QUADRATIC NONLINEAR DERIVATIVE SCHRÖDINGER EQUATIONS - PART 2

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ABSTRACT. In this paper we consider the local well-posedness theory for the quadratic nonlinear Schrödinger equation with low regularity initial data in the case when the nonlinearity contains derivatives. We work in $2 + 1$ dimensions and prove a local well-posedness result for small initial data with low regularity.

1. INTRODUCTION

This work is concerned with the initial value problem for the nonlinear Schrödinger equations which generically have the form:

$$(1) \quad \begin{cases} iu_t - \Delta u = P(u, \bar{u}, \nabla u, \nabla \bar{u}), & t \in \mathbb{R}, x \in \mathbb{R}^n, \\ u(x, 0) = u_0(x) \end{cases}$$

where $u : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{C}$ and $P : \mathbb{C}^{2n+2} \rightarrow \mathbb{C}$ is a polynomial.

We are interested in the theory of local well-posedness for this problem in Sobolev spaces. We considered the same problem in [1], which also contains a more complete introduction to the subject.

The problem (1) becomes more difficult once we consider quadratic and higher order nonlinearities. In this case the most general result known is due to Kenig, Ponce and Vega; see [8]:

Theorem 1. *Assume that P has no constant or linear terms. Then there exist $s = s(n, P) > 0$ and $m = m(n, P) > 0$ such that $\forall u_0 \in H^s(\mathbb{R}^n) \cap L^2(\mathbb{R}^n : |x|^{2m} dx)$ the problem (1) has a unique solution in $C([0, T] : H^s \cap L^2(\mathbb{R}^n : |x|^{2m} dx))$ where $T = T(\|u\|_{H^s \cap L^2(\mathbb{R}^n : |x|^{2m} dx)})$.*

If P does not contain quadratic terms, then the above authors also obtain a similar result without involving any decay; see [8].

If the nonlinearity in (1) contains derivatives, the problem becomes more delicate. One of the reasons is the loss of derivative on the right hand side of the equation. Another is the need of some decay on the initial data. This is motivated by an early result due to Mizohata, see [9], which proves that for the problem

$$(2) \quad \begin{cases} iu_t - \Delta u = b_1(x) \nabla u, & t \in \mathbb{R}, x \in \mathbb{R}^n, \\ u(x, 0) = u_0(x) \end{cases}$$

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the following condition on b_1 is necessary for the L^2 well-posedness theory:

$$(3) \quad \sup_{x \in \mathbb{R}^n, \omega \in \mathbb{S}^{n-1}, R > 0} \left| \operatorname{Re} \int_0^R b_1(x + r\omega) \cdot \omega dr \right| < \infty.$$

We have remarked in [1] that the use of decay of type $L^2(|x|^m dx)$ is not the most appropriate for the Schrödinger equation since this structure is not conserved under the linear flow.

The goal of the paper is to establish the lowest Sobolev regularity the initial data can have so that we have well-posedness. When asking this question, one should be more specific about the type of the equation and the dimension of the space.

The quadratic terms in P are the first ones to be understood. The quadratic nonlinearities without derivatives have been studied in [4], [6], [3], [2] and the references therein, and the results obtained are quite satisfactory.

If the nonlinearity contains terms with derivatives, then the problem is called the derivative nonlinear Schrödinger equation (D-NLS). To formalize the problem we introduce the bilinear forms:

$$(4) \quad B^1(u, v) = \sum_{i=1}^n c_i u v_{x_i} \quad B^2(u, v) = \sum_{i,j=1}^n c_{ij} u_{x_i} v_{x_j}$$

where c_i, c_{ij} are complex constants. The two types of bilinear forms correspond to the problem with one differentiated term and one undifferentiated, respectively, to the problem with both differentiated terms. The general quadratic (D-NLS) takes one of the two forms:

$$(5) \quad \begin{cases} iu_t - \Delta u = B^1((u, \bar{u}), (u, \bar{u})) \\ u(x, 0) = u_0(x), \end{cases}$$

and

$$(6) \quad \begin{cases} iu_t - \Delta u = B^2((u, \bar{u}), (u, \bar{u})) \\ u(x, 0) = u_0(x). \end{cases}$$

Therefore we allow the nonlinearity to contain conjugate terms too; we can formalize that by defining:

$$(7) \quad B^i((u, \bar{u}), (v, \bar{v})) = B_1^i(u, v) + B_2^i(\bar{u}, v) + B_3^i(u, \bar{v}) + B_4^i(\bar{u}, \bar{v})$$

where B_α^i is of type B^i as defined in (4) for all $i = 1, 2$ and $\alpha = 1, 4$.

The scaling exponents for these problems are $s_c = \frac{n}{2} - 1$ for (5), and $s_c = \frac{n}{2}$ for (6). The best result we know is due to Chihara; see [5]: if $m = \frac{n}{2} + 2$ and $s = \frac{n}{2} + 4$, then the quadratic (D-NLS) is locally well-posed (this statement is in the spirit of Theorem 1). In the particular case when the nonlinearity is of the form $B^i(\bar{u}, \bar{u})$, Gruenrock was able to establish the local well-posedness for (5) and (6) for $u_0 \in H^s$ with $s > s_c$; see [7]. This was possible since the quadratic nonlinearity with both terms conjugated behaves the best with respect to the geometry of the problem. This corresponds to the fact that for the bilinear terms of type $B^i(\bar{u}, \bar{v})$ one has access to much better estimates. As a consequence we mutually agree that the factor B_4^i is absent in (7) and seek improvements for the result in [5].

The analysis of the problem brings the conclusion that the “worst” interactions are the orthogonal ones, i.e. those between waves which travel in orthogonal directions. Therefore the problem becomes more interesting in dimensions 2 or higher.

This is why in this work we specialize to the case of two-dimensional quadratic (D-NLS).

In [1] we have obtained the following result:

Theorem 2. *Assume $n = 2$. Given any $s > s_c$ and $T > 0$, there exists $\delta > 0$ such that for every $u_0 \in \mathcal{D}'\mathcal{R}H^s$ with $\delta_0 = \|u_0\|_{\mathcal{D}'\mathcal{R}H^s} < \delta$, the quadratic (D-NLS), in the form (5) or (6), has a unique solution u in $C([0, T] : \mathcal{D}'\mathcal{R}H^s) \cap \mathcal{RD}'Z^s$ with Lipschitz dependence on the initial data.*

The definition of Z^s will come up in the current paper. The space $\mathcal{D}'\mathcal{R}H^s$ (and the same for $\mathcal{RD}'Z^s$) is an improvement of H^s in the sense that the functions there have some spherical symmetry and some decay besides the H^s structure.

Two major questions arise once we acknowledge this result. One is to try to obtain a result without involving any spherical symmetry, and the other one is to remove the smallness condition on the initial data.

The current paper answers the first issue. We essentially prove that for any $s > s_c + 1$ the quadratic (D-NLS) is locally well-posed for small $u_0 \in \mathcal{DH}^s$ (we provide a precise definition of $\mathcal{DH}^s$ in section 2).

In the following we make the result we obtain more precise.

We denote by $\chi_{[0, T]}$ a smooth approximation of the characteristic function of $[0, T]$ such that $\chi_{[0, T]}(t) = 1$, $\forall t \in [0, T]$. We will always consider $\chi_{[0, T]}$ as a function of time; in other words, by $\chi_{[0, T]}$ we mean $\chi_{[0, T]}(t)$.

We dedicate section 2 to the definition of the spaces $\mathcal{DZ}^s$ (for the solutions) and $\mathcal{DW}^s$ (for the nonlinearity). These spaces satisfy the linear estimate:

Theorem 3. *If $g \in \mathcal{DH}^s$ and $f \in \mathcal{DW}^s$, then the solution of*

$$(8) \quad \begin{cases} iu_t - \Delta u = f, \\ u(x, 0) = g(x) \end{cases}$$

satisfies $\chi_{[0, 1]}u \in \mathcal{DZ}^s \cap C_t\mathcal{DH}_x^s$ with an estimate

$$(9) \quad \|\chi_{[0, 1]}u\|_{\mathcal{DZ}^s \cap L_t^\infty \mathcal{DH}_x^s} \lesssim \|g\|_{\mathcal{DH}^s} + \|f\|_{\mathcal{DW}^s}.$$

The basic spaces we use in this paper, i.e. Z^s and W^s , are the same as in [1]; hence, if we ignore the $\mathcal{D}$ part, we can claim the result of Theorem 3 on behalf of Theorem 3 in [1]. However the type of decay we use in this paper is more general than the one used in [1]. The conservation of decay in the linear estimates can be easily adapted to those proofs since the decay is properly scaled for the Schrödinger equation. We leave this as an exercise.

The bilinear estimate is the next key result:

Theorem 4. *If $s > s_c + 1$, we have the bilinear estimate*

$$(10) \quad \|B^i((u, \bar{u}), (v, \bar{v}))\|_{\mathcal{DW}^s} \leq C_s \|u\|_{\mathcal{DZ}^s} \|v\|_{\mathcal{DZ}^s}.$$

Once we have the above two results, a standard fixed point argument, see section 2 in [1], gives us the main result:

Theorem 5. *Assume $n = 2$. Given any $s > s_c + 1$ and $T > 0$, there exists $\delta > 0$ such that for every $u_0 \in \mathcal{DH}^s$ with $\delta_0 = \|u_0\|_{\mathcal{DH}^s} < \delta$, the problem (5), respectively (6), has a unique solution u in $C([0, T] : \mathcal{DH}^s) \cap \mathcal{DZ}^s$ with Lipschitz dependence on the initial data.*

The general approach of this result is similar to the one in Part 1. The most difficult part of the problem is establishing the bilinear estimates (10). We essentially tailor our spaces in order to fit these estimates. Also, since we import the linear estimates from [1], the only result we need to prove is the one in Theorem 10.

We start with $X^{s, \frac{1}{2}, 1}$ as the candidate for Z^s and $X^{s, -\frac{1}{2}, 1}$ as a candidate for W^s . The bilinear estimates work fine as long as we recover information which is at some distance from the paraboloid ($\tau = \xi^2$), but they fail very close to the paraboloid since one catches a logarithm of the high frequency. To remedy this we introduce a more delicate decomposition of the part of the Fourier space which is at a distance less than 1 from the paraboloid. More exactly we use a wave packet decomposition and we measure the packets in $L_t^\infty L_x^2$. Then the target space W^s is also modified at a distance less than 1 from the paraboloid; i.e., we also have a wave packet decomposition and the packets are measured in $L_t^1 L_x^2$. We have to recover an L_t^1 structure on the packets for $B(u, v)$, and this is why we need to involve the extra decay.

All along the argument we do involve decay in the bilinear estimates, and this is why our spaces will be of types DZ^s and DW^s . See section 2 for the definitions.

The spaces we use in this paper are in some way the counterpart of the ones involved in dealing with the wave maps equation; see [11] and [10]. Our spaces are a bit more difficult since they involve phase-space localization.

The result in the current paper is not a trivial reproduction of the argument in [1], for the case when one removes spherical symmetry. There are fundamental technical differences which are hard to explain in just a few words. We would like to bring to the attention of the reader that the decay type we introduce in this paper (see section 2) is very close to the decay along lines as the Mizohata condition requires; see (3). In [1] we have used a hypoelliptic operator to quantify decay, and that was very well fitted for a problem with some symmetry on the solution.

We conclude the introduction with a few open problems. We think that the regularity threshold $s = s_c + 1$ is sharp for (5) and (6). In this paper we provide only the positive result; the negative one is subject to current research.

We believe that our result should extend to all dimensions $n \geq 3$ since we do not foresee any new obstacles. A brief follow-up in this direction may be useful.

Another interesting problem is to remove the smallness assumption on the initial data. This is also a subject of current research.

2. DEFINITION OF THE SPACES

For each u we denote by $\mathcal{F}u = \hat{u}$ the Fourier transform of u . This is always taken with respect to all the variables, unless otherwise specified.

Throughout the paper $A \lesssim B$ means $A \leq CB$ for some constant C which is independent of any possible variable in our problem. We say $A \approx B$ if $A \leq CB \leq C^2A$ for the same constant C . We say that we localize at frequency 2^i to mean that, in the support of the localized function, $|(\xi, \tau)| \in [2^{i-1}, 2^{i+1}]$.

In the Schrödinger equation, time and space scale in a different way, and this suggests defining the norm for (ξ, τ) by $|(\xi, \tau)| = (|\tau| + \xi^2)^{\frac{1}{2}}$. In dealing with the quadratic nonlinearity without derivatives the Bourgain spaces $X^{s, b}$ proved to be very useful. They are defined in the following way:

$$X^{s, b} = \{f \in S'; \langle (\xi, \tau) \rangle^s \langle \tau - \xi^2 \rangle^b \hat{f} \in L^2\}.$$

Here and thereafter $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$ where $|x|$ is the norm of x . We will employ frequency localized versions of $X^{s, \frac{1}{2}}$ which are constructed according to weights present in its definition.

Consider $\varphi_0 : [0, \infty) \rightarrow \mathbb{R}$ to be a nonnegative smooth function such that $\varphi_0(x) = 1$ on $[0, 1]$ and $\varphi_0(x) = 0$ if $x \geq 2$. Then for each $i \geq 1$ we define $\varphi_i : [0, \infty) \rightarrow \mathbb{R}$ by $\varphi_i(x) = \varphi_0(2^{-i}x) - \varphi_0(2^{-i+1}x)$. We define the operators S_i , to localize at frequency 2^i , by

$$\mathcal{F}(S_i f) = \hat{f}_i = \varphi_i(|(\xi, \tau)|) \cdot \hat{f}(\xi, \tau).$$

For $d \in I_i = \{2^{-i}, 2^{-i+1}, \dots, 2^{i+2}\}$ we define

$$\varphi_{i,d}(\xi, \tau) = \varphi_i(|(\xi, \tau)|) \cdot \varphi_{i+\ln_2 d}(|\tau - \xi^2|).$$

There is one simple reason to choose to work with d in this way rather than working with 2^d . If $|(\xi, \tau)| \approx 2^i$, then $|\tau - \xi^2| \approx |(\tau, \xi)|d((\xi, \tau), P) \approx 2^i d((\xi, \tau), P)$ (away from zero). Hence one should think of d as the distance to P since the support of $\varphi_{i,d}$ is approximately the set

$$\{(\xi, \tau) : |(\xi, \tau)| \approx 2^i, d((\tau, \xi), P) \approx d\} \approx \{(\xi, \tau) : |(\xi, \tau)| \approx 2^i, |\tau - \xi^2| \approx d2^i\}.$$

It is easy to notice that

$$\sum_{d \in I_i} \varphi_{i,d}(\xi, \tau) = \varphi_i(|(\xi, \tau)|), \quad \forall (\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R}$$

We define the operators $S_{i,d}$ by $S_{i,d}f = f_{i,d} = \check{\varphi}_{i,d} * S_i f$ and we have $f_i = \sum_{d \in I_i} f_{i,d}$. In the support of $\hat{f}_{i,d}$ we have $1 + |\tau - \xi^2| \approx 2^i d$.

Sometimes it is useful to localize in a linear way rather than a dyadic way. In these cases we localize with respect to the value of $|\tau - \xi^2|$ instead; we will make this clear when we need it.

For each dyadic value $d \in I_i$ we introduce the operators which localize at distances less than and greater than d from P :

$$S_{i,\leq d}f = f_{i,\leq d} = \sum_{d' \in I_i: d' \leq d} f_{i,d'} \quad \text{and} \quad S_{i,\geq d}f = f_{i,\geq d} = f_i - f_{i,\leq d}.$$

The part of $\hat{f}$ which is at a distance less than 1 from P plays an important role, and this is why we define the global operators:

$$S_{\cdot, \leq 1}f = f_{\cdot, \leq 1} = \sum_{i=0}^{\infty} f_{i, \leq 1} \quad \text{and} \quad S_{\cdot, \geq 1}f = f_{\cdot, \geq 1} = \sum_{i=0}^{\infty} f_{i, \geq 1}.$$

We denote by A_i the support in $\mathbb{R}^2 \times \mathbb{R}$ of $\varphi_i(|(\xi, \tau)|)$ and by $A_{i,d}$ the support of $\varphi_{i,d}$. In a similar way we can define $A_{i,\leq d}$, respectively $A_{i,\geq d}$, to be the support of the operators $S_{i,\leq d}$, respectively $S_{i,\geq d}$.

For functions whose Fourier transform is supported in A_i we define, for any $1 \leq p \leq \infty$,

$$\begin{aligned} \|f\|_{X_i^{0, \frac{1}{2}, p}}^p &= \sum_{d \in I_i} \|S_{i,d}f\|_{X^{0, \frac{1}{2}}}^p, \\ \|f\|_{X^{s, \frac{1}{2}, p}}^2 &= \sum_i 2^{2is} \|f_i\|_{X_i^{0, \frac{1}{2}, p}}^2. \end{aligned}$$

For technical purposes we need localized versions of these spaces, such as $X_{i,d}^{s,\frac{1}{2}} = \{f \in X^{s,\frac{1}{2}} : \hat{f} \text{ supported in } A_{i,d}\}$ and, similarly, $X_{i,\leq d}^{s,\frac{1}{2},p}$ and $X_{i,\geq d}^{s,\frac{1}{2},p}$.

$X^{s,\frac{1}{2},1}$ is our first candidate for the space of solutions. Our computations indicate that it is the right space to measure only the part of the solution whose support in the Fourier space is at a distance greater than 1 from P , i.e. the $S_{\geq 1}$ part of our solutions.

The $S_{\leq 1}$ part of the solutions can be measured in $X^{s,\frac{1}{2},\infty}$ plus an additional structure whose construction is described below.

We define the following lattice in the plane $\tau = 0$:

$$\Xi = \{\xi = (r, \theta) : r = n, \theta = \frac{\pi k}{2n}, n, k \text{ positive integers}\}.$$

Ξ is like a lattice in polar coordinates. It has the properties that the distance between any two points is at least 1 and that for every $\eta \in \mathbb{R}^2$ there is a $\xi \in \Xi$ such that $|\xi - \eta| \leq 1$. For each $\xi \in \Xi$ we build a nonnegative function ϕ_ξ to be a smooth approximation of the characteristic function of the cube of size 1 in $\mathbb{R}^2$ centered at ξ and satisfying the natural partition property:

$$\sum_{\xi \in \Xi} \phi_\xi = 1.$$

We can easily impose uniform bounds on the derivatives of the system $(\phi_\xi)_{\xi \in \Xi}$. For each $\xi \in \Xi$ we define:

$$f_\xi = \check{\phi}_\xi * f \quad \text{and} \quad f_{\xi, \leq 1} = \check{\phi}_\xi * f_{\leq 1}.$$

The convolution above is performed with respect to the x variable, i.e. it does not involve the t variable. The support of $\hat{f}_{\xi, \leq 1}$ is like a parallelepiped having center $(\xi, \xi^2) \in P$ and sizes $\approx |\xi|$ in the τ direction and 1 in the other two directions (normal to P and the completing third one).

