SPINOR L -FUNCTIONS FOR GENERIC CUSP FORMS ON $GS(2)$ BELONGING TO PRINCIPAL SERIES REPRESENTATIONS

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ABSTRACT. Let $G = GS(2)$ be the symplectic group with similitude of degree two, which is defined over $\mathbf{Q}$. For a generic cusp form F on the adelicized group $G_{\mathbf{A}}$ whose archimedean type is a principal series representation, we show that its spinor L -function is continued to an entire function and satisfies the functional equation.

INTRODUCTION

This paper is a continuation of [Mo1]. Let us explain the main result of [Mo1]. Let Π be a cuspidal automorphic representation of the symplectic group $G = GS(2)$ with similitude of degree two defined over $\mathbf{Q}$. Suppose that Π is *generic*, that is, Π has a non-vanishing global Whittaker model. Then each local component Π_v has a local Whittaker model. In particular, by a theorem of Kostant [Ko, Theorem 6.8.1], the representation Π_{∞} of $G_{\mathbf{R}}$ must be large in the sense of Vogan [V]. By [V, Theorem 6.2 (f)], an irreducible large representation Π_{∞} of $G_{\mathbf{R}}$ is equivalent to one of the following four kinds of representations:

- (1) a (limit of a) large discrete series representation of $G_{\mathbf{R}}$;
- (2) an irreducible principal series representation $I(\sigma, \nu)$ induced from the minimal parabolic subgroup $P_{\min}$ of $G_{\mathbf{R}}$ whose restriction to $Sp(2, \mathbf{R})$ is also irreducible (see subsection 1.3 for the precise definition of $I(\sigma, \nu)$);
- (3) an irreducible generalized principal series representation $I(P_1; \sigma_1, \nu_1)$ induced from the maximal parabolic subgroup P_1 corresponding to the long root;
- (4) an irreducible generalized principal series representation $I(P_2; \sigma_2, \nu_2)$ induced from the other maximal parabolic subgroup P_2 whose restriction to $Sp(2, \mathbf{R})$ is also irreducible,

where σ_i is a (limit of a) discrete series representation of the semisimple part of P_i ($i = 1, 2$). The main theorem of [Mo1] is the following: if Π_{∞} belongs to either the first or the third class, then the completed spinor L -function $\Lambda(s, \Pi)$ for Π is continued to an entire function and satisfies the expected functional equation. The

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purpose of this paper is to establish similar results if Π_∞ belongs to the second class.

Our basic strategy is the same as in the previous paper – the use of Novodvorsky’s zeta integral [No, §1]. Since the non-archimedean local theory of Novodvorsky’s zeta integral is enough developed for our purpose in [No], [Bu], and [TB1] (cf. [Mo1, §2]), our task is to control the local zeta integrals at the real place. More precisely, we shall derive explicit formulae of local Whittaker functions on $G_{\mathbf{R}}$ belonging to the principal series representations. Here we use the differential equations for the Whittaker functions constructed by Miyazaki and Oda [Mi-O1]. Our formulae are given by Mellin-Barnes type integrals. By virtue of such integral expressions, we can compute the archimedean local zeta integrals quite efficiently.

Recently we know that Asgari and Shahidi [A-S, Proposition 3.8] obtained a result more general than ours by using the Langlands-Shahidi method. In particular they do not impose any restrictions on Π_∞ . We believe that our explicit computation of the local zeta integrals has independent interest. We also note that Takloo-Bighash [TB2] investigates Novodvorsky’s local zeta integrals by using theta correspondence. On the other hand, our method as well as that of Asgari and Shahidi does not work for *non-generic* cusp forms. In such a case, we have to use another way of analyzing the spinor L -function (e.g. Andrianov [An], Kohnen and Skoruppa [K-S], Murase and Sugano [M-S]). We mention that Hori [Ho] and Miyazaki [Mi] treat the spinor L -function attached to certain non-holomorphic cusp forms on $GS(2)$ by extending the method of Andrianov.

This paper is organized as follows. In the first section, we state our main results. In the second section, we give the integral expression of the local Whittaker functions on $GS(2, \mathbf{R})$. In the third section, we compute the archimedean local zeta integrals explicitly. The main result is proved in the final section.

Notation and conventions. (i) For each place v of the field $\mathbf{Q}$ of rational numbers, we denote by $\mathbf{Q}_v$ the completion of $\mathbf{Q}$ at v . The module of an element $x \in \mathbf{Q}_v$ is denoted by $|x|_v$. For a finite place p of $\mathbf{Q}$, $\mathbf{Z}_p$ stands for the ring of integers in $\mathbf{Q}_p$. The adèle ring (resp. the idele group) is denoted by $\mathbf{A}$ (resp. $\mathbf{A}^\times$). The module of an element $x \in \mathbf{A}^\times$ is denoted by $|x|_{\mathbf{A}}$. Unless otherwise stated, we understand that all the measures on locally compact unimodular groups are Haar measures.

(ii) As usual we set $\Gamma_{\mathbf{R}}(s) := \pi^{-s/2} \Gamma(s/2)$ ($s \in \mathbf{C}$), where $\Gamma(s)$ is the Gamma function.

For complex numbers α_i ($1 \leq i \leq r$) and β_j ($1 \leq j \leq s$), we set

$$\Gamma[\alpha_1, \alpha_2, \dots, \alpha_r] := \prod_{i=1}^r \Gamma(\alpha_i), \quad \Gamma \left[\begin{smallmatrix} \alpha_1, \alpha_2, \dots, \alpha_r \\ \beta_1, \beta_2, \dots, \beta_s \end{smallmatrix} \right] := \prod_{i=1}^r \Gamma(\alpha_i) / \prod_{j=1}^s \Gamma(\beta_j).$$

(iii) We denote by $L(\sigma)$ the vertical path of integration in the complex plane $\mathbf{C}$ from $\sigma - \sqrt{-1}\infty$ to $\sigma + \sqrt{-1}\infty$ ($\sigma \in \mathbf{R}$).

(iv) Let $f : L \rightarrow \mathbf{C}$ be a C^∞ -function on a Lie group L . For each element $X \in \mathfrak{l} := \text{Lie}(L)$, we set

$$[R_X f](x) := \left. \frac{d}{dt} \right|_{t=0} f(x \exp(tX)), \quad X \in \mathfrak{l}, \quad x \in L.$$

This action of $\mathfrak{l}$ can be extended to that of the universal enveloping algebra $U(\mathfrak{l})$ of $\mathfrak{l}$. We also write $f(x; X)$ for $[R_X f](x)$ ($X \in U(\mathfrak{l})$).

1. PRELIMINARIES AND THE MAIN THEOREM

In this section we introduce basic ingredients of this paper and state our main result.

1.1. The group $GLSp(2)$. Let $G = GLSp(2)$ be the symplectic group with similitude of degree two, which is defined by

$$G := \{g \in GL(4) \mid {}^t g J_4 g = \nu(g) J_4 \text{ for some } \nu(g) \in \mathbf{G}_m\}, \quad J_4 := \left(\begin{array}{c|c} & 1_2 \\ \hline & -1_2 \end{array} \right).$$

We regard G as an algebraic group defined over $\mathbf{Q}$. For any $\mathbf{Q}$ -algebra R , the group of R -valued points of G is denoted by G_R . We adopt the same convention for algebraic subgroups of G . We take a maximal unipotent subgroup N of G defined over $\mathbf{Q}$ as follows:

$$N := \left\{ n(x_0, x_1, x_2, x_3) := \left(\begin{array}{cc|cc} 1 & & x_1 & x_2 \\ & 1 & x_2 & x_3 \\ \hline & & 1 & \\ & & & 1 \end{array} \right) \left(\begin{array}{c|c} 1 & x_0 \\ \hline & 1 \\ \hline & 1 \\ & -x_0 & 1 \end{array} \right) \in G \right\}.$$

1.2. The Lie group $GLSp(2, \mathbf{R})$. We fix some notation concerning the Lie groups $G := G_{\mathbf{R}} = GLSp(2, \mathbf{R})$ and $G_0 := Sp(2, \mathbf{R}) = \{g \in G \mid \nu(g) = 1\}$, which is basically the same notation as in [Mi-O1], [Mi-O2], and [Mol]. A maximal compact subgroup K (resp. K_0) of G (resp. G_0) is given by $K := G_{\mathbf{R}} \cap O(4)$ (resp. $K_0 := Sp(2, \mathbf{R}) \cap O(4)$). The group K_0 is isomorphic to the unitary group $U(2) := \{g \in GL(2, \mathbf{C}) \mid {}^t \bar{g} g = I_2\}$ of degree two. Define an isomorphism $\kappa : U(2) \cong K_0$ by

$$\kappa : U(2) \ni A + \sqrt{-1}B \mapsto k_{A,B} := \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K_0 \quad (A, B \in M(2, \mathbf{R})).$$

We write the Lie algebras of G , G_0 , and K_0 by $\mathfrak{g}$, $\mathfrak{g}_0$, and $\mathfrak{k}$, respectively. We use the same symbol for a continuous representation of $G_{\mathbf{R}}$ (resp. G_0) and its underlying $(\mathfrak{g}, K)$ -module (resp. $(\mathfrak{g}_0, K_0)$ -module) if there is no fear of confusion. For an arbitrary Lie subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, we denote its complexification $\mathfrak{l} \otimes \mathbf{C}$ by $\mathfrak{l}_{\mathbf{C}}$. We write the dual space $\text{Hom}_{\mathbf{C}}(\mathfrak{l}_{\mathbf{C}}, \mathbf{C})$ of $\mathfrak{l}_{\mathbf{C}}$ by $\mathfrak{l}_{\mathbf{C}}^*$. The differential κ_* of κ defines an isomorphism of Lie algebras: $\kappa_* : \mathfrak{gl}(2, \mathbf{C}) \cong \mathfrak{k}_{\mathbf{C}}$.

The simple Lie algebra $\mathfrak{g}_0$ has a compact Cartan subalgebra $\mathfrak{h} := \mathbf{R}T_1 \oplus \mathbf{R}T_2$, where

$$T_1 := \kappa_* \left(\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & 0 \end{pmatrix} \right); \quad T_2 := \kappa_* \left(\begin{pmatrix} 0 & 0 \\ 0 & \sqrt{-1} \end{pmatrix} \right).$$

Define a $\mathbf{C}$ -basis $\{e_1, e_2\}$ of $\mathfrak{h}_{\mathbf{C}}^*$ by $e_i(T_j) = \sqrt{-1}\delta_{ij}$ ($1 \leq i, j \leq 2$). Then the root system $\Delta = \Delta(\mathfrak{g}_{0, \mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ for the pair $(\mathfrak{g}_{0, \mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ is given by $\Delta(\mathfrak{g}_{0, \mathbf{C}}, \mathfrak{h}_{\mathbf{C}}) = \{\pm 2e_1, \pm 2e_2, \pm(e_1 \pm e_2)\}$. For each root $\alpha \in \Delta$, we define the corresponding root space $\mathfrak{g}_{0, \mathbf{C}}^{\alpha}$ by

$$\mathfrak{g}_{0, \mathbf{C}}^{\alpha} := \{X \in \mathfrak{g}_{0, \mathbf{C}} \mid [H, X] = \alpha(H)X \quad (\forall H \in \mathfrak{h}_{\mathbf{C}})\}.$$

Denote by Δ_c (resp. Δ_{nc}) the set of compact roots in Δ : $\Delta_c = \{\pm(e_1 - e_2)\}$ (resp. $\Delta_{nc} = \Delta \setminus \Delta_c$). We take a positive system Δ^+ of Δ as $\Delta^+ := \{2e_1, e_1 + e_2, 2e_2, e_1 - e_2\}$. Then the sets of compact and non-compact positive roots in Δ^+ are given by $\Delta_c^+ = \Delta_c \cap \Delta^+$ and $\Delta_{nc}^+ = \Delta_{nc} \cap \Delta^+$, respectively. We set $\mathfrak{p}_{\pm} := \bigoplus_{\alpha \in \Delta_{nc}^{\pm}} \mathfrak{g}_{0, \mathbf{C}}^{\pm\alpha}$.

For each symmetric matrix $A = {}^tA \in M_2(\mathbf{R})$, we define an element $p_+(A)$ (resp. $p_-(A)$) of $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) by

$$p_{\pm}(A) := \left(\begin{array}{c|c} A & \pm\sqrt{-1}A \\ \hline \pm\sqrt{-1}A & -A \end{array} \right) \in \mathfrak{p}_{\pm}.$$

Then we can take the root vectors $X_{(\alpha_1, \alpha_2)} \in \mathfrak{g}_{0, \mathbf{C}}$ corresponding to the non-compact roots $\alpha_1 e_1 + \alpha_2 e_2 = (\alpha_1, \alpha_2) \in \Delta_{nc}$ as follows:

$$X_{\pm(2,0)} := p_{\pm} \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right); \quad X_{\pm(1,1)} := p_{\pm} \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right); \quad X_{\pm(0,2)} := p_{\pm} \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

The irreducible finite-dimensional representations of K_0 are parameterized by their highest weights relative to Δ_c^+ . For each dominant integral weight $q = (q_1, q_2) = q_1 e_1 + q_2 e_2 \in \mathfrak{h}_{\mathbf{C}}^*$ ($q_i \in \mathbf{Z}, q_1 \geq q_2$), we denote the corresponding irreducible finite-dimensional representation by $\tau_{(q_1, q_2)}$.

