ON EMBEDDING ALL n -MANIFOLDS INTO A SINGLE $(n + 1)$ -MANIFOLD

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ABSTRACT. For each composite number $n \neq 2^k$, there does not exist a single connected closed $(n + 1)$ -manifold such that any smooth, simply-connected, closed n -manifold can be topologically flatly embedded into it. There is a single connected closed 5-manifold W such that any simply-connected, 4-manifold M can be topologically flatly embedded into W if M is either closed and indefinite, or compact and with non-empty boundary.

1. INTRODUCTION AND SOME PREREQUISITES

The celebrated Whitney embedding theorem states that any smooth n -dimensional manifold can be embedded smoothly into the Euclidean space $\mathbb{R}^{2n}$, or equivalently, any smooth n -dimensional manifold can be embedded into any $2n$ -dimensional manifold. It is natural to consider the following general problem:

Find the smallest non-negative integer e_n , such that any n -dimensional connected, closed manifold can be (topologically flatly or smoothly) embedded into a single connected, closed manifold of dimension $n + e_n$.

We say that M can be *topologically flatly* embedded into W if $M \times [-1, 1]$ can be topologically embedded into W .

Clearly $0 \leq e_n \leq n$.

Examples. The only known e_i 's are for $i = 0, 1, 2, 3$:

- (i) $e_0 = 0$ since a connected 0-manifold is a point;
- (ii) $e_1 = 0$ since a connected, closed 1-manifold is homeomorphic to S^1 ;
- (iii) $e_2 = 1$ since any orientable, connected, closed surface embeds into S^3 , and any non-orientable, connected, closed surface can be presented as a connected sum of an orientable, connected, closed surface and $\mathbb{RP}^2$ or $\mathbb{RP}^2 \# \mathbb{RP}^2$; hence each connected closed surface can be embedded into $S^3 \# \mathbb{RP}^3 \# \mathbb{RP}^3 \cong \mathbb{RP}^3 \# \mathbb{RP}^3$.
- (iv) $e_3 = 2$ since each closed 3-manifold embeds into S^5 , due to Hirsch in the orientable case ([Hi], also [Ki]), and due to Rohlin and Wall independently in the non-orientable case ([Ro], [Wa]; see [WZ] for a proof using only Dehn surgery). Kawauchi [Ka] showed that there does not exist a single oriented, connected, closed

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4-manifold such that any connected, closed 3-manifold can be topologically flatly embedded into it by constructing signature invariants from infinite cyclic coverings, and then Shiomi [Shi] showed that Kawauchi's result is still true if the target 4-manifold is allowed to be non-orientable.

It is reasonable to guess that, in general, $e_n > 1$. The main result of the present paper claims that this is the case for “most” n , and we prove

Theorem 1.1. *If n is a composite number and is not a power of 2, then there does not exist a connected, closed $(n + 1)$ -manifold W , such that any smooth, simply-connected, closed n -manifold M can be topologically flatly embedded into W .*

To prove Theorem 1.1, it is enough to prove there does not exist such an oriented W , since if any smooth, simply-connected, closed n -manifold M can be topologically flatly embedded into a non-orientable $(n + 1)$ -manifold W , then M can also be topologically flatly embedded into the orientable double cover of W because M is simply-connected.

Besides [Ka] and [Shi], there are several recent papers ([EL], [Fa1], and [Fa2]), which study whether a 3-manifold M embeds into connected sums of $\mathbb{CP}^2$. This relates to finding uncountably many smooth structures on the 4-manifold $M \times \mathbb{R}$ (see [Di], [Fa1] and [Fa2]). Fang's observation (cf. [Fa2]) that there exists an obstruction to the embedding of 3-manifolds into a simply-connected definite 4-manifold was the first inspiration for our proof of Theorem 1.1.

We do not know if Theorem 1.1 still holds for all remaining positive integers $n \geq 5$. When $n = 3$, the only simply-connected, closed 3-manifold is S^3 if the Poincaré Conjecture is valid. When $n = 4$, the situation is still not clear, and we have the following (for terminologies involved, see Section 5).

Theorem 1.2. (a) *There exists an oriented, connected, closed 5-manifold W , such that for any simply-connected, closed 4-manifold M with indefinite intersection form, M can be topologically flatly embedded into W .*

(b) *Any simply-connected, compact topological 4-manifold M with $\partial M \neq \emptyset$ admits a topologically flat embedding into $S^2 \tilde{\times} S^3$, the non-trivial S^3 -bundle over S^2 .*

Remark 1.3. There are indefinite 4-manifolds with arbitrarily large signatures. Theorem 1.2 (a) tells us that the signature of 4-manifolds cannot be an obstruction for our codimension 1 embeddings. Note that any oriented, connected, closed 4-manifold with non-zero signature does not bound a compact orientable 5-manifold. It follows that the first Betti number of W in (a) must be positive and that the condition $\partial M \neq \emptyset$ in (b) cannot be removed.

The organization of the paper and an outline of the proofs.

Roughly speaking, we prove Theorem 1.1 as follows. First assume that there exists such a manifold W . In Proposition 2.1, we produce a function φ for each embedding $M^n \rightarrow W$. Proposition 3.3 shows that cup products admitting such a function φ are rare. Then we construct an M^n whose cup product doesn't admit such a function φ , thus get a contradiction.

In Section 2, for any fixed $(n+1)$ -manifold W , Proposition 2.1 claims that for each topologically flat codimension 1 embedding $M^n \rightarrow W$ and each factorization $n = pq$, there is a homological obstruction to the embedding in terms of $H^{p+1}(W; \mathbb{R})$, $H^p(M; \mathbb{R})$, the q -multiple cup product on $H^p(M; \mathbb{R})$, and so on. Since n is not a

prime number and is not a power of 2, we can assume that the factorization $n = pq$ has been chosen so that $p \geq 2$ and $q \geq 3$, and q is odd.

Let $(\wedge^q \mathbb{R}^m)^*$ (resp. $(\vee^q \mathbb{R}^m)^*$) be the vector space of all skew-symmetric (resp. symmetric) q -multilinear functions on the m -dimensional real vector space $\mathbb{R}^m$. In Section 3 we study *special* (see Definition in Section 3) functions in $(\wedge^q \mathbb{R}^m)^*$ (resp. $(\vee^q \mathbb{R}^m)^*$), an example of which is the q -multiple cup product on $H^p(M; \mathbb{R})$ appearing in Proposition 2.1. Proposition 3.3 says that, when $q \geq 3$ is odd and m is sufficiently large, such functions are contained in a proper closed subset of $(\wedge^q \mathbb{R}^m)^*$ (resp. $(\vee^q \mathbb{R}^m)^*$).

In Section 4, we first choose a suitable rational non-special function provided by Proposition 3.3, and then Proposition 4.2 claims that there is a commutative graded algebra $A = \bigoplus_{i=0}^q A_{ip}$ over $\mathbb{Q}$ satisfying Poincaré duality so that the chosen non-special function is the q -multiple cup product on A_p . Next we verify that A can be realized as the rational cohomology ring of a simply-connected closed n -manifold M by invoking a celebrated result of Sullivan. Hence the q -multiple cup product on $H^p(M; \mathbb{R})$ is a non-special function, and therefore M cannot be topologically flatly embedded into W by Proposition 2.1, which finishes the proof of Theorem 1.1.