The next concern is how to measure $f_{\xi, \leq 1}$. Let's denote by $(Q^m)_{m \in \mathbb{Z}^2}$ the standard partition of $\mathbb{R}^2$ in cubes of size 1; i.e. Q^m is centered at $m = (m_1, m_2) \in \mathbb{Z}^2$, has its sides parallel to the standard coordinate axis and has size 1. For each $\xi \in \Xi$, $m \in \mathbb{Z}^2$ and $l \in \mathbb{Z}$ we define the tubes:

$$\begin{aligned} T_\xi^{m,l} &= \bigcup_{t \in [l, l+1]} (Q^m - 2t\xi) \times \{t\} \\ &= \{(x - 2t\xi_1, y - 2t\xi_2, t) : (x, y) \in Q^m \text{ and } t \in [l, l+1]\}. \end{aligned}$$

Then, for each $\xi \in \Xi$, we define the space Y_ξ by the following norm:

$$\|f\|_{Y_\xi}^2 = \sum_{(m,l) \in \mathbb{Z}^3} \|f\|_{L_t^\infty L_x^2(T_\xi^{m,l})}^2.$$

We have $f = \sum_{\xi \in \Xi} f_\xi$ and then we define the space Y^s by the norm:

$$\|f\|_{Y^s}^2 = \sum_{\xi \in \Xi} \langle \xi \rangle^{2s} \|f_\xi\|_{Y_\xi}^2.$$

We define also the localized versions $Y_i = \{f \in Y^0; \hat{f} \text{ supported in } A_i\}$ and $Y_{i,\leq d} = \{f \in Y^0; \hat{f} \text{ supported in } A_{i,\leq d}\}$, the last one being defined for any $d \in I_i$ with $d \leq 1$.

To bring everything together, define Z^s to be

$$Z^s = \{f \in S' : \|f_{\cdot, \geq 1}\|_{X^{s, \frac{1}{2}, 1}} + \|f_{\cdot, \leq 1}\|_{Y^s} + \|f_{\cdot, \leq 1}\|_{X^{s, \frac{1}{2}, \infty}} < \infty\}$$

with the obvious norm.

Our spaces are equipped with an additional decay structure which we describe below. For each i , let Q_i^m be a system of cubes of size 2^i which form a partition of $\mathbb{R}^2$; we choose them so that their sides are parallel to the coordinate axes and the center of Q_i^m is $(2^i m_1, 2^i m_2)$. Let $L = \{(x, y) : ax + by = 0\}$ be the equation of a line passing through the origin and denote by $\bar{n}$ the normal unit vector to L . For each $k \in \mathbb{Z}$, we define $L_i^k = \{(x', y') : (x', y') = (x, y) + k2^i \bar{n}; (x, y) \in L\}$ to be the line parallel to L and at distance $2^i k$ from L . If $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ we introduce the norm

$$\|f\|_{\mathcal{D}_i L^2} = \sup_L \sup_{k \in \mathbb{Z}} \left(\sum_{m: Q_i^m \cap L_i^k \neq \emptyset} \|f\|_{L^2(Q_i^m)} \right).$$

In the above norm we sum up the $\|f\|_{L^2(Q_i^m)}$ in l^1 over those Q_i^m 's which intersect L_i^k , and then we take a supremum with respect to k . In the end we take a supremum with respect to all lines L .

Our decay space is defined by the norm:

$$\|f\|_{\mathcal{D}H^s}^2 = \sum_i 2^{2is} (\|f_i\|_{\mathcal{D}_i L^2}^2 + \|f_i\|_{L^2}^2).$$

If $f : \mathbb{R}^3 \rightarrow \mathbb{C}$ (we include the time dependent functions), we define

$$\|f\|_{\mathcal{D}_i L^2} = \sup_L \sup_{k \in \mathbb{Z}} \left(\sum_{m: Q_i^m \cap L_i^k \neq \emptyset} \|f\|_{L^2(Q_i^m \times \mathbb{R})} \right).$$

For f such that $\hat{f}$ is supported in $A_{i,d}$ we define

$$\|f\|_{\mathcal{D}_i X_{i,d}^{s, \frac{1}{2}}} = 2^{js} (2^j d)^{\frac{1}{2}} (\|f\|_{\mathcal{D}_i L^2} + \|f\|_{L^2}),$$

and the decay version of $X^{s, \frac{1}{2}, p}$, $\mathcal{D}X^{s, \frac{1}{2}, p}$ is defined by the norm

$$\|f\|_{\mathcal{D}X^{s, \frac{1}{2}, 1}}^2 = \sum_i \left(\sum_d \|f_{i,d}\|_{\mathcal{D}_i X_{i,d}^{s, \frac{1}{2}}}^p \right)^{\frac{2}{p}}.$$

For $f \in Y_i$ we define

$$\|f\|_{\mathcal{D}_i Y_i} = \sup_L \sup_{k \in \mathbb{Z}} \left(\sum_{m: Q_i^m \cap L_i^k \neq \emptyset} \left(\sum_{\xi: |\xi| \approx 2^i} \|\chi_{Q_i^m \times \mathbb{R}} f_\xi\|_{Y_\xi}^2 \right)^{\frac{1}{2}} \right) + \|f\|_{Y}.$$

The decay version of Y^s , $\mathcal{D}Y^s$ is defined by the norm

$$\|f\|_{\mathcal{D}Y^s}^2 = \sum_i 2^{2is} \|f_i\|_{\mathcal{D}_i Y_i}^2.$$

To bring everything together, define $\mathcal{D}Z^{s,N}$ to be

$$\mathcal{D}Z^s = \{f \in S' : \|f_{\cdot, \geq 1}\|_{\mathcal{D}X^{s, \frac{1}{2}, 1}} + \|f_{\cdot, \leq 1}\|_{\mathcal{D}Y^s} + \|f_{\cdot, \leq 1}\|_{\mathcal{D}X^{s, \frac{1}{2}, \infty}} < \infty\}.$$

So far we have built the spaces suitable for the solution of (1). We need also a space for the right hand side of the equation; see Theorem 3.

We can easily define $X^{s, -\frac{1}{2}, p}$ by simply replacing $\frac{1}{2}$ with $-\frac{1}{2}$ in the definition of $X^{s, \frac{1}{2}, p}$. Then we define $\mathcal{Y}^s$ by

$$\|f\|_{\mathcal{Y}^s}^2 = \sum_{\xi \in \Xi} \langle \xi \rangle^{2s} \|f_\xi\|_{\mathcal{Y}_\xi}^2$$

where $\mathcal{Y}_\xi$ is defined as follows:

$$\|f\|_{\mathcal{Y}_\xi}^2 = \sum_{(m, l) \in \mathbb{Z}^3} \|f\|_{L_t^1 L_x^2(T_\xi^{m, l})}^2.$$

Notice that $(\mathcal{Y}_\xi)^* = Y_\xi$ since we will use this later for duality purposes.

We introduce W^s , defined by the norm

$$\|f\|_{W^s} = \inf\{\|f_1\|_{\mathcal{Y}^s} + \|f_2\|_{X^{s, -\frac{1}{2}, 1}}; f = f_1 + f_2\}.$$

We measure the right hand side of (8) in

$$W^s = \{f \in S' : \|f_{\cdot, \leq 1}\|_{W^s}^2 + \|f_{\cdot, \geq 1}\|_{X^{s, -\frac{1}{2}, 1}}^2 < \infty\}.$$

As before, we can define $\mathcal{D}X^{s, -\frac{1}{2}, 1}$, $\mathcal{D}\mathcal{Y}^s$ and $\mathcal{D}W^s$.

Besides $X^{s, b}$ we need the conjugate $\bar{X}^{s, b}$, which is defined as follows:

$$\bar{X}^{s, b} = \{f \in S'; \langle (\xi, \tau) \rangle^s \langle \tau + \xi^2 \rangle^b \hat{f} \in L^2\}.$$

We can define all the other elements the same way as above by simply placing a bar on each space and operator, while replacing everywhere $|\tau - \xi^2|$ with $|\tau + \xi^2|$ and P with $\bar{P} = \{(\xi, \tau) : \tau + \xi^2 = 0\}$.

We record the following important facts:

$$f \in X^{s, b} \iff \bar{f} \in \bar{X}^{s, b} \quad \text{and} \quad (X^{0, \frac{1}{2}})^* = X^{0, -\frac{1}{2}}.$$

Before we start we need to introduce some new localization operators. For each $i \in \mathbb{N}$ we define a refined lattice:

$$(11) \quad \Xi^i = \{\xi = (n2^{-i}, \theta) : \theta = \frac{\pi}{2} \frac{l}{n}; n \in \mathbb{N}, l \in \mathbb{Z}\}.$$

For each $\xi \in \Xi^i$ we build the corresponding ϕ_ξ^i to be a smooth approximation of the characteristic function of the cube centered at ξ and with size 2^{-i} . We also assume that the system $(\phi_\xi^i)_{\xi \in \Xi^i}$ forms a partition of unity in $\mathbb{R}^2$.

For each $l \in \mathbb{Z}$ we can easily construct a function $\chi_{[l-\frac{1}{2}, l+\frac{1}{2}]}$ to be a smooth approximation of the characteristic function of the interval $[l-\frac{1}{2}, l+\frac{1}{2}]$ and such that the system $(\chi_{[l-\frac{1}{2}, l+\frac{1}{2}]})_{l \in \mathbb{Z}}$ forms a partition of unity in $\mathbb{R}$. For any $\xi \in \Xi^i$ with $|\xi| \leq 2^{i+1}$ we consider those $l \in \mathbb{Z}$ with the property $|(\xi, l)| \approx 2^i$ and define the operators:

$$\hat{f}_{\xi, l} = \phi_{[l-\frac{1}{2}, l+\frac{1}{2}]}(\tau) \phi_\xi^i(\xi) \hat{f}(\xi, \tau).$$

The support of $\hat{f}_{\xi, l}$ is approximately a tube centered at (ξ, l) and of size $2^{-i} \times 2^{-i} \times 1$, the last one being in the τ direction. Since the distance of these tubes will play an important role, sometimes it would be convenient if we were able to work with $(f_{\xi, \xi^2+l})_{\xi \in \Xi^i, l \in \mathbb{Z}}$ instead. The only problem is that it is not guaranteed that $\xi^2 \in \mathbb{Z}$ for all $\xi \in \Xi^i$. Of course we could change the way we cut in the τ direction, but this would complicate notation even more. We choose instead to ignore that ξ^2 may not be an integer, and go on and use g_{ξ, ξ^2+l} . It will be obvious from the argument that this does not affect in any way the rigorousness of the proof. The last notation we introduce is $f_{\xi, \xi^2 \pm l} = f_{\xi, \xi^2+l} + f_{\xi, \xi^2-l}$.

For $d \leq 2^{i-2}$ we obtain a new decomposition of $g_{i,d}$:

$$(12) \quad g_{i,d} = \sum_{k=2^{i-1}d}^{2^{i+1}d} \sum_{\xi \in \Xi^i} g_{\xi, \xi^2 \pm k}.$$

Notice that the ξ 's $\in \Xi^i$ involved in the above summation have $|\xi| \approx 2^i$.

For the part of $\hat{g}$ supported away from P we obtain a different decomposition:

$$(13) \quad g_{i, \geq 2^{i-2}} = \sum_n \sum_{\xi \in \Xi^i} \sum_{l \in I_\xi} g_{\xi, l}$$

where $I_\xi = \{l \in \mathbb{Z} : 2^{2i-2} \leq |l - \xi^2| \leq 2^{2i+2}\}$. The ξ 's $\in \Xi^i$ involved in the above summation have $|\xi| \leq 2^{i+1}$.

3. BILINEAR ESTIMATES IN $X^{s, \frac{1}{2}, 1}$

In this section we derive the bilinear estimates for $B^i(u, v)$ and $B^i(u, \bar{v})$, $i = 1, 2$, in $X^{s, \frac{1}{2}, 1}$, where B^i is of type (4).¹ We introduce the additional bilinear form $\tilde{B}(u, v) = u \cdot v$. If $\hat{u}$ is localized in A_i we use the estimate $\|\nabla u\|_{L^2} \leq 2^i \|u\|_{L^2}$. $X^{s, \pm \frac{1}{2}, 1}$ are L^2 -like on dyadic pieces; hence if $\hat{u}$ is localized in A_i and $\hat{v}$ is localized in A_j we use the estimates:

$$(14) \quad \|\tilde{B}(u, v)\|_X \leq C \|u\|_{X'} \|v\|_{X''} \Rightarrow \|B^k(u, v)\|_X \leq 2^{(k-1)i+j} C \|u\|_{X'} \|v\|_{X''},$$

$$(15) \quad \|\tilde{B}(u, \bar{v})\|_X \leq C \|u\|_{X'} \|v\|_{X''} \Rightarrow \|B^k(u, \bar{v})\|_X \leq 2^{(k-1)i+j} C \|u\|_{X'} \|v\|_{X''}.$$

Here X, X', X'' are of type $X^{s, \pm \frac{1}{2}, p}$. The constant C may depend on u, v , more exactly on their localizations. The key thing is once we have estimates for $\tilde{B}$, we obtain estimates for B^k , $k = 1, 2$, by simply bringing in the correction factor of $2^{(k-1)i+j}$. In order to simplify the exposure, we decided to derive all the estimates for B^2 , drop the upper index and simply use B instead. In other words, from here on,

$$B(u, v) = \sum_{i,j=1}^n c_{ij} u_{x_i} v_{x_j}.$$

One can easily check that we obtain in a similar way all the estimates needed for $B^1(u, v)$ and $B^1(u, \bar{v})$.

Another thing to keep in mind is that we apply duality along the proof, and anytime we do it we mean it in the context of $\tilde{B}$, not B .

The main results we claim are listed in the following theorem.

Theorem 6. *a) If $i \leq j$, we have the following estimates:*

$$(16) \quad \|B(u, v)\|_{\mathcal{D}_k X_k^{s, -\frac{1}{2}}} \lesssim j 2^{(2-s)i} 2^{(k-j)s} \|u\|_{\mathcal{D}_i X_i^{s, \frac{1}{2}}} \|v\|_{\mathcal{D}_j X_j^{s, \frac{1}{2}}}.$$

The above estimates hold true if $B(u, v)$ is replaced by $B(\bar{u}, v)$ or $B(u, \bar{v})$.

b) If $5i \leq j$, we have the following estimates:

$$(17) \quad \|B(u, v)\|_{\mathcal{D}_j X_{j, \geq 2^{-i}}^{s, -\frac{1}{2}}} \lesssim 2^{(2-s)i} i \|u\|_{\mathcal{D}_i X_i^{s, \frac{1}{2}}} \|v\|_{\mathcal{D}_j X_{j, \geq 2^{-i}}^{s, \frac{1}{2}}}.$$

The above estimates hold true if $B(u, v)$ is replaced by $B(\bar{u}, v)$ or $B(u, \bar{v})$.

¹Recall that due to the result in [7] we know that the bilinear forms $B^i(\bar{u}, \bar{v})$ behave much better; hence we do not seek improvements in those cases.

3.1. Basic estimates. We start with a simple result stating how two parabolas interact under convolution. We need a few technical definitions.

Throughout this section functions are defined on Fourier space (they should be thought of as Fourier transforms). This is why we use the standard coordinates (ξ, τ) .

For each $c \in \mathbb{R}$ denote by $P_c = \{(\xi, \tau) : \tau - \xi^2 = c\}$ and by $\bar{P}_c = \{(\xi, \tau) : \tau + \xi^2 = c\}$. For simplicity $P = P_0$ and $\bar{P} = \bar{P}_0$.

Denote by $\delta_{P_c} = \delta_{\tau - \xi^2 = c}$ the standard surface measure associated with the parabola P_c . With respect to this measure, the restriction of f to P_c has norm

$$\|f\|_{L^2(P_c)} = \left(\int f^2(\xi, \xi^2 + c) \sqrt{1 + 4|\xi|^2} d\xi \right)^{\frac{1}{2}}.$$

The first result was derived in Part 1; see the corresponding section there.

Proposition 1. *Let $f \in L^2(P^1)$ and $g \in L^2(P^2)$ such that f is localized at frequency 2^i and g at frequency 2^j . We have*

$$(18) \quad \|f\delta_{P^1} * g\delta_{P^2}\|_{L^2} \lesssim 2^{\min(i,j)} \|f\|_{L^2(P^1)} \|g\|_{L^2(P^2)}$$

where $P^1 \in \{P_{c_1}, \bar{P}_{c_1}\}$ and $P^2 \in \{P_{c_2}, \bar{P}_{c_2}\}$.

The second result will replace the corresponding one in Part 1 for the case when we do not have any symmetry involved.

Proposition 2. *We assume that we are in the same setup as in Proposition 1. In addition we assume $|c_1|, |c_2| \leq d$. Then*

$$(19) \quad \|f\delta_{P_{c_1}} * g\delta_{P_{c_2}}\|_{L^2(|\xi| \approx 2^k, |\tau - \xi^2| \leq d)} \lesssim d^{\frac{1}{2}} 2^{\frac{\max(i,j) - k}{2}} \|f\|_{L^2(P_{c_1})} \|g\|_{L^2(P_{c_2})}.$$

The result holds true if any P_{c_i} is replaced by $\bar{P}_{c_i}$, $i = 1, 2$.

Proof. We notice that it is enough to prove the result under the hypothesis that $d \leq 2^{i+j-4}$, since otherwise the result in (18) is stronger. Without losing generality we can assume $c_1 = c_2 = 0$ and $i \leq j$.

$f\delta_P * g\delta_P$. It is enough to estimate $|(f\delta_{\tau=\xi^2} * g\delta_{\tau=\xi^2})h|$ for any $h \in L^2$ supported in the region $|\tau - \xi^2| \leq d$. For any such h we have

$$\begin{aligned} & (f\delta_{\tau=\xi^2} * g\delta_{\tau=\xi^2})h \\ &= \int f(\xi)g(\eta)h(\xi + \eta, \xi^2 + \eta^2) \sqrt{1 + 4\xi^2} \sqrt{1 + 4\eta^2} d\xi d\eta. \end{aligned}$$

Since h is supported in a region $|\tau - \xi^2| \leq d$ we need the following condition on the variables inside the integral: $|(\xi + \eta)^2 - (\xi^2 + \eta^2)| \leq d$ or $2|\xi||\eta| \cos \theta \leq d$ where θ is the angle between ξ and η . Hence $\cos \theta \leq 2^{-i-j}d$, which implies $|\theta - \frac{\pi}{2}| \leq 2^{-i-j}d$. This suggests decomposing:

$$[0, 2\pi] = \bigcup_{l \in J_{i,j,d}} I_l = \bigcup_{l=1}^{2^{i+j+2}d^{-1}} [(l - \frac{1}{2})2^{-i-j}d\frac{\pi}{2}, (l + \frac{1}{2})2^{-i-j}d\frac{\pi}{2}],$$

in other words to split $[0, 2\pi]$ into a disjoint union of intervals of size $2^{-i-j}d\frac{\pi}{2}$.

