L}$ over $\hat{h}$, we have an isomorphism $\tilde{h}^*\mathcal{L} \cong \mathcal{L}$ over the identity on M , which implies $\hat{h}^*\eta = \eta$. Since $\hat{h}$ is a fiber preserving map over the identity of T^{2n} , we have $\hat{h}^* = id$. $\square$

By Proposition 7.2, the map $\hat{h}$ is diffeotopic to a symplectomorphism Φ on (M, ω_K) :

$$\hat{h} \simeq \Phi: (M, \omega_K) \rightarrow (M, \omega_K)$$

for any sufficiently large $K > 0$.

Next we compute $\mathcal{F}_{Q_{K,k}}(\Phi)$ using Proposition 7.5. Our setting is $\hat{\phi} = \hat{h}$, $\phi = id_{T^{2n}}$ in Proposition 7.5. Put $y_i = h_i^{U*}u$ and $Y = \sum_{i=1}^n y_i$ in $H^1(T^{2n}; \mathbb{R})$. Let $\tilde{\eta} = c_1(\mathcal{L}_{\hat{h}}^*) \in H^2(M_{\hat{h}}; \mathbb{R})$ be the first Chern class of the dual line bundle to $\mathcal{L}_{\hat{h}}$. We consider the $\mathbb{R}$ -isomorphism (6.1) $\kappa_{\hat{h}}: H^*(S^1; \mathbb{R}) \otimes H^*(M; \mathbb{R}) \xrightarrow{\cong} H^*(M_{\hat{h}}; \mathbb{R})$ with the homomorphism

$$\begin{aligned} s: H^*(M; \mathbb{R}) &= H^*(T^{2n}; \mathbb{R})[\eta]/\sim \ni a \cdot \eta^n \\ &\mapsto (p^*a) \cdot \tilde{\eta}^n \in H^*(M_{\hat{h}}; \mathbb{R}) = H^*(S^1 \times T^{2n}; \mathbb{R})[\tilde{\eta}]/\sim, \end{aligned}$$

where $p: S^1 \times T^{2n} \rightarrow T^{2n}$ is the projection. We have the following commutative diagram:

$$\begin{array}{ccc} H^*(S^1\mathbb{R}) \otimes H^*(M; \mathbb{R}) & \xrightarrow{\kappa_{\hat{h}}} & H^*(M_{\hat{h}}; \mathbb{R}) = H^*(S^1 \times T^{2n}; \mathbb{R})[\tilde{\eta}]/\sim \\ \uparrow 1 \otimes (\pi|_M)^* & & \uparrow \pi^* \\ H^*(S^1; \mathbb{R}) \otimes H^*(T^{2n}; \mathbb{R}) & \xrightarrow{\kappa} & H^*(S^1 \times T^{2n}; \mathbb{R}). \end{array}$$

Similarly as before, elements of $H^*(S^1; \mathbb{R})$ and $H^*(T^{2n}; \mathbb{R})$ are considered as sitting in $H^*(M_{\hat{h}}; \mathbb{R})$ in the obvious way, so we have

$$H^*(S^1; \mathbb{R}) \otimes H^*(T^{2n}; \mathbb{R}) \ni t^* \otimes a \mapsto t^*a (= t^* \cup a) \in H^*(M_{\hat{h}}; \mathbb{R})$$

in the diagram above.

Put $\hat{\omega}_K := \kappa_{\hat{h}}(\omega_K) \in H^2(M_{\hat{h}}; \mathbb{R})$; then we have

$$\hat{\omega}_K = \tilde{\eta} + K\beta$$

because of $\omega_K = \eta + K\beta$ in $H^2(M; \mathbb{R})$.

Next we compute

$$\begin{aligned} & Q_K((T^M M_{\hat{h}}, \sigma_{\hat{h}} \oplus \pi^* \beta_{id}), \hat{\omega}_K) \\ &= c(T^M M_{\hat{h}}, \sigma_{\hat{h}} \oplus \pi^* \beta_{id}) - \left(1 + \frac{\hat{\omega}_K - K\hat{c}_1}{1 - nK}\right)^n \exp \frac{\hat{c}_1 - n\hat{\omega}_K}{1 - nK + \hat{\omega}_K - K\hat{c}_1}, \end{aligned}$$

where $\hat{c}_1 = c_1(T^M M_{\hat{h}}, \sigma_{\hat{h}} \oplus \pi^* \beta_{id})$. Using Lemma 8.1 and an isomorphism $T^M M_{\hat{h}} \cong T^{\mathbb{C}P} M_{\hat{h}} \oplus \pi^* T^B B_{id}$, where $T^B B_{id} = (T^{2n}) \times S^1$ is considered as a trivial $U(n)$ -bundle, we have