1.3. The principal series representations of $GS(2, \mathbf{R})$. We fix a minimal parabolic subgroup $P_{\min}$ of G with a Langlands decomposition $P_{\min} = MAN$ as follows:

$$M := \langle \gamma_i \mid i = 0, 1, 2 \rangle; \quad A := \{ \text{diag}(a_0 a_1, a_0 a_2, a_1^{-1}, a_2^{-1}) \mid a_i > 0 \}; \quad N := \mathbf{N}_{\mathbf{R}},$$

where we define the generators γ_i of M by

$$\gamma_0 = \text{diag}(-1, -1, 1, 1), \quad \gamma_1 = \text{diag}(-1, 1, -1, 1), \quad \gamma_2 = \text{diag}(1, -1, 1, -1).$$

Take a character $\sigma : M \rightarrow \mathbf{C}^{\times}$. For each $\nu = (\nu_0, \nu_1, \nu_2) \in \mathbf{C}^3$, we define a character $\exp(\nu)$ of A by

$$\exp(\nu)(a) := \prod_{i=0}^2 a_i^{\nu_i}, \quad \text{for } a = \text{diag}(a_0 a_1, a_0 a_2, a_1^{-1}, a_2^{-1}) \in A.$$

Then the modulus character of P is given by $\exp(2\rho)$, where $\rho = (3/2, 2, 1) \in \mathbf{C}^3$. We call an induced representation

$$I(\sigma, \nu) := C^{\infty}\text{-Ind}_{P_{\min}}^G (\sigma \otimes \exp(\nu + \rho) \otimes 1_N)$$

the *principal series representation* of G . The representation space of $I(\sigma, \nu)$ is given by

$$\{ F : G \longrightarrow \mathbf{C} \mid C^{\infty}\text{-class}, F(mang) = \sigma(m) \exp(\nu + \rho)(a) F(g), \\ \forall (m, a, n, g) \in M \times A \times N \times G \},$$

on which G acts by right translation. For $(\nu_1, \nu_2) \in \mathbf{C}^2$ in a general position, the representation $I(\sigma, \nu)$ of $\mathbf{G}_{\mathbf{R}}$ is irreducible. Since the location of K_0 -types in $I(\sigma, \nu)$ varies depending on the character σ of M , we separate our argument as follows: we say that we are in **case 1** (resp. **case 2**, **case 3**, and **case 4**) if the irreducible $(\mathfrak{g}, K)$ -module Π_{∞} is equivalent to $I(\sigma, \nu)$ with $\sigma(\gamma_1) = \sigma(\gamma_2) = 1$ (resp. $\sigma(\gamma_1) = \sigma(\gamma_2) = -1$, $\sigma(\gamma_1) = -\sigma(\gamma_2) = 1$, and $\sigma(\gamma_1) = -\sigma(\gamma_2) = -1$). Further we divide each **case k** ($1 \leq k \leq 4$) into two subclasses **case k-(i)** and **case k-(ii)** according as $\sigma(\gamma_0) = 1$ or $\sigma(\gamma_0) = -1$.

Next we describe equivalence between the principal series representations. Let $W(G, MA)$ be the Weyl group of G for the Cartan subgroup MA , which is generated

by the following two elements:

$$w_1 := \left(\begin{array}{c|c} 1 & \\ \hline 1 & 1 \\ \hline & 1 \end{array} \right), \quad w_2 := \left(\begin{array}{c|c} 1 & \\ \hline & -1 \\ \hline 1 & 1 \end{array} \right).$$

For each $w \in W(G, MA)$, we define characters $w\sigma : M \rightarrow \mathbf{C}^\times$ and $\exp(w\nu) : A \rightarrow \mathbf{C}^\times$ by $(w\sigma)(m) := \sigma(w^{-1}mw)$ and $\exp(w\nu)(a) := \exp(\nu)(w^{-1}aw)$. Then we have

Lemma 1.1. (i) *If a principal series $I(\sigma, \nu)$ is irreducible as a $(\mathfrak{g}_0, K_0)$ -module, then so is $I(w\sigma, w\nu)$ for each $w \in W(G, MA)$.*

(ii) *Suppose that principal series representations $I(\sigma, \nu)$ and $I(\sigma', \nu')$ of G are irreducible as $(\mathfrak{g}_0, K_0)$ -modules. Then $I(\sigma, \nu)$ is equivalent to $I(\sigma', \nu')$ as $(\mathfrak{g}, K)$ -modules if and only if there exists an element $w \in W(G, MA)$ such that $w\sigma = \sigma'$ and $w\nu = \nu'$.*

Proof. (i) This is immediate from [Kn, Proposition 10.18]. (ii) Consider, for example, the case where $\sigma(\gamma_j) = \sigma'(\gamma_j) = -1$ ($j = 1, 2$). Suppose that there exists a $(\mathfrak{g}, K_0)$ -isomorphism $\Phi : I(\sigma, \nu) \rightarrow I(\sigma', \nu')$. Then there exists an element $w \in W(G, MA)$ such that $w\nu = \nu'$ and $(w\sigma)(\gamma_i) = \sigma'(\gamma_i)$ ($i = 1, 2$) by [Kn, Proposition 10.18]. Let $v_0, v'_0 \in I(\sigma, \nu)$ be as in (2.10) in the next section. It follows from the irreducibility of $I(\sigma, \nu)$ that Φ commutes with the action of K if and only if $\Phi(\gamma_0 v_0) = \gamma_0 \Phi(v_0)$. By (2.11b) in the next section, we have

$$\begin{aligned} \Phi(\gamma_0 v_0) &= \sigma(\gamma_0) \Phi(v'_0) = \frac{\sigma(\gamma_0)}{\nu_1 \nu_2} (X_{(-2,0)} \cdot X_{(0,-2)} - \frac{1}{4} X_{(-1,-1)}^2) \Phi(v_0), \\ \gamma_0 \Phi(v_0) &= \frac{\sigma'(\gamma_0)}{\nu_1^w \nu_2^w} (X_{(-2,0)} \cdot X_{(0,-2)} - \frac{1}{4} X_{(-1,-1)}^2) \Phi(v_0), \end{aligned}$$

where we write $w\nu = (\nu_0^w, \nu_1^w, \nu_2^w)$. Note that $\nu_1 \nu_2 \neq 0$ by the irreducibility of $I(\sigma, \nu)$ as a $(\mathfrak{g}_0, K_0)$ -module. From this, we can easily check that $\Phi(\gamma_0 v_0) = \gamma_0 \Phi(v_0)$ holds if and only if $w\sigma(\gamma_0) = \sigma'(\gamma_0)$. The other pairs (σ, σ') can be treated similarly. The details will be left to the reader. $\square$

From Lemma 1.1, we may concentrate our attention on the **cases 1, 2-(i), and 3-(i)**.

1.4. The main result. The space of cusp forms on $G_{\mathbf{A}}$ is denoted by $\mathcal{A}^{cusp}(G_{\mathbf{Q}} \backslash G_{\mathbf{A}})$. Let $\Pi = \otimes'_v \Pi_v$ be a cuspidal automorphic representation of $G_{\mathbf{A}}$. We regard Π as a subspace of $\mathcal{A}^{cusp}(G_{\mathbf{Q}} \backslash G_{\mathbf{A}})$. Define a character $\psi = \prod_v \psi_v : \mathbf{N}_{\mathbf{A}} \rightarrow \mathbf{C}^{(1)}$ of $\mathbf{N}_{\mathbf{A}}$ by

$$\psi(n(x_0, x_1, x_2, x_3)) := \mathbf{e}_{\mathbf{A}}(x_0 + x_3).$$

Here $\mathbf{e}_{\mathbf{A}} : \mathbf{A}/\mathbf{Q} \rightarrow \mathbf{C}^{(1)}$ is the additive character of $\mathbf{A}$ characterized by $\mathbf{e}_{\mathbf{A}}(t_\infty) = \exp(2\pi\sqrt{-1}t_\infty)$ ($t_\infty \in \mathbf{R}$). Note that our definition of ψ differs from that in the previous paper [Mo1]. For an automorphic form F on $G_{\mathbf{A}}$, the *global Whittaker function* $\mathcal{W}_F$ attached to F is defined by

$$\mathcal{W}_F(g) := \int_{\mathbf{N}_{\mathbf{Q}} \backslash \mathbf{N}_{\mathbf{A}}} F(n g) \psi(n^{-1}) dn, \quad g \in G_{\mathbf{A}}.$$

Here dn stands for the usual $\mathbf{N}_{\mathbf{A}}$ -invariant measure on $\mathbf{N}_{\mathbf{Q}} \backslash \mathbf{N}_{\mathbf{A}}$.

In this paper we focus our attention on the cuspidal automorphic representation Π of $\mathbf{G}_{\mathbf{A}}$ satisfying the following fundamental assumptions **A.1** and **A.2**:

A.1 For some cusp form F in Π , the global Whittaker function $\mathcal{W}_F$ attached to F does not vanish.

A.2 The $(\mathfrak{g}, K)$ -module Π_{∞} is equivalent to an irreducible principal series representation $I(\sigma, \nu)$, which is also irreducible as a $(\mathfrak{g}_0, K_0)$ -module.

We denote the central character of Π by ω_{Π} , that is, $\omega(z1_4) = \omega_{\Pi}(z)\text{id}_{\Pi}$ ($z \in \mathbf{A}^{\times}$). Define a complex number $\omega_{\infty} \in \mathbf{C}$ by $\omega_{\Pi}(t) = t^{\omega_{\infty}}$ ($t > 0$). Then we have $\omega_{\infty} = 2\nu_0 - \nu_1 - \nu_2$. By the assumption **A.1**, for each finite place $p < \infty$, the irreducible admissible representation Π_p of $\mathbf{G}_{\mathbf{Q}_p}$ has a local Whittaker model. Hence we can attach the local L -factor $L(s, \Pi_p)$ and ϵ -factor $\epsilon(s, \Pi_p, \psi_p)$ to Π_p as in [Mo1, Proposition 4] (cf. [TB1, Theorem 3.6]). Define the spinor L -function $L(s, \Pi)$ of Π by $L(s, \Pi) := \prod_{p < \infty} L(s, \Pi_p)$, where p runs through all the finite places of $\mathbf{Q}$. As was remarked in [Mo1, Proposition 3], the product converges absolutely for $\text{Re}(s) > (5 - \text{Re}(\omega_{\infty}))/2$. We define the Gamma factor $L(s, \Pi_{\infty})$ of $\Pi_{\infty} = I(\sigma, \nu)$ by

$$(1.1) \quad \begin{aligned} L(s, \Pi_{\infty}) := & \Gamma_{\mathbf{R}}\left(s + \frac{\omega_{\infty} + \nu_1 + \nu_2}{2} + \delta_1\right) \Gamma_{\mathbf{R}}\left(s + \frac{\omega_{\infty} - \nu_1 + \nu_2}{2} + \delta_2\right) \\ & \times \Gamma_{\mathbf{R}}\left(s + \frac{\omega_{\infty} + \nu_1 - \nu_2}{2} + \delta_3\right) \Gamma_{\mathbf{R}}\left(s + \frac{\omega_{\infty} - \nu_1 - \nu_2}{2} + \delta_4\right), \end{aligned}$$

where $\delta_i \in \{0, 1\}$ is given by

$$(-1)^{\delta_1} = \sigma(\gamma_0), \quad (-1)^{\delta_2} = \sigma(\gamma_0\gamma_1), \quad (-1)^{\delta_3} = \sigma(\gamma_0\gamma_2), \quad (-1)^{\delta_4} = \sigma(\gamma_0\gamma_1\gamma_2).$$

Then we have the completed spinor L -function $\Lambda(s, \Pi)$:

$$\Lambda(s, \Pi) := L(s, \Pi_{\infty}) \times L(s, \Pi).$$

The ϵ -factor $\epsilon(s, \Pi_{\infty}, \psi_{\infty})$ for Π_{∞} is defined by

$$(1.2) \quad \epsilon(s, \Pi_{\infty}, \psi_{\infty}) := (\sqrt{-1})^{\delta_1 + \delta_2 + \delta_3 + \delta_4} = \begin{cases} 1 & \text{Case 1,} \\ -1 & \text{Cases 2, 3, 4.} \end{cases}$$

The local L - and ϵ -factors of Π_{∞} given by (1.1) and (1.2) coincide with those defined from the Langlands parameter of Π_{∞} (cf. [Bo]). The global ϵ -factor $\epsilon(s, \Pi)$ is given by $\epsilon(s, \Pi) := \prod_v \epsilon(s, \Pi_v, \psi_v)$, where the product is taken over all the places v of $\mathbf{Q}$. For a cusp form $F \in \Pi$, we define a cusp form $\tilde{F}$ as in the previous paper [Mo1, p. 905]. Then $\Pi^{\vee} := \{\tilde{F} \mid F \in \Pi\}$ is a cuspidal automorphic representation of $\mathbf{G}_{\mathbf{A}}$ and is equivalent to the restricted tensor product $\otimes'_v \Pi_v^{\vee}$, where $\Pi_v^{\vee}$ stands for the contragredient representation of Π_v . This follows from [TB1, Proposition 2.3] and the assumption **A.2**. Since $\Pi^{\vee}$ also satisfies the assumptions **A.1** and **A.2**, we can define the spinor L -function $L(s, \Pi^{\vee}) = \prod_{p < \infty} L(s, \Pi_p^{\vee})$ and the completed L -function $\Lambda(s, \Pi^{\vee})$. Now we state the main result of this paper:

Theorem 1.2. Suppose that Π is a cuspidal automorphic representation of $\mathbf{G}_{\mathbf{A}}$ satisfying the assumptions **A.1** and **A.2**. Then

- (i) the completed L -function $\Lambda(s, \Pi)$ of Π is continued to an entire function of $s \in \mathbf{C}$;
- (ii) we have the functional equation

$$\Lambda(s, \Pi) = \epsilon(s, \Pi) \Lambda(1 - s, \Pi^{\vee}).$$

2. EXPLICIT FORMULAE OF LOCAL WHITTAKER FUNCTIONS ON $GS(2, \mathbf{R})$

In this section, we shall give an explicit integral expression of local Whittaker functions belonging to an irreducible principal series representation $\Pi_\infty = I(\sigma, \nu)$ of $\mathbf{G}_{\mathbf{R}} = GS(2, \mathbf{R})$, which enables us to compute Novodvorsky's local zeta integrals in the next section. In the first subsection, we collect basic facts on local Whittaker functions on $\mathbf{G}_{\mathbf{R}}$. Then we shall give the explicit formula for each case.