Correspondingly we split

$$f = \sum_{l \in J_{i,j,d}} f_l \quad \text{and} \quad g = \sum_{l \in J_{i,j,d}} g_l$$

such that f_l is the part of f localized in $A_l = \{\xi : \arg \xi \in I_l\}$ and similarly for g .

If $\arg \xi \in I_l$ and $\arg \eta \in I_{l'}$ and we want them to belong to the domain of integration above, we need to impose $|l - l'| = 2^{i+j}d^{-1}$ (modulo $2^{i+j+2}d^{-1}$). For each l there are two l' 's with this property. We simplify more and consider that there is only one $l^\perp = l + 2^{i+j}d^{-1}$ (modulo $2^{i+j+2}d^{-1}$) with this property; one can easily complete the argument with both values. Then we have

$$\begin{aligned} & (f\delta_{\tau=\xi^2} * g\delta_{\tau=\eta^2})h \\ &= \sum_{l \in J_{i,j,d}} \int f_l(\xi)g_{l^\perp}(\eta)h(\xi + \eta, \xi^2 + \eta^2)\sqrt{1+4\xi^2}\sqrt{1+4\eta^2}d\xi d\eta. \end{aligned}$$

Now that we have a sharp angular localization, we complete it with a norm localization which should be consistent with the angular one:

$$f_l = \sum_m f_{l,m} \quad \text{and} \quad g_{l^\perp} = \sum_n g_{l^\perp,n}.$$

For the low frequency, things are simple: $f_{l,m}$ is the part of f_l localized in the set $A_{l,m}^i = \{\xi \in A_l : |\xi| \in [2^{-i}d(m - \frac{1}{2}), 2^{-i}d(m + \frac{1}{2})]\}$ and notice that this is consistent with the arc length size localization of $g_{l^\perp}$ (which is $2^{-i}d$).

For the high frequency we should do something similar: one would like to localize $|\eta|$ in intervals of size $2^{-j}d$. The only problem we encounter is that if $i \ll j$ and d is small, then we may see the curvature of the circle and the support of $g_{l^\perp,n}$ cannot be approximated by a rectangle.

In order to fix this we chose $g_{l^\perp,n}$ to be the part of g_l localized in $A_{l^\perp,n}^j = \{\eta \in A_{l^\perp} : \eta \cdot v_{l^\perp} \in [2^{-j}d(n - \frac{1}{2}), 2^{-j}d(n + \frac{1}{2})]\}$; we denoted by $v_{l^\perp} = (\cos(l^\perp 2^{-i-j}d\frac{\pi}{2}), \sin(l^\perp 2^{-i-j}d\frac{\pi}{2}))$ and $v_l = (\cos(l 2^{-i-j}d\frac{\pi}{2}), \sin(l 2^{-i-j}d\frac{\pi}{2}))$ (we will need the second one later).

This way the supports of $A_{l,m}^i$ and $B_{l^\perp,n}^j$ are rectangles of sizes $2^{-i}d \times 2^{-j}d$.

The crucial property is that the sum sets of the supports, namely $A_{l,m}^i + A_{n,l^\perp}^j = \{\xi + \eta : \xi \in A_{l,m}^i \text{ and } \eta \in A_{n,l^\perp}^j\}$, are disjoint with respect to the pair (m, n) . This is mainly because the sum set $A_{l,m}^i + A_{n,l^\perp}^j$ is approximately a rectangle of size $2^{-i}d \times 2^{-j}d$ and whose center has coordinates $(2^{-j}dn, 2^{-i}dm)$ with respect to the base $(v_l, v_{l^\perp})$. Let's denote by $h_{m,n}$ the part of h which is supported in this set (more precisely the projection of the support on the ξ space should be supported there). Hence we can write

$$\begin{aligned} & (f\delta_{\tau=\xi^2} * g\delta_{\tau=\eta^2})h \\ &= \sum_{l \in J_{i,j,d}} \sum_m \sum_n \int f_{l,m}(\xi)g_{l^\perp,n}(\eta)h_{m,n}(\xi + \eta, \xi^2 + \eta^2)\sqrt{1+4\xi^2}\sqrt{1+4\eta^2}d\xi d\eta \end{aligned}$$

and then, for fixed l , we can estimate

$$\begin{aligned} |(f_l \delta_{\tau=\xi^2} * g_{l^\perp} \delta_{\tau=\xi^2})h| &\lesssim \left(\sum_m \|f_{l,m}\|_{L^2(P)}^2 \right)^{\frac{1}{2}} \left(\sum_n \|g_{l^\perp,n}\|_{L^2(P)}^2 \right)^{\frac{1}{2}} \\ &\cdot \left(\sum_{m,n} \int h_{m,n}^2(\xi + \eta, \xi^2 + \eta^2) \sqrt{1+4\xi^2} \sqrt{1+4\eta^2} d\xi d\eta \right)^{\frac{1}{2}}. \end{aligned}$$

We introduce the change of variables $(\xi_1, \eta_1, \eta_2) \rightarrow (\zeta_1, \zeta_2, \zeta_3)$:

$$(20) \quad \begin{cases} \xi_1 + \eta_1 = \zeta_1, \\ \xi_2 + \eta_2 = \zeta_2, \\ \xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2 = \zeta_3, \end{cases}$$

whose Jacobian is $\frac{1}{2}(\eta_1 - \xi_1)^{-1}$. If we were to integrate over a region where $|\eta_1 - \xi_1| \geq \frac{|\eta - \xi|}{\sqrt{2}} \geq 2^{k-2}$,² then we would get $|\eta_1 - \xi_1|^{-1} \lesssim 2^{-k}$; hence

$$\begin{aligned} &\int h_{m,n}^2(\xi + \eta, \xi^2 + \eta^2) \sqrt{1+4\xi^2} \sqrt{1+4\eta^2} d\xi d\eta \\ &\approx 2^{i+j-k} \int h_{m,n}^2(\zeta_1, \zeta_2, \zeta_3) d\zeta_1 d\zeta_2 d\zeta_3 \leq d 2^{j-k} \|h_{m,n}\|^2. \end{aligned}$$

In the last estimate we have used the fact that we integrate over a domain where $\Delta\xi_2 \approx 2^{-i}d$. If we were to integrate over the domain where $|\eta_2 - \xi_2| \geq \frac{|\eta - \xi|}{\sqrt{2}}$, then we would switch the role of ξ_1 and ξ_2 in the change of variables. Hence we can conclude the above computation with

$$|(f_l \delta_{\tau=\xi^2} * g_{l^\perp} \delta_{\tau=\xi^2})h| \lesssim d^{\frac{1}{2}} 2^{\frac{j-k}{2}} \|f_l\|_{L^2(P)} \|g_{l^\perp}\|_{L^2(P)} \|h\|_{L^2}.$$

In the end we perform the summation with respect to l to obtain

$$\begin{aligned} |(f \delta_{\tau=\xi^2} * g \delta_{\tau=\xi^2})h| &\lesssim \sum_l d^{\frac{1}{2}} 2^{\frac{j-k}{2}} \|f_l\|_{L^2(P)} \|g_{l^\perp}\|_{L^2(P)} \|h\|_{L^2} \\ &\lesssim d^{\frac{1}{2}} 2^{\frac{j-k}{2}} \left(\sum_l \|f_l\|_{L^2(P)}^2 \right)^{\frac{1}{2}} \left(\sum_l \|g_{l^\perp}\|_{L^2(P)}^2 \right)^{\frac{1}{2}} \|h\|_{L^2} \\ &\lesssim d^{\frac{1}{2}} 2^{\frac{j-k}{2}} \|f\|_{L^2(P)} \|g\|_{L^2(P)} \|h\|_{L^2}. \end{aligned}$$

Since this holds true for any $h \in L^2$ supported in $|\tau - \xi^2| \leq d$, we can conclude with the result of the proposition.

$f\delta_{\bar{P}} * g\delta_P, f\delta_P * g\delta_{\bar{P}}, f\delta_{\bar{P}} * g\delta_{\bar{P}}$. The argument is similar to the above one. $\square$

3.2. Bilinear estimates on dyadic regions. For a bilinear estimate we use the notation

$$\mathcal{X} \cdot \mathcal{Y} \rightarrow \mathcal{Z},$$

which means that we seek for an estimate $\|B(u, v)\|_{\mathcal{Z}} \leq C \|u\|_{\mathcal{X}} \cdot \|v\|_{\mathcal{Y}}$. Here the constant C may depend on some variables, such as the frequency when the functions are localized.

²The estimate $2^k \lesssim |\xi - \eta|$ is weaker in this case; the sharp one is $2^k \lesssim |\xi + \eta|$. We use this since when estimating $f\delta_{\bar{P}} * g\delta_P$ one encounters the second term.

A standard way of writing down each case looks like:

$$X_{i,d_1}^{0,\frac{1}{2}} \cdot X_{j,d_2}^{0,\frac{1}{2}} \rightarrow X_{j,d_3}^{0,-\frac{1}{2}}.$$

This means that for $u \in X_{i,d_1}^{0,\frac{1}{2}}$ and $v \in X_{j,d_2}^{0,\frac{1}{2}}$ we estimate the part of $B(u, v)$ (or $\tilde{B}(u, v)$) whose Fourier transform is supported in A_{j,d_3} . Formally we estimate $\mathcal{F}^{-1}(\chi_{A_{j,d_3}} \mathcal{F}(B(u, v)))$. This is going to be the only kind of “abuse” in notation which we make throughout the paper, i.e. considering $\|B(u, v)\|_{X_{j,d_3}^{s,\frac{1}{2}}}$ even if $\mathcal{F}(B(u, v))$ is not supported in A_{j,d_3} . We choose to do this so that we do not have to relocalize every time in A_{j,d_3} .

Sometimes we prove estimates via duality or conjugation:

$$X \cdot Y \rightarrow Z \iff X \cdot (Z)^* \rightarrow (Y)^* \quad \text{and} \quad X \cdot Y \rightarrow Z \iff \bar{X} \cdot \bar{Y} \rightarrow \bar{Z}.$$

Proposition 3. *Assume $1 \leq i \leq j$. Then we have the estimates*

$$(21) \quad \|B(u, v)\|_{X_{j,d_3}^{0,-\frac{1}{2}}} \lesssim 2^{\frac{3i}{2}} (\max(d_2, d_3))^{-\frac{1}{2}} \|u\|_{X_{i,d_1}^{0,\frac{1}{2}}} \|v\|_{X_{j,d_2}^{0,\frac{1}{2}}},$$

$$(22) \quad \|B(u, v)\|_{X_{j,d_3}^{0,-\frac{1}{2}}} \lesssim 2^{\frac{i+j}{2}} \|u\|_{X_{i,d_1}^{0,\frac{1}{2}}} \|v\|_{X_{j,d_2}^{0,\frac{1}{2}}}.$$

If $|i - j| \leq 1$ and $k \leq j - 1$, then we have the estimates

$$(23) \quad \|B(u, v)\|_{X_{k,d_3}^{0,-\frac{1}{2}}} \lesssim 2^{\frac{k}{2}+j} (\max(d_1, d_2))^{-\frac{1}{2}} \|u\|_{X_{i,d_1}^{0,\frac{1}{2}}} \|v\|_{X_{j,d_2}^{0,\frac{1}{2}}},$$

$$(24) \quad \|B(u, v)\|_{X_{k,d_3}^{0,-\frac{1}{2}}} \lesssim 2^{\frac{3j-k}{2}} \|u\|_{X_{i,d_1}^{0,\frac{1}{2}}} \|v\|_{X_{j,d_2}^{0,\frac{1}{2}}}$$

where the parameters are restricted by $k \leq j - 5 \Rightarrow d_3 \leq 2^{k-2}$.

All of the above estimates hold true, with the same restrictions, if $B(u, v)$ is replaced by $B(\bar{u}, v)$ or $B(u, \bar{v})$.

Proof. We should make some comments about the statement above. If $i \leq j - 2$, then the result is localized at frequency $\approx 2^j$. There is something to estimate only if $k = j, j \pm 1$. The estimates for the case $k = j$ are generic; hence we will prove them only.

It is only when $i = j - 1, j$ that we have parts of the result at lower frequencies and then we have to provide estimates for all $k \leq j + 1$.

We deal first with the case when we measure the outcome at the high frequency, and at the end we deal with the case when we have $i = j - 1, j$ and we have to measure the outcome at lower frequencies.

We need to transform the estimates on paraboloids into estimates on dyadic pieces. If we localize in a region where $|\xi| \approx 2^k$, the parabolas P_c make an angle of $\approx 2^{-k}$ with the τ axis, so we have the following relation between measures:

$$d\xi d\tau \approx 2^{-k} dP_c dc.$$

If $d \leq 2^{i-3}$, then in $A_{i,d}$ we have $|\xi| \approx 2^i$. Therefore for $l \leq i - 3$,

$$(25) \quad \|u\|_{X_{i,2^l}^{0,\pm\frac{1}{2}}}^2 \approx 2^{\pm(l+i)} \int_{b=2^{l-1}}^{2^{l+1}} \|\hat{u}\|_{L^2(P_{(b2^i)})}^2 db.$$

At this time we are ready to start the estimates.

$$X_{i,d_1}^{0,\frac{1}{2}} \cdot X_{j,d_2}^{0,\frac{1}{2}} \rightarrow X_{j,d_3}^{0,-\frac{1}{2}}.$$

Case 1: $d_1 \leq 2^{i-3}, d_2, d_3 \leq 2^{j-3}$.

If $d_2 \leq d_3$, then we can apply the result of (18) to evaluate

$$\begin{aligned} \|\hat{u} * \hat{v}\|_{L^2} &\lesssim \int_{I_1} \int_{I_2} \|\hat{u} \delta_{P_{b_1 2^i}} * \hat{v} \delta_{P_{b_2 2^j}}\|_{L^2} db_1 db_2 \\ &\lesssim \int_{I_1} \int_{I_2} 2^i \|\hat{u}\|_{L^2(P_{b_1 2^i})} \|\hat{v}\|_{L^2(P_{b_2 2^j})} db_1 db_2 \\ &\lesssim 2^i \left(\int_{I_1} (1 + b_1 2^i)^{-1} db_1 \right)^{\frac{1}{2}} \|u\|_{X_{i,d_1}^{0,\frac{1}{2}}} \left(\int_{I_2} (1 + b_2 2^j)^{-1} db_2 \right)^{\frac{1}{2}} \|v\|_{X_{j,d_2}^{0,\frac{1}{2}}} \\ &\approx 2^{\frac{i-j}{2}} \|u\|_{X_{i,d_1}^{0,\frac{1}{2}}} \|v\|_{X_{j,d_2}^{0,\frac{1}{2}}}. \end{aligned}$$

Here we used the fact that $I_1 \approx [\frac{d_1}{2}, 2d_1]$, which gives us $\int (1 + b_1 2^i)^{-1} db_1 \approx 2^{-i}$. We do the same thing for the integral with respect to b_2 . (25) gives us

$$\|\tilde{B}(u, v)\|_{X_{j,d_3}^{0,-\frac{1}{2}}} \lesssim (2^j d_3)^{-\frac{1}{2}} \|\hat{u} * \hat{v}\|_{L^2} \lesssim 2^{\frac{i}{2}-j} d_3^{-\frac{1}{2}} \|u\|_{X_{i,d_1}^{0,\frac{1}{2}}} \|v\|_{X_{j,d_2}^{0,\frac{1}{2}}}.$$

If $d_3 \leq d_2$, then, by duality, we have

$$X_{i,d_1}^{0,\frac{1}{2}} \cdot \bar{X}_{j,d_3}^{0,\frac{1}{2}} \rightarrow \bar{X}_{j,d_2}^{0,-\frac{1}{2}} \Leftrightarrow \bar{X}_{i,d_1}^{0,\frac{1}{2}} \cdot X_{j,d_3}^{0,\frac{1}{2}} \rightarrow X_{j,d_2}^{0,-\frac{1}{2}}.$$

The proof of the last estimate is similar to the one above since (18) allows us to use δ_{P_c} instead of δ_{P_c} . This justifies (21).

If $\max(2^i d_1, 2^j d_2, 2^j d_3) = 2^j d_3$ and we run the same argument, but use (19) instead, we obtain the estimate (22). If $\max(2^i d_1, 2^j d_2, 2^j d_3) = 2^j d_2$, then we obtain the result by duality as indicated above. If $\max(2^i d_1, 2^j d_2, 2^j d_3) = 2^i d_1$, then by duality the estimate is equivalent to

$$\bar{X}_{j,d_3}^{0,\frac{1}{2}} \cdot X_{j,d_2}^{0,\frac{1}{2}} \rightarrow \bar{X}_{i,d_1}^{0,-\frac{1}{2}} \Leftrightarrow X_{j,d_3}^{0,\frac{1}{2}} \cdot \bar{X}_{j,d_2}^{0,\frac{1}{2}} \rightarrow X_{i,d_1}^{0,-\frac{1}{2}}.$$

This estimate is derived in a similar manner by using (19) in the case when the output is localized at lower frequencies.

Case 2: $d_1 \geq 2^{i-2}, d_2, d_3 \leq 2^{j-3}$.

