ASYMPTOTIC BEHAVIOUR OF CODIMENSIONS OF P. I. ALGEBRAS SATISFYING CAPELLI IDENTITIES

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ABSTRACT. Let A be a p. i. algebra with 1 in characteristic zero, satisfying a Capelli identity. Then the cocharacter sequence $c_n(A)$ is asymptotic to a function of the form $an^g\ell^n$, where $\ell \in \mathbb{N}$ and $g \in \mathbb{Z}$.

The second author conjectured that if A is any p. i. algebra in characteristic zero, with cocharacter sequence $c_n(A)$, then the asymptotic behaviour of this sequence should be given by

$$c_n(A) \simeq a \cdot n^t \cdot \ell^n,$$

where ℓ is a non-negative integer, t is an integer or a half-integer, and a belongs to $\mathbb{Q}[\sqrt{2\pi}, \sqrt{b}]$, for some $0 < b \in \mathbb{Z}$. We call ℓ^n the exponential part and $a \cdot n^t$ the rational part of the codimension growth.

Giambruno and Zaicev made progress on this conjecture by proving that for some non-negative integer ℓ ,

$$f_1(n)\ell^n \leq c_n(A) \leq f_2(n)\ell^n,$$

where $f_1(n)$ and $f_2(n)$ are Laurent polynomials, but not necessarily of the same degree.

In this paper we make some progress towards verifying the above conjecture by proving the following theorem.

Theorem 1. *Let A be a p.i. algebra with 1 satisfying a Capelli identity. Then*

$$c_n(A) \simeq a \cdot n^g \cdot \ell^n$$

where, furthermore, $g \in \frac{1}{2}\mathbb{Z}$. Also, the constant a here is a sum of Selberg-type integrals; see for example Theorem 36 below.

This theorem can help to determine whether or not the generating function $f(x) = \sum_{n \geq 0} c_n(A)x^n$ of the codimensions is algebraic. For example, if $g \in \mathbb{Z}$ and is negative, then $f(x)$ is *not* algebraic; see Lemma 3.2 in [1].

This application raises the following further question about Theorem 1: when is $g < 0$? The techniques of [6] imply the following: Let A_j , $j = 1, 2$, be p. i. algebras with T -ideals of identities $id(A_j) = I_j$. This is known to be the case, for example,

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if A_j is the algebra of $n \times n$ matrices. Assume $c_n(A_i) \simeq a_j \cdot n^{g_j} \cdot \ell_j^n$ with $g_1, g_2 < 0$. Then the algebras with T -ideals of identities equal to $I_1 I_2$ and $I_1 \cap I_2$ also have that same property $g < 0$. However, the comment at the end of the introduction implies that one can construct many algebras with $g \geq 0$.

The Giambruno–Zaicev theorem was proven in two stages: In [10] they proved it in the special case that A is finitely generated, or more generally that A satisfies a Capelli identity, and in [11] they proved the general case. In the current paper we will prove a slightly weaker version of the conjecture in the case that A satisfies a Capelli identity and has a unit. We will show that $c_n(A) \simeq a n^t \ell^n$, where ℓ is a non-negative integer, t is an integer or half integer but we don't have control over the constant a . Getting such control would involve evaluating a certain integral, which we describe in the second section. Under the weaker assumption that $c_n(A)$ is an increasing sequence, we prove that $a_1 n^t \ell^n \leq c_n(A) \leq a_2 n^t \ell^n$.

The proof combines and adapts ideas from the first author's paper [5] and the second author's paper [14]. The main idea is that for any p. i. algebra, $c_n(A) = \sum m_\lambda d_\lambda$, where d_λ is the degree of the S_n -character corresponding to λ , and m_λ is the multiplicity of that character in the cocharacter sequence. If A satisfies a Capelli identity, then m_λ will be zero unless λ has height bounded by some constant k . For such a λ , the multiplicities were estimated in [5]. We think of the multiplicities as functions from $\mathbb{N}^k \rightarrow \mathbb{N}$. In [5] we showed that $\mathbb{N}^k$ could be partitioned into a finite number of regions such that m_λ was equal to a polynomial on each. In section 1 we show that as far as the asymptotic behaviour of c_n is concerned, we need only consider regions of the form

$$\{v_0 + \alpha_1 v_1 + \cdots + \alpha_r v_r \mid \alpha_1, \dots, \alpha_r \geq 0\} \cap (\underline{i} + (d\mathbb{Z})^k),$$

where the v_i are all partitions, all except v_0 are linearly independent and have height at most the exponent $\ell \leq k$, and v_1 is the partition (1^ℓ) , thought of as the vector $(1, \dots, 1, 0, \dots, 0)$ (ℓ 1's and $k - \ell$ 0's). In the next section, we compute $\sum m_\lambda d_\lambda$ over one such region, adapting the arguments from [14]. In that work the second author computes the asymptotic behaviour of $\sum d_\lambda^\beta$, where λ runs over partitions of n with at most ℓ parts. The sum we need to compute differs both in the region we are summing over and the function we are summing, but for all that, the computations from [14] turn out to be just what we need here. The main result of this section is that $\sum m_\lambda d_\lambda$ over such a domain is asymptotic to a function of the form $a n^t \ell^n$, where a , t and ℓ all have the required form. Finally, in the last section we deal with the technical problems involved in summing the $\sum m_\lambda d_\lambda$ from the various domains.

Based on these computations we present a conjecture.

Conjecture. Let $f(x_1, \dots, x_k)/g(x_1, \dots, x_k)$ be a rational function such that the numerator and denominator are each symmetric polynomials with integer coefficients, and such that the denominator is a product of terms of the form $(1 - x_1^{a_1} \cdots x_k^{a_k})$. We expand the fraction as a series in Schur functions,

$$f(x_1, \dots, x_k)/g(x_1, \dots, x_k) = \sum_{n=0}^{\infty} \sum_{\lambda \in \text{Par}(n)} m_\lambda S_\lambda(x_1, \dots, x_k)$$

so that m_λ is the coefficient of the Schur function $S_\lambda(x_1, \dots, x_k)$. Define c_n to be $\sum_{\lambda \in \text{Par}(n)} m_\lambda d_\lambda$. Then there exists a modulus d and constants a_i, t_i and ℓ_i where

$i = 0, \dots, d-1$ such that, if we restrict n to be congruent to $i \bmod d$, then c_n has asymptotic behaviour

$$c_n \simeq a_i n^{t_i} \ell_i^n.$$

Moreover, each t_i will be either an integer or a half-integer and each ℓ_i will be the d^{th} -root of a rational number.

We close this introduction with a lower bound on the degree of the rational part of $c_n(A)$.

Theorem 2. *Let A be a p. i. algebra satisfying a Capelli identity with $c_n(A) \simeq an^t \ell^n$. Let $1 \leq s \leq 4$ be minimal such that ℓ can be written as a sum of s squares. Then $t \geq \frac{3}{2}s - 1 - \frac{1}{2}\ell$.*

Proof. Let I be the ideal of identities of A , and denote by I_a the ideal of identities of $a \times a$ matrices. It is implicit in [10] and explicit in Theorems 2.5 and 2.8 of [7] that there exist $a_1, \dots, a_k$ such that $I \subseteq I_{a_1} \cdots I_{a_k}$ and such that the generic algebra with identities $I_{a_1} \cdots I_{a_k}$ has the same exponential rate of growth as A . Since this algebra will have cocharacter sequence less than or equal to that of A , it suffices to prove our lower bound in the case of $I = I_{a_1} \cdots I_{a_k}$. It follows from Theorem 1.4 of [6] that the exponential rate of growth of such an algebra is $\ell = a_1^2 + \cdots + a_k^2$ and that the degree of the rational part is

$$t = k - 1 + \sum -\frac{1}{2}(a_i^2 - 1) = \frac{3}{2}k - 1 - \frac{1}{2}\ell.$$

We minimize this sum by making k as small as possible. $\square$

It turns out that there is no upper bound on the degree of the rational part. Here is an example taken from [13]. It generalizes easily to any value of ℓ . Let A be the subalgebra of $k \times k$ matrices spanned by e_{11} and all e_{ij} with $i < j$. Then all identities of A are consequences of $[x_1, x_2]x_3 \cdots x_k$ and $c_n(A) \simeq an^{k-1}$. So, the exponent $k-1$ can be made as large as desired while the exponential rate of growth is equal to 1.

1. THE MULTIPLICITIES m_λ

We first describe the relevant theorem from [5]. The space $\mathbb{R}^k$ can be partitioned into a finite number of subsets such that the multiplicity function m_λ restricted to each of these subsets is equal to a polynomial in the coordinates. In order to describe this decomposition, it is useful to have a definition.

