LINEAR FUNCTIONS ON THE CLASSICAL MATRIX GROUPS

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ABSTRACT. Let M be a random matrix in the orthogonal group $\mathcal{O}_n$, distributed according to Haar measure, and let A be a fixed $n \times n$ matrix over $\mathbb{R}$ such that $\text{Tr}(AA^t) = n$. Then the total variation distance of the random variable $\text{Tr}(AM)$ to a standard normal random variable is bounded by $\frac{2\sqrt{3}}{n-1}$, and this rate is sharp up to the constant. Analogous results are obtained for M a random unitary matrix and A a fixed $n \times n$ matrix over $\mathbb{C}$. The proofs are applications of a new abstract normal approximation theorem which extends Stein's method of exchangeable pairs to situations in which continuous symmetries are present.

1. INTRODUCTION

Let $\mathcal{O}_n$ denote the group of $n \times n$ orthogonal matrices, and let M be distributed according to Haar measure on $\mathcal{O}_n$. Let A be a fixed $n \times n$ matrix over $\mathbb{R}$, subject to the condition that $\text{Tr}(AA^t) = n$, and let $W = \text{Tr}(AM)$. D'Aristotile, Diaconis, and Newman showed in [4] that

$$\sup_{\substack{A: \text{Tr}(AA^t)=n \\ -\infty < x < \infty}} |\mathbb{P}(W \leq x) - \Phi(x)| \rightarrow 0$$

as $n \rightarrow \infty$. Their argument uses classical methods involving sub-subsequences and tightness, and cannot be improved to yield a theorem for finite n . Theorem 4 below gives an explicit rate of convergence of the law of W to the standard normal distribution in the total variation metric on probability measures, specifically,

$$(1) \quad d_{TV}(\mathbb{L}_W, \mathfrak{N}(0, 1)) \leq \frac{2\sqrt{3}}{n-1}$$

for all $n \geq 2$.

The history of this problem begins with the following theorem, first given rigorous proof by Borel in [2]: let X be a random vector on the unit sphere S^{n-1} , and let X_1 be the first coordinate of X . Then $\mathbb{P}(\sqrt{n}X_1 \leq t) \rightarrow \Phi(t)$ as $n \rightarrow \infty$, where $\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-\frac{x^2}{2}} dx$. Since the first column of a Haar distributed orthogonal matrix is uniformly distributed on the unit sphere, Borel's theorem follows from Theorem 4 by taking $A = \sqrt{n} \oplus \mathbf{0}$, where $\mathbf{0}$ is the $n-1 \times n-1$ matrix with all zeros. Borel's theorem was generalized in one direction by Diaconis and Freedman [8], who proved the convergence of the first k coordinates of $\sqrt{n}X$ to independent standard

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normal random variables in total variation distance for $k = o(n)$; [8] also contains a detailed history of this problem. This line of research was further developed in [7], where a total variation bound was given between an $r \times r$ block of a random orthogonal matrix and an $r \times r$ matrix of independent standard Gaussians, for $r = O(n^{1/3})$. This was later improved by Jiang (see [12]) to $r = O(n^{1/2})$, which he proved was sharp. In the same paper, Jiang also showed that given a sequence of Haar distributed random matrices $\{M_n\}$, there is a sequence of Gaussian matrices $\{Y_n\}$ with Y_j defined on the same probability space as M_j such that if

$$\epsilon_n = \max_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m_n}} |\sqrt{n}M_{ij} - Y_{ij}|$$

with $m_n = O(\frac{n}{\log n})$, then $\epsilon_n \rightarrow 0$ in probability as $n \rightarrow \infty$. Thus an $n \times m_n$ block of a Haar distributed matrix can be approximated by a Gaussian matrix ‘in probability’. Theorem 4 gives another sense in which a random orthogonal matrix is close to a matrix of independent normals by giving a uniform bound of distance to normal over all linear combinations of entries of M .

Another special case of Theorem 4 is $A = I$, so that $W = \text{Tr}(M)$. Diaconis and Mallows (see [5]) first proved that $\text{Tr}(M)$ is approximately normal; Stein [15] and Johansson [13] later independently obtained fast rates of convergence to normal of $\text{Tr}(M^k)$ for fixed k , with Johansson’s rates an improvement on Stein’s. In studying eigenvalues of random orthogonal matrices, Diaconis and Shahshahani [9] extended this to show that the joint limiting distribution of $\text{Tr}(M), \text{Tr}(M^2), \dots, \text{Tr}(M^k)$ converges to that of independent normal variables as $n \rightarrow \infty$, for k fixed.

