STABILITY AND INSTABILITY OF LIAPUNOV-SCHMIDT AND HOPF BIFURCATION FOR A FREE BOUNDARY PROBLEM ARISING IN A TUMOR MODEL

AVNER FRIEDMAN AND BEI HU

ABSTRACT. We consider a free boundary problem for a system of partial differential equations, which arises in a model of tumor growth. For any positive number R there exists a radially symmetric stationary solution with free boundary $r = R$. The system depends on a positive parameter μ , and for a sequence of values $\mu_2 < \mu_3 < \cdots$ there also exist branches of symmetric-breaking stationary solutions, parameterized by ε , $|\varepsilon|$ small, which bifurcate from these values. In particular, for $\mu = \mu(\varepsilon)$ near μ_2 the free boundary has the form $r = R + \varepsilon Y_{2,0}(\theta) + O(\varepsilon^2)$ where $Y_{2,0}$ is the spherical harmonic of mode $(2, 0)$. It was recently proved by the authors that the stationary solution is asymptotically stable for any $0 < \mu < \mu_*$, but linearly unstable if $\mu > \mu_*$, where $\mu_* = \mu_2$ if $R > \bar{R}$ and $\mu_* < \mu_2$ if $R < \bar{R}$; $\bar{R} \approx 0.62207$. In this paper we prove that for $R > \bar{R}$ each of the stationary solutions which bifurcates from $\mu = \mu_2$ is linearly stable if $\varepsilon > 0$ and linearly unstable if $\varepsilon < 0$. We also prove, for $R < \bar{R}$, that the point $\mu = \mu_*$ is a Hopf bifurcation, in the sense that the linearized time-dependent problem has a family of solutions which are asymptotically periodic in t .

1. STATEMENT OF THE PROBLEM

Consider the following problem: Find 3-dimensional domains $\Omega(t)$ and functions σ, p such that:

$$\begin{aligned} (1.1) \quad & \sigma_t - \Delta \sigma + \sigma = 0 \quad \text{in } \Omega(t), \, t > 0, \\ (1.2) \quad & \Delta p = -\mu(\sigma - \tilde{\sigma}) \quad \text{in } \Omega(t), \, t > 0 \quad (0 < \tilde{\sigma} < 1), \\ (1.3) \quad & \sigma = 1 \quad \text{on } \partial\Omega(t), \, t > 0, \\ (1.4) \quad & p = \kappa \quad \text{on } \partial\Omega(t), \, t > 0 \end{aligned}$$

where κ is the mean curvature ($\kappa > 0$ if $\Omega(t)$ is a ball), and

$$(1.5) \quad V_n = -\frac{\partial p}{\partial n} \quad \text{on } \partial\Omega(t), \, t > 0,$$

Received by the editors February 24, 2005 and, in revised form, September 26, 2005 and September 6, 2006.

2000 *Mathematics Subject Classification*. Primary 35R35, 35K55; Secondary 35Q80, 35C20, 92C37.

Key words and phrases. Free boundary problems, stationary solution, stability, instability, bifurcation, symmetry-breaking, tumor cell.

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where $\partial/\partial n$ is the derivative along the outward normal $\vec{n}$ and V_n is the velocity of the free boundary $\partial\Omega(t)$ in the direction $\vec{n}$, and

$$(1.6) \quad \begin{aligned} \partial\Omega(0) : r &= R_0(\theta, \phi), \\ \sigma|_{t=0} &= \sigma_0(r, \theta, \phi) \end{aligned}$$

are given initial conditions.

The system represents a simple model of tumor growth where the tumor region $\Omega(t)$ contains just one type of cells with density c ; σ is the concentration of nutrients, and p is the pressure which results from cell proliferation. It is assumed that the cell density is linearly dependent on the concentration of nutrients, and, for simplicity, we take $c = \sigma$. It is also assumed in this model that the cell density grows according to the proliferation rate $\mu(\sigma - \tilde{\sigma})$ where $\tilde{\sigma}$ is a constant smaller than the concentration of nutrients on the boundary, that is, $\tilde{\sigma} < 1$.

Many mathematical models of tumor growth, which consider the tumor tissue as a density of proliferating cells, have been developed and studied in many papers; see [5, 6, 7, 8, 14, 15, 18, 19] and the references given there.

Assuming that the tumor region is a porous medium, so that Darcy's law $\vec{V} = -\nabla p$ holds, the law of conservation of mass $\text{div } \vec{V} = \mu(\sigma - \tilde{\sigma})$ yields the equation (1.2). Furthermore, the continuity condition $\vec{V} \cdot \vec{n} = V_n$ yields the relation (1.5). Finally, the cell-to-cell adhesiveness suggests the relation (1.4) (see [2], [3], [4]).

It is known [14] that, under the previous assumption $0 < \tilde{\sigma} < 1$, there exists a unique radially symmetric stationary solution $(\sigma_S(r), p_S(r), R_S)$, and it is given by

$$(1.7) \quad \begin{aligned} \sigma_S(r) &= \frac{R_S}{\sinh R_S} \frac{\sinh r}{r}, \quad p_S(r) = C - \mu\sigma_S(r) + \frac{\mu}{6}\tilde{\sigma}r^2 \\ \text{where } C &= \frac{1}{R_S} + \mu - \frac{\mu}{6}\tilde{\sigma}R_S^2, \end{aligned}$$

and R_S is uniquely determined by the equation

$$(1.8) \quad \frac{1}{R_S^2}(R_S \coth R_S - 1) = \frac{\tilde{\sigma}}{3}.$$

It was also proved in [10] that there exists a family of symmetry-breaking bifurcation branches of stationary solutions with free boundaries and μ 's given by

$$(1.9) \quad r = R_S + \varepsilon Y_{n,0}(\theta) + O(\varepsilon^2), \quad \mu = \mu_n + O(\varepsilon), \quad n \geq 2$$

where $Y_{n,0}(\theta)$ is the spherical harmonic of order $(n, 0)$, $|\varepsilon|$ is small,

$$(1.10) \quad \mu_n(R_S) = \frac{n[n(n+1) - 2]I_{1/2}(R_S)}{2R_S^3 I_{3/2}(R_S)[I_{5/2}(R_S)/I_{3/2}(R_S) - I_{n+3/2}(R_S)/I_{n+1/2}(R_S)]},$$

and $\mu_n(R_S) < \mu_{n+1}(R_S)$. We refer to these stationary bifurcations as Liapunov-Schmidt bifurcations.

The terms $O(\varepsilon^2)$, $O(\varepsilon)$ in (1.9) can actually be expanded into power series in ε .

By *asymptotic stability* of the stationary solution (1.7) we mean the following: Given any initial values

$$(1.11) \quad \begin{aligned} R_0(\theta, \phi) &= R_S + \eta \rho_0(\theta, \phi), \\ \sigma_0(r, \theta, \phi) &= \sigma_S(r) + \eta w_0(r, \theta, \phi), \end{aligned}$$

if $|\eta|$ is sufficiently small, then there exists a solution to (1.1)–(1.6) for all $t > 0$, and

$$(1.12) \quad \partial\Omega(t) \rightarrow \text{sphere } |x - a| = R_S \quad \text{as } t \rightarrow \infty,$$

for some center a ; more precisely, as $t \rightarrow \infty$, the solution $(\sigma, p, \Omega(t))$ converges to the radially symmetric stationary solution centered at a .

Recently [12, 13] the authors have proved that the radially symmetric solutions (1.7) are asymptotically stable as long as

$$0 < \mu < \mu_*(R_S),$$

where $\mu_*(R_S)$ is a function satisfying:

$$(1.13) \quad \begin{aligned} \mu_*(R_S) &< \mu_2(R_S) & \text{if } R_S < \bar{R}, \\ \mu_*(R_S) &= \mu_2(R_S) & \text{if } R_S = \bar{R} \end{aligned}$$

and $\bar{R} \approx 0.62207$. It was also proved in [12] that the convergence in (1.12) (in case $\mu < \mu_*(R_S)$) is exponentially fast.

In addition to the concept of asymptotic stability, we introduce the concepts of *linear stability* and *linear instability*. We linearize the system (1.1)–(1.6) about (σ_S, p_S, R_S) by writing

$$\sigma = \sigma_S + \varepsilon w, \quad p = p_S + \varepsilon q, \quad R = R_S + \varepsilon \rho$$

and collecting only the ε -order terms.

Because the original problem is invariant under translation of the origin, if we want to keep the origin fixed, we need to impose some constraints on the initial data. For the linearized problem these constraints are satisfied if

$$(1.14) \quad \begin{aligned} &\rho_0(\theta, \phi) \text{ and } w_0(r, \theta, \phi) \text{ do not contain spherical harmonics of} \\ &\text{order 1 : } Y_{1,m}(\theta, \phi) \quad (m = -1, 0, 1). \end{aligned}$$

In that case we say that the stationary solution (σ_S, p_S, R_S) is *linearly stable* if $\rho(\theta, \phi, t) \rightarrow 0$ as $t \rightarrow \infty$. In the general case, by *linear stability* we mean that for any initial data $\rho_0(\theta, \phi)$ and $w_0(r, \theta, \phi)$,

$$\rho(\theta, \phi, t) - \sum_{m=-1}^1 a_m Y_{1,m}(\theta, \phi) \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

for some constants a_m ; the a_m vanish if (1.14) is satisfied.

If $\rho(\theta, \phi, t) \rightarrow \infty$ for a sequence $t = t_m \rightarrow \infty$, then we say (σ_S, p_S, R_S) is *linearly unstable*.

In [12] we proved that if $\mu > \mu_*(R_S)$, then (σ_S, p_S, R_S) is linearly unstable.

The concept of linear stability/instability can be defined with respect to any stationary solution, not just radially symmetric ones. In this paper we shall consider the linear stability/instability of the bifurcation branch (1.9) for $n = 2$ in case $\mu_*(R_S) = \mu_2(R_S)$ and for a somewhat restricted version of the linearization

FIGURE 1. The free boundary is given by $r = R_S + \varepsilon Y_{2,0} + O(\varepsilon^2)$.

procedure, as will be explained in Section 5. We shall prove:

$$(1.15) \quad \text{If } \mu_*(R_S) = \mu_2(R_S), \text{ then the stationary solutions (1.9) with } \varepsilon > 0 \\ \text{are linearly stable, and the solutions with } \varepsilon < 0 \text{ are linearly unstable.}$$

Since we shall also prove that the $O(\varepsilon)$ term in the second equality in (1.9) is strictly monotone decreasing in ε , we can summarize the result (1.15) and the results of [12, 13] mentioned above in Figure 1.

Usually, one expects the bifurcation branches of the linear stability/instability in the diagram in Figure 1 reversed, namely, linear stability for $\mu(\varepsilon) > \mu_2$, instability for $\mu(\varepsilon) < \mu_2$. However, we note that the linear stability to be established in this paper is not the standard one since for each initial perturbation we also change the location of the origin of the nonlinear problem (see Theorem 10.1) and only then we linearize the system.

We shall also consider in this paper the case $\mu_*(R_S) < \mu_2(R_S)$ and prove the following Hopf bifurcation result:

$$(1.16) \quad \text{If } \mu_*(R_S) < \mu_2(R_S), \text{ then there exist periodic solutions to the} \\ \text{linearized problem with free boundary of the form } \rho(\theta, \phi, t) = \\ C_1 \sin \beta t + C_2 \cos \beta t + \sum_{m=-1}^1 D_m Y_{1,m}(\theta, \phi); \text{ the constant } \beta \text{ is de-} \\ \text{termined as a solution of a special transcendental equation. Fur-} \\ \text{thermore, any solution to the linearized problem converges expo-} \\ \text{nentially fast to one of these periodic solutions.}$$

The first step in the proof of (1.15) is to compute the sign of the coefficient μ_{21} in the expansion

$$(1.17) \quad \mu = \mu_2 + \varepsilon \mu_{21} + O(\varepsilon^2);$$

we shall prove that

$$(1.18) \quad \mu_{21} < 0.$$

In the process of proving (1.18) we shall give a new and simpler proof than [10] of the existence of the bifurcation branches (1.9), (1.10) for n even. The proof uses various calculations from [12] and [10] as well as the following Crandall-Rabinowitz theorem.

Theorem 1.1 (see [9, Theorem 1.7]). *Let X, Y be real Banach spaces and $F(x, \mu)$ a C^p map, $p \geq 3$, of a neighborhood $(0, \mu_0)$ in $X \times \mathbb{R}$ into Y . Suppose*

- (i) $F(0, \mu) = 0$ for all μ in a neighborhood of μ_0 ,
- (ii) $\text{Ker } F_x(0, \mu_0)$ is a one dimensional space, spanned by x_0 ,
- (iii) $\text{Im } F_x(0, \mu_0) = Y_1$ has codimension 1,
- (iv) $F_{\mu x}(0, \mu_0)x_0 \notin Y_1$.

Then $(0, \mu_0)$ is a bifurcation point of the equation $F(x, \mu) = 0$ in the following sense: In a neighborhood of $(0, \mu_0)$ the set of solutions of $F(x, \mu) = 0$ consists of two C^{p-2} smooth curves Γ_1 and Γ_2 which intersect only at the point $(0, \mu_0)$; Γ_1 is the curve $(0, \mu)$ and Γ_2 can be parameterized as follows:

$$\begin{aligned} \Gamma_2 : (x(\varepsilon), \mu(\varepsilon)), \quad |\varepsilon| \text{ small}, \\ (x(0), \mu(0)) = (0, \mu_0), \quad x'(0) = x_0. \end{aligned}$$

In Section 2 we collect various formulas for the Bessel functions of the second kind $I_m(\xi)$, and for the spherical harmonic functions $Y_{n,m}(\theta, \phi)$. In Section 3 we prove the existence of symmetry breaking bifurcation branches of solutions as described in (1.9)–(1.10), using Theorem 1.1. To do that, we define an appropriate function F , compute its Fréchet derivatives, and derive various expansions around a stationary solution up to second order. In Section 4 we compute the coefficient μ_{21} . In Section 5, we derive the linearized problem and in Section 6 we take its Laplace transform and formally obtain explicit formulas for the Laplace transform $\hat{\rho}$ of the free boundary. We proceed in Section 7 to establish estimates on the coefficients $\hat{\rho}_{n,0}$ in the expansion

$$\hat{\rho} = \sum_{n=0}^{\infty} \hat{\rho}_{n,0} Y_{n,0},$$

and in Section 8 we justify the formal calculations of Section 6. Next, in Sections 9 and 10 we prove the instability and stability results for the first Liapunov-Schmidt bifurcation, as stated in (1.15). Finally, in Section 11 we prove the Hopf bifurcation result stated in (1.16).

Remark 1.1. The results of this paper extend immediately to the more general case where in (1.4) κ is replaced by $\gamma \cdot \kappa$; in this case all the analysis remains the same except that the bifurcation parameter μ is replaced everywhere by μ/γ . Our results also extend to the case where in (1.1) we replace σ_t by $c\sigma_t$ and in (1.4) we replace κ by $\gamma \cdot \kappa$. Indeed the transformation $t \rightarrow c \cdot \tau$, $c \cdot p \rightarrow p$ reduces this more general case to the case where (1.1), (1.3) hold, in (1.2) μ is replaced by μ/c , and in (1.4) κ is replaced by γ/c , so that again the bifurcation parameter is μ/γ .

2. BESSEL FUNCTIONS AND SPHERICAL HARMONICS

In the sequel we shall need several properties of the modified Bessel function $I_m(\xi)$, where ξ is a complex variable and m is any nonnegative real number. For convenience we collect all these facts in this section. The following relations are

well known:

$$(2.1) \quad I_m(\xi) = \left(\frac{\xi}{2}\right)^m \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(m+k+1)} \left(\frac{\xi}{2}\right)^{2k},$$

$$(2.2) \quad I_{1/2}(\xi) = \sqrt{\frac{2}{\pi\xi}} \sinh \xi,$$

$$(2.3) \quad I_{3/2}(\xi) = \sqrt{\frac{2}{\pi\xi}} \left\{ -\frac{\sinh \xi}{\xi} + \cosh \xi \right\},$$

$$(2.4) \quad I_{5/2}(\xi) = \sqrt{\frac{2}{\pi\xi}} \left\{ \left[\frac{3}{\xi^2} + 1 \right] \sinh \xi - \frac{3}{\xi} \cosh \xi \right\},$$

$$(2.5) \quad I_m''(\xi) + \frac{1}{\xi} I_m'(\xi) - \left(1 + \frac{m^2}{\xi^2}\right) I_m(\xi) = 0,$$

$$(2.6) \quad I_m'(\xi) + \frac{m}{\xi} I_m(\xi) = I_{m-1}(\xi), \quad m \geq 1,$$

$$(2.7) \quad I_m'(\xi) - \frac{m}{\xi} I_m(\xi) = I_{m+1}(\xi), \quad m \geq 0,$$

$$(2.8) \quad I_{m-1}(\xi) - I_{m+1}(\xi) = \frac{2m}{\xi} I_m(\xi), \quad m \geq 1,$$

$$(2.9) \quad I_m(\xi) = \sqrt{\frac{2}{\pi\xi}} e^{\xi} \left[1 - \frac{4m^2 - 1}{8\xi} + O(|\xi|^{-2}) \right],$$

$$|\arg(\xi)| < \frac{\pi}{2} - \delta, \quad |\xi| \rightarrow +\infty.$$

By (2.8) we have, for any complex number s ,

$$(2.10) \quad \frac{d}{dr} \left[\frac{I_{n+1/2}(r\sqrt{s+1})}{r^{1/2}} \right] = \frac{n}{r^{3/2}} I_{n+1/2}(r\sqrt{s+1}) + \frac{\sqrt{s+1}}{r^{1/2}} I_{n+3/2}(r\sqrt{s+1}).$$

By differentiating (2.10) in r and using (2.6) and (2.7), we obtain

$$(2.11) \quad \frac{d^2}{dr^2} \left[\frac{I_{n+1/2}(r\sqrt{s+1})}{r^{1/2}} \right] = \left(\frac{s+1}{r^{1/2}} + \frac{n(n-1)}{r^{5/2}} \right) I_{n+1/2}(r\sqrt{s+1}) - \frac{2\sqrt{s+1}}{r^{3/2}} I_{n+3/2}(r\sqrt{s+1}).$$

We obtain the equality (2.10) from [12, (3.23)], and (2.11) by differentiating (2.10) in r and using (2.6) and (2.7).

We introduce the functions

$$(2.12) \quad P_n(\xi) = \frac{I_{n+3/2}(\xi)}{\xi I_{n+1/2}(\xi)}, \quad n = 0, 1, 2, 3, \dots$$

Then ([12, Lemma 2.1]),

$$(2.13) \quad P_n(\xi) = \frac{1}{\xi^2 P_{n+1}(\xi) + (2n+3)},$$

$$(2.14) \quad P_n(0) = \frac{1}{2n+3},$$

$$(2.15) \quad \frac{d}{d\xi} P_n(\xi) = \frac{1}{\xi} - \frac{2n+3}{\xi} P_n(\xi) - \xi P_n^2(\xi),$$

$$(2.16) \quad |P_n(\xi)| \leq \sqrt{2} P_n(|\xi|) \leq \frac{\sqrt{2}}{2n+3} \quad \text{for } |\arg(\xi)| \leq \pi/4.$$

We also have, for any small $\delta > 0$,

$$(2.17) \quad |P_n(\xi)| \leq C(\delta)P_n(|\xi|) \quad \text{for } |\arg(\xi)| \leq \frac{\pi}{2} - \delta;$$

indeed, one can prove (2.17) by repeating the proof of (2.16) in [12, Lemma 2.1], noting that $\cos 2\theta \geq -\cos 2\delta > -1 + \delta$ if $|\theta| \leq \frac{\pi}{2} - \delta$.

For real $r > 0$,

$$(2.18) \quad P_n(r) > P_{n+1}(r), \quad n \geq 0,$$

$$(2.19) \quad \frac{d}{dr}P_n(r) < 0,$$

$$(2.20) \quad \frac{2n+5}{r^2 + (2n+3)(2n+5)} < P_n(r) < \frac{r^2 + (2n+5)(2n+7)}{2(2n+5)r^2 + (2n+3)(2n+5)(2n+7)},$$

$$(2.21) \quad \frac{1}{r} - \frac{n+1}{r^2} < P_n(r) < \frac{1}{r} - \frac{n+1}{r^2} + \frac{n^2+n+1}{2r^3},$$

$$(2.22) \quad P_n(r) < \frac{1}{r};$$

the inequalities (2.19)–(2.21) are taken from [12, Lemma 2.3] and the proof of (2.22) follows by comparison of $P_n(r)$ with $1/r$, using the differential equation (2.15).

For simplicity of notation, we shall usually denote R_S by R throughout this paper. From (1.8) and [12, (2.11)] we have

$$(2.23) \quad \frac{\tilde{\sigma}}{3} = P_0(R).$$

Using the P_n function we can write (1.10) in the form

$$(2.24) \quad \mu_n(R) = n \left(\frac{n(n+1)}{2} - 1 \right) \frac{1}{R^5 P_0(R) [P_1(R) - P_n(R)]}.$$

Lemma 2.1.

$$(2.25) \quad \lambda \equiv (\sigma_S)_r(R) = RP_0(R),$$

$$(2.26) \quad \frac{\partial^2 \sigma_S}{\partial r^2} \Big|_{r=R} = 1 - 2P_0(R),$$

$$(2.27) \quad \frac{\partial^3 \sigma_S}{\partial r^3} \Big|_{r=R} = RP_0(R)[1 - 2P_1(R)],$$

and

$$(2.28) \quad \frac{\partial p_S}{\partial r} \Big|_{r=R} = 0,$$

$$(2.29) \quad \frac{\partial^2 p_S}{\partial r^2} \Big|_{r=R} = -\mu[1 - 3P_0(R)] = -\mu R^2 P_0(R)P_1(R),$$

$$(2.30) \quad \frac{\partial^3 p_S}{\partial r^3} \Big|_{r=R} = -\mu RP_0(R)[1 - 2P_1(R)].$$

Proof. The equality (2.25) follows directly by differentiating $\sigma_S(r)$ (in (1.7)) and using (2.3). Using (2.25) and the equation for σ_S , we get

$$(\sigma_S)_{rr}(R) = -\frac{2}{R}(\sigma_S)_r(R) + \sigma_S(R) = 1 - 2P_0(R),$$

which establishes (2.26). Differentiating the equation for σ_S in r , we obtain

$$\begin{aligned} \frac{\partial^3 \sigma_S}{\partial r^3} \Big|_{r=R} &= -\frac{2}{R}(\sigma_S)_{rr}(R) + \left(\frac{2}{R^2} + 1\right)(\sigma_S)_r(R) \\ &= -\frac{2}{R}(1 - 2P_0(R)) + \left(\frac{2}{R^2} + 1\right)RP_0(R) \\ &= -\frac{2}{R}(1 - 3P_0(R)) + RP_0(R) \\ &= RP_0(R)[1 - 2P_1(R)] \quad (\text{by (2.13) with } n = 0). \end{aligned}$$

The equation (2.28) follows from the boundary condition of the stationary solution $p_S(r)$. Using (1.7), (2.23) and (2.26), we get

$$\frac{\partial^2 p_S(R)}{\partial r^2} = \mu \left(-(\sigma_S)_{rr}(R) + \frac{1}{3}\tilde{\sigma} \right) = \mu [3P_0(R) - 1],$$

which implies (2.29), by (2.13). Finally, (1.7) and (2.27) yield the relation (2.30). $\square$

We recall that the Laplace operator on a unit sphere has the form

$$(2.31) \quad \Delta_\omega S(\theta, \phi) = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial S}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 S}{\partial \phi^2} \quad (\omega = (\theta, \phi)).$$

In particular,

$$(2.32) \quad \Delta_\omega Y_{n,m}(\theta, \phi) = -n(n+1)Y_{n,m}(\theta, \phi) \quad (-n \leq m \leq n)$$

where $Y_{n,m}$ is the spherical harmonic of order (n, m) , so that

$$(2.33) \quad -\Delta(r^n Y_{n,m}) = 0.$$

Using (2.5) we find that

$$(2.34) \quad -\Delta \left(\frac{I_{n+1/2}(r)}{r^{1/2}} Y_{n,m} \right) + \left(\frac{I_{n+1/2}(r)}{r^{1/2}} Y_{n,m} \right) = 0.$$

We recall that

$$(2.35) \quad Y_{0,0} = \frac{1}{2\sqrt{\pi}},$$

$$(2.36) \quad Y_{2,0}(\theta) = \frac{1}{4} \sqrt{\frac{5}{\pi}} (3 \cos^2 \theta - 1),$$

$$(2.37) \quad Y_{4,0}(\theta) = \frac{3}{16\sqrt{\pi}} (35 \cos^4 \theta - 30 \cos^2 \theta + 3).$$

The following identities can be verified by direct computations:

$$(2.38) \quad Y_{2,0}^2 = \frac{1}{\sqrt{\pi}} \left[\frac{3}{7} Y_{4,0} + \frac{\sqrt{5}}{7} Y_{2,0} + \frac{1}{2} Y_{0,0} \right],$$

$$(2.39) \quad \left(\frac{\partial Y_{2,0}}{\partial \theta} \right)^2 = \frac{3}{\sqrt{\pi}} \left[-\frac{4}{7} Y_{4,0} + \frac{\sqrt{5}}{7} Y_{2,0} + Y_{0,0} \right].$$

Recall [17, page 305] that for any $n \geq 0$,

$$(2.40) \quad Y_{n,0}(\theta) = \sqrt{\frac{2n+1}{4\pi}} L_n(\cos \theta),$$

where $L_n(x)$ are the Legendre polynomials:

$$(2.41) \quad L_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, \quad L_n(-x) = (-1)^n L_n(x).$$

Clearly,

$$(2.42) \quad \text{span}\{L_0(x), L_1(x), L_2(x), \dots, L_n(x)\} = \text{span}\{1, x, x^2, \dots, x^n\}.$$

By (2.40) and (2.41), we see that $Y_{n,0}(\theta)$ is an even function if n is even and an odd function if n is odd. It is well known ([17, page 306]) that, for any integer $n \geq 0$,

$$(2.43) \quad Y_{2,0}Y_{n,0} = \sum_{j=0}^{\infty} c_{nj}Y_{j,0}.$$

Since $L_n(x)$ is a polynomial of degree n , $P_2(x)P_n(x)$ is a polynomial of degree $n+2$, so by (2.42) we must have

$$c_{nj} = 0 \quad \text{for } j \geq n+3.$$

Similarly, if we write

$$(2.44) \quad (Y_{2,0})_{\theta}(Y_{n,0})_{\theta} = \sum_{j=0}^{\infty} d_{nj}Y_{j,0},$$

then the above argument shows that $d_{nj} = 0$ for $j \geq n+3$.