2.1. Local Whittaker functions. We recall some basic facts on local Whittaker functions for real reductive groups, in our case, $\mathbf{G}_{\mathbf{R}}$. Firstly we introduce the space

$$C^\infty(\mathbf{N}_{\mathbf{R}} \backslash \mathbf{G}_{\mathbf{R}}; \psi_\infty) := \{W : \mathbf{G}_{\mathbf{R}} \rightarrow \mathbf{C} \mid C^\infty\text{-class}, \\ W(ng) = \psi_\infty(n)W(g), \quad (n, g) \in \mathbf{N}_{\mathbf{R}} \times \mathbf{G}_{\mathbf{R}}\},$$

on which the group $\mathbf{G}_{\mathbf{R}}$ acts by right translation. The restriction of a global Whittaker function $\mathcal{W}_F$ to $\mathbf{G}_{\mathbf{R}}$ is of moderate growth; that is, there exist constants $C > 0, M > 0$ such that $|\mathcal{W}_F(g)| < C\|g\|^M$ ($g \in \mathbf{G}_{\mathbf{R}}$), where we set $\|g\| := \max\{g_{i,j}, (g^{-1})_{i,j} \mid 1 \leq i, j \leq 4\}$. Hence we also consider the subspace

$$C_{mg}^\infty(\mathbf{N}_{\mathbf{R}} \backslash \mathbf{G}_{\mathbf{R}}; \psi_\infty) := \{W \in C^\infty(\mathbf{N}_{\mathbf{R}} \backslash \mathbf{G}_{\mathbf{R}}; \psi_\infty) \mid W \text{ is of moderate growth}\}$$

of $C^\infty(\mathbf{N}_{\mathbf{R}} \backslash \mathbf{G}_{\mathbf{R}}; \psi_\infty)$. Wallach's multiplicity one theorem [Wa, Theorem 8.8 (1)] asserts that for an arbitrary irreducible $(\mathfrak{g}, K)$ -module π_∞ we have

$$(2.1) \quad \dim_{\mathbf{C}} \operatorname{Hom}_{\mathfrak{g}, K}(\pi_\infty, C_{mg}^\infty(\mathbf{N}_{\mathbf{R}} \backslash \mathbf{G}_{\mathbf{R}}; \psi_\infty)) \leq 1.$$

If there is a non-zero intertwining operator $\Psi \in \operatorname{Hom}_{\mathfrak{g}, K}(\pi_\infty, C_{mg}^\infty(\mathbf{N}_{\mathbf{R}} \backslash \mathbf{G}_{\mathbf{R}}; \psi_\infty))$, then we say π_∞ is *generic* and call the image $W_v := \Psi(v)$ of $v \in \pi_\infty$ the *local Whittaker function corresponding to $v \in \pi_\infty$* . We set $\operatorname{Wh}(\pi_\infty, \psi_\infty) := \{W_v \mid v \in \pi_\infty\}$ and call it the *Whittaker model of π_∞ with respect to ψ_∞* . For each vector $v \in \pi_\infty$, we define a function $\phi_v(a)$ on $A_0 := \{\operatorname{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_i > 0\}$ by

$$W_v(a) = a_1^2 a_2^{-1} \phi_v(a), \quad a = \operatorname{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \in A_0.$$

Here we note the following:

Lemma 2.1. *For each local Whittaker function $W_v \in \operatorname{Wh}(\pi_\infty, \psi_\infty)$, its restriction $W_v|_{A_0}$ to A_0 can be expressed as*

$$(2.2) \quad W_v|_{A_0}(\operatorname{diag}(a_1, a_2, a_1^{-1}, a_2^{-1})) \\ = a_1^2 a_2^{-1} \cdot \int_{L(\sigma_1)} \frac{ds_1}{2\pi\sqrt{-1}} \left(\frac{\pi a_1}{a_2}\right)^{-s_1} \int_{L(\sigma_2)} \frac{ds_2}{2\pi\sqrt{-1}} (\pi a_2^2)^{-s_2} V_v(s_1, s_2),$$

where $V_v(s_1, s_2)$ is a holomorphic function on a region $\{(s_1, s_2) \in \mathbf{C}^2 \mid \operatorname{Re}(s_j) > \sigma_j - \epsilon \ (j = 1, 2)\}$ for some $\sigma_j \in \mathbf{R}$ and $\epsilon > 0$.

Proof. By a well-known result of Harish-Chandra ([HC, Lemma 14, page 15]), $W_v(g)$ is uniformly of moderate growth; that is, there exists a constant $M > 0$ such that the functions

$$|W_v(g; X)| \cdot \|a\|^{-M}$$

on $\mathbf{G}_{\mathbf{R}}$ are bounded for all $X \in U(\mathfrak{g}_0)$ (M is independent of X). On the other hand, we have

$$W_v(a; X) = (\psi_\infty)_*(\operatorname{Ad}(a)X)W_v(a), \quad X \in \mathfrak{n} := \operatorname{Lie}(\mathbf{N}_{\mathbf{R}}).$$

Here $(\psi_\infty)_* \in \mathfrak{n}_{\mathbf{C}}^*$ stands for the differential of the character $\psi_\infty : \mathbf{N}_{\mathbf{R}} \rightarrow \mathbf{C}^{(1)}$. Hence the function

$$\left(\frac{\pi a_1}{a_2}\right)^l (\pi a_2^2)^m \cdot W_v(a) \cdot \|a\|^{-M}$$

on A_0 is bounded for each $l, m \geq 0$. It is easy to see that

$$\|a\| \leq \pi^{3/2} \cdot \max\left\{\frac{\pi a_1}{a_2}, \left(\frac{\pi a_1}{a_2}\right)^{-1}\right\} \cdot \max\{\pi a_2^2, (\pi a_2^2)^{-1}\}^{1/2}, \quad \forall a \in A_0.$$

Therefore, for each $l, m \geq 0$, there exists a constant $C_{l,m} > 0$ such that the inequality

$$|W_v(a)| \leq C_{l,m} \left(\frac{\pi a_1}{a_2}\right)^{-l} \max\left\{\frac{\pi a_1}{a_2}, \left(\frac{\pi a_1}{a_2}\right)^{-1}\right\}^M (\pi a_2^2)^{-m} \max\{\pi a_2^2, (\pi a_2^2)^{-1}\}^{M/2}$$

holds. This estimate implies that the integral

$$\int_0^\infty d^\times a_1 \int_0^\infty d^\times a_2 (a_1^2 a_2)^{-1} W_v(a) \left(\frac{\pi a_1}{a_2}\right)^{\sigma_1} (\pi a_2^2)^{\sigma_2}$$

converges absolutely for $\sigma_1 > M + 2$ and $\sigma_2 > (M + 3)/2$. Now Lemma 2.1 follows from the Mellin inversion formula. $\square$

We also recall the dimension of the Whittaker functionals *without* moderate growth condition for a (not necessarily irreducible) principal series representation $I(\sigma, \nu)$. It follows from [G-W, Theorem 5.2] (cf. [Wa, 8.7]), that the natural injective map

$$\mathrm{Hom}_{\mathfrak{g}, K}(I(\sigma, \nu), C^\infty(\mathbf{N}_{\mathbf{R}} \backslash \mathbf{G}_{\mathbf{R}}; \psi_\infty)) \rightarrow \mathrm{Hom}_{\mathfrak{n}}(I(\sigma, \nu), (\psi_\infty)_*)$$

is an isomorphism. By [Ko, Theorem 5.5], the dimension of the latter space is $\sharp W(G_0, A_0) = 8$, the order of the Weyl group for G_0 . Therefore we have

$$(2.3) \quad \dim_{\mathbf{C}} \mathrm{Hom}_{\mathfrak{g}, K}(I(\sigma, \nu), C^\infty(\mathbf{N}_{\mathbf{R}} \backslash \mathbf{G}_{\mathbf{R}}; \psi_\infty)) = 8.$$

By Lemma 2.1, in order to know the values $W_v(a)$ of a Whittaker function on A_0 , we have only to determine the function $V_v(s_1, s_2)$. For this purpose, it is convenient to introduce some products of Gamma functions:

$$\begin{aligned} & Q_0(t_1, t_2; \alpha_1, \alpha_2, \alpha_3, \alpha_4; \nu_1, \nu_2) \\ &:= \Gamma\left[\frac{t_1}{2} + \frac{\nu_1 - \nu_2}{4} + \alpha_1, \frac{t_1}{2} - \frac{\nu_1 - \nu_2}{4} + \alpha_2, \frac{t_2}{2} + \frac{\nu_1 + \nu_2}{4} + \alpha_3, \frac{t_2}{2} - \frac{\nu_1 + \nu_2}{4} + \alpha_4\right]; \\ & R_0(s, t) \equiv R_0(s_1, s_2, t_1, t_2) := \Gamma\left[\frac{s_1}{2}, \frac{s_1 - t_1 - t_2}{2}, \frac{s_2 - t_1}{2}, \frac{s_2 - t_2}{2}\right]. \end{aligned}$$

Moreover we set

$$Q_0(t) \equiv Q_0(t_1, t_2) \equiv Q_0(t_1, t_2; \nu_1, \nu_2) := Q_0(t_1, t_2; 0, 0, 0, 0; \nu_1, \nu_2).$$

2.2. Case 1. In this subsection, we suppose that Π_∞ is equivalent to an irreducible principal series representation $I(\sigma, \nu)$ with $\sigma(\gamma_i) = 1$ ($1 \leq i \leq 2$). Let us define three vectors v_0, v'_1 , and v''_1 in $I(\sigma, \nu)$ by

$$(2.4) \quad v_0(k_{A,B}) = 1 \ (\forall k_{A,B} \in K_0), \quad v'_1(g) := v_0(g; X_{(1,1)}), \quad v''_1(g) := v_0(g; X_{(-1,-1)}).$$

Then we have the following explicit formula for the values of the functions W_v ($v \in \{v_0, v'_1, v''_1\}$) on A_0 :

Proposition 2.2. *For $v \in \{v_0, v'_1, v''_1\}$, the restrictions $W_v|_{A_0}$ of the Whittaker functions corresponding to v are given by (2.2) with*

$$(2.5) \quad V_v(s_1, s_2) = \int_{L(\tau_1)} \frac{dt_1}{2\pi\sqrt{-1}} \int_{L(\tau_2)} \frac{dt_2}{2\pi\sqrt{-1}} U_v(s_1, s_2, t_1, t_2),$$

where we set

$$U_{v_0}(s_1, s_2, t_1, t_2) := C \cdot Q_0(t_1, t_2; \nu_1, \nu_2) R_0(s_1, s_2, t_1, t_2)$$

and

$$U_{v'_1}(s_1, s_2, t_1, t_2) = U_{v''_1}(s_1, s_2, t_1, t_2) := 4\sqrt{-1}U_{v_0}(s_1 + 1, s_2, t_1, t_2)$$

with a constant $C \in \mathbf{C}^\times$. Here $\sigma_j, \tau_j \in \mathbf{R}$ in (2.2) and (2.5) are taken so that

$$\sigma_1 > \tau_1 + \tau_2, \quad \sigma_2 > \max\{\tau_1, \tau_2\} \quad \tau_1 > |\operatorname{Re}(\nu_1 - \nu_2)|/2, \quad \tau_2 > |\operatorname{Re}(\nu_1 + \nu_2)|/2.$$

Proof.

Step 1. Since it is easy to see that

$$W_{v'_1}(a) = W_{v''_1}(a) = 4\sqrt{-1}\left(\frac{\pi a_1}{a_2}\right)W_{v_0}(a),$$

we may concentrate our attention on $W_{v_0}(a)$. Miyazaki and Oda [Mi-O1, Theorem 10.1] constructed the following system of partial differential equations satisfied by $\phi_{v_0}(a)$:

$$(2.6) \quad \left\{ \partial_1^2 + \partial_2^2 - 8\left(\frac{\pi a_1}{a_2}\right)^2 - 16\pi^2 a_2^4 \right\} \phi_{v_0}(a) = (\nu_1^2 + \nu_2^2) \phi_{v_0}(a);$$

$$(2.7) \quad \left\{ (\partial_1^2 - 1)(\partial_2^2 - 1 - 16\pi^2 a_2^4) + 8\left(\frac{\pi a_1}{a_2}\right)^2 (\partial_1 + 1)(\partial_2 - 1) + 16\left(\frac{\pi a_1}{a_2}\right)^4 \right\} \phi_{v_0}(a)$$

$$(2.8) \quad = (\nu_1^2 - 1)(\nu_2^2 - 1) \phi_{v_0}(a),$$

where we set $\partial_i := a_i \frac{\partial}{\partial a_i}$. Note that there are some misprints in [Mi-O1]. We thank Takuya Miyazaki for informing us of the misprints.

Step 2. Next we shall prove that the natural injection from the intertwining space $\operatorname{Hom}_{\mathfrak{g}, K}(\Pi_\infty, C^\infty(\mathbf{N}_{\mathbf{R}} \backslash \mathbf{G}_{\mathbf{R}}; \psi_\infty))$ to the space $\mathcal{S}(\Pi_\infty; \psi_\infty)$ of C^∞ -solutions of the system of the differential equations (2.6) and (2.7) is an isomorphism. Firstly we note that the space $\mathcal{S}(\Pi_\infty; \psi_\infty)$ is at most 8-dimensional. Indeed, if we define $\phi^{(k)}(a)$ ($0 \leq k \leq 7$) by $\phi^{(4i+j)}(a) = \partial_1^i \partial_2^j \phi_{v_0}(a)$ ($0 \leq i \leq 1, 0 \leq j \leq 3$), then we know from the equations (2.6) and (2.7) that there exists a set of functions $M_{k,l}^1(a), M_{k,l}^2(a) \in \mathbf{C}[a_1^{\pm 1}, a_2^{\pm 1}]$ ($0 \leq k, l \leq 7$) such that

$$\partial_1 \phi^{(l)}(a) = \sum_{k=0}^7 M_{k,l}^1(a) \phi^{(k)}(a), \quad \partial_2 \phi^{(l)}(a) = \sum_{k=0}^7 M_{k,l}^2(a) \phi^{(k)}(a).$$

Hence we have $\dim_{\mathbf{C}} \mathcal{S}(\Pi_\infty; \psi_\infty) \leq 8$ (cf. [Kn, Theorems B.8 and B.9]). In view of (2.3), we have

$$8 = \dim_{\mathbf{C}} \operatorname{Hom}_{\mathfrak{g}, K}(\Pi_\infty, C^\infty(\mathbf{N}_{\mathbf{R}} \backslash \mathbf{G}_{\mathbf{R}}; \psi_\infty)) \leq \dim_{\mathbf{C}} \mathcal{S}(\Pi_\infty; \psi_\infty) \leq 8.$$

Hence we have the assertion.