Definition 3. Given $v_1, \dots, v_r$ vectors in $\mathbb{R}^k$ the linear cone $C(v_1, \dots, v_r)$ is defined by

$$C(v_1, \dots, v_r) = \{\alpha_1 v_1 + \cdots + \alpha_r v_r \mid \alpha_1, \dots, \alpha_r \geq 0\}.$$

Moreover, given an additional vector v_0 the affine linear cone $C(v_0; v_1, \dots, v_r)$ is defined to be $v_0 + C(v_1, \dots, v_r)$. This is a translation of $C(v_1, \dots, v_r)$ by the vector v_0 . If in addition the vectors $v_1, \dots, v_r$ are linearly independent we call $C(v_1, \dots, v_r)$ a simplicial cone and $C(v_0; v_1, \dots, v_r)$ a simplicial affine cone, and we call $v_1, \dots, v_r$ the basis in either case. Finally, if the v_i can all be taken to have coordinates in $\mathbb{Q}$ we call the resulting cone rational.

Lemma 4. *Any (rational or affine) cone is a finite union of (rational or affine) simplicial cones with disjoint interiors.*

Proof. Let d be the dimension of the span of $v_1, \dots, v_r$. We use induction on d . If $d = 1$, then the cone is a ray and so is already simplicial. If $d > 1$, then the boundary of C is a union of $(d - 1)$ -dimensional cones, so it is a finite union of simplicial cones $v_0 + C_1, \dots, v_0 + C_m$. Let v be a vector with rational coordinates such that $v_0 + v$ is in the interior of C . Then each of the C_i together with v span a simplicial cone C'_i . Since the C_i have disjoint interiors, so do the C'_i ; and since C is convex, the union of the C'_i gives all of C . $\square$

Here then is the theorem we need from [5].

Theorem 5. *Given a p. i. algebra A with cocharacter supported in the strip of height k , there exists a positive integer d and a partition of $(\mathbb{R}^+)^k$ into regions $R_1, \dots, R_m$, each a finite intersection of rational affine linear cones such that m_λ is given by a polynomial on each $R_a \cap (\underline{i} + (d\mathbb{Z})^k)$, where $\underline{i} \in \{0, \dots, d - 1\}^k$.*

Note that here and throughout we use $\mathbb{R}^+$ instead of $\mathbb{R}^{\geq 0}$ for the non-negative real numbers.

Here is an example to illustrate the theorem; see [4] and [9]. It involves trace cocharacters instead of ordinary cocharacters, but the theory is the same.

Theorem. *Let $\lambda = (\lambda_1, \lambda_2)$ be a partition of height at most 2. Then the multiplicity of χ^λ in the trace cocharacter sequence of 3×3 matrices depends on whether $\lambda_1 > 2\lambda_2$. If this is the case, then*

$$m_\lambda = \frac{\lambda_1^2 \lambda_2^5}{17280} - \frac{11\lambda_1 \lambda_2^6}{103680} + \frac{71\lambda_2^7}{1451520} + \mathcal{O}(n^6)$$

and if not, then

$$m_\lambda = \frac{\lambda_1^7}{1451520} - \frac{\lambda_1^6 \lambda_2}{103680} + \frac{\lambda_1^5 \lambda_2^2}{17280} - \frac{\lambda_1^4 \lambda_2^3}{5184} + \frac{\lambda_1^3 \lambda_2^4}{2592} - \frac{7\lambda_1^2 \lambda_2^5}{17280} + \frac{7\lambda_1 \lambda_2^6}{34560} - \frac{19\lambda_2^7}{483840} + \mathcal{O}(n^6).$$

Moreover, each of the two $\mathcal{O}(n^6)$ terms is given by 36 (not necessarily distinct) polynomials, depending on λ_1 and $\lambda_2 \bmod 6$.

Note that the set of partitions is divided into partitions with $\lambda_1 \geq 2\lambda_2$ and $\lambda_1 \leq 2\lambda_2$. These two sets are each *cones*: The former is $C((2, 1), (1, 0))$ and the latter is $C((2, 1), (1, 1))$, and within each of these cones, the partitions are further divided up according to which lattice $(a, b) + 6\mathbb{Z} \times 6\mathbb{Z}$ they belong to.

Definition 6. Given a real function f on $D \subseteq \mathbb{Z}^k$ such that there exists a modulus d and such that f restricted to each $D \cap (\underline{i} + (d\mathbb{Z}))^k$ is equal to a polynomial function for each $\underline{i} \in \{0, \dots, d - 1\}^k$, we will say that f is almost polynomial.

In Theorem 5, each R_i is a finite intersection of affine cones. Our next goal is to show that each such intersection can be written as a finite union of simplicial cones.

Definition 7. A polyhedral set in $\mathbb{R}^k$ is an intersection of finitely many half-spaces. If the half-spaces can all be defined by equations with rational coefficients, we call the polyhedral set rational. Note that the intersection of the half-spaces $a_1 x_1 + \dots + a_k x_k \geq b$ and $a_1 x_1 + \dots + a_k x_k \leq b$ is an $(n - 1)$ -dimensional hyperplane, and so a polyhedral set may be contained in a hyperplane of dimension less than k . The dimension of a polyhedral set is the dimension of the smallest hyperplane containing it. (We use the term hyperplane to refer to a translate of a subspace of $\mathbb{R}^k$, not necessarily of dimension $k - 1$.) Note that an intersection of affine cones

FIGURE 1. Proof of Lemma 10

is a polyhedral set. Given $S \subseteq \mathbb{R}^k$ we denote by $\text{conv}(S)$ the convex closure of S , i.e., the smallest convex set containing S .

Lemma 8. *Let $F \subseteq \mathbb{R}^k$ be a d -dimensional rational polyhedral set contained in the d -dimensional hyperplane Π and $p \in \mathbb{R}^k - \Pi$. Then there exist finitely many disjoint rational polyhedral sets each with the same dimension as F and together containing all of the integer points of $\text{conv}(p, F)$.*

Proof. Let Π' be a $(k-1)$ -dimensional rational hyperplane containing Π and not containing p . Let Π' have equation $a_1x_1 + \cdots + a_kx_k = b$, where the a_i and b are all integers and the a_i are relatively prime. Let $p = (p_1, \dots, p_k)$ and let $a_1p_1 + \cdots + a_kp_k = b'$. Note that $\text{conv}(p, F)$ is contained between these two hyperplanes. But there are only finitely many integers b_j between b and b' (inclusive) and so only finitely many hyperplanes parallel to Π' containing integer points. Each of these hyperplanes intersects $\text{conv}(F, p)$ in a polyhedral set having the same dimension as F . $\square$

We now need a general fact from convex set theory. Let $S \subseteq \mathbb{R}^k$ be any infinite, closed, convex set, and let p and q be two points of S . Given a $v \in \mathbb{R}^k$ the ray $p + \mathbb{R}^+v$ will be contained in S if and only if the ray $q + \mathbb{R}^+v$ is contained in S . Based on this fact, the characteristic cone of S is defined to be

$$\text{cc}(S) = \{v \mid p + \mathbb{R}^+v \subseteq S\}.$$

An important theorem about $\text{cc}(S)$ is that it is indeed a cone and if S is a polyhedral set, then it will be a linear cone; see for example section 2.5 of [12] or section 1.5 of [16]. We record this as a lemma.

Lemma 9. *Let $S \subseteq \mathbb{R}^k$ be any infinite, closed, convex set, and let p be a point of S . Then the set of rays with endpoint p which are completely contained in S forms a cone.*

Lemma 10. *Let F be a rational polyhedral set in $\mathbb{R}^k$. Then there exist finitely many rational affine linear cones whose interiors are disjoint and whose union contains all but finitely many of the integer points of F .*

Proof. We use induction on $d = \dim F$ to prove the lemma. The case of $d = 1$ is easy since a convex one-dimensional set is a line segment or a ray. A line segment will contain only finitely many integer points, and a ray is an affine linear cone.

In the general case, let F be a rational polyhedral set of dimension d and let p be a rational point in the interior of F ; see Figure 1. The $(d-1)$ -dimensional

faces of F are called facets and their union is the boundary of F . Let $F_1, \dots, F_a$ be the facets of F and for each i let $P_i = \text{conv}(F_i, p)$. For example, in Figure 1, the two-dimensional figure F is bounded by the one-dimensional facets $F_1, \dots, F_4$. Note that if the facet F_i is contained by the $(d-1)$ -dimensional hyperplane Π_i , then $p \in \mathbb{R}^k - \Pi_i$; hence Lemma 8 can be applied. By that lemma, the integer points of P_i are covered by finitely many $(d-1)$ -dimensional rational polyhedral sets. So there exist polyhedral sets of dimension $d-1$, P_{ij} , such that $\cup_j P_{ij}$ has the same integer points as P_i . By induction each P_{ij} is the union of disjoint rational affine cones. In the example, P_2 and P_3 are finite and will have only finitely many integer points, and these points are not covered by cones in this example. The regions P_1 and P_4 are infinite, and Lemma 8 implies that in each there are a finite number of rays containing all of their rational points.