Lemma 2.2. *There holds*

$$(2.45) \quad c_{nj} = d_{nj} = 0 \quad \text{if } j+n = \text{odd},$$

$$(2.46) \quad c_{nj} = d_{nj} = 0 \quad \text{if } |n-j| \geq 3,$$

and

$$c_{jn} = c_{nj}, \quad d_{jn} + d_{nj} = 6c_{nj} \quad \text{for all } n, j;$$

furthermore,

$$(2.47) \quad \sum_{j=0}^{n+2} (c_{nj})^2 \leq \frac{5}{4\pi}$$

and

$$(2.48) \quad \sum_{j=0}^{n+2} (d_{nj})^2 \leq \frac{45}{16\pi} n(n+1).$$

Proof. If n is even and j is odd, then the product $Y_{2,0}Y_{2,n}$ is an even function while $Y_{j,0}$ is odd, so that $c_{nj} = \langle Y_{2,0}Y_{n,0}, Y_{j,0} \rangle = 0$. One can similarly prove (2.45) for all the remaining cases.

Clearly

$$c_{nj} = \langle Y_{2,0}Y_{n,0}, Y_{j,0} \rangle = \langle Y_{2,0}Y_{j,0}, Y_{n,0} \rangle = c_{jn}.$$

By symmetry, $c_{nj} = c_{jn} = 0$ if $n \geq j+3$.

For d_{nj} we have

$$\begin{aligned} d_{nj} + d_{jn} &= \langle (Y_{2,0})_\theta (Y_{n,0})_\theta, Y_{j,0} \rangle + \langle (Y_{2,0})_\theta (Y_{j,0})_\theta, Y_{n,0} \rangle \\ &= 2\pi \int_0^\pi (Y_{n,0} Y_{j,0})_\theta (Y_{2,0})_\theta \sin \theta d\theta \\ &= -2\pi \int_0^\pi Y_{n,0} Y_{j,0} \frac{\partial}{\partial \theta} [(Y_{2,0})_\theta \sin \theta] d\theta \\ &= 12\pi \int_0^\pi Y_{n,0} Y_{j,0} Y_{2,0} \sin \theta d\theta \end{aligned}$$

by (2.31), (2.32), so that $d_{nj} + d_{jn} = 6c_{nj}$. It follows that if $n \geq j + 3$, then $d_{nj} = -d_{jn} + 6c_{nj} = 0$.

By (2.43) and the orthonormality of $Y_{n,m}$,

$$\sum_{j=0}^{n+2} (c_{nj})^2 = 2\pi \int_0^\pi [Y_{2,0}(\theta) Y_{n,0}(\theta)]^2 \sin \theta d\theta \leq \max_{0 \leq \theta \leq \pi} |Y_{2,0}(\theta)|^2 \leq \frac{5}{4\pi}.$$

From the equation $\Delta_\omega Y_{n,0} + n(n+1)Y_{n,0} = 0$, we obtain, after multiplying by $Y_{n,0} \sin \theta$ and integrating with respect to θ ,

$$2\pi \int_0^\pi \left(\frac{\partial Y_{n,0}(\theta)}{\partial \theta} \right)^2 \sin \theta d\theta = n(n+1) 2\pi \int_0^\pi (Y_{n,0}(\theta))^2 \sin \theta d\theta = n(n+1),$$

so that by (2.43) and the orthonormality of $Y_{n,m}$,

$$\sum_{j=0}^{n+2} (d_{nj})^2 = 2\pi \int_0^\pi \left[\frac{\partial Y_{2,0}}{\partial \theta} \frac{\partial Y_{n,0}}{\partial \theta} \right]^2 \sin \theta d\theta \leq n(n+1) \max_{0 \leq \theta \leq \pi} \left| \frac{\partial Y_{2,0}}{\partial \theta} \right|^2,$$

from which (2.48) follows. $\square$

Using (2.38), (2.39) we can compute the coefficients c_{nj} and d_{nj} for $n = 2$:

$$(2.49) \quad c_{20} = \frac{1}{2\sqrt{\pi}}, \quad c_{22} = \frac{\sqrt{5}}{7\sqrt{\pi}}, \quad c_{24} = \frac{3}{7\sqrt{\pi}},$$

$$(2.50) \quad d_{20} = \frac{3}{\sqrt{\pi}}, \quad d_{22} = \frac{3\sqrt{5}}{7\sqrt{\pi}}, \quad d_{24} = -\frac{12}{7\sqrt{\pi}}.$$

3. THE BIFURCATION PROBLEM

In this section we give an alternative proof to [10] regarding the existence of bifurcation branches of symmetry-breaking solutions of the form (1.9). We want to use Theorem 1.1, and we therefore begin by defining a suitable function F .

Consider a family of domains with boundaries $\partial\Omega_\varepsilon : r = R_S + \tilde{R}(\theta, \phi)$ where $\tilde{R}(\theta, \phi) = \varepsilon S(\theta, \phi)$. Let (σ, p) form the solution of

$$(3.1) \quad -\Delta\sigma + \sigma = 0 \quad \text{in } \Omega_\varepsilon,$$

$$(3.2) \quad -\Delta p = \mu(\sigma - \tilde{\sigma}) \quad \text{in } \Omega_\varepsilon \quad (\tilde{\sigma} < 1),$$

$$(3.3) \quad \sigma = 1 \quad \text{on } \partial\Omega_\varepsilon,$$

$$(3.4) \quad p = \kappa \quad \text{on } \partial\Omega_\varepsilon.$$

We define F by

$$(3.5) \quad F(\tilde{R}, \mu) = \frac{\partial p}{\partial n} \Big|_{\partial \Omega_\varepsilon}.$$

Then $(\sigma, p, R_S + \tilde{R})$ is a stationary solution of (1.1)–(1.5) if and only if $F(\tilde{R}, \mu) = 0$.

The function $S(\theta, \phi)$ may be viewed as a function defined on the unit sphere

$$\Sigma = \{x : |x| = 1\}.$$

We shall later on assume that $S(\theta, \phi)$ is in $C^{m+\alpha}(\Sigma)$, that is, S is $C^{m+\alpha}$ as a function defined on Σ ; note that this does not mean that $S(\theta, \phi)$ is in $C^{m+\alpha}$ in the variable (θ, ϕ) .

As in [10] we introduce the Banach spaces

$$X^{m+\alpha} = \left\{ \tilde{R} \in C^{m+\alpha}(\Sigma), \tilde{R} \text{ is } \pi\text{-periodic in } \theta, 2\pi\text{-periodic in } \phi \right\},$$

$$X_1^{m+\alpha} = \text{closure of the linear space spanned by}$$

$$\{Y_{j,0}(\theta), j = 0, 1, 2, \dots\} \text{ in } X^{m+\alpha},$$

$$X_2^{m+\alpha} = \text{closure of the linear space spanned by}$$

$$\{Y_{j,0}(\theta), j = 0, 2, 4, 6, \dots\} \text{ in } X^{m+\alpha}.$$

We shall later on view $F(\tilde{R}, \mu)$ as a mapping from $X^{m+3+\alpha}$ to $X^{m+\alpha}$ (or from $X_j^{m+3+\alpha}$ to $X_j^{m+\alpha}$).

As mentioned in Section 2, we shall usually denote R_S simply by R .

In order to apply Theorem 1.1, we need to compute the Fréchet derivatives of F . To do that, we need the expansion of (σ, p) of order ε . We can formally write, for any $\mu > 0$,

$$(3.6) \quad \sigma = \sigma_S + \varepsilon \sigma_1 + O(\varepsilon^2),$$

$$(3.7) \quad p = p_S + \varepsilon p_1 + O(\varepsilon^2),$$

$$(3.8) \quad \partial \Omega_\varepsilon : r = R + \varepsilon S(\theta, \phi).$$

Then, formally,

$$(3.9) \quad \begin{aligned} \varepsilon \sigma_1 \Big|_{\partial B_R} &= \varepsilon \sigma_1 \Big|_{\partial \Omega_\varepsilon} + O(\varepsilon^2) \\ &= [1 - \sigma_S(R + \varepsilon S)] + O(\varepsilon^2) = -\varepsilon \lambda S + O(\varepsilon^2), \end{aligned}$$

where $\lambda = (\sigma_S)_r(R)$. Substituting (3.6) and (3.9) into (3.1) and (3.3) and dropping the $O(\varepsilon^2)$ terms, we find that σ_1 satisfies

$$(3.10) \quad -\Delta \sigma_1 + \sigma_1 = 0 \quad \text{in } B_R,$$

$$(3.11) \quad \sigma_1 = -\lambda S(\theta, \phi) \quad \text{on } \partial B_R.$$

By [16],

$$(3.12) \quad \kappa = \frac{1}{R} - \frac{\varepsilon}{R^2} \left(S + \frac{1}{2} \Delta_\omega S \right) + O(\varepsilon^2);$$

in the subsequent Lemma 4.1 we shall prove a more refined formula for κ . We next expand the boundary condition for p . Since $\frac{\partial p_S(R)}{\partial r} = 0$, the term $-(p_S)_r(R)S(\theta, \phi)$ vanishes. From (3.2), (3.4) we then obtain

$$(3.13) \quad -\Delta p_1 = \mu \sigma_1 \quad \text{in } B_R,$$

$$(3.14) \quad p_1 = -\frac{1}{R^2} \left(S + \frac{1}{2} \Delta_\omega S \right) \quad \text{on } \partial B_R.$$

If we write

$$(3.15) \quad S(\theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^n a_{n,m} Y_{n,m}(\theta, \phi),$$

then we can solve (σ_1, p_1) explicitly by

$$(3.16) \quad \sigma_1(r, \theta, \phi) = - \sum_{n=0}^{\infty} \sum_{m=-n}^n \lambda a_{n,m} \frac{I_{n+1/2}(r)}{r^{1/2}} \frac{R^{1/2}}{I_{n+1/2}(R)} Y_{n,m}(\theta, \phi)$$

and

$$(3.17) \quad p_1(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^n a_{n,m} \frac{r^n}{R^n} \left[\frac{1}{R^2} \left(\frac{n(n+1)}{2} - 1 \right) - \mu \lambda \right] Y_{n,m}(\theta, \phi) - \mu \sigma_1(r, \theta, \phi).$$

Indeed, this is easily verified using (2.33), (2.34).

Lemma 3.1. *If $S \in C^{3+\alpha}(\Sigma)$ and σ is a solution of (3.1) and (3.3), then*

$$(3.18) \quad \|\sigma - \sigma_S\|_{C^{3+\alpha}(\overline{\Omega}_\varepsilon)} \leq C|\varepsilon| \|S\|_{C^{3+\alpha}(\Sigma)},$$

where the constant C is independent of ε .

Proof. The function $\sigma_S(r)$ is well defined for all r , and it is clear that

$$-\Delta(\sigma - \sigma_S) + (\sigma - \sigma_S) = 0 \quad \text{in } \Omega_\varepsilon.$$

On the boundary $\partial\Omega_\varepsilon = \{r = R + \varepsilon S(\theta, \phi)\}$, we have

$$(3.19) \quad \begin{aligned} \sigma(R + \varepsilon S, \theta, \phi) - \sigma_S(R + \varepsilon S) &= 1 - \sigma_S(R + \varepsilon S) \\ &= \sigma_S(R) - \sigma_S(R + \varepsilon S). \end{aligned}$$

We can also differentiate this relation three times along $\partial\Omega_\varepsilon$, and thus we get

$$(3.20) \quad \|\sigma - \sigma_S\|_{C^{3+\alpha}(\partial\Omega_\varepsilon)} \leq C|\varepsilon| \|S\|_{C^{3+\alpha}(\Sigma)}.$$

Since $\partial\Omega_\varepsilon$ is in $C^{3+\alpha}$ uniformly in ε , we can use the Schauder estimates to obtain the estimates (3.18). $\square$

We next want to rigorously establish the equations (3.6), (3.7) by estimating the $O(\varepsilon^2)$ terms in the $C^{3+\alpha}$ -norm. To do that we shall first transform Ω_ε onto the ball B_R by the Hanzawa transformation.

The Hanzawa transformation is a diffeomorphism defined by

$$(r, \theta, \phi) = H_\varepsilon(r', \theta', \phi') \equiv (r' + \chi(R - r')\varepsilon S(\theta', \phi'), \theta', \phi'),$$

where

$$\chi \in C^\infty, \quad \chi(z) = \begin{cases} 0 & \text{if } |z| \geq 3\delta_0/4 \\ 1 & \text{if } |z| < \delta_0/4 \end{cases}, \quad \left| \frac{d^k \chi}{dz^k} \right| \leq C/\delta_0^k.$$

Observe that H_ε maps B_R onto Ω_ε while keeping the ball $\{r < R - (3\delta_0/4)\}$ fixed. The inverse Hanzawa transformation H_ε^{-1} maps Ω_ε onto B_R .

Let σ_1 be the solution of (3.10)–(3.11). By the Schauder estimates $\sigma_1 \in C^{3+\alpha}(\overline{B}_R)$ and $\|\sigma_1\|_{C^{3+\alpha}(\overline{B}_R)} \leq C\|S\|_{C^{3+\alpha}(\Sigma)}$. Setting

$$(3.21) \quad \tilde{\sigma}_1(r, \theta, \phi; \varepsilon) = \sigma_1(H_\varepsilon^{-1}(r, \theta, \phi)) \quad \text{in } \Omega_\varepsilon,$$

we have

$$(3.22) \quad -\Delta \tilde{\sigma}_1 + \tilde{\sigma}_1 = \tilde{f} \quad \text{in } \Omega_\varepsilon,$$

$$(3.23) \quad \tilde{\sigma}_1 = -\lambda S(\theta, \phi) \quad \text{on } \partial\Omega_\varepsilon,$$

where $\frac{1}{\varepsilon}\tilde{f}$ is a linear combination of products of at most second order derivatives of σ_1 and S (see [10, Section 3]), and

$$(3.24) \quad \|\tilde{f}\|_{C^{1+\alpha}(\overline{\Omega}_\varepsilon)} \leq C|\varepsilon| \|S\|_{C^{3+\alpha}(\Sigma)}.$$

Thus the function

$$(3.25) \quad \psi = \sigma - \sigma_S - \varepsilon\tilde{\sigma}_1$$

is well defined in Ω_ε and satisfies

$$(3.26) \quad -\Delta\psi + \psi = -\varepsilon\tilde{f} \quad \text{in } \Omega_\varepsilon,$$

$$(3.27) \quad \psi = -g \quad \text{on } \partial\Omega_\varepsilon,$$

where

$$g(\theta, \phi) = \sigma_S(R + \varepsilon S(\theta, \phi)) - 1 - \lambda S(\theta, \phi).$$

Similarly to (3.20), we have $\|g\|_{C^{3+\alpha}(\Sigma)} \leq C|\varepsilon|^2 \|S\|_{C^{3+\alpha}(\Sigma)}$. Hence, the Schauder estimates yield the following result.

Lemma 3.2.

$$(3.28) \quad \|\sigma - \sigma_S - \varepsilon\tilde{\sigma}_1\|_{C^{3+\alpha}(\overline{\Omega}_\varepsilon)} \leq C|\varepsilon|^2 \|S\|_{C^{3+\alpha}(\Sigma)}.$$

We now turn to the solution p of (3.2), (3.4). In view of Lemma 3.2,

$$(3.29) \quad -\Delta p = \mu(\sigma_S - \tilde{\sigma} + \varepsilon\tilde{\sigma}_1) + \varepsilon^2 k \quad \text{in } \Omega_\varepsilon,$$

where

$$(3.30) \quad \|k\|_{C^{3+\alpha}(\overline{\Omega})} \leq C\|S\|_{C^{3+\alpha}(\Sigma)}.$$

We want to compute $\partial p / \partial n$ in terms of $S(\theta, \phi)$.

By the proof of (3.12) in [16],

$$(3.31) \quad \left\| \kappa - \frac{1}{R} + \frac{\varepsilon}{R^2} \left(S + \frac{1}{2} \Delta_\omega S \right) \right\|_{C^{1+\alpha}(\Sigma)} \leq C|\varepsilon|^2 \|S\|_{C^{3+\alpha}(\Sigma)}.$$

Let p_1 be the solution of (3.13) and (3.14), and define

$$\tilde{p}_1(r, \theta, \phi; \varepsilon) = p_1(H_\varepsilon^{-1}(r, \theta, \phi)) \quad \text{in } \Omega_\varepsilon.$$

By the same procedure as before, using (3.31) we get

$$\begin{aligned} \|p_1\|_{C^{1+\alpha}(\overline{B}_R)} &\leq C\|S\|_{C^{3+\alpha}(\Sigma)}, \\ \|p - p_S - \varepsilon\tilde{p}_1\|_{C^{1+\alpha}(\partial\Omega_\varepsilon)} &\leq C|\varepsilon|^2 \|S\|_{C^{3+\alpha}(\Sigma)}, \end{aligned}$$

and then, by (3.29), (3.30) and the Schauder estimates,

$$(3.32) \quad \|p - p_S - \varepsilon\tilde{p}_1\|_{C^{1+\alpha}(\overline{\Omega}_\varepsilon)} \leq C|\varepsilon|^2 \|S\|_{C^{3+\alpha}(\Sigma)}.$$

Since

$$\frac{\partial p_S}{\partial r}(R) = 0,$$

we have

$$(3.33) \quad \left. \frac{\partial(p_S + \varepsilon\tilde{p}_1)}{\partial n} \right|_{r=R+\varepsilon S(\theta, \phi)} = \varepsilon \left(\frac{\partial^2 p_S(R)}{\partial r^2} S + \frac{\partial p_1(R, \theta, \phi)}{\partial r} \right) + O(|\varepsilon|^2 \|S\|_{C^{3+\alpha}(\Sigma)}),$$

and so, by (3.32),

$$(3.34) \quad \frac{\partial p}{\partial n} \Big|_{r=R+\varepsilon S(\theta, \phi)} = \varepsilon \left(\frac{\partial^2 p_S(R)}{\partial r^2} S + \frac{\partial p_1(R, \theta, \phi)}{\partial r} \right) + O(|\varepsilon|^2 \|S\|_{C^{3+\alpha}(\Sigma)}).$$

The relation (3.34) shows that the mapping $(\tilde{R}, \mu) \rightarrow F(\tilde{R}, \mu)$ from $X^{m+3+\alpha}$ to $X^{m+\alpha}$ is bounded if $m = 0$, and the same argument shows that the same is true for any $m \geq 0$. A similar argument shows that this mapping is Fréchet differentiable in $(\tilde{R}, \mu)$; furthermore $\partial F(\tilde{R}, \mu)/\partial \tilde{R}$ ($\partial F(\tilde{R}, \mu)/\partial \mu$) is obtained by solving a linearized problem about $(\tilde{R}, \mu)$ with respect to $\tilde{R}$ (μ). By using the Schauder estimates we can then further obtain differentiability of $F(\tilde{R}, \mu)$ to any order.

From (3.34) it follows that the Fréchet derivative of $F(\tilde{R}, \mu)$ in $\tilde{R}$, at $(0, \mu)$, is given by

$$(3.35) \quad [F_{\tilde{R}}(0, \mu)]S = \frac{\partial^2 p_S(R)}{\partial r^2} S + \frac{\partial p_1(R, \theta, \phi)}{\partial r}.$$

To compute the right-hand side, we note that $\frac{\partial^2 p_S(R)}{\partial r^2}$ was already computed in (2.29), so that it remains to compute $\frac{\partial p_1(R, \theta, \phi)}{\partial r}$. We first compute $\partial \sigma_1(R, \theta, \phi)/\partial r$. By (3.16) we have

$$(3.36) \quad \begin{aligned} \frac{\partial \sigma_1}{\partial r} \Big|_{r=R} &= -\lambda \sum_{n=0}^{\infty} \sum_{m=-n}^n a_{n,m} \frac{\partial}{\partial r} \left[\frac{I_{n+1/2}(r)}{r^{1/2}} \frac{R^{1/2}}{I_{n+1/2}(R)} \right]_{r=R} Y_{n,m}(\theta, \phi) \\ &= -\lambda \sum_{n=0}^{\infty} \sum_{m=-n}^n a_{n,m} \left[\frac{n}{R} + \frac{I_{n+3/2}(R)}{I_{n+1/2}(R)} \right] Y_{n,m}(\theta, \phi) \quad \text{by (2.10),} \end{aligned}$$

so that, by (3.17),

$$(3.37) \quad \frac{\partial p_1}{\partial r} \Big|_{r=R} = \sum_{n=0}^{\infty} \sum_{m=-n}^n a_{n,m} \frac{n}{R} \left[\frac{1}{R^2} \left(\frac{n(n+1)}{2} - 1 \right) - \mu \lambda \right] Y_{n,m}(\theta, \phi) - \mu \frac{\partial \sigma_1}{\partial r} \Big|_{r=R}.$$

Hence

$$(3.38) \quad \frac{\partial p_1}{\partial r} \Big|_{r=R} = \sum_{n=0}^{\infty} \sum_{m=-n}^n a_{n,m} \left[\frac{n}{R^3} \left(\frac{n(n+1)}{2} - 1 \right) + \mu R^2 P_0(R) P_n(R) \right] Y_{n,m}.$$

Substituting (2.29) and (3.38) into (3.35) we get

$$(3.39) \quad \begin{aligned} [F_{\tilde{R}}(0, \mu)]S &= \sum_{n=0}^{\infty} \sum_{m=-n}^n a_{n,m} \left[\frac{n}{R^3} \left(\frac{n(n+1)}{2} - 1 \right) \right. \\ &\quad \left. + \mu R^2 P_0(R) P_n(R) - \mu R^2 P_0(R) P_1(R) \right] Y_{n,m}(\theta, \phi); \end{aligned}$$

in particular,

$$(3.40) \quad [F_{\tilde{R}}(0, \mu)]Y_{n,m} = \left[\frac{n}{R^3} \left(\frac{n(n+1)}{2} - 1 \right) - \mu R^2 P_0(R) [P_1(R) - P_n(R)] \right] Y_{n,m}.$$

It follows that, for $n \neq 1$, $[F_{\tilde{R}}(0, \mu)]Y_{n,m} = 0$ if and only if $\mu = \mu_n(R)$, where $\mu_n(R)$ is defined by (2.24), or equivalently, by (1.10). Thus the operator $A_\mu = [F_{\tilde{R}}(0, \mu)]$

maps $X_1^{3+\alpha}$ to X_1^α , and

$$(3.41) \quad \ker A_\mu = \begin{cases} \{Y_{1,0}\} & \text{if } \mu \neq \mu_2, \mu_3, \dots, \\ \text{span}\{Y_{n,0}, Y_{1,0}\} & \text{if } \mu = \mu_n. \end{cases}$$

If we work with the space $X_1^{3+\alpha}$, then $\dim(\ker A_{\mu_n}) = 2$. In order to satisfy the requirement (ii) of Theorem 1.1 we shall henceforth work with the space $X_2^{3+\alpha}$. The operators are still well defined by the Algebra property of $X_2^{j+\alpha}$. In this $X_2^{3+\alpha}$ space, $\dim(\ker A_\mu) = 1$ if $\mu = \mu_n$. If n is even and ≥ 2 , then clearly the direct sum $\text{Im } A_{\mu_n} \oplus \{Y_{n,0}\}$ is the whole space X_2^α and therefore $\text{codim}(A_{\mu_n}) = 1$. Differentiating (3.40) in μ , we obtain

$$(3.42) \quad [F_{\tilde{R}\mu}(0, \mu_n)]S = \sum_{n=0}^{\infty} \sum_{m=-n}^n a_{n,m} R^2 P_0(R) [P_n(R) - P_1(R)] Y_{n,m}.$$

In particular,

$$(3.43) \quad [F_{\tilde{R}\mu}(0, \mu_n)]Y_{n,0} = R^2 P_0(R) [P_n(R) - P_1(R)] Y_{n,0} \notin \text{Im } A_{\mu_n}.$$

We can now apply the Crandall-Rabinowitz theorem to conclude:

Theorem 3.3. *The points $(0, \mu_n)$ (n even and ≥ 2) are bifurcation points for the problem (3.1)–(3.4), and the corresponding free boundaries are of the form*

$$(3.44) \quad r = R + \varepsilon Y_{n,0}(\theta) + o(\varepsilon).$$

In a similar way, we can consider $F(\tilde{R}, \mu)$ as a mapping from $X_2^{m+3+\alpha}$ to $X_2^{m+\alpha}$ for any $m > 1$ and then replace $o(\varepsilon)$ by an expansion to order ε^m with error $o(\varepsilon^m)$.

Remark 3.1. Theorem 3.3 establishes bifurcation only for even $n \geq 2$. For any odd $n \geq 3$ we may apply the Crandall-Rabinowitz theorem in a more delicate manner, working with the space

$$X = M^{k+\alpha} = \text{closure of the linear space spanned by} \\ \{Y_{j,0}(\theta), j = 0, 2, 3, 4, 5, 6, \dots\} \text{ in } X^{k+\alpha}$$

for $k \geq 3$. Here we face the problem that F does not map $M^{k+\alpha}$ into $M^{k-3+\alpha}$, but we could shift the center of the system in order to eliminate the mode of $Y_{1,0}$ and thus make the modified F map $M^{k+\alpha}$ into $M^{k-3+\alpha}$. This shift is of higher order than $\varepsilon Y_{1,0}$, and its existence should follow by a fixed point theorem; cf. an analogous situation in Remark 1 in [10]. The mapping $\tilde{F}$ composed of F followed by this shift of the origin will map $M^{k+\alpha}$ onto $M^{k-3+\alpha}$, and the Crandall-Rabinowitz theorem could then be applied to $\tilde{F}$. The details of this approach would involve complicated computations and will not be pursued here.