Step 3. By (2.1), we have only to check that our formula in the proposition belongs to $\mathcal{S}(\Pi_\infty; \psi_\infty)$ and defines a function on $\mathbf{G}_{\mathbf{R}}$ of moderate growth. The second assertion can be easily verified by using Stirling's formula. The equation (2.6) follows from the following computation:

$$\begin{aligned}
& (s_1^2 - 2s_1s_2 + 2s_2^2)V_{v_0}(s_1, s_2) - 4V_{v_0}(s_1 + 2, s_2) - 8V_{v_0}(s_1, s_2 + 2) \\
&= \int dt_1 \int dt_2 \{s_1^2 - 2s_1s_2 + 2s_2^2 - s_1(s_1 - t_1 - t_2) - 2(s_2 - t_1)(s_2 - t_2)\} \\
&\quad \times Q_0(t)R_0(s, t) \\
&= \int dt_1 \int dt_2 [-(s_1 - t_1 - t_2)\{(s_2 - t_1) + (s_2 - t_2)\} + t_1^2 + t_2^2] Q_0(t)R_0(s, t) \\
&= \int dt_1 \int dt_2 Q_0(t) \{-4R_0(s_1, s_2, t_1 - 2, t_2) - 4R_0(s_1, s_2, t_1, t_2 - 2) \\
&\quad + (t_1^2 + t_2^2)R_0(s_1, s_2, t_1, t_2)\} \\
&= \int dt_1 \int dt_2 \{-4Q_0(t_1 + 2, t_2) - 4Q_0(t_1, t_2 + 2) + (t_1^2 + t_2^2)Q_0(t_1, t_2)\} R_0(s, t) \\
&= \int dt_1 \int dt_2 \left\{ -\left(t_1 + \frac{\nu_1 - \nu_2}{2}\right)\left(t_1 - \frac{\nu_1 - \nu_2}{2}\right) \right. \\
&\quad \left. - \left(t_2 + \frac{\nu_1 + \nu_2}{2}\right)\left(t_2 - \frac{\nu_1 + \nu_2}{2}\right) + (t_1^2 + t_2^2) \right\} Q_0(t)R_0(s, t) \\
&= \frac{1}{2}(\nu_1^2 + \nu_2^2)V_{v_0}(s_1, s_2).
\end{aligned}$$

Here and below we use an abbreviated expression:

$$(2.9) \quad \int dt_1 \int dt_2 = \int_{L(\tau_1)} \frac{dt_1}{2\pi\sqrt{-1}} \int_{L(\tau_2)} \frac{dt_2}{2\pi\sqrt{-1}}.$$

In the fourth equality of the above computation, we make changes of variables $t_1 \mapsto t_1 + 2$ and $t_2 \mapsto t_2 - 2$. Similarly, the equation (2.7) follows from

$$\begin{aligned}
& (s_1^2 - 1)[\{(s_1 - 2s_2)^2 - 1\}V_{v_0}(s_1, s_2) - 16V_{v_0}(s_1, s_2 + 2)] \\
&\quad - 8(s_1 + 1)(s_1 - 2s_2 + 1)V_{v_0}(s_1 + 2, s_2) + 16V_{v_0}(s_1 + 4, s_2) \\
&= \int dt_1 \int dt_2 [(s_1 - t_1 - t_2 + 2)(s_1 - t_1 - t_2) \\
&\quad \times \{(s_2 - t_1 + 2)(s_2 - t_1) + (s_2 - t_2 + 2)(s_2 - t_2) - 2(s_2 - t_1)(s_2 - t_2)\} \\
&\quad - 2(s_1 - t_1 - t_2)\{(s_2 - t_1)((t_1 - 1)^2 - t_2^2) - (s_2 - t_2)(t_1^2 - (t_2 - 1)^2)\} \\
&\quad + (t_1^2 - t_2^2)^2 - 2(t_1^2 + t_2^2) + 1] Q_0(t)R_0(s, t). \\
&= \int dt_1 \int dt_2 Q_0(t) [16\{R_0(s_1, s_2, t_1 - 4, t_2) + R_0(s_1, s_2, t_1, t_2 - 4) \\
&\quad - 2R_0(s_1, s_2, t_1 - 2, t_2 - 2)\} - 8((t_1 - 1)^2 - t_2^2)R_0(s_1, s_2, t_1 - 2, t_2) \\
&\quad + 8(t_1^2 - (t_2 - 1)^2)R_0(s_1, s_2, t_1, t_2 - 2) + \{(t_1^2 - t_2^2)^2 - 2(t_1^2 + t_2^2) + 1\}R_0(s, t)]
\end{aligned}$$

$$\begin{aligned}
&= \int dt_1 \int dt_2 \left[16 \{ Q_0(t_1 + 4, t_2) + Q_0(t_1, t_2 + 4) - 2Q_0(t_1 + 2, t_2 + 2) \} \right. \\
&\quad \left. - 8((t_1 + 1)^2 - t_2^2)Q_0(t_1 + 2, t_2) + 8(t_1^2 - (t_2 + 1)^2)Q_0(t_1, t_2 + 2) \right. \\
&\quad \left. + \{ (t_1^2 - t_2^2)^2 - 2(t_1^2 + t_2^2) + 1 \} Q_0(t_1, t_2) \right] R_0(s, t) \\
&= (\nu_1^2 - 1)(\nu_2^2 - 1)V_{v_0}(s_1, s_2).
\end{aligned}$$

□

2.3. Case 2. In this subsection, we suppose that Π_∞ is equivalent to an irreducible principal series representation $I(\sigma, \nu)$ with $\sigma(\gamma_i) = -1$ ($1 \leq i \leq 2$). We take two vectors v_0 and v'_0 in Π_∞ characterized by

$$(2.10) \quad v_0(k_{A,B}) = \det(A + \sqrt{-1}B), \quad v'_0(k_{A,B}) = \det(A - \sqrt{-1}B), \quad \forall k_{A,B} \in K_0.$$

Then $\mathbf{C} \cdot v_0$ (resp. $\mathbf{C} \cdot v'_0$) is a multiplicity one K_0 -type $\tau_{(1,1)}$ (resp. $\tau_{(-1,-1)}$) occurring in Π_∞ . The following proposition gives an explicit formula for the values of the functions W_{v_0} and $W_{v'_0}$ on A_0 :

Proposition 2.3. *For $v \in \{v_0, v'_0\}$, the restrictions $W_v|_{A_0}$ of Whittaker functions corresponding to v are given by the formulae (2.2) and (2.5) with*

$$\begin{aligned}
&U_{v_0}(s_1, s_2, t_1, t_2) \\
&= C \cdot \{ Q_0(t_1, t_2 - 1; \nu_1, \nu_2) + Q_0(t_1 - 1, t_2; \nu_1, \nu_2) \} R_0(s_1 + 1, s_2 + 1, t_1, t_2); \\
&U_{v'_0}(s_1, s_2, t_1, t_2) \\
&= C \cdot \{ Q_0(t_1, t_2 - 1; \nu_1, \nu_2) - Q_0(t_1 - 1, t_2; \nu_1, \nu_2) \} R_0(s_1 + 1, s_2 + 1, t_1, t_2).
\end{aligned}$$

Here $C \in \mathbf{C}^\times$ is a constant common to v_0 and v'_0 , and $\sigma_j, \tau_j \in \mathbf{R}$ in (2.2) and (2.5) are taken so that

$$\begin{aligned}
\sigma_1 + 1 &> \tau_1 + \tau_2, & \sigma_2 + 1 &> \max\{\tau_1, \tau_2\}, \\
\tau_1 - 1 &> |\operatorname{Re}(\nu_1 - \nu_2)|/2, & \tau_2 - 1 &> |\operatorname{Re}(\nu_1 + \nu_2)|/2.
\end{aligned}$$

Proof.

Step 1. Firstly we show that, for all $g \in \mathbf{G}_\mathbf{R}$,

$$(2.11a) \quad v'_0(g; X_{(2,0)} \cdot X_{(0,2)} - \frac{1}{4}X_{(1,1)}^2) = \nu_1 \nu_2 v_0(g);$$

$$(2.11b) \quad v_0(g; X_{(-2,0)} \cdot X_{(0,-2)} - \frac{1}{4}X_{(-1,-1)}^2) = \nu_1 \nu_2 v'_0(g).$$

It is easy to see that $\Pi_\infty(X_{(2,0)} \cdot X_{(0,2)} - \frac{1}{4}X_{(1,1)}^2)v'_0$ is annihilated by $\kappa_*\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$ and $\kappa_*\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$ and has weight $e_1 + e_2$ with respect to $\mathfrak{h}_\mathbf{C}$. Hence it is a constant multiple of v_0 . Since

$$\begin{aligned}
X_{(2,0)} \cdot X_{(0,2)} - \frac{1}{4}X_{(1,1)}^2 &\equiv (H_1 - 1) \cdot \{ H_2 + \kappa_*\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) \} + H_2 \cdot \kappa_*\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \\
&\quad + \kappa_*\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \cdot \kappa_*\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) \pmod{\mathfrak{n}U(\mathfrak{g})}
\end{aligned}$$

with $H_1 := \operatorname{diag}(1, 0, -1, 0)$ and $H_2 := \operatorname{diag}(0, 1, 0, -1) \in \mathfrak{g}$, we have

$$v'_0(1_4; X_{(2,0)} \cdot X_{(0,2)} - \frac{1}{4}X_{(1,1)}^2) = (\nu_1 + 2 - 1)(\nu_2 + 1 - 1) - (\nu_2 + 1) + 1 = \nu_1 \nu_2.$$

This proves (2.11a). We can check (2.11b) in the same way. From (2.11), we obtain the following differential equations for $\phi_{v_0}(a)$ and $\phi_{v'_0}(a)$:

$$(2.12) \quad \begin{cases} \left\{ \partial_1 \partial_2 + 4 \left(\frac{\pi a_1}{a_2} \right)^2 - 4\pi a_2^2 \partial_1 \right\} \phi_{v'_0}(a) = \nu_1 \nu_2 \phi_{v_0}(a); \\ \left\{ \partial_1 \partial_2 + 4 \left(\frac{\pi a_1}{a_2} \right)^2 + 4\pi a_2^2 \partial_1 \right\} \phi_{v_0}(a) = \nu_1 \nu_2 \phi_{v'_0}(a). \end{cases}$$

On the other hand, by [Mi-O1, Theorem 10.1], we have the following differential equations for $\phi_{v_0}(a)$ and $\phi_{v'_0}(a)$ arising from the Casimir element of $\mathfrak{g}_0$:

$$(2.13) \quad \begin{cases} \left\{ \partial_1^2 + \partial_2^2 - 8 \left(\frac{\pi a_1}{a_2} \right)^2 - 16(\pi a_2^2)^2 + 8\pi a_2^2 - (\nu_1^2 + \nu_2^2) \right\} \phi_{v_0}(a) = 0; \\ \left\{ \partial_1^2 + \partial_2^2 - 8 \left(\frac{\pi a_1}{a_2} \right)^2 - 16(\pi a_2^2)^2 - 8\pi a_2^2 - (\nu_1^2 + \nu_2^2) \right\} \phi_{v'_0}(a) = 0. \end{cases}$$

Step 2. As in the proof of Proposition 2.2, we shall prove that the natural injection from the intertwining space $\text{Hom}_{\mathfrak{g},K}(\Pi_\infty, C^\infty(\mathbf{N}_{\mathbf{R}} \backslash \mathbf{G}_{\mathbf{R}}; \psi_\infty))$ to the space $\mathcal{S}(\Pi_\infty; \psi_\infty)$ of C^∞ -solutions of the system of the differential equations (2.12) and (2.13) is an isomorphism. For this purpose, consider the set $\mathcal{F}$ of the following eight functions on A_0 :

$$\begin{array}{cccc} \phi_{v_0}(a), & \partial_1 \phi_{v_0}(a), & \partial_2 \phi_{v_0}(a), & \partial_1^2 \phi_{v_0}(a), \\ \phi_{v'_0}(a), & \partial_1 \phi_{v'_0}(a), & \partial_2 \phi_{v'_0}(a), & \partial_1^2 \phi_{v'_0}(a). \end{array}$$

We know from (2.12) and (2.13) that, for each $\phi(a) \in \mathcal{F}$, the functions $\partial_i \phi(a)$ can be written as a linear combination of the functions in $\mathcal{F}$ with coefficients in $\mathbf{C}[a_1^{\pm 1}, a_2^{\pm 1}]$. This implies that $\dim_{\mathbf{C}} \mathcal{S}(\Pi_\infty; \psi_\infty) \leq 8$. Hence in view of (2.3), we have an isomorphism

$$\text{Hom}_{\mathfrak{g},K}(\Pi_\infty, C^\infty(\mathbf{N}_{\mathbf{R}} \backslash \mathbf{G}_{\mathbf{R}}; \psi_\infty)) \cong \mathcal{S}(\Pi_\infty; \psi_\infty).$$

Step 3. By the previous two steps, it remains to show that the formula in the proposition satisfies (2.12) and (2.13). The systems of equations (2.12) and (2.13) are equivalent to the systems

$$(2.14a) \quad \left\{ \partial_1 \partial_2 + 4 \left(\frac{\pi a_1}{a_2} \right)^2 - \nu_1 \nu_2 \right\} \phi_{v_0+v'_0}(a) + 4\pi a_2^2 \partial_1 \phi_{v_0-v'_0}(a) = 0,$$

$$(2.14b) \quad \left\{ \partial_1 \partial_2 + 4 \left(\frac{\pi a_1}{a_2} \right)^2 + \nu_1 \nu_2 \right\} \phi_{v_0-v'_0}(a) + 4\pi a_2^2 \partial_1 \phi_{v_0+v'_0}(a) = 0,$$

and

$$(2.15a) \quad \left\{ \partial_1^2 + \partial_2^2 - 8 \left(\frac{\pi a_1}{a_2} \right)^2 - 16(\pi a_2^2)^2 - (\nu_1^2 + \nu_2^2) \right\} \phi_{v_0+v'_0}(a) + 8\pi a_2^2 \phi_{v_0-v'_0}(a) = 0,$$

$$(2.15b) \quad \left\{ \partial_1^2 + \partial_2^2 - 8 \left(\frac{\pi a_1}{a_2} \right)^2 - 16(\pi a_2^2)^2 - (\nu_1^2 + \nu_2^2) \right\} \phi_{v_0-v'_0}(a) + 8\pi a_2^2 \phi_{v_0+v'_0}(a) = 0,$$

respectively. The equations (2.14a) and (2.15a) can be verified from the following computations:

$$\begin{aligned} & \{s_1(s_1 - 2s_2) + \nu_1 \nu_2\} V_{v_0+v'_0}(s_1, s_2) - 4V_{v_0+v'_0}(s_1 + 2, s_2) + 4s_1 V_{v_0-v'_0}(s_1, s_2 + 1) \\ &= \int dt_1 \int dt_2 \left[\{s_1(s_1 - 2s_2) + \nu_1 \nu_2 - (s_1 + 1)(s_1 + 1 - t_1 - t_2)\} \right. \\ & \quad \left. \times Q_0(t_1, t_2 - 1) R_0(s_1 + 1, s_2 + 1, t_1, t_2) + 4s_1 Q_0(t_1 - 1, t_2) R_0(s_1 + 1, s_2 + 2, t_1, t_2) \right] \end{aligned}$$