If $d_2, d_3 \leq 2^{2i-j-3}$, then we can deduce (22) as above, via duality:

$$X_{i,d_1}^{0,\frac{1}{2}} \cdot X_{j,d_2}^{0,\frac{1}{2}} \rightarrow X_{j,d_3}^{0,-\frac{1}{2}} \Leftrightarrow X_{j,d_3}^{0,\frac{1}{2}} \cdot \bar{X}_{j,d_2}^{0,\frac{1}{2}} \rightarrow X_{i,d_1}^{0,-\frac{1}{2}}.$$

In this case we also have that (22) is stronger than (21). If $\max(d_2, d_3) \geq 2^{2i-j-3}$ and $d_2 \leq d_3$, then we use a different argument. We decompose $A_{j,d_2} = A'_{j,d_2} \cap A''_{j,d_2}$ into two disjoint sets, where A'_{j,d_2} contains the elements from A_{j,d_2} with large ξ_1 , while A''_{j,d_2} contains the elements from A_{j,d_2} with large ξ_2 . The key property we want from the two sets is that A'_{j,d_2} has size d_2 in the direction of ξ_1 , while A''_{j,d_2} has size d_2 in the direction of ξ_2 . Then we have:

$$\begin{aligned} \|\tilde{B}(u, v)\|_{L^2} &\lesssim \|\hat{u} * (\chi_{A'_{j,d_2}} \hat{v})\|_{L^2} + \|\hat{u} * (\chi_{A''_{j,d_2}} \hat{v})\|_{L^2} \\ &\lesssim \|\hat{u}\|_{L^1_{\xi_2, \tau} L^2_{\xi_1}} \|\chi_{A'_{j,d_2}} \hat{v}\|_{L^2_{\xi_2, \tau} L^1_{\xi_1}} + \|\hat{u}\|_{L^1_{\xi_1, \tau} L^2_{\xi_2}} \|\chi_{A''_{j,d_2}} \hat{v}\|_{L^2_{\xi_1, \tau} L^1_{\xi_2}} \\ &\lesssim 2^{\frac{3i}{2}} \|\hat{u}\|_{L^2} d_2^{\frac{1}{2}} \|\hat{v}\|_{L^2} \lesssim 2^{\frac{i-j}{2}} \|u\|_{X_{i,d_1}^{0,\frac{1}{2}}} \|v\|_{X_{j,d_2}^{0,\frac{1}{2}}}. \end{aligned}$$

In the above computations we have used the size of the support of the involved functions as described above, while for $\hat{u}$ we simply used that it has sizes $2^i \times 2^i \times 2^{2i}$. If we rewrite the above estimate in $X_{j,d_3}^{0,-\frac{1}{2}}$ we obtain (21), which is stronger than (22) in this case. If $d_3 \leq d_2$ we obtain the estimate via duality.

Case 3: $\max(d_2, d_3) \geq 2^{j-2}$.

If $d_2 \leq d_3$ and $d_2 \leq 2^{j-3}$, then we have two possibilities:

- $d_1 \leq 2^{i-3}$. Then the argument written down in Case 1 is valid for (21), which is stronger than (22).

- $d_1 \geq 2^{i-2}$. Then the last argument from Case 2 is valid for (21), which is stronger than (22).

If $d_3 \leq d_2$ and $d_3 \leq 2^{j-3}$, then the estimates follow by duality. If $d_2, d_3 \geq 2^{j-2}$, we use a simpler argument:

$$\|\hat{u}\|_{L^1} \lesssim 2^{\frac{3}{2}i} d_1^{\frac{1}{2}} \|\hat{u}\|_{L^2} \approx 2^i \|u\|_{X_{i,d_1}^{0,\frac{1}{2}}}.$$

Then we continue with

$$\begin{aligned} \|\tilde{B}(u, v)\|_{X_{j,d_3}^{0,-\frac{1}{2}}} &\approx 2^{-j} \|\hat{u} * \hat{v}\|_{L^2(A_{j,d_3})} \\ &\lesssim 2^{-j} \|\hat{u}\|_{L^1} \cdot \|\hat{v}\|_{L^2} \lesssim 2^{i-2j} \|u\|_{X_{j,d_1}^{0,\frac{1}{2}}} \|v\|_{X_{j,d_2}^{0,\frac{1}{2}}}. \end{aligned}$$

Notice that, since $i \leq j$, the first estimate is stronger than the second; hence we do not have anything else to prove.

$$\bar{X}_{i,d_1}^{0,\frac{1}{2}} \cdot X_{j,d_2}^{0,\frac{1}{2}} \rightarrow X_{j,d_3}^{0,-\frac{1}{2}}.$$

These estimates are implicitly proved in the previous group of estimates.

$$X_{i,d_1}^{0,\frac{1}{2}} \cdot \bar{X}_{j,d_2}^{0,\frac{1}{2}} \rightarrow X_{j,d_3}^{0,-\frac{1}{2}}.$$

If $j-5 \leq i \leq j$, then the estimate is similar to the one in $\bar{X}_{i,d_1}^{0,\frac{1}{2}} \cdot X_{j,d_2}^{0,\frac{1}{2}} \rightarrow X_{j,d_3}^{0,-\frac{1}{2}}$ since we have comparable interacting frequencies.

The nontrivial case is $i \leq j-6$. It is not possible that $d_2, d_3 \leq 2^{j-10}$, since functions in $\bar{X}_{i,d_1}^{0,\frac{1}{2}}$ have their Fourier transform supported in a region with $\tau < -2^{2j-4}$ and functions in $X_{j,d_3}^{0,-\frac{1}{2}}$ have their Fourier transform supported in a region with $\tau > 2^{2j-4}$; an easy computation shows that, by convolution, the Fourier transform of a function in $X_{i,d_1}^{0,\frac{1}{2}}$ cannot move the first support to the second one.

If $\min(d_2, d_3) \geq 2^{j-3}$, then arguments similar to the ones we have used before will give the desired estimates.

High - High interactions with output at low frequencies. We have to deal with estimates of type $X_{i,d_1}^{0,\frac{1}{2}} \cdot X_{j,d_2}^{0,\frac{1}{2}} \rightarrow X_{k,d_3}^{0,-\frac{1}{2}}$, for $i = j-1, j$ and $k \leq j+1$. The estimate for the case $i = j$ is generic; hence we will work only this one out. In order to see more easily the duality, we choose to replace k by i (this new “ i ” is different than the one before) and look for an estimate of type

$$X_{j,d_1}^{0,\frac{1}{2}} \cdot X_{j,d_2}^{0,\frac{1}{2}} \rightarrow X_{i,d_3}^{0,-\frac{1}{2}}.$$

Conjugation and duality give us

$$X_{i,d_3}^{0,\frac{1}{2}} \cdot \bar{X}_{j,d_2}^{0,\frac{1}{2}} \rightarrow X_{j,d_1}^{0,-\frac{1}{2}} \Rightarrow \bar{X}_{i,d_3}^{0,\frac{1}{2}} \cdot X_{j,d_2}^{0,\frac{1}{2}} \rightarrow \bar{X}_{j,d_1}^{0,-\frac{1}{2}} \Rightarrow X_{j,d_1}^{0,\frac{1}{2}} \cdot X_{j,d_2}^{0,\frac{1}{2}} \rightarrow X_{i,d_3}^{0,-\frac{1}{2}}$$

and this is enough to justify the estimates. Similarly we obtain the estimates:

$$X_{j,d_1}^{0,\frac{1}{2}} \cdot \bar{X}_{j,d_2}^{0,\frac{1}{2}} \rightarrow X_{i,d_3}^{0,-\frac{1}{2}}.$$

□

3.3. Bilinear estimates in $X^{s, \frac{1}{2}}$ involving decay. The results in Proposition 3 indicate that we cannot recover a full derivative in the case when, see (21) and (22), d_2 and d_3 are small. This is an indication that we have to include the additional decay structure in order to complete the section.

Proposition 4. *If $i \leq j$, then we have the following estimates hold true:*

$$(26) \quad \|B(u, v)\|_{X_{j, d_3}^{0, -\frac{1}{2}}} \lesssim 2^{2i} \|u\|_{\mathcal{D}X_{i, d_1}^{0, \frac{1}{2}}} \|v\|_{X_{j, d_2}^{0, \frac{1}{2}}}.$$

The above estimate holds true if $B(u, v)$ is replaced by $B(\bar{u}, v)$ or $B(u, \bar{v})$.

Proof. The estimates are weaker than those in (21) and (22) unless $d_2, d_3 \leq 2^{-i-10}$ and $3i \leq j$. For interacting frequencies $(\xi, \tau_1) \in A_{i, d_1}$, $(\eta, \tau_2) \in A_{j, d_2}$ such that $(\xi + \eta, \tau_1 + \tau_2) \in A_{j, d_3}$ we have

$$\tau_1 + \tau_2 - (\xi + \eta)^2 = \tau_1 - \xi^2 + \tau_2 - \eta^2 - 2\xi \cdot \eta;$$

therefore $|\xi \cdot \eta| \lesssim \max(2^i d_1, 2^j d_2, 2^j d_3) \lesssim 2^{j-i}$, which implies that the angle θ between ξ and η should satisfy $|\theta \pm \frac{\pi}{2}| \lesssim 2^{-2i}$.

We consider first the case $d_1 \leq 2^{i-3}$. It is enough to estimate $\langle f \cdot g, h \rangle_{L^2} = \langle \hat{f} * \hat{g}, \hat{h} \rangle_{L^2}$ for $f \in \mathcal{D}X_{i, d_1}^{0, \frac{1}{2}}$, $g \in X_{j, d_2}^{0, \frac{1}{2}}$ and $h \in (X_{j, d_3}^{0, -\frac{1}{2}})^* = \bar{X}_{j, d_3}^{0, \frac{1}{2}}$.

We define $\Theta_p = \{\eta \in \Xi, |\eta| \approx 2^j : \arg \eta \in [(p - \frac{1}{2})2^{-2i} \frac{\pi}{2}, (p + \frac{1}{2})2^{-2i} \frac{\pi}{2}]\}$ and by f_p we denote the part of f whose Fourier transform is localized in Θ_p . If $p^\perp = p \pm 2^{2i}$, then from the above observations it follows that

$$\langle f \cdot g, h \rangle_{L^2} = \sum_p \langle f_{p^\perp} \cdot g_p, h \rangle_{L^2}.$$

Moreover, the part of h which counts in $\langle f_{p^\perp} \cdot g_p, h \rangle_{L^2}$ has also a similar localization, which is very close to the one of h_p . This is due to the fact that the low frequency $f_{p^\perp}$ cannot significantly change the support of the high frequency g_p on the Fourier space. We only need to have this information in a qualitative way rather than a quantitative one.

For each p , we define the orthonormal basis $x_p^1 = (\cos(p2^{-2i} \frac{\pi}{2}), \sin(p2^{-2i} \frac{\pi}{2}))$ and $x_p^2 = (-\sin(p2^{-2i} \frac{\pi}{2}), \cos(p2^{-2i} \frac{\pi}{2}))$ for $\mathbb{R}^2$. We denote by ξ_p^1, ξ_p^2 the corresponding basis on the Fourier side.

For each $m \in Z$ we define the rectangles $R_p^m \subset \mathbb{R}^2$ centered at $(m_1 2^j, m_2 2^i)$ with respect to the basis (x^p, y^p) and of sizes $2^j \times 2^i$ ($x^p \times y^p$ directions).

$\hat{g}_p$ is supported in a “curved” parallelepiped whose sizes are larger than the dual sizes of $Q_p^m \times \mathbb{R}$; the size of its support in the direction of ξ_p^1 is d_2 ; hence we can conclude

$$\sum_m \|g_p\|_{L_{x_p^1}^\infty L_{x_p^2, t}^2(R_p^m \times \mathbb{R})}^2 \lesssim d_2 \|g_p\|_{L^2}^2.$$

A similar result holds for h (here it is key that $\hat{h}$ has a similar angular localization to $\hat{g}_p$):

$$\sum_m \|\bar{h}_p\|_{L_{x_p^1}^\infty L_{x_p^2, t}^2(R_p^m \times \mathbb{R})}^2 \lesssim d_3 \|\bar{h}_p\|_{L^2}^2.$$

This gives us an $l_m^1(L_{x_p^1}^\infty L_{x_p^2, t}^1(R_p^m \times \mathbb{R}))$ structure for $g_p \bar{h}_p$. Therefore we need to estimate f in $l_m^\infty(L_{x_p^1}^1 L_{x_p^2, t}^1(R_p^m \times \mathbb{R}))$.

Each R_p^m can be written as $R_p^m \subset \bigcup_{\bar{m}: Q_i^{\bar{m}} \cap L_i^k \neq \emptyset} Q_i^{\bar{m}}$. We can be more precise: the line L generating L_i^k goes in the direction of x^p , and we can restrict indexes $\bar{m}$ to a set of cardinality $\approx 2^{j-i}$. Then we have

$$\begin{aligned} \|f_{p^\perp}\|_{L_{x_p^1}^1 L_{x_p^2, t}^\infty(R_p^m \times \mathbb{R})} &\lesssim \sum_{\bar{m}} \|f_{p^\perp}\|_{L_{x_p^1}^1 L_{x_p^2, t}^\infty(Q_i^{\bar{m}} \times \mathbb{R})} \\ &\lesssim \sum_{\bar{m}} \sum_{\xi \in \Xi^i} \sum_k \|f_{\xi, \xi^2+k}\|_{L_{y^p, t}^\infty L_{x^p}^1(Q_i^{\bar{m}} \times \mathbb{R})} \lesssim 2^i \sum_{\bar{m}} \sum_{\xi \in \Xi^i} \sum_k \|f_{\xi, \xi^2+k}\|_{L^\infty(Q_i^{\bar{m}} \times \mathbb{R})} \\ &\lesssim 2^i (2^{3i} d_1)^{\frac{1}{2}} \sum_{\bar{m}} \left(\sum_{\xi \in \Xi^i} \sum_k \|f_{\xi, \xi^2+k}\|_{L^\infty(Q_i^{\bar{m}} \times \mathbb{R})}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

In the above estimate we have used the fact that we have about $2^i d_1$ values for k and about 2^{2i} for the ξ 's; in Ξ^i we actually have about 2^{4i} ξ 's, but in the decomposition of $f_{p^\perp}$ we use only about 2^{2i} due to the angular localization. Next we use (41) to obtain

$$\|f\|_{L_{x_p^1}^1 L_{x_p^2, t}^\infty(R_p^m \times \mathbb{R})} \lesssim (2^{3i} d_1)^{\frac{1}{2}} \|f\|_{\mathcal{D}_i L^2} \lesssim 2^i \|f\|_{\mathcal{D} X_{i, d_1}^{0, \frac{1}{2}}}.$$

Hence we obtain the estimate

$$|\langle f \cdot g_p, h_p \rangle_{L^2}| \lesssim 2^i \|f\|_{\mathcal{D} X_{i, d_1}^{0, \frac{1}{2}}} d_2^{\frac{1}{2}} \|g_p\|_{L^2} d_3^{\frac{1}{2}} \|\bar{h}_p\|_{L^2}.$$

Summing with respect to p we obtain

$$|\langle f \cdot g, h \rangle_{L^2}| \lesssim 2^i \|f\|_{\mathcal{D} X_{i, d_1}^{0, \frac{1}{2}}} d_2^{\frac{1}{2}} \|g\|_{L^2} d_3^{\frac{1}{2}} \|\bar{h}\|_{L^2} \approx 2^{i-j} \|f\|_{\mathcal{D} X_{i, d_1}^{0, \frac{1}{2}}} \|g\|_{X_{j, d_2}^{0, \frac{1}{2}}} \|h\|_{\bar{X}_{j, d_3}^{0, \frac{1}{2}}}.$$

This translates into:

$$\|\tilde{B}(f, g)\|_{X_{j, d_3}^{0, \frac{1}{2}}} \lesssim 2^{i-j} \|f\|_{\mathcal{D} X_{i, d_1}^{0, \frac{1}{2}}} \|g\|_{X_{j, d_2}^{0, \frac{1}{2}}}.$$

Finally, by using (14) we obtain (26).

One can easily see from the above argument that we can easily carry on the same proof for the case when we deal with $B(f, g)$. As for $B(f, \bar{g})$, this was written just for the sake of completeness, since there is nothing to estimate there. The supports of high frequencies $\bar{A}_{j, d_2}$ and A_{j, d_3} are too far away to be linked via the convolution with the low frequency since we are in the case $3i \leq j$.

The proof for the case $d_1 \geq 2^{i-2}$ is completely similar to the previous one, just that we work with $g_{\xi, l}$ instead of g_{ξ, ξ^2+k} and use (44) instead of (41). $\square$

We can put the results of Propositions 3 and 4 together to obtain:

Proposition 5. *a) If $i \leq j$, we have the following estimates:*

$$(27) \quad \|B(u, v)\|_{X_{j, d_3}^{0, -\frac{1}{2}}} \lesssim 2^{2i} \|u\|_{\mathcal{D}_i X_{i, d_1}^{0, \frac{1}{2}}} \|v\|_{X_{j, d_2}^{0, \frac{1}{2}}}.$$

The above estimate holds true if $B(u, v)$ is replaced by $B(\bar{u}, v)$ or $B(u, \bar{v})$.

b) Assume $|i - j| \leq 1$. Then we have the following estimates:

$$(28) \quad \|B(u, v)\|_{X_{k, d_3}^{0, -\frac{1}{2}}} \lesssim 2^{\frac{k+3j}{2}} \|u\|_{\mathcal{D}_i X_{i, d_1}^{0, \frac{1}{2}}} \|v\|_{X_{j, d_2}^{0, \frac{1}{2}}}.$$

The above estimate holds true if $B(u, v)$ is replaced by $B(\bar{u}, v)$ or $B(u, \bar{v})$.

3.4. Abstract result. Before we turn to completing the bilinear estimates by proving the conservation of decay, we need to prepare some theoretical facts.

We fix λ to be positive. The arguments below are independent of the size of λ ; hence, later on, we have the freedom to apply the results we obtain for various values of λ .

Let $(Q^m)_{m \in \mathbb{Z}^n}$ be a partition of $\mathbb{R}^n$ (physical space) in disjoint cubes of size λ . We assume that Q^m is centered at λm . Similarly, let $(R^p)_{p \in \mathbb{Z}^n}$ be a partition of $\mathbb{R}^n$ (frequency space) in disjoint cubes of size λ^{-1} . We assume that R^p is centered at $\lambda^{-1}p$.

Let χ_{Q^m} be a smooth approximation of the characteristic function of Q^m in the following sense: χ_{Q^m} is a nonnegative function essentially supported³ in Q^m such that $\|\frac{\partial^\alpha \chi_{Q^m}}{\partial x^\alpha}\|_{L^\infty} \lesssim \lambda^{-|\alpha|}, \forall \alpha \in \mathbb{N}^n$. We also want

$$\sum_m \chi_{Q^m}(x) = 1 \quad \forall x \in \mathbb{R}^n.$$

We say that $\tilde{\chi}_{Q^m}$ is a generalized characteristic function of Q^m if $\tilde{\chi}_{Q^m}$ is essentially supported in Q^m and $\|\frac{\partial^\alpha \tilde{\chi}_{Q^m}}{\partial x^\alpha}\|_{L^\infty} \lesssim \lambda^{-|\alpha|}, \forall \alpha \in \mathbb{N}^n$. The simplest examples of generalized characteristic functions of Q^m are $\lambda^{|\alpha|} \frac{\partial^\alpha \chi_{Q^m}}{\partial x^\alpha}$ for any $\alpha \in \mathbb{N}^n$. A key property of generalized functions is that

$$(29) \quad |\tilde{\chi}_{Q^m}(x)| \lesssim \sum_{e \in I} \chi_{Q^{m+e}}(x)$$

where $I = \{v \in \mathbb{Z}^n : |v_i| \leq 1 \forall i = 1, \dots, n\}$.