Section 5 is devoted to the case $n = 4$, and we get Theorem 1.2 by explicit constructions and the classification theorem for simply-connected, closed 4-manifolds.

In the remainder of this section, we recall and fix needed notation and conventions in tensor algebra based on [Sh].

Prerequisites on tensor algebra. Let V be an m -dimensional vector space over a field $\mathbb{F}$ of characteristic zero. Denote by $\otimes^p V$ the p -fold tensor product of V , and by $\otimes V = \bigoplus_p (\otimes^p V)$ the tensor algebra.

The p th *exterior power* of V is denoted by $\wedge^p V$. The product operation $\wedge : (\wedge^p V) \times (\wedge^q V) \rightarrow \wedge^{p+q} V$ can be extended bilinearly to general elements of the vector space $\wedge V = \bigoplus_p (\wedge^p V) = \mathbb{F} \oplus V \oplus \wedge^2 V \oplus \dots \oplus \wedge^m V$ to make $\wedge V$ into an algebra over $\mathbb{F}$, called the *exterior algebra* on V .

Similarly the p th *symmetric power* of V is denoted by $\vee^p V$. The product operation $\vee : (\vee^p V) \times (\vee^q V) \rightarrow \vee^{p+q} V$ can be extended bilinearly to general elements of the vector space $\vee V = \bigoplus_p (\vee^p V) = \mathbb{F} \oplus V \oplus \vee^2 V \oplus \dots$ to make $\vee V$ into an infinite-dimensional algebra over $\mathbb{F}$, called the *symmetric algebra* on V .

Let $\times^p V$ denote the Cartesian product $V \times V \times \dots \times V$ of p copies of V . Each multilinear function $F : \times^p V \rightarrow \mathbb{F}$ determines a unique linear function $L : \otimes^p V \rightarrow \mathbb{F}$ such that for all $v_1, \dots, v_p \in V$, $L(v_1 \otimes \dots \otimes v_p) = F(v_1, \dots, v_p)$. The correspondence $F \leftrightarrow L$ establishes a natural isomorphism between the vector space $L(\times^p V; \mathbb{F})$ of all multilinear functions and the dual space $(\otimes^p V)^*$ of $\otimes^p V$. For any skew-symmetric (resp. symmetric) multilinear function $F : \times^p V \rightarrow \mathbb{F}$, there exists a unique linear function $L_s : \wedge^p V$ (resp. $\vee^p V$) $\rightarrow \mathbb{F}$ such that for all $v_1, \dots, v_p \in V$, $L_s(v_1 \wedge \dots \wedge v_p) = F(v_1, \dots, v_p)$ (resp. $L_s(v_1 \vee \dots \vee v_p) = F(v_1, \dots, v_p)$). The correspondence $F \leftrightarrow L_s$ establishes a natural isomorphism between the vector space $\text{Sk}(\times^p V; \mathbb{F})$ (resp. $\text{Sym}(\times^p V; \mathbb{F})$) of all skew-symmetric (resp. symmetric) multilinear functions and the dual space $(\wedge^p V)^*$ (resp. $(\vee^p V)^*$) of $\wedge^p V$ (resp. $\vee^p V$). Thus we may consider a skew-symmetric (resp. symmetric) multilinear function $F : \times^p V \rightarrow \mathbb{F}$ as an element of $(\wedge^p V)^*$ (resp. $(\vee^p V)^*$), and vice versa.

Let $\{e_1, \dots, e_m\}$ be the standard basis for the real vector space $\mathbb{R}^m$; i.e. $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ has components 0 except for its i -th component, which is equal to 1. Each element $F \in (\wedge^q \mathbb{R}^m)^*$ (resp. $(\vee^q \mathbb{R}^m)^*$) is determined by $\binom{m}{q}$ (resp.

$\binom{m+q-1}{q}$) real numbers $(F_{i_1 \dots i_q})$, where $F_{i_1 \dots i_q} = F(e_{i_1}, \dots, e_{i_q})$ for $1 \leq i_1 < \dots < i_q \leq m$ (resp. $1 \leq i_1 \leq \dots \leq i_q \leq m$). Thus $\dim(\wedge^q \mathbb{R}^m)^* = \binom{m}{q}$ and $(\vee^q \mathbb{R}^m)^* = \binom{m+q-1}{q}$.

2. HOMOLOGICAL OBSTRUCTIONS FOR EMBEDDINGS

From now on all manifolds appearing in this paper are considered to be oriented and connected. $\beta_i(X)$ denotes the i -th Betti number of a compact manifold X .

The following proposition sets up a property on the cohomology ring of the n -manifold M if it can be embedded into the $(n+1)$ -manifold W .

Proposition 2.1. *Let M^n and W^{n+1} be closed and connected, oriented manifolds.*

If M^n topologically flatly embeds in W^{n+1} , then for any integer factorization $n = pq$, where $p, q > 0$, there exists a subspace V of $H^p(M; \mathbb{R})$ and a linear transformation $\varphi : V \rightarrow H^p(M; \mathbb{R})$ such that

- (i) φ has no fixed non-zero vectors,
- (ii) $\dim V \geq \frac{1}{2}(\beta_p(M) - \beta_{p+1}(W))$, and
- (iii) for any $x_1, \dots, x_q \in V$, $x_1 \cup \dots \cup x_q = \varphi(x_1) \cup \dots \cup \varphi(x_q)$.

Proof. All homology and cohomology groups in this proof are with coefficients in $\mathbb{R}$. We divide the proof into two cases.

From the definition of a topologically flat embedding, we can identify a closed tubular neighborhood of M in W with $M \times [-1, 1]$.

Case 1. Suppose $W \setminus M$ is not connected. Denote the closures of the two components of $W \setminus M$ by W'_1 and W'_2 , and let W_i be the union of W'_i and $M \times (-1, 1)$. Then $\{W_1, W_2\}$ is an excisive couple and from the Mayer-Vietoris sequence, we obtain $H^p(W_1) \oplus H^p(W_2) \xrightarrow{f} H^p(M) \xrightarrow{\delta^*} H^{p+1}(W)$. Here the first map is given by $f(w_1, w_2) = i_1^*(w_1) - i_2^*(w_2)$ for $w_1 \in H^p(W_1), w_2 \in H^p(W_2)$, where i_1 and i_2 are inclusions $M \hookrightarrow W_1$ and $M \hookrightarrow W_2$ respectively. Clearly $\dim \operatorname{im}(f) = \dim \ker(\delta^*) \geq \dim H^p(M) - \dim H^{p+1}(W) = \beta_p(M) - \beta_{p+1}(W)$. Since $\dim \operatorname{im}(i_1^*) + \dim \operatorname{im}(i_2^*) \geq \dim \operatorname{im}(f)$, we may set $V = \operatorname{im}(i_1^*)$ and assume that $\dim V \geq \frac{1}{2}(\beta_p(M) - \beta_{p+1}(W))$.