$$\begin{aligned}
&= \int dt_1 \int dt_2 Q_0(t_1, t_2 - 1) \left[\{s_1(s_1 - 2s_2) + \nu_1\nu_2 - (s_1 + 1)(s_1 + 1 - t_1 - t_2)\} \right. \\
&\quad \left. \times R_0(s_1 + 1, s_2 + 1, t_1, t_2) + 4s_1 R_0(s_1 + 1, s_2 + 2, t_1 + 1, t_2 - 1) \right] \\
&= \int dt_1 \int dt_2 \left[(s_1 + 1 - t_1 - t_2) \{ (s_2 + 1 - t_2) - (s_2 + 1 - t_1) \} + t_1^2 - (t_2 - 1)^2 + \nu_1\nu_2 \right] \\
&\quad \times Q_0(t_1, t_2 - 1) R_0(s_1 + 1, s_2 + 1, t_1, t_2) \\
&= \int dt_1 \int dt_2 Q_0(t_1, t_2 - 1) \left[4R_0(s_1 + 1, s_2 + 1, t_1, t_2 - 2) - 4R_0(s_1 + 1, s_2 + 1, t_1 - 2, t_2) \right. \\
&\quad \left. + \{t_1^2 - (t_2 - 1)^2 + \nu_1\nu_2\} R_0(s_1 + 1, s_2 + 1, t_1, t_2) \right] \\
&= \int dt_1 \int dt_2 \left[4Q_0(t_1, t_2 + 1) - 4Q_0(t_1 + 2, t_2 - 1) \right. \\
&\quad \left. + \{t_1^2 - (t_2 - 1)^2 + \nu_1\nu_2\} Q_0(t_1, t_2 - 1) \right] R_0(s_1 + 1, s_2 + 1, t_1, t_2) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
&\{s_1^2 + (s_1 - 2s_2)^2 - (\nu_1^2 + \nu_2^2)\} V_{v_0+v'_0}(s_1, s_2) \\
&\quad - 8V_{v_0+v'_0}(s_1 + 2, s_2) - 16V_{v_0+v'_0}(s_1, s_2 + 2) + 8V_{v_0-v'_0}(s_1, s_2 + 1) \\
&= \int dt_1 \int dt_2 \left[\{ -2(s_1 + 1 - t_1 - t_2)(2s_2 + 2 - t_1 - t_2) + 2(t_1^2 + t_2^2) - (\nu_1^2 + \nu_2^2) \right. \\
&\quad \left. + 2(-2s_2 - 1) \} Q_0(t_1, t_2 - 1) R_0(s_1 + 1, s_2 + 1, t_1, t_2) \right. \\
&\quad \left. + 8Q_0(t_1 - 1, t_2) R_0(s_1 + 1, s_2 + 2, t_1, t_2) \right] \\
&= \int dt_1 \int dt_2 \left[Q_0(t_1, t_2 - 1) \{ -2(s_1 + 1 - t_1 - t_2)(2s_2 + 2 - t_1 - t_2) + 2(t_1^2 + t_2^2) \right. \\
&\quad \left. - (\nu_1^2 + \nu_2^2) + 2(-2s_2 - 1) + 4(s_2 - t_2 + 1) \} R_0(s_1 + 1, s_2 + 1, t_1, t_2) \right] \\
&= \int dt_1 \int dt_2 \left[-8Q_0(t_1 + 2, t_2 - 1) - 8Q_0(t_1, t_2 + 1) + \{2t_1^2 + 2(t_2 - 1)^2 - (\nu_1^2 + \nu_2^2)\} \right. \\
&\quad \left. \times Q_0(t_1, t_2 - 1) \right] R_0(s_1 + 1, s_2 + 1, t_1, t_2) \\
&= 0,
\end{aligned}$$

respectively. Here we use the abbreviation (2.9). We can verify the equations of (2.14b) and (2.15b) in the same manner. $\square$

2.4. Case 3. In this subsection, we suppose that Π_∞ is equivalent to an irreducible principal series representation $I(\sigma, \nu)$ with $\sigma(\gamma_1) = -\sigma(\gamma_2) = 1$. Then the K_0 -types $\tau_{(1,0)}$ and $\tau_{(0,-1)}$ occur in the representation $I(\sigma, \nu)$ with multiplicity one. We fix a basis $\{v_1, v_0\}$ (resp. $\{v'_1, v'_0\}$) of $\tau_{(1,0)}$ (resp. $\tau_{(0,-1)}$) by

$$(2.16) \quad (v_1(k_{A,B}), v_0(k_{A,B})) = (0, 1)(A + \sqrt{-1}B), \quad k_{A,B} \in K_0;$$

$$(2.17) \quad (v'_1(k_{A,B}), v'_0(k_{A,B})) = (1, 0) \det(A + \sqrt{-1}B)^{-1} (A + \sqrt{-1}B), \quad k_{A,B} \in K_0.$$

Then we know from [Mi-O1, Lemmas 3.9 and 3.10] that for all $g \in \mathbf{G}_{\mathbf{R}}$,

$$(2.18) \quad \begin{aligned} \frac{1}{2}v'_1(g; X_{(1,1)}) - v'_0(g; X_{(2,0)}) &= \nu_2 v_1(g); \\ v'_1(g; X_{(0,2)}) - \frac{1}{2}v'_0(g; X_{(1,1)}) &= \nu_2 v_0(g); \\ \frac{1}{2}v_1(g; X_{(-1,-1)}) + v_0(g; X_{(0,-2)}) &= \nu_2 v'_1(g); \\ -v_1(g; X_{(-2,0)}) - \frac{1}{2}v_0(g; X_{(-1,-1)}) &= \nu_2 v'_0(g). \end{aligned}$$

Proposition 2.4. *For $v \in \{v_1 \pm v'_0, v_0 \pm v'_1\}$, the restrictions $W_v|_{A_0}$ of Whittaker functions corresponding to v are given by the formulae (2.2) and (2.5) with*

$$\begin{aligned} U_{v_1+v'_0}(s_1, s_2, t_1, t_2) &= C \cdot Q_0(t_1, t_2; 0, \frac{1}{2}, \frac{1}{2}, 0; \nu_1, \nu_2) R_1(s_1, s_2, t_1, t_2); \\ U_{v_0-v'_1}(s_1, s_2, t_1, t_2) &= \frac{\sqrt{-1}}{2}(-s_1 + \nu_2 + 1) U_{v_1+v'_0}(s_1 - 1, s_2, t_1, t_2); \\ U_{v_1-v'_0}(s_1, s_2, t_1, t_2) &= C \cdot Q_0(t_1, t_2; \frac{1}{2}, 0, 0, \frac{1}{2}; \nu_1, \nu_2) R_1(s_1, s_2, t_1, t_2); \\ U_{v_0+v'_1}(s_1, s_2, t_1, t_2) &= \frac{\sqrt{-1}}{2}(-s_1 - \nu_2 - 1) U_{v_1-v'_0}(s_1 + 1, s_2, t_1, t_2), \end{aligned}$$

where we set

$$R_1(s, t) \equiv R_1(s_1, s_2, t_1, t_2) := \Gamma\left[\frac{s_1}{2} + \frac{1}{2}, \frac{s_1 - t_1 - t_2}{2}, \frac{s_2 - t_1}{2}, \frac{s_2 - t_2}{2}\right].$$

Here $C \in \mathbf{C}^\times$ is a constant common to $v_1 + v'_0$ and $v_1 - v'_0$ and $\sigma_j, \tau_j \in \mathbf{R}$ in (2.2) and (2.5) are taken so that

$$\begin{aligned} \sigma_1 &> \max\{\tau_1 + \tau_2 + 1, 0\}, & \sigma_2 &> \max\{\tau_1, \tau_2\}, \\ \tau_1 + 1 &> |\operatorname{Re}(\nu_1 - \nu_2)|/2, & \tau_2 + 1 &> |\operatorname{Re}(\nu_1 + \nu_2)|/2. \end{aligned}$$

Proof.

Step 1. It follows from (2.18) that

$$(2.19a) \quad 2\sqrt{-1}\left(\frac{\pi a_1}{a_2}\right)\phi_{v'_1}(a) - \partial_1\phi_{v'_0}(a) = \nu_2\phi_{v_1}(a);$$

$$(2.19b) \quad (\partial_2 - 4\pi a_2^2)\phi_{v'_1}(a) - 2\sqrt{-1}\left(\frac{\pi a_1}{a_2}\right)\phi_{v'_0}(a) = \nu_2\phi_{v_0}(a);$$

$$(2.19c) \quad 2\sqrt{-1}\left(\frac{\pi a_1}{a_2}\right)\phi_{v_1}(a) + (\partial_2 + 4\pi a_2^2)\phi_{v_0}(a) = \nu_2\phi_{v'_1}(a);$$

$$(2.19d) \quad -\partial_1\phi_{v_1}(a) - 2\sqrt{-1}\left(\frac{\pi a_1}{a_2}\right)\phi_{v_0}(a) = \nu_2\phi_{v'_0}(a).$$

In addition to (2.19), we have the following differential equations arising from the Casimir element ([Mi-O1, Theorem 11.3]):

$$(2.20a) \quad P\phi_{v_1}(a) + 4\sqrt{-1}\left(\frac{\pi a_1}{a_2}\right)\phi_{v_0}(a) = 0;$$

$$(2.20b) \quad -4\sqrt{-1}\left(\frac{\pi a_1}{a_2}\right)\phi_{v_1}(a) + (P + 8\pi a_2^2)\phi_{v_0}(a) = 0;$$

$$(2.20c) \quad (P - 8\pi a_2^2)\phi_{v'_1}(a) + 4\sqrt{-1}\left(\frac{\pi a_1}{a_2}\right)\phi_{v'_0}(a) = 0;$$

$$(2.20d) \quad -4\sqrt{-1}\left(\frac{\pi a_1}{a_2}\right)\phi_{v'_1}(a) + P\phi_{v'_0}(a) = 0,$$

where we set

$$P := \partial_1^2 + \partial_2^2 - 8\left(\frac{\pi a_1}{a_2}\right)^2 - 16(\pi a_2^2)^2 - (\nu_1^2 + \nu_2^2).$$

Step 2. We claim that the space $S(\Pi_\infty; \psi_\infty)$ of C^∞ -solutions of the system of the differential equations (2.19) and (2.20) is at most 8-dimensional. As in the proof of Proposition 2.2, this can be easily verified by considering the following eight functions on A_0 :

$$\begin{aligned} \phi_{v_0}(a), \quad \phi_{v_1}(a), \quad \partial_1 \phi_{v_0}(a), \quad \partial_2 \phi_{v_1}(a), \\ \phi_{v'_0}(a), \quad \phi_{v'_1}(a), \quad \partial_2 \phi_{v'_0}(a), \quad \partial_1 \phi_{v'_1}(a). \end{aligned}$$

Hence we have an isomorphism $\text{Hom}_{\mathfrak{g}, K}(\Pi_\infty, C^\infty(\mathbf{N}_{\mathbf{R}} \backslash \mathbf{G}_{\mathbf{R}}; \psi_\infty)) \cong S(\Pi_\infty; \psi_\infty)$ as in the proof of Proposition 2.2.

Step 3. As in the cases 1 and 2, it remains to show that the formula in the proposition satisfies (2.19) and (2.20). We rewrite the system of the equations (2.19) in terms of the new unknown functions $\phi_v(a)$ ($v \in \{v_0 \pm v'_1, v_1 \pm v'_0\}$). Firstly, the system of the equations (2.19a) and (2.19d) is equivalent to the system

$$(2.21a) \quad -2\sqrt{-1}\left(\frac{\pi a_1}{a_2}\right)\phi_{v_0-v'_1}(a) = (\partial_1 + \nu_2)\phi_{v_1+v'_0}(a);$$

$$(2.21b) \quad -2\sqrt{-1}\left(\frac{\pi a_1}{a_2}\right)\phi_{v_0+v'_1}(a) = (\partial_1 - \nu_2)\phi_{v_1-v'_0}(a).$$

From this, we know that $\phi_{v_0 \pm v'_1}(a)$ is determined by $\phi_{v_1 \mp v'_0}(a)$. Under the relation (2.21), the system of the equations (2.19b) and (2.19c) is equivalent to the system

$$(2.22a) \quad \left\{(\partial_1 + \nu_2)(\partial_2 + \nu_2 + 1) + 4\left(\frac{\pi a_1}{a_2}\right)^2\right\}\phi_{v_1+v'_0}(a) = -4\pi a_2^2(\partial_1 - \nu_2)\phi_{v_1-v'_0}(a);$$

$$(2.22b) \quad -4\pi a_2^2(\partial_1 + \nu_2)\phi_{v_1+v'_0}(a) = \left\{(\partial_1 - \nu_2)(\partial_2 - \nu_2 + 1) + 4\left(\frac{\pi a_1}{a_2}\right)^2\right\}\phi_{v_1-v'_0}(a).$$

Next we rewrite the system (2.20) in terms of the new unknown functions. Under the relations (2.21), the system of the equations (2.20a) and (2.20d) is equivalent to the system

$$(2.23a) \quad \left\{(\partial_1 - 1)^2 + \partial_2^2 - 8\left(\frac{\pi a_1}{a_2}\right)^2 - 16(\pi a_2^2)^2 - \nu_1^2 - (\nu_2 + 1)^2\right\}\phi_{v_1+v'_0}(a) = 0;$$

$$(2.23b) \quad \left\{(\partial_1 - 1)^2 + \partial_2^2 - 8\left(\frac{\pi a_1}{a_2}\right)^2 - 16(\pi a_2^2)^2 - \nu_1^2 - (\nu_2 - 1)^2\right\}\phi_{v_1-v'_0}(a) = 0.$$

On the other hand, the system of the equations (2.20b) and (2.20c) is equivalent to the system (2.22) under (2.21) and (2.23). Up to this point, the system of the equations (2.19) and (2.20) is equivalent to the system of the equations (2.22) and (2.23). The verification of (2.23) is analogous to that of (2.7) in the proof of Proposition 2.2. We shall check the equation (2.22a). For brevity, we put

$$Q_1(t_1, t_2) := Q_0(t_1, t_2; 0, \frac{1}{2}, \frac{1}{2}, 0; \nu_1, \nu_2), \quad Q_2(t_1, t_2) := Q_0(t_1, t_2; \frac{1}{2}, 0, 0, \frac{1}{2}; \nu_1, \nu_2).$$