The other source of motivation for theorems like Theorem 4 is Hoeffding’s combinatorial central limit theorem [11], which can be stated as follows. Let $A = (a_{ij})$ be a fixed $n \times n$ matrix over $\mathbb{R}$, normalized to have row and column sums equal to zero and $\frac{1}{n-1} \sum_{i,j} a_{ij}^2 = 1$. Let π be a random permutation in S_n , and let $W(\pi) = \sum_i a_{i\pi(i)}$. Then, under certain conditions on A , W is approximately normal. Later, Bolthausen [1] proved an explicit rate of convergence via Stein’s method. Note that if

$$M_{ij} = \begin{cases} 1, & \pi(j) = i, \\ 0, & \text{otherwise,} \end{cases}$$

then $W = \text{Tr}(AM)$, and so Hoeffding’s theorem is really a theorem about the distribution of linear functions on the set of permutation matrices.

The unitary group is another source of many important applications; see, e.g. [6]. In Section 4, the random variable $\text{Tr}(AM)$ for A a fixed matrix over $\mathbb{C}$ and M a random unitary matrix distributed according to Haar measure on $\mathcal{U}_n$ is considered. The main theorem of the section, Theorem 6, gives a bound on the total variation distance of $\text{Re}[\text{Tr}(AM)]$ to standard normal analogous to that of Theorem 4; this can be viewed as a theorem about real-linear functions on $\mathcal{U}_n$. Corollary 7 shows that in the limit, the complex random variable $\text{Tr}(AM)$ is close to standard complex normal. The methods used here cannot be used directly to prove the convergence of $\text{Tr}(AM)$ to the standard complex normal; they work for approximation of real-valued random variables only. A version of the present methods in a multivariate context is forthcoming in [3], which includes a rate of convergence for Corollary 7.

Notation and conventions. The total variation distance $d_{TV}(\mu, \nu)$ between the measures μ and ν on $\mathbb{R}$ is defined by

$$d_{TV}(\mu, \nu) = \sup_A |\mu(A) - \nu(A)|,$$

where the supremum is over measurable sets A . This is equivalent to

$$d_{TV}(\mu, \nu) = \frac{1}{2} \sup_f \left| \int f(t) d\mu(t) - \int f(t) d\nu(t) \right|,$$

where the supremum is taken over continuous functions which are bounded by 1 and vanish at infinity; this is the definition used in what follows. The total variation distance between two random variables X and Y is defined to be the total variation distance between their distributions:

$$d_{TV}(X, Y) = \sup_A |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)| = \frac{1}{2} \sup_f |\mathbb{E}f(X) - \mathbb{E}f(Y)|.$$

We will use $\mathfrak{N}(\mu, \sigma^2)$ to denote the normal distribution on $\mathbb{R}$ with mean μ and variance σ^2 .

2. AN ABSTRACT NORMAL APPROXIMATION THEOREM

In this section, a general approach for normal approximation to random variables with continuous symmetries is developed. The ideas which give rise to Theorem 1 below first appeared in Stein [15], where fast rates of convergence to Gaussian (as $n \rightarrow \infty$) were obtained for $\text{Tr}(M^k)$, with $k \in \mathbb{N}$ fixed and M a random $n \times n$ orthogonal matrix.

Theorem 1. *Suppose that (W, W_ϵ) is a family of exchangeable pairs defined on a common probability space with $\mathbb{E}W = 0$ and $\mathbb{E}W^2 = \sigma^2$. Suppose that there are functions α and β with*

$$\mathbb{E}|\alpha(\sigma^{-1}W)| < \infty, \quad \mathbb{E}|\beta(\sigma^{-1}W)| < \infty,$$

a random variable E , and a constant λ such that

(i)

$$\frac{1}{\epsilon^2} \mathbb{E}[W_\epsilon - W | W] = -\lambda W + o(1)\alpha(W),$$

(ii)

$$\frac{1}{\epsilon^2} \mathbb{E}[(W_\epsilon - W)^2 | W] = 2\lambda\sigma^2 + E\sigma^2 + o(1)\beta(W),$$

(iii)

$$\frac{1}{\epsilon^2} \mathbb{E}|W_\epsilon - W|^3 = o(1),$$

where $o(1)$ refers to the limit as $\epsilon \rightarrow 0$, with the implied constants deterministic.

Then

$$d_{TV}(W, Z) \leq \frac{1}{\lambda} \mathbb{E}|E|,$$

where $Z \sim \mathfrak{N}(0, \sigma^2)$.