However, the union of the P_i may not cover all of F . The points of $F - \cup P_i$ consist of the union of all rays with endpoint p which do not intersect the boundary of F . Hence it is the union of all rays starting at p and completely contained in F . In the example, $F - \cup P_i$ is a 2-dimensional cone. In general, we may apply Lemma 9 to conclude that it is an affine cone. Hence F will be the union of this cone with the P_i , and since there are finitely many affine cones containing all but finitely many of the integer points of $\cup P_i$, this completes the proof. $\square$

Corollary 11. *Let F be a rational polyhedral set in $\mathbb{R}^k$. Then there exist finitely many disjoint rational, affine simplicial cones whose union contains all but finitely many of the integer points of R .*

Proof. In Lemma 10, if two of the cones intersect along their boundary we replace one by a slightly smaller cone and then use Lemma 4 to replace the cones by simplicial ones. $\square$

In light of this corollary, we may now reformulate Theorem 5.

Corollary 12. *Given a p. i. algebra A with cocharacter supported in the strip of height k , there exist a positive integer d and a partition of $(\mathbb{R}^+)^k$ into disjoint rational, affine simplicial cones $C_1, \dots, C_m$ and a finite number of points such that m_λ is given by a polynomial on each $C_a \cap (\underline{i} + (d\mathbb{Z})^k)$, where $\underline{i} \in \{0, \dots, d-1\}^k$.*

One of the main results of [14] is that if λ has all parts close to equal, then d_λ has exponential behaviour k^n . Lemma 16 below shows that if none of the basis vectors is $(1, \dots, 1)$, then the exponential behaviour of d_λ is strictly less.

Remark 13. Given $v \in \mathbb{R}^k$, denote $v = (v^{(1)}, \dots, v^{(k)})$. It is a partition if $v \in \mathbb{Z}^k$ and $v^{(1)} \geq \dots \geq v^{(k)} \geq 0$. Given $u = (u^{(1)}, \dots, u^{(k)}) \in \mathbb{R}^k$ with $u^{(1)}, \dots, u^{(k)} \geq 0$, denote $|u| = u^{(1)} + \dots + u^{(k)}$. Let $C = C(v_0; v_1, \dots, v_r)$ be a cone, (w.l.o.g.) $v_1, \dots, v_r \neq 0$, and let $u = (u^{(1)}, \dots, u^{(k)}) \in C$, so $u = v_0 + \sum_{i=1}^r t_i v_i$; if $v_0, \dots, v_r$ are partitions, then $u^{(1)} \geq \dots \geq u^{(k)} \geq 0$ and $|u| = |v_0| + \sum_{i=1}^r t_i |v_i|$. Moreover, since $v_1, \dots, v_r \neq 0$, $|u| \rightarrow \infty$ whenever $\sum_i t_i \rightarrow \infty$. If the partition v_i is not proportional to $(1, \dots, 1)$, then there exist $0 \leq \delta_i < 1$ such that $v_i^{(k)} \leq (\delta_i/k)|v_i|$.

Lemma 14. *Let $C = C(v_0; v_1, \dots, v_r) \subseteq \mathbb{R}^k$ be a cone, $v_1, \dots, v_r \neq 0$ partitions where none is proportional to $(1, \dots, 1)$. There exist $0 \leq \delta < 1$ such that for all $u = (u^{(1)}, \dots, u^{(k)}) = v_0 + \sum_{i=1}^r t_i v_i \in C$, $u^{(k)} \leq (\delta/k)|u|$, provided that $\sum_i t_i$ is large enough.*

Proof. Write $v_i = (v_i^{(1)}, \dots, v_i^{(k)})$, so $|v_i| = v_i^{(1)} + \dots + v_i^{(k)}$, $i = 1, \dots, r$. For each i there exists $0 \leq \delta_i < 1$ such that $v_i^{(k)} \leq (\delta_i/k)|v_i|$.

Let $\delta' = \max\{\delta_i \mid i = 1, \dots, r\}$, so $0 \leq \delta' < 1$. Choose some $\delta' < \delta < 1$, for example $\delta = (\delta' + 1)/2$. Now let $u = (u^{(1)}, \dots, u^{(k)}) = v_0 + \sum_{i=1}^r t_i v_i \in C$. Then

$$u^{(k)} = v_0^{(k)} + \sum_{i=1}^r t_i v_i^{(k)} \leq v_0^{(k)} + \sum_{i=1}^r t_i \frac{\delta_i}{k} |v_i| \leq v_0^{(k)} + \frac{\delta'}{k} \sum_{i=1}^r t_i |v_i| = v_0^{(k)} + \frac{\delta'}{k} |u|.$$

If $\sum_{i=1}^r t_i$ is large, then also $\sum_{i=1}^r t_i |v_i|$ becomes large and $v_0^{(k)}$ is negligible, so $v_0^{(k)} + \frac{\delta'}{k} |u| \leq \frac{\delta}{k} |u|$. Therefore $u^{(k)} \leq \frac{\delta}{k} |u|$. $\square$

Lemma 15. *Let $0 \leq \delta < 1$ and consider partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ satisfying $\lambda_k \leq (\delta/k)|\lambda|$. Then, as $|\lambda| \rightarrow \infty$, the exponential growth of $d_\lambda (= f^\lambda)$ is bounded by $(k - \epsilon)^{|\lambda|}$ for some $\epsilon > 0$.*

Proof. Let $n = |\lambda|$. By the hook formula,

$$(1) \quad d_\lambda = \frac{n!}{\prod h_{ij}} \leq \frac{n!}{\lambda_1! \cdots \lambda_k!}.$$

Write $\lambda_k = (\frac{1}{k} - x) \cdot n$, so $0 < (1 - \delta)/k \leq x$. Clearly, $\lambda_1 + \dots + \lambda_{k-1} = n - \lambda_k = (\frac{k-1}{k} + x) \cdot n$. Fixing λ_k , it is well known that the product $\lambda_1! \cdots \lambda_{k-1}!$ minimizes when $\lambda_1 \approx \dots \approx \lambda_{k-1}$, so when $\lambda_1 \approx \dots \approx \lambda_{k-1} \approx (\frac{1}{k} + \frac{x}{k-1}) \cdot n$, in which case $\frac{n!}{\prod h_{ij}} \leq \frac{n!}{\lambda_1! \cdots \lambda_k!}$ maximizes. Hence, by (1),

$$d_\lambda \leq \frac{n!}{\lambda_1! \cdots \lambda_k!},$$

$$(2) \quad \lambda_1 \approx \dots \approx \lambda_{k-1} \approx \left(\frac{1}{k} + \frac{x}{k-1}\right) \cdot n, \quad \lambda_k = \left(\frac{1}{k} - x\right) \cdot n.$$

Apply Stirling's formula $m! \approx \sqrt{2\pi} \cdot e^{-m} m^m \sqrt{m}$ to the r.h.s. of (2). After discarding a few terms of the order of magnitude of $\sqrt{n}$ we have $d_\lambda \lesssim (1/g(x))^n$ where

$$g(x) = \left(\frac{1}{k} + \frac{x}{k-1}\right)^{(k-1)(\frac{1}{k} + \frac{x}{k-1})} \left(\frac{1}{k} - x\right)^{(\frac{1}{k} - x)}.$$

It is easy to show that $g(x)$ monotonically increases: let $h(x) = \ln g(x)$; then $h'(x) = \ln[(\frac{1}{k} + \frac{x}{k-1})/(\frac{1}{k} - x)] > 0$. Thus $1/g(x)$ decreases with x . Since $g(0) = 1/k$, therefore for $0 < x \leq 1/k$, $g(x) > 1/k$ and we can write $1/g(x) = k - \epsilon$ for some $\epsilon > 0$. $\square$

As a corollary of Lemmas 14 and 15 we now have

Lemma 16. *Let $C = C(v_0; v_1, \dots, v_r) \subseteq \mathbb{R}^k$ be an affine linear cone in which the basis vectors are all partitions and none is a multiple of $(1, \dots, 1)$. Then there exists an $\epsilon > 0$ and a polynomial $p(x_1, \dots, x_k)$ such that for every partition $\lambda \in C$, $d_\lambda \leq p(\lambda)(k - \epsilon)^{|\lambda|}$.*

Given $0 \leq \ell \leq k$ we let $\mathbb{R}^{k,\ell} \subseteq \mathbb{R}^k$ be the vectors with the last $k - \ell$ coordinates all equal to zero and we let $1_{k,\ell}$ be the vector in $\mathbb{R}^{k,\ell}$ with first ℓ coordinates equal to 1. We now have this corollary.