Then we have

$$\begin{aligned}
& (-s_1 + \nu_2)(s_1 - 2s_2 + \nu_2 + 1)V_{v_1+v'_0}(s_1, s_2) + V_{v_1+v'_0}(s_1 + 2, s_2) \\
&= \int dt_1 \int dt_2 Q_1(t_1, t_2) \left[4R_1(s_1, s_2, t_1 - 2, t_2) + 4R_1(s_1, s_2, t_1, t_2 - 2) \right. \\
&\quad \left. + \{2(t_1 + t_2 - \nu_2)s_2 - (t_1 + t_2)(t_1 + t_2 + 1) + \nu_2(\nu_2 + 1)\} R_1(s, t) \right] \\
&= \int dt_1 \int dt_2 \left[4Q_1(t_1 + 2, t_2) + 4Q_1(t_1, t_2 + 2) \right. \\
&\quad \left. + \{2(t_1 + t_2 - \nu_2)s_2 - (t_1 + t_2)(t_1 + t_2 + 1) + \nu_2(\nu_2 + 1)\} 4Q_1(t_1, t_2) \right] R_1(s, t) \\
&= \int dt_1 \int dt_2 \left\{ 2(t_1 + t_2 - \nu_2)s_2 - 2t_1 t_2 - \frac{1}{2}\nu_1^2 + \frac{1}{2}\nu_2^2 \right\} Q_1(t_1, t_2) R_1(s, t)
\end{aligned}$$

and

$$\begin{aligned}
& V_{v_1-v'_0}(s_1, s_2 + 1) \\
&= \int dt_1 \int dt_2 Q_2(t_1 + 1, t_2 + 1) R_1(s_1, s_2 + 1, t_1 + 1, t_2 + 1) \\
&= -\frac{1}{4} \int dt_1 \int dt_2 \left(t_1 + \frac{\nu_1 - \nu_2}{2} \right) \left(t_2 + \frac{-\nu_1 - \nu_2}{2} \right) Q_1(t_1, t_2) R_1(s_1, s_2 + 1, t_1 + 1, t_2 + 1).
\end{aligned}$$

By combining these two formulae, we have

$$\begin{aligned}
& (-s_1 + \nu_2)(s_1 - 2s_2 + \nu_2 + 1)V_{v_1+v'_0}(s_1, s_2) + V_{v_1+v'_0}(s_1 + 2, s_2) \\
&\quad + 4(-s_1 - \nu_2)V_{v_1-v'_0}(s_1, s_2 + 1) \\
&= \int dt_1 \int dt_2 \left[\{ (2t_2 - \nu_1 - \nu_2)(s_2 - t_1) + (2t_1 - \nu_1 - \nu_2)(s_2 - t_2) \} \frac{1}{2}(s_1 - t_1 - t_2 - 2) \right. \\
&\quad \left. - \left(t_1 + \frac{\nu_1 - \nu_2}{2} \right) \left(t_2 + \frac{-\nu_1 - \nu_2}{2} \right) (t_1 + t_2 + 2 + \nu_2) \right] Q_1(t_1, t_2) \\
&\quad \times R_1(s_1, s_2 + 1, t_1 + 1, t_2 + 1) \\
&= \int dt_1 \int dt_2 Q_1(t_1, t_2) \left[2(2t_2 - \nu_1 - \nu_2)R_1(s_1, s_2 + 1, t_1 - 1, t_2 + 1) \right. \\
&\quad \left. + 2(2t_1 - \nu_1 - \nu_2)R_1(s_1, s_2 + 1, t_1 + 1, t_2 - 1) \right. \\
&\quad \left. - \left(t_1 + \frac{\nu_1 - \nu_2}{2} \right) \left(t_2 + \frac{-\nu_1 - \nu_2}{2} \right) (t_1 + t_2 + 2 + \nu_2) R_1(s_1, s_2 + 1, t_1 + 1, t_2 + 1) \right] \\
&= \int dt_1 \int dt_2 \left[2(2t_2 - \nu_1 - \nu_2 - 2)Q_1(t_1 + 2, t_2) + 2(2t_1 - \nu_1 - \nu_2 + 2)Q_1(t_1, t_2 + 2) \right. \\
&\quad \left. - \left\{ \left(t_1 + \frac{\nu_1 - \nu_2}{2} \right) \left(t_2 + \frac{-\nu_1 - \nu_2}{2} \right) (t_1 + t_2 + 2 + \nu_2) \right\} Q_1(t_1, t_2) \right] \\
&\quad \times R_1(s_1, s_2 + 1, t_1 + 1, t_2 + 1) \\
&= 0.
\end{aligned}$$

The verification of (2.22b) is similar. $\square$

Remarks. (i) Let v_0 be the K_0 -spherical vector in $I(\sigma, \nu)$ ($\sigma(\gamma_1) = \sigma(\gamma_2) = 1$). Niwa [Ni] obtained another kind of integral expression for the Whittaker function W_{v_0} by using the theta correspondence between $O(2, 2)$ and $Sp(2, \mathbf{R})$. We first discovered the formula in Proposition 2.2 (i) by computing the double Mellin transform of Niwa's formula.

(ii) The first author [I] obtained yet another kind of integral expressions for the Whittaker functions belonging to the principal series representation $I(\sigma, \nu)$ by evaluating the so-called Jacquet integral ([J]).

3. COMPUTATION OF THE ARCHIMEDEAN LOCAL ZETA INTEGRALS

In this section we compute Novodvorsky's archimedean local zeta integrals by using the Mellin-Barnes type integral formulae for Whittaker functions, which are obtained in the previous section.

3.1. Some preliminary computation. Suppose that the representation Π_∞ of $G_{\mathbf{R}}$ is equivalent to an irreducible principal series representation $I(\sigma, \nu)$. For each local Whittaker function $W_v \in \text{Wh}(\Pi_\infty, \psi_\infty)$ corresponding to a vector $v \in \Pi_\infty$, we define Novodvorsky's archimedean local zeta integral associated to W_v by

$$Z_N^{(\infty)}(s, W_v) := \int_{\mathbf{R}^\times} d^\times y \int_{\mathbf{R}} dx W_v(X(x, y; 1)) |y|^{s-3/2}.$$

Here we set

$$X(x, y; y_1) := \left(\begin{array}{c|cc} yy_1 & & \\ \hline y & & \\ \hline & y_1^{-1} & \\ x & & 1 \end{array} \right).$$

As in the previous paper [Mol] (cf. [H-M]), it is useful to introduce an auxiliary variable $y_1 > 0$:

$$Z_N^{(\infty)}(s, y_1; W_v) := \int_{\mathbf{R}^\times} d^\times y \int_{\mathbf{R}} dx W_v(X(x, y; y_1)) |y|^{s-3/2}.$$

We also set

$$Z_N^\pm(s, y_1; W_v) := \int_0^\infty d^\times y \int_{\mathbf{R}} dx W_v(X(\pm x, \pm y; y_1)) y^{s-3/2}.$$

Then we have

$$\begin{aligned} (3.1) \quad Z_N^{(\infty)}(s, y_1; W_v) &= Z_N^+(s, y_1; W_v) + Z_N^-(s, y_1; W_v) \\ &= Z_N^+(s, y_1; W_v) + Z_N^+(s, y_1; W_{\Pi_\infty(\gamma_0)v}). \end{aligned}$$

The following lemma is the common starting point of our evaluation.

Lemma 3.1. *Let $W_v \in \text{Wh}(\Pi_\infty, \psi_\infty)$ be the local Whittaker function associated to a vector $v \in \Pi_\infty$. We express the restriction $W_v|_{A_0}$ to A_0 as in Lemma 2.1. For $y_1 > 0$, put*

$$Z(s, y_1; p_i; W_v) := \int_0^\infty d^\times y \int_{\mathbf{R}} dx W_v(Y(x, y; y_1)) p_i(x) \exp\left(\frac{2\pi\sqrt{-1}xy}{1+x^2}\right) y^{s-3/2}$$

with

$$Y(x, y; y_1) := \text{diag}(\sqrt{y}y_1, \sqrt{y/(1+x^2)}, 1/(\sqrt{y}y_1), \sqrt{(1+x^2)/y}) \in A_0,$$

where we set

$$p_1(x) = 1, \quad p_2(x) = \frac{1}{\sqrt{1+x^2}}, \quad \text{and} \quad p_3(x) = \frac{\sqrt{-1}x}{\sqrt{1+x^2}}.$$

Then the integral $Z(s, y_1; p_i; W_v)$ converges absolutely for $\operatorname{Re}(s) \gg 0$ and equals

$$\frac{1}{2} \int_{L(\sigma_1)} \frac{ds_1}{2\pi\sqrt{-1}} \int_{L(\sigma_2)} \frac{ds_2}{2\pi\sqrt{-1}} \pi^{-s-s_1+1/2} y_1^{2-s_1} V_v(s_1, s_2) \gamma_i(s_1, s_2)$$

with

$$\gamma_1(s_1, s_2) = \Gamma\left[\frac{s-s_2}{2}, \frac{s_1-s_2-s}{2} \right], \quad \gamma_2(s_1, s_2) = \Gamma\left[\frac{s-s_2}{2}, \frac{s_1-s_2-s+1}{2} \right],$$

and

$$\gamma_3(s_1, s_2) = (-1) \Gamma\left[\frac{s-s_2+1}{2}, \frac{s_1-s_2-s}{2} \right].$$

Here the paths $L(\sigma_j)$ ($j = 1, 2$) of integration are taken so that all the poles of $V_v(s_1, s_2) \gamma_i(s_1, s_2)$ lie on the right.

Proof. We first prove the absolute convergence. By the proof of Lemma 2.1, there exists a constant $M > 0$ such that for each $l, m \geq 0$ we can take a constant $C > 0$ satisfying the inequality

$$|W_v(Y(x, y; y_1))| \leq C(y_1 \sqrt{1+x^2})^{-l} \left(\frac{y}{1+x^2}\right)^{-m} \|Y(x, y; y_1)\|^M, \\ \forall x \in \mathbf{R}, \forall y > 0, \forall y_1 > 0.$$

By putting $l = 3m$, we know that for each $m \geq 0$ and $y_1 > 0$ there exists a constant $C_1 > 0$ such that

$$|W_v(Y(x, y; y_1))| \leq C_1 y^{-m} (1+x^2)^{-m/2} \|Y(x, y; y_1)\|^M, \quad \forall x \in \mathbf{R}, \forall y > 0.$$

Moreover for each $y_1 > 0$ we can take a constant $C_2 > 0$ such that

$$\|Y(x, y; y_1)\| \leq C_2 \cdot \max\{y, y^{-1}\} \cdot \sqrt{1+x^2}, \quad \forall x \in \mathbf{R}, \forall y > 0.$$

Now it is easy to see that the integral $Z(s, y_1; p_i; W_v)$ converges absolutely for $\operatorname{Re}(s) \gg 0$.

We substitute $(1+x^2)y$ for y and carry out the integration with respect to x by means of the formula

$$\int_{\mathbf{R}} (1+x^2)^{-\rho} \left(\frac{1}{\sqrt{-1}x}\right) \exp(2\pi\sqrt{-1}xy) dx = \frac{2\pi^\rho y^{\rho-1/2}}{\Gamma(\rho)} \begin{pmatrix} K_{\rho-1/2}(2\pi y) \\ -K_{\rho-3/2}(2\pi y) \end{pmatrix}$$

for $\operatorname{Re}(\rho) > 0$ and $y > 0$ ([Er, p. 119]). Then $Z(s, y_1; p_1; W_v)$ becomes

$$\int_{L(\sigma_1)} \frac{ds_1}{2\pi\sqrt{-1}} \int_{L(\sigma_2)} \frac{ds_2}{2\pi\sqrt{-1}} \frac{2\pi^{-s-s_1/2-s_2+1/2} y_1^{2-s_1}}{\Gamma(-s + \frac{s_1+1}{2})} V_v(s_1, s_2) \\ \times \int_0^\infty d^\times y K_{-s+s_1/2}(2\pi y) y^{s_1/2-s_2}.$$

Therefore the formula

$$\int_0^\infty K_\nu(2\pi y) y^s d^\times y = 2^{-2} \pi^{-s} \Gamma\left[\frac{s+\nu}{2}, \frac{s-\nu}{2}\right], \quad \operatorname{Re}(s) > |\operatorname{Re}(\nu)|$$

leads the assertion. We can compute $Z(s, y_1; p_i; W_v)$ ($i = 2, 3$) in the same manner. $\square$

We also recall Barnes' lemma, which plays a central role in our computation.

Lemma 3.2 ([W-W, p. 289]). *For complex numbers a, b, c, d satisfying $a + c, a + d, b + c, b + d \notin \mathbf{Z}_{\leq 0}$, we have*

$$\frac{1}{2\pi\sqrt{-1}} \int_{-\sqrt{-1}\infty}^{\sqrt{-1}\infty} \Gamma[a + s, b + s, c - s, d - s] ds = \Gamma \left[\begin{matrix} a + c, a + d, b + c, b + d \\ a + b + c + d \end{matrix} \right].$$

Here the path of integration is curved, if necessary, to ensure that the poles of $\Gamma(c - s)\Gamma(d - s)$ lie on the right of the path and the poles of $\Gamma(a + s)\Gamma(b + s)$ lie on the left.

3.2. Case 1. In this subsection, we suppose that Π_∞ is equivalent to an irreducible principal series representation $I(\sigma, \nu)$ with $\sigma(\gamma_1) = \sigma(\gamma_2) = 1$. Let $v_0, v'_1, v''_1 \in \Pi_\infty$ be as in (2.4). Then we have

Lemma 3.3. (i) $\Pi_\infty(\gamma_0)v_0 = \sigma(\gamma_0)v_0$. (ii) $\Pi_\infty(\gamma_0)v'_1 = \sigma(\gamma_0)v''_1$.

Proof. (i) Let us compare the values at $k_{A,B} \in K_0$. Since

$$(\Pi_\infty(\gamma_0)v_0)(k_{A,B}) = v_0(k_{A,B} \gamma_0) = v_0(\gamma_0 k_{A,-B}) = \sigma(\gamma_0)v_0(k_{A,-B}) = \sigma(\gamma_0)v_0(k_{A,B}),$$

we have the assertion.

(ii) By (i), we get

$$\begin{aligned} \Pi_\infty(\gamma_0)v'_1 &= \Pi_\infty(\gamma_0)\Pi_\infty(X_{1,1})v_0 = \Pi_\infty(X_{-1,-1})\Pi(\gamma_0)v_0 \\ &= \Pi_\infty(X_{-1,-1})\sigma(\gamma_0)v_0 = \sigma(\gamma_0)v''_1. \end{aligned}$$

□

We compute the local zeta integral associated to W_{v_0} or $W_{v'_1}$ according as $\sigma(\gamma_0) = 1$ or $\sigma(\gamma_0) = -1$.