Remarks. (i) A straightforward computation using condition (i) shows that the random variable E of condition (ii) has $\mathbb{E}E = 0$.

- (ii) The factor of $\frac{1}{\epsilon^2}$ in each of the three expressions above could be replaced by a general function $f(\epsilon)$. In practice, W_ϵ is typically constructed such that $W_\epsilon - W = O(\epsilon)$. This makes it clear that $f(\epsilon) = \frac{1}{\epsilon^2}$ is the suitable choice for condition (ii). It is less clear that $f(\epsilon) = \frac{1}{\epsilon^2}$ is the suitable choice for condition (i). In the applications given here, while $W_\epsilon - W = O(\epsilon)$, symmetry conditions imply that

$$\mathbb{E}[W_\epsilon - W|W] = O(\epsilon^2).$$

Before beginning the proof, some background on Stein's method is helpful. The following lemma is key.

Lemma 2 (Stein). *Let $Z \sim \mathfrak{N}(0, 1)$. Then*

- (i) *For all $f \in C_o^1(\mathbb{R})$,*

$$\mathbb{E}[f'(Z) - Zf(Z)] = 0.$$

- (ii) *If Y is a random variable such that*

$$\mathbb{E}[f'(Y) - Yf(Y)] = 0$$

for all $f \in C_b^1(\mathbb{R})$, then $L(Y) = L(Z)$; i.e., Y is also distributed as a standard Gaussian random variable.

- (iii) *For $g : \mathbb{R} \rightarrow \mathbb{R}$ with $\mathbb{E}g(Z) < \infty$ given, the function*

$$(2) \quad U_o g(t) = e^{t^2/2} \int_{-\infty}^t [g(x) - \mathbb{E}g(Z)] e^{-x^2/2} dx$$

is a solution to the differential equation

$$f'(x) - xf(x) = g(x) - \mathbb{E}g(Z).$$

The lemma says that the standard Gaussian distribution γ on $\mathbb{R}$ is the unique distribution with the property that $\int_{\mathbb{R}} (f'(x) - xf(x)) d\gamma(x)$ is always zero. The idea of Stein's method is that if W is a random variable such that $\mathbb{E}[f'(W) - Wf(W)]$ is always *small*, then the distribution of W is close to γ . There are several approaches to bounding this quantity; the approach taken here is modelled on the method of exchangeable pairs (see [14]). In any of the approaches, the following bounds on U_o are useful.

Lemma 3 (Stein). *Let U_o be the operator defined in equation (2). Then*

- (i) $\|U_o g\|_\infty \leq \sqrt{\frac{\pi}{2}} \|g - \mathbb{E}g(Z)\|_\infty \leq \sqrt{2\pi} \|g\|_\infty.$
(ii) $\|(U_o g)'\|_\infty \leq 2 \|g - \mathbb{E}g(Z)\|_\infty \leq 4 \|g\|_\infty.$
(iii) $\|(U_o g)''\|_\infty \leq 2 \|g'\|_\infty.$

With this background, the proof of Theorem 1 is straightforward.

Proof of Theorem 1. By considering $\sigma^{-1}W$ instead of W , we may without loss assume that $\sigma = 1$. For $g \in C_o^\infty(\mathbb{R})$ fixed, let f be the solution given in equation (2) to the differential equation

$$f'(x) - xf(x) = g(x) - \mathbb{E}g(Z).$$

Fix ϵ . By the exchangeability of (W, W_ϵ) ,

$$\begin{aligned} 0 &= \mathbb{E}[(W_\epsilon - W)(f(W_\epsilon) + f(W))] \\ (3) \quad &= \mathbb{E}[(W_\epsilon - W)(f(W_\epsilon) - f(W)) + 2(W_\epsilon - W)f(W)] \\ &= \mathbb{E}[\mathbb{E}[(W_\epsilon - W)^2|W] f'(W) + 2\mathbb{E}[(W_\epsilon - W)|W] f(W) + R], \end{aligned}$$

where R is the error in the derivative approximation. By Taylor's theorem and Lemma 3,

$$|R| \leq \frac{\|f''\|_\infty}{2} |W_\epsilon - W|^3 \leq \|g'\|_\infty |W_\epsilon - W|^3,$$

and so by condition (iii),

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} \mathbb{E}|R| = 0.$$

Dividing both sides of (3) by $2\lambda\epsilon^2$ and taking the limit as $\epsilon \rightarrow 0$, conditions (i) and (ii) yield

$$0 = \mathbb{E} \left[f'(W) - Wf(W) + \frac{E}{2\lambda} f'(W) \right] = \mathbb{E} \left[g(W) - g(Z) + \frac{E}{2\lambda} f'(W) \right].$$