Corollary 17. *Let $C = C(v_0; v_1, \dots, v_r)$ be an affine linear cone in $\mathbb{R}^k$ in which $v_1, \dots, v_r \in \mathbb{R}^{k, \ell}$ and none of $v_1, \dots, v_r$ is a multiple of $1_{k, \ell}$. Then there exists an $\epsilon > 0$ and a polynomial $p(x_1, \dots, x_k)$ such that for every partition $\lambda \in C$, $d_\lambda \leq p(\lambda)(\ell - \epsilon)^{|\lambda|}$.*

In [10] Giambruno and Zaicev proved that for any p. i. algebra A satisfying a Capelli identity of degree k , the sequence $\{\sqrt[n]{c_n(A)}\}$ has a limit $\ell = e(A)$ which is a non-negative integer less than or equal to k . (Giambruno and Zaicev denote the limit by e for exponential rate of growth, but we need to reserve e for $2.71 \dots$.) In the course of proving this (Theorem 3 in their paper), they also proved that there was a constant K (in their paper $K = dq - d$) such that for every n , the cocharacter sequence contains χ^λ , $\lambda \in \text{Par}(n)$, with non-zero multiplicity and λ close to a rectangle of height ℓ , in the sense that $\sum_{i=1}^\ell |\lambda_i - \frac{n}{\ell}| \leq K$, and $\sum_{i>\ell} \lambda_i \leq K$, and that it doesn't contain any χ^λ with $\lambda = (\lambda_1, \lambda_2, \dots)$ and with $\lambda_{\ell+1}$ large. Translating these two facts into the language of Theorem 5 yields the following lemma. We attribute it to Giambruno and Zaicev since it is immediate from their work. To state the lemma more easily, let $C = C(v_0; v_1, \dots, v_r)$ be a simplicial cone. We will say that C is in the support of m_λ if m_λ is a non-zero polynomial on the intersection of C with some lattice $\underline{z} = (d\mathbb{Z})^k$.

Lemma 18 (Giambruno and Zaicev). *Let A be a p. i. algebra which satisfies a Capelli identity and with exponential rate of growth ℓ of the codimensions. Let C be a simplicial cone in the support of m_λ , as above. Then each of the basis elements of C is a partition of height at most ℓ . Moreover, there exists at least one simplicial cone C in the support of m_λ with basis element (1^ℓ) .*

Proof. Let $C = C(v_0; v_1, \dots, v_t)$ be a simplicial cone in the support of m_λ . So, on the intersection of C and a lattice, the multiplicity function equals a non-zero polynomial $p(\lambda_1, \dots, \lambda_k)$. Let $q(\alpha_1, \dots, \alpha_t)$ be the multiplicity of $\lambda = \alpha_1 v_1 + \dots + \alpha_t v_t$. Note that each λ_j equals $\sum_i \alpha_i v_{j,i}$, where $v_{j,i}$ is the i^{th} coordinate of v_j , and so q is obtained from p by a linear substitution. Hence q is also a non-zero polynomial. Assume by way of contradiction that v_1 is a partition of height $\geq \ell + 1$. Write q as $\sum_i \alpha_1^i q_i(\alpha_2, \dots, \alpha_t)$. For any choice of $\alpha_2, \dots, \alpha_t$ in the lattice, there would exist infinitely many values of α_1 in the lattice such that $v_0 + \alpha_1 v_1 + \dots + \alpha_t v_t$ is not a partition with $\lambda_{\ell+1}$ small. By Giambruno and Zaicev's theorem, the multiplicity m_λ would be zero in these cases. Hence, by a Vandermonde argument each q_i would be zero. This would imply that m_λ is identically zero.

On the other hand, if there were no simplicial cones with (1^ℓ) as a basis element, then the exponential rate of growth would be less than ℓ , by Corollary 17. $\square$

Theorem 19. *Let A be a p. i. algebra which satisfies a Capelli identity of degree k such that $e(A) = \ell$. Then there exists a set of affine simplicial cones $C_1, \dots, C_a \subseteq \mathbb{R}^k$, each with $1_{k, \ell}$ as a basis vector and with all basis vector partitions in $\mathbb{R}^{k, \ell}$ such that*

$$c_n(A) \simeq \sum_{i=1}^a \{m_\lambda d_\lambda \mid \lambda \in \text{Par}(n) \cap C_i\}$$

and such that the multiplicity function m_λ restricted to each C_i is almost polynomial.

Proof. Combining Theorem 5 with Corollary 11 shows that $\text{Par}(n)$ can be written as a union of a finite number of disjoint rational affine simplicial cones $\{C'_i\}$ containing all partitions of height at most k such that m_λ is almost polynomial in each one of them. Hence,

$$c_n(A) \simeq \sum_i \{m_\lambda d_\lambda \mid \lambda \in \text{Par}(n) \cap C'_i\}.$$

If $1_{k,\ell}$ is not a basis element of any such C'_i , then by Corollary 17 the sum over C'_i is at most $Cn^t(\ell - \epsilon)^n$ for appropriate constants. But the Giambruno and Zaicev lemma shows that $c_n(A)$ is bounded below by a function of the form $Cn^t\ell^n$ and so the sum of over the C'_i not containing $1_{k,\ell}$ may be ignored. $\square$

2. THE SUMS $\sum m_\lambda d_\lambda$ IN CONES

In this section we fix an affine simplicial cone $\bar{C} = C(v_0; v_1, \dots, v_r) \subset \mathbb{R}^k$ whose basis vectors are partitions contained in $\mathbb{R}^{k,\ell}$ and the first one is $v_1 = 1_{k,\ell}$, a lattice $\underline{i} + (d\mathbb{Z})^\ell$, and a non-negative polynomial function with rational coefficients, m_λ . It is our goal to estimate

$$\sum \{d_\lambda m_\lambda \mid \lambda \in \bar{C} \cap (\underline{i} + (d\mathbb{Z})^\ell) \cap \text{Par}(n)\}.$$

For convenience we modify this slightly by letting C be the corresponding linear cone $C = C(v_1, \dots, v_r)$ and replacing $\underline{i}$ by $\underline{i} - v_0$. This changes the sum to

$$(3) \quad \sum \{d_{\lambda+v_0} m_{\lambda+v_0} \mid \lambda \in C \cap (\underline{i} + (d\mathbb{Z})^\ell) \cap \text{Par}(n - |v_0|)\}.$$

Our computation of this sum will imitate the computations in sections 1 and 2 of [14]. In that paper the asymptotic behavior of $\sum d_\lambda^\beta$ is computed, where λ was restricted to have at most ℓ non-zero parts. In adapting the proof to the current situation we need to deal with the facts that our λ is additionally restricted to be in an affine cone and a lattice, and that the sum is of a polynomial times d_λ instead of a power of d_λ .

It is possible that some $\lambda_i = \lambda_j$ for every $\lambda \in C$. As in [14], we define $\theta_1, \dots, \theta_p$ to be maximal such that

$$(4) \quad \lambda_1 = \dots = \lambda_{\theta_1}, \lambda_{\theta_1+1} = \dots = \lambda_{\theta_1+\theta_2}, \dots \text{ for all } \lambda \in C.$$

Remark 20. Note that the sum $\sum m_\lambda d_\lambda$ with the above restrictions on λ will be empty unless the sum of the parts of $\underline{i}$ is congruent to $n \bmod d$, which we take to be the case.

Generalizing the notation $\Lambda_\ell(n)$ from [14] we define $\Lambda_\ell^\theta(n)$ to be the set of all partitions of n with height at most ℓ that satisfy equation (4). The degrees d_λ of such partitions are estimated in [14], and we now recall some notation from that paper.

Definition 21. Given a partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of n , define c_i by $\lambda_i = \frac{n}{\ell} + c_i \sqrt{n}$, $i = 1, \dots, \ell$. Let $\bar{D}(c) = \prod \{(c_i - c_j) \mid i < j, c_i \neq c_j\}$, $E(c) = \prod \{(i - j) \mid j < i, c_i = c_j\}$, and $\gamma_\ell = (2\pi)^{-\frac{\ell-1}{2}} \ell^{\frac{1}{2}\ell^2}$.

Let $\lambda \in \Lambda_\ell^\theta(n)$ with each $\lambda_{\theta_1+\dots+\theta_i} \neq \lambda_{\theta_1+\dots+\theta_{i+1}}$. Then

$$\bar{D}(c) = D^\theta(c) = \prod \{(c_{\theta_1+\dots+\theta_i} - c_{\theta_1+\dots+\theta_{i+1}}) \mid i < j\}$$

and $E(c) = (\theta_1 - 1)! \dots (\theta_p - 1)!$

Lemma 22 (= Generalization 1.2 of [14]). *With notation as in the previous definition, assume that $\lambda \in \Lambda_\ell^\theta(n)$ is such that each $|c_i| \leq a$ and that each $c_{\theta_1+\dots+\theta_i} - c_{\theta_1+\dots+\theta_{i+1}}$ is greater than or equal to some fixed $\delta > 0$. Then*

$$d_\lambda \simeq \gamma_\lambda D^\theta(c) \left(\prod (\theta_i - 1)! \right) e^{-\frac{\ell}{2}c^2} n^{-\frac{\ell^2-1}{2}} \sqrt{n}^{\binom{\ell}{2} - \sum_i \binom{\theta_i}{2}} \ell^n,$$

where by c^2 we mean $c_1^2 + \dots + c_\ell^2$.

