LIMIT THEOREMS FOR FREE MULTIPLICATIVE CONVOLUTIONS

HARI BERCOVICI AND JIUN-CHAU WANG

ABSTRACT. We determine the distributional behavior for products of free random variables in a general infinitesimal triangular array. The main theorems in this paper extend a result for measures supported on the positive half-line, and provide a new limit theorem for measures on the unit circle with nonzero first moment.

1. INTRODUCTION

Given two probability measures μ, ν on $\mathbb{R}_+ = (0, +\infty)$, we will denote by $\mu \otimes \nu$ their classical multiplicative convolution, and by $\mu \boxtimes \nu$ their free multiplicative convolution. Thus, $\mu \otimes \nu$ is the distribution of XY , where X and Y are classically independent positive random variables with distributions μ and ν , respectively. Analogously, $\mu \boxtimes \nu$ is the distribution of $X^{1/2}YX^{1/2}$, where X and Y are freely independent positive random variables with distributions μ and ν . A triangular array $\{\nu_{nk} : n \geq 1, 1 \leq k \leq k_n\}$ of probability measures on $\mathbb{R}_+$ is said to be *infinitesimal* if

$$\lim_{n \rightarrow \infty} \max_{1 \leq k \leq k_n} \nu_{nk}(\{t \in \mathbb{R}_+ : |t - 1| \geq \varepsilon\}) = 0,$$

for every $\varepsilon > 0$. Given such an array, one is interested in the asymptotic behavior of the measures

$$\mu_n = \nu_{n1} \otimes \nu_{n2} \otimes \cdots \otimes \nu_{nk_n}$$

and

$$\nu_n = \nu_{n1} \boxtimes \nu_{n2} \boxtimes \cdots \boxtimes \nu_{nk_n}.$$

The case of μ_n is completely understood and is reduced to the theory of addition of independent random variables by a logarithmic change of variables. However, the free case ν_n does not simply reduce to the additive theory by this change of variables.

The problem was first addressed in [4], where a triangular array such that $\nu_{n1} = \nu_{n2} = \cdots = \nu_{nk_n}$ for all n was considered. In this case, necessary and sufficient conditions were found for the weak convergence of the measures ν_n . In particular, it was shown that the sequence ν_n converges weakly if μ_n converges, but not conversely.

In this paper we will find necessary and sufficient conditions for the weak convergence of ν_n without any further assumptions on the infinitesimal array. We

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also prove analogous results for infinitesimal triangular arrays on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. In both cases the possible limit of ν_n is $\boxtimes$ -infinitely divisible as shown in [8] and [2].

The additive version of our results was studied earlier. Thus, consider an array $\{\mu_{nk} : n \geq 1, 1 \leq k \leq k_n\}$ of probability measures on $\mathbb{R}$. Infinitesimality in this case means that

$$\lim_{n \rightarrow \infty} \max_{1 \leq k \leq k_n} \mu_{nk}(\{t \in \mathbb{R} : |t| \geq \varepsilon\}) = 0, \quad \varepsilon > 0.$$

Denote by

$$\lambda_n = \mu_{n1} * \mu_{n2} * \cdots * \mu_{nk_n}$$

the classical additive convolutions, and by

$$\rho_n = \mu_{n1} \boxplus \mu_{n2} \boxplus \cdots \boxplus \mu_{nk_n}$$

the free additive convolutions of these measures. When $\mu_{n1} = \mu_{n2} = \cdots = \mu_{nk_n}$ for all n , it was shown in [3] that λ_n converges weakly if and only if ρ_n converges weakly. This result was extended to arbitrary infinitesimal arrays by Chistyakov and Götze in [9]. These authors made heavy use of analytic subordination (first proved for the free additive convolution in [11] generically and in [7] for the general case; cf. also [12, 13], [1] and [8] for different approaches). Our methods do not require analytic subordination and are close to the original approach in [4].

The remainder of this paper is organized as follows. In Section 2, we describe the analytical apparatus necessary for the calculation of free multiplicative convolutions. We also describe the $\boxtimes$ -infinitely divisible measures on $\mathbb{R}_+$ and $\mathbb{T}$ and some useful approximation results. In Section 3 we give the convergence criteria for arrays on $\mathbb{R}_+$, and in Section 4, we prove the analogous result for $\mathbb{T}$.

2. PRELIMINARIES

The analogue of the Fourier transform for multiplicative free convolutions was discovered by Voiculescu [10] (see also [5, 6]). Denote by $\mathcal{M}_+$ the collection of Borel probability measures defined on $\mathbb{R}_+$, and by $\mathcal{M}_{\mathbb{T}}^{\times}$ the Borel probability measures ν supported on the circle $\mathbb{T}$ with nonzero first moment, i.e. $\int_{\mathbb{T}} t d\nu(t) \neq 0$.

Given $\nu \in \mathcal{M}_+$, one defines the analytic function ψ_{ν} by

$$\psi_{\nu}(z) = \int_0^{\infty} \frac{tz}{1-tz} d\nu(t), \quad z \in \mathbb{C} \setminus (0, +\infty).$$

The function ψ_{ν} is univalent in the left half-plane $i\mathbb{C}^+$, and $\psi_{\nu}(i\mathbb{C}^+)$ is a region contained in the circle with diameter $(-1, 0)$; moreover, $\psi_{\nu}(i\mathbb{C}^+) \cap (-\infty, 0) = (-1, 0)$. Setting $\Omega_{\nu} = \psi_{\nu}(i\mathbb{C}^+)$, one defines the S -transform of the measure ν to be

$$S_{\nu}(z) = \frac{1+z}{z} \psi_{\nu}^{-1}(z), \quad z \in \Omega_{\nu}.$$

The remarkable property of the S -transform is that for $\mu, \nu \in \mathcal{M}_+$, one has

$$S_{\mu \boxplus \nu}(z) = S_{\mu}(z) S_{\nu}(z),$$

for every z in a neighborhood of $(-1, 0)$.

For $\nu \in \mathcal{M}_{\mathbb{T}}^{\times}$, the function ψ_{ν} is defined by the formula given above (with the integral calculated over $\mathbb{T}$), but its domain of definition is now the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The function ψ_{ν} has an inverse in a neighborhood of zero since $\psi'_{\nu}(0) = \int_{\mathbb{T}} t d\nu(t) \neq 0$. The corresponding S -transform is defined in

a neighborhood of zero. It is sometimes convenient to use a variation of the S -transform:

$$\Sigma_\nu(z) = S_\nu\left(\frac{z}{1-z}\right).$$

If $\nu \in \mathcal{M}_\mathbb{T}^\times$, the function Σ_ν is also defined in a neighborhood of zero, and

$$\Sigma_{\mu \boxtimes \nu}(z) = \Sigma_\mu(z) \Sigma_\nu(z), \quad \mu, \nu \in \mathcal{M}_\mathbb{T}^\times,$$

for all z in a neighborhood of zero where all functions involved are defined.