For any $x_1, \dots, x_q \in V$, there exist $y_1, \dots, y_q \in H^p(W_1)$ such that $x_i = i_1^*(y_i)$, $1 \leq i \leq q$. Since $M = \partial W'_1$, we have $i_{1*}[M] = 0 \in H_n(W_1)$, where $[M]$ denotes the fundamental class of M . So $\langle x_1 \cup \dots \cup x_q, [M] \rangle = \langle i_1^*(y_1 \cup \dots \cup y_q), [M] \rangle = \langle y_1 \cup \dots \cup y_q, i_{1*}[M] \rangle = 0$. Therefore for any $x_1, \dots, x_q \in V$, $x_1 \cup \dots \cup x_q = 0$.

The result is thus proved by letting φ be the zero map.

Case 2. Suppose $W \setminus M$ is connected. Let $W_1 = M \times (-1, 1)$ and $W_2 = W \setminus (M \times [-\frac{1}{2}, \frac{1}{2}])$. Clearly, $W_1 \cap W_2 = M \times (-1, -\frac{1}{2}) \cup M \times (\frac{1}{2}, 1)$.

Again by the Mayer-Vietoris sequence, we get the exact sequence $H^p(W_1) \oplus H^p(W_2) \rightarrow H^p(W_1 \cap W_2) \rightarrow H^{p+1}(W)$, which reduces to $H^p(M) \oplus H^p(W_2) \xrightarrow{f} H^p(M) \oplus H^p(M) \xrightarrow{\delta^*} H^{p+1}(W)$. Now f can be written as $f(x, w) = (x - i_0^*(w), x - i_1^*(w))$, where i_0 and i_1 are inclusions $M \times \{-1\} \hookrightarrow W_2$ and $M \times \{1\} \hookrightarrow W_2$, respectively. By the exactness, $\dim \operatorname{im}(f) = \dim \ker(\delta^*) \geq \dim(H^p(M) \oplus H^p(M)) - \dim H^{p+1}(W) = 2\beta_p(M) - \beta_{p+1}(W)$.

Let $\Delta = \{(x, x) | x \in H^p(M)\} \subset H^p(M) \oplus H^p(M)$ and $U = \{(i_0^*(w), i_1^*(w)) | w \in H^p(W_2)\}$. Then $\Delta + U = \operatorname{im}(f)$ and therefore $\dim(\Delta + U) = \dim \operatorname{im}(f) \geq 2\beta_p(M) - \beta_{p+1}(W)$. Since $\dim \Delta = \beta_p(M)$, there is a subspace S of U with $\dim S \geq \beta_p(M) - \beta_{p+1}(W)$ such that $S \cap \Delta = \{(0, 0)\}$. Let $\pi_1, \pi_2 : H^p(M) \oplus H^p(M) \rightarrow H^p(M)$ be the projections of $H^p(M) \oplus H^p(M)$ onto its first and second factors, respectively.

Since $\dim \pi_1(S) + \dim \pi_2(S) \geq \dim S$, without loss of generality, we may assume that $\dim \pi_1(S) \geq \frac{1}{2} \dim S \geq \frac{1}{2}(\beta_p(M) - \beta_{p+1}(W))$.

It follows from $\partial(W_2 \setminus M \times (-1, 1)) = -(M \times \{-1\}) \cup (M \times \{1\})$ that $i_{0*}[M] = i_{1*}[M]$. Thus for any $w_1, \dots, w_q \in H^p(W_2)$, $\langle w_1 \cup \dots \cup w_q, i_{0*}[M] - i_{1*}[M] \rangle = 0$, and this implies $\langle i_0^*(w_1) \cup \dots \cup i_0^*(w_q), [M] \rangle = \langle i_1^*(w_1) \cup \dots \cup i_1^*(w_q), [M] \rangle$. In particular, if $(u_1, v_1), \dots, (u_q, v_q) \in S$, then

$$(2.1) \quad u_1 \cup \dots \cup u_q = v_1 \cup \dots \cup v_q.$$

Let $\{y_1, \dots, y_r\}$ be a basis for $\pi_1(S)$. We can find $z_1, \dots, z_r \in H^p(M)$ such that $(y_j, z_j) \in S$, $1 \leq j \leq r$. Then there is a unique linear transformation $\varphi : \pi_1(S) \rightarrow H^p(M)$ such that $\varphi(y_j) = z_j$, $1 \leq j \leq r$.

One can write any $y \in \pi_1(S)$ as $y = \sum_j c_j y_j$. Then $\varphi(y) = \sum_j c_j z_j$. Thus $(y, \varphi(y)) \in S$. Since $S \cap \Delta = \{(0, 0)\}$, if $y \neq 0$, then $\varphi(y) \neq y$. Now setting $V = \pi_1(S)$, one gets the desired result from equation (2.1). $\square$

3. SPECIAL MULTILINEAR FUNCTIONS

Proposition 2.1 motivates the following definition.

Definition 3.1. Let F be an element of $(\wedge^q \mathbb{R}^m)^*$ (resp. $(\vee^q \mathbb{R}^m)^*$). We say that F is *special* if there exist a subspace U of $\mathbb{R}^m$ with $\dim U \geq \frac{m}{3}$ and a linear map $\varphi : U \rightarrow \mathbb{R}^m$ with no fixed non-zero vectors such that for all $x_1, \dots, x_q \in U$, $F(x_1, \dots, x_q) = F(\varphi(x_1), \dots, \varphi(x_q))$. In the equation, we consider F as a skew-symmetric (resp. symmetric) multilinear function (cf. Section 1).

Example 3.2. Suppose M^n topologically flatly embeds in W^{n+1} , $n = pq$, $p, q > 0$, and $\beta_p(M) \geq 3\beta_{p+1}(W)$. Then by Proposition 2.1, the q -multiple cup product on $H^p(M, \mathbb{R})$ is special.

In this section, we will show

Proposition 3.3. Suppose $q \geq 3$ is an odd integer. If m is sufficiently large, then there exists a proper closed subset X_m of $(\wedge^q \mathbb{R}^m)^*$ (resp. $(\vee^q \mathbb{R}^m)^*$) such that if $F \in (\wedge^q \mathbb{R}^m)^*$ (resp. $F \in (\vee^q \mathbb{R}^m)^*$) is special, then $F \in X_m$.

Remark 3.4. If $q \geq 2$ is even, then each $F \in (\wedge^q \mathbb{R}^m)^*$ (resp. $F \in (\vee^q \mathbb{R}^m)^*$) is special. We may simply take $U = \mathbb{R}^m$ and $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^m$ to be the map sending x to $-x$ for all $x \in \mathbb{R}^m$.

Before proving Proposition 3.3, we shall discuss two useful lemmas.

It is a classical result about the Jordan normal form that (cf. Theorem 3.4.5 in [HJ]) any real square matrix is similar to a block diagonal matrix, where each block takes one of the following two forms:

$$J_k(a) = \begin{bmatrix} a & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & a \end{bmatrix}, C_l(a, b) = \begin{bmatrix} a & b & 1 & 0 \\ -b & a & 0 & 1 \\ & \ddots & \ddots & \\ & & \ddots & 1 & 0 \\ & & & \ddots & 0 & 1 \\ & & & & a & b \\ & & & & -b & a \end{bmatrix}.$$

The sizes of these two square matrices are k and $2l$ respectively, where k and l are positive integers, a and $b \neq 0$ are real numbers.