In order to take the v_0 into account we use this lemma.

Lemma 23. *Let $1 \leq \ell \leq k$, let ν be a fixed partition of height k and denote by $\tilde{\nu}$ the sum $\sum_{i>\ell} \nu_i$. Let $\lambda \in \Lambda_\ell^\theta(n)$ be such that each $\lambda_{\theta_1+\dots+\theta_i} - \lambda_{\theta_1+\dots+\theta_{i+1}}$ goes to infinity. As n goes to infinity,*

$$\frac{d_{\lambda+\nu}}{d_\lambda} \simeq b n^{\tilde{\nu}},$$

where $b \in \mathbb{Q}$ is a constant.

Proof. By the Young–Frobenius formula (or by the hook formula),

$$d_\lambda = \frac{|\lambda|!}{\prod_{i=1}^k (\lambda_i + k - i)!} \prod_{1 \leq i < j \leq k} \lambda_i - \lambda_j + j - i,$$

and similarly

$$d_{\lambda+\nu} = \frac{|\lambda+\nu|!}{\prod_{i=1}^k (\lambda_i + \nu_i + k - i)!} \prod_{1 \leq i < j \leq k} \lambda_i + \nu_i - \lambda_j - \nu_j + j - i.$$

Compare corresponding terms in the ratio $d_{\lambda+\nu}/d_\lambda$. The fraction $|\lambda+\nu|!/|\lambda|!$ is asymptotic to $n^{|\nu|}$; for $i = 1, \dots, \ell$ the fraction $(\lambda_i + k - i)!/(\lambda_i + \nu_i + k - i)!$ is asymptotic to $\lambda_i^{-\nu_i}$; and for $i > \ell$ it is the rational number $(k - i)!/(\nu_i + k - i)!$ (independent of λ). Since $|c_i| < a$, $\lambda_i \simeq n/\ell$, $i = 1, \dots, \ell$, so

$$\lambda_1^{-\nu_1} \dots \lambda_\ell^{-\nu_\ell} \simeq \left(\frac{\ell}{n} \right)^{\nu_1 + \dots + \nu_\ell}.$$

Multiplying by $n^{|\nu|}$ gives a rational number times $n^{\tilde{\nu}}$. Finally, estimate

$$A := \prod_{1 \leq i < j \leq k} \frac{\lambda_i + \nu_i - \lambda_j - \nu_j + j - i}{\lambda_i - \lambda_j + j - i}.$$

If $i \leq \ell$, then $\lambda_i - \lambda_j \rightarrow \infty$; hence

$$\frac{\lambda_i + \nu_i - \lambda_j - \nu_j + j - i}{\lambda_i - \lambda_j + j - i} \simeq 1.$$

If $i > \ell$, then $\lambda_i = \lambda_j = 0$, and the corresponding term is a rational number. It follows that $A \simeq r$ for some $r \in \mathbb{Q}$. The lemma now follows. $\square$

We now recall and modify some more definitions from [14].

Definition 24. Let $\Lambda_\ell^\theta(n, a)$ be the partitions in $\Lambda_\ell^\theta(n)$ with each $|c_i| \leq a$ and $\Lambda_\ell^\theta(n, a, \delta)$ be the partitions in $\Lambda_\ell^\theta(n, a)$ with each $c_{\theta_1+\dots+\theta_i} - c_{\theta_1+\dots+\theta_{i+1}} \geq \delta$.

Combining this definition with the previous two lemmas yields:

Corollary 25. *If $\lambda \in \Lambda_\ell^\theta(n, a, \delta)$, then*

$$d_{\lambda+v_0} \simeq KD^\theta(c)e^{-\frac{\ell}{2}c^2}n^{\bar{v}_0-\frac{\ell^2-1}{2}}\sqrt{n}^{\binom{\ell}{2}-\sum_i\binom{\theta_i}{2}}\ell^n,$$

where K is a constant in $\mathbb{Q}(\sqrt{\pi})$ depending only on ℓ , θ and v_0 .

Recall that C is the simplicial cone $C = (v_1, \dots, v_r)$ where $v_1 = 1_{k,\ell}$. Since $v_1, \dots, v_r$ are linearly independent and by Lemma 18 they belong to $\mathbb{R}^{k,\ell}$, r will be less than or equal to $\ell = \dim \mathbb{R}^{k,\ell}$, since elements of $\mathbb{R}^{k,\ell}$ have k coordinates, but the last $k - \ell$ coordinates are $= 0$.

Let $\lambda \in C$, so it is a positive combination of the v_i 's, and write that combination in the form

$$\lambda = \left(\frac{n}{\ell} + \alpha_1\sqrt{n}\right)v_1 + \alpha_2\sqrt{n}v_2 + \dots + \alpha_r\sqrt{n}v_r.$$

In order for the sum to be in C we require that $\alpha_i \geq 0$ for $i = 2, \dots, r$ and $\frac{n}{\ell} + \alpha_1\sqrt{n} \geq 0$. Given the partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$, write $\lambda_i = \frac{n}{\ell} + c_i\sqrt{n}$. Comparing to the previous description we have

$$\left(\frac{n}{\ell} + \alpha_1\sqrt{n}\right)v_1 + \alpha_2\sqrt{n}v_2 + \dots + \alpha_r\sqrt{n}v_r = \left(\frac{n}{\ell} + c_1\sqrt{n}, \dots, \frac{n}{\ell} + c_r\sqrt{n}\right).$$

Keeping in mind that $v_1 = 1_{k,\ell}$, this implies that

$$\alpha_1v_1 + \dots + \alpha_rv_r = (c_1, \dots, c_\ell)$$

and so we define

$$(5) \quad T(\alpha_1, \dots, \alpha_r) = \sum_{i=1}^r \alpha_i v_i = (c_1, \dots, c_\ell).$$

Let τ_n be given by

$$\tau_n(\alpha) = \frac{n}{\ell}v_1 + \sqrt{n}T(\alpha) = \frac{n}{\ell}v_1 + \sqrt{n}\sum_{i=1}^r \alpha_i v_i.$$

Since $C(v_1, \dots, v_r)$ is assumed to be simplicial, the v_i are linearly independent and so T and hence τ_n are one-to-one. Note that $|v_1| = \ell$ and $\tau_n(\alpha)$ will have parts sum to n precisely when $\sum \alpha_i |v_i| = 0$, where by $|v_i|$ we mean the sum of the parts of v_i . This is equivalent to

$$\alpha_1 = -\sum_{i=2}^r \frac{|v_i|}{\ell} \alpha_i.$$

Definition 26. Define $Q(a) \subseteq (\mathbb{R}^+)^{r-1}$ as follows. Let $(x_2, \dots, x_r) \in (\mathbb{R}^+)^{r-1}$ and denote $x_1 = x_1(x) = -\sum_{i=2}^r \frac{|v_i|}{\ell} x_i$. By (5), $T(x_1(x), x) = T(x_1, x_2, \dots, x_r) = (c_1, \dots, c_\ell)$. Given $a > 0$, define $Q(a)$ via: $x = (x_2, \dots, x_r) \in Q(a)$ iff $T(x_1(x), x) = (c_1, \dots, c_\ell)$ and all $|c_i| \leq a$. Continuing in this vein, note that $c_1 = \dots = c_{\theta_1}$, $c_{\theta_1+1} = \dots = c_{\theta_1+\theta_2}$, ..., in the image of T and so define $Q(a, \delta)$ to be the subset of $Q(a)$ with the additional conditions $c_{\theta_1+\dots+\theta_i} - c_{\theta_1+\dots+\theta_i+1} \geq \delta$.

Remark 27. Note that both $Q(a)$ and $Q(a, a^{-r})$ tend to $(\mathbb{R}^+)^{r-1}$ as $a \rightarrow \infty$.