The weak convergence of probability measures can be translated in terms of their S -transforms.

Theorem 2.1 ([5, 6]). (1) *Given $\{\nu_n\}_{n=1}^\infty \subset \mathcal{M}_+$, the sequence $\{\nu_n\}_{n=1}^\infty$ converges weakly to a measure $\nu \in \mathcal{M}_+$ if and only if there exist two positive numbers $0 < b < a < 1$ such that the disk D with the diameter $(-a, -b)$ is contained in Ω_{ν_n} for all n , and the sequence $\{S_{\nu_n}\}_{n=1}^\infty$ converges uniformly on D to a function S .*

(2) *Given $\{\nu_n\}_{n=1}^\infty \subset \mathcal{M}_\mathbb{T}^\times$, the sequence $\{\nu_n\}_{n=1}^\infty$ converges weakly to a measure $\nu \in \mathcal{M}_\mathbb{T}^\times$ if and only if there exists a neighborhood of zero $K \subset \mathbb{D}$ such that for all Σ_{ν_n} defined in K , the sequence $\{\Sigma_{\nu_n}\}_{n=1}^\infty$ converges uniformly on K to a function Σ .*

Moreover, if (1) is satisfied, then $S = S_\nu$, and if (2) is satisfied, then $\Sigma = \Sigma_\nu$.

An array $\{\nu_{nk}\}_{n,k} \subset \mathcal{M}_\mathbb{T}^\times$ is infinitesimal if

$$\lim_{n \rightarrow \infty} \max_{1 \leq k \leq k_n} \nu_{nk}(\{t \in \mathbb{T} : |\arg t| \geq \varepsilon\}) = 0,$$

for every $\varepsilon > 0$; here the principal value of the argument is used. The following proposition gives an approximation of the S -transform (see Theorem 3.1 in [4] and Theorem 1.1 and Theorem 2.1 in [2]).

Proposition 2.2. *For $0 < b < a < 1$ and $\varepsilon \in (0, 1)$, define $\overline{D}$ to be the closed disk with diameter $[-a, -b]$, and set $K_\varepsilon = \{z \in \mathbb{C} : |z| \leq \varepsilon\}$.*

(1) *If an array $\{\nu_{nk}\}_{n,k} \subset \mathcal{M}_+$ is infinitesimal, then the functions $S_{\nu_{nk}}$ are defined in $\overline{D}$ for sufficiently large n , and we have*

$$S_{\nu_{nk}}(z) = 1 + \left[\int_0^\infty \frac{1-t}{1+z-tz} d\nu_{nk}(t) \right] (1 + u_{nk}(z)),$$

for all $z \in \overline{D}$, where $\lim_{n \rightarrow \infty} \max_{1 \leq k \leq k_n} |u_{nk}(z)| = 0$ uniformly on $\overline{D}$.

(2) *If an array $\{\nu_{nk}\}_{n,k} \subset \mathcal{M}_\mathbb{T}^\times$ is infinitesimal, then $S_{\nu_{nk}}$ are defined in K_ε when n is large, and we have*

$$S_{\nu_{nk}}(z) = 1 + \left[\int_\mathbb{T} \frac{1-t}{1+z-tz} d\nu_{nk}(t) \right] (1 + v_{nk}(z)),$$

for all $z \in K_\varepsilon$, where $\lim_{n \rightarrow \infty} \max_{1 \leq k \leq k_n} |v_{nk}(z)| = 0$ uniformly on K_ε .

A measure $\nu \in \mathcal{M}_+$ is said to be $\otimes$ -infinitely divisible if, for each $n \in \mathbb{N}$, there exists a measure $\nu_n \in \mathcal{M}_+$ such that

$$\nu = \underbrace{\nu_n \otimes \nu_n \otimes \cdots \otimes \nu_n}_{n \text{ times}}.$$

The notion of $\boxtimes$ -infinite divisibility is defined analogously. The study of $\otimes$ -infinitely divisible measures on $\mathbb{R}_+$ reduces (by a change of variable) to the study of the usual

*-infinitely divisible measures on $\mathbb{R}$. The Fourier transform needs to be replaced by the *Mellin-Fourier* transform of a measure $\nu \in \mathcal{M}_+$ defined by

$$\Phi_\nu(s) = \int_0^\infty t^{is} d\nu(t), \quad s \in \mathbb{R}.$$

The fundamental property of the Mellin-Fourier transform is that

$$\Phi_{\mu \otimes \nu}(s) = \Phi_\mu(s) \Phi_\nu(s).$$

A $\otimes$ -infinitely divisible measure $\nu \in \mathcal{M}_+$ has the Mellin-Fourier transform

$$\Phi_\nu(s) = \exp \left[i\lambda s + \int_0^\infty \left(t^{-is} - 1 + \frac{is \log t}{\log^2 t + 1} \right) \frac{\log^2 t + 1}{\log^2 t} d\rho(t) \right], \quad s \in \mathbb{R},$$

where $\lambda \in \mathbb{R}$ and ρ is a finite positive Borel measure on $\mathbb{R}_+$. We use the notation $\nu_{\otimes}^{\lambda, \rho}$ to denote the $\otimes$ -infinitely divisible measure determined by λ and ρ . For $\boxtimes$ -infinite divisibility we have the following formulas as in [5, 6]. A measure $\nu \in \mathcal{M}_+$ is $\boxtimes$ -infinitely divisible if and only if there exist $\gamma \in \mathbb{R}$ and a finite positive Borel measure σ on the compact space $[0, +\infty]$ such that $S_\nu(z) = \exp(v_{\gamma, \sigma}(z))$, where $v_{\gamma, \sigma}$ is given by

$$v_{\gamma, \sigma} \left(\frac{z}{1-z} \right) = \gamma - \sigma(\{+\infty\})z + \int_{[0, +\infty)} \frac{1+tz}{z-t} d\sigma(t), \quad z \in \mathbb{C} \setminus [0, 1].$$

A measure $\nu \in \mathcal{M}_{\mathbb{T}}^\times$ is $\boxtimes$ -infinitely divisible if and only if there exist $\gamma \in \mathbb{R}$ and a finite positive Borel measure σ on $\mathbb{T}$ such that $\Sigma_\nu(z) = \exp(u_{\gamma, \sigma}(z))$, where $u_{\gamma, \sigma}$ is given by

$$u_{\gamma, \sigma}(z) = -i\gamma + \int_{\mathbb{T}} \frac{1+tz}{1-tz} d\sigma(t), \quad z \in \mathbb{D}.$$

We denote by $\nu_{\boxtimes}^{\gamma, \sigma}$ the $\boxtimes$ -infinitely divisible measure determined by γ and σ . There is a unique $\boxtimes$ -infinitely divisible measure m on $\mathbb{T}$ such that its first moment is zero. This is the Haar, or normalized arclength measure.