For $n = 2$, the bifurcation branch $(\tilde{R}(\varepsilon), \mu(\varepsilon))$ satisfies $F(\tilde{R}(\varepsilon), \mu(\varepsilon)) = 0$, $(\tilde{R}(0), \mu(0)) = (0, \mu_2)$, and

$$\tilde{R}(\varepsilon) = R + \varepsilon Y_{2,0}(\theta) + o(\varepsilon).$$

It will be important to compute $\mu'(0)$.

4. THE COMPUTATION OF $\mu'(0)$

In order to compute $\mu'(0)$, we need to derive, for each fixed μ , the ε^2 -order expansion in the special case $S = Y_{2,0}(\theta)$:

$$(4.1) \quad \sigma(r, \theta, \phi) = \sigma_S(r) + \varepsilon \sigma_1(r, \theta) + \varepsilon^2 \sigma_2(r, \theta) + O(\varepsilon^3) \quad \text{in } \Omega_\varepsilon,$$

$$(4.2) \quad p(r, \theta, \phi, \mu) = p_S(r, \mu) + \varepsilon p_1(r, \theta, \mu) + \varepsilon^2 p_2(r, \theta, \mu) + O(\varepsilon^3) \quad \text{in } \Omega_\varepsilon.$$

Lemma 4.1. *If $\partial\Omega_\varepsilon : r = R + \varepsilon S(\theta, \phi)$, where $S \in C^2(\Sigma)$, then*

$$(4.3) \quad \kappa \Big|_{r=R+\varepsilon S(\theta, \phi)} = \frac{1}{R} - \frac{\varepsilon}{R^2} \left(S + \frac{1}{2} \Delta_\omega S \right) + \frac{\varepsilon^2}{R^3} \left(S^2 + S \Delta_\omega S \right) + O(\varepsilon^3).$$

Proof. We use the notation $\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi$ for the unit normal vectors in the r, θ, ϕ directions, respectively. Then, written in the rectangular coordinates in $\mathbb{R}^3$,

$$\begin{aligned} \vec{e}_r &= (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \\ \vec{e}_\theta &= (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta), \\ \vec{e}_\phi &= (-\sin \phi, \cos \phi, 0), \end{aligned}$$

and the gradient is given by

$$\nabla_x = \vec{e}_r \partial_r + \vec{e}_\theta \frac{1}{r} \partial_\theta + \vec{e}_\phi \frac{1}{r \sin \theta} \partial_\phi \equiv \vec{e}_r \partial_r + \frac{1}{r} \nabla_\omega.$$

If a surface is given by $r = R + \tilde{R}(\theta, \phi)$, or, alternatively, by $\xi(r, \theta, \phi) = 0$ where $\xi(r, \theta, \phi) = R + \tilde{R}(\theta, \phi) - r$, then the normal vector is given by $\nabla_x \xi / |\nabla_x \xi|$, or

$$\vec{n} = \frac{1}{\sqrt{1 + |\nabla_\omega \tilde{R}|^2 / (R + \tilde{R})^2}} \left(\vec{e}_r - \frac{\tilde{R}_\theta}{R + \tilde{R}} \vec{e}_\theta - \frac{\tilde{R}_\phi}{(R + \tilde{R}) \sin \theta} \vec{e}_\phi \right),$$

and the mean curvature is then $-\frac{1}{2} \operatorname{div} \frac{\nabla_x \xi}{|\nabla_x \xi|} \Big|_{\xi=0}$, or

$$\begin{aligned} \kappa \Big|_{r=R+\tilde{R}(\theta, \phi)} &= -\frac{1}{2} \operatorname{div} \left(\frac{\nabla_x (\tilde{R} - r)}{|\nabla_x (\tilde{R} - r)|} \right) \Big|_{r=R+\tilde{R}(\theta, \phi)} \\ &= \frac{1}{2} \operatorname{div} \left[\frac{1}{\sqrt{1 + |\nabla_\omega \tilde{R}|^2 / r^2}} \left(\vec{e}_r - \frac{\tilde{R}_\theta}{r} \vec{e}_\theta - \frac{\tilde{R}_\phi}{r \sin \theta} \vec{e}_\phi \right) \right]_{r=R+\tilde{R}(\theta, \phi)}. \end{aligned}$$

By direct computations,

$$\begin{aligned} \operatorname{div} \vec{e}_r &= \frac{1}{r} \vec{e}_\theta \cdot \partial_\theta \vec{e}_r + \frac{1}{r \sin \theta} \vec{e}_\phi \cdot \partial_\phi \vec{e}_r = \frac{2}{r}, \\ \operatorname{div} \vec{e}_\theta &= \frac{1}{r} \vec{e}_\theta \cdot \partial_\theta \vec{e}_\theta + \frac{1}{r \sin \theta} \vec{e}_\phi \cdot \partial_\phi \vec{e}_\theta = \frac{1}{r} \cot \theta, \\ \operatorname{div} \vec{e}_\phi &= \frac{1}{r} \vec{e}_\theta \cdot \partial_\theta \vec{e}_\phi + \frac{1}{r \sin \theta} \vec{e}_\phi \cdot \partial_\phi \vec{e}_\phi = 0. \end{aligned}$$

Using the formulas (2.31) and $\operatorname{div}(f\vec{m}) = f \operatorname{div} \vec{m} + \nabla_x f \cdot \vec{m}$, we obtain

$$\begin{aligned} \kappa \Big|_{r=R+\tilde{R}(\theta,\phi)} &= \left\{ \frac{1}{\sqrt{1+|\nabla_\omega \tilde{R}|^2/r^2}} \left(\frac{1}{r} - \frac{\cot \theta}{2r^2} \tilde{R}_\theta \right) \right. \\ &\quad + \frac{1}{2} \left[\frac{\partial}{\partial r} \left(\frac{1}{\sqrt{1+|\nabla_\omega \tilde{R}|^2/r^2}} \right) - \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{\sqrt{1+|\nabla_\omega \tilde{R}|^2/r^2}} \frac{\tilde{R}_\theta}{r} \right) \right. \\ &\quad \left. \left. - \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left(\frac{1}{\sqrt{1+|\nabla_\omega \tilde{R}|^2/r^2}} \frac{\tilde{R}_\phi}{r \sin \theta} \right) \right] \right\}_{r=R+\tilde{R}(\theta,\phi)} \\ &= \left\{ \frac{1}{2} \frac{\partial}{\partial r} \left(\frac{1}{\sqrt{1+|\nabla_\omega \tilde{R}|^2/r^2}} \right) + \frac{1}{r} \frac{1}{\sqrt{1+|\nabla_\omega \tilde{R}|^2/r^2}} \right. \\ &\quad - \frac{\tilde{R}_\theta}{2r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sqrt{1+|\nabla_\omega \tilde{R}|^2/r^2}} \right) - \frac{\tilde{R}_\phi}{2r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \left(\frac{1}{\sqrt{1+|\nabla_\omega \tilde{R}|^2/r^2}} \right) \\ &\quad \left. - \frac{1}{2r^2 \sqrt{1+|\nabla_\omega \tilde{R}|^2/r^2}} \Delta_\omega \tilde{R} \right\}_{r=R+\tilde{R}(\theta,\phi)} \end{aligned}$$

or

$$\begin{aligned} \kappa \Big|_{r=R+\tilde{R}(\theta,\phi)} &= \left\{ \frac{1}{r} \left(1 + \frac{|\nabla_\omega \tilde{R}|^2}{r^2} \right)^{-3/2} \left[1 + \frac{3}{2} \frac{|\nabla_\omega \tilde{R}|^2}{r^2} \right] \right. \\ &\quad - \frac{\tilde{R}_\theta}{2r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sqrt{1+|\nabla_\omega \tilde{R}|^2/r^2}} \right) \\ &\quad - \frac{\tilde{R}_\phi}{2r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \left(\frac{1}{\sqrt{1+|\nabla_\omega \tilde{R}|^2/r^2}} \right) \\ &\quad \left. - \frac{1}{2r^2 \sqrt{1+|\nabla_\omega \tilde{R}|^2/r^2}} \Delta_\omega \tilde{R} \right\}_{r=R+\tilde{R}(\theta,\phi)}. \end{aligned} \quad (4.4)$$

Taking $\tilde{R} = \varepsilon S(\theta, \phi)$ with small $|\varepsilon|$, we obtain

$$\kappa \Big|_{r=R+\varepsilon S(\theta,\phi)} = \frac{1}{R+\varepsilon S} - \frac{\varepsilon}{2(R+\varepsilon S)^2} \Delta_\omega S + O(\varepsilon^3).$$

Since

$$\begin{aligned} \frac{1}{R+\varepsilon S} &= \frac{1}{R} - \frac{\varepsilon}{R^2} S + \frac{\varepsilon^2}{R^3} S^2 + O(\varepsilon^3), \\ \frac{\varepsilon}{2(R+\varepsilon S)^2} &= \frac{\varepsilon}{2R^2} - \frac{2\varepsilon^2}{R^3} S + O(\varepsilon^2), \end{aligned}$$

(4.3) follows. □

Notice that for $\partial\Omega_\varepsilon : r = R + \varepsilon Y_{2,0}$, we have from (3.16), (3.17),

$$(4.5) \quad \sigma_1(r, \theta) = -\lambda \frac{I_{5/2}(r)}{r^{1/2}} \frac{R^{1/2}}{I_{5/2}(R)} Y_{2,0}(\theta) \quad (\lambda = (\sigma_S)_r(R)),$$

$$(4.6) \quad p_1(r, \theta, \mu) = \left\{ \frac{r^2}{R^2} \left(\frac{2}{R^2} - \mu\lambda \right) + \mu\lambda \frac{I_{5/2}(r)}{r^{1/2}} \frac{R^{1/2}}{I_{5/2}(R)} \right\} Y_{2,0}(\theta).$$

From (4.1), (4.5) we formally derive for σ_2 the equations:

$$(4.7) \quad -\Delta\sigma_2 + \sigma_2 = 0 \quad \text{in } B_R,$$

$$(4.8) \quad \sigma_2 = -\frac{1}{2}(\sigma_S)_{rr}(R) Y_{2,0}^2(\theta) - (\sigma_1)_r(R, \theta) Y_{2,0}(\theta) \quad \text{on } \partial B_R.$$

Taking $S = Y_{2,0}$ in (4.3) and using (2.32), we obtain the formula

$$(4.9) \quad \kappa \Big|_{r=R+\varepsilon Y_{2,0}} = \frac{1}{R} + \varepsilon \frac{2}{R^2} Y_{2,0} - \varepsilon^2 \frac{5}{R^3} Y_{2,0}^2 + O(\varepsilon^3).$$

Using this expansion and (4.2), (4.6), we can also formally derive the equations for p_2 :

$$(4.10) \quad -\Delta p_2 = \mu\sigma_2 \quad \text{in } B_R,$$

$$(4.11) \quad p_2 = -\frac{5}{R^3} Y_{2,0}^2(\theta) - \frac{1}{2}(p_S)_{rr}(R) Y_{2,0}^2(\theta) - (p_1)_r(R, \theta) Y_{2,0}(\theta) \quad \text{on } \partial B_R.$$

We next prove a formula which will be important in the sequel.

Lemma 4.2.

$$(4.12) \quad \frac{1}{2} Y_{2,0} F_{\tilde{R}\tilde{R}}(0, \mu) Y_{2,0} = \left\{ \frac{\partial p_2}{\partial r} \Big|_{r=R} + Y_{2,0} \frac{\partial^2 p_1}{\partial r^2} \Big|_{r=R} + \frac{1}{2} (Y_{2,0})^2 \frac{\partial^3 p_S}{\partial r^3} \Big|_{r=R} - \frac{1}{R^2} \frac{\partial Y_{2,0}}{\partial \theta} \frac{\partial p_1}{\partial \theta} \Big|_{r=R} \right\}.$$

Proof. Since the free boundary $r = R + \varepsilon Y_{2,0}(\theta)$ is independent of ϕ , the function p is also independent of ϕ , and from (4.2) we get

$$\begin{aligned} \frac{\partial p}{\partial r} \Big|_{r=R+\varepsilon Y_{2,0}} &= \frac{\partial p}{\partial r} \Big|_{r=R} + \varepsilon Y_{2,0} \frac{\partial^2 p}{\partial r^2} \Big|_{r=R} + \frac{1}{2} (\varepsilon Y_{2,0})^2 \frac{\partial^3 p}{\partial r^3} \Big|_{r=R} + O(\varepsilon^3) \\ &= \varepsilon \left(\frac{\partial p_1}{\partial r} \Big|_{r=R} + Y_{2,0} \frac{\partial^2 p_S}{\partial r^2} \Big|_{r=R} \right) \\ &\quad + \varepsilon^2 \left(\frac{\partial p_2}{\partial r} \Big|_{r=R} + Y_{2,0} \frac{\partial^2 p_1}{\partial r^2} \Big|_{r=R} + \frac{1}{2} (Y_{2,0})^2 \frac{\partial^3 p_S}{\partial r^3} \Big|_{r=R} \right) + O(\varepsilon^3) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial p}{\partial \theta} \Big|_{r=R+\varepsilon Y_{2,0}} &= \frac{\partial p}{\partial \theta} \Big|_{r=R} + \varepsilon Y_{2,0} \frac{\partial^2 p}{\partial \theta \partial r} \Big|_{r=R} + O(\varepsilon^2) \\ &= \varepsilon \frac{\partial p_1}{\partial \theta} \Big|_{r=R} + O(\varepsilon^2). \end{aligned}$$

Using these results, we can compute

$$\begin{aligned}
 F(\varepsilon Y_{2,0}, \mu) &= \frac{\partial p}{\partial n} \Big|_{r=R+\varepsilon Y_{2,0}} = \nabla p \cdot \vec{n} \\
 &= \left(\vec{e}_r \frac{\partial p}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial p}{\partial \theta} \right) \cdot \left(\vec{e}_r - \varepsilon \frac{1}{R} \frac{\partial Y_{2,0}}{\partial \theta} \vec{e}_\theta + O(\varepsilon^2) \right) \\
 (4.13) \quad &= \varepsilon \left[\frac{\partial p_1}{\partial r} \Big|_{r=R} + Y_{2,0} \frac{\partial^2 p_S}{\partial r^2} \Big|_{r=R} \right] \\
 &\quad + \varepsilon^2 \left\{ \frac{\partial p_2}{\partial r} \Big|_{r=R} + Y_{2,0} \frac{\partial^2 p_1}{\partial r^2} \Big|_{r=R} + \frac{1}{2} (Y_{2,0})^2 \frac{\partial^3 p_S}{\partial r^3} \Big|_{r=R} \right. \\
 &\quad \left. - \frac{1}{R^2} \frac{\partial Y_{2,0}}{\partial \theta} \frac{\partial p_1}{\partial \theta} \Big|_{r=R} \right\} + O(\varepsilon^3).
 \end{aligned}$$

On the other hand, since $F(0, \mu) \equiv 0$, Taylor's expansion gives

$$F(\varepsilon Y_{2,0}, \mu) = \varepsilon F_{\bar{R}}(0, \mu) Y_{2,0} + \varepsilon^2 \frac{1}{2} Y_{2,0} F_{\bar{R}\bar{R}}(0, \mu) Y_{2,0} + O(\varepsilon^3).$$

Comparing the ε^2 terms in this formula with the ε^2 terms in (4.13), we get the relation (4.12). $\square$

At this point we want to rigorously establish the ε^2 -order expansions for σ and p in the special case of $\partial\Omega_\varepsilon : r = R + \varepsilon Y_{2,0}$.

Lemma 4.3. *If*

$$(4.14) \quad -\Delta\sigma + \sigma = 0 \quad \text{in } \Omega_\varepsilon,$$

$$(4.15) \quad \sigma = 1 \quad \text{on } \partial\Omega_\varepsilon,$$

where $\partial\Omega_\varepsilon : r = R + \varepsilon Y_{2,0}$, then

$$(4.16) \quad \|\sigma - [\sigma_S + \varepsilon\sigma_1 + \varepsilon^2\sigma_2]\|_{C^{3+\alpha}(\bar{\Omega}_\varepsilon)} \leq \text{const} \cdot |\varepsilon|^3,$$

where σ_1 is the solution of

$$\begin{aligned}
 -\Delta\sigma_1 + \sigma_1 &= 0 \quad \text{in } \mathbb{R}^3, \\
 \sigma_1 &= -\lambda Y_{2,0}(\theta) \quad \text{on } \partial B_R,
 \end{aligned}$$

given by (4.5) and σ_2 is the solution of (4.7), (4.8), given by

$$\begin{aligned}
 (4.17) \quad \sigma_2 &= \left[3P_0(R) + R^2 P_0(R) P_2(R) - \frac{1}{2} \right] \frac{1}{\sqrt{\pi}} \left[\frac{3}{7} \frac{I_{9/2}(r)}{r^{1/2}} \frac{R^{1/2}}{I_{9/2}(R)} Y_{4,0}(\theta) \right. \\
 &\quad \left. + \frac{\sqrt{5}}{7} \frac{I_{5/2}(r)}{r^{1/2}} \frac{R^{1/2}}{I_{5/2}(R)} Y_{2,0}(\theta) + \frac{1}{2} \frac{I_{1/2}(r)}{r^{1/2}} \frac{R^{1/2}}{I_{1/2}(R)} Y_{0,0} \right].
 \end{aligned}$$

Proof. Note that σ_S , σ_1 and σ_2 are defined (by their explicit formulas) for all $r > 0$, and they satisfy the same elliptic equations $-\Delta u + u = 0$ for all $r > 0$. By (3.36) (with $a_{n,m} = 1$ for $(n, m) = (2, 0)$ and $a_{n,m} = 0$ otherwise) and (2.25),

$$\begin{aligned}
 (4.18) \quad -\frac{\partial \sigma_1(R, \theta)}{\partial r} &= \lambda \left[\frac{2}{R} + R P_2(R) \right] Y_{2,0}(\theta) \\
 &= [2P_0(R) + R^2 P_0(R) P_2(R)] Y_{2,0}(\theta).
 \end{aligned}$$

Using also (2.26), we can rewrite the boundary condition (4.8) as

$$(4.19) \quad \sigma_2 = \left[3P_0(R) + R^2 P_0(R) P_2(R) - \frac{1}{2} \right] Y_{2,0}^2(\theta) \quad \text{on } \partial B_R.$$

Recalling the formula (2.38) for $Y_{2,0}^2$, we can verify that σ_2 satisfies the boundary conditions (4.17) at $r = R$. Since σ_2 also satisfies the elliptic equation (4.7), we deduce, using (2.34), that σ_2 has the form (4.17).

The function

$$\psi = \sigma(r, \theta, \phi) - \{\sigma_S(r) + \varepsilon\sigma_1 + \varepsilon^2\sigma_2\}$$

satisfies the differential equation

$$(4.20) \quad -\Delta\psi + \psi = 0 \quad \text{in } \Omega_\varepsilon.$$

Since also $\|\psi\|_{C^{3+\alpha}(\Sigma)} = O(\varepsilon^3)$ on $\partial\Omega_\varepsilon$, the inequality (4.16) follows by the Schauder estimates. $\square$

Next we establish the second order expansion for p :

Lemma 4.4. *If*

$$(4.21) \quad -\Delta p = \mu(\sigma - \tilde{\sigma}) \quad \text{in } \Omega_\varepsilon,$$

$$(4.22) \quad p = \kappa \quad \text{on } \partial\Omega_\varepsilon,$$

where $\partial\Omega_\varepsilon : r = R + \varepsilon Y_{2,0}$, then

$$(4.23) \quad \|p - [p_S + \varepsilon p_1 + \varepsilon^2 p_2(r, \theta, \phi)]\|_{C^{1+\alpha}(\overline{\Omega}_\varepsilon)} \leq \text{const} \cdot |\varepsilon|^3,$$

where p_1 is given in (4.6), and p_2 is the solution of (4.10)–(4.11), given by

$$(4.24) \quad \begin{aligned} p_2 = & \left(-\frac{9}{R^3} + \frac{3}{2}\mu_2 P_0(R) \right) \frac{1}{\sqrt{\pi}} \left[\frac{3}{7} \frac{r^4}{R^4} Y_{4,0}(\theta) \right. \\ & \left. + \frac{\sqrt{5}}{7} \frac{r^2}{R^2} Y_{2,0}(\theta) + \frac{1}{2} Y_{0,0} \right] - \mu_2 \sigma_2; \end{aligned}$$

here σ_2 is given by (4.17).

Proof. The proof is similar to that of Lemma 4.3. Using the second order expansion for σ and the mean curvature κ , we find that p_2 satisfies (4.10)–(4.11), where the first term of the boundary condition comes from the ε^2 -terms in the expansion of curvature in κ , and the last two terms come from the deformation from $r = R$ to $r = R + \varepsilon Y_{2,0}$.

To solve for p_2 , we observe that

$$\Delta(p_2 + \mu\sigma_2) = 0 \quad \text{in } B_R,$$

and, by (4.11) and (4.8),

$$\begin{aligned} p_2 + \mu\sigma_2 = & -\left(\frac{5}{R^3} + \frac{1}{2}(p_S)_{rr}(R) + \frac{\mu}{2}(\sigma_S)_{rr}(R) \right) Y_{2,0}^2(\theta) \\ & -[(p_1)_r(R, \theta) + \mu(\sigma_1)_r(R, \theta)] Y_{2,0}(\theta) \quad \text{on } \partial B_R. \end{aligned}$$

From (2.26) and (2.29), we have

$$\frac{1}{2}(p_S)_{rr}(R) + \frac{\mu}{2}(\sigma_S)_{rr}(R) = \frac{1}{2}\mu P_0(R),$$

and from (3.37) (with $a_{n,m} = 1$ if $(n, m) = (2, 0)$ and $a_{n,m} = 0$ otherwise) and (2.25),

$$\begin{aligned} (p_1)_r(R, \theta) + \mu(\sigma_1)_r(R, \theta) &= \frac{2}{R} \left[\frac{2}{R^2} - \mu\lambda \right] Y_{2,0}(\theta) \\ &= \left[\frac{4}{R^3} - 2\mu P_0(R) \right] Y_{2,0}(\theta). \end{aligned}$$

Thus we can rewrite the boundary condition for $p_2 + \mu\sigma_2$ as

$$p_2 + \mu\sigma_2 = \left(-\frac{9}{R^3} + \frac{3}{2}\mu P_0(R)\right)Y_{2,0}^2(\theta) \quad \text{on } \partial B_R.$$

Using (2.38) and recalling (2.33), (2.34), we find that

$$p_2 + \mu\sigma_2 = \left(-\frac{9}{R^3} + \frac{3}{2}\mu P_0(R)\right)\frac{1}{\sqrt{\pi}}\left[\frac{3}{7}\frac{r^4}{R^4}Y_{4,0}(\theta) + \frac{\sqrt{5}}{7}\frac{r^2}{R^2}Y_{2,0}(\theta) + \frac{1}{2}Y_{0,0}\right],$$

and the assertion (4.24) then follows by recalling (4.17). Finally, the estimate (4.23) follows from the Schauder estimates. $\square$

Later on we shall need the following result.