$$\begin{aligned} c(T^M M_{\hat{h}}, \sigma_{\hat{h}} \oplus \pi^* \beta_{id}) &= c(T^{\mathbb{C}P} M_{\hat{h}} \oplus \pi^* T^B B_{id}) = c(T^{\mathbb{C}P} M_{\hat{h}}) \\ &= c(\mathcal{L}_{\hat{h}} \otimes \pi_h^* V_h) = c\left(\bigoplus_{i=1}^n \mathcal{L}_{\hat{h}} \otimes \pi_h^*(L_i)_{h_i}\right) \\ &= \prod_{i=1}^n \{1 + \tilde{\eta} + t^* \cup h_i^{U*} u + c_1(L_i)\} \\ &= \prod_{i=1}^n (1 + \tilde{\eta} + t^* y_i + \gamma_i) \end{aligned}$$

in $H^*(M_{\hat{h}}; \mathbb{R}) = H^*(S^1 \times T^{2n}; \mathbb{R})[\tilde{\eta}]/\sim$. In particular, we have

$$\hat{c}_1 = n\tilde{\eta} + t^* \cup Y + \beta.$$

Put

$$\zeta_i = t^* y_i + \gamma_i;$$

then we have

$$\zeta_i^2 = (t^* y_i + \gamma_i)^2 = 2t^* y_i \gamma_i$$

and

$$\zeta_i^3 = \dots = \zeta_i^j = \dots = 0 \quad (j \geq 3).$$

Let $\bar{\sigma}_i = \bar{\sigma}_i(\zeta_1, \dots, \zeta_n)$ be the i -th fundamental symmetric polynomial of ζ_j 's for $1 \leq i \leq n$. Put $\bar{s}_\nu = \sum_{j=1}^n \zeta_j^\nu$; then Newton's formulas imply

$$\bar{\sigma}_{\nu-2} \bar{s}_2 - \bar{\sigma}_{\nu-1} \bar{s}_1 + \nu \bar{\sigma}_\nu = 0$$

for $\nu = 1, \dots, n$, where $\bar{\sigma}_{-1} = 0$ and $\bar{\sigma}_0 = 1$. This equality also holds for $\nu = n+1, n+2, \dots$ in $H^*(T^{2n}, \mathbb{R})$.

Put

$$f(t) = \sum_{\nu=0}^{\infty} \bar{\sigma}_{\nu} t^{\nu} = \bar{\sigma}_0 + \bar{\sigma}_1 t + \dots + \bar{\sigma}_{\nu} t^{\nu} + \dots, \quad f(0) = \bar{\sigma}_0 = 1;$$

then we have

$$f'(t) - \bar{s}_1 f(t) + \bar{s}_2 t f(t) = \sum_{\nu=1}^{\infty} (\bar{\sigma}_{\nu} \nu - \bar{s}_1 \bar{\sigma}_{\nu-1} + \bar{s}_2 \bar{\sigma}_{\nu-2}) t^{\nu-1} = 0.$$

This differential equation with $f(0) = 1$ is solved as

$$f(t) = \exp \left(\bar{s}_1 t - \bar{s}_2 \frac{t^2}{2} \right).$$

Since we have

$$\begin{aligned} \bar{s}_1 &= \sum_{i=1}^n \zeta_i = \sum_{i=1}^n (t^* y_i + \gamma_i) = t^* Y + \beta, \\ \bar{s}_2 &= \sum_{i=1}^n \zeta_i^2 = \sum_{i=1}^n 2t^* y_i \gamma_i = 2t^* Z, \end{aligned}$$

where

$$Z = \sum_{i=1}^n y_i \gamma_i,$$

we obtain

$$\begin{aligned} \prod_{i=1}^n (1 + \tilde{\eta} + t^* y_i + \gamma_i) &= \prod_{i=1}^n (1 + \tilde{\eta} + \zeta_i) \\ &= (1 + \tilde{\eta})^n + (1 + \tilde{\eta})^{n-1} \bar{\sigma}_1 + \dots + (1 + \tilde{\eta}) \bar{\sigma}_{n-1} + \bar{\sigma}_n \\ &= (1 + \tilde{\eta})^n f \left(\frac{1}{1 + \tilde{\eta}} \right) \\ &= (1 + \tilde{\eta})^n \exp \left(\frac{t^* Y + \beta}{1 + \tilde{\eta}} - \frac{t^* Z}{(1 + \tilde{\eta})^2} \right) \\ &= (1 + \tilde{\eta})^{n-2} \exp \frac{\beta}{1 + \tilde{\eta}} \left[(1 + \tilde{\eta})^2 + t^* \{ (1 + \tilde{\eta}) Y - Z \} \right]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\left(1 + \frac{\hat{\omega}_K - K \hat{c}_1}{1 - nK} \right)^n \exp \frac{\hat{c}_1 - n \hat{\omega}_K}{1 - nK + \hat{\omega}_K - K \hat{c}_1} \\ &= \left(1 + \tilde{\eta} - \frac{K}{1 - nK} t^* Y \right)^n \exp \frac{t^* Y + (1 - nK) \beta}{1 - nK + (1 - nK) \tilde{\eta} - K t^* Y} \\ &= (1 + \tilde{\eta})^{n-2} \exp \frac{\beta}{1 + \tilde{\eta}} \cdot \left((1 + \tilde{\eta})^2 + \frac{K \beta t^* Y}{1 - nK} + (1 + \tilde{\eta}) t^* Y \right), \end{aligned}$$