We conclude this section with a result which will be used repeatedly.

Lemma 2.3. *Consider a sequence $\{r_n\}_{n=1}^\infty \subset \mathbb{R}$ and two triangular arrays $\{z_{nk} : n \geq 1, 1 \leq k \leq k_n\}$, $\{w_{nk} : n \geq 1, 1 \leq k \leq k_n\}$ of complex numbers. Assume that*

- (1) $\Im w_{nk} \geq 0$, for $n \geq 1$ and $1 \leq k \leq k_n$.
- (2) $z_{nk} = w_{nk}(1 + \varepsilon_{nk})$, where

$$\varepsilon_n = \max_{1 \leq k \leq k_n} |\varepsilon_{nk}|$$

converges to zero as $n \rightarrow \infty$.

- (3) There exists a positive constant M such that for sufficiently large n ,

$$|\Re w_{nk}| \leq M \Im w_{nk}.$$

Then the sequence $\{r_n + \sum_{k=1}^{k_n} z_{nk}\}_{n=1}^\infty$ converges if and only if the sequence $\{r_n + \sum_{k=1}^{k_n} w_{nk}\}_{n=1}^\infty$ converges. Moreover, the two sequences have the same limit.

Proof. The assumptions on $\{z_{nk}\}_{n,k}$ and $\{w_{nk}\}_{n,k}$ imply

$$(2.1) \quad \left| \left(r_n + \sum_{k=1}^{k_n} z_{nk} \right) - \left(r_n + \sum_{k=1}^{k_n} w_{nk} \right) \right| \leq 2(1+M)\varepsilon_n \left(\sum_{k=1}^{k_n} \Im w_{nk} \right)$$

and

$$(2.2) \quad (1 - \varepsilon_n - M\varepsilon_n) \left(\sum_{k=1}^{k_n} \Im w_{nk} \right) \leq \left| \sum_{k=1}^{k_n} \Im z_{nk} \right|,$$

for sufficiently large n . If the sequence $\{r_n + \sum_{k=1}^{k_n} z_{nk}\}_{n=1}^\infty$ converges to a complex number z , (2.2) implies that $\{\sum_{k=1}^{k_n} \Im w_{nk}\}_{n=1}^\infty$ is bounded, and then (2.1) shows that the sequence $\{r_n + \sum_{k=1}^{k_n} w_{nk}\}_{n=1}^\infty$ also converges to z . Conversely, if $\{r_n + \sum_{k=1}^{k_n} w_{nk}\}_{n=1}^\infty$ converges to z , then the sequence $\{\sum_{k=1}^{k_n} \Im w_{nk}\}_{n=1}^\infty$ is bounded and hence by (2.1) the sequence $\{r_n + \sum_{k=1}^{k_n} z_{nk}\}_{n=1}^\infty$ converges to z as well. $\square$

3. FREE MULTIPLICATIVE CONVOLUTION ON $\mathbb{R}_+$

Given an infinitesimal triangular array $\{\nu_{nk} : 1 \leq k \leq k_n, n \in \mathbb{N}\} \subset \mathcal{M}_+$ and $\tau > 0$, define positive numbers

$$b_{nk} = \exp \left(\int_{e^{-\tau}}^{e^{\tau}} \log t \, d\nu_{nk}(t) \right),$$

and measures ν_{nk}° by

$$d\nu_{nk}^\circ(t) = d\nu_{nk}(b_{nk}t).$$

Obviously, $\max_{1 \leq k \leq k_n} |b_{nk} - 1| \rightarrow 0$ as $n \rightarrow \infty$, and hence the array $\{\nu_{nk}^\circ\}_{n,k}$ is also infinitesimal. Define

$$g_{nk}(w) = \int_0^\infty \frac{t^2 - 1}{t^2 + 1} d\nu_{nk}^\circ \left(\frac{1}{t} \right) + \int_0^\infty \left[\frac{1 + tw}{w - t} \right] \frac{(t - 1)^2}{t^2 + 1} d\nu_{nk}^\circ \left(\frac{1}{t} \right),$$

for $w \in \mathbb{C} \setminus [0, +\infty)$. Note that $g_{nk}(\overline{w}) = \overline{g_{nk}(w)}$ and $\Im g_{nk}(w) \leq 0$ for all w such that $\Im w > 0$.

Lemma 3.1. *For every compact set $K \subset \mathbb{C}^+ \cap (i\mathbb{C}^+)$ there exists a positive constant $M = M(\tau, K)$ such that for sufficiently large n , we have*

$$|\Re g_{nk}(w)| \leq M |\Im g_{nk}(w)|, \quad w \in K, 1 \leq k \leq k_n.$$

Proof. We assume for convenience that $\tau = 1$. No generality is lost since one can make a linear change of variable to modify the value of τ . By a change of variable we have

$$\int_0^\infty \frac{t^2 - 1}{t^2 + 1} d\nu_{nk}^\circ \left(\frac{1}{t} \right) = \int_{-\infty}^\infty \frac{1 - e^{2x}}{1 + e^{2x}} d\rho_{nk}(x + a_{nk})$$

and

$$\int_0^\infty \frac{(t - 1)^2}{t^2 + 1} d\nu_{nk}^\circ \left(\frac{1}{t} \right) = \int_{-\infty}^\infty \frac{(1 - e^x)^2}{1 + e^{2x}} d\rho_{nk}(x + a_{nk}),$$

where the probability measure ρ_{nk} is defined as $d\rho_{nk}(x) = d\nu_{nk}(e^x)$, and $a_{nk} = \int_{|x| < 1} x d\rho_{nk}(x)$. Note that the family $\{\rho_{nk}\}_{n,k}$ is now an infinitesimal family of probability measures on $\mathbb{R}$, and hence

$$\lim_{n \rightarrow \infty} \max_{1 \leq k \leq k_n} |a_{nk}| = 0.$$

We proceed by rewriting

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1-e^{2x}}{1+e^{2x}} d\rho_{nk}(x+a_{nk}) &= \int_{|x|<1} \left[\frac{1-e^{2(x-a_{nk})}}{1+e^{2(x-a_{nk})}} + (x-a_{nk}) \right] d\rho_{nk}(x) \\ &\quad - \int_{|x|\geq 1} a_{nk} d\rho_{nk}(x) + \int_{|x|\geq 1} \frac{1-e^{2(x-a_{nk})}}{1+e^{2(x-a_{nk})}} d\rho_{nk}(x). \end{aligned}$$