Lemma 4.5. *For $\mu = \mu_2$,*

$$(4.25) \quad \left\langle \frac{1}{2}Y_{2,0}F_{\bar{R}\bar{R}}(0, \mu_2)Y_{2,0}, Y_{2,0} \right\rangle = \frac{\sqrt{5}}{7\sqrt{\pi}}\mu_2 RP_0(R) \left\{ -\frac{P_1(R) + P_2(R)}{2} - R^2 P_2^2(R) + R^2 P_1(R)P_2(R) \right\}.$$

Proof. We start by computing the scalar product of $Y_{2,0}$ with each of the four terms on the right-hand side of (4.12). For the second term, we have, by (4.6), (2.11) with $n = 2$, and (2.12),

$$(4.26) \quad Y_{2,0} \frac{\partial^2 p_1}{\partial r^2} \Big|_{r=R} = \left(\frac{4}{R^4} + \mu_2 \lambda [1 - 2P_2(R)] \right) Y_{2,0}^2,$$

so that, using (2.38) and $\langle Y_{n,0}, Y_{m,0} \rangle = \delta_n^m$ ($\delta_n^m = 1$ if $n = m$ and $\delta_n^m = 0$ if $n \neq m$), we obtain

$$(4.27) \quad \left\langle Y_{2,0} \frac{\partial^2 p_1}{\partial r^2} \Big|_{r=R}, Y_{2,0} \right\rangle = \left(\frac{4}{R^4} + \mu_2 RP_0(R)[1 - 2P_2(R)] \right) \frac{\sqrt{5}}{7\sqrt{\pi}}.$$

For the third term, we get, using (2.38) and (2.30),

$$(4.28) \quad \left\langle \frac{1}{2}(Y_{2,0})^2 \frac{\partial^3 p_S}{\partial r^3} \Big|_{r=R}, Y_{2,0} \right\rangle = \frac{\mu_2}{2} RP_0(R)[2P_1(R) - 1] \frac{\sqrt{5}}{7\sqrt{\pi}}.$$

For the fourth term in the braces, we obtain, since $p_1 = (2/R^2)Y_{2,0}(\theta)$ on ∂B_R ,

$$(4.29) \quad -\frac{1}{R^2} \frac{\partial Y_{2,0}}{\partial \theta} \frac{\partial p_1}{\partial \theta} \Big|_{r=R} = -\frac{2}{R^4} \left(\frac{\partial Y_{2,0}}{\partial \theta} \right)^2 \quad (\text{by (4.6)}),$$

and using (2.39) we find that

$$(4.30) \quad \left\langle -\frac{1}{R^2} \frac{\partial Y_{2,0}}{\partial \theta} \frac{\partial p_1}{\partial \theta} \Big|_{r=R}, Y_{2,0} \right\rangle = -\frac{2}{R^4} \frac{3\sqrt{5}}{7\sqrt{\pi}}.$$

The sum of the right-hand sides of (4.27), (4.28), and (4.30) is equal to

$$(4.31) \quad \begin{aligned} &= \frac{\sqrt{5}}{7\sqrt{\pi}} \left\{ -\frac{2}{R^4} + \mu_2 RP_0(R) \left[\frac{1}{2} + P_1(R) - 2P_2(R) \right] \right\} \\ &= \frac{\mu_2 \sqrt{5}}{7\sqrt{\pi}} RP_0(R) \left\{ -\frac{1}{2}[P_1(R) - P_2(R)] + \frac{1}{2} + P_1(R) - 2P_2(R) \right\} \\ &\quad (\text{by (2.24)}) \\ &= \frac{\mu_2 \sqrt{5}}{7\sqrt{\pi}} RP_0(R) \left\{ \frac{1}{2} + \frac{1}{2}P_1(R) - \frac{3}{2}P_2(R) \right\}. \end{aligned}$$

We finally compute the scalar product of $Y_{2,0}$ with the first term on the right-hand side of (4.12). By (4.24),

$$\left. \frac{\partial(p_2 + \mu_2 \sigma_2)}{\partial r} \right|_{r=R} = \left(-\frac{9}{R^3} + \frac{3}{2} \mu_2 P_0(R) \right) \frac{1}{\sqrt{\pi}} \left[\frac{12}{7R} Y_{4,0}(\theta) + \frac{2\sqrt{5}}{7R} Y_{2,0}(\theta) \right].$$

Since, by (4.17) and (2.10), (2.12),

$$\begin{aligned} \mu_2 \left. \frac{\partial \sigma_2}{\partial r} \right|_{r=R} &= \mu_2 \left[3P_0(R) + R^2 P_0(R) P_2(R) - \frac{1}{2} \right] \frac{1}{\sqrt{\pi}} \left[\frac{3}{7} \left(\frac{4}{R} + R P_4(R) \right) Y_{4,0}(\theta) \right. \\ &\quad \left. + \frac{\sqrt{5}}{7} \left(\frac{2}{R} + R P_2(R) \right) Y_{2,0}(\theta) + \frac{1}{2} R P_0(R) Y_{0,0} \right], \end{aligned}$$

we get

$$\begin{aligned} \left. \frac{\partial p_2}{\partial r} \right|_{r=R} &= \frac{\Lambda_1}{\sqrt{\pi}} \left[\frac{12}{7R} Y_{4,0}(\theta) + \frac{2\sqrt{5}}{7R} Y_{2,0}(\theta) \right] \\ &\quad + \frac{\Lambda_2}{\sqrt{\pi}} \left[\frac{3}{7} R P_4(R) Y_{4,0}(\theta) + \frac{\sqrt{5}}{7} R P_2(R) Y_{2,0}(\theta) + \frac{1}{2} R P_0(R) Y_{0,0} \right], \end{aligned}$$

where

$$(4.32) \quad \Lambda_1 = \left\{ -\frac{9}{R^3} - \frac{3}{2} \mu_2 P_0(R) - \mu_2 R^2 P_0(R) P_2(R) + \frac{\mu_2}{2} \right\},$$

$$(4.33) \quad \Lambda_2 = -\mu_2 \left\{ 3P_0(R) + R^2 P_0(R) P_2(R) - \frac{1}{2} \right\}.$$

Hence

$$(4.34) \quad \left\langle \frac{\partial p_2}{\partial r}(R, \theta), Y_{2,0} \right\rangle = \Lambda_1 \frac{2\sqrt{5}}{7R\sqrt{\pi}} + \Lambda_2 \frac{\sqrt{5}}{7\sqrt{\pi}} R P_2(R).$$

Combining (4.34) with (4.31), the scalar product of $Y_{2,0}$ with the left-hand side of (4.12) takes the form:

$$\begin{aligned} (4.35) \quad &\left\langle \frac{1}{2} Y_{2,0} F_{\tilde{R}\tilde{R}}(0, \mu_2) Y_{2,0}, Y_{2,0} \right\rangle \\ &= \frac{\sqrt{5}}{7\sqrt{\pi}} \left\{ \mu_2 R P_0(R) \left[\frac{1}{2} + \frac{1}{2} P_1(R) - \frac{3}{2} P_2(R) \right] + \frac{2}{R} \Lambda_1 + R P_2(R) \Lambda_2 \right\}. \end{aligned}$$

We proceed to compute Λ_1 and Λ_2 . Using (2.24), (2.13) with $n = 0$, we obtain

$$\begin{aligned} \Lambda_1 &= -\mu_2 \frac{9}{4} R^2 P_0(R) [P_1(R) - P_2(R)] - \mu_2 R^2 P_0(R) P_2(R) + \frac{\mu_2}{2} [1 - 3P_0(R)] \\ &= \mu_2 R^2 P_0(R) \left(-\frac{9}{4} [P_1(R) - P_2(R)] - P_2(R) + \frac{1}{2} P_1(R) \right) \\ &= \mu_2 R^2 P_0(R) \left(-\frac{7}{4} P_1(R) + \frac{5}{4} P_2(R) \right), \end{aligned}$$

so that

$$\frac{2}{R} \Lambda_1 = \mu_2 R P_0(R) \left(-\frac{7}{2} P_1(R) + \frac{5}{2} P_2(R) \right).$$

Similarly, using (2.13) with $n = 0$, we get

$$\Lambda_2 = -\mu_2 \left\{ \frac{3}{2} P_0(R) + R^2 P_0(R) P_2(R) - \frac{R^2}{2} P_0(R) P_1(R) \right\}.$$

Thus, the right-hand side of (4.35) is equal to

$$\begin{aligned}
& \frac{\sqrt{5}}{7\sqrt{\pi}}\mu_2 R P_0(R) \left\{ \left[\frac{1}{2} + \frac{1}{2}P_1(R) - \frac{3}{2}P_2(R) \right] + \left(-\frac{7}{2}P_1(R) + \frac{5}{2}P_2(R) \right) \right. \\
& \quad \left. - \left\{ \frac{3}{2}P_2(R) + R^2 P_2^2(R) - \frac{R^2}{2}P_2(R)P_1(R) \right\} \right\} \\
&= \frac{\sqrt{5}}{7\sqrt{\pi}}\mu_2 R P_0(R) \left\{ \frac{1}{2} - 3P_1(R) - \frac{1}{2}P_2(R) - R^2 P_2^2(R) + \frac{1}{2}R^2 P_1(R)P_2(R) \right\} \\
&= \frac{\sqrt{5}}{7\sqrt{\pi}}\mu_2 R P_0(R) \left\{ -\frac{1}{2}P_1(R) - \frac{1}{2}P_2(R) - R^2 P_2^2(R) + R^2 P_1(R)P_2(R) \right\},
\end{aligned}$$

where we used (2.13) with $n = 1$ in deriving the last equality. $\square$

Lemma 4.6. For $R > 0$,

$$(4.36) \quad -\frac{1}{2}P_1(R) - \frac{1}{2}P_2(R) - R^2 P_2^2(R) + R^2 P_1(R)P_2(R) < 0.$$

Proof. Using (2.15) we find that the function $f(r) \equiv r^2[P_1(r) - P_2(r)] - 1$ satisfies

$$\begin{aligned}
& \frac{df(r)}{dr} + \left(\frac{3}{r} + r[P_1(r) + P_2(r)] \right) f(r) \\
&= 2r[P_1(r) - P_2(r)] + r^2 \left\{ -\frac{5}{r}P_1(r) + \frac{7}{r}P_2(r) - r[P_1^2(r) - P_2^2(r)] \right\} \\
& \quad + \left(\frac{3}{r} + r[P_1(r) + P_2(r)] \right) f(r) \\
&= -\frac{3}{r} - r[P_1(r) - P_2(r)] < -\frac{3}{r} \quad (\text{by (2.18)}).
\end{aligned}$$

Since clearly also

$$f(r) < 0 \quad \text{for small } r > 0,$$

it follows that $f(r) < 0$ for all $0 < r < \infty$. In particular,

$$(4.37) \quad R^2[P_1(R) - P_2(R)] < 1,$$

so that

$$(4.38) \quad 2R^2 P_2(R)[P_1(R) - P_2(R)] < 2P_2(R) < P_1(R) + P_2(R),$$

which implies (4.36). $\square$

We are now ready to establish the main theorem of this section:

Theorem 4.7. For the bifurcation branch $(\tilde{R}(\varepsilon), \mu(\varepsilon))$ emanating from $\mu = \mu_2$,

$$\begin{aligned}
(4.39) \quad \frac{d\mu}{d\varepsilon} \Big|_{\varepsilon=0} &= \mu_2^2 \frac{\sqrt{5}}{28\sqrt{\pi}} R^4 P_0(R) \left\{ -\frac{1}{2}P_1(R) - \frac{1}{2}P_2(R) \right. \\
& \quad \left. - R^2 P_2^2(R) + R^2 P_1(R)P_2(R) \right\} < 0.
\end{aligned}$$

Proof. Along the bifurcation branch emanating from $\mu = \mu_2$, we have $r = R + \tilde{R}$, where

$$\tilde{R} = \varepsilon Y_{2,0}(\theta) + \varepsilon^2 z_2(\theta) + O(\varepsilon^3)$$

and $F(\tilde{R}(\varepsilon), \mu(\varepsilon)) = 0$.

By Taylor's expansion,

$$(4.40) \quad 0 = F(0, \mu(\varepsilon)) + F_{\tilde{R}}(0, \mu(\varepsilon))\tilde{R} + \frac{1}{2}\tilde{R}F_{\tilde{R}\tilde{R}}(0, \mu(\varepsilon))\tilde{R} + O(\varepsilon^3).$$

The first term of (4.40) on the right-hand side vanishes since $F(0, \mu) = 0$ for any μ . To compute the second term on the right-hand side, we write $\mu(\varepsilon) = \mu_2 + \varepsilon\mu'(0) + O(\varepsilon^2)$, so that

$$\begin{aligned} F_{\tilde{R}}(0, \mu(\varepsilon))\tilde{R} &= F_{\tilde{R}}(0, \mu_2)\tilde{R} + \varepsilon\mu'(0)F_{\tilde{R}\mu}(0, \mu_2)\tilde{R} + O(\varepsilon^3) \\ &= \varepsilon F_{\tilde{R}}(0, \mu_2)Y_{2,0} + \varepsilon^2 F_{\tilde{R}}(0, \mu_2)z_2 + \varepsilon^2\mu'(0)F_{\tilde{R}\mu}(0, \mu_2)Y_{2,0} + O(\varepsilon^3) \\ &= \varepsilon^2 F_{\tilde{R}}(0, \mu_2)z_2 + \varepsilon^2\mu'(0)F_{\tilde{R}\mu}(0, \mu_2)Y_{2,0} + O(\varepsilon^3), \end{aligned}$$

where we used the fact that $Y_{2,0} \in \ker F_{\tilde{R}}(0, \mu_2)$.

The third term on the right-hand side of (4.40) is clearly equal to

$$\varepsilon^2 \frac{1}{2} Y_{2,0} F_{\tilde{R}\tilde{R}}(0, \mu_2) Y_{2,0} + O(\varepsilon^3).$$

Collecting ε^2 -terms in (4.40), we obtain

$$(4.41) \quad F_{\tilde{R}}(0, \mu_2)z_2 + \mu'(0)F_{\tilde{R}\mu}(0, \mu_2)Y_{2,0} + \frac{1}{2}Y_{2,0}F_{\tilde{R}\tilde{R}}(0, \mu_2)Y_{2,0} = 0.$$

By (3.42) with $S = Y_{2,0}$ we further have

$$(4.42) \quad F_{\tilde{R}\mu}(0, \mu_2)Y_{2,0} = -R^2 P_0(R) [P_1(R) - P_2(R)] Y_{2,0}.$$

Note, by (3.39) (with $\mu = \mu_2$ so that the coefficient of $Y_{2,0}$ vanishes in (3.39)), $F_{\tilde{R}}(0, \mu_2)z_2$ is orthogonal to $Y_{2,0}$. Hence if we take the scalar product of the left-hand side of (4.41) with $Y_{2,0}$ and use (4.42), we obtain

$$(4.43) \quad \mu'(0)R^2 P_0(R) [P_1(R) - P_2(R)] = \left\langle \frac{1}{2} Y_{2,0} F_{\tilde{R}\tilde{R}}(0, \mu_2) Y_{2,0}, Y_{2,0} \right\rangle.$$

Using (2.24) with $n = 2$ and recalling (4.25), the assertion (4.39) easily follows. $\square$

5. THE LINEARIZED PROBLEM

We denote the stationary solution of the bifurcation branch emanating from μ_2 by $(\sigma(\varepsilon), p(\varepsilon), \mu(\varepsilon), \partial\Omega(\varepsilon))$, so that

$$(5.1) \quad \begin{aligned} \partial\Omega(\varepsilon) : r &= R_S + \tilde{R}(\varepsilon)(\theta), \\ \sigma(\varepsilon)(r, \theta) &= \sigma_S(r) + \varepsilon\sigma_1(r, \theta) + O(\varepsilon^2), \\ p(\varepsilon)(r, \theta) &= p_S(r, \mu_2) + \varepsilon p_1(r, \theta) + O(\varepsilon^2), \end{aligned}$$

and

$$(5.2) \quad \tilde{R}(\varepsilon)(\theta) = \varepsilon Y_{2,0}(\theta) + O(\varepsilon^2),$$

$$(5.3) \quad \mu(\varepsilon) = \mu_2 + \mu'(0)\varepsilon + O(\varepsilon^2),$$

where $\mu'(0)$ is given by (4.39). Note that since now $\mu = \mu(\varepsilon)$ is a function of ε , the p_1 here is different from the p_1 in Section 4. For clarity we shall denote the p_1 from Section 4 by $p_1(r, \theta, \mu_2)$, so that the present p_1 is given by

$$(5.4) \quad p_1(r, \theta) = p_1(r, \theta, \mu_2) + \mu'(0) \frac{\partial p_S}{\partial \mu}.$$

By (4.5), (4.6),

$$(5.5) \quad \sigma_1(r, \theta) = -\lambda \frac{I_{5/2}(r)}{r^{1/2}} \frac{R^{1/2}}{I_{5/2}(R)} Y_{2,0}(\theta) \quad (\lambda = (\sigma_S)_r(R)),$$

$$(5.6) \quad \begin{aligned} p_1(r, \theta) &= p_1(r, \theta, \mu_2) + \mu'(0) \frac{\partial p_S}{\partial \mu} \\ &= \left\{ \frac{r^2}{R^2} \left(\frac{2}{R^2} - \mu_2 \lambda \right) + \mu_2 \lambda \frac{I_{5/2}(r)}{r^{1/2}} \frac{R^{1/2}}{I_{5/2}(R)} \right\} Y_{2,0}(\theta) \\ &\quad + \mu'(0) \left\{ -\sigma_S(r) + \frac{1}{6} \tilde{\sigma} r^2 \right\}. \end{aligned}$$

We also have

$$F(\tilde{R}(\varepsilon), \mu(\varepsilon)) = 0 \text{ for small } |\varepsilon|.$$

We now fix ε , with $|\varepsilon|$ small. As in [12, Section 3], we can derive the linearized system near the stationary solution $(\sigma(\varepsilon), p(\varepsilon), \mu(\varepsilon), \partial\Omega(\varepsilon))$ by substituting

$$\begin{aligned} \sigma(r, \theta, \phi, t) &= \sigma(\varepsilon)(r, \theta) + \eta w(r, \theta, \phi, t), \\ p(r, \theta, \phi, t) &= p(\varepsilon)(r, \theta) + \eta q(r, \theta, \phi, t), \\ \partial\Omega(\varepsilon) : r &= R + \tilde{R}(\varepsilon)(\theta) + \eta \rho(\theta, \phi, t) \end{aligned}$$

into the system (1.1)–(1.6) and collecting the terms of order η . Before writing down the linearized system, let us examine the linearization of the boundary condition (1.5). It is easily seen that

$$(5.7) \quad V_n = \eta \left(1 + \frac{|\nabla_\omega \tilde{R}|^2}{(R + \tilde{R})^2} \right)^{-1/2} \rho_t + O(\eta^2).$$

We also have

$$(5.8) \quad \begin{aligned} \frac{\partial p}{\partial n} \Big|_{r=R+\tilde{R}+\eta\rho} &= \eta \rho \frac{\partial}{\partial r} \nabla_x p(\varepsilon)(r, \theta) \cdot \vec{n} \Big|_{r=R+\tilde{R}} + \eta \frac{\partial q}{\partial n} \Big|_{r=R+\tilde{R}} \\ &\quad - \eta \left(1 + \frac{|\tilde{R}_\theta|^2}{(R + \tilde{R})^2} \right)^{-1/2} \frac{\rho_\theta}{(R + \tilde{R})^2} \frac{\partial p(\varepsilon)}{\partial \theta} \Big|_{r=R+\tilde{R}} + O(\eta^2). \end{aligned}$$

To prove (5.8) we compute

$$\begin{aligned} &\left(1 + \frac{|\nabla_\omega(\tilde{R} + \eta\rho)|^2}{(R + \tilde{R} + \eta\rho)^2} \right)^{-1/2} \\ &= \left(1 + \frac{|\nabla_\omega \tilde{R}|^2}{(R + \tilde{R})^2} + \frac{2\eta \nabla_\omega \tilde{R} \nabla_\omega \rho}{(R + \tilde{R})^2} - \frac{2\eta \rho |\nabla_\omega \tilde{R}|^2}{(R + \tilde{R})^3} + O(\eta^2) \right)^{-1/2} \\ &= \left(1 + \frac{|\nabla_\omega \tilde{R}|^2}{(R + \tilde{R})^2} \right)^{-1/2} \\ &\quad - \eta \left(1 + \frac{|\nabla_\omega \tilde{R}|^2}{(R + \tilde{R})^2} \right)^{-3/2} \left[\frac{\nabla_\omega \tilde{R} \nabla_\omega \rho}{(R + \tilde{R})^2} - \frac{\rho |\nabla_\omega \tilde{R}|^2}{(R + \tilde{R})^3} \right] + O(\eta^2). \end{aligned}$$

Since $\tilde{R}$ is independent of ϕ ,

$$\begin{aligned}
& \vec{n} \Big|_{r=R+\tilde{R}+\eta\rho} \\
&= \left(1 + \frac{|\nabla_\omega(\tilde{R} + \eta\rho)|^2}{(R + \tilde{R} + \eta\rho)^2}\right)^{-1/2} \left[\vec{e}_r - \frac{\tilde{R}_\theta + \eta\rho_\theta}{R + \tilde{R} + \eta\rho} \vec{e}_\theta - \frac{\eta\rho_\phi}{(R + \tilde{R} + \eta\rho) \sin \theta} \vec{e}_\phi \right] \\
&= \vec{n} \Big|_{r=R+\tilde{R}} - \eta \left(1 + \frac{|\tilde{R}_\theta|^2}{(R + \tilde{R})^2}\right)^{-3/2} \\
&\quad \times \left[\frac{\tilde{R}_\theta \rho_\theta}{(R + \tilde{R})^2} - \frac{\rho |\tilde{R}_\theta|^2}{(R + \tilde{R})^3} \right] \left(\vec{e}_r - \frac{\tilde{R}_\theta}{R + \tilde{R}} \vec{e}_\theta \right) \\
&\quad - \eta \left(1 + \frac{|\tilde{R}_\theta|^2}{(R + \tilde{R})^2}\right)^{-1/2} \left(\frac{\rho_\theta}{R + \tilde{R}} \vec{e}_\theta + \frac{\rho_\phi}{(R + \tilde{R}) \sin \theta} \vec{e}_\phi \right) + O(\eta^2).
\end{aligned}$$

Next, from the equation

$$\begin{aligned}
0 &= F(\tilde{R}(\varepsilon), \mu(\varepsilon)) = \nabla_x p(\varepsilon)(r, \theta) \cdot \vec{n} \Big|_{r=R+\tilde{R}} \\
&= \left(1 + \frac{|\tilde{R}_\theta|^2}{(R + \tilde{R})^2}\right)^{-1/2} \left[\frac{\partial}{\partial r} p(\varepsilon)(r, \theta) - \frac{\tilde{R}_\theta}{(R + \tilde{R})^2} \frac{\partial}{\partial \theta} p(\varepsilon)(r, \theta) \right]_{r=R+\tilde{R}},
\end{aligned}$$

we deduce that $\nabla_x p(\varepsilon)(r, \theta) \cdot \left(\vec{e}_r - \frac{\tilde{R}_\theta}{(R + \tilde{R})^2} \vec{e}_\theta \right) = 0$ on $r = R + \tilde{R}$, so that

$$\begin{aligned}
& \nabla_x p(\varepsilon)(r, \theta) \Big|_{r=R+\tilde{R}} \cdot \left(\vec{n} \Big|_{r=R+\tilde{R}+\eta\rho} - \vec{n} \Big|_{r=R+\tilde{R}} \right) \\
&= -\eta \left(1 + \frac{|\tilde{R}_\theta|^2}{(R + \tilde{R})^2}\right)^{-1/2} \frac{\rho_\theta}{R + \tilde{R}} \frac{\partial p(\varepsilon)}{\partial \theta} \Big|_{r=R+\tilde{R}} + O(\eta^2).
\end{aligned}$$

From this expression for $\nabla_x p$ and the expression derived earlier for $\vec{n}$ we get

$$\begin{aligned}
(5.9) \quad \frac{\partial p}{\partial n} \Big|_{r=R+\tilde{R}+\eta\rho} &= \nabla_x \left[p(\varepsilon)(r, \theta) + \eta q \right] \cdot \vec{n} \Big|_{r=R+\tilde{R}+\eta\rho} \\
&= \nabla_x p(\varepsilon)(r, \theta) \Big|_{r=R+\tilde{R}} + \eta \rho \frac{\partial}{\partial r} \nabla_x p(\varepsilon)(r, \theta) \Big|_{r=R+\tilde{R}},
\end{aligned}$$

from which (5.8) follows.

The following system is the linearization of (1.1)–(1.6) about $(\sigma(\varepsilon), p(\varepsilon), \Omega(\varepsilon))$:

$$(5.10) \quad w_t - \Delta w + w = 0 \quad \text{in } \Omega(\varepsilon) \times \{t > 0\},$$

$$(5.11) \quad w(R + \tilde{R}(\varepsilon)(\theta), \theta, \phi, t) = -\frac{\partial \sigma(\varepsilon)}{\partial r} \Big|_{r=R+\tilde{R}(\varepsilon)(\theta)} \rho, \quad t > 0,$$

$$(5.12) \quad \Delta q = -\mu(\varepsilon)w \quad \text{in } \Omega(\varepsilon) \times \{t > 0\},$$

$$(5.13) \quad q = \frac{\partial \kappa}{\partial \tilde{R}} \Big|_{r=R+\tilde{R}(\varepsilon)} \rho \quad \text{on } \partial\Omega(\varepsilon), \quad t > 0,$$

$$\begin{aligned}
(5.14) \quad \rho_t &= -\left(1 + \frac{(\tilde{R}(\varepsilon)_\theta)^2}{(R + \tilde{R}(\varepsilon))^2}\right)^{1/2} \left\{ \rho \frac{\partial}{\partial r} \nabla_x p(\varepsilon)(r, \theta) \cdot \vec{n} \Big|_{r=R+\tilde{R}(\varepsilon)} \right. \\
&\quad \left. + \frac{\partial q}{\partial n} \Big|_{r=R+\tilde{R}(\varepsilon)} \right\} + \frac{\rho_\theta}{(R + \tilde{R}(\varepsilon))^2} \frac{\partial}{\partial \theta} p(\varepsilon) \Big|_{r=R+\tilde{R}(\varepsilon)},
\end{aligned}$$

where (5.14) follows from (5.7), (5.8) and $\frac{\partial \kappa}{\partial \tilde{R}} \Big|_{r=R+\tilde{R}} \rho$ is the Fréchet derivative of the curvature with respect to the boundary deformation in the direction ρ at $r = R + \tilde{R}$.