Definition 28. Let

$$S_{\overline{C}}(n) = \sum \{m_{v_0+\lambda} d_{v_0+\lambda} \mid \lambda \in \text{Par}(n) \cap C \cap (\underline{i} + (d\mathbb{Z})^\ell)\}.$$

Two related sums are

$$S_{\overline{C}}(n, a, \theta) = \sum \{m_{v_0+\lambda} d_{v_0+\lambda} \mid \lambda \in \Lambda_{\ell}^{\theta}(n, a) \cap C \cap (\underline{i} + (d\mathbb{Z})^{\ell})\}$$

and

$$S_{\overline{C}}(n, a, \delta, \theta) = \sum \{m_{v_0+\lambda} d_{v_0+\lambda} \mid \lambda \in \Lambda_{\ell}^{\theta}(n, a, \delta, \theta) \cap C \cap (\underline{i} + (d\mathbb{Z})^{\ell})\}.$$

Remark 29. Note by equation (3) that $S_C(n)$ represents the contribution of the characters in the cone C intersecting the lattice $\underline{i} + (d\mathbb{Z})^{\ell}$ to the codimension, and recall by Remark 20 that $\text{Par}(n) \cap (\underline{i} + (d\mathbb{Z})^{\ell})$ will be empty unless $n \equiv |i| \pmod{d}$. Hence, if $n \not\equiv |i| \pmod{d}$, each of $S_{\overline{C}}(n)$, $S_{\overline{C}}(n, a)$, etc. equals zero.

We note that m_{λ} is assumed to be a non-negative polynomial function of $\lambda_1, \dots, \lambda_{\ell}$ with rational coefficients. Let the vector v_i have coordinates v_{ij} . Then it follows from the definition of τ_n that if $\lambda = \tau_n(\alpha)$, then each λ_j equals $v_{1j} \frac{n}{\ell} + \sqrt{n} \sum_i \alpha_i v_{ij}$. Let the multiplicity function m_{λ} be equal to the polynomial $m(x_1, \dots, x_{\ell})$ on the intersection of the cone and the lattice. By substitution,

$$m_{v_0+\lambda} = m(v_0 + \frac{n}{\ell} v_1 + \sqrt{n} \sum \alpha_i v_i),$$

and so each x_j is replaced by $v_{0j} + v_{1j} \frac{n}{\ell} + \sum_i \alpha_i v_{ij}$ and so

$$m_{v_0+\lambda} \simeq n^{\mu} g(\alpha_1, \dots, \alpha_r),$$

where μ is an integer or a half-integer and g is a polynomial with rational coefficients. The following is a direct analogue of Lemma 2.2 of [14] and the proof is the same.

Lemma 30. *Referring to Remarks 20 and 29, let $n \equiv |\underline{i}| \pmod{d}$. Then, as $n \rightarrow \infty$ (namely, $n = |\underline{i}| + q \cdot d$ and $q \rightarrow \infty$), the sum $S_{\overline{C}}(n, a, \delta, \theta)$ is asymptotic to*

$$\sum n^{\mu} f(\alpha) e^{-\frac{\ell}{2} T(\alpha)^2} n^{\tilde{v}_0 - \frac{\ell^2 - 1}{2}} \sqrt{n}^{\binom{\ell}{2} - \sum_j \binom{\theta_j}{2}} \ell^{n + |v_0|},$$

where the sum is over $(\alpha_2, \dots, \alpha_r) \in Q(a, \delta)$, $\alpha_1 = -\sum \frac{|v_i|}{\ell} \alpha_i$ such that $\tau_n(\alpha) \in \underline{i} + (d\mathbb{Z})^{\ell}$, and where $f(\alpha) = g(\alpha) K D^{\theta}(T(\alpha))$.

Note that $Q(a)$ and $Q(a, \delta)$ are bounded $(r-1)$ -dimensional surfaces and we can consider their volumes. The proofs of Lemmas 2.3 and 2.4 of [14] now go over essentially unchanged.

Lemma 31 (= Lemma 2.4 of [14]). *If $\delta = \frac{1}{a^r}$, then*

$$\lim_{a \rightarrow \infty} \text{vol}_{r-1}(Q(n, a) - Q(n, a, \delta)) = 0.$$

Following [14] we let $D_{\delta}(c)$ equal $\prod \{(c_i - c_j) \mid i < j, c_i - c_j \geq \delta\}$.

We now record the analogues of the next three lemmas of [14]. The proofs are essentially unchanged, although since we are restricting to partitions in the intersection of the cone C with the lattice $(\underline{i} + (d\mathbb{Z})^{\ell})$ we replace the condition that $\frac{n}{\ell} + c_j \sqrt{n} \geq 0$ must be an integer with $\tau_n(\alpha) \in (\underline{i} + (d\mathbb{Z})^{\ell})$ and, in order for the intersection to be non-empty, we require $n \equiv |\underline{i}| \pmod{d}$ as in the previous lemma.

Lemma 32 (= Proposition 2.5 of [14]). *Let $n \equiv |\underline{i}| \pmod{d}$ and let*

$$u(n, a) = \sum g(\alpha) D_{a^{-r}}(T(\alpha)) e^{-\frac{\ell}{2} (T(\alpha)^2)}$$

be summed over $(\alpha_2, \dots, \alpha_r) \in Q(n, a) - Q(n, a, a^{-r})$ and $\alpha_1 = -\sum \frac{|v_i|}{\ell} \alpha_i$ such that $\tau_n(\alpha) \in (\underline{i} + (d\mathbb{Z})^\ell)$. As $n \rightarrow \infty$ (namely, $n = |\underline{j}| + q \cdot d$ and $q \rightarrow \infty$),

$$\lim_{a \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{u(n, a)}{\sqrt{n}^{r-1}} = 0.$$

Now, we may choose a small cube U in $Q(1)$ with volume V_1 such that

$$g(\alpha) D_{a^{-r}}(T(\alpha)) e^{-\frac{\ell}{2}(T(\alpha)^2)} \geq m > 0 \quad \text{on } U.$$

We now have this analogue of Lemmas 2.6 and 2.7 of [14].

Lemma 33 (= Lemma 2.6 of [14]). *If $a \geq 1$ and a is so large that $Q(a, a^{-r})$ contains U , then $\lim_{n \rightarrow \infty} \frac{v(n, a)}{\sqrt{n}^{r-1}} \geq mV_1$, where*

$$v(n, a) = \sum g(\alpha) D_{a^{-r}}(T(\alpha)) e^{-\frac{\ell}{2}(T(\alpha)^2)}$$

summed over $(\alpha_2, \dots, \alpha_r) \in Q(n, a, a^{-r})$ and $\alpha_1 = -\sum \frac{|v_i|}{\ell} \alpha_i$ such that $\tau_n(\alpha) \in (\underline{i} + (d\mathbb{Z})^\ell)$.

Lemma 34 (= Lemma 2.7 of [14]). *Let*

$$w(n, a) = g(\alpha) D_\delta(T(\alpha)) e^{-\frac{\ell^3}{3}(T(\alpha)^2)}$$

be summed over $(\alpha_2, \dots, \alpha_r) \notin Q(n, a)$ and $\alpha_1 = -\sum \frac{|v_i|}{\ell} \alpha_i$ such that $\tau_n(\alpha) \in (\underline{i} + (d\mathbb{Z})^\ell)$. Then

$$\lim_{a \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{w(n, a)}{\sqrt{n}^{r-1}} = 0.$$

Just as in [14] we can combine these lemmas together with Lemma 1.6 of that work to get the following analogue of Theorem 2.8 of [10]:

Theorem 35. *Let $n \equiv |\underline{j}| \pmod{d}$, as above, and let $\sigma(n, a) = \frac{S_{\overline{C}}(n)}{S_{\overline{C}}(n, a, a^{-r})}$. Then, as $n \rightarrow \infty$ (namely, $n = |\underline{j}| + q \cdot d$ and $q \rightarrow \infty$), $\lim_{a \rightarrow \infty} \lim_{n \rightarrow \infty} \sigma(n, a) = 1$.*

As in [14], the limit of $\frac{v(n, a)}{\sqrt{n}^{r-1}}$, defined in Lemma 33, converges to a certain integral. To see this, we need some concepts from lattices. The volume of a lattice

$$\mathcal{L} = b_0 + b_a \mathbb{Z} + \dots + b_n \mathbb{Z} \subset \mathbb{R}^n$$

is defined to be the volume of the parallelepiped

$$\text{vol}(\mathcal{L}) = \text{vol}\{x_1 b_1 + \dots + x_n b_n \mid 0 \leq x_i \leq 1, i = 1, \dots, n\}.$$

Now let $D \subseteq \mathbb{R}^n$ be a “nice” domain, $f : D \rightarrow \mathbb{R}$ a continuous function, and $\mathcal{L}$ an n -dimensional lattice with small volume. Then

$$\sum_{x \in \mathcal{L} \cap D} f(x) \approx \frac{1}{v} \int_D f(x) dx,$$

and the two sides become equal as $\text{vol}(\mathcal{L}) \rightarrow 0$.