It is easy to see that

$$\left| 1 - e^{2(x-a_{nk})} + (x-a_{nk}) + (x-a_{nk})e^{2(x-a_{nk})} \right| \leq 60(x-a_{nk})^2,$$

for $|x| < 1$. Consequently, for all n and k we have

$$\left| \int_{|x|<1} \left[\frac{1-e^{2(x-a_{nk})}}{1+e^{2(x-a_{nk})}} + (x-a_{nk}) \right] d\rho_{nk}(x) \right| \leq 60 \int_{-\infty}^{\infty} \frac{(1-e^x)^2}{1+e^{2x}} d\rho_{nk}(x+a_{nk}).$$

Since the family $\{\rho_{nk}\}_{n,k}$ is infinitesimal, there exists $N \in \mathbb{N}$ such that

$$|a_{nk}| \leq \frac{1}{2},$$

for all $n \geq N$, $1 \leq k \leq k_n$. Note that

$$\frac{(1-e^x)^2}{1+e^{2x}} \geq \frac{(1-\sqrt{e})^2}{1+e} \quad \text{and} \quad 5(1-e^x)^2 \geq |1-e^{2x}|,$$

for all $|x| \geq \frac{1}{2}$. We deduce that for $n \geq N$, $1 \leq k \leq k_n$ we have

$$\begin{aligned} \left| \int_{|x|\geq 1} a_{nk} d\rho_{nk}(x) \right| &\leq \int_{|x|\geq 1} d\rho_{nk}(x) \\ &\leq \frac{1+e}{(1-\sqrt{e})^2} \int_{-\infty}^{\infty} \frac{(1-e^x)^2}{1+e^{2x}} d\rho_{nk}(x+a_{nk}) \end{aligned}$$

and

$$\left| \int_{|x|\geq 1} \frac{1-e^{2(x-a_{nk})}}{1+e^{2(x-a_{nk})}} d\rho_{nk}(x) \right| \leq 5 \int_{-\infty}^{\infty} \frac{(1-e^x)^2}{1+e^{2x}} d\rho_{nk}(x+a_{nk}).$$

Therefore, for sufficiently large n ,

$$\left| \int_0^{\infty} \frac{t^2-1}{t^2+1} d\nu_{nk}^{\circ} \left(\frac{1}{t} \right) \right| \leq 74 \int_0^{\infty} \frac{(t-1)^2}{t^2+1} d\nu_{nk}^{\circ} \left(\frac{1}{t} \right).$$

The compactness of the set K implies the existence of positive constants M_1 and M_2 such that

$$\left| \Re \left[\frac{1+tw}{w-t} \right] \right| \leq M_1$$

and

$$\left| \Im \left[\frac{1+tw}{w-t} \right] \right| = -\Im \left[\frac{1+tw}{w-t} \right] \geq M_2,$$

for all $t \in (0, +\infty)$ and $w \in K$. Hence, we have for sufficiently large n and for $w \in K$,

$$\begin{aligned} |\Re g_{nk}(w)| &= \left| \int_0^\infty \frac{t^2 - 1}{t^2 + 1} d\nu_{nk}^\circ \left(\frac{1}{t} \right) + \int_0^\infty \Re \left[\frac{1 + tw}{w - t} \right] \frac{(t - 1)^2}{t^2 + 1} d\nu_{nk}^\circ \left(\frac{1}{t} \right) \right| \\ &\leq 74 \int_0^\infty \frac{(t - 1)^2}{t^2 + 1} d\nu_{nk}^\circ \left(\frac{1}{t} \right) + M_1 \int_0^\infty \frac{(t - 1)^2}{t^2 + 1} d\nu_{nk}^\circ \left(\frac{1}{t} \right) \\ &\leq \frac{(74 + M_1)}{M_2} \int_0^\infty M_2 \frac{(t - 1)^2}{t^2 + 1} d\nu_{nk}^\circ \left(\frac{1}{t} \right) \\ &\leq -\frac{(74 + M_1)}{M_2} \int_0^\infty \Im \left[\frac{1 + tw}{w - t} \right] \frac{(t - 1)^2}{t^2 + 1} d\nu_{nk}^\circ \left(\frac{1}{t} \right) \\ &= \frac{74 + M_1}{M_2} |\Im g_{nk}(w)|. \end{aligned}$$

The result follows with $M = (74 + M_1)/M_2$. $\square$

Fix a closed disk $\overline{D} \subset i\mathbb{C}^+$ with diameter $[-a, -b]$, where $0 < b < a < 1$. By Proposition 2.2, $S_{\nu_{nk}}$ is defined in $\overline{D}$ for large n . Setting $w = z/(1 + z)$, and using the identity

$$\frac{(w - 1)(t - 1)}{w - t} = \frac{t^2 - 1}{t^2 + 1} + \left[\frac{1 + tw}{w - t} \right] \frac{(t - 1)^2}{t^2 + 1},$$

we see that the function $S_{\nu_{nk}^\circ}$ admits the following approximation:

$$S_{\nu_{nk}^\circ} \left(\frac{w}{1 - w} \right) - 1 = g_{nk}(w) \left(1 + u_{nk} \left(\frac{w}{1 - w} \right) \right),$$

in another closed disk $\overline{D}_0 = \{z/(1 + z) : z \in \overline{D}\}$ with real center, and

$$\lim_{n \rightarrow \infty} \max_{1 \leq k \leq k_n} \left| u_{nk} \left(\frac{w}{1 - w} \right) \right| = 0,$$

uniformly for all $w \in \overline{D}_0$. Note that

$$S_{\nu_{nk}^\circ} \left(\frac{w}{1 - w} \right) = b_{nk} S_{\nu_{nk}} \left(\frac{w}{1 - w} \right).$$

The infinitesimality of the array $\{\nu_{nk}\}_{n,k}$ also shows that $S_{\nu_{nk}}(z)$ converges uniformly in k and $z \in \overline{D}$ to 1 as $n \rightarrow \infty$; indeed, $S_{\delta_1} \equiv 1$. Hence, for sufficiently large n , the principal branch of $\log S_{\nu_{nk}}(z)$ is defined in $\overline{D}$. Furthermore, since

$$\log w = w - 1 + o(|w - 1|),$$

as $w \rightarrow 1$, it is easy to see from Lemma 3.1 and Lemma 2.3 that we have the following result. Fix a real number γ , and a finite positive Borel measure σ on $[0, +\infty]$.