We next compute more explicitly the expressions $\frac{\partial \kappa}{\partial \tilde{R}}$, $\frac{\partial}{\partial r} \nabla_x p \cdot \vec{n}$ and $\frac{\partial q}{\partial n}$ which appear in (5.14). To compute $\frac{\partial \kappa}{\partial \tilde{R}} \Big|_{r=R+\tilde{R}(\varepsilon)} \rho$, we use (4.3) with $\varepsilon S = \tilde{R}(\varepsilon) + \eta \rho$ and $|\eta| \ll |\varepsilon|$ (notice also that $|\tilde{R}(\varepsilon)| \leq \text{const} \cdot |\varepsilon|$) to obtain

$$\begin{aligned} & \kappa \Big|_{r=R+\tilde{R}+\eta \rho} - \kappa \Big|_{r=R+\tilde{R}} \\ &= -\frac{\eta}{R^2} \left(\rho + \frac{1}{2} \Delta_\omega \rho \right) + \frac{\eta}{R^3} \left(2\tilde{R} \rho + \tilde{R} \Delta_\omega \rho + \rho \Delta_\omega \tilde{R} \right) + O(\varepsilon^2 \eta) + O(\eta^2), \end{aligned}$$

where $O(\varepsilon^2 \eta)$ comes from the computation used to prove (4.3). Recalling that $\tilde{R} = \varepsilon Y_{2,0} + O(\varepsilon^2)$, we get

$$(5.15) \quad \frac{\partial \kappa}{\partial \tilde{R}} \Big|_{r=R+\tilde{R}(\varepsilon)} \rho = -\frac{1}{R^2} \left(\rho + \frac{1}{2} \Delta_\omega \rho \right) + \varepsilon \frac{Y_{2,0}}{R^3} \left(-4\rho + \Delta_\omega \rho \right) + O(\varepsilon^2).$$

To compute $\frac{\partial}{\partial r} \nabla_x p \cdot \vec{n}$, note that

$$\begin{aligned} (5.16) \quad \vec{n} \Big|_{r=R+\tilde{R}} &= \left(1 + \frac{|\tilde{R}_\theta|^2}{(R+\tilde{R})^2} \right)^{-1/2} \left[\vec{e}_r - \frac{\tilde{R}_\theta}{R+\tilde{R}} \vec{e}_\theta \right] \\ &= \vec{e}_r - \frac{\varepsilon}{R} \frac{\partial Y_{2,0}}{\partial \theta} \vec{e}_\theta + O(\varepsilon^2). \end{aligned}$$

Hence

$$\begin{aligned} (5.17) \quad & \frac{\partial}{\partial r} \nabla_x p(\varepsilon)(r, \theta) \cdot \vec{n} \Big|_{r=R+\tilde{R}(\varepsilon)} \\ &= \left(\frac{\partial \nabla_x p(\varepsilon)}{\partial r} \Big|_{r=R} + \varepsilon Y_{2,0} \frac{\partial^2 \nabla_x p(\varepsilon)}{\partial r^2} \Big|_{r=R} \right) \cdot \left(\vec{e}_r - \frac{\varepsilon}{R} \frac{\partial Y_{2,0}}{\partial \theta} \vec{e}_\theta \right) + O(\varepsilon^2) \\ &= \frac{\partial^2 p_S(R)}{\partial r^2} + \varepsilon \left(\frac{\partial^2 p_1(R, \theta)}{\partial r^2} + \frac{\partial^3 p_S(R)}{\partial r^3} Y_{2,0} \right) + O(\varepsilon^2). \end{aligned}$$

Finally

$$\begin{aligned} (5.18) \quad & \frac{\partial q}{\partial n} \Big|_{r=R+\tilde{R}(\varepsilon)} = \nabla_x q \cdot \vec{n} \Big|_{r=R+\tilde{R}(\varepsilon)} \\ &= \left(\nabla_x q \Big|_{r=R} + \varepsilon Y_{2,0} \frac{\partial \nabla_x q}{\partial r} \Big|_{r=R} \right) \cdot \left(\vec{e}_r - \frac{\varepsilon}{R} \frac{\partial Y_{2,0}}{\partial \theta} \vec{e}_\theta \right) + O(\varepsilon^2) \\ &= \frac{\partial q}{\partial r} \Big|_{r=R} + \varepsilon \frac{\partial^2 q}{\partial r^2} Y_{2,0} \Big|_{r=R} - \frac{\varepsilon}{R^2} \frac{\partial Y_{2,0}}{\partial \theta} \frac{\partial q}{\partial \theta} \Big|_{r=R} + O(\varepsilon^2). \end{aligned}$$

We would like to establish linear stability or instability for the stationary solution $(\sigma(\varepsilon), p(\varepsilon), \partial \Omega(\varepsilon))$. The difficulty in doing so arises from the fact that this stationary solution has only been determined up to order ε . One could probably extend the solution as an analytic function in ε , as in [16], but it is difficult to compute the coefficients of ε^n for large n . We therefore restrict ourselves to the system (5.10)–(5.14) after we drop out all the $O(\varepsilon^2)$ -terms, that is, to the system obtained from (5.10)–(5.14) by keeping only the terms of orders ε^0 and ε^1 . (Such an approximation

was also done in [11] for a different free boundary problem.) Thus we wish to consider the system

$$(5.19) \quad w_t - \Delta w + w = 0 \quad \text{in } B_R \times \{t > 0\},$$

$$(5.20) \quad \left[w + \varepsilon Y_{2,0} \frac{\partial w}{\partial r} \right]_{r=R} = - \left[\frac{\partial \sigma_S}{\partial r} + \varepsilon \frac{\partial \sigma_1}{\partial r} + \varepsilon Y_{2,0} \frac{\partial^2 \sigma_S}{\partial r^2} \right]_{r=R} \rho, \quad t > 0,$$

$$(5.21) \quad \Delta q = -[\mu_2 + \varepsilon \mu'(0)]w \quad \text{in } B_R \times \{t > 0\},$$

$$(5.22) \quad \left[q + \varepsilon Y_{2,0} \frac{\partial q}{\partial r} \right]_{r=R} = -\frac{1}{R^2} \left(\rho + \frac{1}{2} \Delta_\omega \rho \right) + \varepsilon \frac{Y_{2,0}}{R^3} \left(-4\rho + \Delta_\omega \rho \right) \quad \text{on } \partial B_R, \quad t > 0,$$

$$(5.23) \quad \rho_t = - \left\{ \frac{\partial^2 p_S(R)}{\partial r^2} + \varepsilon \left(\frac{\partial^2 p_1(R, \theta)}{\partial r^2} + \frac{\partial^3 p_S(R)}{\partial r^3} Y_{2,0} \right) \right\} \rho - \frac{\partial q}{\partial r} \Big|_{r=R} - \varepsilon Y_{2,0} \frac{\partial^2 q}{\partial r^2} \Big|_{r=R} + \varepsilon \frac{1}{R^2} \frac{\partial Y_{2,0}}{\partial \theta} \frac{\partial q}{\partial \theta} \Big|_{r=R} + \varepsilon \frac{1}{R^2} \frac{\partial \rho}{\partial \theta} \frac{\partial p_1}{\partial \theta} \Big|_{r=R}.$$

We want to further simplify this system by breaking it into two coupled subsystems and dropping the $O(\varepsilon^2)$ terms that occur in this process. The first subsystem (for (w_1, q_1)) corresponds to the zero order terms in (5.19)–(5.23) and the second subsystem (for (w_2, q_2)) corresponds to the ε -order terms, and

$$\begin{aligned} w &= w_1 + \varepsilon w_2, \\ q &= q_1 + \varepsilon q_2. \end{aligned}$$

The boundary conditions for w_1 and w_2 take the form

$$\begin{aligned} w_1 \Big|_{r=R} &= -\frac{\partial \sigma_S(R)}{\partial r} \rho, \\ \left[w_2 + \varepsilon Y_{2,0} \frac{\partial w_2}{\partial r} \right]_{r=R} &= - \left[\frac{\partial \sigma_1(R, \theta)}{\partial r} + \frac{\partial^2 \sigma_S(R)}{\partial r^2} Y_{2,0} \right] \rho - \frac{\partial w_1(R, \theta, t)}{\partial r} Y_{2,0}. \end{aligned}$$

Since $w = w_1 + \varepsilon w_2$, $\varepsilon Y_{2,0} \frac{\partial w_2}{\partial r}$ will contribute an $O(\varepsilon^2)$ term to the solution w and, as before, we drop such a term. Similarly we can obtain boundary conditions for q_1, q_2 and drop the $O(\varepsilon^2)$ terms from the boundary conditions; we also drop the $O(\varepsilon^2)$ terms from the differential equations for q_1 and q_2 .

We then obtain the following system:

$$(5.24) \quad (w_1)_t - \Delta w_1 + w_1 = 0 \quad \text{in } B_R \times \{t > 0\},$$

$$(5.25) \quad (w_2)_t - \Delta w_2 + w_2 = 0 \quad \text{in } B_R \times \{t > 0\},$$

$$(5.26) \quad w_1 = -\frac{\partial \sigma_S(R)}{\partial r} \rho \quad \text{on } \partial B_R \times \{t > 0\},$$

$$(5.27) \quad w_2 = - \left[\frac{\partial \sigma_1(R, \theta)}{\partial r} + \frac{\partial^2 \sigma_S(R)}{\partial r^2} Y_{2,0} \right] \rho - \frac{\partial w_1(R, \theta, t)}{\partial r} Y_{2,0} \quad \text{on } \partial B_R \times \{t > 0\},$$

$$(5.28) \quad \Delta q_1 = -\mu_2 w_1 \quad \text{in } B_R \times \{t > 0\},$$

$$(5.29) \quad \Delta q_2 = -\mu_2 w_2 - \mu'(0)w_1 \quad \text{in } B_R \times \{t > 0\},$$

$$(5.30) \quad q_1 = -\frac{1}{R^2} \left(\rho + \frac{1}{2} \Delta_\omega \rho \right) \quad \text{on } \partial B_R \times \{t > 0\},$$

$$(5.31) \quad q_2 = \frac{Y_{2,0}}{R^3} \left(-4\rho + \Delta_\omega \rho \right) - \frac{\partial q_1(R, \theta, t)}{\partial r} Y_{2,0} \quad \text{on } \partial B_R \times \{t > 0\},$$

$$(5.32) \quad \rho_t = - \left\{ \frac{\partial^2 p_S(R)}{\partial r^2} + \varepsilon \left(\frac{\partial^2 p_1(R, \theta)}{\partial r^2} + \frac{\partial^3 p_S(R)}{\partial r^3} Y_{2,0} \right) \right\} \rho \\ - \frac{\partial q_1}{\partial r} \Big|_{r=R} - \varepsilon \left[\frac{\partial q_2}{\partial r} + \frac{\partial^2 q_1}{\partial r^2} Y_{2,0} \right]_{r=R} \\ + \varepsilon \frac{1}{R^2} \frac{\partial Y_{2,0}}{\partial \theta} \frac{\partial q_1}{\partial \theta} \Big|_{r=R} + \varepsilon \frac{1}{R^2} \frac{\partial \rho}{\partial \theta} \frac{\partial p_1}{\partial \theta} \Big|_{r=R}.$$

This is the final form of the ε -order part of the linearized problem (5.10)–(5.14), and it is this system that we intend to consider for stability or instability. This system may also be viewed as the linearized problem of the original problem obtained by taking $\eta = \varepsilon^2$ and discarding all the $O(\eta)$ terms.

Until now we have taken the perturbation functions around the stationary solution to be functions of (r, θ, ϕ) . In the sequel we shall focus only on functions of (r, θ) . The method we use can be extended to the perturbation by functions of (r, θ, ϕ) ; however the computation becomes very messy and shall not be pursued here.

The system (5.24)–(5.32) is complemented with the initial conditions, which we shall take to be independent of ϕ :

$$(5.33) \quad w_1 \Big|_{t=0} = w^{1,0}(r, \theta) = \sum_{n=0}^{\infty} w_{n,0}^{1,0}(r) Y_{n,0}(\theta),$$

$$(5.34) \quad w_2 \Big|_{t=0} = 0,$$

$$(5.35) \quad \rho \Big|_{t=0} = \rho_0(\theta) = \sum_{n=0}^{\infty} \rho_{n,0}^0 Y_{n,0}(\theta),$$

where $w_{n,0}^{1,0}(r)$ and $\rho_{n,0}^0$ are given. Then w_i , q_i and ρ are also independent of ϕ , and, in particular,

$$(5.36) \quad \rho = \sum_{n=0}^{\infty} \rho_{n,0}(t) Y_{n,0}(\theta);$$

the $\rho_{n,0}(t)$ are functions depending also on ε .

6. THE LAPLACE TRANSFORM OF THE LINEARIZED PROBLEM

We shall denote the Laplace transform of a function $\Phi(t)$ by $\widehat{\Phi}(s)$, that is,

$$\widehat{\Phi}(s) = \int_0^{\infty} e^{-st} \Phi(t) dt.$$

In order to prove stability or instability, for the linearized problem (5.24)–(5.35), we proceed as in [12] by first formally taking the Laplace transform of the system

(5.24)–(5.32). We write

$$(6.1) \quad \rho = \sum_{n=0}^{\infty} \rho_{n,0}(t) Y_{n,0}(\theta),$$

$$(6.2) \quad w_j = \sum_{n=0}^{\infty} w_{n,0}^j(r, t) Y_{n,0}(\theta) \quad (j = 1, 2),$$

$$(6.3) \quad q_j = \sum_{n=0}^{\infty} q_{n,0}^j(r, t) Y_{n,0}(\theta) \quad (j = 1, 2),$$

and introduce the Laplace transforms:

$$(6.4) \quad \hat{\rho} = \sum_{n=0}^{\infty} \hat{\rho}_{n,0}(s) Y_{n,0}(\theta),$$

$$(6.5) \quad \hat{w}_j = \sum_{n=0}^{\infty} \hat{w}_{n,0}^j(r, s) Y_{n,0}(\theta) \quad (j = 1, 2),$$

$$(6.6) \quad \hat{q}_j = \sum_{n=0}^{\infty} \hat{q}_{n,0}^j(r, s) Y_{n,0}(\theta) \quad (j = 1, 2).$$

Introducing the function

$$(6.7) \quad h_n(s) \equiv h_n(s, \mu_2, R) = \frac{1}{\mu_2} \left[\frac{s}{R^2 P_0(R)} + \frac{n}{R^5 P_0(R)} \left(\frac{n(n+1)}{2} - 1 \right) \right] \\ - P_1(R) + P_n(R\sqrt{s+1})$$

as in [12, (3.26)], we shall prove that the $\hat{\rho}_{n,0}(s)$ satisfy a system of equations

$$(6.8) \quad h_n(s) \hat{\rho}_{n,0}(s) + \varepsilon \sum_{j=n-2}^{n+2} B_{j,n}(s) \hat{\rho}_{j,0}(s) = F_n(s) + \varepsilon G_n(s),$$

where, by convention, $B_{j,n} = 0$ if either j or n is negative. We shall compute $B_{j,n}$ in terms of the coefficients of the system (5.24)–(5.32), and F_n , G_n in terms of the initial data $w_{n,0}^{1,0}$, $\rho_{n,0}^0$. We perform the computation in four steps.

Step (i). We first evaluate the coefficients which involve the stationary solution. From (5.6),

$$\frac{\partial^2 p_1(R, \theta)}{\partial r^2} = \frac{\partial^2 p_1(R, \theta, \mu_2)}{\partial r^2} + \mu'(0) \left(-\frac{\partial^2 \sigma_S(R)}{\partial r^2} + \frac{\tilde{\sigma}}{3} \right);$$

the first term on the right-hand side is computed by (4.26), and the second term by (2.26) and (2.23). Using also (2.13) with $n = 0$, we obtain

$$(6.9) \quad \frac{\partial^2 p_1(R, \theta)}{\partial r^2} = \left(\frac{4}{R^4} + \mu_2 \lambda [1 - 2P_2(R)] \right) Y_{2,0} - \mu'(0) R^2 P_0(R) P_1(R).$$

From (4.18) and the boundary condition $p_1 = (2/R^2) Y_{2,0}(\theta)$ on ∂B_R (see (4.6)), we obtain

$$(6.10) \quad \frac{\partial \sigma_1(R, \theta)}{\partial r} = -\lambda \left(\frac{2}{R} + R P_2(R) \right) Y_{2,0},$$

$$(6.11) \quad \frac{\partial p_1(R, \theta)}{\partial \theta} = \frac{2}{R^2} \frac{\partial Y_{2,0}}{\partial \theta}.$$

By (2.30), (6.9) and (2.24),

$$\begin{aligned} & \frac{\partial^2 p_1(R, \theta)}{\partial r^2} + \frac{\partial^3 p_S(R)}{\partial r^3} Y_{2,0} \\ &= \left(\frac{4}{R^4} + 2\mu_2 R P_0(R) [P_1(R) - P_2(R)] \right) Y_{2,0} - \mu'(0) R^2 P_0(R) P_1(R) \\ &= \frac{12}{R^4} Y_{2,0} - \mu'(0) R^2 P_0(R) P_1(R); \end{aligned}$$

therefore, by Lemma 2.2,

$$\begin{aligned} \left(\frac{\partial^2 p_1(R, \theta)}{\partial r^2} + \frac{\partial^3 p_S(R)}{\partial r^3} Y_{2,0} \right) \rho &= \frac{12}{R^4} \sum_{j=0}^{\infty} \left[\sum_{n=j-2}^{j+2} c_{nj} \rho_{n,0}(t) \right] Y_{j,0} \\ &\quad - \mu'(0) R^2 P_0(R) P_1(R) \sum_{n=0}^{\infty} \rho_{n,0}(t) Y_{n,0}. \end{aligned}$$

This implies, after switching the indices n and j , that

$$\begin{aligned} (6.12) \quad & \left\langle \left(\frac{\partial^2 p_1(R, \theta)}{\partial r^2} + \frac{\partial^3 p_S(R)}{\partial r^3} Y_{2,0} \right) \hat{\rho}, Y_{n,0} \right\rangle \\ &= \frac{12}{R^4} \sum_{j=n-2}^{n+2} c_{jn} \hat{\rho}_{j,0}(s) - \mu'(0) R^2 P_0(R) P_1(R) \hat{\rho}_{n,0}(s). \end{aligned}$$

Similarly, by (5.30),

$$\begin{aligned} \frac{(Y_{2,0})_{\theta}}{R^2} \frac{\partial q_1(R, \theta)}{\partial \theta} &= \frac{1}{R^4} \sum_{n=0}^{\infty} \left(\frac{n(n+1)}{2} - 1 \right) \rho_{n,0}(t) (Y_{n,0})_{\theta} (Y_{2,0})_{\theta} \\ &= \frac{1}{R^4} \sum_{n=0}^{\infty} \left(\frac{n(n+1)}{2} - 1 \right) \rho_{n,0}(t) \left(\sum_{j=0}^{n+2} d_{nj} Y_{j,0} \right) \\ &= \frac{1}{R^4} \sum_{n=0}^{\infty} \left[\sum_{j=n-2}^{n+2} \left(\frac{j(j+1)}{2} - 1 \right) d_{jn} \rho_{j,0}(t) \right] Y_{n,0}, \end{aligned}$$

so that

$$(6.13) \quad \left\langle \frac{(Y_{2,0})_{\theta}}{R^2} \frac{\partial q_1(R, \theta)}{\partial \theta}, Y_{n,0} \right\rangle = \frac{1}{R^4} \sum_{j=n-2}^{n+2} \left(\frac{j(j+1)}{2} - 1 \right) d_{jn} \rho_{j,0}(t).$$

Next, by (6.11),

$$\begin{aligned} \frac{\rho_{\theta}}{R^2} \frac{\partial p_1(R, \theta)}{\partial \theta} &= \frac{2}{R^4} \sum_{n=0}^{\infty} \rho_{n,0}(t) (Y_{n,0})_{\theta} (Y_{2,0})_{\theta} \\ &= \frac{2}{R^4} \sum_{n=0}^{\infty} \rho_{n,0}(t) \left(\sum_{j=0}^{n+2} d_{nj} Y_{j,0} \right) \\ &= \frac{2}{R^4} \sum_{n=0}^{\infty} \left[\sum_{j=n-2}^{n+2} d_{jn} \rho_{j,0}(t) \right] Y_{n,0}, \end{aligned}$$

and therefore

$$(6.14) \quad \left\langle \frac{\hat{\rho}_{\theta}}{R^2} \frac{\partial p_1(R, \theta)}{\partial \theta}, Y_{n,0} \right\rangle = \frac{2}{R^4} \sum_{j=n-2}^{n+2} d_{jn} \hat{\rho}_{j,0}(s).$$

Step (ii). We next compute $\widehat{w}_1$ and $\widehat{q}_1$ in the same way as in [12]. As in [12],

$$(6.15) \quad -\Delta \widehat{w}_{n,0}^1 + \left(\frac{n(n+1)}{r^2} + (s+1) \right) \widehat{w}_{n,0}^1 = w_{n,0}^1(r, 0) \quad \text{in } B_R,$$

$$(6.16) \quad \widehat{w}_{n,0}^1(R, s) = -\lambda \widehat{\rho}_{n,0}(s),$$

so that, by [12, (3.17)],

$$(6.17) \quad \widehat{w}_{n,0}^1 = -\lambda \widehat{\rho}_{n,0}(s) \frac{I_{n+1/2}(r\sqrt{s+1})}{r^{1/2}} \frac{R^{1/2}}{I_{n+1/2}(R\sqrt{s+1})} + \xi_{1n}^1,$$

where ξ_{1n}^1 satisfies

$$(6.18) \quad \begin{aligned} -\Delta \xi_{1n}^1 + \left(\frac{n(n+1)}{r^2} + (s+1) \right) \xi_{1n}^1 &= w_{n,0}^1(r, 0) \quad \text{in } B_R, \\ \xi_{1n}^1(R, s) &= 0. \end{aligned}$$

Also, by [12, (3.20)],

$$(6.19) \quad \begin{aligned} \widehat{q}_{n,0}^1 &= \frac{r^n}{R^n} \left[\frac{1}{R^2} \left(\frac{n(n+1)}{2} - 1 \right) - \frac{\mu_2 \lambda}{s+1} \right] \widehat{\rho}_{n,0} - \frac{\mu_2}{s+1} \widehat{w}_{n,0} + \psi_{1n} \\ &= \left\{ \frac{r^n}{R^n} \left[\frac{1}{R^2} \left(\frac{n(n+1)}{2} - 1 \right) - \frac{\mu_2 \lambda}{s+1} \right] \right. \\ &\quad \left. + \frac{\mu_2 \lambda}{s+1} \frac{I_{n+1/2}(r\sqrt{s+1})}{r^{1/2}} \frac{R^{1/2}}{I_{n+1/2}(R\sqrt{s+1})} \right\} \widehat{\rho}_{n,0} \\ &\quad - \frac{\mu_2}{s+1} \xi_{1n}^1 + \psi_{1n}^1, \end{aligned}$$

where ψ_{1n}^1 is the solution of

$$(6.20) \quad \begin{aligned} -\Delta \psi_{1n}^1 + \frac{n(n+1)}{r^2} \psi_{1n}^1 &= \frac{\mu_2}{s+1} w_{n,0}^1(r, 0) \quad \text{in } B_R, \\ \psi_{1n}^1|_{r=R} &= 0. \end{aligned}$$

Using (2.10), (2.11) and (2.12) in (6.19), we get

$$(6.21) \quad \begin{aligned} \frac{\partial \widehat{q}_{n,0}^1}{\partial r} \Big|_{r=R} &= \left\{ \frac{n}{R^3} \left(\frac{n(n+1)}{2} - 1 \right) + \mu_2 \lambda R P_n(R\sqrt{s+1}) \right\} \widehat{\rho}_{n,0} \\ &\quad + \frac{d}{dr} \left[-\frac{\mu_2}{s+1} \xi_{1n}^1 + \psi_{1n}^1 \right]_{r=R}, \end{aligned}$$

$$(6.22) \quad \begin{aligned} \frac{\partial^2 \widehat{q}_{n,0}^1}{\partial r^2} \Big|_{r=R} &= \left\{ \frac{n(n-1)}{R^4} \left(\frac{n(n+1)}{2} - 1 \right) - 2\mu_2 \lambda P_n(R\sqrt{s+1}) \right. \\ &\quad \left. + \mu_2 \lambda \right\} \widehat{\rho}_{n,0} + \frac{d^2}{dr^2} \left[-\frac{\mu_2}{s+1} \xi_{1n}^1 + \psi_{1n}^1 \right]_{r=R}. \end{aligned}$$

Since

$$\frac{\partial^2 \widehat{q}_1}{\partial r^2} \Big|_{r=R} Y_{2,0} = \sum_{n=0}^{\infty} \frac{\partial^2 \widehat{q}_{n,0}^1}{\partial r^2} \Big|_{r=R} Y_{n,0} Y_{2,0} = \sum_{n=0}^{\infty} \left[\sum_{j=0}^{n+2} c_{nj} \frac{\partial^2 \widehat{q}_{n,0}^1}{\partial r^2} \Big|_{r=R} \right] Y_{j,0},$$

(6.22) implies (after switching the indices n and j) that

$$\begin{aligned}
 \left\langle \frac{\partial^2 \widehat{q}_1}{\partial r^2} \right|_{r=R} Y_{2,0}, Y_{n,0} \rangle &= \sum_{j=n-2}^{n+2} c_{jn} \frac{\partial^2 \widehat{q}_{j,0}^1}{\partial r^2} \Big|_{r=R} \\
 (6.23) \quad &= \sum_{j=n-2}^{n+2} \left\{ \frac{j(j-1)}{R^4} \left(\frac{j(j+1)}{2} - 1 \right) - 2\mu_2 \lambda P_j(R\sqrt{s+1}) \right. \\
 &\quad \left. + \mu_2 \lambda \right\} c_{jn} \widehat{\rho}_{j,0} + \sum_{j=n-2}^{n+2} c_{jn} \frac{d^2}{dr^2} \left[-\frac{\mu_2}{s+1} \xi_{1j}^1 + \psi_{1j}^1 \right]_{r=R}.
 \end{aligned}$$

Step (iii). The only remaining coefficient in (5.32) that is yet to be computed is $\partial q_2(R, \theta, t)/\partial r$.