Rearranging and applying the bound on $\|f'\|$ from Lemma 3 yields

$$|\mathbb{E}g(W) - \mathbb{E}g(Z)| \leq \frac{2\|g\|_\infty}{\lambda} \mathbb{E}|E|.$$

Since $C_o^\infty(\mathbb{R})$ is dense (with respect to the supremum norm) in the class of bounded continuous functions vanishing at infinity, this completes the proof. $\square$

3. THE ORTHOGONAL GROUP

This section is mainly devoted to the proof of the following theorem.

Theorem 4. *Let A be a fixed $n \times n$ matrix over $\mathbb{R}$ such that $\text{Tr}(AA^t) = n$, $M \in \mathcal{O}_n$ distributed according to Haar measure, and $W = \text{Tr}(AM)$. Let Z be a standard normal random variable. Then for $n > 1$,*

$$(4) \quad d(W, Z)_{TV} \leq \frac{2\sqrt{3}}{n-1}.$$

The bound in Theorem 4 is sharp up to the constant; consider the matrix $A = \sqrt{n} \oplus \mathbf{0}$ where $\mathbf{0}$ is the $n-1 \times n-1$ matrix with all zeros. For this A , Theorem 4 reproves the following theorem, proved in [8] with slightly worse constant.

Theorem 5. *Let $\mathbf{x} \in \sqrt{n}S^{n-1}$ be uniformly distributed, and let Z be a standard normal random variable. Then*

$$d_{TV}(x_1, Z) \leq \frac{2\sqrt{3}}{n-1}.$$

It is shown in [8] that the order of this error term is correct.

Proof of Theorem 4. First note that one can assume without loss of generality that A is diagonal: let $A = UDV$ be the singular value decomposition of A . Then $W = \text{Tr}(UDVM) = \text{Tr}(DVMU)$, and the distribution of VMU is the same as the distribution of M by the translation invariance of Haar measure.

Now define the pair (W, W_ϵ) for each ϵ as follows. Choose $H = (h_{ij}) \in \mathcal{O}(n)$ according to Haar measure, independent of M , and let $M_\epsilon = HA_\epsilon H^t M$, where

$$A_\epsilon = \begin{bmatrix} \sqrt{1-\epsilon^2} & \epsilon & & & \\ -\epsilon & \sqrt{1-\epsilon^2} & & & \\ & & 1 & & \\ & & & \ddots & \\ & 0 & & & 1 \end{bmatrix},$$

thus M_ϵ can be thought of as a small random rotation of M . Let $W_\epsilon = W(M_\epsilon)$; (W, W_ϵ) is an exchangeable pair by construction.

It is convenient to rewrite M_ϵ as follows. Let I_2 be the 2×2 identity matrix, let K be the $n \times 2$ matrix consisting of the first two columns of H , and let

$$C_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} M_\epsilon &= M + K[(\sqrt{1 - \epsilon^2} - 1)I_2 + \epsilon C_2]K^t M \\ &= M + K\left[\left(-\frac{\epsilon^2}{2} + O(\epsilon^4)\right)I_2 + \epsilon C_2\right]K^t M, \end{aligned}$$

and so

$$(5) \quad W_\epsilon - W = \epsilon \left[\left(-\frac{\epsilon}{2} + O(\epsilon^3) \right) \text{Tr}(AKK^t M) + \text{Tr}(AKC_2K^t M) \right].$$

Now, the distribution of H is unchanged by multiplying a fixed row or column by -1 and H is orthogonal, thus $\mathbb{E}h_{ij}h_{k\ell} = \frac{1}{n}\delta_{ik}\delta_{j\ell}$. This implies that

$$\mathbb{E}[KK^t] = \frac{2}{n}I_n$$

and

$$\mathbb{E}[KC_2K^t] = 0;$$

combining this with (5) yields:

$$\begin{aligned} &\frac{n}{\epsilon^2} \mathbb{E}[(W_\epsilon - W)|W] \\ &= -\frac{n}{2} \mathbb{E}[\mathbb{E}[\text{Tr}(AKK^t M)|M] | W] + \frac{n}{\epsilon} \mathbb{E}[\mathbb{E}[\text{Tr}(AKC_2K^t M)|M] | W] + O(\epsilon) \\ &= -\mathbb{E}[\mathbb{E}[\text{Tr}(AM)|M] | W] + O(\epsilon) \\ &= -W + O(\epsilon), \end{aligned}$$

where the independence of M and H has been used to get the third line, and the implied constants in the $O(\epsilon)$ here and in what follows may depend on n . Condition (i) of Theorem 1 is thus satisfied with $\lambda = \frac{1}{n}$.