Lemma 3.2. *Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of positive real numbers. Then the sequence of functions $\{-\log \alpha_n + \sum_{k=1}^{k_n} \log S_{\nu_{nk}}(z)\}_{n=1}^\infty$ converges to $v_{\gamma, \sigma}(z)$ uniformly for all $z \in \overline{D}$ as $n \rightarrow \infty$ if and only if*

$$\lim_{n \rightarrow \infty} \left(-\log \alpha_n + \sum_{k=1}^{k_n} [g_{nk}(w) - \log b_{nk}] \right) = v_{\gamma, \sigma} \left(\frac{w}{1 - w} \right)$$

uniformly for all $w \in \overline{D}_0$.

Theorem 3.3. *For an infinitesimal array $\{\nu_{nk}\}_{n,k} \subset \mathcal{M}_+$ and a sequence $\{\alpha_n\}_{n=1}^\infty \subset \mathbb{R}_+$, the following two assertions are equivalent:*

- (1) *The sequence $\nu_{n1} \boxtimes \nu_{n2} \boxtimes \cdots \boxtimes \nu_{nk_n} \boxtimes \delta_{\alpha_n}$ converges weakly to $\nu_{\boxtimes}^{\gamma, \sigma}$.*
- (2) *The sequence of measures*

$$d\sigma_n(t) = \sum_{k=1}^{k_n} \frac{(t-1)^2}{t^2+1} d\nu_{nk}^\circ\left(\frac{1}{t}\right)$$

converges weakly in $[0, +\infty]$ to σ , and the sequence

$$\gamma_n = -\log \alpha_n + \sum_{k=1}^{k_n} \left[\int_0^\infty \frac{t^2-1}{t^2+1} d\nu_{nk}^\circ\left(\frac{1}{t}\right) - \log b_{nk} \right]$$

converges to γ as $n \rightarrow \infty$.

Proof. Assume (1) holds. From Theorem 2.1, there exists a closed disk $\overline{D}$ with real center such that

$$\lim_{n \rightarrow \infty} \left(\frac{1}{\alpha_n} \prod_{k=1}^{k_n} S_{\nu_{nk}}(z) \right) = S_{\nu_{\boxtimes}^{\gamma, \sigma}}(z) = \exp(v_{\gamma, \sigma}(z))$$

uniformly on the disk $\overline{D}$. We may choose $\overline{D}$ small enough so that $\exp(v_{\gamma, \sigma}(z))$ is in $-i\mathbb{C}^+$ on $\overline{D}$. Applying the principal branch of the logarithm function, we deduce that

$$\lim_{n \rightarrow \infty} \left(-\log \alpha_n + \sum_{k=1}^{k_n} \log S_{\nu_{nk}}(z) \right) = v_{\gamma, \sigma}(z),$$

uniformly on $\overline{D}$. Thus, Lemma 3.2 implies that

$$(3.1) \quad \lim_{n \rightarrow \infty} \left(-\log \alpha_n + \sum_{k=1}^{k_n} [g_{nk}(w) - \log b_{nk}] \right) = v_{\gamma, \sigma}\left(\frac{w}{1-w}\right)$$

uniformly on $\overline{D_0} = \{z/(1+z) : z \in \overline{D}\}$. Note that

$$(3.2) \quad -\log \alpha_n + \sum_{k=1}^{k_n} [g_{nk}(w) - \log b_{nk}] = \gamma_n + \int_0^\infty \frac{1+tw}{w-t} d\sigma_n(t).$$

Considering the imaginary part of the equation (3.1), we have

$$(3.3) \quad -\Im w \int_{[0, +\infty)} \frac{1+t^2}{|w-t|^2} d\sigma(t) = -\lim_{n \rightarrow \infty} \Im w \int_{(0, +\infty)} \frac{1+t^2}{|w-t|^2} d\sigma_n(t).$$

Note that the function $t \mapsto \frac{1+t^2}{|w-t|^2}$ is bounded away from zero and infinity for all $w \in \overline{D_0}$; moreover, if $\Im w \neq 0$, then (3.3) shows that

$$\sup_n \sigma_n((0, +\infty)) < +\infty,$$

and hence the family $\{\sigma_n\}_{n=1}^\infty$ has a weak cluster point σ' on the compact space $[0, +\infty]$. Then (3.3) shows that $\sigma' = \sigma$, and consequently the measures σ_n converge weakly to σ on $[0, +\infty]$ as $n \rightarrow \infty$. Then it is easy to see from (3.1) and (3.2) that $\lim_{n \rightarrow \infty} \gamma_n = \gamma$.

Conversely, assume (2) holds. The infinitesimality of the array $\{\nu_{nk}\}_{n,k}$ implies that there exist $a', b' \in (0, 1)$ with $b' < a'$ such that $S_{\nu_{nk}}$ are defined in $\overline{D'}$, the closed

disk with diameter $[-a', -b']$, for sufficiently large n . Let $\overline{D'_0} = \{z/(1+z) : z \in \overline{D'}\}$ and observe that there exists a positive constant $M = M(a', b')$ such that

$$\left| \frac{1+tw}{w-t} \right| \leq M, \quad w \in \overline{D'_0}, t \in (0, +\infty).$$