We sort the blocks described above into 6 forms: **1:** $J_1(1)$, **2:** $J_1(-1)$, **3:** $J_1(a)$, $|a| > 1$, **4:** $J_1(a)$, $|a| < 1$, **5:** $J_k(a)$, $k > 1$, **6:** $C_l(a, b)$. By using the Jordan form, we can find a suitable basis for $\mathbb{R}^m$ under which the linear map φ mentioned in the above definition has a simple matrix representation.

We use $\text{span}\{v_1, \dots, v_k\}$ to denote the subspace of $\mathbb{R}^m$ generated by vectors $v_1, \dots, v_k \in \mathbb{R}^m$, and $\lfloor c \rfloor$ to denote the greatest integer $\leq c$ for $c \in \mathbb{R}$.

Lemma 3.5. *Let U be a subspace of $\mathbb{R}^m$, $\varphi : U \rightarrow \mathbb{R}^m$ be a linear map with no fixed non-zero vectors, and $l = \lfloor \frac{\dim U}{18} \rfloor$.*

There exists a basis $\{v_1, \dots, v_m\}$ for $\mathbb{R}^m$ such that $V = \text{span}\{v_1, \dots, v_l\}$ is a subspace of U and the matrix of $\varphi|V$ with respect to this basis is $\begin{bmatrix} \Lambda \\ \Gamma \end{bmatrix}$, where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_l)$ is a diagonal matrix with $\lambda_{i_1} \dots \lambda_{i_q} \neq 1$ for any odd integer q and $1 \leq i_1 \leq \dots \leq i_q \leq l$.

Proof. Choose a basis $\{u_1, \dots, u_k\}$ for U and complete it to a basis $\{u_1, \dots, u_m\}$ for $\mathbb{R}^m$, where $k = \dim U$. Denote the matrix of φ with respect to this basis by $\begin{bmatrix} J \\ * \end{bmatrix}$, where J is a $k \times k$ square matrix. We may choose $\{u_1, \dots, u_k\}$ so that J is in real Jordan canonical form. The Jordan blocks have 6 forms as mentioned above. For some i_0 , the total size t of blocks of form i_0 in J should satisfy $t \geq \frac{k}{6}$. We change the order of $u_1, \dots, u_k$ so that in the block expression for J , blocks of form i_0 appear first.

Suppose $i_0 = 2, 3$ or 4 . Choose $\{v_1, \dots, v_m\}$ to be just $\{u_1, \dots, u_m\}$. For the matrix representation $\begin{bmatrix} \Lambda \\ \Gamma \end{bmatrix}$ of $\varphi|V$ under this basis, as $l \leq t$ and each Jordan block of form i_0 is a 1×1 matrix, Λ is a diagonal matrix, $\text{diag}(\lambda_1, \dots, \lambda_l)$. For any $1 \leq i_1 \leq \dots \leq i_q \leq l$, if $i_0 = 2$, then $\lambda_{i_1} \dots \lambda_{i_q} = -1$ since q is odd; if $i_0 = 3$, then $|\lambda_{i_1} \dots \lambda_{i_q}| > 1$; and if $i_0 = 4$, then $|\lambda_{i_1} \dots \lambda_{i_q}| < 1$. In any case, $\lambda_{i_1} \dots \lambda_{i_q} \neq 1$.

Suppose $i_0 = 1, 5$ or 6 . If there exists a subspace V of U such that

$$(3.1) \quad \dim V = l \text{ and } \varphi(V) \cap V = \{0\},$$

then we can choose a basis $\{v_1, \dots, v_l\}$ for V , a basis for $\varphi(V)$, and extend the union of these two bases to a basis $\{v_1, \dots, v_m\}$ for $\mathbb{R}^m$. As $\varphi(V) \cap V = \{0\}$, the matrix representation for $\varphi|V$ must be $\begin{bmatrix} 0 \\ \Gamma \end{bmatrix}$, which satisfies the requirement.

Now we only need to find a subspace V with property (3.1). Set $U' = \text{span}\{u_1, \dots, u_t\}$ and denote the matrix of $\varphi|U'$ with respect to the basis $\{u_1, \dots, u_m\}$ by $\begin{bmatrix} J_0 \\ * \end{bmatrix}$.

Suppose first that J_0 contains only blocks of the same form $J_1(1)$. Let $W = \text{span}\{u_{t+1}, \dots, u_m\}$. Then $\mathbb{R}^m = U' \oplus W$. Since $J_0 = I$, for $u \in U'$, $\varphi(u) = u + w$ for some $w \in W$. Suppose that $x \in \varphi(U') \cap U'$. Then $x \in U'$ and there exists $u \in U'$ such that $\varphi(u) = x$. But $\varphi(u) = u + w = x + 0$, so $\varphi(u) = u$, and hence $u = 0$, since φ only fixes the zero vector. Therefore $\varphi(U') \cap U' = \{0\}$, and we may simply shrink U' to V with $\dim V = l$ as $t \geq l$.

Suppose now J_0 contains only blocks of the same forms $J_k(a)$, $k > 1$, or $C_l(a, b)$. Fix a block B_j , let $\{w_1, \dots, w_{k_j}\}$ be the subset of the basis $\{u_1, \dots, u_t\}$ that

corresponds to the block B_j . Note that $k_j \geq 2$. Set $W_j = \text{span}\{w_1, \dots, w_{k_j}\}$, $W_{j,0} = \text{span}\{w_i \mid i \text{ even}\}$ and $W_{j,1} = \text{span}\{w_i \mid i \text{ odd}\}$. Clearly, $W_j = W_{j,0} \oplus W_{j,1}$ and $\dim W_{j,0} \geq \frac{k_j}{3}$.

Denote the projection of $\mathbb{R}^m$ onto $W_{j,1}$ by p_1 . (This projection is well defined as there is a natural decomposition $\mathbb{R}^m = W_j \oplus W'$, where $W' = \text{span}(\{u_1, \dots, u_m\} \setminus \{w_1, \dots, w_{k_j}\})$.) If $B_j = C_{\frac{k_j}{2}}(a, b)$, where k_j is necessarily even, then $p_1\varphi(w_{2i}) = bw_{2i-1}$, $1 \leq i \leq \frac{k_j}{2}$, and if $B_j = J_{k_j}(a)$, then $p_1\varphi(w_{2i}) = w_{2i-1}$, $1 \leq i \leq \lfloor \frac{k_j}{2} \rfloor$. By these two equations, we know $\varphi(W_{j,0}) \cap W_{j,0} = \{0\}$. Similarly, if we take V' to be the direct sum $\bigoplus_j W_{j,0}$ for all blocks in J_0 , we will have $\varphi(V') \cap V' = \{0\}$ and $\dim V' \geq \frac{t}{3} \geq \frac{\dim U}{18}$. We may simply shrink V' to V with $\dim V = l$. $\square$

The next lemma uses the transcendence degree of a field extension, which follows from Chap. VIII, Theorem 1.1 of [La].