Recall now that A is assumed to be diagonal. The second condition of Theorem 1 can also be verified using the expression in (5) as follows:

$$\begin{aligned} (6) \quad &\frac{n}{2\epsilon^2} \mathbb{E}[(W_\epsilon - W)^2 | W] \\ &= \frac{n}{2} \mathbb{E}[\mathbb{E}[(\text{Tr}(AKC_2K^t M))^2 | M] | W] + O(\epsilon) \\ &= \frac{n}{2} \mathbb{E} \left[\sum_{i,j} \sum_{\substack{i' \neq i \\ j' \neq j}} m_{i'i} m_{j'j} a_{ii} a_{jj} \mathbb{E}[(h_{i1}h_{i'2} - h_{i2}h_{i'1})(h_{j1}h_{j'2} - h_{j2}h_{j'1}) | M] \right] \\ &\quad + O(\epsilon), \end{aligned}$$

where the conditions on i' and j' are justified as the expression inside the expectation is identically zero when either $i = i'$ or $j = j'$.

Standard techniques are available for computing the mixed moments of entries of H ; see e.g. [10], section 4.2. Using these techniques and the independence of M and H gives that for $i' \neq i$ and $j' \neq j$,

$$\begin{aligned}
 (7) \quad & \mathbb{E} [(h_{i1}h_{i'2} - h_{i2}h_{i'1})(h_{j1}h_{j'2} - h_{j2}h_{j'1})|M] \\
 &= \mathbb{E} [(h_{i1}h_{i'2} - h_{i2}h_{i'1})(h_{j1}h_{j'2} - h_{j2}h_{j'1})] \\
 &= \frac{2}{n(n-1)} [\delta_{ij}\delta_{i'j'} - \delta_{ij'}\delta_{ji'}];
 \end{aligned}$$

putting this into (6) yields

$$\begin{aligned}
 & \frac{n}{2\epsilon^2} \mathbb{E} [(W_\epsilon - W)^2|W] \\
 &= \frac{1}{n-1} \sum_{i,j} \sum_{\substack{i' \neq i \\ j' \neq j}} m_{i'i} m_{j'j} a_{ii} a_{jj} [\delta_{i'j'} \delta_{ij} - \delta_{ij'} \delta_{ji'}] + O(\epsilon) \\
 &= \frac{1}{n-1} \left[\sum_i a_{ii}^2 [(M^t M)_{ii} - m_{ii}^2] - \sum_{i,i' \neq i} (MA)_{ii'} (MA)_{i'i} \right] + O(\epsilon) \\
 &= \frac{1}{n-1} \left[n - \sum_i a_{ii}^2 m_{ii}^2 - \left[\text{Tr}((MA)^2) - \sum_i a_{ii}^2 m_{ii}^2 \right] \right] + O(\epsilon) \\
 &= 1 + \frac{1}{n-1} [1 - \text{Tr}((AM)^2)] + O(\epsilon),
 \end{aligned}$$

thus

$$(8) \quad \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} \mathbb{E} [(W_\epsilon - W)^2|W] = \frac{2}{n} + \frac{2}{n(n-1)} [1 - \text{Tr}((AM)^2)],$$

and so

$$(9) \quad E = \frac{2}{n(n-1)} [1 - \text{Tr}((AM)^2)].$$

Finally, (5) gives immediately that

$$\mathbb{E} [|W_\epsilon - W|^3|W] = O(\epsilon^3).$$

It remains to bound $n\mathbb{E}|E|$. Let $\sum'$ stand for summing over distinct indices.