Thus, in view of (3.2), we deduce that (3.1) holds pointwise in $\overline{D'_0}$. Since $\Im g_{nk}(w) \leq 0$ for $w \in D'_0 \cap i\mathbb{C}^+$, the family $\{-\log \alpha_n + \sum_{k=1}^{k_n} [g_{nk}(w) - \log b_{nk}]\}_{n=1}^\infty$ is normal in $D'_0 \cap i\mathbb{C}^+$. Moreover, note that $g_{nk}(\overline{w}) = \overline{g_{nk}(w)}$ and

$$v_{\gamma, \sigma} \left(\frac{\overline{w}}{1-\overline{w}} \right) = \overline{v_{\gamma, \sigma} \left(\frac{w}{1-w} \right)},$$

for $w \in \overline{D'_0}$. Therefore, as an application of Montel's theorem, we conclude that (3.1) holds uniformly on compact subsets of $\overline{D'_0}$. From Lemma 3.2 we conclude that there exists a smaller closed disk $\overline{D''} \subset \overline{D'}$ with real center, such that

$$\lim_{n \rightarrow \infty} \left(-\log \alpha_n + \sum_{k=1}^{k_n} \log S_{\nu_{nk}}(z) \right) = v_{\gamma, \sigma}(z),$$

uniformly on $\overline{D''}$. Applying the exponential, we obtain

$$\lim_{n \rightarrow \infty} \left(\frac{1}{\alpha_n} \prod_{k=1}^{k_n} S_{\nu_{nk}}(z) \right) = S_{\nu_{\boxtimes}^{\gamma, \sigma}}(z) = \exp(v_{\gamma, \sigma}(z))$$

uniformly on $\overline{D''}$. Therefore (1) follows from Theorem 2.1. $\square$

It has been pointed out in [4] that the weak convergence criteria for products of free and independent random variables are not equivalent. Nevertheless, the following correspondence is true.

Corollary 3.4. *Given an infinitesimal array $\{\nu_{nk}\}_{n,k} \subset \mathcal{M}_+$ and a sequence $\{\alpha_n\}_{n=1}^\infty \subset \mathbb{R}_+$, the following two statements are equivalent:*

- (1) *The sequence $\nu_{n1} \boxtimes \nu_{n2} \boxtimes \cdots \boxtimes \nu_{nk_n} \boxtimes \delta_{\alpha_n}$ converges weakly to $\nu_{\boxtimes}^{\gamma, \sigma}$ and $\sigma(\{0\}) = \sigma(\{\infty\}) = 0$.*
- (2) *The sequence $\nu_{n1} \otimes \nu_{n2} \otimes \cdots \otimes \nu_{nk_n} \otimes \delta_{\alpha_n}$ converges weakly to $\nu_{\otimes}^{\lambda, \rho}$.*

If conditions (1) and (2) are satisfied, then the measures σ and ρ are related by

$$d\sigma(t) = \frac{\log^2 t + 1}{\log^2 t} \frac{(t-1)^2}{t^2 + 1} d\rho(t)$$

and

$$\gamma - \lambda = \int_0^\infty \left(\frac{t^2 - 1}{t^2 + 1} + \frac{\log^2 t}{\log^2 t + 1} \right) \frac{\log^2 t + 1}{\log^2 t} d\rho(t).$$

Proof. The proof is identical with that of Theorem 4.2 in [4]. $\square$

4. FREE MULTIPLICATIVE CONVOLUTION ON $\mathbb{T}$

Fix an infinitesimal array $\{\nu_{nk} : n \geq 1, 1 \leq k \leq k_n\} \subset \mathcal{M}_{\mathbb{T}}^\times$ and $\tau \in (0, \pi)$. Consider the centering constant

$$b_{nk} = \exp \left(\int_{|\arg t| < \tau} \log t d\nu_{nk}(t) \right),$$

and the centered measure ν_{nk}° obtained as follows:

$$d\nu_{nk}^\circ(t) = d\nu_{nk}(b_{nk}t).$$

Here, as before, $\log t = i \arg t$ represents the principal branch of $\log t$. We have $\max_{1 \leq k \leq k_n} |\arg b_{nk}| \rightarrow 0$ as $n \rightarrow \infty$, and hence the array $\{\nu_{nk}^\circ\}_{n,k}$ is also infinitesimal. Define

$$h_{nk}(z) = -i \int_{\mathbb{T}} \Im t d\nu_{nk}^\circ(t) + \int_{\mathbb{T}} \frac{1+tz}{1-tz} (1 - \Re t) d\nu_{nk}^\circ(t), \quad z \in \mathbb{D}.$$

Note that $\Re h_{nk}(z) \geq 0$ for all $z \in \mathbb{D}$.

Lemma 4.1. *For every compact neighborhood of zero $K \subset \mathbb{D}$, there exists a positive constant $M = M(\tau, K)$ such that for sufficiently large n , we have*

$$|\Im h_{nk}(z)| \leq M \Re h_{nk}(z), \quad z \in K, 1 \leq k \leq k_n.$$

Proof. We may again assume that $\tau = 1$. Define probability measures ρ_{nk} on $\mathbb{R}$ such that $\rho_{nk}(\sigma) = \nu_{nk}(e^{i\sigma})$ if $\sigma \in [-\pi, \pi)$, and $\rho_{nk}(\sigma) = 0$ when $\sigma \cap [-\pi, \pi) = \emptyset$. Changing variables, we have

$$\int_{\mathbb{T}} \Im t d\nu_{nk}^\circ(t) = \int_{-\infty}^{\infty} \sin x d\rho_{nk}(x + a_{nk})$$

and

$$\int_{\mathbb{T}} (1 - \Re t) d\nu_{nk}^\circ(t) = \int_{-\infty}^{\infty} (1 - \cos x) d\rho_{nk}(x + a_{nk}),$$

where $a_{nk} = \int_{|x| < 1} x d\rho_{nk}(x) = \int_{|\arg t| < 1} \arg t d\nu_{nk}(t)$. The infinitesimality of the array $\{\nu_{nk}\}_{n,k}$ implies that for sufficiently large n ,

$$\max_{1 \leq k \leq k_n} |a_{nk}| \leq \frac{1}{10}.$$

By using the elementary inequalities

$$|\sin x - x| \leq 2(1 - \cos x), \quad -2 \leq x \leq 2,$$

and

$$\frac{1}{10} + |\sin x| \leq 10(1 - \cos x), \text{ where } \frac{9}{10} \leq |x| \leq \pi + \frac{9}{10},$$

we have

$$\begin{aligned} \left| \int_{\mathbb{T}} \Im t d\nu_{nk}^\circ(t) \right| &\leq \left| \int_{|x| < 1} [\sin(x - a_{nk}) - (x - a_{nk})] d\rho_{nk}(x) \right| \\ &\quad + \left| \int_{[-\pi, -1] \cup [1, \pi)} a_{nk} d\rho_{nk}(x) \right| \\ &\quad + \left| \int_{[-\pi, -1] \cup [1, \pi)} \sin(x - a_{nk}) d\rho_{nk}(x) \right| \\ &\leq 12 \int_{\mathbb{T}} (1 - \Re t) d\nu_{nk}^\circ(t), \end{aligned}$$

for large n and $1 \leq k \leq k_n$. Also, from the compactness of the set K , there exist two positive constants M_1 and M_2 such that

$$\left| \Re \left[\frac{1+tz}{1-tz} \right] \right| = \Re \left[\frac{1+tz}{1-tz} \right] \geq M_1$$

and

$$\left| \Im \left[\frac{1+tz}{1-tz} \right] \right| \leq M_2,$$

for all $t \in \mathbb{T}$ and $z \in K$. The result follows as in the proof of Lemma 3.1, with $M = (12 + M_2)/M_1$. $\square$