Lemma 3.6. *Let $g_1(x_1, \dots, x_r), \dots, g_s(x_1, \dots, x_r)$ be rational functions in r variables $x_1, \dots, x_r$ with coefficients in $\mathbb{R}$. If $s > r$, then there exists a non-zero polynomial P in s variables such that $P(g_1(x_1, \dots, x_r), \dots, g_s(x_1, \dots, x_r)) = 0$.*

Proof of Proposition 3.3. Suppose that $F \in (\wedge^q \mathbb{R}^m)^*$ is special. By definition, there exists a subspace U of $\mathbb{R}^m$ with $\dim U \geq \frac{m}{3}$ and a linear map $\varphi : U \rightarrow \mathbb{R}^m$ with no fixed non-zero vectors such that for all $x_1, \dots, x_q \in U$,

$$(3.2) \quad F(x_1, \dots, x_q) = F(\varphi(x_1), \dots, \varphi(x_q)).$$

The strategy for the proof is roughly as follows. We will use equation (3.2) to express the coordinates of F in the standard basis for $(\wedge^q \mathbb{R}^m)^*$. These coordinates will be expressed as rational functions of some variables, and the number of variables used will be less than $\dim(\wedge^q \mathbb{R}^m)^*$ when m is large enough. Thus according to Lemma 3.6, all special F lie in the zero set of a single non-zero polynomial, which is a proper closed subset of $(\wedge^q \mathbb{R}^m)^*$.

For the subspace U and the linear map φ , choose a basis $\{v_1, \dots, v_m\}$ for $\mathbb{R}^m$ with the properties described in Lemma 3.5. Let $\{e_1, \dots, e_m\}$ be the standard basis for $\mathbb{R}^m$. Set $v_j = \sum_{i=1}^m \tau_{ij} e_i$, define $F_{i_1 \dots i_q} = F(e_{i_1}, \dots, e_{i_q})$ and $f_{i_1 \dots i_q} = F(v_{i_1}, \dots, v_{i_q})$, $1 \leq i_1 < \dots < i_q \leq m$. Clearly, each $F_{i_1 \dots i_q}$ is a rational function of the variables τ_{ij} and $f_{j_1 \dots j_q}$.

Obviously, Lemma 3.5 still holds for any integer l less than $\lfloor \frac{\dim U}{18} \rfloor$. Now $l = \lfloor \frac{\dim U}{18} \rfloor \geq \lfloor \frac{m}{54} \rfloor$, and here we always set $l = \lfloor \frac{m}{54} \rfloor$. Recall that $V = \text{span}\{v_1, \dots, v_l\} \subset U$, and the matrix of $\varphi|_V$ with respect to the basis $\{v_1, \dots, v_m\}$ is $\begin{bmatrix} \Lambda \\ \Gamma \end{bmatrix}$, where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_l)$ and $\Gamma = (\gamma_{ij})_{(m-l) \times l}$.

Define $R = \{(i_1, \dots, i_q) \mid 1 \leq i_1 < \dots < i_q \leq m\}$, $R_1 = \{(i_1, \dots, i_q) \mid 1 \leq i_1 < \dots < i_q \leq l\}$ and $R' = R \setminus R_1$. For any $(i_1, \dots, i_q) \in R_1$, by (3.2),

$$(3.3) \quad f_{i_1 \dots i_q} = F(\varphi(v_{i_1}), \dots, \varphi(v_{i_q})) = \lambda_{i_1} \dots \lambda_{i_q} f_{i_1 \dots i_q} + \sum_{(j_1, \dots, j_q) \in R'} c_{i_1 \dots i_q, j_1 \dots j_q} f_{j_1 \dots j_q},$$

where each coefficient $c_{i_1 \dots i_q, j_1 \dots j_q}$ is a polynomial of λ_i 's and γ_{ij} 's. It then follows from $\lambda_{i_1} \dots \lambda_{i_q} \neq 1$ that $f_{i_1 \dots i_q}$ can be expressed as a rational function of other variables appearing in (3.3). Note that the expression of this rational function does not depend on F .

Now each $F_{i_1 \dots i_q}$ is a rational function in the variables τ_{ij} ($1 \leq i, j \leq m$), λ_i ($1 \leq i \leq l$), γ_{ij} ($1 \leq i \leq m-l, 1 \leq j \leq l$) and $f_{i_1 \dots i_q}$ ($(i_1, \dots, i_q) \in R'$). The total number of variables used here is $m^2 + l + (m-l)l + \binom{m}{q} - \binom{l}{q}$, which is less than $\binom{m}{q}$ for sufficiently large m as $q \geq 3$ and $l = \lfloor \frac{m}{54} \rfloor$.

Thus by Lemma 3.6, there exists a non-zero polynomial P in the variables $F_{i_1 \dots i_q}$, where $1 \leq i_1 < \dots < i_q \leq m$, such that $P = 0$ for any special function F . Now X_m can be chosen to be the zero set of P , which is obviously a proper closed set.

For the situation $F \in (\vee^q \mathbb{R}^m)^*$, we only need in the proof above to substitute $(\wedge^q \mathbb{R}^m)^*$ by $(\vee^q \mathbb{R}^m)^*$, substitute $\binom{m}{q}, \binom{l}{q}$ by $\binom{m+q-1}{q}, \binom{l+q-1}{q}$, respectively, and change the restriction $i_1 < \dots < i_q$ to $i_1 \leq \dots \leq i_q$ in the definitions of R and R_1 . $\square$

4. MAIN THEOREM

In this section we will prove Theorem 1.1. As mentioned in the introduction, we only need to prove this when the target W is oriented.

For a graded algebra $A = \bigoplus_{i=0}^{\infty} A_i$, over $\mathbb{Q}$, we always assume its multiplication $\cup$ is associative, $A_0 = \mathbb{Q}$ and $1 \in \mathbb{Q} = A_0$ is the unit element of the algebra A . A is called *commutative* if for all $a \in A_r, b \in A_s$, $a \cup b = (-1)^{rs}(b \cup a)$.

The rational cohomology ring $H^*(X; \mathbb{Q})$ of a path-connected topological space X is a commutative graded algebra over $\mathbb{Q}$ (cf. [Sp], pp. 263-264).

Definition 4.1. Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a commutative graded algebra over $\mathbb{Q}$. Assume that there exists a positive integer n such that $A_n \cong \mathbb{Q}$, $A_i = 0$ for $i > n$ and A_i is finite dimensional for $i < n$. Let φ_i be the map from A_i to $\text{Hom}(A_{n-i}, A_n)$ defined by $\varphi_i(u)(v) = u \cup v$, $\forall u \in A_i, v \in A_{n-i}$. We say that A satisfies *Poincaré duality* if φ_i is an isomorphism from A_i to $\text{Hom}(A_{n-i}, A_n)$ for all $0 \leq i \leq n$.

Proposition 4.2. Suppose that p is a positive integer, and $q \geq 3$ is an odd integer. Let $F \in (\wedge^q V)^*$ if p is odd and $F \in (\vee^q V)^*$ if p even, where V is a finite dimensional vector space over $\mathbb{Q}$.

There exists a commutative graded algebra $A = \bigoplus_{i=0}^{pq} A_i$ satisfying Poincaré duality such that

- (i) $A_p = V$, $A_{pq} = \mathbb{Q}$, and $A_i \neq 0$ only if i is a multiple of p ,
- (ii) for all $v_1, \dots, v_q \in A_p = V$, $F(v_1, \dots, v_q) = v_1 \cup \dots \cup v_q$.