$$\begin{aligned}
 (10) \quad \mathbb{E} [\text{Tr}((AM)^2)] &= \mathbb{E} \left[\sum_{i,j} a_{ii} a_{jj} m_{ij} m_{ji} \right] \\
 &= \frac{1}{n} \sum_i a_{ii}^2 = 1,
 \end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E}[(\operatorname{Tr}((AM)^2))^2] \\
&= \mathbb{E}\left[\left(\sum_{i,j} a_{ii}a_{jj}m_{ij}m_{ji}\right)\left(\sum_{k,l} a_{kk}a_{ll}m_{kl}m_{lk}\right)\right] \\
&= \sum_{i,j,k,l} a_{ii}a_{jj}a_{kk}a_{ll} \left[\frac{n+1}{(n-1)n(n+2)} [\delta_{ij}\delta_{kl}(1-\delta_{ik}) + \delta_{ik}\delta_{jl}(1-\delta_{ij}) \right. \\
&\quad \left. + \delta_{il}\delta_{jk}(1-\delta_{ij})] + \frac{3}{n(n+2)} \mathbb{I}(i=j=k=l) \right] \\
&= \frac{n+1}{n(n-1)(n+2)} \left(\sum'_{i,k} a_{ii}^2 a_{kk}^2 + \sum'_{i,j} a_{ii}^2 a_{jj}^2 + \sum'_{i,j} a_{ii}^2 a_{jj}^2 \right) \\
&\quad + \frac{3}{n(n+2)} \sum_i a_{ii}^4.
\end{aligned}$$

Now,

$$\sum'_{i,j} a_{ii}^2 a_{jj}^2 = \sum_i a_{ii}^2 (n - a_{ii}^2) = n^2 - \sum_i a_{ii}^4.$$

Applying the above gives

$$\begin{aligned}
(11) \quad \mathbb{E}[(\operatorname{Tr}((AM)^2))^2] &= \frac{3(n+1)n^2}{(n-1)n(n+2)} - \frac{3(n+1)}{(n-1)n(n+2)} \sum_i a_{ii}^4 + \frac{3}{n(n+2)} \sum_i a_{ii}^4 \\
&\leq 3 + \frac{6}{(n-1)(n+2)}.
\end{aligned}$$

Putting these estimates into Theorem 1 gives:

$$(12) \quad d_{TV}(W, Z) \leq \frac{2\sqrt{2 + \frac{6}{(n-1)(n+2)}}}{(n-1)}.$$

Noting that $\frac{6}{(n-1)(n+2)} \leq 1$ for $n \geq 3$ and that the bound in Theorem 4 is trivially true for $n = 2$ completes the proof. $\square$

4. THE UNITARY GROUP

Now let $M \in \mathcal{U}_n$ be distributed according to Haar measure, let A be an $n \times n$ matrix over $\mathbb{C}$, and let $W = \operatorname{Tr}(AM)$. In [4] it was shown that if $M = \Gamma + i\Lambda$ and A and B are fixed real diagonal matrices with $\operatorname{Tr}(AA^t) = \operatorname{Tr}(BB^t) = n$, then $\operatorname{Tr}(A\Gamma) + i\operatorname{Tr}(B\Lambda)$ converges in distribution to a standard complex normal random variable. This implies in particular that $\operatorname{Re}(W)$ converges in distribution to $\mathfrak{N}(0, \frac{1}{2})$. The main theorem of this section gives a rate of this convergence in total variation distance.

A more natural question might be the convergence of W to a standard complex random variable. As this is a multivariate problem, Theorem 1 cannot be applied. A multivariate version of Theorem 1 is forthcoming in [3], which also includes a rate of convergence of W to a standard complex Gaussian random variable.

Theorem 6. *With M , A , and W as above, let W_θ be the inner product of W with the unit vector making angle θ with the real axis. Then*

$$(13) \quad d_{TV} \left(W_\theta, \mathfrak{N} \left(0, \frac{1}{2} \right) \right) \leq \frac{c}{n}$$

for a constant c which is independent of θ .

The constant c is asymptotically equal to $2\sqrt{2}$; for $n \geq 8$ it can be taken to be 4.

Proof. To prove the theorem, first note that it suffices to consider the case $\theta = 0$, that is, to prove that

$$d_{TV} \left(\operatorname{Re}(W), \mathfrak{N} \left(0, \frac{1}{2} \right) \right) \leq \frac{c}{n}.$$

The theorem then follows as stated since the distribution of W is invariant under multiplication by any complex number of unit modulus. Also, A can again be assumed diagonal with positive real entries by the singular value decomposition.

The proof is almost identical to the orthogonal case. Let $H \in \mathcal{U}_n$ be a random unitary matrix, independent of M , and let $M_\epsilon = HA_\epsilon H^* M$, where A_ϵ is as in the orthogonal case.