Suppose $K \subset \mathbb{D}$ is a neighborhood of zero. The infinitesimality of the array $\{\nu_{nk}\}_{n,k}$ implies that $S_{\nu_{nk}}(z)$ converges uniformly in k and $z \in K$ to 1 as $n \rightarrow \infty$, and hence for sufficiently large n , $\Sigma_{\nu_{nk}}(z)$ and the principal branch of $\log \Sigma_{\nu_{nk}}(z)$ are defined in $K' = \{z/(1+z) : z \in K\}$.

Fix a real number γ and a finite positive Borel measure σ on $\mathbb{T}$.

Lemma 4.2. *Let $\{\lambda_n\}_{n=1}^\infty \subset \mathbb{T}$. Then*

$$\lim_{n \rightarrow \infty} \exp \left(-\log \lambda_n + \sum_{k=1}^{k_n} \log \Sigma_{\nu_{nk}}(z) \right) = \Sigma_{\nu_{\boxtimes}^{\gamma, \sigma}}(z)$$

uniformly on K' if and only if

$$\lim_{n \rightarrow \infty} \exp \left(-\log \lambda_n + \sum_{k=1}^{k_n} [h_{nk}(z) - \log b_{nk}] \right) = \Sigma_{\nu_{\boxtimes}^{\gamma, \sigma}}(z)$$

uniformly on K' .

Proof. From Proposition 2.2, we have the following approximation for the function $S_{\nu_{nk}^\circ}$:

$$S_{\nu_{nk}^\circ}(z) = 1 + \left[\int_{\mathbb{T}} \frac{1-t}{1+z-tz} d\bar{\nu}_{nk}(t) \right] (1 + v_{nk}(z)), \quad z \in K,$$

where

$$v_n(z) = \max_{1 \leq k \leq k_n} |v_{nk}(z)|$$

satisfies $\lim_{n \rightarrow \infty} v_n(z) = 0$ uniformly in K . Introducing a change of variable $z \mapsto \frac{z}{1-z}$ and using the identity

$$\frac{(1-t)(1-z)}{1-tz} = -i\Im t + \frac{1+tz}{1-tz}(1 - \Re t),$$

we conclude that

$$b_{nk} \Sigma_{\nu_{nk}}(z) = \Sigma_{\nu_{nk}^\circ}(z) = 1 + h_{nk}(z) \left(1 + v_{nk} \left(\frac{z}{1-z} \right) \right), \quad z \in K'.$$

Lemmas 4.1 and 2.3 imply that for any sequence of purely imaginary numbers $\{r_n\}_{n=1}^\infty$, the sequence $\{r_n + \sum_{k=1}^{k_n} [\Sigma_{\nu_{nk}^\circ}(z) - 1]\}_{n=1}^\infty$ converges uniformly on K' if and only if the sequence $\{r_n + \sum_{k=1}^{k_n} h_{nk}(z)\}_{n=1}^\infty$ converges uniformly on K' . Moreover, the two sequences have the same limit. Since $\log w/(w-1) \rightarrow 1$ as $w \rightarrow 1$, we can replace $\Sigma_{\nu_{nk}^\circ}(z) - 1$ by $\log \Sigma_{\nu_{nk}^\circ}(z)$. The result follows by choosing $r_n = -\log \lambda_n - \sum_{k=1}^{k_n} \log b_{nk}$. $\square$

Theorem 4.3. *For an infinitesimal array $\{\nu_{nk}\}_{n,k} \subset \mathcal{M}_{\mathbb{T}}^\times$ and a sequence $\{\lambda_n\}_{n=1}^\infty \subset \mathbb{T}$, the following assertions are equivalent:*

- (1) *The sequence $\nu_{n1} \boxtimes \nu_{n2} \boxtimes \cdots \boxtimes \nu_{nk_n} \boxtimes \delta_{\lambda_n}$ converges weakly to $\nu_{\boxtimes}^{\gamma, \sigma}$.*

(2) *The sequence of measures*

$$d\sigma_n(t) = \sum_{k=1}^{k_n} (1 - \Re t) d\nu_{nk}^\circ(t)$$

converges weakly on $\mathbb{T}$ to σ , and the limit

$$\lim_{n \rightarrow \infty} e^{i\gamma_n} = e^{i\gamma}$$

exists, where

$$\gamma_n = \arg \lambda_n + \sum_{k=1}^{k_n} \left[\int_{\mathbb{T}} \Im t d\nu_{nk}^\circ(t) + \arg b_{nk} \right].$$

Proof. Assume (1) holds. From Theorem 2.1, there exists $\varepsilon \in (0, 1)$ such that all $\Sigma_{\nu_{nk}}$ are defined in $K'_\varepsilon = \{z/(1+z) : |z| \leq \varepsilon\}$, and we have

$$\lim_{n \rightarrow \infty} \left(\frac{1}{\lambda_n} \prod_{k=1}^{k_n} \Sigma_{\nu_{nk}}(z) \right) = \Sigma_{\nu_{\boxtimes}^\sigma}(z) = e^{u_{\gamma, \sigma}(z)}$$

uniformly on K'_ε . Hence, by Lemma 4.2 and the definition of $u_{\gamma, \sigma}(z)$, we have

(4.1)

$$\lim_{n \rightarrow \infty} \exp \left(-\log \lambda_n + \sum_{k=1}^{k_n} [h_{nk}(z) - \log b_{nk}] \right) = \exp \left(-i\gamma + \int_{\mathbb{T}} \left[\frac{1+tz}{1-tz} \right] d\sigma(t) \right)$$

uniformly on K'_ε . Taking the absolute value on both sides of (4.1), we deduce that

$$(4.2) \quad \lim_{n \rightarrow \infty} \exp \left(\Re \left[\sum_{k=1}^{k_n} h_{nk}(z) \right] \right) = \exp \left(\int_{\mathbb{T}} \Re \left[\frac{1+tz}{1-tz} \right] d\sigma(t) \right)$$

uniformly on K'_ε . Note that

$$(4.3) \quad -\log \lambda_n + \sum_{k=1}^{k_n} [h_{nk}(z) - \log b_{nk}] = -i\gamma_n + \int_{\mathbb{T}} \left[\frac{1+tz}{1-tz} \right] d\sigma_n(t).$$