Proof. First assume that p is odd. Let $q = 2r + 1$. We construct A by setting

$$A_i = \begin{cases} \wedge^s V & \text{if } i = sp \text{ and } 0 \leq s \leq r, \\ (\wedge^{2r+1-s} V)^* & \text{if } i = sp \text{ and } r+1 \leq s \leq 2r+1, \\ 0 & \text{otherwise.} \end{cases}$$

Now suppose $u \in A_{sp}, v \in A_{tp}$ ($0 \leq s, t \leq 2r+1$). We define $u \cup v$, the product of u and v , as follows. (We assume that $s+t \leq 2r+1$, since otherwise $u \cup v = 0$.)

Case 1. If $s+t \leq r$, then $u \in A_{sp} = \wedge^s V, v \in A_{tp} = \wedge^t V$, and we define $u \cup v \in A_{(s+t)p} = \wedge^{s+t} V$ to be $u \wedge v$.

Case 2. If $s \leq r, t \leq r$ and $s+t \geq r+1$, then we define $u \cup v \in A_{(s+t)p} = (\wedge^{2r+1-s-t} V)^*$ by the equation $(u \cup v)(w) = F(u \wedge v \wedge w)$ for $w \in \wedge^{2r+1-s-t} V$.

Case 3. If $s \leq r, t \geq r+1$ and $s+t \leq 2r+1$, then $u \in A_{sp} = \wedge^s V, v \in A_{tp} = (\wedge^{2r+1-t} V)^*$, and we define $u \cup v \in A_{(s+t)p} = (\wedge^{2r+1-s-t} V)^*$ by the equation $(u \cup v)(w) = v(w \wedge u)$ for $w \in \wedge^{2r+1-s-t} V$.

Case 4. If $s \geq r+1$, $t \leq r$ and $s+t \leq 2r+1$, then $u \in A_{sp} = (\wedge^{2r+1-s}V)^*$, $v \in A_{tp} = \wedge^t V$, and we define $u \cup v \in A_{(s+t)p} = (\wedge^{2r+1-s-t}V)^*$ by the equation $(u \cup v)(w) = u(v \wedge w)$ for $w \in \wedge^{2r+1-s-t}V$.

A routine calculation shows that this product makes A a commutative graded algebra satisfying Poincaré duality and conditions (i) except that $A_{pq} = \mathbb{Q}^*$. By the canonical isomorphism $f: \mathbb{Q}^* \rightarrow \mathbb{Q}$ (i.e. for $w \in \mathbb{Q}^*$, $f(w) = w(1)$), we can identify A_{pq} with $\mathbb{Q}$. Then it is easy to see that condition (ii) holds.

If p is even, we can just replace all $\wedge$ in the above construction by $\vee$. $\square$

Proof of Theorem 1.1. Suppose such an $(n+1)$ -manifold W exists. Since n is a composite number and is not a power of 2, we have a factorization $n = pq$, where p is an integer ≥ 2 and q is an odd integer ≥ 3 . By Proposition 3.3, we can select an integer $m \geq 3\beta_{p+1}(W)$ such that all special functions F sit inside a proper closed subset X_m of $(\wedge^q \mathbb{R}^m)^*$ (when p is odd) or $(\vee^q \mathbb{R}^m)^*$ (when p is even). Since X_m is proper closed, we can pick an $F \notin X_m$ such that $F(e_{i_1}, \dots, e_{i_q}) \in \mathbb{Q}$ for any $1 \leq i_1, \dots, i_q \leq m$.

Taking $V = \mathbb{Q}^m$, by Proposition 4.2, there exists a commutative graded algebra A satisfying Poincaré duality and conditions (i) and (ii). $p \geq 2$ guarantees that $A_1 = 0$. By Theorem (13.2) from [Su], there exists a simply-connected, closed, smooth n -manifold M such that the rational cohomology ring $H^*(M; \mathbb{Q})$ is isomorphic to A . Denote this isomorphism by $h_{\mathbb{Q}}$, and define $h_{\mathbb{R}} = h_{\mathbb{Q}} \otimes \text{id} : H^*(M; \mathbb{R}) = H^*(M; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R} \rightarrow A \otimes_{\mathbb{Q}} \mathbb{R}$. Clearly $h_{\mathbb{R}}$ gives isomorphisms $H^p(M; \mathbb{R}) \rightarrow \mathbb{R}^m$ and $H^n(M; \mathbb{R}) \rightarrow \mathbb{R}$. Condition (ii) in Proposition 4.2 can be transformed to

$$(4.1) \quad F(h_{\mathbb{R}}(x_1), \dots, h_{\mathbb{R}}(x_q)) = h_{\mathbb{R}}(x_1 \cup \dots \cup x_q).$$

Note that $\beta_p(M) = m \geq 3\beta_{p+1}(W)$; hence $\frac{1}{2}(\beta_p(M) - \beta_{p+1}(W)) \geq \frac{m}{3}$. Since M can be topologically flatly embedded into W , by Proposition 2.1, there exists a subspace V of $H^p(M; \mathbb{R})$ and a linear transformation $\varphi : V \rightarrow H^p(M; \mathbb{R})$ with no fixed non-zero vectors such that $\dim V \geq \frac{1}{2}(\beta_p(M) - \beta_{p+1}(W)) \geq \frac{m}{3}$ and for any $x_1, \dots, x_q \in V$,

$$(4.2) \quad x_1 \cup \dots \cup x_q = \varphi(x_1) \cup \dots \cup \varphi(x_q).$$

Define $U = h_{\mathbb{R}}(V)$, $\psi : U \rightarrow \mathbb{R}^m$ to be $\psi = h_{\mathbb{R}} \circ \varphi \circ h_{\mathbb{R}}^{-1}$. Then $\dim U \geq \frac{m}{3}$ and $\psi : U \rightarrow \mathbb{R}^m$ is a linear map with no fixed non-zero vectors. By equations (4.1) and (4.2), for any $y_1, \dots, y_q \in U$, we have $F(y_1, \dots, y_q) = F(\psi(y_1), \dots, \psi(y_q))$. Thus F is special. But $F \notin X_m$, a contradiction. $\square$

5. EMBEDDING OF 4-MANIFOLDS INTO 5-MANIFOLDS

In this section we prove Theorem 1.2. Let us first recall some basic knowledge about the intersection forms of 4-manifolds. For any compact, connected, oriented 4-manifold M , the cup product

$$\cup : H^2(M, \partial M) \times H^2(M, \partial M) \rightarrow H^4(M, \partial M), (\alpha, \beta) \mapsto \alpha \cup \beta$$

gives a symmetric bilinear form Q_M over $\mathbb{Z}$ on $H_2(M)$ through the isomorphisms $H^2(M, \partial M) \cong H_2(M)$ and $H^4(M, \partial M) \cong \mathbb{Z}$. Clearly, $Q_M(a, b) = 0$ if a or b is a torsion element. So Q_M descends to a symmetric bilinear form on $H_2(M)/\text{Torsion} \cong \mathbb{Z}^r$.

For a given integral symmetric bilinear form Q on a finitely generated free abelian group A , the *rank* $\text{rk}(Q)$ of Q is the dimension of A . Extend and diagonalize Q

over $A \otimes_{\mathbb{Z}} \mathbb{R}$; the number of positive entries and the number of negative entries are denoted by b_2^+ and b_2^- respectively, and the difference $b_2^+ - b_2^-$ is the *signature* $\sigma(Q)$ of Q .