In a similar way we define the system $(\chi_{R^p})_{p \in \mathbb{Z}^n}$; there is only one difference: $\|\frac{\partial^\alpha \chi_{R^p}}{\partial \xi^\alpha}\|_{L^\infty} \lesssim \lambda^{|\alpha|}, \forall \alpha \in \mathbb{N}^n$. Next we define a generalized characteristic function of R^p . This time the simplest examples of generalized characteristic functions of R^p are $\lambda^{-|\alpha|} \frac{\partial^\alpha \chi_{R^p}}{\partial \xi^\alpha}$ for any $\alpha \in \mathbb{N}^n$. These generalized characteristic functions enjoy a similar property to (29).

It is important to make the following convention. While the systems $(\chi_{Q^m})_{m \in \mathbb{Z}^n}$ and $(\chi_{R^p})_{p \in \mathbb{Z}^n}$ are fixed, the generalized characteristic functions can be arbitrary.

Proposition 6. *We have the following estimates:*

$$(30) \quad \|\tilde{\chi}_{Q^m}(x) \tilde{\chi}_{R^p}(D) \tilde{\chi}_{Q^{m'}}(x) f\|_{L^2} \lesssim C_N \langle m - m' \rangle^{-N} \sum_{|\alpha|+|\beta|=N} \|\tilde{\chi}_{R^p}^\alpha(D) \tilde{\chi}_{Q^{m'}}^\beta(x) f\|_{L^2},$$

$$(31) \quad \sum_p \|\tilde{\chi}_{Q^m}(x) \tilde{\chi}_{R^p}(D) f\|_{L^2}^2 \lesssim C_N \sum_{m'} \langle m - m' \rangle^{-N} \|\chi_{Q^{m'}}(x) f\|_{L^2}^2,$$

$$(32) \quad \|\tilde{\chi}_{R^p}(D) \tilde{\chi}_{Q^m}(x) \tilde{\chi}_{R^{p'}}(D) \chi_{Q^{m'}}(x) f\|_{L^2} \lesssim C_N \langle m - m' \rangle^{-N} \langle p - p' \rangle^{-N} \sum_{|\alpha|+|\beta| \leq 2N} \|\tilde{\chi}_{R^p}^\alpha(D) \tilde{\chi}_{Q^{m'}}^\beta(x) f\|_{L^2}$$

where $\tilde{\chi}_{R^p}^\alpha$ is a generalized characteristic function of R^p which depends on $\tilde{\chi}_{R^p}$ and α and $\tilde{\chi}_{Q^{m'}}^\beta$ is a generalized characteristic function of $Q^{m'}$ which depends on $\tilde{\chi}_{Q^{m'}}$ and β .

³There are various ways to define the concept of essentially supported. In this paper we make the convention that a function is essentially supported in a rectangle R if it is supported in the double of R .

Remark. The exact expression of $\tilde{\chi}_{R^p}^\alpha$ or $\tilde{\chi}_{Q^{m'}}^\beta$ is not important since these terms will be dealt with via the estimate (29).

Proof. For (30) we need to estimate $\|\lambda^{-|\gamma|}(x - \lambda m')^\gamma \tilde{\chi}_{R^p}(D) \tilde{\chi}_{Q^{m'}}(x) f\|_{L^2}$. We start with the commutator identity:

$$\begin{aligned} & \lambda^{-|\gamma|}(x - \lambda m')^\gamma \tilde{\chi}_{R^p}(D) \tilde{\chi}_{Q^{m'}}(x) f \\ &= \sum_{\alpha+\beta=\gamma} \lambda^{-|\alpha|} \frac{\partial^\alpha \tilde{\chi}_{R^p}}{\partial \xi^\alpha} \lambda^{-|\beta|} (x - \lambda m')^\beta \tilde{\chi}_{Q^{m'}}(x) f. \end{aligned}$$

Then we notice that $\tilde{\chi}_{Q^{m'}}^\beta = \lambda^{-|\beta|} (x - \lambda m')^\beta \tilde{\chi}_{Q^{m'}}$ is a generalized characteristic function of $Q^{m'}$ and $\tilde{\chi}_{R^p}^\alpha = \lambda^{-|\alpha|} \frac{\partial^\alpha \tilde{\chi}_{R^p}}{\partial \xi^\alpha}$ is a generalized characteristic function of R^p . We denote by γ^i the vector in $\mathbb{N}^n$ whose i -th component is 1 and the rest are 0. For $\langle m - m' \rangle \geq 2$ we have:

$$\begin{aligned} & \|\langle m - m' \rangle^N \tilde{\chi}_{Q^m}(x) \tilde{\chi}_{R^p}(D) \tilde{\chi}_{Q^{m'}}(x) f\|_{L^2} \\ & \lesssim \sum_i \|\langle m_i - m'_i \rangle^N \tilde{\chi}_{Q^m}(x) \tilde{\chi}_{R^p}(D) \tilde{\chi}_{Q^{m'}}(x) f\|_{L^2} \\ & \lesssim \sum_i \|\lambda^{-N} (x - \lambda m')^{N\gamma^i} \tilde{\chi}_{R^p}(D) \tilde{\chi}_{Q^{m'}}(x) f\|_{L^2} \\ & \lesssim \sum_i \sum_{\alpha+\beta=N} \|\tilde{\chi}_{R^p}^\alpha(D) \tilde{\chi}_{Q^{m'}}^\beta(x) f\|_{L^2}. \end{aligned}$$

In order to prove (31) proceed as follows:

$$\begin{aligned} \sum_p \|\tilde{\chi}_{Q^m}(x) \tilde{\chi}_{R^p}(D) f\|_{L^2}^2 & \lesssim \sum_p \left(\sum_{m'} \|\tilde{\chi}_{Q^m}(x) \tilde{\chi}_{R^p}(D) \tilde{\chi}_{Q^{m'}}(x) f\|_{L^2} \right)^2 \\ & \lesssim C_N^2 \sum_p \left(\sum_{m'} \langle m - m' \rangle^{-N} \sum_{|\alpha|+|\beta|=N} \|\tilde{\chi}_{R^p}^\alpha(D) \tilde{\chi}_{Q^{m'}}^\beta(x) f\|_{L^2} \right)^2 \\ & \lesssim C_N \sum_{m'} \langle m - m' \rangle^{-N} \sum_p \sum_{|\alpha|+|\beta|=N} \|\tilde{\chi}_{R^p}^\alpha(D) \tilde{\chi}_{Q^{m'}}^\beta(x) f\|_{L^2}^2 \\ & \lesssim C_N \sum_{m'} \langle m - m' \rangle^{-N} \|\chi_{Q^{m'}}(x) f\|_{L^2}^2. \end{aligned}$$

In the last estimate we used twice (once in frequency and once in space) the property (29).

The proof of (32) is a direct consequence of (30) and of its analogue:

$$\|\tilde{\chi}_{R^p}(x) \tilde{\chi}_{Q^m}(D) \tilde{\chi}_{R^{p'}}(x) h\|_{L^2} \lesssim C_N \langle p - p' \rangle^{-N} \sum_{|\alpha|+|\beta|=N} \|\tilde{\chi}_{Q^m}^\alpha(D) \tilde{\chi}_{R^{p'}}^\beta(x) h\|_{L^2}.$$

The proof of this estimate is similar to the one we provided for (30). If we take in this estimate $h = \tilde{\chi}_{Q^{m'}} f$, we obtain

$$\begin{aligned} & \|\tilde{\chi}_{R^p}(x) \tilde{\chi}_{Q^m}(D) \tilde{\chi}_{R^{p'}}(x) \tilde{\chi}_{Q^{m'}} f\|_{L^2} \\ & \lesssim C_N \langle p - p' \rangle^{-N} \sum_{|\alpha|+|\beta|=N} \|\tilde{\chi}_{Q^m}^\alpha(D) \tilde{\chi}_{R^{p'}}^\beta(x) \tilde{\chi}_{Q^{m'}} f\|_{L^2}. \end{aligned}$$

Then we apply (30) for each of the terms $\|\tilde{\chi}_{Q^m}^\alpha(D)\tilde{\chi}_{R^{p'}}^\beta(x)\tilde{\chi}_{Q^{m'}}f\|_{L^2}$ and conclude with the claim in (32). $\square$

3.5. Conservation of decay in bilinear estimates. This section would be extremely long and tedious if we were to carry out all the computations. This is why we will just indicate the main ideas. In principle, things should be simple. In the bilinear estimates we used the $\mathcal{D}$ property only on the low frequency; hence the result should inherit the $\mathcal{D}$ property from the high frequency, which looks reasonable if the interaction is localized at the high frequency too. In the case of high-high to low frequency, there is enough room to transform the $\mathcal{D}$ structure at high frequency into one at low frequency.

The section is dedicated to proving the following result:

Proposition 7. *a) If $i \leq j$, we have the following estimates:*

$$(33) \quad \|B(u, v)\|_{\mathcal{D}_j X_{j,d_3}^{0,-\frac{1}{2}}} \lesssim 2^{2i} \|u\|_{\mathcal{D}_i X_{i,d_1}^{0,\frac{1}{2}}} \|v\|_{\mathcal{D}_j X_{j,d_2}^{0,\frac{1}{2}}}.$$

The above estimates hold true if $B(u, v)$ is replaced by $B(\bar{u}, v)$ or $B(u, \bar{v})$.

b) If $|i - j| \leq 1$, we have the following estimates:

$$(34) \quad \|B(u, v)\|_{\mathcal{D}_k X_{k,d_3}^{0,-\frac{1}{2}}} \lesssim 2^{\frac{k+3j}{2}} \|u\|_{\mathcal{D}_i X_{i,d_1}^{0,\frac{1}{2}}} \|v\|_{\mathcal{D}_j X_{j,d_2}^{0,\frac{1}{2}}}.$$

The above estimates hold true if $B(u, v)$ is replaced by $B(\bar{u}, v)$ or $B(u, \bar{v})$.

To simplify the exposition, we choose the work with $B(u, v) = \nabla u \cdot \nabla v$ throughout the proof. This does not restrict in any way the generality of the argument.

Proof of Theorem 7. a) We estimated $\|B(u, v)\|_{X_{j,d_3}^{0,\frac{1}{2}}}$ for $u \in \mathcal{D}_i X_i^{0,\frac{1}{2},1}$ and $v \in X_{j,d_2}^{0,\frac{1}{2}}$; see (27). Given now the fact that $v \in \mathcal{D}_j X_{j,d_2}^{0,\frac{1}{2}}$, we want to estimate $\|B(u, v)\|_{\mathcal{D}_j X_{j,d_3}^{0,\frac{1}{2}}}$.

One has to start with an estimate for $\chi_{Q_j^m} B(u, v)_{j,d_3}$ and try to commute $\chi_{Q_j^m}$ all the way next to v . This will be done in two steps: first commute $\chi_{Q_j^m}$ with the localization φ_{j,d_3} and second with the ∇ . On the physical side, we deal with the system Q_j^m , while on the frequency side we deal with A_{j,d_3} , which has a τ component too and has sizes greater than the dual ones, namely 2^{-j} , in the ξ directions.

In the same spirit with (30) we can prove

$$(35) \quad \|\chi_{Q_j^m} h_{j,d_3}\|_{L^2} \lesssim \sum_{m'} C_N \langle m - m' \rangle^{-N} \sum_{|\alpha|+|\beta|} \|\tilde{\varphi}_{j,d_3}^\alpha(D) \tilde{\chi}_{Q_j^{m'}}^\beta h\|_{L^2}.$$

Here $\tilde{\varphi}_{j,d_3}^\alpha$ are generalized characteristic functions of the set A_{j,d_3} in the following sense: $\tilde{\varphi}_{j,d_3}^\alpha$ is supported in A_{j,d_3} and $\|\tilde{\varphi}_{j,d_3}^\alpha\|_{L^\infty} \leq C_\alpha$.

Matters are reduced to dealing with $\tilde{\chi}_{Q_j^m} B(u, v)$ for an arbitrary generalized characteristic function of $\tilde{\chi}_{Q_j^m}$. An exact calculus gives us

$$\tilde{\chi}_{Q_j^m} \nabla v = \nabla(\tilde{\chi}_{Q_j^m} v) - \nabla(\tilde{\chi}_{Q_j^m}) v.$$

We observe that it is enough to deal with the term $\nabla u \nabla(\tilde{\chi}_{Q_j^m} v)$. If we succeed in obtaining the right estimates and then are able to sum them with respect to m (over the above mentioned domain), then we will definitely be able to treat the

term $\nabla u \cdot \nabla(\tilde{\chi}_{Q_j^m})v$ for the following reasons: there is no ∇ on v and in addition $\|\nabla \tilde{\chi}_{Q_j^m}\|_{L^\infty} \leq 2^{-j}$, so we are better off with a factor of 2^{-2j} .

The main problem we encounter in dealing with $B(u, \tilde{\chi}_{Q_j^m}v)$ is that $\tilde{\chi}_{Q_j^m}$ is not localized anymore in A_{j,d_2} as v is, which means we cannot apply directly the bilinear estimates derived before. On the other hand, $\tilde{\chi}_{Q_j^m}v$ is highly localized in A_{j,d_2} in the following sense:

$$(36) \quad \|\tilde{\chi}_{Q_j^m}v\|_{L_{k,d}^2} \lesssim C_P 2^{-|k-j|P} \max\left(\frac{d_2}{d}, \frac{d}{d_2}\right)^{-P} \sum_{m'} \langle m - m' \rangle^{-P} \|\chi_{Q_j^{m'}}v\|_{L^2}.$$

We go ahead with the rest of the argument and leave the proof of this estimate for the end of the section. If we take $P \geq 3$, use the fact that $\tilde{\varphi}_{j,d_3}^\alpha$ is supported in A_{j,d_3} and $\|\tilde{\varphi}_{j,d_3}^\alpha\|_{L^\infty} \leq C_\alpha$ and use the bilinear estimates (27) we can obtain

$$\begin{aligned} (2^j d_3)^{\frac{1}{2}} \|\tilde{\varphi}_{j,d_3}^\alpha(D)B(u, \tilde{\chi}_{Q_j^m}v)\|_{L^2} &\lesssim \sum_{k,d} \|B(u, \tilde{\chi}_{Q_j^m}v)_{k,d}\|_{X_{j,d_3}^{0,\frac{1}{2}}} \\ &\lesssim C 2^{2i} \|u\|_{\mathcal{D}_i X_{i,d_1}^{0,\frac{1}{2}}} (2^j d_2)^{\frac{1}{2}} \sum_{m'} \langle m - m' \rangle^{-P} \|\chi_{Q_j^{m'}}v\|_{L^2}. \end{aligned}$$

Now we can also use the estimate in (35) and, if $N, P \geq 3$, we obtain

$$\begin{aligned} &\sum_{m: Q_j^m \cap L_j^k \neq \emptyset} (2^j d_3)^{\frac{1}{2}} \|\chi_{Q_j^m} B(u, v)_{j,d_3}\|_{L^2} \\ &\lesssim C 2^{2i} \|u\|_{\mathcal{D}_i X_{i,d_1}^{0,\frac{1}{2}}} \sum_{m: Q_j^m \cap L_j^k \neq \emptyset} \sum_{m'} \sum_{m''} (2^j d_2)^{\frac{1}{2}} \langle m - m' \rangle^{-N} \\ &\quad \cdot \langle m' - m'' \rangle^{-P} \|\chi_{Q_j^{m''}}v\|_{L^2} \\ &\lesssim C 2^{2i} \|u\|_{\mathcal{D}_i X_{i,d_1}^{0,\frac{1}{2}}} (2^j d_2)^{\frac{1}{2}} \sup_{k'} \sum_{m'': Q_j^{m''} \cap L_j^{k'} \neq \emptyset} \|\chi_{Q_j^{m''}}v\|_{L^2} \\ &\lesssim C 2^{2i} \|u\|_{\mathcal{D}_i X_{i,d_1}^{0,\frac{1}{2}}} \|v\|_{\mathcal{D}_j X_{j,d_2}^{0,\frac{1}{2}}}. \end{aligned}$$

Taking a supremum with respect to all L_j^k in the above inequality gives us the claim in (33).