Let I_2 be the 2×2 identity matrix, let K be the $n \times 2$ matrix consisting of the first two columns of H , and let C_2 be as before. Then

$$(14) \quad \begin{aligned} W_\epsilon - W &= \operatorname{Tr} \left(\left(-\frac{\epsilon^2}{2} + O(\epsilon^4) \right) AKK^* M + \epsilon AKC_2 K^* M \right) \\ &= \epsilon \left[\left(-\frac{\epsilon}{2} + O(\epsilon^3) \right) \operatorname{Tr} (AKK^* M) + \operatorname{Tr} (AKC_2 K^* M) \right]. \end{aligned}$$

Let $W^r = \operatorname{Re}(W)$ and $W_\epsilon^r = \operatorname{Re}(W_\epsilon)$. As in the orthogonal case, to verify the conditions of Theorem 1, various mixed moments of the entries of H are needed. The relevant unitary integrals can also be found in [10], section 4.2. They imply in particular that

$$(15) \quad \mathbb{E} [(KK^*)_{ij}] = \frac{2}{n} \delta_{ij},$$

$$(16) \quad \mathbb{E} [(KC_2 K^*)_{ij}] = 0,$$

thus

$$(17) \quad \lim_{\epsilon \rightarrow 0} \frac{n}{\epsilon^2} \mathbb{E} [W_\epsilon^r - W^r | W] = -W^r;$$

condition (i) is satisfied with $\lambda = \frac{1}{n}$. Also by (14),

$$(18) \quad \begin{aligned} &\lim_{\epsilon \rightarrow 0} \frac{n}{2\epsilon^2} \mathbb{E} [(W_\epsilon^r - W^r)^2 | W] \\ &= \lim_{\epsilon \rightarrow 0} \frac{n}{2} \mathbb{E} \left[(\operatorname{Re} (\operatorname{Tr} (AKC_2 K^* M)))^2 | W \right] \\ &= \frac{n}{4} \operatorname{Re} \mathbb{E} \left[\sum_{i,j,k,l} a_{ii} m_{ji} a_{kk} m_{lk} (h_{i1} \bar{h}_{j2} - h_{i2} \bar{h}_{j1}) (h_{k1} \bar{h}_{l2} - h_{k2} \bar{h}_{l1}) \right. \\ &\quad \left. + a_{ii} m_{ji} a_{kk} \bar{m}_{lk} (h_{i1} \bar{h}_{j2} - h_{i2} \bar{h}_{j1}) (\bar{h}_{k1} h_{l2} - \bar{h}_{k2} h_{l1}) \right] | W \Bigg]. \end{aligned}$$

Using the formulae from [10], it is straightforward to show that

$$(19) \quad \begin{aligned} & \mathbb{E}[(h_{i1}\bar{h}_{j2} - h_{i2}\bar{h}_{j1})(h_{k1}\bar{h}_{l2} - h_{k2}\bar{h}_{l1})] \\ &= -\frac{2\delta_{il}\delta_{jk}(1-\delta_{ij})}{(n-1)(n+1)} + \frac{2\delta_{ij}\delta_{kl}(1-\delta_{ik})}{(n-1)n(n+1)} - \frac{2\mathbb{I}(i=j=k=l)}{n(n+1)} \end{aligned}$$

and

$$(20) \quad \begin{aligned} & \mathbb{E}[(h_{i1}\bar{h}_{j2}h_{i2}\bar{h}_{j1})(\bar{h}_{k1}h_{l2} - \bar{h}_{k2}h_{l1})] \\ &= \frac{2(\delta_{ik}\delta_{jl}(1-\delta_{ij}))}{(n-1)(n+1)} - \frac{2(\delta_{ij}\delta_{kl}(1-\delta_{ik}))}{n(n-1)(n+1)} + \frac{2\mathbb{I}(i=j=k=l)}{n(n+1)}. \end{aligned}$$

Recall that $\sum'_{i,j}$ stands for summing over all pairs (i, j) where i and j are distinct.

Putting (19) and (20) into (18) and using the independence of M and H gives:

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \frac{n}{2\epsilon^2} \mathbb{E}[(W_\epsilon^r - W^r)^2 | W] \\ &= \frac{n}{2(n-1)(n+1)} \operatorname{Re} \mathbb{E} \left[\sum_{i,j,k,\ell} a_{ii}m_{ji}a_{kk}m_{\ell k} (-\delta_{i\ell}\delta_{jk}(1-\delta_{ij}) \right. \\ & \quad \left. + \frac{1}{n}\delta_{ij}\delta_{k\ell}(1-\delta_{ik}) - \left(\frac{n-1}{n}\right)\mathbb{I}(i=j=k=\ell)) \right. \\ & \quad \left. + \sum_{i,j,k,\ell} a_{ii}m_{ji}a_{kk}m_{\ell k} (\delta_{ik}\delta_{j\ell}(1-\delta_{ij}) \right. \\ & \quad \left. - \frac{1}{n}\delta_{ij}\delta_{k\ell}(1-\delta_{ik}) + \left(\frac{n-1}{n}\right)\mathbb{I}(i=j=k=l)) \right] W \Bigg] \\ &= \frac{n}{2(n-1)(n+1)} \operatorname{Re} \mathbb{E} \left[-\sum'_{i,j} a_{ii}a_{jj}m_{ij}m_{ji} + \frac{1}{n}\sum'_{i,k} a_{ii}a_{kk}m_{ii}m_{kk} \right. \\ & \quad \left. - \left(\frac{n-1}{n}\right)\sum_i a_{ii}^2 m_{ii}^2 + \sum'_{i,j} a_{ii}^2 |m_{ji}|^2 \right. \\ & \quad \left. - \frac{1}{n}\sum'_{i,k} a_{ii}a_{kk}m_{ii}\bar{m}_{kk} + \frac{n-1}{n}\sum_i a_{ii}^2 |m_{ii}|^2 \right] W \Bigg] \\ &= \frac{n}{2(n-1)(n+1)} \operatorname{Re} \mathbb{E} \left[-\left(\operatorname{Tr}((AM)^2) - \sum_i (AM)_{ii}^2\right) \right. \\ & \quad \left. + \frac{1}{n}\left(W^2 - \sum_i (AM)_{ii}^2\right) - \left(\frac{n-1}{n}\right)\sum_i (AM)_{ii}^2 \right. \\ & \quad \left. + \sum_i a_{ii}^2(1 - |m_{ii}|^2) \right] \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{n} \left(|W|^2 - \sum_i a_{ii}^2 |m_{ii}|^2 \right) \\
 & + \frac{n-1}{n} \sum_i a_{ii}^2 |m_{ii}|^2 \left| W \right| \\
 & = \frac{1}{2} + \frac{1}{2(n-1)(n+1)} \\
 & + \frac{n}{2(n-1)(n+1)} \operatorname{Re} \mathbb{E} \left[-\operatorname{Tr}((AM)^2) + \frac{W^2 - |W|^2}{n} \left| W \right| \right].
 \end{aligned}$$

Condition (2) of Theorem 1 is thus satisfied with

$$(21) \quad nE = \frac{1}{2(n-1)(n+1)} + \frac{n}{2(n-1)(n+1)} \operatorname{Re} \mathbb{E} \left[-\operatorname{Tr}((AM)^2) + \frac{W^2 - |W|^2}{n} \left| W \right| \right].$$

It remains to estimate $n\mathbb{E}|E|$. First,

$$\begin{aligned}
 \mathbb{E} |\operatorname{Tr}((AM)^2)| &= \mathbb{E} \sqrt{\sum_{i,j,k,l} a_{ii} a_{jj} m_{ij} m_{ji} a_{kk} a_{ll} \overline{m}_{kl} \overline{m}_{lk}} \\
 &\leq \sqrt{\sum_{i,j,k,l} a_{ii} a_{jj} a_{kk} a_{ll} \mathbb{E} [m_{ij} m_{ji} \overline{m}_{kl} \overline{m}_{lk}]} \\
 &= \sqrt{\frac{2n^2}{(n-1)(n+1)} - \frac{2}{(n-1)n(n+1)} \left(\sum_i a_{ii}^4 \right)} \\
 &\leq \sqrt{2 + \frac{1}{n^2 - 1}},
 \end{aligned}$$

using the formulae of [10] to evaluate the integrals.

Next,

$$\begin{aligned}
 \mathbb{E}|W|^2 &= \mathbb{E} \left[\sum_{i,j} a_{ii} a_{jj} m_{ii} \overline{m}_{jj} \right] \\
 &= \frac{1}{n} \sum_i a_{ii}^2 \\
 &= 1.
 \end{aligned}$$

Putting these estimates into (21) proves the theorem. $\square$

Theorem 6 yields the following bivariate corollary, which can also be seen as a corollary of the main unitary lemma of [4].

Corollary 7. *For each n , let M_n be a random $n \times n$ unitary matrix, A_n an $n \times n$ matrix over $\mathbb{C}$ with $\operatorname{Tr}(A_n A_n^*) = n$, and $W_n = \operatorname{Tr}(A_n M_n)$. Then the distribution of W_n converges to the standard complex normal distribution in the weak-star topology.*

Proof. The result follows immediately from Theorem 6 by considering the characteristic function of W_n . $\square$

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