We say Q is *indefinite* if both b_2^+ and b_2^- are positive. We say Q is *even* if $Q(a, a)$ is even for any $a \in A$ and *odd* otherwise. By choosing a basis for A , one can express Q as a symmetric matrix. If $\det Q = \pm 1$, we say Q is *unimodular*. From now on, we will not specify the basis for A and just denote symmetric bilinear forms by symmetric matrices.

For the intersection form Q_M , its signature is defined to be the *signature* $\sigma(M)$ of the manifold M . It follows from Poincaré duality that Q_M is unimodular if M is closed.

Let S and S' be two matrices that give integral symmetric bilinear forms on A and A' through a basis for A and a basis for A' . The union of these two bases forms a basis for $A \oplus A'$. Under this basis, the matrix $\text{diag}(S, S')$ gives a bilinear form on $A \oplus A'$, which is denoted by $S \oplus S'$. For simplicity, $S \oplus S \oplus \dots \oplus S$, which contains r copies of S , is denoted by rS . Note that $Q_M \oplus Q_{M'}$ is the intersection form of $M \# M'$ for closed 4-manifolds M and M' .

The following lemma tells us that any simply-connected indefinite closed 4-manifold can be built up by a number of fixed 4-manifolds through the connected sum operation. Let $\#rM$ be the connected sum of r copies of M for an integer $r > 0$ and $\#0M^4 = S^4$.

Lemma 5.1. *There exist oriented closed connected 4-manifolds $M_i, 1 \leq i \leq 7$, such that for any simply-connected closed 4-manifold M with indefinite intersection form, M is homeomorphic to $\#k_1M_1\#k_2M_2\#\dots\#k_7M_7$ for some non-negative integers k_i .*

Proof. All the indefinite unimodular forms can be classified as follows (cf. [GS], Theorem 1.2.21). If Q_M is odd, then it is isomorphic to $b_2^+[+1] \oplus b_2^-[-1]$. If Q_M is even, then it is isomorphic to $c_1E_8 \oplus c_2H$, where c_1, c_2 are integers, $c_2 \geq 0$, E_8 is an even form with $\text{rk}(E_8) = \sigma(E_8) = 8$, and $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

In [Fr], Freedman proved the classification result for simply-connected 4-manifolds as follows. For every integral, unimodular symmetric bilinear form Q , there exists a simply-connected, closed, topological 4-manifold M such that $Q_M \cong Q$. If Q is even, this manifold is unique up to homeomorphism. If Q is odd, there are exactly two different homeomorphism types of such manifolds. At most one of these two homeomorphism types carries a smooth structure.

Now we need to consider two cases.

Case 1. Q_M is odd, i.e. $Q_M \cong p[+1] \oplus q[-1]$.

$\mathbb{CP}^2$ is a smooth manifold with intersection form $[+1]$. If we reverse the orientation of $\mathbb{CP}^2$, the resulting manifold $\overline{\mathbb{CP}}^2$ has intersection form $[-1]$. By Freedman's theorem above, there are also non-smoothable manifolds X_1 and X_{-1} that have intersection forms $[+1]$ and $[-1]$, respectively.

Note that the two manifolds with the same odd intersection form can be distinguished by Kirby-Siebenmann triangulation invariants $\text{KS} \in \mathbb{Z}_2$ (cf. [KS]). One has non-zero invariant and cannot be smooth, while the other has zero invariant and is stably smoothable. Thus $\text{KS}(X_1) = \text{KS}(X_{-1}) = 1$.

The Kirby-Siebenmann invariant is additive in the sense that (cf. [FQ], Section 10.2B) if there is a homeomorphism of boundaries $\partial M \cong \partial N$, then

$\text{KS}(M \cup_{\partial} N) = \text{KS}(M) + \text{KS}(N)$. Combined with the fact that D^4 is smoothable, we know that $\text{KS}(M_1 \# M_2) = \text{KS}(M_1) + \text{KS}(M_2)$ for connected, closed, topological 4-manifolds M_1 and M_2 .

Clearly the form $p[+1] \oplus q[-1]$ can be realized by both $X_1 \# (p-1)\mathbb{CP}^2 \# q\overline{\mathbb{CP}}^2$ (if $p > 0$) and $X_{-1} \# p\mathbb{CP}^2 \# (q-1)\overline{\mathbb{CP}}^2$ (if $q > 0$). These two manifolds have non-trivial Kirby-Siebenmann invariant and are therefore non-smoothable. On the other hand, the smooth manifold $\#p\mathbb{CP}^2 \# q\overline{\mathbb{CP}}^2$ also has intersection form $p[+1] \oplus q[-1]$.

Now no matter whether M is smooth or not, it must be homeomorphic to the connected sum of some copies of $M_1 = \mathbb{CP}^2$, $M_2 = \overline{\mathbb{CP}}^2$, $M_3 = X_1$ and $M_4 = X_{-1}$.

Case 2. Q_M is even, i.e. $Q_M \cong pE_8 \oplus q(-E_8) \oplus rH$, where p, q and r are non-negative integers.

Let X_{E_8} and X_{-E_8} be the simply-connected 4-manifolds with intersection forms E_8 and $-E_8$, respectively. Clearly $S^2 \times S^2$ has intersection form H . Then $pX_{E_8} \# qX_{-E_8} \# r(S^2 \times S^2)$ has the same intersection form as M , and by Freedman's theorem, they are homeomorphic.

Therefore, we can finish the proof by taking $M_5 = X_{E_8}$, $M_6 = X_{-E_8}$, and $M_7 = S^2 \times S^2$. $\square$

When M is a 4-manifold with boundary, we can glue M and $-M$ together by the identity map between ∂M and $\partial(-M)$. The resulting manifold is called the *double* of M and denoted by DM .

Lemma 5.2. *For any simply-connected, compact topological 4-manifold M with $\partial M \neq \emptyset$, its double DM is homeomorphic to either $\#k\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ or $\#kS^2 \times S^2$ for some non-negative integer k .*

Proof. Clearly DM is a simply-connected, closed 4-manifold with $\sigma(DM) = \sigma(M) + \sigma(-M) = 0$ (cf. [Ki], Chapter II, Theorem 5.3). By the classification result of symmetric bilinear forms (cf. the first paragraph in the proof of Lemma 5.1), the intersection form Q of DM is either $k[+1] \oplus k[-1]$ or kH , depending on whether Q is odd or even.

When Q is odd, DM and the smooth manifold $\#k\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ have the same intersection form. As $\text{KS}(DM) = \text{KS}(M) + \text{KS}(-M) = 0$, they are homeomorphic. When Q is even, DM and $\#kS^2 \times S^2$ have the same intersection form and are therefore homeomorphic. $\square$

The next lemma is the key observation in proving Theorem 1.2.