To apply this theory to our case, we start with the lattice

$$\mathcal{L}_0 = \{(\alpha_2, \dots, \alpha_r) \mid \alpha_1 v_1 + \dots + \alpha_r v_r \in (d\mathbb{Z})^\ell\},$$

where, as usual, $\alpha_1 = -\sum \frac{|\alpha_i|}{\ell}$. Let $\text{vol}(\mathcal{L}_0) = V$. Next define

$$\mathcal{L} = \{(\alpha_2, \dots, \alpha_r) \mid \tau_n(\alpha) \in \vec{v} + (d\mathbb{Z})^\ell\}.$$

Then $\text{vol}(\mathcal{L}) = \frac{V}{\sqrt{n^{r-1}}}$ and so $\lim_{n \rightarrow \infty} \frac{v(n,a)}{\sqrt{n^{r-1}}}$ converges to the integral

$$\frac{1}{V} \int \cdots \int_{Q(a, a^{-r})} g(x_1, \dots, x_r) D^\theta(T(x_1, \dots, x_r)) e^{-\frac{\ell}{2} T(x_1, \dots, x_r)^2} dx_2 \cdots dx_r,$$

where $x_1 = -\sum \frac{|v_i|}{\ell} x_i$. Taking the limit as $a \rightarrow \infty$ gives the integral

$$I(C) = \frac{1}{V} \int_0^\infty \cdots \int_0^\infty g(x_1, \dots, x_r) D^\theta(T(x_1, \dots, x_r)) e^{-\frac{\ell}{2} T(x_1, \dots, x_r)^2} dx_2 \cdots dx_r.$$

Then, analogously to Theorem 2.10 of [14], we have

Theorem 36. *Let $n \equiv |\underline{i}| \pmod{d}$, as above. Then, as $n \rightarrow \infty$,*

$$S_{\overline{C}}(n) \simeq I(C) K \sqrt{n}^{\binom{\ell}{2} - \sum_i \binom{\theta_i}{2} + r - 1} n^{\tilde{v}_0 - \frac{t^2 - 1}{2}} \ell^n.$$

In particular, for these values of n , $S_{\overline{C}}(n)$ is asymptotic to a constant times n to the power of an integer or half-integer times a positive integer to the power of n .

In order to get more information about the constant factor, we would need to know more about the integral $I(C)$. We close this section with a conjecture.

Conjecture. *The integral $I(C)$ is in $\mathbb{Q}[\sqrt{a_1}, \dots, \sqrt{a_n}, \sqrt{2\pi}]$ for some integers $a_1, \dots, a_n$, under the hypothesis that g is a polynomial with rational coefficients and that the vectors v_i used to define T and θ have rational coordinates.*

3. LAST STEPS OF THE PROOF

In the previous section, we estimated $\sum m_\lambda d_\lambda$ for λ in a simplicial cone C , assuming m_λ to be polynomial. In general, $c_n(A)$ will be a sum of such and we now deal with the problem of taking the sum.

Definition 37. Let $w_1 + C_1, \dots, w_a + C_a$ be rational affine simplicial cones as in Theorem 19. I.e., it is a maximal list such that each C_j contains $1_{k,\ell}$ as a basis element and m_λ is almost polynomial on each C_j ; see Definition 6. By Theorem 19 those cones not containing $1_{k,\ell}$ in their basis don't contribute to the asymptotics of $c_n(A)$ and so we may disregard them.

Let d be a modulus such that m_λ equals a polynomial on each $w_j + C_j \cap (\underline{i} + (d\mathbb{N})^k)$, for $\underline{i} \in (d\mathbb{N})^k$. Denote this polynomial by $m(\underline{i}, j)(\lambda)$. As in Remark 29, this intersection of a cone and a lattice will contain partitions of n if and only if $n \equiv |\underline{i}| \pmod{d}$. Denote by $\mathcal{S}(n)$ the set of all $\underline{i} \in (\mathbb{Z}_d)^k$ and $1 \leq j \leq a$ such that

$$\text{Par}(n) \cap (\underline{i} + (d\mathbb{Z})^k) \cap C_j \neq \emptyset.$$

Now, using this notation,

$$c_n(A) \simeq \sum_{(\underline{i}, j) \in \mathcal{S}(n)} \sum \{m_\lambda d_\lambda | \lambda \in \text{Par}(n) \cap (\underline{i} + (d\mathbb{Z})^k) \cap C_j\} = \sum_{(\underline{i}, j) \in \mathcal{S}(n)} S_{C_j}(n - |\underline{i}|).$$

Combining this with Theorem 36 now gives

Lemma 38. *Let d be as in Definition 37 so that m_λ is a polynomial on each $(\underline{i} + (d\mathbb{Z})^k) \cap C_j$ where $1 \leq j \leq a$ and $\underline{i} \in \{0, \dots, d-1\}^k$. Let $0 \leq m \leq d-1$, and let $n \equiv m \pmod{d}$ go to infinity, i.e., $n = qk + m$ and $q \rightarrow \infty$. Then there are constants a_m and t_m , the latter in $\frac{1}{2}\mathbb{Z}$, such that, for these n , the codimension sequence of A satisfies $c_n(A) \simeq a_m n^{t_m} \ell^n$.*

In order to prove that the t_m are equal we need only assume that the c_n are non-decreasing. This is true whenever the algebra A has 1, but it is also true under a weaker hypothesis, such as $xA \neq 0$ for every $0 \neq x \in A$.

Lemma 39. *With notation as in the previous lemma, if $xA \neq 0$ for all $0 \neq x \in A$, then all t_m are equal.*

Proof. The map $f(x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n)x_{n+1}$ takes non-identities of A to non-identities. It follows that the codimension sequence is non-decreasing. If not all t_m were equal, there would be an m such that, say, $t_m > t_{m+1}$. But this would imply that for all $n \equiv m \pmod d$,

$$\frac{c_{n+1}(A)}{c_n(A)} \simeq \frac{a_{m+1}(n+1)^{t_{m+1}}\ell^{n+1}}{a_m n^{t_m}\ell^n} \longrightarrow 0,$$

contradicting the fact that the $c_n(A)$ are non-decreasing. $\square$

In this case we have

$$a_1 n^t \ell^n \leq c_n(A) \leq a_2 n^t \ell^n$$

for some $0 < a_1 \leq a_2$.

Drensky showed in [8] that the cocharacter sequence of A is Young derived, whenever A has a unit. We record that theorem by way of reminding the reader of the definition of Young derived sequences.

Theorem 40. *Let A be a p. i. algebra with unit. Then there exists a sequence of S_n characters $\{\psi_n\}_{n=0}^\infty$ such that for all n , $\chi_n(A) = \sum \chi^{(n-i)} \otimes \psi_i$. Let each $\chi_n(A) = \sum m_\lambda \chi^\lambda$ and each $\psi_n = \sum m'_\lambda \chi^\lambda$. Then, using Young's rule, the relation between $\chi(A)$ and $\{\psi\}$ can be restated as*

$$m_\lambda = \sum \{m'_\nu \mid \nu \subseteq \lambda, \lambda/\nu \text{ a horizontal strip}\}.$$

Letting $c'_n(A)$ be the degree of ψ_n we get

$$(6) \quad c_n(A) = \sum_{i=0}^n \binom{n}{i} c'_i$$

We now need to verify that $c'_n(A)$ has growth properties similar to those of $c_n(A)$ as described in Theorem 36. This will be Lemma 43. In order to prove this, we need to show that the sequence $\{\psi_n\}$ has the same properties of $\chi_n(A)$ that we used in the proof of Theorem 36. First note that $\{\psi_n\}$ is supported by partitions of height at most k , since the cocharacter sequence of A is. The next ingredient we need is Theorem 5. This theorem is a consequence of the fact that the Poincaré series for (the generic algebra of) A is a nice rational function, see [5], which is based on [3]. A nice rational function is one in which the numerator and denominator are polynomials with integer coefficients and the denominator can be written as a product of terms of the form $(1 - u)$, where u is a monic monomial. Section 2 of [5] shows that if a nice rational function is also a symmetric function, then the coefficients of the Schur functions are as in Theorem 5. Let m'_λ denote the multiplicity of χ^λ in $\{\psi_n\}$, so $\psi_n = \sum m'_\lambda \chi^\lambda$. Define $g(x_1, \dots, x_k)$ to be $\sum m'_\lambda S_\lambda(x_1, \dots, x_k)$, where $S_\lambda(x_1, \dots, x_k)$ are Schur functions. Then Drensky proved that g is related to the Poincaré series of A via

$$P_k(A) = \frac{g(x_1, \dots, x_k)}{(1 - x_1) \cdots (1 - x_k)}.$$

Since $P_k(A)$ is a nice rational function, $g(x)$ will be one also, and it follows from the techniques of [5] that the m'_λ satisfy the conclusion of Theorem 5.