We start with the computation of w_2 . Since, by (2.26) and (6.10),

$$\frac{\partial \sigma_1(R, \theta)}{\partial r} + \frac{\partial^2 \sigma_S(R)}{\partial r^2} Y_{2,0} = [1 - 4P_0(R) - R^2 P_0(R) P_2(R)] Y_{2,0},$$

the boundary condition (5.27) can be written as

$$\begin{aligned}
 w_2 \Big|_{r=R} &= - \left[\frac{\partial \sigma_1(R, \theta)}{\partial r} + \frac{\partial^2 \sigma_S(R)}{\partial r^2} Y_{2,0} \right] \left(\sum_{n=0}^{\infty} \rho_{n,0}(t) Y_{n,0} \right) \\
 &\quad - \left(\sum_{n=0}^{\infty} \frac{\partial w_{n,0}^1(R, t)}{\partial r} Y_{n,0} \right) Y_{2,0} \\
 (6.24) \quad &= \sum_{n=0}^{\infty} \left\{ - [1 - 4P_0(R) - R^2 P_0(R) P_2(R)] \rho_{n,0}(t) Y_{n,0} \right. \\
 &\quad \left. - \frac{\partial w_{n,0}^1(R, t)}{\partial r} Y_{n,0} \right\} Y_{2,0}.
 \end{aligned}$$

If we rewrite the product $Y_{2,0} Y_{n,0}$ as a linear combination of $Y_{j,0}(\theta)$, by (2.43), then we get

$$\begin{aligned}
 w_2 \Big|_{r=R} &= \sum_{j=0}^{\infty} \sum_n c_{nj} \left\{ - [1 - 4P_0(R) - R^2 P_0(R) P_2(R)] \rho_{n,0}(t) \right. \\
 &\quad \left. - \frac{\partial w_{n,0}^1(R, t)}{\partial r} \right\} Y_{j,0}.
 \end{aligned}$$

Taking the Laplace transform and noting, by (6.17) and (2.10), that

$$(6.25) \quad \frac{\partial \widehat{w}_{n,0}^1}{\partial r} \Big|_{r=R} = -\lambda \widehat{\rho}_{n,0} \left(\frac{n}{R} + (s+1) R P_n(R\sqrt{s+1}) \right) + \frac{d}{dr} \xi_{1n}^1 \Big|_{r=R},$$

we obtain

$$\begin{aligned}
 \widehat{w}_2 \Big|_{r=R} &= \sum_{j=0}^{\infty} \sum_{n=j-2}^{j+2} c_{nj} \left\{ - [1 - 4P_0(R) - R^2 P_0(R) P_2(R)] \widehat{\rho}_{n,0} \right. \\
 &\quad \left. + \lambda \widehat{\rho}_{n,0} \left(\frac{n}{R} + (s+1) R P_n(R\sqrt{s+1}) \right) - \frac{d}{dr} \xi_{1n}^1 \Big|_{r=R} \right\} Y_{j,0}.
 \end{aligned}$$

After switching the summation indices j and n , we get

$$(6.26) \quad \begin{aligned} \widehat{w}_{n,0}^2(R, s) = & \sum_{j=n-2}^{n+2} c_{jn} \left\{ -[1 - 4P_0(R) - R^2 P_0(R) P_2(R)] \right. \\ & \left. + \lambda \left(\frac{j}{R} + (s+1) R P_j(R\sqrt{s+1}) \right) \right\} \widehat{\rho}_{j,0} - \sum_{j=n-2}^{n+2} c_{jn} \frac{d}{dr} \xi_{1j}^1 \Big|_{r=R}. \end{aligned}$$

Since $\widehat{w}^2$ is a solution of $s\widehat{w}^2 - \Delta\widehat{w}^2 + \widehat{w}^2 = 0$ (here we use (5.34)), its n -mode has the form

$$(6.27) \quad \widehat{w}_{n,0}^2(r, s) = \widehat{w}_{n,0}^2(R, s) \frac{I_{n+1/2}(r\sqrt{s+1})}{r^{1/2}} \frac{R^{1/2}}{I_{n+1/2}(R\sqrt{s+1})},$$

so that

$$(6.28) \quad \frac{\partial \widehat{w}_{n,0}^2(R, s)}{\partial r} = \widehat{w}_{n,0}^2(R, s) \left(\frac{n}{R} + (s+1) R P_n(R\sqrt{s+1}) \right).$$

We next compute the boundary condition for $\widehat{q}_2$. By (5.31) and (6.21),

$$(6.29) \quad \begin{aligned} \widehat{q}_2 \Big|_{r=R} = & \left(\sum_{n=0}^{\infty} \frac{-4 - n(n+1)}{R^3} \widehat{\rho}_{n,0}(s) Y_{n,0} \right) Y_{2,0} - \frac{\partial \widehat{q}_1(R, \theta, s)}{\partial r} Y_{2,0} \\ = & \left\{ \sum_{n=0}^{\infty} \left[-\frac{n}{R^3} \left(\frac{n(n+1)}{2} - 1 \right) - \mu_2 \lambda R P_n(R\sqrt{s+1}) \right. \right. \\ & \left. \left. + \frac{-4 - n(n+1)}{R^3} \right] \widehat{\rho}_{n,0}(s) Y_{n,0} \right. \\ & \left. - \sum_{n=0}^{\infty} \frac{d}{dr} \left[\frac{-\mu_2}{s+1} \xi_{1n}^1 + \psi_{1n}^1 \right]_{r=R} Y_{n,0} \right\} Y_{2,0} \\ = & \sum_{n=0}^{\infty} \sum_{j=0}^{n+2} \left\{ \left[-\frac{n}{R^3} \left(\frac{n(n+1)}{2} - 1 \right) - \mu_2 \lambda R P_n(R\sqrt{s+1}) \right. \right. \\ & \left. \left. + \frac{-4 - n(n+1)}{R^3} \right] \widehat{\rho}_{n,0}(s) c_{nj} - \frac{d}{dr} \left[\frac{-\mu_2}{s+1} \xi_{1n}^1 + \psi_{1n}^1 \right]_{r=R} c_{nj} \right\} Y_{j,0}, \end{aligned}$$

so that (after switching n and j)

$$(6.30) \quad \begin{aligned} \widehat{q}_{n,0}^2(R, s) = & \sum_{j=n-2}^{n+2} \left\{ -\frac{j}{R^3} \left(\frac{j(j+1)}{2} - 1 \right) - \mu_2 \lambda R P_j(R\sqrt{s+1}) \right. \\ & \left. + \frac{-4 - j(j+1)}{R^3} \right\} c_{jn} \widehat{\rho}_{j,0} - \sum_{j=n-2}^{n+2} \frac{d}{dr} \left[\frac{-\mu_2}{s+1} \xi_{1j}^1 + \psi_{1j}^1 \right]_{r=R} c_{jn}. \end{aligned}$$

In order to compute $\widehat{q}_{n,0}^2(r, s)$ we introduce the function $\psi = \widehat{q}_{n,0}^2 + \frac{\mu_2}{s+1} \widehat{w}_{n,0}^2 + \frac{\mu'(0)}{s+1} \widehat{w}_{n,0}^1$. Since

$$(6.31) \quad -\Delta\psi + \frac{n(n+1)}{r^2} \psi = \frac{\mu'(0)}{s+1} w_{n,0}^1(r, 0) \quad \text{in } B_R,$$

we can write

$$(6.32) \quad \psi(r, s) = \frac{r^n}{R^n} \psi(R, s) + \psi_{1n}^2$$

where

$$(6.33) \quad \begin{aligned} -\Delta \psi_{1n}^2 + \frac{n(n+1)}{r^2} \psi_{1n}^2 &= \frac{\mu'(0)}{s+1} w_{n,0}^1(r, 0) \quad \text{in } B_R, \\ \psi_{1n}^2|_{r=R} &= 0. \end{aligned}$$

Hence

$$\begin{aligned} \hat{q}_{n,0}^2(r, s) &= \frac{r^n}{R^n} \left(\hat{q}_{n,0}^2(R, s) + \frac{\mu_2}{s+1} \hat{w}_{n,0}^2(R, s) + \frac{\mu'(0)}{s+1} \hat{w}_{n,0}^1(R, s) \right) \\ &\quad - \frac{\mu_2}{s+1} \hat{w}_{n,0}^2(r, s) - \frac{\mu'(0)}{s+1} \hat{w}_{n,0}^1(r, s) + \psi_{1n}^2. \end{aligned}$$

Using (6.25), (6.28) and (5.26) (according to which $\hat{w}_{n,0}^1 = -\lambda \hat{\rho}_{n,0}$) we then get

$$(6.34) \quad \begin{aligned} \frac{\partial \hat{q}_{n,0}^2(R, s)}{\partial r} &= \frac{n}{R} \hat{q}_{n,0}^2(R, s) - \mu_2 \hat{w}_{n,0}^2(R, s) R P_n(R\sqrt{s+1}) \\ &\quad + \mu'(0) \lambda \hat{\rho}_{n,0} R P_n(R\sqrt{s+1}) + \frac{d}{dr} \left[\psi_{1n}^2 - \frac{\mu'(0)}{s+1} \frac{d}{dr} \xi_{1n}^1 \right]_{r=R}. \end{aligned}$$

Substituting (6.26) and (6.30) into (6.34), we find that

$$(6.35) \quad \frac{\partial \hat{q}_{n,0}^2(R, s)}{\partial r} = \sum_{j=n-2}^{n+2} k_{n,0}^{2,j}(s) c_{jn} \hat{\rho}_{j,0} + \mu'(0) \lambda \hat{\rho}_{n,0} R P_n(R\sqrt{s+1}) + g_{n,0}^2(s),$$

where

$$(6.36) \quad \begin{aligned} k_{n,0}^{2,j}(s) &= \left\{ \frac{n}{R} \left[-\frac{j}{R^3} \left(\frac{j(j+1)}{2} - 1 \right) - \mu_2 \lambda R P_j(R\sqrt{s+1}) \right. \right. \\ &\quad \left. \left. + \frac{-4-j(j+1)}{R^3} \right] \right. \\ &\quad \left. - \mu_2 R P_n(R\sqrt{s+1}) \left[-[1 - 4P_0(R) - R^2 P_0(R) P_2(R)] \right. \right. \\ &\quad \left. \left. + \lambda \left(\frac{j}{R} + (s+1) R P_j(R\sqrt{s+1}) \right) \right] \right\} \end{aligned}$$

and

$$(6.37) \quad \begin{aligned} g_{n,0}^2(s) &= -\frac{n}{R} \sum_{j=n-2}^{n+2} \frac{d}{dr} \left[\frac{-\mu_2}{s+1} \xi_{1j}^1 + \psi_{1j}^1 \right]_{r=R} c_{jn} \\ &\quad + \mu_2 R P_n(R\sqrt{s+1}) \sum_{j=n-2}^{n+2} c_{jn} \frac{d}{dr} \xi_{1j}^1 \Big|_{r=R} \\ &\quad + \frac{d}{dr} \left[\psi_{1n}^2 - \frac{\mu'(0)}{s+1} \frac{d}{dr} \xi_{1n}^1 \right]_{r=R}. \end{aligned}$$

It is clear that $g_{n,0}^2(s)$ depends only on the initial data.

Step (iv). Taking the Laplace transform in (5.32) we get

$$(6.38) \quad \begin{aligned} 0 &= \left\{ s \hat{\rho}_{n,0}(s) - \rho_{n,0}(0) + \frac{\partial^2 p_S(R)}{\partial r^2} \hat{\rho}_{n,0} + \frac{\partial \hat{q}_{n,0}^1}{\partial r} \Big|_{r=R} \right\} \\ &\quad + \varepsilon \left\langle \left(\frac{\partial^2 p_1(R, \theta)}{\partial r^2} + \frac{\partial^3 p_S(R)}{\partial r^3} Y_{2,0} \right) \hat{\rho}, Y_{n,0} \right\rangle \\ &\quad + \varepsilon \left\langle \frac{\partial \hat{q}_1(R, \theta)}{\partial \theta} \frac{(Y_{2,0})_\theta}{R^2}, Y_{n,0} \right\rangle + \varepsilon \left\langle \frac{\partial p_1(R, \theta)}{\partial \theta} \frac{\hat{\rho}_\theta}{R^2}, Y_{n,0} \right\rangle \\ &\quad + \varepsilon \left\langle \frac{\partial^2 \hat{q}_1}{\partial r^2} \Big|_{r=R} Y_{2,0}, Y_{n,0} \right\rangle + \varepsilon \frac{\partial \hat{q}_{n,0}^2}{\partial r} \Big|_{r=R}. \end{aligned}$$

The last two terms in the braces were computed in (2.29) and (6.21); they are the same terms as in [12, (3.24)]. Recalling the definition of $h_n(s, \mu_2, R)$ in (6.7), the expression in braces in (6.38) takes the form

$$\mu_2 R^2 P_0(R) h_n(s, \mu_2, R) \widehat{\rho}_{n,0} - \rho_{n,0}(0) - \frac{d}{dr} \left[\frac{\mu_2}{s+1} \xi_{1n}^1 - \psi_{1n}^1 \right]_{r=R}.$$

The second term on the right-hand side of (6.38) was computed in (6.12), the third term in (6.13), the fourth term in (6.14), the fifth term in (6.23), and the sixth term in (6.35). Combining these results and comparing with (6.8), we obtain the following:

Lemma 6.1. *The equation (6.8) holds with*

$$\begin{aligned} \mu_2 R^2 P_0(R) B_{j,n}(s) &= \delta_n^j \mu'(0) R^2 P_0(R) [P_n(R\sqrt{s+1}) - P_1(R)] \\ &\quad + \left\{ \frac{12}{R^4} c_{jn} + \frac{1}{R^4} \left(\frac{j(j+1)}{2} - 1 \right) d_{jn} \right. \\ &\quad + \frac{2}{R^4} d_{jn} + k_{n,0}^{2,j}(s) c_{jn} + \left[\frac{j(j-1)}{R^4} \left(\frac{j(j+1)}{2} - 1 \right) \right. \\ &\quad \left. \left. - 2\mu_2 \lambda P_j(R\sqrt{s+1}) + \mu_2 \lambda \right] c_{jn} \right\} \\ &\quad (\delta_n^j = 1 \text{ if } n = j, \delta_n^j = 0 \text{ if } n \neq j), \end{aligned} \tag{6.39}$$

$$(6.40) \quad \mu_2 R^2 P_0(R) F_n(s) = \rho_{n,0}(0) + \frac{d}{dr} \left[\frac{\mu_2}{s+1} \xi_{1n}^1 - \psi_{1n}^1 \right]_{r=R},$$

$$(6.41) \quad \mu_2 R^2 P_0(R) G_n(s) = -g_{n,0}^2(s) - \sum_{j=n-2}^{n+2} c_{jn} \frac{d^2}{dr^2} \left[-\frac{\mu_2}{s+1} \xi_{1j}^1 + \psi_{1j}^1 \right]_{r=R}.$$

7. ESTIMATES OF $B_{2,2}(0)$, $B_{n,j}(s)$, $F_n(s)$ AND $G_n(s)$

Lemma 7.1.

$$(7.1) \quad B_{2,2}(0) > 0.$$

Proof. From (6.36) we have

$$\begin{aligned} &k_{2,0}^{2,2}(0) \\ &= -\frac{18}{R^4} - 2\mu_2 R P_0(R) P_2(R) \\ &\quad + \mu_2 R P_2(R) \left\{ [1 - 4P_0(R) - R^2 P_0(R) P_2(R)] - [2P_0(R) + R^2 P_0(R) P_2(R)] \right\} \\ &= -\frac{9}{2} \mu_2 R P_0(R) [P_1(R) - P_2(R)] \\ &\quad + \mu_2 R P_2(R) [1 - 6P_0(R) - 2R^2 P_0(R) P_2(R)] \quad (\text{by (2.24)}) \\ &= -\frac{9}{2} \mu_2 R P_0(R) [P_1(R) - P_2(R)] \\ &\quad + \mu_2 R P_2(R) [R^2 P_0(R) P_1(R) - 3P_0(R) - 2R^2 P_0(R) P_2(R)] \quad (\text{by (2.13)}) \\ &= \mu_2 R P_0(R) \left\{ -\frac{9}{2} [P_1(R) - P_2(R)] - 3P_2(R) + R^2 P_1(R) P_2(R) - 2R^2 P_2^2(R) \right\}. \end{aligned}$$

Since $d_{22} = 3c_{22}$,

$$\begin{aligned}
 & \mu_2 R^2 P_0(R) B_{2,2}(0) \\
 &= \mu'(0) R^2 P_0(R) [P_2(R) - P_1(R)] + k_{2,0}^{2,2}(0) c_{22} \\
 & \quad + \left\{ \frac{28}{R^4} + \mu_2 R P_0(R) [1 - 2P_2(R)] \right\} c_{22} \\
 (7.2) \quad &= \mu'(0) R^2 P_0(R) [P_2(R) - P_1(R)] + k_{2,0}^{2,2}(0) c_{22} \\
 & \quad + \mu_2 R P_0(R) \left\{ 7[P_1(R) - P_2(R)] + 1 - 2P_2(R) \right\} c_{22} \quad (\text{by (2.24)}) \\
 &= \mu'(0) R^2 P_0(R) [P_2(R) - P_1(R)] + \mu_2 R P_0(R) c_{22} \left\{ \frac{5}{2} [P_1(R) - P_2(R)] \right. \\
 & \quad \left. + 1 - 5P_2(R) + R^2 P_1(R) P_2(R) - 2R^2 P_2^2(R) \right\}.
 \end{aligned}$$

Since $\mu'(0) < 0$ (by Theorem 4.7) and $P_1(R) > P_2(R)$ (by (2.18)), we have

$$(7.3) \quad \mu'(0) R^2 P_0(R) [P_2(R) - P_1(R)] > 0.$$

Hence if we prove that the quantity in the braces on the right-hand side of (7.2) is positive, then (7.1) will follow from (7.2). But this is indeed the case since, by (2.14), (2.18) and (2.19),

the quantity in the braces of (7.2)

$$\begin{aligned}
 &= 1 - 5P_1(R) + \frac{15}{2} [P_1(R) - P_2(R)] + R^2 P_1(R) P_2(R) - 2R^2 P_2^2(R) \\
 &= \frac{15}{2} [P_1(R) - P_2(R)] + 2R^2 P_1(R) P_2(R) - 2R^2 P_2^2(R) \quad (\text{by (2.13)}) \\
 &= \frac{15}{2} [P_1(R) - P_2(R)] + 2R^2 P_2(R) [P_1(R) - P_2(R)] > 0.
 \end{aligned}$$

This completes the proof. $\square$

Lemma 7.2.

$$(7.4) \quad B_{j,n}(s) = 0 \quad \text{if } j + n = \text{odd}.$$

Proof. The proof is immediate from (6.39) and (2.45). $\square$

From (7.4) it follows that odd and even terms in (6.8) actually decouple.

Lemma 7.3. *If*

$$(7.5) \quad \begin{aligned}
 & w_1 \Big|_{t=0} \equiv w_2 \Big|_{t=0} \equiv 0, \\
 & \rho_{n,0} \Big|_{t=0} \equiv 0 \quad \text{for } n \neq 2, \quad \rho_{2,0} \Big|_{t=0} = \mu_2 R^2 P_0(R),
 \end{aligned}$$

then (6.8) reduces to

$$(7.6) \quad h_n(s) \hat{\rho}_{n,0}(s) = \delta_n^2 - \varepsilon \sum_{j=n-2}^{n+2} B_{j,n}(s) \hat{\rho}_{j,0}(s), \quad n = 0, 2, 4, \dots,$$

where $\hat{\rho}_{n,0}(s) = 0$ for n odd.

Proof. Under the assumptions (7.5), $\xi_{1n}^1 = \psi_{1n}^1 = \psi_{1n}^2 = 0$ and $g_{n,2} = 0$, so that, $G_n(s) = 0$ by (6.41) and $F_n(s) = \delta_n^2$ by (6.40). $\square$

Lemma 7.4.

$$(7.7) \quad |B_{j,n}(s)| \leq C(n+1)^3 \quad \text{for } \operatorname{Re} s > -\delta_1,$$

for some positive constant δ_1 .

Proof. By Lemma 2.2, $|c_{nj}| \leq 1$ and $|d_{nj}| \leq n+1$, and $B_{j,n}(s) = 0$ if $|j-n| \geq 3$ or if $j+n$ is odd. From (6.36) we see that the factor $(s+1)$ in $k_{n,0}^{2,j}$ appears only in $(s+1)P_n(R\sqrt{s+1})P_j(R\sqrt{s+1})$ and is therefore also bounded, by (2.16) and (2.22).

Hence

$$\left| k_{n,0}^{2,j}(s) + \frac{n}{R^4} \frac{j^2(j+1)}{2} \right| \leq C(n+1)(j+1)^2.$$

From (6.39) we also have

$$\left| \mu_2 R^2 P_0(R) B_{j,n}(s) - \left(k_{n,0}^{2,j}(s) + \frac{j(j-1)}{R^4} \frac{j(j+1)}{2} \right) c_{jn} \right| \leq C(j+1)^3.$$

If $c_{jn} \neq 0$, then $|n-j| \leq 2$ so that

$$\left| \left(-\frac{n}{R^4} \frac{j^2(j+1)}{2} + \frac{j(j-1)}{R^4} \frac{j(j+1)}{2} \right) c_{jn} \right| = \frac{j^2(j+1)|j-n+1|}{R^4} \leq \frac{C(n+1)^3}{R^4},$$

and combining this with the preceding two estimates, (7.7) follows. $\square$

For $n \neq 2$, we can divide the equation (6.8) by h_n and take the inverse Laplace transform in (6.8) to obtain

$$(7.8) \quad \rho_{n,0}(t) + \varepsilon \sum_{j=n-2}^{n+2} (b_{j,n} * \rho_{j,0})(t) = f_n(t),$$

where $b_{j,n}(t)$ is the inverse Laplace transform of $B_{j,n}(s)/h_n(s)$ and $f_n(t)$ is the inverse Laplace transform of $\{F_n(s) + \varepsilon G_n(s)\}/h_n(s)$.

For $n = 2$, we write (6.8) as

$$\{h_2(s) + \varepsilon B_{2,2}(s)\} \widehat{\rho}_{2,0}(s) + \varepsilon \{B_{0,2}(s) \widehat{\rho}_{0,0}(s) + B_{4,2}(s) \widehat{\rho}_{4,0}(s)\} = F_2(s) + \varepsilon G_2(s).$$

If we divide by $h_2(s) + \varepsilon B_{2,2}(s)$ and then take the inverse Laplace transform, we obtain

$$(7.9) \quad \rho_{2,0}(t) + \varepsilon [(b_{0,2} * \rho_{0,0})(t) + (b_{4,2} * \rho_{4,0})(t)] = f_2(t),$$

where $b_{j,2}(t)$ is the inverse Laplace transform of $B_{j,2}(s)/[h_2(s) + \varepsilon B_{2,2}(s)]$ and $f_2(t)$ is the inverse Laplace transform of $\{F_n(s) + \varepsilon G_n(s)\}/[h_2(s) + \varepsilon B_{2,2}(s)]$.

Lemma 7.5. *If the functions in (5.33), (5.35) satisfy the regularity conditions $w^{1,0} \in H^4(B_R)$, $\rho_0 \in H^{7/2}(\Sigma)$ and $\|w^{1,0}\|_{H^4(B_R)} + \|\rho_0\|_{H^{7/2}(\Sigma)}$ is small, then the following inequalities hold: For $n \neq 2$:*

$$(7.10) \quad |b_{j,n}(t)| \leq C_0 \left\{ (n+1)^3 e^{-\delta_1(n^3+1)t} + (n+1)^2 e^{-\delta_1(n^2+1)t} \right\},$$

$$(7.11) \quad |f_n(t)| \leq (n+1)^{-3} e^{-\delta_1 t},$$

and for $n = 2$, if $\varepsilon > 0$, then

$$(7.12) \quad |b_{j,2}(t)| \leq C_0 e^{-2\delta_1|\varepsilon|t}, \quad j = 0, 4,$$

$$(7.13) \quad |f_2(t)| \leq e^{-2\delta_1|\varepsilon|t},$$

for some $\delta_1 > 0$. If we drop the assumption that $\|w^{1,0}\|_{H^4(B_R)} + \|\rho_0\|_{H^{7/2}(\Sigma)}$ is small, then we need to add a constant factor in (7.11), (7.13).

If we drop the assumption that $\|w^{1,0}\|_{H^4(B_R)} + \|\rho_0\|_{H^{7/2}(\Sigma)}$ is small, then we need to add a constant factor in (7.11) and (7.13).

Proof. By examining the structure of $B_{j,n}(s)$ (using also (6.36)) we find that

$$(7.14) \quad \begin{aligned} B_{j,n}(s) = & M_{j,n}^1 + M_{j,n}^2 P_j(R\sqrt{s+1}) + M_{j,n}^3 P_n(R\sqrt{s+1}) \\ & + M_{j,n}^4 (s+1) P_j(R\sqrt{s+1}) P_n(R\sqrt{s+1}), \end{aligned}$$

where $M_{j,n}^m$ ($m = 1, 2, 3, 4$, $|j-n| \leq 2$) are constants such that

$$\begin{aligned} |M_{j,n}^1| &\leq C(n+1)^3, \\ |M_{j,n}^2| + |M_{j,n}^3| &\leq C(n+1), \\ |M_{j,n}^4| &\leq C. \end{aligned}$$

Thus [13, Lemma 5.1] can be directly applied to the inverse Laplace transform of $B_{j,n}(s)/h_n(s)$ for the first three terms in (7.14). To finish the proof, it suffices to establish that the inverse Laplace transform of

$$(7.15) \quad (s+1)P_j(R\sqrt{s+1})P_n(R\sqrt{s+1})/h_n(s) \quad \text{for } j = n-2, n, n+2$$

is bounded by the right-hand side of (7.10). This function is holomorphic for $\operatorname{Re} s > -\delta_1(n^2+1)$.

From (2.13) we have

$$\xi^2 P_n(\xi) P_{n+1}(\xi) = 1 - (2n+3)P_n(\xi), \quad \xi^2 P_{n+1}(\xi) P_{n+2}(\xi) = 1 - (2n+5)P_{n+1}(\xi).$$

The quotient of these two equalities gives

$$\frac{P_{n+2}(\xi)}{P_n(\xi)} = \frac{1 - (2n+5)P_{n+1}(\xi)}{1 - (2n+3)P_n(\xi)},$$

which can be rewritten as

$$(7.16) \quad P_{n+2}(\xi) - P_n(\xi) = (2n+3)P_n(\xi)P_{n+2}(\xi) - (2n+5)P_n(\xi)P_{n+1}(\xi),$$

and using (2.17) and (2.22), we obtain

$$(7.17) \quad |P_{n+2}(\xi) - P_n(\xi)| \leq \frac{C(n+1)}{|\xi|^2} \quad \text{for } |\arg(\xi)| \leq \frac{3\pi}{8}.$$

We can compute from (2.2), (2.3), and (2.4) that

$$(7.18) \quad \left| P_0(\xi) - \frac{1}{\xi} \right| + \left| P_1(\xi) - \frac{1}{\xi} \right| \leq \frac{C}{|\xi|^2} \quad \text{for } |\arg(\xi)| \leq \frac{3\pi}{8}.$$

We then deduce by induction (using (7.17)) that

$$(7.19) \quad \left| P_n(\xi) - \frac{1}{\xi} \right| \leq \frac{C(n+1)^2}{|\xi|^2} \quad \text{for } |\arg(\xi)| \leq \frac{3\pi}{8}.$$

It follows that, for $|n-j| \leq 2$,

$$(7.20) \quad \begin{aligned} \left| P_j(\xi)P_n(\xi) - \frac{1}{\xi^2} \right| &\leq \left| \left(P_j(\xi) - \frac{1}{\xi} \right) P_n(\xi) \right| + \left| \frac{1}{\xi} \left(P_n(\xi) - \frac{1}{\xi} \right) \right| \\ &\leq \frac{C(n+1)^2}{|\xi|^3} \quad \text{for } |\arg(\xi)| \leq \frac{3\pi}{8}. \end{aligned}$$

By repeating the proof of [13, Lemma 5.1] we can replace in (7.15) the function $h_n(s)$ with $c_1 s + c(n)$ (c_1 and $c(n)$ are positive and $c(n) \approx \text{const} \cdot n^3$ for large n), incurring an error that can be estimated by the right-hand side of (7.10).