Lemma 5.3. *For any oriented closed connected 4-manifold M , there exists an oriented closed connected 5-manifold W such that for any positive integer r , $\#rM$ can be topologically flatly embedded into W .*

Proof. First let us recall the definition of surgery operations. (For details, see [GS].) Let M be a closed n -manifold and let $\varphi : S^k \times D^{n-k} \rightarrow M$ ($0 \leq k \leq n$) be an embedding. We can remove $\varphi(\text{int}(S^k \times D^{n-k}))$ from M and replace it with $D^{k+1} \times S^{n-k-1}$ by gluing along $\varphi|_{S^k \times S^{n-k-1}}$. This procedure is called a *surgery* by φ on M . Clearly, the new manifold is still closed.

Choose an embedding $\varphi : S^0 \times D^4 \rightarrow M^4$, and denote the images of φ , which are two disjoint 4-balls, by D' and D^* respectively. Set $M^o = M - \text{int}(D' \cup D^*)$. Then the surgery by φ on M gives a manifold $M' = M^o \cup S^3 \times D^1$. Let $p : S^3 \times D^1 \rightarrow D^1$ be the projection onto the second factor. From now on, we regard D^1 as $[0, 1]$ and

FIGURE 1

S^1 as $\mathbb{R}/\mathbb{Z}$. Define $q : D^1 \rightarrow S^1$ to be $q(x) = [x]$. Clearly $q \circ p$ can be extended to a map $f : M' \rightarrow S^1$ as we can let the whole M^o map to $[0]$ in S^1 . Figure 1 illustrates M' and the map f , where M^o maps to the big dot on the circle.

Let $W' = M' \times S^1$, and $g : W' \rightarrow T^2 = S^1 \times S^1$ be defined by $g(x, y) = (f(x), y)$. Choose a small closed disk D in T^2 centered at $([\frac{1}{4}], [0])$ and identify it with the standard 2-ball D^2 . Then there is a homeomorphism from $g^{-1}(D)$ to $S^3 \times D^2$. Thus we can apply a surgery operation which cuts out $\text{int}(g^{-1}(D))$ from W' and glues back $D^4 \times S^1$. We will show that the resulting manifold W has the desired property.

Let $h : W \rightarrow T^2$ be given by g on $W' - \text{int}(g^{-1}(D))$ and the projection onto the factor $\partial D = S^1$ in the piece $D^4 \times S^1$. For a point $q \in T^2$, we have $h^{-1}(q) = M^o$ if $q = ([0], [y])$, $h^{-1}(q) = \emptyset$ if $q \in \text{int}(D)$, $h^{-1}(q) = D^4$ if $q \in \partial D$ and $h^{-1}(q) = S^3$ otherwise.

For any positive integer r , define $C_r = \{([\frac{1}{4} + rt], [t]) | 0 \leq t \leq 1\} \subset T^2$ and $X_r = h^{-1}(C_r)$. We want to show that X_r is $\#rM$. The T^2 along with the curve C_r are shown in Figure 2, where the shaded region denotes the disk D .

FIGURE 2

The curve C_r intersects the longitude $\{([0], [y]) | 0 \leq y \leq 1\}$ in r points, say successively $a_1, \dots, a_r$, as t increases. Name the two points of $C_r \cap \partial D$ by a_0 and

a_{r+1} . Note that $h^{-1}(a_i)$ ($1 \leq i \leq r$) is a copy of M^o , both $h^{-1}(a_0)$ and $h^{-1}(a_{r+1})$ are D^4 , while for each sub-arc of C_r that connects a_i and a_{i+1} ($0 \leq i \leq r$), its pre-image under h is a copy of $S^3 \times D^1$. Figure 3 illustrates X_r .

FIGURE 3

For the given manifold M , if $M - \text{int}(D')$ and $M - \text{int}(D^*)$ are glued to $S^3 \times I$ along the orientation reversing homeomorphism $\psi_1 : \partial D' \rightarrow -(S^3 \times \{0\})$ and $\psi_2 : \partial D^* \rightarrow S^3 \times \{1\}$, respectively, then one gets a closed manifold N . In fact, N is just $M \# M$ as $S^3 \times \{\frac{1}{2}\}$ cuts N into two components which are homeomorphic to $M - \text{int}(D')$ and $M - \text{int}(D^*)$, respectively. It can be seen from Figure 3 that the adjacent copies of M^o are connected to $S^3 \times D^1$ in the manner described by ψ_1 and ψ_2 above, while the two D^4 fill the holes of D^* and D' in the first and the last copies of M^o . Therefore X_r is homeomorphic to $\#rM$.

Choose a very small $\varepsilon > 0$ and set $N = \{([\frac{1}{4} + rt], [t + s]) | 0 \leq t \leq 1, -\varepsilon \leq s \leq \varepsilon\} \subset T^2$. All the points in N with fixed s form a curve C_r^s in T^2 . The above argument for C_r applies to this curve as well. So $h^{-1}(C_r^s)$ is still $\#rM$. Note that $h^{-1}(C_r^s)$ vary continuously on s . Therefore $h^{-1}(N)$ gives an embedding of $(\#rM) \times [-\varepsilon, \varepsilon]$ in W . $\square$

Lemma 5.4. *If there are topologically flat embeddings $M_1^4 \hookrightarrow W_1^5$ and $M_2^4 \hookrightarrow W_2^5$, then $M_1 \# M_2$ can be topologically flatly embedded into $W_1 \# W_2$.*

Proof. Choose D_i^4 in M_i . Since $M_i \times I$ is a part of W_i , $D_i^4 \times I$ is a 5-ball in W_i ($i = 1, 2$). Cut out these two 5-balls from the W_i , respectively, and glue their boundaries. Then we have an embedding of $(M_1 \# M_2) \times I$ into $W_1 \# W_2$. $\square$

Finally we are ready to prove Theorem 1.2.

Proof. (a) Choose M to be M_i , $1 \leq i \leq 7$ respectively as given in Lemma 5.1 and construct corresponding 5-manifolds W_i by Lemma 5.3. It follows from Lemma 5.4 that the desired manifold W can be chosen to be $W_1 \# W_2 \# \dots \# W_7$.

(b) Clearly M embeds in DM .

It is well known (cf. [GS], Section 4.6) that $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \cong S^2 \tilde{\times} S^2$ and that $S^2 \tilde{\times} S^2 \# S^2 \tilde{\times} S^2 \cong S^2 \times S^2 \# S^2 \tilde{\times} S^2$, where $S^2 \tilde{\times} S^2$ denotes the unique non-trivial S^2 -bundle over S^2 . Combined with Lemma 5.2, DM is homeomorphic to either $\#kS^2 \times S^2$ or $\#(k-1)S^2 \times S^2 \# S^2 \tilde{\times} S^2$.

$S^2 \tilde{\times} S^2$ can be embedded into $S^2 \tilde{\times} S^3$ because the inclusion $\text{SO}(3) \hookrightarrow \text{SO}(4)$ induces the isomorphism $\pi_1(\text{SO}(3)) \cong \pi_1(\text{SO}(4)) \cong \mathbb{Z}_2$. Also, $S^2 \times S^2$ can be embedded into $\mathbb{R}^5$ through the standard inclusion $S^2 \times D^3 \hookrightarrow \mathbb{R}^5$. Therefore by Lemma 5.4, both $\#kS^2 \times S^2$ and $\#(k-1)S^2 \times S^2 \# S^2 \tilde{\times} S^2$ can be embedded into $S^2 \tilde{\times} S^3$. $\square$

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