We owe the reader the proof of (36). For simplicity let us assume that $k = j$ and that $d, d_2 \leq 2^{j-2}$. One can easily reproduce the argument we provide below for the general case. We have:

$$\begin{aligned} \|\tilde{\chi}_{Q_j^m}v\|_{L_{k,d}^2}^2 &\approx \sum_{\xi} \sum_k \|\varphi_{\xi, \xi^2+k}(D) \tilde{\chi}_{Q_j^m}v\|_{L^2}^2 \\ &\lesssim \sum_{\xi} \sum_k \left(\sum_{\bar{\xi}} \sum_{\bar{k}} \sum_{m'} \|\varphi_{\xi, \xi^2+k}(D) \tilde{\chi}_{Q_j^m} \varphi_{\bar{\xi}, \bar{\xi}^2+\bar{k}}(D) (\chi_{Q_j^{m'}}v)\|_{L^2} \right)^2. \end{aligned}$$

At this time we can invoke the result in (32) in the following context: Q_j^m is the system of cubes in the physical space of size 2^i , and $A_{\xi}^k = \{(\eta, \tau) : |\eta - \xi| \leq 2^{-i+1}, |\tau - \xi^2 - k| \leq \frac{1}{2}\}$ is the system of rectangles in the frequency space of size $2^{-i} \times 2^{-i} \times 1$. Since $\tilde{\chi}_{Q_j^m}$ is independent of t , we can ignore the τ component and

then we are in the setup of the result in (32); therefore,

$$\begin{aligned} & \|\varphi_{\xi, \xi^2+k}(D)\tilde{\chi}_{Q_j^m}\varphi_{\bar{\xi}, \bar{\xi}^2+\bar{k}}(D)\chi_{Q_j^{m'}}v\|_{L^2} \\ & \lesssim \langle 2^j(\xi - \bar{\xi}) \rangle^{-P} \langle \xi^2 + k - \bar{\xi}^2 - \bar{k} \rangle^{-P} \langle m - m' \rangle^{-P} \sum_{|\alpha|+|\beta|\leq 2P} \|\tilde{\varphi}_{\bar{\xi}, \bar{\xi}^2+\bar{k}}^\alpha(D)\tilde{\chi}_{Q_j^{m'}}^\beta v\|_{L^2}. \end{aligned}$$

The term $\langle \xi^2 + k - \bar{\xi}^2 - \bar{k} \rangle^{-P}$ cannot be justified via (32); instead we make a simple remark: if $\langle \xi^2 + k - \bar{\xi}^2 - \bar{k} \rangle \geq 2$, then the actual term $\varphi_{\xi, \xi^2+k}(D)\tilde{\chi}_{Q_j^m}\varphi_{\bar{\xi}, \bar{\xi}^2+\bar{k}}(D)\chi_{Q_j^{m'}}v$ equals 0 since the multiplication with $\chi_{\tilde{Q}_j^n}$ does not change the τ component of the support on the Fourier side. Then we can continue with

$$\begin{aligned} \|\tilde{\chi}_{Q_j^m}v\|_{L_{j,d}^2}^2 & \lesssim \sum_{\xi} \sum_k \sum_{\bar{\xi}} \sum_{\bar{k}} \sum_{m'} \langle 2^j(\xi - \bar{\xi}) \rangle^{-P} \langle \xi^2 + k - \bar{\xi}^2 - \bar{k} \rangle^{-P} \langle m - m' \rangle^{-P} \\ & \quad \cdot \sum_{|\alpha|+|\beta|\leq 2P} \|\tilde{\varphi}_{\bar{\xi}, \bar{\xi}^2+\bar{k}}^\alpha(D)\tilde{\chi}_{Q_j^{m'}}^\beta v\|_{L^2}^2 \\ & \lesssim \max\left(\frac{d_2}{d}, \frac{d}{d_2}\right)^{-P} \sum_{|\alpha|+|\beta|\leq 2P} \sum_{\bar{\xi}} \sum_{\bar{k}} \sum_{m'} \langle m - m' \rangle^{-P} \|\tilde{\varphi}_{\bar{\xi}, \bar{\xi}^2+\bar{k}}^\alpha(D)\tilde{\chi}_{Q_j^{m'}}^\beta v\|_{L^2}^2 \\ & \lesssim \max\left(\frac{d_2}{d}, \frac{d}{d_2}\right)^{-P} \sum_{m'} \langle m - m' \rangle^{-P} \|\tilde{\chi}_{Q_j^{m'}}v\|_{L^2}^2. \end{aligned}$$

b) We estimated $\|B(u, v)\|_{X_{k,d_3}^{0, \frac{1}{2}}}$ for $u \in X_i^{0, \frac{1}{2}, 1}$ and $v \in X_{j,d_2}^{0, \frac{1}{2}}$, where $|i - j| \leq 1$; see (28). Given now the fact that $v \in \mathcal{D}_j X_{j,d_2}^{0, \frac{1}{2}}$ we want to estimate $\|B(u, v)\|_{\mathcal{D}_k X_{k,d_3}^{0, \frac{1}{2}}}$. A straightforward computation gives us that

$$\|h\|_{\mathcal{D}_k L^2} \lesssim 2^{\frac{j-k}{2}} \|h\|_{\mathcal{D}_j L^2}.$$

This has to do with the fact that in a cube Q_j^m we fit $\approx 2^{j-k}$ cubes $Q_k^{m'}$ on a straight line. Hence we can go ahead and estimate $\|B(u, v)\|_{\mathcal{D}_j X_{k,d_3}^{0, \frac{1}{2}}}$ and bring the correction of $2^{\frac{j-k}{2}}$ at the end. Once we are in this setup we can reproduce the same argument as in part a), since v comes with a $\mathcal{D}_j$ structure. $\square$

3.6. Bilinear estimates on frequency dyadic regions. In the end we want to obtain bilinear estimates on dyadic regions with respect to the frequency only.

Proof of Theorem 6. a) We deal first with the case when the outcome is localized at high frequency. Fixing d_3 and making use of (33) we estimate

$$\begin{aligned} \|B(u, v)\|_{\mathcal{D}_j X_{j,d_3}^{0, -\frac{1}{2}}} & \lesssim \sum_{d_1, d_2} \|B(u_{\cdot, d_1}, v_{\cdot, d_2})\|_{\mathcal{D}_j X_{j,d_3}^{0, -\frac{1}{2}}} \\ & \lesssim 2^{2i} \sum_{d_1, d_2} \|u_{\cdot, d_1}\|_{\mathcal{D}_i X_{i,d_1}^{0, \frac{1}{2}}} \|v_{\cdot, d_2}\|_{\mathcal{D}_j X_{j,d_2}^{0, \frac{1}{2}}} \lesssim 2^{2i} \|u\|_{\mathcal{D}_i X_i^{0, \frac{1}{2}, 1}} \|v\|_{\mathcal{D}_j X_j^{0, \frac{1}{2}, 1}}. \end{aligned}$$

Summing up with d_3 and passing to general s gives us (16).

In the case when $|i - j| \leq 1$ and $k \leq j - 5$, we estimate in the same way, this time making use of (34), to obtain (17).

b) We decompose

$$v_{\cdot, \geq 2^{-i}} = \sum_{2^{-i} \leq d' \leq 2^i} v_{\cdot, d'} + \sum_{d_2 \geq 2^{i+1}} v_{\cdot, d_2}$$

and notice that $\hat{u} * \sum_{2^{-i} \leq d' \leq 2^i} \hat{v}_{\cdot, d'}$ is essentially localized at a distance less than 2^i from P while $\hat{u} * \hat{v}_{\cdot, d_2}$ is localized essentially at a distance d_2 from P for any $d_2 \geq 2^{i+1}$. This happens because $\hat{u}$ is localized at frequency 2^i .

We fix $d_3 \geq 2^{-i}$ and, as in part a), we estimate

$$\|B(u, \sum_{2^{-i} \leq d' \leq 2^i} v_{\cdot, d'})\|_{X_{j, d_3}^{0, -\frac{1}{2}}} \lesssim 2^{2i} \|u\|_{X_i^{0, \frac{1}{2}, 1}} \left\| \sum_{2^{-i} \leq d' \leq 2^i} v_{\cdot, d'} \right\|_{X_j^{0, \frac{1}{2}, 1}}.$$

In a similar manner we can conclude that for any $d_2 \geq 2^i$ we obtain

$$\|B(u, v_{\cdot, d_2})\|_{X_{j, d_2}^{0, -\frac{1}{2}}} \lesssim 2^{2i} \|u\|_{X_i^{0, \frac{1}{2}, 1}} \|v_{\cdot, d_2}\|_{X_j^{0, \frac{1}{2}, 1}}.$$

Taking into account the observation above the localization of the interactions, we sum with respect to d_3 , for $2^{-i} \leq d_3 \leq 2^i$ and then with respect to $d_2 \geq 2^i$, to obtain

$$\|B(u, v)\|_{X_{j, \leq 2^{-i}}^{0, -\frac{1}{2}, 1}} \lesssim 2^{2i} \|u\|_{X_i^{0, \frac{1}{2}, 1}} \|v\|_{X_{j, \leq 2^{-i}}^{0, \frac{1}{2}, 1}}.$$

Passing to general s gives us (17). $\square$

4. BILINEAR ESTIMATES INVOLVING THE $\mathbf{Y}$ SPACES

In the previous section we have just seen that the theory of bilinear estimates cannot be completely closed in the $X^{s, \frac{1}{2}, 1}$ spaces. This is the reason for introducing a more refined structure to measure our solutions, namely the wave-packet one. We concluded that the interactions causing problems in the $X^{s, \frac{1}{2}, 1}$ theory are the low-high ones. This is why we need to complete Theorem 6 with a result for this particular case.

Theorem 7. *Assume we have $5i \leq j$. We have the bilinear estimates*

$$(37) \quad \|B(u, v)\|_{\mathcal{DW}_j^s} \lesssim i 2^{(2-s)i} \|u\|_{\mathcal{DZ}_i^s} \|v\|_{\mathcal{DZ}_j^s}.$$

The estimate remains valid if $B(u, v)$ is replaced by $B(\bar{u}, v)$ or $B(u, \bar{v})$.

In what follows we make a few important remarks for the rest of this section.

The result in (17) shows that it is fine to use the $X^{s, \frac{1}{2}, 1}$ structure to measure the low frequency and part of the high frequency (both input and output) at a distance greater than 2^{-i} from P . Thus we shall obtain estimates for

$$(38) \quad X_i^{0, \frac{1}{2}, 1} \cdot Y_{j, \leq 2^{-i}} \rightarrow \mathcal{Y}_{j, \leq 2^{-i}} + X_{j, \geq 2^{-i}}^{0, -\frac{1}{2}, 1}; \quad X_i^{0, \frac{1}{2}, 1} \cdot X_{j, \geq 2^{-i}}^{0, \frac{1}{2}, 1} \rightarrow \mathcal{Y}_{j, \leq 2^{-i}}.$$

We also need the corresponding estimates when we involve conjugates of these spaces. The condition $5i \leq j$ implies that the low frequency does not see the curvature of the parabola at the high frequency; in other words, the parabola at high frequency is flat in these interactions. This is why the estimates for $B(\bar{u}_i, v_j)$ are similar to the ones for $B(u_i, v_j)$.

If we have to deal with $B(u_i, \bar{v}_j)$, a simple geometric argument shows that the interaction is localized at high frequency and in a region with $\tau \leq 0$. This makes these estimates weaker than the ones in (38).

Remark 1. Once we get one of the estimates in (38), we trivially get the corresponding ones with conjugate spaces.

We have to conserve decay in these estimates. First we prove

$$\|B(u, v)\|_{\mathcal{RW}_j^s} \lesssim i^{\frac{3}{2}} 2^{(1-s)i} \|u\|_{\mathcal{RDZ}_i^s} \|v\|_{\mathcal{RZ}_j^s}$$

and the similar ones. Then we obtain the estimates with decay on all terms by a similar argument as in section 3.5.

Remark 2. We first prove the estimates without involving decay on the bilinear term and on the high frequency. But we do involve decay on the low frequency.

This being said, we can start the preparations for this section.

4.1. Basic estimates. This section is concerned with providing results of type $Y \cdot \mathcal{DL}^2 \rightarrow \mathcal{Y}$, $Y \cdot \mathcal{DL}^2 \rightarrow L^2$ and $L^2 \cdot \mathcal{DL}^2 \rightarrow \mathcal{Y}$.

Lemma 1. *Let $g \in L^2$ such that $\hat{g}$ is supported in a tube of size $2^{-i} \times 2^{-i} \times 1$. We have the estimate*

$$(39) \quad \sum_{m,l} \|g\|_{L^\infty(Q_i^{m,l})}^2 \lesssim 2^{-2i} \|g\|_{L^2}^2.$$

Proof. The support of $\hat{g}$ is a tube with volume 2^{-2i} ; therefore we have

$$\|g\|_{L^\infty(Q_i^{m,l})} \lesssim 2^{-i} \sum_{(m',l') \in \mathbb{Z}^3} C_N \langle (m,l) - (m',l') \rangle^{-N} \|g\|_{L^2(Q_i^{m',l'})}.$$

If we choose $N \geq 4$, then we use Cauchy-Schwarz and get

$$\|g\|_{L^\infty(Q_i^{m,l})}^2 \lesssim 2^{-2i} \sum_{(m',l') \in \mathbb{Z}^3} C_N^2 \langle (m,l) - (m',l') \rangle^{-N} \|g\|_{L^2(Q_i^{m',l'})}^2.$$

We can perform the summation with respect to (m,l) :

$$\sum_{(m,l)} \|g\|_{L^\infty(Q_i^{m,l})}^2 \lesssim 2^{-2i} \sum_{m,l} \sum_{m',l'} \langle (m,l) - (m',l') \rangle^{-N} \|g\|_{L^2(Q_i^{m',l'})}^2 \lesssim 2^{-2i} \|g\|_{L^2}^2.$$

In the last line we use again the fact that if $N \geq 4$, then we have

$$\sum_{m,l} \langle (m,l) - (m',l') \rangle^{-N} \lesssim 1.$$

This is enough to justify the claim. $\square$

Lemma 2. *Let $\hat{g}$ be supported in $A_{i,d}$ where $d \leq 2^{i-2}$. For any $p \in \mathbb{Z}$ we have*

$$(40) \quad \sum_{\xi} \sum_{k=2^{i-1}d}^{2^{i+1}d} \|\chi_{Q_i^m} g_{\xi, \xi^2+k}\|_{L^\infty}^2 \lesssim C_N 2^{-i} \sum_{m'} \langle m - m' \rangle^{-N} \|\chi_{Q_i^{m'}} g\|_{L^2}^2,$$

$$(41) \quad \sum_{m: Q_i^m \cap L_i^p \neq \emptyset} \left(\sum_{\xi} \sum_{k=2^{i-1}d}^{2^{i+1}d} \|\chi_{Q_i^m} g_{\xi, \xi^2+k}\|_{L^\infty}^2 \right)^{\frac{1}{2}} \lesssim 2^{-i} \|g\|_{\mathcal{D}_i L^2}.$$

Proof. The support of $\hat{g}_{\xi, \xi^2+k}$ is $2^{-i} \times 2^{-i} \times 1$; hence

$$\|\chi_{Q_i^m} g_{\xi, \xi^2+k}\|_{L^\infty} \lesssim C_N 2^{-i} \sum_{m'} \langle m - m' \rangle^{-N} \|\chi_{Q_i^{m'}} g_{\xi, \xi^2+k}\|_{L^2}.$$

Then, (40) amounts to proving

$$(42) \quad \sum_{\xi} \sum_k \|\chi_{Q_i^m} g_{\xi, \xi^2+k}\|_{L^2}^2 \lesssim C_N \sum_{m'} \langle m - m' \rangle^{-N} \|\chi_{Q_i^{m'}} g\|_{L^2}.$$

For fixed ξ we have the obvious bound:

$$\sum_k \|\chi_{Q_i^m} g_{\xi, \xi^2+k}\|_{L^2}^2 \approx \|\chi_{Q_i^m} \sum_k g_{\xi, \xi^2+k}\|_{L^2}^2$$

since $\chi_{Q_i^m}$ is a cut in the x space while $(\varphi_{\xi, \xi^2+k})_k$ is a cut in the τ direction. Hence it is enough to prove (42) in the particular case $d = 2^{-i}$ (i.e. $k = 0$):

$$\sum_{\xi} \|\chi_{Q_i^m} g_{\xi, \xi^2}\|_{L^2}^2 \lesssim C_N \sum_{m'} \langle m - m' \rangle^{-N} \|\chi_{Q_i^{m'}} g\|_{L^2}.$$

We can write

$$\chi_{Q_i^m} g_{\xi, \xi^2} = \chi_{Q_i^m} \left(\sum_{m'} \chi_{Q_i^{m'}} g \right)_{\xi, \xi^2}.$$

Invoking the results (and notation) from section 3.4, see also the adjustments in section 3.5, we claim

$$(43) \quad \|\chi_{Q_i^m} (\chi_{Q_i^{m'}} g)_{\xi, \xi^2}\|_{L^2} \lesssim C_N \langle m - m' \rangle^{-N} \sum_{|\alpha|+|\beta| \leq N} \|\tilde{\varphi}_{\xi, \xi^2}^\alpha(D) \tilde{\chi}_{Q_i^{m'}}^\beta(g)\|_{L^2}.$$

Then we sum with respect to m' and use Cauchy-Schwarz to obtain

$$\|\chi_{Q_i^m} g_{\xi, \xi^2}\|_{L^2}^2 \lesssim C_N \langle m - m' \rangle^{-N} \sum_{|\alpha|+|\beta| \leq N} \|\tilde{\varphi}_{\xi, \xi^2}^\alpha(D) \tilde{\chi}_{Q_i^{m'}}^\beta(g)\|_{L^2}^2.$$

Recalling (29), both in space and frequency, we continue with

$$\begin{aligned} \sum_{\xi} \|\chi_{Q_i^m} g_{\xi, \xi^2}\|_{L^2}^2 &\lesssim C_N \langle m - m' \rangle^{-N} \sum_{|\alpha|+|\beta| \leq N} \sum_{\xi} \|\tilde{\varphi}_{\xi, \xi^2}^\alpha(D) \tilde{\chi}_{Q_i^{m'}}^\beta(g)\|_{L^2}^2 \\ &\lesssim C_N \langle m - m' \rangle^{-N} \sum_{\alpha} \|\chi_{Q_i^{m'}} g\|_{L^2}^2. \end{aligned}$$

(41) is a consequence of (40). $\square$

In a similar way we can prove the following result:

Lemma 3. *If $\hat{g}$ is supported in $A_{i,d}$ for $d \geq 2^{i-2}$, then for any $p \in \mathbb{Z}$,*

$$(44) \quad \sum_{m: Q_i^m \cap L_i^p \neq \emptyset} \left(\sum_{\xi} \sum_{l=2^{2i-2}d}^{2^{2i+2}} \|\chi_{Q_i^m} g_{\xi, l}\|_{L^\infty}^2 \right)^{\frac{1}{2}} \lesssim 2^{-i} \|g\|_{\mathcal{D}_i L^2}.$$

For each $\alpha \in Z^2$ we define $A_\alpha = \{m \in Z^2 : m_1 \in [2^j(2\alpha - 1), 2^j(2\alpha + 1)], m_2 \in [2^i(2\alpha - 1), 2^i(2\alpha + 1)]\}$. We have the following result:

Lemma 4. *The families $(T_\eta^{m,l})_{m \in A_\alpha}$ and $(T_{\eta+\xi}^{m,l})_{m \in A_\beta}$ contain disjoint tubes unless $|\alpha - \beta| = \max(|\alpha_1 - \beta_1|, |\alpha_2 - \beta_2|) \leq 2$; in other words, if $T_{\eta+\xi}^{m,l} \cap T_{\eta+\xi}^{m',l} \neq \emptyset$, where $m \in A_\alpha$ and $m' \in A_\beta$, then $|\alpha - \beta| \leq 2$.*

Proof. It is enough to prove the result in the case $l = 0$. Let us assume that there is $(x, t) \in T_{\eta+\xi}^{m,0} \cap T_{\eta+\xi}^{m',0}$, where $m \in A_\alpha$ and $m' \in A_\beta$. Then

$$\|x - m + t\eta\| \leq \sqrt{2} \quad \text{and} \quad |x - m' + t(\xi + \eta)| \leq \sqrt{2},$$

which implies $\|m - m' + t\xi\| \leq 2\sqrt{2}$. Recalling that $t \in [0, 1]$, $\|\xi\| \approx 2^i$, $i < j$ and the definition of A_α, A_β we obtain the claim. $\square$

Lemma 5. *For each m, m' there is essentially only one m'' such that $Q_i^{m,l} \cap T_{\eta}^{m',l} \cap T_{\eta+\xi}^{m'',l} \neq \emptyset$; more precisely, there are at most 5 m'' 's with this property.*

Proof. The underlying idea is that the intersection $T_{\eta}^{m',l} \cap T_{\eta+\xi}^{m'',l}$ is a subtube of size 2^{j-i} in the long direction and $2^{j-i} \geq 2^i$, the latter being the size of the cube $Q_i^{m,l}$. One can formalize an explicit proof. $\square$

For each α , we define $B_\alpha = \{m : Q_i^{m,l} \cap T_{\eta}^{m',l} \neq \emptyset \text{ for } m' \in A_\alpha\}$. Notice that the family of tubes $(T_{\eta}^{m',l})_{m' \in A_\alpha}$ fills up a parallelepiped of size $2^{j+1} \times 2^{i+1} \times 1$ (the last one in the t direction) and the longest side is in the direction of η . Hence if L is the line in $\mathbb{R}^2$ passing through the origin in the direction of η , then there is a $k \in \mathbb{Z}$ such that

$$(45) \quad B_\alpha \subset \{m : Q_i^m \cap (L_i^{k-1} \cup L_i^k \cup L_i^{k+1}) \neq \emptyset\}.$$

We conclude with the main result of this section.