We write the integral

$$\frac{1}{2\pi i} \int_{-\delta_1(n^2+1)-i\infty}^{-\delta_1(n^2+1)+i\infty} \frac{(s+1)P_j(R\sqrt{s+1})P_n(R\sqrt{s+1})}{c_1 \cdot s + c(n)} e^{st} ds$$

as the sum of the integrals for $|\tau| \leq \delta_1(n^2+1)$ and $|\tau| \geq \delta_1(n^2+1)$. The integral for $|\tau| \leq \delta_1(n^2+1)$ is bounded by $C(n+1)^2 e^{-\delta_1(n^2+1)t}$, as seen by using the uniform bounds from (2.16). For the integral on $|\tau| \geq \delta_1(n^2+1)$, we replace $R^2 P_n(R\sqrt{s+1})P_j(R\sqrt{s+1})$ by $1/(s+1)$, incurring an error which is bounded by the right-hand side of (7.20); the resulting error in the integral for $|\tau| > \delta_1(n^2+1)$ is bounded by the right-hand side of (7.10) (here we use the fact that if $\xi^2 = R^2(s+1)$ and $s = -\delta_1(n^2+1) + i\tau$, then $|\arg(\xi)| \leq \frac{3\pi}{8}$ if $|\tau| \geq \delta_1(n^2+1)$). The integral obtained after this replacement can be computed explicitly, and it is clearly bounded by the right-hand side of (7.10).

We next establish (7.11). Observe that $F_n(s)$ and $G_n(s)$ are determined by only the initial data $\rho_{n,0}(0)$, the $\xi_{1j}^1(s)$ from (6.18), the $\psi_{1j}^1(s)$ from (6.20), and $\xi_{1j}^2(s)$ from (6.33). All these functions are uniquely determined by the initial data $w_{n,0}^1(r, 0)$. By applying [12, Lemmas 3.2 and 3.3], we can then estimate all the terms in $F_n(s)$, $G_n(s)$, except for

$$(7.21) \quad \frac{d^2}{dr^2} \left[-\frac{\mu_2}{s+1} \xi_{1j}^1 + \psi_{1j}^1 \right]_{r=R}.$$

To estimate this term we extend [12, Lemmas 3.2 and 3.3] by repeating their proof with one higher order derivative; this requires one higher order derivative (in r) on the initial data, which is provided by the regularity assumptions in Lemma 7.5 and [16, Lemma 8.2]. If $n = 2$ and $\varepsilon > 0$, then $h_2(s) + \varepsilon B_{22}(s) = h_2'(0)s + B_{22}(0)\varepsilon + O(|s|^2 + \varepsilon^2)$ and we can therefore repeat the previous proof of (7.10), (7.11) and derive the estimate (7.12), (7.13). $\square$

8. RIGOROUS DERIVATION OF (7.8)–(7.9)

The Laplace transforms performed in Section 6 were carried out in a formal way. In order to justify this procedure we must prove that the solution of (5.24)–(5.35) exists for all $t > 0$ and that ρ satisfies, for instance, the inequality

$$(8.1) \quad \sup_{0 < t < \infty} e^{-\alpha t} \left(\|\rho\|_{H^{7/2}(\Sigma)} + \|\rho_t\|_{H^{1/2}(\Sigma)} \right) < \infty$$

for some $\alpha > 0$.

In this section we want to derive (7.8)–(7.9) rigorously for a solution of (5.24)–(5.35) which is only defined on a finite time interval:

Theorem 8.1. *Assume that $w_0 \in H^4(B_R)$, $\rho_0 \in H^{7/2}(\Sigma)$ and let (w_i, q_i, ρ) be a solution of (5.24)–(5.35) in a finite time interval $0 < t < t_0$. Then the formulas (7.8)–(7.9) are valid in the same interval $0 < t < t_0$.*

Proof. Let $T_1 = t_0$ and T be any number smaller than T_1 . We introduce a cut-off function $\zeta \in C^\infty[0, \infty)$ such that

$$(8.2) \quad \zeta(t) \equiv 1 \quad \text{if } 0 < t < T, \quad \zeta(t) \equiv 0 \quad \text{if } T+1 < t < \infty$$

and multiply the system (5.24)–(5.35) by $\zeta(t)$. Setting

$$(8.3) \quad \tilde{\rho}(\theta, t) = \zeta(t)\rho(\theta, t),$$

$$(8.4) \quad \tilde{w}_j(r, \theta, t) = \zeta(t)w_j(r, \theta, t) \quad (j = 1, 2),$$

$$(8.5) \quad \tilde{q}_j(r, \theta, t) = \zeta(t)q_j(r, \theta, t) \quad (j = 1, 2),$$

we get

$$(8.6) \quad (\tilde{w}_1)_t - \Delta \tilde{w}_1 + \tilde{w}_1 = k_1(r, \theta, t) \quad \text{in } B_R \times \{t > 0\},$$

$$(8.7) \quad (\tilde{w}_2)_t - \Delta \tilde{w}_2 + \tilde{w}_2 = k_2(r, \theta, t) \quad \text{in } B_R \times \{t > 0\},$$

$$(8.8) \quad \tilde{w}_1 = -\frac{\partial \sigma_S(R)}{\partial r} \tilde{\rho} \quad \text{on } \partial B_R \times \{t > 0\},$$

$$(8.9) \quad \tilde{w}_2 = -\left[\frac{\partial \sigma_1(R, \theta)}{\partial r} + \frac{\partial^2 \sigma_S(R)}{\partial r^2} Y_{2,0} \right] \tilde{\rho} - \frac{\partial \tilde{w}_1(R, \theta, t)}{\partial r} Y_{2,0} \\ \text{on } \partial B_R \times \{t > 0\},$$

$$(8.10) \quad \Delta \tilde{q}_1 = -\mu_2 \tilde{w}_1 \quad \text{in } B_R \times \{t > 0\},$$

$$(8.11) \quad \Delta \tilde{q}_2 = -\mu_2 \tilde{w}_2 - \mu'(0) \tilde{w}_1 \quad \text{in } B_R \times \{t > 0\},$$

$$(8.12) \quad \tilde{q}_1 = -\frac{1}{R^2} \left(\tilde{\rho} + \frac{1}{2} \Delta_\omega \tilde{\rho} \right) \quad \text{on } \partial B_R \times \{t > 0\},$$

$$(8.13) \quad \tilde{q}_2 = \frac{Y_{2,0}}{R^3} \left(-4\tilde{\rho} + \Delta_\omega \tilde{\rho} \right) - \frac{\partial \tilde{q}_1(R, \theta, t)}{\partial r} Y_{2,0} \quad \text{on } \partial B_R \times \{t > 0\},$$

$$(8.14) \quad \tilde{\rho}_t = -\left\{ \frac{\partial^2 p_S(R)}{\partial r^2} + \varepsilon \left(\frac{\partial^2 p_1(R, \theta)}{\partial r^2} + \frac{\partial^3 p_S(R)}{\partial r^3} Y_{2,0} \right) \right\} \tilde{\rho} + k_3(\theta, t) \\ - \frac{\partial \tilde{q}_1}{\partial r} \Big|_{r=R} - \varepsilon \left[\frac{\partial \tilde{q}_2}{\partial r} + \frac{\partial^2 \tilde{q}_1}{\partial r^2} Y_{2,0} \right]_{r=R} \\ + \varepsilon \frac{1}{R^2} \frac{\partial Y_{2,0}}{\partial \theta} \frac{\partial \tilde{q}_1}{\partial \theta} \Big|_{r=R} + \varepsilon \frac{1}{R^2} \frac{\partial \tilde{\rho}}{\partial \theta} \frac{\partial p_1}{\partial \theta} \Big|_{r=R},$$

where

$$(8.15) \quad k_j(r, \theta, t) = \zeta'(t)w_j(r, \theta, t) \quad (j = 1, 2), \quad k_3(\theta, t) = \zeta'(t)\rho(\theta, t).$$

We regard k_j as inhomogeneous terms and proceed to perform the Laplace transform as in Section 6. We shall only describe here the changes that need to be done in the calculations of Section 6.

In Step (i) from (6.9)–(6.14), there are no changes at all.

In Step (ii), an extra term $\hat{u}_{k_1, n}$ due to the inhomogeneous right-hand side should be added to $\hat{w}_{n,0}^1$ in (6.17); we can find out that $u_{k_1, n}$ satisfies:

$$(8.16) \quad (u_{k_1, n})_t - \Delta u_{k_1, n} + \left(\frac{n(n+1)}{r^2} + 1 \right) u_{k_1, n} = \langle k_1, Y_{n,0} \rangle \quad \text{in } B_R, \\ u_{k_1, n} = 0 \quad \text{on } \partial B_R, \\ u_{k_1, n} \Big|_{t=0} = 0,$$

and since $k_1(r, \theta, t) \equiv 0$ for $0 < t < T$, by the maximum principle,

$$(8.17) \quad u_{k_1, n} \equiv 0 \quad \text{for } 0 < t < T.$$

This term is carried over to (6.19)–(6.22), resulting in two extra terms in (6.23):

$$(8.18) \quad \left[-\frac{\mu_2}{s+1} \hat{u}_{k_1, n} + \psi_{k_1, n}(s) \right]_{r=R},$$

where

$$(8.19) \quad \begin{aligned} -\Delta\psi_{k_1,n} + \frac{n(n+1)}{r^2}\psi_{k_1,n} &= \frac{\mu_2}{s+1}\langle \widehat{k}_1, Y_{n,0} \rangle \quad \text{in } B_R, \\ \psi_{k_1,n} &= 0 \quad \text{on } \partial B_R. \end{aligned}$$

Continuing with the computation, we find that the following extra terms appear in (6.23):

$$(8.20) \quad \sum_{j=n-2}^{n+2} c_{jn} \frac{d^2}{dr^2} \left[-\frac{\mu_2}{s+1} \widehat{u}_{k_1,n} + \psi_{k_1,n}(s) \right]_{r=R}.$$

In Step (iii), a term is added to (6.26) in the form

$$(8.21) \quad -\sum_{j=n-2}^{n+2} c_{jn} \frac{d}{dr} \widehat{u}_{k_1,n} \Big|_{r=R}.$$

All the added terms show up in (6.27)–(6.34) through the expression $\widehat{w}_{n,0}^2(R, s)$. Furthermore, an extra term is added to (6.27) due to the inhomogeneous term k_2 ; it is $\widehat{u}_{k_2,n}$, the Laplace transform of the solution $u_{k_2,n}$:

$$(8.22) \quad \begin{aligned} (u_{k_2,n})_t - \Delta u_{k_2,n} + \left(\frac{n(n+1)}{r^2} + 1 \right) u_{k_2,n} &= \langle k_2, Y_{n,0} \rangle \quad \text{in } B_R, \\ u_{k_2,n} &= 0 \quad \text{on } \partial B_R, \\ u_{k_2,n} \Big|_{t=0} &= 0, \end{aligned}$$

and by the maximum principle,

$$(8.23) \quad u_{k_2,n} \equiv 0 \quad \text{for } 0 < t < T.$$

Altogether we obtain in (6.35) the extra terms

$$(8.24) \quad \begin{aligned} &-\frac{n}{R} \sum_{j=n-2}^{n+2} \frac{d}{dr} \left[\frac{-\mu_2}{s+1} \widehat{u}_{k_1,j} + \psi_{k_1,j} \right]_{r=R} c_{jn} \\ &+ \mu_2 R P_n(R\sqrt{s+1}) \sum_{j=n-2}^{n+2} c_{jn} \frac{d}{dr} \widehat{u}_{k_1,n} \Big|_{r=R} \\ &+ \frac{d}{dr} \left[\psi_{k_2,n} - \frac{\mu'(0)}{s+1} \widehat{u}_{k_1,n} \right]_{r=R} - \frac{\mu_2}{s+1} \frac{d}{dr} \widehat{u}_{k_2,n} \Big|_{r=R}, \end{aligned}$$

where

$$(8.25) \quad \begin{aligned} -\Delta\psi_{k_2,n} + \frac{n(n+1)}{r^2}\psi_{k_2,n} &= \frac{\mu'(0)}{s+1}\langle \widehat{k}_1, Y_{n,0} \rangle + \frac{\mu_2}{s+1}\langle \widehat{k}_2, Y_{n,0} \rangle \quad \text{in } B_R, \\ \psi_{k_2,n} &= 0 \quad \text{on } \partial B_R. \end{aligned}$$

□

Lemma 8.2. *The inverse Laplace transforms of $\psi_{k_1,n}$ and $\psi_{k_2,n}$ are well defined, and if we denote them by $v_{k_1,n}(t)$ and $v_{k_2,n}(t)$, then*

$$(8.26) \quad v_{k_1,n}(t) \equiv v_{k_2,n}(t) \equiv 0 \quad \text{for } 0 < t < T.$$

Proof. Clearly, $v_{k_1,n}(t)$ satisfies

$$(8.27) \quad \begin{aligned} -\Delta v_{k_1,n} + \frac{n(n+1)}{r^2} v_{k_1,n} &= \mu_2 e^{-t} * \langle k_1, Y_{n,0} \rangle \quad \text{in } B_R, \\ v_{k_1,n} &= 0 \quad \text{on } \partial B_R. \end{aligned}$$

From the previous consideration we know that $k_1 = 0$ for $0 < t < T$, so that for any function $b(\cdots, t)$

$$(k_1 * b)(t) = \int_0^t k_1(\cdots, \tau) b(\cdots, t - \tau) d\tau = 0 \quad \text{if } 0 < t < T.$$

Thus the right-hand side of (8.27) is zero when $0 < t < T$. It follows that $v_{k_1, n}(t) \equiv 0$ for $0 < t < T$. The proof for $v_{k_2, n}$ is similar. $\square$

Proof of Theorem 8.1 (continued). The equation for ρ involves k_3 , which can be treated in a similar manner to k_1 and k_2 . Thus the truncated system (as a result of the cut-off function $\zeta(t)$) coincides with the original system for $t < T$, and of course we can take the Laplace transform of this system. We can then proceed as in Section 6 to derive equations as in Lemma 6.1 with several extra terms on the right-hand side of (6.8), given explicitly by (8.18) and (8.24) and a term involving k_3 . We now proceed as in Section 8 to derive (7.8), (7.9) for $\tilde{\rho}$, with several extra terms on the right-hand side. These terms take the form

$$u_{k_j, n} * b_{j, n} \quad (j = 1, 2), \quad v_{k_j, n} * \tilde{b}_{j, n} \quad (j = 1, 2), \quad \langle k_3, Y_{n, 0} \rangle * \bar{b}_n,$$

for some functions $b_{j, n}, \tilde{b}_{j, n}, \bar{b}_n$, their r derivatives on the boundary, etc., these terms must vanish on the interval $[0, T]$.

Thus we derived (7.8)–(7.9) from (5.24)–(5.35) without the assumption that the solution is global in time. $\square$

9. LINEAR INSTABILITY FOR $\varepsilon < 0$

It is proved in [12, Section 4] that if $\mu = \mu_2$ (which occurs when $R > \bar{R}$), then the roots for $h_n(s)$ for $n \neq 1, 2$ lie in $\text{Re } s < 0$. Thus any possible instability in the system (6.8) may arise only from the modes $n = 1, 2$.

It is also proved in [12, (4.5), (3.27)] that $h_1(0) = h_2(0) = 0$, and except for $s = 0$, all remaining roots of h_1 and h_2 lie in $\text{Re } s < 0$. It is then clear from the structure of (6.8) that the ε -order terms in modes $n = 1, 2$ will decide the stability.

We recall from [12, Lemma 4.7] that

$$(9.1) \quad \frac{dh_2(0)}{ds} > 0.$$

If $\varepsilon < 0$, then for the initial data as in Lemma 6.3, the function

$$(9.2) \quad A(s) = \frac{1}{h_2(s) + \varepsilon B_{2,2}(s)}$$

will have a zero at

$$(9.3) \quad s = -\varepsilon \frac{B_{2,2}(0)}{h_2'(0)} + O(\varepsilon^2) \equiv \delta_0(\varepsilon) > 0,$$

and we therefore expect linear instability. In fact we shall prove the following:

Theorem 9.1. *If $\varepsilon < 0$, then, for the initial data as in (7.5), there exists a solution of (5.24)–(5.33) for which*

$$(9.4) \quad |\rho_{2,0}(t_n)| \rightarrow \infty \text{ for a sequence } t_n \rightarrow \infty;$$

hence the bifurcation branch with $\varepsilon < 0$ is linearly unstable.

Proof. Denote by $\mathcal{K}$ the metric space of sequences $\{v_n(s)\}_{n=0,2,4,\dots}$ of functions holomorphic in $\{|s - \delta_0(\varepsilon)| > \frac{1}{2}\delta_0(\varepsilon), \operatorname{Re} s > 0\}$, satisfying

$$(9.5) \quad \begin{aligned} |v_2(s) - A(s)| &\leq \frac{|\varepsilon|^{-1/2}}{1 + |s|^2} \quad \text{if } |s - \delta_0(\varepsilon)| \geq \frac{1}{2}\delta_0(\varepsilon), \operatorname{Re} s > 0, \\ |v_n(s)| &\leq \frac{|\varepsilon|^{(n-5)/3}}{1 + |s|^2} \quad \text{if } |s - \delta_0(\varepsilon)| \geq \frac{1}{2}\delta_0(\varepsilon), \operatorname{Re} s > 0, n \geq 4, \\ |v_0(s)| &\leq \frac{|\varepsilon|^{-1/3}}{1 + |s|^2} \quad \text{if } |s - \delta_0(\varepsilon)| \geq \frac{1}{2}\delta_0(\varepsilon), \operatorname{Re} s > 0, \end{aligned}$$

with the distance between two sequences $\{v_n(s)\}_{n=0,2,4,\dots}$ and $\{\bar{v}_n(s)\}_{n=0,2,4,\dots}$ defined by

$$\sup_{\{|s - \delta_0(\varepsilon)| \geq \frac{1}{2}\delta_0(\varepsilon), \operatorname{Re} s > 0\}} \left\{ \sup_{n \geq 4} |\varepsilon|^{-(n-5)/3} (1 + |s|^2) |v_n(s) - \bar{v}_n(s)|, \right. \\ \left. |\varepsilon|^{-1/2} (1 + |s|^2) |v_2(s) - \bar{v}_2(s)|, |\varepsilon|^{-1/3} (1 + |s|^2) |v_0(s) - \bar{v}_0(s)| \right\}.$$

We introduce a mapping $\mathcal{M}$ from $\{v_n(s)\}_{n=0,2,4,\dots}$ to $\{u_n(s)\}_{n=0,2,4,\dots}$ by

$$(9.6) \quad u_2(s) - A(s) = -\varepsilon \left\{ \frac{B_{0,2}(s)v_0(s)}{h_2(s) + \varepsilon B_{2,2}(s)} + \frac{B_{4,2}(s)v_4(s)}{h_2(s) + \varepsilon B_{2,2}(s)} \right\},$$

$$(9.7) \quad h_n(s)u_n(s) = -\varepsilon \sum_{j=n-2}^{n+2} B_{j,n}(s)v_j(s), \quad n \neq 2.$$

We shall prove that $\mathcal{M}$ is a contraction mapping in $\mathcal{K}$, and it thus has a unique fixed point, which is then the unique solution of (7.6) in $\mathcal{K}$.

As proved in [12, Section 4] for $n \neq 2$, $|h_n(s)| \geq c(n)$ in the region $\{|s - \delta_0(\varepsilon)| \geq \frac{1}{2}\delta_0(\varepsilon), \operatorname{Re} s > 0\}$, where $c(n)$ is positive and $c(n) \approx \operatorname{const} \cdot n^3$ for large n . If $n \geq 6$, the function $v_2(s)$ does not appear in (9.7), and therefore

$$(9.8) \quad \begin{aligned} |u_n(s)| &= |\varepsilon| \left| \sum_{j=n-2}^{n+2} \frac{1}{h_n(s)} B_{j,n}(s)v_j(s) \right| \\ &\leq |\varepsilon| C \sum_{j=n-2}^{n+2} \frac{|\varepsilon|^{(j-5)/3}}{1 + |s|^2} \leq \frac{|\varepsilon|^{(n-7)/3}}{1 + |s|^2} \cdot C|\varepsilon| \\ &< \frac{|\varepsilon|^{(n-5)/3}}{1 + |s|^2} \quad \text{if } |s - \delta_0(\varepsilon)| \geq \frac{1}{2}\delta_0(\varepsilon), \operatorname{Re} s > 0, \end{aligned}$$

if $|\varepsilon|$ is small such that $C|\varepsilon|^{1/3} < 1$.

By (9.2), (9.3), on $|s - \delta_0(\varepsilon)| = \frac{1}{2}\delta_0(\varepsilon)$ we have $1/A(s) = h'_2(0)s + \varepsilon B_{2,2}(0) + O(\varepsilon^2) \sim \operatorname{const} \cdot |\varepsilon|$ as $\varepsilon \rightarrow 0$, so that

$$(9.9) \quad \begin{aligned} |v_2(s)| &\leq |v_2(s) - A(s)| + |A(s)| < \frac{|\varepsilon|^{-1/2}}{1 + |s|^2} + C \frac{|\varepsilon|^{-1}}{1 + |s|} < C \frac{|\varepsilon|^{-1}}{1 + |s|} \\ &\quad \text{for } |s - \delta_0(\varepsilon)| \geq \frac{1}{2}\delta_0(\varepsilon), \operatorname{Re} s > 0. \end{aligned}$$

Using this inequality, noticing also that $|h_4(s)| \geq c(1 + |s|)$ for $\{|s - \delta_0(\varepsilon)| \geq \frac{1}{2}\delta_0(\varepsilon), \operatorname{Re} s > 0\}$, we obtain the same estimates for $u_4(s)$ as in (9.8).

In the case $n = 0$, we can use (7.7), (9.9) and $|h_0(s)| \geq c(1 + |s|)$ for $\{|s - \delta_0(\varepsilon)| \geq \frac{1}{2}\delta_0(\varepsilon), \operatorname{Re} s > 0\}$ to deduce from (9.7) that

$$(9.10) \quad \begin{aligned} |u_0(s)| &\leq C|\varepsilon| \left\{ \frac{|\varepsilon|^{-1/3}}{1 + |s|^2} + \frac{|\varepsilon|^{-1}}{1 + |s|^2} \right\} \\ &\leq \frac{C}{1 + |s|^2} < \frac{|\varepsilon|^{-1/3}}{1 + |s|^2} \quad \text{for } |s - \delta_0(\varepsilon)| \geq \frac{1}{2}\delta_0(\varepsilon), \operatorname{Re} s > 0. \end{aligned}$$

Finally, for $n = 2$, notice that the right-hand side of (9.7) does not involve v_2 , whereas v_0 and v_4 are bounded by $|\varepsilon|^{-1/3}/(1 + |s|^2)$. Hence

$$(9.11) \quad \begin{aligned} |u_2(s) - A(s)| &\leq \frac{C}{|h_2(s) + \varepsilon B_{2,2}(s)|} \frac{|\varepsilon|^{2/3}}{1 + |s|^2} \leq \frac{C}{|\varepsilon|} \frac{|\varepsilon|^{2/3}}{1 + |s|^2} \\ &\leq \frac{|\varepsilon|^{-1/2}}{1 + |s|^2} \quad \text{for } |s - \delta_0(\varepsilon)| \geq \frac{1}{2}\delta_0(\varepsilon), \operatorname{Re} s > 0. \end{aligned}$$

We have thus proved that $\mathcal{M}$ maps $\mathcal{K}$ into itself. The proof that $\mathcal{M}$ is a contraction follows by a similar argument. We conclude that $\mathcal{M}$ has a unique fixed point in $\mathcal{K}$, and this is the unique solution of (7.6) in the region $|s - \delta_0(\varepsilon)| \geq \frac{1}{2}\delta_0(\varepsilon), \operatorname{Re} s > 0$.

The inverse Laplace transform of $A(s)$ is well defined, as shown in [12]. The estimate (9.5) allows us to define the inverse Laplace transform for the functions $u_2(s) - A(s)$ and $u_n(s)$ ($n \neq 2$) and these are holomorphic in $\operatorname{Re} s > a$ for any $a > \delta_0(\varepsilon)$.

We can then define the inverse Laplace transform (with $\operatorname{Re} s = a$, $a > \delta_0(\varepsilon)$) of $\hat{\rho}$, and then define w_1, w_2, q_1, q_2 as solutions of the equations (5.24)–(5.31). As in [12], we can then prove that (w, q, ρ) is the solution of the linearized problem (5.24)–(5.32). We now make the simple but important observation that if ρ is bounded, then its Laplace transform $\hat{\rho}$ is a holomorphic function in $\operatorname{Re} s > 0$ and so are the elements in the sequence $\{\hat{\rho}_{n,0}\}_{n=0,2,4,\dots}$ of its expansion into a series of spherical harmonics. Since $\{\hat{\rho}_{n,0}\}_{n=0,2,4,\dots}$ must agree with the fixed point on the circle $|s - \delta_0(\varepsilon)| = \frac{1}{2}\delta_0(\varepsilon)$,

$$(9.12) \quad |\hat{\rho}_{2,0}(s) - A(s)| \leq |\varepsilon|^{-1/2} \quad \text{if } |s - \delta_0(\varepsilon)| = \frac{1}{2}\delta_0(\varepsilon),$$

$$(9.13) \quad |\hat{\rho}_{n,0}(s)| \leq |\varepsilon|^{(n-5)/3} \quad \text{if } |s - \delta_0(\varepsilon)| = \frac{1}{2}\delta_0(\varepsilon), n \geq 4,$$

$$(9.14) \quad |\hat{\rho}_{n,0}(s)| \leq |\varepsilon|^{-1/3} \quad \text{if } |s - \delta_0(\varepsilon)| = \frac{1}{2}\delta_0(\varepsilon), n = 0, 2.$$

Recalling that $A(s) \sim \operatorname{const} |\varepsilon|$ on $|s - \delta_0(\varepsilon)| = \frac{1}{2}\delta_0(\varepsilon)$,

$$|\hat{\rho}_{2,0}(s) - A(s)| \leq |\varepsilon|^{-1/2} < |A(s)| \quad \text{on } |s - \delta_0(\varepsilon)| = \frac{1}{2}\delta_0(\varepsilon),$$

and this implies that inside the disk $|s - \delta_0(\varepsilon)| < \frac{1}{2}\delta_0(\varepsilon)$,

$$(9.15) \quad \begin{aligned} &(\text{number of poles} - \text{number of roots}) \text{ of } \hat{\rho}_{2,0}(s) \\ &= (\text{number of poles} - \text{number of roots}) \text{ of } A(s) = 1. \end{aligned}$$

Thus $\hat{\rho}_{2,0}(s)$ has one pole $\alpha + i\beta$ with $\alpha > 0$, which contradicts the assumption that $\rho(t)$ is a bounded function.