Proposition 3.4. *Let W_{v_0} and $W_{v'_1}$ be the Whittaker functions given in Proposition 2.2 with $C = 1$ and $L(s, \Pi_\infty)$ the Gamma factor defined in (1.1).*

(i) *If $\sigma(\gamma_0) = 1$, then the integral $Z_N^{(\infty)}(s, y_1; W_{v_0})$ converges absolutely for $\text{Re}(s) \gg 0$ and*

$$\frac{Z_N^{(\infty)}(s, y_1; W_{v_0})}{L(s, \Pi_\infty)} = 2^4 \pi^{1/2} y_1^{2-s-\omega_\infty/2} G(y_1, s + \frac{\omega_\infty}{2}; 0, 0, 0, 0; 0, 0).$$

(ii) *If $\sigma(\gamma_0) = -1$, then the integral $Z_N^{(\infty)}(s, y_1; W_{v'_1})$ converges absolutely for $\text{Re}(s) \gg 0$ and*

$$\frac{Z_N^{(\infty)}(s, y_1; W_{v'_1})}{L(s, \Pi_\infty)} = 2^4 \pi^{5/2} y_1^{2-s-\omega_\infty/2} G(y_1, s + \frac{\omega_\infty}{2}; 0, 0, 0, 0; 1, 1).$$

Here we set

(3.2)

$$\begin{aligned} G(y_1, s; \alpha_1, \alpha_2, \alpha_3, \alpha_4; \beta_1, \beta_2) &:= \int_{L(\sigma_1)} \frac{ds_1}{2\pi\sqrt{-1}} (\pi y_1)^{-2s_1} \\ &\times \Gamma \left[\begin{matrix} s_1 + \frac{\nu_1 - \nu_2}{4} + \frac{\alpha_1}{2}, & s_1 - \frac{\nu_1 - \nu_2}{4} + \frac{\alpha_2}{2}, & s_1 + \frac{\nu_1 + \nu_2}{4} + \frac{\alpha_3}{2}, & s_1 - \frac{\nu_1 + \nu_2}{4} + \frac{\alpha_4}{2} \\ s_1 + \frac{s}{2} + \frac{\beta_1}{2}, & s_1 + \frac{1-s}{2} + \frac{\beta_2}{2} \end{matrix} \right], \end{aligned}$$

where the path $L(\sigma_1)$ of integration is taken so that all the poles of the integrand lie on the right.

Proof. (i) By (3.1) and Lemma 3.3(i), we have $Z_N^{(\infty)}(s, y_1; W_{v_0}) = 2Z_N^+(s, y_1; W_{v_0})$. Because of the Iwasawa decomposition

$$\begin{pmatrix} y & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & xy/(1+x^2) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y/\sqrt{1+x^2} & 0 \\ 0 & \sqrt{1+x^2} \end{pmatrix} \frac{1}{\sqrt{1+x^2}} \begin{pmatrix} 1 & -x \\ x & 1 \end{pmatrix}$$

for each $y > 0$ and $x \in \mathbf{R}$, we have

$$W_{v_0}(X(x, y; y_1)) = y^{\omega_\infty/2} \exp\left(\frac{2\pi\sqrt{-1}xy}{1+x^2}\right) W_{v_0}(Y(x, y; y_1)).$$

From now on we assume that $\omega_\infty = 0$, because the reduction to this case is easy.

Then, by virtue of the explicit formula of $W_{v_0}|_{A_0}$ in Proposition 2.2 and Lemma 3.1, we know that $Z_N^{(\infty)}(s, y_1; W_{v_0})$ is equal to

$$\begin{aligned} & \frac{1}{(2\pi\sqrt{-1})^4} \int ds_1 \int ds_2 \int dt_1 \int dt_2 \pi^{-s-s_1+1/2} y_1^{2-s_1} Q_0(t_1, t_2) \\ & \times \Gamma\left[\frac{s_1}{2}, \frac{s_1-t_1-t_2}{2}, \frac{s_2-t_1}{2}, \frac{s_2-t_2}{2}, \frac{s-s_2}{2}, \frac{s_1-s_2-s}{2}\right]. \end{aligned}$$

Here and below we frequently omit the paths of integration.

By using Lemma 3.2, we carry out the integration with respect to s_2 to get

$$\begin{aligned} Z_N^{(\infty)}(s, y_1; W_{v_0}) &= \frac{2}{(2\pi\sqrt{-1})^3} \int ds_1 \pi^{-s-s_1+1/2} y_1^{2-s_1} \Gamma\left[\frac{s_1}{2}, \frac{s_1-t_1-t_2}{2}\right] \\ & \times \int dt_1 \int dt_2 Q_0(t_1, t_2) \Gamma\left[\frac{s-t_1}{2}, \frac{s-t_2}{2}, \frac{s_1-s-t_1}{2}, \frac{s_1-s-t_2}{2}\right]. \end{aligned}$$

We use Lemma 3.2 again to perform the integrations with respect to t_1 and t_2 . Finally, making a change of variable $s_1 \mapsto 2s_1 + s$, we arrive at the desired formula.

(ii) By Lemma 3.3 (ii), we have

$$Z_N^{(\infty)}(s, y_1; W_{v'_1}) = Z_N^+(s, y_1; W_{v'_1}) - Z_N^+(s, y_1; W_{v''_1}).$$

The rest of the computation is quite analogous to (i). $\square$

Remark 1. (i) If $\sigma(\gamma_0) = -1$, then Lemma 3.3 implies that $Z_N^{(\infty)}(s, y_1; W_{v_0}) = 0$. Therefore we have to consider the vector $v'_1 \in \Pi_\infty$ so as to get a non-vanishing local zeta integral.

(ii) In the above computation, K -Bessel functions come out from the integration with respect to the variable x . The resulting integral is quite similar to Novodvorsky's zeta integral for $GSp(2) \times GL(2)$ ([No, §3], [Bu, §3]), which is computed in [Ni]. Hence it is possible to reduce our computation to that of [Ni].

3.3. Case 2. In this subsection, we suppose that Π_∞ is equivalent to an irreducible principal series representation $I(\sigma, \nu)$ with $\sigma(\gamma_1) = \sigma(\gamma_2) = -1$. Let $v_0, v'_0 \in \Pi_\infty$ be as in (2.10). Then we have

Proposition 3.5. *Let W_{v_0} be the Whittaker function given in Proposition 2.3 with $C = 1$ and $L(s, \Pi_\infty)$ the Gamma factor defined in (1.1). Then the integral $Z_N^{(\infty)}(s, y_1; W_{v_0})$ converges absolutely for $\text{Re}(s) \gg 0$.*

Proposition 3.6. (i) *Moreover, if $\sigma(\gamma_0) = 1$, then*

$$\frac{Z_N^{(\infty)}(s, y_1; W_{v_0})}{L(s, \Pi_\infty)} = 2^4 \pi^{3/2} y_1^{2-s-\omega_\infty/2} G(y_1, s + \frac{\omega_\infty}{2}; 0, 0, 1, 1; 1, 1).$$

(ii) If $\sigma(\gamma_0) = -1$, then

$$\frac{Z_N^{(\infty)}(s, y_1; W_{v_0})}{L(s, \Pi_\infty)} = 2^4 \pi^{3/2} y_1^{2-s-\omega_\infty/2} G(y_1, s + \frac{\omega_\infty}{2}; 1, 1, 0, 0; 1, 1).$$

Here we use the same symbol $G(y_1, s; \dots)$ as in Proposition 3.4.

Proof. We prove only (i), because we can prove (ii) in the same manner. As in the proof of Proposition 3.4, we may assume $\omega_\infty = 0$. In a way similar to Lemma 3.3, we can prove $\Pi_\infty(\gamma_0)v_0 = \sigma(\gamma_0)v'_0$. Hence we have

$$(3.3) \quad Z_N^{(\infty)}(s, y_1; W_{v_0}) = Z_N^+(s, y_1; W_{v_0}) + \sigma(\gamma_0)Z_N^+(s, y_1; W_{v'_0}).$$

Notice that

$$\begin{aligned} W_{v_0}(X(x, y; y_1)) &= \exp\left(\frac{2\pi\sqrt{-1}xy}{1+x^2}\right) \cdot \frac{1-\sqrt{-1}x}{\sqrt{1+x^2}} \cdot W_{v_0}(Y(x, y; y_1)); \\ W_{v'_0}(X(x, y; y_1)) &= \exp\left(\frac{2\pi\sqrt{-1}xy}{1+x^2}\right) \cdot \frac{1+\sqrt{-1}x}{\sqrt{1+x^2}} \cdot W_{v'_0}(Y(x, y; y_1)). \end{aligned}$$

Then, by virtue of Proposition 2.3 and Lemma 3.1, we know that (3.3) becomes

$$\begin{aligned} &\frac{1}{(2\pi\sqrt{-1})^4} \int ds_1 \int ds_2 \int dt_1 \int dt_2 \pi^{-s-s_1+1/2} y_1^{2-s_1} \\ &\times \left(Q_0(t_1, t_2 - 1) \Gamma\left[\frac{s_1+1}{2}, \frac{s_1-t_1-t_2+1}{2}, \frac{s_2-t_1+1}{2}, \frac{s_2-t_2+1}{2}, \frac{s-s_2}{2}, \frac{s_1-s_2-s+1}{2}\right] \right. \\ &\quad \left. + Q_0(t_1 - 1, t_2) \Gamma\left[\frac{s_1+1}{2}, \frac{s_1-t_1-t_2+1}{2}, \frac{s_2-t_1+1}{2}, \frac{s_2-t_2+1}{2}, \frac{s-s_2}{2}, \frac{s_1-s_2-s+1}{2}\right] \right). \end{aligned}$$

By means of Lemma 3.2, we get

$$\begin{aligned} Z_N^{(\infty)}(s, y_1; W_{v_0}) &= \frac{2}{(2\pi\sqrt{-1})^3} \int ds_1 \int dt_1 \int dt_2 \pi^{-s-s_1+1/2} y_1^{2-s_1} Q_0(t_1, t_2) \\ &\times \left(\Gamma\left[\frac{s-t_1+1}{2}, \frac{s-t_2}{2}, \frac{s_1-s-t_1+2}{2}, \frac{s_1-s-t_2+1}{2}, \frac{s_1-t_1-t_2}{2}, \frac{s_1+1}{2}\right] \right. \\ &\quad \left. + \Gamma\left[\frac{s-t_1+1}{2}, \frac{s-t_2+2}{2}, \frac{s_1-s-t_1}{2}, \frac{s_1-s-t_2+1}{2}, \frac{s_1-t_1-t_2}{2}, \frac{s_1+1}{2}\right] \right) \\ &= \frac{2}{(2\pi\sqrt{-1})^3} \int ds_1 \int dt_1 \int dt_2 \pi^{-s-s_1+1/2} y_1^{2-s_1} Q_0(t_1, t_2) \\ &\times \Gamma\left[\frac{s-t_1+1}{2}, \frac{s-t_2}{2}, \frac{s_1-s-t_1}{2}, \frac{s_1-s-t_2+1}{2}, \frac{s_1-t_1-t_2}{2}, \frac{s_1+1}{2}\right]. \end{aligned}$$

Here, in the first equality, we make the changes of variables $t_2 \mapsto t_2 + 1$ in the former term and $t_1 \mapsto t_1 + 1$ in the latter term. Finally, applying Lemma 3.2 twice, we obtain the assertion. $\square$

3.4. Case 3. In this subsection, we suppose that Π_∞ is equivalent to an irreducible principal series representation $I(\sigma, \nu)$ with $\sigma(\gamma_1) = -\sigma(\gamma_2) = 1$. We take the set of vectors $\{v_1, v_0, v'_1, v'_0\}$ as in (2.16). Then we have

Proposition 3.7. *Let W_{v_0} and W_{v_1} be the Whittaker functions given in Proposition 2.4 with $C = 1$ and $L(s, \Pi_\infty)$ the Gamma factor defined in (1.1). Then the*

integrals $Z_N^{(\infty)}(s, y_1; W_v)$ ($v \in \{v_0, v_1\}$) converge absolutely for $\operatorname{Re}(s) \gg 0$. Moreover we have

(i) if $\sigma(\gamma_0) = 1$, then

$$\begin{aligned} & \left(\frac{Z_N^{(\infty)}(s, y_1; W_{v_1})}{Z_N^{(\infty)}(s, y_1; W_{v_0})} / L(s, \Pi_\infty) \right) \\ &= 2^4 \pi^{3/2} y_1^{2-s-\omega_\infty/2} \left(\frac{G(y_1, s + \frac{\omega_\infty}{2}; 1, 0, 0, 1; 1, 0)}{(-\sqrt{-1})G(y_1, s + \frac{\omega_\infty}{2}; 0, 1, 1, 0; 0, 1)} \right); \end{aligned}$$

(ii) if $\sigma(\gamma_0) = -1$, then

$$\begin{aligned} & \left(\frac{Z_N^{(\infty)}(s, y_1; W_{v_1})}{Z_N^{(\infty)}(s, y_1; W_{v_0})} / L(s, \Pi_\infty) \right) \\ &= 2^4 \pi^{3/2} y_1^{2-s-\omega_\infty/2} \left(\frac{G(y_1, s + \frac{\omega_\infty}{2}; 0, 1, 1, 0; 1, 0)}{(-\sqrt{-1})G(y_1, s + \frac{\omega_\infty}{2}; 1, 0, 0, 1; 0, 1)} \right). \end{aligned}$$

Here we use the same symbol $G(y_1, s; \dots)$ as in Proposition 3.4.