Moreover, the real part of the function $-\log \lambda_n + \sum_{k=1}^{k_n} [h_{nk}(z) - \log b_{nk}]$ is the Poisson integral of the measure $d\sigma_n(\frac{1}{t})$, and hence (4.2) uniquely determines the weak cluster point of $\{\sigma_n\}_{n=1}^\infty$ which is σ . We therefore conclude the weak convergence of the sequence $\{\sigma_n\}_{n=1}^\infty$. Moreover, consider $z = 0$ in (4.1) and (4.2) to deduce that

$$\lim_{n \rightarrow \infty} \frac{e^{i\gamma}}{e^{i\gamma_n}} = 1,$$

as desired.

The converse implication is fairly easy now, since one can basically reverse the steps to reach the statement (1) by using Lemma 4.2 and the fact that $\{-\log \lambda_n + \sum_{k=1}^{k_n} [h_{nk}(z) - \log b_{nk}]\}_{n=1}^\infty$ is normal in $\mathbb{D}$. Therefore the details of the proof are left to the reader. $\square$

The previous result does not cover the possibility that the measures $\nu_{n1} \boxtimes \nu_{n2} \boxtimes \cdots \boxtimes \nu_{nk_n} \boxtimes \delta_{\lambda_n}$ might converge to the Haar measure m . We address now this special case. Let us also note for further use the equality

$$\Sigma_\nu(0) = \frac{1}{\int_{\mathbb{T}} t d\nu(t)}, \quad \nu \in \mathcal{M}_{\mathbb{T}}^\times.$$

Theorem 4.4. *For an infinitesimal array $\{\nu_{nk}\}_{n,k} \subset \mathcal{M}_{\mathbb{T}}^{\times}$ and a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{T}$, the following assertions are equivalent:*

- (1) *The sequence $\nu_{n1} \boxtimes \nu_{n2} \boxtimes \cdots \boxtimes \nu_{nk_n} \boxtimes \delta_{\lambda_n}$ converges weakly to m .*
- (2)

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{k_n} \int_{\mathbb{T}} (1 - \Re t) d\nu_{nk}^{\circ}(t) = +\infty.$$

Proof. Assume (2) holds. Define

$$\nu_n = \nu_{n1} \boxtimes \nu_{n2} \boxtimes \cdots \boxtimes \nu_{nk_n} \boxtimes \delta_{\lambda_n}, \quad n \in \mathbb{N}.$$

The compactness of $\mathbb{T}$ implies that $\{\nu_n\}_{n=1}^{\infty}$ is tight. Suppose ν is a weak cluster point of $\{\nu_n\}_{n=1}^{\infty}$. From the free multiplicative analogue of Hinčin's theorem (Theorem 2.1 in [2]), the measure ν is $\boxtimes$ -infinitely divisible. By passing to a subsequence, we may assume that ν_n converges weakly to ν as $n \rightarrow \infty$. By (4.3), we can reformulate the statement (2) as follows:

$$\lim_{n \rightarrow \infty} \Re \sum_{k=1}^{k_n} h_{nk}(0) = +\infty.$$

Then the inequality (2.2) implies that

$$\lim_{n \rightarrow \infty} \Re \sum_{k=1}^{k_n} \log \Sigma_{\nu_{nk}}(0) = +\infty,$$

and consequently we deduce that

$$\begin{aligned} \left| \int_{\mathbb{T}} t d\nu(t) \right| &= \lim_{n \rightarrow \infty} \left| \int_{\mathbb{T}} t d\nu_n(t) \right| \\ &= \lim_{n \rightarrow \infty} \frac{1}{\left| \prod_{k=1}^{k_n} \Sigma_{\nu_{nk}}(0) \right|} \\ &= \lim_{n \rightarrow \infty} \exp \left(-\Re \sum_{k=1}^{k_n} \log \Sigma_{\nu_{nk}}(0) \right) = 0. \end{aligned}$$

Therefore, the $\boxtimes$ -infinitely divisible measure ν has zero first moment, and hence we conclude that $\nu = m$. Moreover, the full sequence ν_n converges weakly to m since $\{\nu_n\}_{n=1}^{\infty}$ has a unique weak cluster point m .

Conversely, assume (1) holds but (2) fails to be true. By passing, if necessary, to a subsequence, we may assume the sequence of measures

$$d\sigma_n(t) = \sum_{k=1}^{k_n} (1 - \Re t) d\nu_{nk}^{\circ}(t)$$

converges weakly to a finite positive Borel measure σ on $\mathbb{T}$. Since the sequence of functions $\{-\log \lambda_n + \sum_{k=1}^{k_n} [h_{nk}(z) - \log b_{nk}]\}_{n=1}^{\infty}$ is normal in $\mathbb{D}$, we may assume, by passing to a further subsequence, that

$$\lim_{n \rightarrow \infty} \left(-\log \lambda_n + \sum_{k=1}^{k_n} [h_{nk}(z) - \log b_{nk}] \right) = f(z)$$

uniformly on compact subsets of $\mathbb{D}$, where the function f is analytic in $\mathbb{D}$. Note that the function f is not identically infinity since

$$\Re f(z) = \lim_{n \rightarrow \infty} \int_{\mathbb{T}} \Re \left[\frac{1+tz}{1-tz} \right] d\sigma_n(t) = \int_{\mathbb{T}} \Re \left[\frac{1+tz}{1-tz} \right] d\sigma(t), \quad z \in \mathbb{D}.$$

Then, as in the proof of Theorem 4.3, we conclude that there exists $\gamma \in \mathbb{R}$ such that

$$\lim_{n \rightarrow \infty} e^{i\gamma_n} = e^{i\gamma},$$

where the number γ_n is defined as in Theorem 4.3. An application of Theorem 4.3 then shows that a subsequence of $\{\nu_n\}_{n=1}^{\infty}$ converges weakly to $\nu_{\boxtimes}^{\gamma, \sigma}$, which contradicts (1). Therefore (2) must be true. $\square$

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DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47405-4301
E-mail address: bercovic@indiana.edu

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47405-4301
E-mail address: jiuwang@indiana.edu