This establishes the assertion (9.4). $\square$

10. LINEAR STABILITY FOR $\varepsilon > 0$

We consider the case $\varepsilon > 0$, and assume that

$$(10.1) \quad \|w^{1,0}\|_{H^4(B_R)} + \|\rho_0\|_{H^{7/2}(\Sigma)} \text{ is finite.}$$

To prove linear stability, it suffices to consider two separate cases: (i) the perturbation functions are even functions of θ , and (ii) the perturbation functions are odd functions of θ .

Theorem 10.1. *If $\varepsilon > 0$, and the initial data are even function in θ satisfying (10.1), then there exists a unique solution of (5.24)–(5.33) such that*

$$(10.2) \quad |\rho_{n,0}(t)| \leq C(n+1)^{-3}e^{-\delta_1|\varepsilon|t} \quad \text{for } n \neq 2,$$

$$(10.3) \quad |\rho_{2,0}(t)| \leq Ce^{-\delta_1|\varepsilon|t},$$

where C is a constant independent of ε .

Proof. As in the proof of Theorem 9.1, it suffices to prove that there exists a unique solution of (7.8), (7.9) satisfying (10.2)–(10.3). Without loss of generality we may assume that the expression in (10.1) is sufficiently small so that (7.10), (7.11) hold; we shall then prove that

$$(10.4) \quad |\rho_{n,0}(t)| \leq (n+1)^{-3}e^{-\delta_1|\varepsilon|t} \quad \text{for } n \neq 2,$$

$$(10.5) \quad |\rho_{2,0}(t)| \leq Ke^{-\delta_1|\varepsilon|t},$$

where $K = 1 + 2C_0/\delta_1$ and C_0 is the constant which appears in (7.10). The proof is again based on the contraction mapping principle. Let $\mathcal{K}$ be the set of sequences $\{\rho_{n,0}(t)\}_{n=0,2,4,\dots}$ satisfying (10.4), (10.5). Given any sequence in $\mathcal{K}$, we substitute it into the ε -order terms in (7.8), (7.9) and solve for a new $\tilde{\rho}_{n,0}$, thus defining a mapping

$$\{\tilde{\rho}_{n,0}(t)\}_{n=0,2,4,\dots} = \mathcal{M}\{\rho_{n,0}(t)\}_{n=0,2,4,\dots}.$$

Using Lemma 7.5, we have, for $n \neq 2$,

$$\begin{aligned} |\tilde{\rho}_{n,0}(t)| &\leq |f_n(t)| + |\varepsilon| \sum_{j=n-2}^{n+2} |(b_{j,n} * \rho_{j,0})(t)| \\ &\leq (n+1)^{-3}e^{-\delta_1 t} + |\varepsilon| \sum_{j=n-2}^{n+2} 2C_0 \left\{ (n+1)^3 e^{-\delta_1(n^3+1)t} \right. \\ &\quad \left. + (n+1)^2 e^{-\delta_1(n^2+1)t} \right\} * \left\{ (n+1)^{-3} (1 + 3^3 K) e^{-\delta_1|\varepsilon|t} \right\} \\ &\leq (n+1)^{-3}e^{-\delta_1 t} + 2C_0(1 + 3^3 K)|\varepsilon|(n+1)^{-3}e^{-\delta_1|\varepsilon|t} \\ &< 2(n+1)^{-3}e^{-\delta_1|\varepsilon|t}, \end{aligned}$$

provided we choose $2C_0(1 + 3^3 K)|\varepsilon| < 1$.

For $n = 2$, we use (7.9) to obtain

$$\begin{aligned} |\tilde{\rho}_{2,0}(t)| &\leq |f_2(t)| + 2C_0|\varepsilon|e^{-2\delta_1|\varepsilon|t} * e^{-\delta_1|\varepsilon|t} \\ &\leq e^{-2\delta_1|\varepsilon|t} + 2C_0|\varepsilon| \frac{1}{\delta_1|\varepsilon|} e^{-\delta_1|\varepsilon|t} \\ &< Ke^{-\delta_1|\varepsilon|t}, \end{aligned}$$

provided we choose $K = 1 + 2C_0/\delta_1$. This proves that $\mathcal{M}$ maps $\mathcal{K}$ into itself. If we define the distance between $\{\rho_{n,0}\}_{n=0,2,4,\dots}$ and $\{\tilde{\rho}_{n,0}\}_{n=0,2,4,\dots}$ by

$$\max \left\{ \sup_{n \neq 2} \sup_{0 \leq t < \infty} (n+1)^3 e^{\delta_1 |\varepsilon| t} |\rho_{n,0}(t) - \tilde{\rho}_{n,0}(t)|, \right. \\ \left. \sup_{0 \leq t < \infty} K^{-1} e^{\delta_1 |\varepsilon| t} |\rho_{2,0}(t) - \tilde{\rho}_{2,0}(t)| \right\},$$

we can then repeat the above argument to show that we have a contraction on this space, and hence there is a unique fixed point for $\mathcal{M}$, which gives a unique solution in $\mathcal{K}$.

If $|\varepsilon|$ is small, then the solution of the system (7.8)–(7.9) for $0 < t < 1$ is unique not only in the space of functions ρ satisfying (10.2), (10.3) for $0 < t < 1$, but also in the larger space of functions $\rho(t, \theta)$ with

$$\int_0^1 \|\rho\|_{L^2(\Sigma)}^2 dt < \infty.$$

Since a solution (5.24)–(5.35) in $0 < t < 1$ must satisfy (7.8)–(7.9) (by Theorem 8.1), we obtain the uniqueness assertion of Theorem 10.1 for $0 < t < 1$. We can now proceed to prove uniqueness for $1 \leq t \leq 2$ (with the same ε), $2 \leq t \leq 3$, etc. $\square$

We next consider the system odd in θ . In this case we only need to consider the system for $n = 1, 3, 5, \dots$.

Theorem 10.2. *Let $\varepsilon > 0$ and let the initial data be odd functions in θ satisfying $w^{1,0} \in H^4(B_R)$ and $\rho_0 \in H^{7/2}(\Sigma)$. Then there exists a point x_0 , depending only on the initial data, such that for the system obtained analogously to (5.24)–(5.35) by linearization about x_0 , there exists a unique solution satisfying (10.2), (10.3).*

Proof. The idea of the proof is essentially the same as in [13, Section 7], but the details are much simpler since here we are dealing with a linear system and not with the product of solutions from the nonlinear deformation. We only outline the proof.

Recall that the roots of $h_{2n+1}(s)$ lie in the half plane $\operatorname{Re} s < 0$, while $h_1(0) = 0$ and all the other roots of $h_1(s)$ lie in $\operatorname{Re} s < 0$. In order to circumvent the difficulty caused by the zero root of $h_1(s)$, we need to make a translation of the center about which we linearize. We have dealt with such a problem already in [12], where we considered the stability of the nonlinear problem (1.1)–(1.6) with respect to the stationary radial solution. The present case is much easier since we are dealing with a linearized problem.

We rewrite the system (6.8) as

$$(10.6) \quad h_n(s) \hat{\rho}_{n,0}(s) = F_n(s) + \varepsilon G_n(s) - \varepsilon \sum_{j=n-2}^{n+2} B_{j,n}(s) \hat{\rho}_{j,0}(s);$$

in particular,

$$(10.7) \quad h_1(s) \hat{\rho}_{1,0}(s) = F_1(s) + \varepsilon G_1(s) - \varepsilon f(s),$$

where

$$(10.8) \quad f(s) = \sum_{j=1}^3 B_{j,1}(s) \hat{\rho}_{j,0}(s).$$

Notice that $F_1(s)$ and $G_1(s)$ depend only on the initial data, but $f(s)$ does not depend explicitly on the initial data.

We shall again use two fixed point theorems as in [13, Section 7]. We look for a solution in the class

$$(10.9) \quad |\rho_{n,0}(t)| \leq n^{-3}e^{-\delta_1 t},$$

where we choose δ_1 such that all the roots of $h_n(s)$ for $n \geq 3$ and for $h_1(s)$ (except for $s = 0$) lie in $\operatorname{Re} s < -2\delta_1$.

Suppose $f(s)$ is a given holomorphic function, uniformly bounded in a neighborhood of $s = 0$. By applying [13, Theorem 2.2] we can make a translation $x \rightarrow x + a$ of the center of the linearization; this translation is equivalent to keeping the origin fixed, but translating the initial data according to [13, (6.4),(6.5)].

The functions on the right-hand side of (10.7) actually depend on a , through their dependence on the initial data according to [13, (6.4),(6.5)], and for clarity we shall write the right-hand side of (10.8) as

$$F_1(s; a) + \varepsilon G_1(s; a) - \varepsilon f(s).$$

We want to solve for a such that

$$(10.10) \quad F_1(0; a) + \varepsilon G_1(0; a) - \varepsilon f(0) = 0.$$

This can be done, as in [13], recalling that $F_1(0, a)$ is given by [13, (6.3)] (when $\varepsilon = 0$) and using a fixed point theorem [13, Theorem 2.2].

Next, given any $\{\rho_{n,0}(t)\}_{n=1,3,5,\dots}$ satisfying (10.9), we define $f(s)$ by (10.8), find a by (10.10), and use this choice of a into F_n and G_n of (10.6).

We then proceed to solve the new $\tilde{\rho}_{n,0}(t)$ as the inverse Laplace transform of $\hat{\tilde{\rho}}_{n,0}(s)$ (which is defined by the right-hand side of (10.6)). Since (10.10) cancels the zero of $h_1(s)$ at $s = 0$, we can proceed as in the proof Theorem 10.1 to obtain the new $\tilde{\rho}_{n,0}(t)$ satisfying (10.9). We can further prove that the map $\mathcal{M} : \{\rho_{n,0}(t)\} \rightarrow \{\tilde{\rho}_{n,0}(t)\}$ is a contraction. Thus we have a fixed point, which corresponds to a solution that converges to zero exponentially fast. For the even terms coming out of this translation of the center (they must be of order ε^2 or higher, by [13, (6.4),(6.5)]), we apply Theorem 10.1. Uniqueness follows as in Theorem 10.1. $\square$

Remark 10.1. Our method of proving linear stability can be extended to allow functions depending also on ϕ . However, in this case, we cannot separate the odd and even modes, and we are forced to combine the proofs of Theorems 10.1 and 10.2, which makes the computation more complicated.

11. HOPF BIFURCATION: LINEARIZED PROBLEM

In this section we consider the case $R_S < \bar{R}$. In this case $\mu^*(R_S) = \mu_0^*(R_S) < \mu_2^*(R_S)$ [12]. We consider the linearized problem about the radially symmetric solution by setting

$$(11.1) \quad \begin{aligned} \sigma(r, \theta, \phi, t) &= \sigma_S(r) + \varepsilon w(r, \theta, \phi, t), \\ p(r, \theta, \phi, t) &= p_S(r) + \varepsilon q(r, \theta, \phi, t), \\ \partial\Omega(t) : r &= R_S + \varepsilon \rho(\theta, \phi, t), \end{aligned}$$

and using the relations

$$\begin{aligned} V_n &= \varepsilon \rho_t + O(\varepsilon^2), \\ \kappa &= \frac{1}{R} - \frac{\varepsilon}{R^2} \left(\rho + \frac{1}{2} \Delta_\omega \rho \right) + O(\varepsilon^2). \end{aligned}$$

Substituting these quantities into (1.1)–(1.5) and collecting only the linear terms in ε , we obtain, as in [12], the linearized system

$$(11.2) \quad w_t - \Delta w + w = 0 \quad \text{in } B_R \times \{t > 0\},$$

$$(11.3) \quad w(R, \theta, \phi, t) = -\lambda \rho(\theta, \phi, t) \quad \text{for } t > 0 \quad \left(\lambda = \frac{\partial}{\partial r} \sigma_S(r) \Big|_{r=R} \right),$$

$$(11.4) \quad \Delta q = -\mu w \quad \text{in } B_R \times \{t > 0\},$$

$$(11.5) \quad q(R, \theta, \phi, t) = -\frac{1}{R^2} \left(\rho + \frac{1}{2} \Delta_\omega \rho \right),$$

and

$$\frac{d\rho}{dt} = -\frac{\partial^2 p_S}{\partial r^2} \Big|_{r=R} \rho - \frac{\partial q}{\partial r} \Big|_{r=R} \quad \text{for } t > 0.$$

Using (1.7) and (1.8) we find that

$$\frac{\partial^2 p_S}{\partial r^2} \Big|_{r=R} = -\mu(1 - \tilde{\sigma}).$$

Thus the equation for ρ can also be written in the form

$$(11.6) \quad \frac{d\rho}{dt} = \mu(1 - \tilde{\sigma})\rho - \frac{\partial q}{\partial r} \Big|_{r=R} \quad \text{for } t > 0.$$

From (1.9) we also have the initial conditions

$$(11.7) \quad w \Big|_{t=0} = w_0(r, \theta, \phi) \quad \text{for } 0 \leq r \leq R, \quad \rho \Big|_{t=0} = \rho_0(\theta, \phi).$$

Theorem 11.1. For $R < \overline{R}$ and $\mu = \mu_0^*(R)$,

(i) there exists a family of periodic solutions of the form:

$$(11.8) \quad \rho(\theta, \phi, t) = \rho(t) = C_1 \sin \beta t + C_2 \cos \beta t + \sum_{m=-1}^1 D_m Y_{1,m}(\theta, \phi)$$

for the linearized problem, where $\pm i\beta$ are roots of $h_0(s)$, and $\beta > 0$;

(ii) any periodic solution to the linearized problem is of the form (11.8);

(iii) for any initial data $w_0 \in L^2(B_R)$, $\rho_0 \in L^2(\Sigma)$, the solution of the linearized problem converges exponentially fast to a periodic solution established in (i), (ii).

Proof. We first prove existence. Since the systems are linear, we can consider each mode separately. We look for solutions of the form

$$(11.9) \quad \begin{aligned} \rho(t, \theta, \phi) &= e^{\nu t} Y_{n,m}(\theta, \phi), \\ w(r, \theta, \phi, t) &= e^{\nu t} u(r) Y_{n,m}(\theta, \phi), \\ q(r, \theta, \phi, t) &= e^{\nu t} \zeta(r) Y_{n,m}(\theta, \phi), \end{aligned}$$

where

$$\begin{aligned} -\Delta u + \left(\frac{n(n+1)}{r^2} + \nu + 1 \right) u &= 0 \quad \text{in } B_R, \\ u &= -\lambda \quad \text{on } \partial B_R, \\ -\Delta \zeta + \frac{n(n+1)}{r^2} \zeta &= \mu u \quad \text{in } B_R, \\ \zeta &= \frac{1}{R^2} \left(\frac{n(n+1)}{2} - 1 \right) \quad \text{on } \partial B_R. \end{aligned}$$

One can verify that this is a solution if and only if $h_n(\nu) = 0$. For $n = 0$, under our assumptions, we have $h_0(\pm i\beta) = 0$ for some $\beta > 0$. Substituting into (11.9) we obtain (i).

We next establish convergence and uniqueness. The Laplace transform has been computed in [12, Section 3], or equivalently, we can use the formulas in Section 5 by setting $q_2 \equiv w_2 \equiv 0$; here, however, we shall allow our functions to depend on (r, θ, ϕ, t) . Since for $n \geq 2$ all the roots of $h_n(s)$ lie in $\operatorname{Re} s < 0$, for mode $n \geq 2$, all solutions converge to zero exponentially fast, and therefore there are no periodic solutions with these modes.

Consider next the modes $n = 0, 1$. We have

$$(11.10) \quad \hat{\rho}_{n,m} = \frac{1}{\mu R^2 P_0(R) h_n(s)} F_{n,m}(s) \quad (-n \leq m \leq n),$$

where

$$(11.11) \quad F_{n,m}(s) = \rho_{n,m}(0) + \frac{d}{dr} \left[\frac{\mu}{s+1} \xi_{1,n,m} - \psi_{1,n,m} \right]_{r=R}$$

and $\xi_{1,n,m}(r, s)$, $\psi_{1,n,m}(r, s)$ are the solutions of

$$(11.12) \quad \begin{aligned} -\Delta \xi_{1,n,m} + \left(\frac{n(n+1)}{r^2} + (s+1) \right) \xi_{1,n,m} &= w_{n,m}(r, 0) \quad \text{in } B_R, \\ \xi_{1,n,m} \Big|_{r=R} &= 0, \end{aligned}$$

and

$$(11.13) \quad \begin{aligned} -\Delta \psi_{1,n,m} + \frac{n(n+1)}{r^2} \psi_{1,n,m} &= \frac{\mu}{s+1} w_{n,m}(r, 0) \quad \text{in } B_R, \\ \psi_{1,n,m} \Big|_{r=R} &= 0. \end{aligned}$$

For $n = 0$, except for $\pm i\beta$ ($\beta > 0$), all other roots of $h_0(s)$ lie in $\operatorname{Re} s < -2\delta_1$. Thus

$$\begin{aligned} \rho_{0,0}(t) &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{1}{\mu R^2 P_0(R) h_0(s)} F_{0,0}(s) e^{st} ds \quad (a > 0) \\ &= \left(\operatorname{Res}_{s=\beta i} + \operatorname{Res}_{s=-\beta i} \right) \frac{F_{0,0}(s) e^{st}}{\mu R^2 P_0(R) h_0(s)} \\ &\quad + \frac{1}{2\pi i} \int_{-\delta_1-i\infty}^{-\delta_1+i\infty} \frac{1}{\mu R^2 P_0(R) h_0(s)} F_{0,0}(s) e^{st} ds \\ &\equiv \rho_P(t) + J(t). \end{aligned}$$

Since $h_0(s)$ has simple roots at $s = \pm i\beta$, $\rho_P(t)$ is of the form $\tilde{C}_1 e^{i\beta t} + \tilde{C}_2 e^{-i\beta t}$ for some complex constants $\tilde{C}_1$ and $\tilde{C}_2$. Since our solution is real, we must have

$$\rho_P(t) = C_1 \sin \beta t + C_2 \cos \beta t$$

for some real constants C_1 and C_2 . As in the proof of [12, Section 5], $J(t)$ converges to 0 exponentially fast. Hence

$$(11.14) \quad |\rho_{0,0}(t) - \rho_P(t)| \leq Ce^{-\delta_1 t} \quad \text{for } t > 0.$$

Let

$$\begin{aligned} w_P(r, t) &= \rho_P(t)u(r), \\ q_P(r, t) &= \rho_P(t)\zeta(r), \end{aligned}$$

where

$$\begin{aligned} -\Delta u + (\omega + 1)u &= 0 \quad \text{in } B_R, \\ u &= -\lambda \quad \text{on } \partial B_R, \\ -\Delta \zeta &= \mu u \quad \text{in } B_R, \\ \zeta &= -\frac{1}{R^2} \quad \text{on } \partial B_R. \end{aligned}$$

Then the function $W = w_{0,0}(r, t) - w_P(r, t)$ and $Q = q_{0,0}(r, t) - q_P(r, t)$ satisfy the system

$$\begin{aligned} W_t - \Delta W &= W = 0 \quad \text{in } B_R \times \{t > 0\}, \\ W(R, t) &= -\lambda[\rho_{0,0}(t) - \rho_P(t)] \quad \text{for } t > 0, \\ \Delta Q &= -\mu W \quad \text{in } B_R \times \{t > 0\}, \\ Q(R, t) &= -\frac{1}{R^2}[\rho_{0,0}(t) - \rho_P(t)]. \end{aligned}$$

Using parabolic estimates we can then also prove that

$$(11.15) \quad |w_{0,0}(r, t) - w_P(r, t)| + |q_{0,0}(r, t) - q_P(r, t)| \leq Ce^{-\delta_1 t} \quad \text{for } t > 0.$$

We can use a similar argument for the mode $n = 1$; in this case $s = 0$ is a simple root for $h_1(s)$ and all other roots lie in $\operatorname{Re} s < -2\delta_1$. The above procedure shows

that all solutions in mode 1 must converge exponentially fast to $\sum_{m=-1}^1 D_m Y_{1,m}(\theta, \phi)$,

for some constants D_m . □

Remark 11.1. The Hopf bifurcation for a nonlinear system with a parameter μ produces a family of periodic solutions for $\mu = \mu(\varepsilon)$ in a neighborhood of the bifurcation point $\mu = \mu_*$ ($\mu_* = \mu(0)$). However, for a linear problem, such as the one considered in Theorem 11.1, periodic solutions (as $t \rightarrow \infty$) can arise only when μ coincides with the bifurcation point μ_* .

ACKNOWLEDGEMENT

This work was partially supported by the National Science Foundation upon agreement No. 0112050.

REFERENCES

- [1] B. BAZALLY AND A. FRIEDMAN, *A free boundary problem for an elliptic-parabolic system: Application to a model of tumor growth*, Comm. Partial Diff. Eq., Vol. 28, 2003, pp. 517-560. MR1976462 (2004c:35420)
- [2] H.M. BYRNE, *The importance of intercellular adhesion in the development of carcinomas*, IMA J. Math. Appl. Med. Biol., Vol. 14, 1997, pp. 305-323.
- [3] H.M. BYRNE, *A weakly nonlinear analysis of a model of avascular solid tumour growth*, J. Math. Biol., Vol. 39, 1999, pp. 59-89. MR1705626 (2000i:92011)

- [4] H.M. BYRNE AND M.A.J. CHAPLAIN, *Modelling the role of cell-cell adhesion in the growth and development of carcinomas*, Mathl. Comput. Modelling, Vol. 12, 1996, pp. 1-17.
- [5] H.M. BYRNE AND M.A.J. CHAPLAIN, *Growth of nonnecrotic tumors in the presence and absence of inhibitors*, Math. Biosci., Vol. 130, 1995, pp. 151-181.
- [6] H.M. BYRNE AND M.A.J. CHAPLAIN, *Growth of nonnecrotic tumors in the presence and absence of inhibitors*, Math. Biosci., Vol. 135, 1996, pp. 187-216.
- [7] M.A.J. CHAPLAIN, *The development of a spatial pattern in a model for cancer growth*, Experimental and Theoretical Advances in Biological Pattern Formation (H.G. Othmer, P.K. Maini, and J.D. Murray, eds), Plenum Press, 1993, pp. 45-60.
- [8] X. CHEN AND A. FRIEDMAN, *A free boundary problem for an elliptic-hyperbolic system: An application to tumor growth*, SIAM J. Math. Anal., Vol. 35, 2003, pp. 974-986. MR2049029 (2005f:35333)
- [9] M.G. CRANDALL AND P.H. RABINOWITZ, *Bifurcation from simple eigenvalues*, J. Functional Analysis, Vol. 8, 1971, pp. 321-340. MR0288640 (44:5836)
- [10] M. FONTELOS AND A. FRIEDMAN, *Symmetry-breaking bifurcations of free boundary problems in three dimensions*, Asymptotic Analysis, Vol. 35, 2003, pp. 187-206. MR2011787 (2005a:35290)
- [11] M. FONTELOS AND A. FRIEDMAN, *Symmetry-breaking bifurcations of charged drops*, Arch. Rat. Mech. Anal., Vol. 172, 2004, pp. 267-294. MR2058166 (2005b:76140)
- [12] A. FRIEDMAN AND BEI HU, *Bifurcation from stability to instability for a free boundary problem arising in a tumor model*, Arch. Rat. Mech. Anal., Vol. 180, No. 2, 2006, pp. 293-330. MR2210911 (2006j:35246)
- [13] A. FRIEDMAN AND BEI HU, *Asymptotic stability for a free boundary problem arising in a tumor model*, J. Diff. Eqn., Vol. 227, No. 2, 2006, pp. 598-639. MR2237681
- [14] A. FRIEDMAN AND F. REITICH, *Analysis of a mathematical model for the growth of tumors*, J. Math. Biology, Vol. 38, 1999, pp. 262-284. MR1684873 (2001f:92011)
- [15] A. FRIEDMAN AND F. REITICH, *Symmetry-breaking bifurcation of analytic solutions to free boundary problems: An application to a model of tumor growth*, Trans. Amer. Math. Soc., Vol. 353, 2001, pp. 1587-1634. MR1806728 (2002a:35208)
- [16] A. FRIEDMAN AND F. REITICH, *Nonlinear stability of a quasi-static Stefan problem with surface tension: A continuation approach*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., Vol. 30(4), 2001, pp. 341-403. MR1895715 (2003e:35326)
- [17] A. GALINDO AND P. PASCUAL, *Quantum Mechanics, Vol. I*, Springer, New York, 1990. MR1079543 (91k:81001)
- [18] H.P. GREENSPAN, *Models for the growth of a solid tumor by diffusion*, Studies Appl. Math., Vol. 52, 1972, pp. 317-340.
- [19] H.P. GREENSPAN, *On the growth and stability of cell cultures and solid tumors*, J. Theoretical Biology, Vol. 56, 1976, pp. 229-242. MR0429164 (55:2183)

DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, 231 WEST 18TH AVENUE,
COLUMBUS, OHIO 43210

E-mail address: `afriedman@mbi.osu.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, INDIANA 46556

E-mail address: `b1hu@nd.edu`