Lemma 6. *We have the estimate*

$$(46) \quad \|f \cdot g\|_{L^2} \lesssim 2^{-\frac{i+j}{2}} \|f\|_{Y_j} \|g\|_{L^2}.$$

Proof. For $m' \in A_\alpha$, we have

$$\begin{aligned} \|f \cdot g\|_{L^2(T_{\eta}^{m',l})}^2 &\lesssim \sum_{m \in B_\alpha} \|f \cdot g\|_{L^2(Q_i^{m,l} \cap T_{\eta}^{m',l})}^2 \\ &\lesssim 2^{i-j} \sum_{m \in B_\alpha} \|f\|_{L_t^\infty L_x^2(T_{\eta}^{m',l})}^2 \|g\|_{L^\infty(Q_i^{m,l})}^2 \\ &\lesssim 2^{i-j} \|f\|_{L_t^\infty L_x^2(T_{\eta}^{m',l})}^2 \sum_{m \in B_\alpha} \|g\|_{L^\infty(Q_i^{m,l})}^2 \lesssim 2^{-i-j} \|f\|_{L_t^\infty L_x^2(T_{\eta}^{m',l})}^2 \|g\|_{L^2}^2. \end{aligned}$$

In the last line we have used the result in (39). We sum the above estimate with respect to (m, l) over Z^3 to obtain (46). $\square$

The next lemma is a geometrical one. We work with $f = f_{\eta, \leq 2^{-i}}$ and $g = g_{\xi^0, l}$, $\xi^0 \in \Xi^i$, $l \in \mathbb{Z}$ where $|\eta| \approx 2^j$ and $|(\xi^0, l)| \approx 2^i$.

Lemma 7. *Assume $d \geq 2^{-i}$. If $\hat{g} * \hat{f}$ is supported in a region where $|\tau - \xi^2| \leq d$, then $|\cos \alpha| \leq |\xi^0|^{-1}d$, where α is the angle between ξ^0 and η .*

Proof. $\hat{f}$ is supported in a region where $|\tau_2 - \eta^2| \leq 2^{-i}|\eta|$, while $\hat{g}$ is supported in a region where $|\xi - \xi^0| \leq 2^{-i}$ and $|\tau_1 - l| \leq \frac{1}{2}$. A generic point in the support of $\hat{f} * \hat{g}$ is of type $(\xi_1 + \xi_2, \tau_1 + \tau_2)$ where (ξ_1, τ_1) is in the support of $\hat{g}$ and (ξ_2, τ_2) is in the support of $\hat{f}$. We want this point to satisfy $|\tau_1 + \tau_2 - (\xi_1 + \xi_2)^2| \leq d$.

We have $|\tau_1 - \xi_1^2| \leq 2^{2i} \leq 2^{j-i}$, $\Delta|\xi_1| \approx 2^{-i}$, $\Delta|\eta| \approx 1$; therefore the condition is equivalent to $|2\eta \cdot \xi^0| \leq 2^j d$. This implies the conclusion of the lemma. $\square$

Lemma 8. *For fixed ξ and k , the interactions $\hat{g}_{\xi, \xi^2+k} * \hat{f}_{\eta, \leq 2^{-i}}$ have disjoint supports with respect to η ; the same is true for $\hat{g}_{\xi, l} * \hat{f}_{\eta, \leq 2^{-i}}$.*

Proof. The sizes of the support of $\hat{g}_{\xi, \xi^2+k}$ are $2^{-i} \times 2^{-i} \times 1$. The support of $\hat{v}_{\eta, \leq 2^{-i}}$ is a parallelepiped of size $2^{-i} \times 1 \times 2^j$ whose longest side is tangent to P . The key property is that we can translate the support of $\hat{u}_{\xi, \leq 2^{-i}}$ so that it is included in the support of $\hat{v}_{\eta, \leq 2^{-i}}$ (by simply translating the center of the first to the center of the second). Therefore the support of $\hat{v}_{\eta, \leq 2^{-i}} * \hat{u}_{\xi, \leq 2^{-i}}$ is a translate of the support of $\hat{v}_{\eta, \leq 2^{-i}}$ by the vector (ξ, ξ^2) . Therefore if we keep ξ and k fixed and take $\eta \neq \eta'$ both in A_ξ , then the supports of $\hat{v}_{\eta, \leq 2^{-i}} * \hat{u}_{\xi, \leq 2^{-i}}$ and $\hat{v}_{\eta', \leq 2^{-i}} * \hat{u}_{\xi, \leq 2^{-i}}$ are disjoint. $\square$

4.2. Estimates: $\mathcal{D}X_i^{0, \frac{1}{2}, 1} \cdot Y_{j, \leq 2^{-i}} \rightarrow \mathcal{Y}_{j, \leq 2^{-i}}$. The main result of this section is the following:

Proposition 8. *We have the estimate*

$$(47) \quad \|v_{j, \leq 2^{-i}} \cdot u_i\|_{\mathcal{Y}_{j, \leq 2^{-i}}} \lesssim 2^{i-j} \|v_{j, \leq 2^{-i}}\|_{Y_j} \cdot \|u_i\|_{\mathcal{D}X^{0, \frac{1}{2}, 1}}.$$

This result is a direct consequence of the following estimates:

$$(48) \quad \|f_{j, \leq 2^{-i}} \cdot g_{i, \leq 2^{i-2}}\|_{\mathcal{Y}_{j, \leq 2^{-i}}} \lesssim 2^{i-j} \|f_{j, \leq 2^{-i}}\|_{Y_j} \cdot \|g_{i, \leq 2^{i-2}}\|_{\mathcal{D}X^{0, \frac{1}{2}, 1}},$$

$$(49) \quad \|f_{j, \leq 2^{-i}} \cdot g_{i, \geq 2^{i-2}}\|_{\mathcal{Y}_{j, \leq 2^{-i}}} \lesssim 2^{i-j} \|f_{j, \leq 2^{-i}}\|_{Y_j} \cdot \|g_{i, \geq 2^{i-2}}\|_{\mathcal{D}X^{0, \frac{1}{2}, 1}},$$

$$(50) \quad \|f_{j, \leq 2^{-i}} \cdot g_i\|_{\mathcal{Y}_{j, \leq 2^{-i}}} \lesssim 2^{i-j} \|f_{j, \leq 2^{-i}}\|_{Y_j} \cdot \|g_i\|_{\mathcal{D}X^{0, \frac{1}{2}, 1}},$$

$$(51) \quad \|f_{j, \leq 2^{-i}} \cdot g_i\|_{\mathcal{D}Y_{j, \leq 2^{-i}}} \lesssim 2^{i-j} \|f_{j, \leq 2^{-i}}\|_{\mathcal{D}Y_j} \cdot \|g_i\|_{\mathcal{D}X^{0, \frac{1}{2}, 1}}.$$

Proof. Throughout this section we use the following decompositions:

$$(52) \quad f_{j, \leq 2^{-i}} = \sum_{\eta \in \Xi} f_{\eta, \leq 2^{-i}},$$

$$(53) \quad g_i = g_{i, \leq 2^{i-2}} + g_{i, \geq 2^{i-2}} = \sum_k \sum_{\xi \in \Xi^i} g_{\xi, \xi^2 \pm k} + \sum_{\xi \in \Xi^i} \sum_{l \in I_\xi} g_{\xi, l}.$$

For more details about the decomposition in (53), see (12) and (13). We do not want to bother about carrying the $\pm$ in $g_{\xi, \xi^2 \pm k}$ in all computations. We choose to work only with the g_{ξ, ξ^2+k} (k will be positive) and completing the argument for both choices of sign is a trivial matter.

We prove first (48). We make use of the decompositions in (52) and (53). We define $\Theta_p = \{\eta \in \Xi, |\eta| \approx 2^j : \arg \eta \in [(p - \frac{1}{2})2^{i-j}\frac{\pi}{2}, (p + \frac{1}{2})2^{i-j}\frac{\pi}{2}]\}$. The size of $\sum_{\eta \in \Theta_p} \hat{f}_{\eta, \leq 2^{-i}}$ in the angular direction is $\approx 2^i$; therefore the interactions $\hat{g}_i * \sum_{\eta \in \Theta_p} \hat{f}_{\eta, \leq 2^{-i}}$ have disjoint support with respect to p . As a consequence:

$$(54) \quad \|S_{j, \leq 2^{-i}}(g_i \cdot f_{j, \leq 2^{-i}})\|_{\mathcal{Y}}^2 \approx \sum_p \|S_{j, \leq 2^{-i}}(g_i \cdot \sum_{\eta \in \Theta_p} f_{\eta, \leq 2^{-i}})\|_{\mathcal{Y}}^2.$$

We decompose

$$\begin{aligned} & S_{j, \leq 2^{-i}}(g_{i, \leq 2^{i-2}} \cdot \sum_{\eta \in \Theta_p} f_{\eta, \leq 2^{-i}}) \\ &= S_{j, \leq 2^{-i}} \left(\sum_{d \leq 2^{i-2}} \sum_{\xi} \sum_{k=2^{i-1}d}^{2^{i+1}d} g_{\xi, \xi^2+k} \right) \left(\sum_{\eta \in \Theta_p} f_{\eta, \leq 2^{-i}} \right). \end{aligned}$$

From Lemma 8 we know that $g_{\xi, \xi^2+k} \cdot f_{\eta, \leq 2^{-i}}$ is supported in $A_{j, \leq 2^{-i}}$ iff $|\cos \alpha| \leq 2^{-2i}$, where α is the angle between ξ and η . The angle between any two η 's in Θ_p is at most $2^{i-j} \leq 2^{-2i}$ and the angle between any two ξ 's in Ξ^i is either at least 2^{-2i} or the same. Therefore all ξ 's involved in the above summation have the same angular localization; we just keep this in mind and not formalize it. What is important is that we sum over a set containing $\approx 2^{2i}$ ξ 's. We continue with

$$\begin{aligned} S_{j, \leq 2^{-i}} & \left(\sum_{\xi} \sum_k g_{\xi, \xi^2+k} \right) \left(\sum_{\eta \in \Theta_p} f_{\eta, \leq 2^{-i}} \right) \\ &= \sum_k \sum_{\xi} \sum_{\eta \in \Theta_p} (g_{\xi, \xi^2+k} \cdot f_{\eta, \leq 2^{-i}})_{\xi+\eta, \leq 2^{-i}} \\ &= \sum_{\alpha} \sum_l \sum_{m \in B^{\alpha}} \sum_{m' \in A^{\alpha}} \sum_k \sum_{\xi} \sum_{\eta \in \Theta_p} (\chi_{Q_i^m} g_{\xi, \xi^2+k} \cdot \chi_{T_{\eta}^{m', l}} f_{\eta, \leq 2^{-i}})_{\xi+\eta, \leq 2^{-i}}. \end{aligned}$$

For fixed α and l , $\sum_{m \in B^{\alpha}} \chi_{T_{\eta}^{m', l}} f_{\eta, \leq 2^{-i}}$ is essentially supported (in the physical space) in a parallelepiped of size $2^i \times 2^j \times 1$ which is independent of $\eta \in \Theta_p$. The position of this parallelepiped is a function of α and l . Hence we have

(55)

$$\begin{aligned} & \|S_{j, \leq 2^{-i}} \left(\sum_{\xi} \sum_k g_{\xi, \xi^2+k} \right) \left(\sum_{\eta \in \Theta_p} f_{\eta, \leq 2^{-i}} \right)\|_{\mathcal{Y}}^2 \\ & \approx \sum_{\alpha} \sum_l \left\| \sum_{m \in B^{\alpha}} \sum_{m' \in A^{\alpha}} \sum_k \sum_{\xi} \sum_{\eta \in \Theta_p} (\chi_{Q_i^m} g_{\xi, \xi^2+k} \cdot \chi_{T_{\eta}^{m', l}} f_{\eta, \leq 2^{-i}})_{\xi+\eta, \leq 2^{-i}} \right\|_{\mathcal{Y}}^2. \end{aligned}$$

We fix α , l and $m \in B_{\alpha}$. Without losing the generality of the argument, we choose $l = 0$. We fix ξ and $k \in [2^{i-1}d, 2^{i+1}d]$. We also want to drop the notation relocalization $(\cdot)_{\eta+\xi, \leq 2^{-i}}$ and we can do that by making the convention that $\chi g_{\xi, \xi^2+k} \cdot \chi f_{\eta}$ has to be measured in $\mathcal{Y}_{\eta+\xi}$.

We continue with

$$\begin{aligned} & \left\| \sum_{m' \in A^{\alpha}} \sum_{\eta} \chi_{Q_i^m} g_{\xi, \xi^2+k} \cdot \chi_{T_{\eta}^{m', 0}} f_{\eta, \leq 2^{-i}} \right\|_{\mathcal{Y}}^2 \\ & \approx \sum_{m' \in A^{\alpha}} \sum_{\eta \in \Theta_p} \sum_{m''} \left\| \chi_{Q_i^m} g_{\xi, \xi^2+k} \cdot \chi_{T_{\eta}^{m', 0}} f_{\eta, \leq 2^{-i}} \right\|_{L_t^1 L_x^2(T_{\eta+\xi}^{m'', 0})}^2. \end{aligned}$$

For fixed m' , let m'' be such that $Q_i^m \cap T_{\eta}^{m', 0} \cap T_{\eta+\xi}^{m'', 0} \neq \emptyset$. The size of this intersection in the direction of t is $\approx 2^{i-j}$; therefore we can estimate

$$\begin{aligned} & \left\| \chi_{Q_i^m} g_{\xi, \xi^2+k} \cdot \chi_{T_{\eta}^{m', 0}} f \right\|_{L_t^1 L_x^2(T_{\eta+\xi}^{m'', 0})} \lesssim 2^{i-j} \left\| \chi_{Q_i^m} g_{\xi, \xi^2+k} \cdot \chi_{T_{\eta}^{m', 0}} f_{\eta, \leq 2^{-i}} \right\|_{L_t^{\infty} L_x^2} \\ & \lesssim 2^{i-j} \left\| \chi_{Q_i^m} g_{\xi, \xi^2+k} \right\|_{L^{\infty}} \left\| \chi_{T_{\eta}^{m', 0}} f_{\eta, \leq 2^{-i}} \right\|_{L_t^{\infty} L_x^2}. \end{aligned}$$

Taking into account the result of Lemma 5 we can perform the $l_{m'}$ summation and obtain

$$\begin{aligned} & \left\| \chi_{Q_i^m} g_{\xi, \xi^2+k} \cdot \sum_{m' \in A_{\alpha}} \chi_{T_{\eta}^{m', 0}} f_{\eta, \leq 2^{-i}} \right\|_{\mathcal{Y}_{\xi+\eta}} \\ & \lesssim 2^{i-j} \left\| \chi_{Q_i^m} g_{\xi, \xi^2+k} \right\|_{L^{\infty}} \left\| \sum_{m' \in A_{\alpha}} \chi_{T_{\eta}^{m', 0}} f_{\eta, \leq 2^{-i}} \right\|_{\mathcal{Y}_{\eta}}. \end{aligned}$$

Next we can perform the l_η^2 summation to obtain

$$\begin{aligned} & \|\chi_{Q_i^m} g_{\xi, \xi^2+k} \cdot \sum_{\eta \in \Theta_p} \sum_{m' \in A_\alpha} \chi_{T_\eta^{m',0}} f_{\eta, \leq 2^{-i}}\|_{Y_{\xi+\eta}} \\ & \lesssim 2^{i-j} \|\chi_{Q_i^m} g_{\xi, \xi^2+k}\|_{L^\infty} \left(\sum_{\eta \in \Theta_p} \left\| \sum_{m' \in A_\alpha} \chi_{T_\eta^{m',0}} f_{\eta, \leq 2^{-i}} \right\|_{Y_\eta}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

We fix $d \in I_i$ and perform the summation with respect to ξ and $k \in [2^{i-1}d, 2^{i+1}d]$:

$$\begin{aligned} & \left\| \sum_{\xi} \sum_k \chi_{Q_i^m} g_{\xi, \xi^2+k} \cdot \sum_{\eta \in \Theta_p} \sum_{m' \in A_\alpha} \chi_{T_\eta^{m',0}} f_{\eta, \leq 2^{-i}} \right\|_{Y_{\xi+\eta}} \\ & \lesssim 2^{i-j} 2^i (2^i d)^{\frac{1}{2}} \left(\sum_{\xi} \sum_k \|\chi_{Q_i^m} g_{\xi, \xi^2+k}\|_{L^\infty}^2 \right)^{\frac{1}{2}} \left(\sum_{\eta \in \Theta_p} \left\| \sum_{m' \in A_\alpha} \chi_{T_\eta^{m',0}} f_{\eta, \leq 2^{-i}} \right\|_{Y_\eta}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

We sum with respect to $m \in B_\alpha$:

$$\begin{aligned} & \left\| \sum_{m \in B_\alpha} \sum_{\xi} \sum_k \chi_{Q_i^m} g_{\xi, \xi^2+k} \cdot \sum_{\eta \in \Theta_p} \sum_{m' \in A_\alpha} \chi_{T_\eta^{m',0}} f_{\eta, \leq 2^{-i}} \right\|_{Y_{\xi+\eta}} \\ & \lesssim 2^{i-j} 2^i (2^i d)^{\frac{1}{2}} \sum_{m \in B_\alpha} \left(\sum_{\xi} \sum_k \|\chi_{Q_i^m} g_{\xi, \xi^2+k}\|_{L^\infty}^2 \right)^{\frac{1}{2}} \\ & \quad \cdot \left(\sum_{\eta \in \Theta_p} \left\| \sum_{m' \in A_\alpha} \chi_{T_\eta^{m',0}} f_{\eta, \leq 2^{-i}} \right\|_{Y_\eta}^2 \right)^{\frac{1}{2}} \\ & \lesssim 2^{i-j} \|g_{i,d}\|_{\mathcal{D}X^{0, \frac{1}{2}}} \left(\sum_{\eta \in \Theta_p} \left\| \sum_{m' \in A_\alpha} \chi_{T_\eta^{m',0}} f_{\eta, \leq 2^{-i}} \right\|_{Y_\eta}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

In the last inequality we have used (41). We sum with respect to d :

$$\begin{aligned} & \left\| \sum_{m \in B_\alpha} \sum_{\xi} \sum_k \chi_{Q_i^m} g_{\xi, \xi^2+k} \cdot \sum_{\eta \in \Theta_p} \sum_{m' \in A_\alpha} \chi_{T_\eta^{m',0}} f_{\eta, \leq 2^{-i}} \right\|_{Y_{\xi+\eta}} \\ & \lesssim 2^{i-j} \|g_{i, \leq 2^{i-2}}\|_{\mathcal{D}X^{0, \frac{1}{2}, 1}} \left(\sum_{\eta \in \Theta_p} \left\| \sum_{m' \in A_\alpha} \chi_{T_\eta^{m',0}} f_{\eta, \leq 2^{-i}} \right\|_{Y_\eta}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Now we make use of (55) and sum with respect to α and l to obtain

$$\begin{aligned} & \|S_{j, \leq 2^{-i}} \left(\sum_{\xi} \sum_k g_{\xi, \xi^2+k} \right) \left(\sum_{\eta \in \Theta_p} f_{\eta, \leq 2^{-i}} \right)\|_{Y}^2 \\ & \lesssim 2^{i-j} \|g_{i, \leq 2^{i-2}}\|_{\mathcal{D}X^{0, \frac{1}{2}, 1}} \left(\sum_{\eta \in \Theta_p} \|f_{\eta, \leq 2^{-i}}\|_{Y_\eta}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

In the end we use (54) to perform the summation with respect to p and obtain the claim in (48).