where we particularly used $(t^*)^2 = 0$. Summing up the above computation, we have

$$\begin{aligned} & Q_K((T^M M_{\tilde{h}}, \sigma_{\tilde{h}} \oplus \pi^* \beta_{id}), \hat{\omega}_K) \\ &= (1 + \tilde{\eta})^{n-2} \exp \frac{\beta}{1 + \tilde{\eta}} [(1 + \tilde{\eta})^2 + t^* \{(1 + \tilde{\eta})Y - Z\}] \\ &\quad - (1 + \tilde{\eta})^{n-2} \exp \frac{\beta}{1 + \tilde{\eta}} \cdot \left((1 + \tilde{\eta})^2 + \frac{K\beta t^* Y}{1 - nK} + (1 + \tilde{\eta})t^* Y \right) \\ &= -t^*(1 + \tilde{\eta})^{n-2} \exp \frac{\beta}{1 + \tilde{\eta}} \cdot \left(Z + \frac{K}{1 - nK} \beta Y \right) \end{aligned}$$

in $H^*(M_{\tilde{h}}; \mathbb{R})$.

By Proposition 7.5 we obtain

$$\begin{aligned} - \sum_{k \geq 0} F_{Q_{K,k}}(\Phi) &\equiv \iota^* \left\{ -(1 + \tilde{\eta})^{n-2} \exp \frac{\beta}{1 + \tilde{\eta}} \cdot \left(Z + \frac{K}{1 - nK} \beta Y \right) \right\} \\ &\quad \text{mod } \bigoplus_{k \geq 0} D_{Q_{K,k}}(M, \omega_K) \\ &= -(1 + \eta)^{n-2} \exp \frac{\beta}{1 + \eta} \cdot \left(Z + \frac{K}{1 - nK} \beta Y \right). \end{aligned}$$

Put

$$(1 + \eta)^{n-2} \exp \frac{\beta}{1 + \eta} = a_0 + a_2 + a_4 + \cdots,$$

where $a_{2k} \in H^{2k}(M; \mathbb{R})$; then we have

$$\mathcal{F}_{Q_{K,k}}([\Phi]) = \left[a_{2k-4} \left(Z + \frac{K}{1 - nK} \beta Y \right) \right] = [a_{2k-4} Z]$$

in $H^{2k-1}(M; \mathbb{R})/D_{Q_{K,k}}(M, \omega_K) = H^{2k-1}(M; \mathbb{R})/a_{2k-4}\beta \cdot H^1(M; \mathbb{R})$. In the case of $k = 2$, it is given by

$$\begin{aligned} \mathcal{F}_{Q_{K,2}} : \mathcal{S}(M, \omega) &\rightarrow H^3(M; \mathbb{R})/\beta \cdot H^1(M; \mathbb{R}) \\ &\cong \{ H^3(T^{2n}; \mathbb{R})/\beta \cdot H^1(T^{2n}; \mathbb{R}) \} \oplus H^1(T^{2n}; \mathbb{R}) \cdot \eta \end{aligned}$$

with

$$(9.2) \quad \mathcal{F}_{Q_{K,2}}([\Phi]) = [Z] = \left[\sum_{i=1}^n y_i \gamma_i \right] = \left[\sum_{i=1}^n (h_i^U)^* u \gamma_i \right].$$

Moreover we have $\mathcal{F}_{Q_{K,k}} = a_{2k-4} \mathcal{F}_{Q_{K,2}}$ for $k \geq 2$, where a_{2k-4} is considered as a well-defined homomorphism from $H^3(M; \mathbb{R})/D_{Q_{K,2}}(M, \omega_K)$ to $H^{2k-1}(M; \mathbb{R})/D_{Q_{K,k}}(M, \omega_K)$. Since there is no condition for the maps h_i^U ($1 \leq i \leq n$) and the corresponding class $[\Phi]$ belongs to $\mathcal{S}_0(M, \omega)$ by Lemma 9.1, it is easy to see that the restriction of $\mathcal{F}_{Q_{K,2}}$ to $\mathcal{S}_0(M, \omega)$ is a nontrivial homomorphism by (9.2). Therefore we have $\iota^*[\mathcal{F}_{Q_{K,2}}] \neq 0$ in $H^1(\mathcal{S}_0(M, \omega_K), H^3(M; \mathbb{R})/D_{Q_{K,2}}(M, \omega_K))$, hence $[\mathcal{F}_{Q_{K,2}}] \neq 0$ in $H^1(\mathcal{S}(M, \omega_K), H^3(M; \mathbb{R})/D_{Q_{K,2}}(M, \omega_K))$. Consequently we have Theorem E.

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