Lemma 41. *Let m'_λ be as above. Then there exists an integer d and a partition of $(\mathbb{R}^+)^k$ into regions $R_1, \dots, R_m$, each a finite intersection of rational affine linear cones such that m'_λ is given by a polynomial on each $R_a \cap (\underline{i} + (d\mathbb{Z})^k)$, where $\underline{i} \in \{0, \dots, d-1\}^k$.*

We now prove these analogues of results that Giambruno and Zaicev proved about the cocharacter sequence of A ; see Lemma 18 and the paragraph preceding it.

Lemma 42. *For $e(A) = \ell$, the multiplicities m'_λ are zero for partitions with λ_ℓ large, and, for every n , there are non-zero m'_λ with λ (a partition of $n \cdot (\ell - 1)$) close to the rectangle of height $(n^{\ell-1})$.*

Proof. Theorem 40 says that m_λ is the sum of all m'_ν such that λ/ν is a horizontal strip. This condition is equivalent to $\lambda_{i+1} \leq \nu_i \leq \lambda_i$ for all i . Now if $\nu = (\nu_1, \dots, \nu_k)$ is a partition of height greater than or equal to ℓ with $m'_\nu > 0$, let λ be the partition obtained from ν by increasing $\nu_{\ell+1}$ to ν_ℓ . Then λ/ν is a horizontal strip and so $m_\lambda \geq m'_\nu > 0$. Since $e(A) = \ell$, there is a constant K such that $\lambda_{\ell+1} \leq K$ whenever $m_\lambda > 0$ (see after Corollary 17). Then the above implies that, also, $\nu_\ell \leq K$ if $m'_\nu > 0$.

Likewise, since m_λ is non-zero for some λ close to (n^ℓ) , there must be a ν with m'_ν non-zero and λ/ν a horizontal strip. Hence each $\nu_i - \nu_{i+1}$ is bounded, and by the first part of the lemma, ν_ℓ is bounded. Hence, ν must be close to $n^{\ell-1}$. $\square$

The results of sections 1 and 2 now apply to c'_n , yielding the following analogue of Lemma 38.

Lemma 43. *Let $0 \leq m \leq d-1$ and let $n \equiv m \pmod{d}$ go to infinity. Then there are constants b_m and t'_m with $t'_m \in \frac{1}{2}\mathbb{Z}$ such that $c'_n \simeq b_m n^{t'_m} (\ell - 1)^n$.*

Note that the $c'_n(A)$ need not be increasing, and so the t'_m need not be equal, so we cannot invoke an analogue of Lemma 39 in this case. For example, if A is the infinite-dimensional Grassmann algebra, then $c'_n(A)$ is 0 or 1 depending on whether n is odd or even, respectively.

We now combine this lemma with equation (6) to prove that the $a_m n^{t'_m}$ are all equal. It follows from that equation that

$$c_n(A) = \sum_{q=0}^n \binom{n}{q} c'_q = \sum_{m=0}^{d-1} \sum_{\substack{0 \leq q \leq n \\ q \equiv m \pmod{d}}} \binom{n}{q} c'_q.$$

Lemma 44. *The codimensions $c_n(A)$ are asymptotic to*

$$\sum_{m=0}^{d-1} \sum_{\substack{0 \leq q \leq n \\ q \equiv m \pmod{d}}} \binom{n}{q} b_m \cdot q^{t'_m} (\ell - 1)^q.$$

Proof. Use Lemma 43. Given $\epsilon > 0$ there exists q_0 such that for $q > q_0$,

$$(1 - \epsilon) b_m \cdot q^{t'_m} (\ell - 1)^q \leq c'_q(A) \leq (1 + \epsilon) b_m \cdot q^{t'_m} (\ell - 1)^q,$$

where $q \equiv m \pmod{d}$, for any m . Now observe that $\sum_{q \leq q_0} \binom{n}{q} q^t (\ell - 1)^q$ and $\sum_{q \leq q_0} \binom{n}{q} c'_q(A)$ are both bounded by a polynomial in n and so may be disregarded. $\square$

Lemma 45. *For each $0 \leq m \leq d - 1$, the sum $\sum \binom{n}{q} b_m \cdot q^{t'_m} (\ell - 1)^q$, summed over $0 \leq q \leq n$, $q \equiv m \pmod{d}$, is asymptotic to $an^{t'_m} \ell^n$, where a is a constant depending on b_m , ℓ and t'_m . In particular the asymptotics do not depend on $n \pmod{d}$.*

Proof. Let ω be a primitive d^{th} root of 1 and note that $\sum_{\alpha=0}^{d-1} \omega^{(q-m)\alpha}$ is zero if $q \not\equiv m \pmod{d}$ and is d if $q \equiv m \pmod{d}$. So, the sum $\sum \binom{n}{q} b_m \cdot q^{t'_m} (\ell - 1)^q$, summed over $0 \leq q \leq n$, $q \equiv m \pmod{d}$ can be rewritten as

$$\frac{1}{d} \sum_{q=0}^n \sum_{\alpha=0}^{d-1} \omega^{(q-m)\alpha} \binom{n}{q} b_m q^{t'_m} (\ell - 1)^q = \frac{1}{d} \sum_{\alpha=0}^{d-1} \omega^{-m\alpha} \sum_{q=0}^n \binom{n}{q} b_m q^{t'_m} (\omega^\alpha (\ell - 1))^q.$$

Now, using Lemma 1.1 of [2] we get that the inner sum is asymptotic to a constant times

$$n^{t'_m} (\omega^\alpha (\ell - 1) + 1)^n.$$

The absolute value of $(\omega^\alpha (\ell - 1) + 1)$ is maximized when $\omega^\alpha (\ell - 1)$ is real and positive, which happens precisely when $\alpha = 0$, and so these terms dominate the sum. The lemma now follows. $\square$

Substituting Lemma 45 into Lemma 43 yields

$$c_n(A) \simeq \sum_{m=0}^{d-1} a_m n^{t'_m} \ell^n.$$

If g is the maximum value of the t'_m and $a = \sum \{b_m | t'_m = g\}$, then this sum is asymptotic to $an^g \ell^n$ and $2g$ is an integer.

REFERENCES

- [1] W. Beckner and A. Regev, Asymptotics and algebraicity of some generating functions, *Adv. Math.* **65** (1987), 1–15. MR893467 (88h:05008)
- [2] W. Beckner and A. Regev, Asymptotic estimates using probability, *Adv. Math.* **138** (1998), 1–14. MR1645060 (99j:05011)
- [3] A. Ya. Belov, Rationality of Hilbert series with respect to free algebras, *Russian Math. Surveys* **52** (1997), 394–395. MR1480146 (98f:16018)
- [4] A. Berele, Approximate multiplicities in the trace cocharacter sequence of two three-by-three matrices, *Comm. Algebra* **25** (1997) 1975–1983. MR1446144 (98j:05129)
- [5] A. Berele, Applications of Belov's theorem to the cocharacter sequence of p. i. algebras, *J. of Alg.* **298** (2006), 208–214. MR2215124 (2006m:16023)
- [6] A. Berele and A. Regev, Codimensions of products and of intersections of verbally prime T -ideals, *Israel J. Math.* **103** (1998), 17–28. MR1613536 (99b:16037)
- [7] A. Berele and A. Regev, Exponential growth for codimensions of some p. i. algebras, *J. of Alg.* **241** (2001), 118–145. MR1838847 (2002k:16046)
- [8] V. Drensky, Codimensions of T -ideals and Hilbert series of relatively free algebras, *J. of Alg.* **89** (1984), 178–223. MR765766 (86b:16010)
- [9] V. Drensky and G. K. Genov, Multiplicities of Schur functions in invariants of two 3×3 matrices, *J. Algebra* **264** (2003), 496–519. MR1981418 (2004c:16037)
- [10] A. Giambruno and M. V. Zaicev, On codimension growth of finitely generated associative algebras, *Adv. Math.* **140** (1998), 145–155. MR1658530 (99k:16049)
- [11] A. Giambruno and M. V. Zaicev, Exponential codimension growth of PI algebras: An exact estimate, *Adv. Math.* **142** (1999), 221–243. MR1680198 (2000a:16048)

- [12] B. Grünbaum, *Convex Polytopes*, Interscience, London, 1967. MR0226496 (37:2085)
- [13] A. Guterman and A. Regev, On the growth of identities, In: *Algebra* (Moscow: 1998), 319–330, de Gruyter, Berlin, 2000. MR1754678 (2001a:16039)
- [14] A. Regev, Asymptotic values for degrees associated with strips of Young diagrams, *Adv. in Math.* **41** (1981), 115–136. MR625890 (82h:20015)
- [15] A. Regev, Algebras satisfying a Capelli identity, *Israel J. Math.* **33** (1979), 149–154. MR571250 (81i:16022)
- [16] G. M. Zeigler, *Lectures on Polytopes*, Graduate Texts in Mathematics **152**, Springer-Verlag, New York, 1995. MR1311028 (96a:52011)

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