The argument for (49) is carried out in the same fashion. We have the estimate (44) to replace (41) in this case.

(50) is the sum of (48) and (49).

(51) is the sum of the variants of (48) and (49) with decay. We sketch the proof for

$$(56) \quad \|v_{j,\leq 2^{-i}} \cdot u_{i,\leq 2^{i-2}}\|_{\mathcal{DY}_{j,\leq 2^{-i}}} \lesssim 2^{i-j} \|v_{j,\leq 2^{-i}}\|_{\mathcal{DY}_j} \cdot \|u_{i,\leq 2^{i-2}}\|_{\mathcal{DX}^{0,\frac{1}{2},1}}.$$

We follow the steps in the proof of (48). For fixed $\tilde{m}$ we decompose

$$\begin{aligned} \chi_{Q_j^{\tilde{m}}} S_{j,\leq 2^{-i}} & \left(\sum_{\xi} \sum_k g_{\xi,\xi^2+k} \right) \left(\sum_{\eta \in \Theta_p} f_{\eta,\leq 2^{-i}} \right) \\ & = \sum_{\alpha} \sum_l \sum_{m \in B^\alpha} \sum_{m' \in A^\alpha} \sum_k \sum_{\xi} \sum_{\eta \in \Theta_p} \chi_{Q_i^m} g_{\xi,\xi^2+k} \cdot \chi_{T_{\eta}^{m',l}} \chi_{Q_j^{\tilde{m}}} f_{\eta,\leq 2^{-i}}. \end{aligned}$$

Then we continue the exact same argument, just that we always replace $f_{\eta,\leq 2^{-i}}$ by $\chi_{Q_j^{\tilde{m}}} f_{\eta,\leq 2^{-i}}$. We end up with

$$\begin{aligned} & \sum_{\eta} \|\chi_{Q_j^{\tilde{m}}} (g_{i,\leq 2^{i-2}} \cdot f_{j,\leq 2^{-i}})_{\eta,\leq 2^{-i}}\|_{Y_{\eta}}^2 \\ & \lesssim 2^{i-j} \|g_{i,\leq 2^{i-2}}\|_{\mathcal{DX}^{0,\frac{1}{2},1}} \left(\sum_{\eta} \|\chi_{Q_j^{\tilde{m}}} f_{\eta,\leq 2^{-i}}\|_{Y_{\eta}}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Summing over the set of $\tilde{m}$ with the property $Q_{\tilde{m}}^j \cap L_i^k \neq \emptyset$ and then taking the supremum with respect to k and L gives us (56). $\square$

4.3. Estimates: $X_i^{0,\frac{1}{2},1} \cdot Y_{j,\leq 2^{-i}} \rightarrow X_{j,\geq 2^{-i}}^{0,\frac{1}{2},1}$. The main estimate in this section is the following:

Proposition 9. *We have the estimate*

$$(57) \quad \|v_{j,\leq 2^{-i}} \cdot u_i\|_{X_{j,\geq 2^{-i}}^{0,-\frac{1}{2}}} \lesssim 2^{i-j} i^{\frac{1}{2}} \|v_{j,\leq 2^{-i}}\|_{Y_j} \cdot \|u_i\|_{X^{0,\frac{1}{2},1}}.$$

The proof of this result is split again into two parts. We claim

$$(58) \quad \|v_{j,\leq 2^{-i}} \cdot u_{i,\leq 2^{i-2}}\|_{X_{j,\geq 2^{-i}}^{0,-\frac{1}{2}}} \lesssim 2^{i-j} i^{\frac{1}{2}} \|v_{j,\leq 2^{-i}}\|_{Y_j} \cdot \|u_{i,\leq 2^{i-2}}\|_{X^{0,\frac{1}{2},1}},$$

$$(59) \quad \|v_{j,\leq 2^{-i}} \cdot u_{i,\geq 2^{i-2}}\|_{X_{j,\geq 2^{-i}}^{0,-\frac{1}{2}}} \lesssim 2^{i-j} i^{\frac{1}{2}} \|v_{j,\leq 2^{-i}}\|_{Y_j} \cdot \|u_{i,\geq 2^{i-2}}\|_{L^2}.$$

Proof. We decompose $v_{j,\leq 2^{-i}}$ as in (52) and $u_{i,\leq 2^{i-2}}$ as in (53). We know from the previous section that

$$(60) \quad \|g_{i,\leq 2^{i-2}} \cdot f_{j,\leq 2^{-i}}\|_{X_{j,\geq 2^{-i}}^{0,\frac{1}{2},1}}^2 \approx \sum_p \|g_{i,\leq 2^{i-2}} \cdot \sum_{\eta \in \Theta_p} f_{\eta,\leq 2^{-i}}\|_{X_{j,\geq 2^{-i}}^{0,\frac{1}{2},1}}^2.$$

For a fixed $d_2 \geq 2^{-i}$, we decompose

$$S_{j,d_2}(g_{i,\leq 2^{-i}} \cdot \sum_{\eta \in \Theta_p} f_{\eta,\leq 2^{-i}}) = S_{j,d_2} \left(\sum_{k=0}^{2^{2i-2}} \sum_{\xi} g_{\xi,\xi^2+k} \right) \left(\sum_{\eta \in \Theta_p} f_{\eta,\leq 2^{-i}} \right).$$

From Lemma 8 we know that $g_{\xi,\xi^2+k} \cdot f_{\eta,\leq 2^{-i}}$ is supported in A_{j,d_2} iff $|\cos \alpha| \leq 2^{-i} d_2$, where α is the angle between ξ and η . The angle between any two η 's in

Θ_p is at most $2^{i-j} \leq 2^{-2i}$, and the angle between any two ξ 's in Ξ^i is either at least 2^{-2i} or the same. Therefore the ξ 's involved in the above summation have an angular localization in a set of cardinality $\approx 2^i d_2$; we just keep this in mind and not formalize it. What is really important is that we sum over a set containing $\approx 2^{3i} d_2$ ξ 's.

For each g_{ξ, ξ^2+k} and $f_{\eta, \leq 2^{-i}}$ we can apply the result in (46):

$$\|f_{\eta, \leq 2^{-i}} \cdot g_{\xi, \xi^2+k}\|_{L^2} \lesssim 2^{-\frac{i+j}{2}} \|f_{\eta, \leq 2^{-i}}\|_{Y_j} \|g_{\xi, \xi^2+k}\|_{L^2}.$$

Using the result in Lemma 8 we can perform the summation with respect to $\eta \in \Theta_p$:

$$\left\| \sum_{\eta \in \Theta_p} f_{\eta, \leq 2^{-i}} \cdot g_{\xi, \xi^2+k} \right\|_{L^2} \lesssim 2^{-\frac{i+j}{2}} \left\| \sum_{\eta \in \Theta_p} f_{\eta, \leq 2^{-i}} \|g_{\xi, \xi^2+k}\|_{L^2} \right\|_{L^2}.$$

Then we can perform the summation with respect to ξ :

$$\left\| \sum_{\eta \in \Theta_p} f_{\eta, \leq 2^{-i}} \cdot g_{i, 2^{-i}k} \right\|_{L^2_{j, d_2}} \lesssim 2^{-\frac{i+j}{2}} (2^{3i} d_2)^{\frac{1}{2}} \left\| \sum_{\eta \in \Theta_p} f_{\eta, \leq 2^{-i}} \|g_{i, 2^{-i}k}\|_{L^2} \right\|_{L^2}$$

followed by the one with respect to k :

$$\left\| \sum_{\eta \in \Theta_p} f_{\eta, \leq 2^{-i}} \cdot g_i \right\|_{L^2_{j, d_2}} \lesssim 2^{-\frac{i+j}{2}} (2^{3i} d_2)^{\frac{1}{2}} \left\| \sum_{\eta \in \Theta_p} f_{\eta, \leq 2^{-i}} \|g_i\|_{X^{0, \frac{1}{2}, 1}} \right\|_{L^2}$$

In the end we perform the summation with respect to p and pass to the $X^{0, \frac{1}{2}}$ norm:

$$\|f_{j, \leq 2^{-i}} \cdot g_i\|_{X^{0, \frac{1}{2}}_{j, d_2}} \lesssim 2^{i-j} \|f_{j, \leq 2^{-i}}\|_{Y_j} \|g_i\|_{X^{0, \frac{1}{2}, 1}}.$$

We sum with respect to d_2 (over a set of cardinality $\approx 2i$) to obtain the statement in (58).

Now we continue with the proof of (59). The approach is similar to the one above, but we still outline the main steps. We decompose $v_{j, \leq 2^{-i}}$ as in (52) and $u_{i, \geq 2^{i-2}}$ as in (53). We know from the previous section that

$$(61) \quad \|g_{i, \geq 2^{i-2}} \cdot f_{j, \leq 2^{-i}}\|_{X^{0, \frac{1}{2}, 1}_{j, \geq 2^{-i}}}^2 \approx \sum_p \|g_{i, \geq 2^{i-2}} \cdot \sum_{\eta \in \Theta_p} f_{\eta, \leq 2^{-i}}\|_{X^{0, \frac{1}{2}, 1}_{j, \geq 2^{-i}}}^2.$$

For a fixed $d_2 \geq 2^{-i}$, we decompose

$$S_{j, d_2}(g_{i, \leq 2^{-i}} \cdot \sum_{\eta \in \Theta_p} f_{\eta, \leq 2^{-i}}) = S_{j, d_2} \left(\sum_l \sum_n \sum_{\xi \in \Xi_n^i} g_{\xi, l} \right) \left(\sum_{\eta \in \Theta_p} f_{\eta, \leq 2^{-i}} \right).$$

From Lemma 8 we know that $g_{\xi, l} \cdot f_{\eta, \leq 2^{-i}}$ is supported in A_{j, d_2} iff $|\cos \alpha| \leq |\xi|^{-1} d_2$, where α is the angle between ξ and η . The angle between any two η 's in Θ_p is at most $2^{i-j} \leq 2^{-2i}$ and the angle between any two ξ 's in Ξ_n^i is either at least $n^{-1} = |\xi|^{-1} 2^{-i}$ or the same. Therefore the ξ 's involved in the above summation have an angular localization in a set of cardinality $\approx 2^i d_2$; we will just keep this in mind and not formalize it. What will be really important is that we sum over a set containing $\approx 2^i d_2$ ξ 's from Ξ_n^i and over a set containing $\approx 2^{3i} d_2$ ξ 's from Ξ^i .

The setup is exactly like in the proof of (58) with l playing the role of k , and the proof can be continued in the same fashion. $\square$

4.4. **Estimates:** $X_i^{0, \frac{1}{2}, 1} \cdot X_{j, \geq 2^{-i}}^{0, \frac{1}{2}, 1} \rightarrow \mathcal{Y}_{j, \leq 2^{-i}}$. The main estimate in this section is the following:

Proposition 10. *We have the estimate*

$$(62) \quad \|v_{j, \geq 2^{-i}} \cdot u_i\|_{\mathcal{Y}_{j, \leq 2^{-i}}} \lesssim 2^{i-j} i^{\frac{1}{2}} \|v_{j, \geq 2^{-i}}\|_{X_{j, \geq 2^{-i}}^{0, \frac{1}{2}, 1}} \cdot \|u_i\|_{X^{0, \frac{1}{2}, 1}}.$$

This can be obtained by duality from $X_i^{0, \frac{1}{2}, 1} \cdot Y_{j, \leq 2^{-i}} \rightarrow X_{j, \geq 2^{-i}}^{0, \frac{1}{2}, 1}$.

4.5. Bilinear estimates on dyadic regions.

Proof of Theorem 7. We decompose

$$\begin{aligned} B(u_i, v_j) &= S_{j, \leq 2^{-i}} B(u_i, v_j) + S_{j, \geq 2^{-i}} B(u_i, v_j) \\ &= S_{j, \leq 2^{-i}} B(u_i, v_{j, \leq 2^{-i}}) + S_{j, \leq 2^{-i}} B(u_i, v_{j, \geq 2^{-i}}) \\ &\quad + S_{j, \geq 2^{-i}} B(u_i, v_{j, \leq 2^{-i}}) + S_{j, \geq 2^{-i}} B(u_i, v_{j, \geq 2^{-i}}). \end{aligned}$$

For the first term we make use of (50) to obtain

$$\begin{aligned} \|S_{j, \leq 2^{-i}} B(u_i, v_{j, \leq 2^{-i}})\|_{\mathcal{Y}^s} &\approx 2^{sj} \|S_{j, \leq 2^{-i}} B(u_i, v_{j, \leq 2^{-i}})\|_{\mathcal{Y}} \\ &\lesssim i^{\frac{1}{2}} 2^i 2^{(s-1)j} \|\nabla v_{j, \leq 2^{-i}}\|_Y \|\nabla u_i\|_{\mathcal{D}X^{0, \frac{1}{2}}} \\ &\lesssim i^{\frac{1}{2}} 2^{2i} 2^{sj} \|v_{j, \leq 2^{-i}}\|_Y \|u_i\|_{\mathcal{D}X^{0, \frac{1}{2}}} \\ &\approx i^{\frac{1}{2}} 2^{(2-s)i} \|v_{j, \leq 2^{-i}}\|_{Y^s} \|u_i\|_{\mathcal{D}X^{s, \frac{1}{2}}} \lesssim i 2^{(2-s)i} \|v_{j, \leq 2^{-i}}\|_{Y^s} \|u_i\|_{\mathcal{D}Z^s}. \end{aligned}$$

For the second term we make use of (62) to obtain

$$\begin{aligned} \|S_{j, \leq 2^{-i}} B(u_i, v_{j, \geq 2^{-i}})\|_{\mathcal{Y}^s} &\approx 2^{sj} \|S_{j, \geq 2^{-i}} B(u_i, v_{j, \geq 2^{-i}})\|_{\mathcal{Y}} \\ &\lesssim i^{\frac{1}{2}} 2^i 2^{(s-1)j} \|\nabla v_{j, \geq 2^{-i}}\|_{X^{0, \frac{1}{2}}} \cdot \|\nabla u_i\|_{\mathcal{D}X^{0, \frac{1}{2}}} \\ &\lesssim i^{\frac{1}{2}} 2^{(2-s)i} \|v_{j, \geq 2^{-i}}\|_{X^{s, \frac{1}{2}}} \cdot \|u_i\|_{\mathcal{D}X^{s, \frac{1}{2}}} \\ &\lesssim i^{\frac{3}{2}} 2^{(2-s)i} \|v_{j, \geq 2^{-i}}\|_{Z^s} \cdot \|u_i\|_{\mathcal{D}Z^s}. \end{aligned}$$

For the third term we make use of (57) to obtain

$$\begin{aligned} \|S_{j, \geq 2^{-i}} B(u_i, v_{j, \leq 2^{-i}})\|_{X^{s, -\frac{1}{2}, 1}} &\approx 2^{sj} \|S_{j, \geq 2^{-i}} B(u_i, v_{j, \leq 2^{-i}})\|_{X^{0, -\frac{1}{2}, 1}} \\ &\lesssim i^{\frac{3}{2}} 2^i 2^{(s-1)j} \|\nabla v_{j, \leq 2^{-i}}\|_Y \|\nabla u_i\|_{\mathcal{D}X^{0, \frac{1}{2}}} \\ &\lesssim i^{\frac{3}{2}} 2^{(2-s)i} \|v_{j, \leq 2^{-i}}\|_{Y^s} \|u_i\|_{\mathcal{D}X^{s, \frac{1}{2}}} \lesssim i^2 2^{(2-s)i} \|v_{j, \leq 2^{-i}}\|_{Y^s} \|u_i\|_{\mathcal{D}Z^s}. \end{aligned}$$

The fourth term had been handled in Theorem 6. By adding all the estimates we obtain

$$\|B(u, v)\|_{W_j^s} \lesssim i^{\frac{3}{2}} 2^{(2-s)i} \|u\|_{\mathcal{D}Z_i^s} \|v\|_{Z_j^s}.$$

In the end we can recover the decay via an argument similar to the one in Proposition 7, part a). One would notice that over there we had to recover decay of type $\mathcal{D}_j$ and all we used is that the high frequency comes with that decay. We already worked out the conservation of decay for the first term; see (51). $\square$

5. BILINEAR ESTIMATES: PROOF OF THEOREM 4

This is a standard argument once we have the bilinear estimates on dyadic pieces; see (16), (17) and (37). For reference one could use Part 1; see the corresponding section there.

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