Proof. We give a proof of (i) only, because (ii) can be proved in the same manner. Firstly, in the same way as in Lemma 3.3, we can easily show that

$$\Pi_\infty(\gamma_0)v_1 = -\sigma(\gamma_0)v'_0, \quad \Pi_\infty(\gamma_0)v_0 = \sigma(\gamma_0)v'_1.$$

Hence we have

$$(3.4) \quad Z_N^{(\infty)}(s, y_1; W_{v_1}) = Z_N^+(s, y_1, W_{v_1}) - \sigma(\gamma_0)Z_N^+(s, y_1, W_{v'_0});$$

$$(3.5) \quad Z_N^{(\infty)}(s, y_1; W_{v_0}) = Z_N^+(s, y_1, W_{v_0}) + \sigma(\gamma_0)Z_N^+(s, y_1, W_{v'_1}).$$

The evaluation of (3.4) can be done in the same way as Proposition 3.4 by using the explicit formulae given in Proposition 2.4. On the other hand, the computation of (3.5) needs a little more effort. Firstly, by the Iwasawa decomposition of $X(x, y; y_1)$, we have

$$\begin{aligned} W_{v_0}(X(x, y; y_1)) &= \exp\left(\frac{2\pi\sqrt{-1}xy}{1+x^2}\right) \cdot \frac{1-\sqrt{-1}x}{\sqrt{1+x^2}} W_{v_0}(Y(x, y; y_1)); \\ W_{v'_1}(X(x, y; y_1)) &= \exp\left(\frac{2\pi\sqrt{-1}xy}{1+x^2}\right) \cdot \frac{1+\sqrt{-1}x}{\sqrt{1+x^2}} W_{v'_1}(Y(x, y; y_1)). \end{aligned}$$

Next by

$$\frac{s_1 \pm (\nu_2 + 1)}{2} = \frac{s_1 - t_1 - t_2 \pm 1}{2} + \left(\frac{t_1}{2} \mp \frac{\nu_1 - \nu_2}{4}\right) + \left(\frac{t_2}{2} \pm \frac{\nu_1 + \nu_2}{4}\right),$$

we decompose $U_{v_0 \pm v'_1}(s_1, s_2, t_1, t_2)$ into three terms and apply Lemma 3.1. Then we get

$$Z_N^{(\infty)}(s, y_1; W_{v_0}) = -\sqrt{-1} \int \frac{ds_1}{2\pi\sqrt{-1}} \pi^{-s-s_1+1/2} y_1^{2-s_1} \Gamma\left[-s + \frac{s_1}{2} + 1\right] \sum_{i=1}^6 T_i(s_1)$$

with

$$T_i(s_1) = \frac{1}{(2\pi\sqrt{-1})^3} \int dt_1 \int dt_2 \int ds_2 T'_i(s_1, s_2, t_1, t_2).$$

Here $T'_i \equiv T'_i(s_1, s_2, t_1, t_2)$ is given by

$$\begin{aligned} T'_1 &= Q_0(t; 0, \frac{1}{2}, \frac{1}{2}, 0)R'(1, 1, 0); & T'_2 &= Q_0(t; 1, \frac{1}{2}, \frac{1}{2}, 0)R'(-1, 1, 0); \\ T'_3 &= Q_0(t; 0, \frac{1}{2}, \frac{1}{2}, 1)R'(-1, 1, 0); & T'_4 &= Q_0(t; \frac{1}{2}, 0, 0, \frac{1}{2})R'(1, 0, 1); \\ T'_5 &= Q_0(t; \frac{1}{2}, 1, 0, \frac{1}{2})R'(-1, 0, 1); & T'_6 &= Q_0(t; \frac{1}{2}, 0, 1, \frac{1}{2})R'(-1, 0, 1), \end{aligned}$$

with

$$\begin{aligned} R'(a, b, c) &\equiv R'(s_1, s_2, t_1, t_2; a, b, c) \\ &:= \Gamma \left[\frac{s_1 - t_1 - t_2 + a}{2}, \frac{s_2 - t_1}{2}, \frac{s_2 - t_2}{2}, \frac{s - s_2 + b}{2}, \frac{s_1 - s_2 - s + c}{2} \right]. \end{aligned}$$

By means of Lemma 3.2, we obtain

$$\begin{aligned} T_1(s_1) &= \frac{2}{(2\pi\sqrt{-1})^2} \int dt_1 \int dt_2 Q_0(t; 0, \frac{1}{2}, \frac{1}{2}, 0)R''(1, 1, 0, 0); \\ T_2(s_1) + T_6(s_1) &= \frac{2}{(2\pi\sqrt{-1})^2} \int dt_1 \int dt_2 Q_0(t; \frac{1}{2}, 0, 1, \frac{1}{2})R''(0, 0, 1, -1); \\ T_4(s_1) &= \frac{2}{(2\pi\sqrt{-1})^2} \int dt_1 \int dt_2 Q_0(t; \frac{1}{2}, 0, 0, \frac{1}{2})R''(0, 0, 1, 1); \\ T_3(s_1) + T_5(s_1) &= \frac{2}{(2\pi\sqrt{-1})^2} \int dt_1 \int dt_2 Q_0(t; \frac{1}{2}, 1, 0, \frac{1}{2})R''(0, 0, -1, 1), \end{aligned}$$

where we put

$$\begin{aligned} R''(a, b, c, d) &\equiv R''(s_1, t_1, t_2; a, b, c, d) \\ &:= \Gamma \left[\frac{s - t_1}{2} + a, \frac{s - t_2}{2} + b, \frac{s_1 - s - t_1}{2} + c, \frac{s_1 - s - t_2}{2} + d \right]. \end{aligned}$$

Here in $T_2(s_1)$ (resp. $T_3(s_1)$), we make the changes of variables $t_1 \rightarrow t_1 - 1$ and $t_2 \rightarrow t_2 + 1$ (resp. $t_1 \rightarrow t_1 + 1$ and $t_2 \rightarrow t_2 - 1$). Next we use Lemma 3.2 again to carry out the integrations with respect to t_1 and t_2 to get

$$\begin{aligned} T_1(s_1) &= 2^3 L(\frac{1}{2}, 1, 1, \frac{1}{2})M(0, 0; 1, 1); & T_2(s_1) + T_6(s_1) &= 2^3 L(\frac{1}{2}, 0, 1, \frac{1}{2})M(1, 0; 1, 1); \\ T_4(s_1) &= 2^3 L(\frac{1}{2}, 0, 0, \frac{1}{2})M(1, 1; 1, 1); & T_3(s_1) + T_5(s_1) &= 2^3 L(\frac{1}{2}, 1, 0, \frac{1}{2})M(0, 1; 1, 1). \end{aligned}$$

Here we put

$$L(a, b, c, d) := \Gamma \left[\frac{s}{2} + \frac{\nu_1 - \nu_2}{4} + a, \frac{s}{2} - \frac{\nu_1 - \nu_2}{4} + b, \frac{s}{2} + \frac{\nu_1 + \nu_2}{4} + c, \frac{s}{2} - \frac{\nu_1 + \nu_2}{4} + d \right]$$

and

$$\begin{aligned} M(a, b, c, d) &:= \Gamma \left[\frac{s_1 - s}{2} + \frac{\nu_1 - \nu_2}{4} + a, \frac{s_1 - s}{2} - \frac{\nu_1 - \nu_2}{4} + \frac{1}{2}, \frac{s_1 - s}{2} + \frac{\nu_1 + \nu_2}{4} + \frac{1}{2}, \frac{s_1 - s}{2} - \frac{\nu_1 + \nu_2}{4} + b, \right. \\ &\quad \left. \frac{s_1}{2} + c, \frac{s_1}{2} + d \right]. \end{aligned}$$

Finally, by using $\Gamma(s+1) = s\Gamma(s)$ repeatedly, we know that

$$\sum_{i=1}^6 T_i(s_1) = 2^3 L\left(\frac{1}{2}, 0, 0, \frac{1}{2}\right) M(0, 0; 0, 0).$$

By a change of variable $s_1 \mapsto 2s_1 + s$, we get the desired expression of $Z_N^{(\infty)}(s, y_1; W_{v_0})$ in the proposition. $\square$

Remark 2. For a Whittaker function $W_v \in \text{Wh}(\Pi_\infty, \psi_\infty)$, we consider the following integral:

$$B_v^{(\infty)}(s; g) := \int_{\mathbf{R}^\times} d^\times y \int_{\mathbf{R}} dx W_v \left(\begin{pmatrix} y & & \\ & y & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & & -1 \\ & & 1 \end{pmatrix} g \right) |y|^{s-3/2}$$

($s \in \mathbf{C}, g \in \mathbf{G}_{\mathbf{R}}$). Note that $Z_N^{(\infty)}(s, y_1; W_v)$ is the restriction of $B_v(s; g)$ to a one-parameter subgroup of $\mathbf{G}_{\mathbf{R}}$. It is easy to see that

$$B_v^{(\infty)}(s; \begin{pmatrix} 1 & & x_1 & x_2 \\ & 1 & x_2 & x_3 \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} u & & \\ & 1 & \\ & & 1 & \\ & & & u \end{pmatrix} g) = e^{2\pi\sqrt{-1}x_2} |u|^{-s+1/2} B_v^{(\infty)}(s; g).$$

Hence we expect that the ratio $Z_N^{(\infty)}(W_v; s; y_1)/L(s, \Pi_\infty)$ in Propositions 3.3, 3.4, or 3.5 represents a value on the one-parameter subgroup of a local Bessel function on $\mathbf{G}_{\mathbf{R}}$ (or a local generalized Whittaker function on $\mathbf{G}_{\mathbf{R}}$ in the terminology of [PS]). We shall confirm this expectation in [Mo2].

4. PROOF OF THE MAIN THEOREM

As in [Mo1], our main theorem follows from the following properties of the archimedean local zeta integrals.

Proposition 4.1. *For each local Whittaker function $W \in \text{Wh}(\Pi_\infty, \psi_\infty)$, we set*

$$\widetilde{W}(g) := \omega_{\Pi_\infty}(\nu(g))^{-1} W(g\eta_\infty), \quad \eta_\infty = \kappa \left(\begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} \right) \in K_0,$$

where $\omega_{\Pi_\infty} : \mathbf{R}^\times \rightarrow \mathbf{C}^\times$ stands for the central character of Π_∞ .

(i) *There exists a local Whittaker function $W \in \text{Wh}(\Pi_\infty, \psi_\infty)$ satisfying the following two conditions:*

- (a) *the local zeta integrals $Z_N^{(\infty)}(s, W)$ and $Z_N^{(\infty)}(s, \widetilde{W})$ converge absolutely for $\text{Re}(s) \gg 0$ and are continued to non-zero meromorphic functions on the whole s -plane;.*
- (b) *the local functional equation*

$$\frac{Z_N^{(\infty)}(1-s, \widetilde{W})}{L(1-s, \Pi_\infty^\vee)} = \epsilon(s, \Pi_\infty, \psi_\infty) \frac{Z_N^{(\infty)}(s, W)}{L(s, \Pi_\infty)}$$

holds.

(ii) For each complex number $s_0 \in \mathbf{C}$, there exists a local Whittaker function $W \in \text{Wh}(\Pi_\infty, \psi_\infty)$ such that the associated local zeta integral $Z_N^{(\infty)}(s, W)$ converges absolutely for $\text{Re}(s) \gg 0$ and the ratio $Z_N^{(\infty)}(s, W)/L(s, \Pi_\infty)$ is continued to an entire function of $s \in \mathbf{C}$ which does not vanish at $s = s_0$.

Proof. Because of Lemma 1.1, we have only to treat the cases 1, 2-(i), and 3 -(i). Moreover we may suppose that $\omega_\infty = 0$. The assertion (ii) follows from Propositions 3.4, 3.5, and 3.6 in the same way as the proof of [Mo1, Proposition 6]. In order to prove (i), we introduce the following integral for $W \in \text{Wh}(\Pi_\infty, \psi_\infty)$:

$$\begin{aligned} \tilde{Z}_N^{(\infty)}(s, y_1; W) &:= \int_{\mathbf{R}^\times} d^\times y \int_{\mathbf{R}} dx W\left(\begin{array}{c|c} y & \\ \hline y y_1^{-1} & \\ \hline x y_1^{-1} & 1 \\ & y_1 \end{array}\right) |y|^{s-3/2} \\ &= y_1^{2s-1} Z_N^{(\infty)}(s, y_1; W). \end{aligned}$$

If we confirm the local functional equation

$$\frac{\tilde{Z}_N^{(\infty)}(1-s, y_1; \widetilde{W}_v)}{L(1-s, \Pi_\infty^\vee)} = \epsilon(s, \Pi_\infty, \psi_\infty) \frac{Z_N^{(\infty)}(s, y_1; W_v)}{L(s, \Pi_\infty)} (\neq 0)$$

for a suitable vector $v \in \Pi_\infty$, we can complete the proof by the same argument as [Mo1, pp. 918–919].

Case 1. For the case $\sigma(\gamma_0) = 1$, the claim is obvious by Proposition 3.4 (i). We consider the case $\sigma(\gamma_0) = -1$. Since $\omega_{\Pi_\infty}(-1) = \sigma(\gamma_1 \gamma_2) = 1$ and v_0 is fixed by K_0 , we have

$$\begin{aligned} \widetilde{W}_{v'_1}(g) &= \omega_{\Pi_\infty}(g) W_{v_0}(g \eta_\infty; X_{(1,1)}) = W_{v_0}(g; \text{Ad}(\eta_\infty) X_{(1,1)}) \\ &= W_{v_0}(g; X_{(1,1)}) = W_{v'_1}(g), \quad g \in G. \end{aligned}$$

Hence Proposition 3.4 (ii) implies the assertion.

Case 2(i). Let $v_0 \in I(\sigma, \nu)$ be as in Proposition 3.5 ($\sigma(\gamma_0) = -\sigma(\gamma_1) = -\sigma(\gamma_2) = 1$). Then we have

$$\widetilde{W}_{v_0}(g) = \omega_{\Pi_\infty}(g) W_{v_0}(g \eta_\infty) = (-1) W_{v_0}(g), \quad \forall g \in G.$$

Hence Proposition 3.5(i) gives the assertion.

Case 3(i). Let $v_0, v_1 \in I(\sigma, \nu)$ be as in Proposition 3.7 ($\sigma(\gamma_0) = \sigma(\gamma_1) = -\sigma(\gamma_2) = 1$). Then we have

$$\begin{pmatrix} \widetilde{W}_{v_1}(g) \\ \widetilde{W}_{v_0}(g) \end{pmatrix} = \omega_{\Pi_\infty}(g) \begin{pmatrix} & -\sqrt{-1} \\ \sqrt{-1} & \end{pmatrix} \begin{pmatrix} W_{v_1}(g) \\ W_{v_0}(g) \end{pmatrix}, \quad \forall g \in G.$$

Since $\omega_{\Pi_\infty}(-1) = \sigma(\gamma_1 \gamma_2) = -1$, $\widetilde{W}_{v_1}(g)$ belongs to the representation $I(\sigma', \nu)$ with $\sigma'(\gamma_0) = -\sigma'(\gamma_1) = \sigma'(\gamma_2) = -1$. Now our assertion can be easily verified from Proposition 3.7 (i) and (ii). $\square$

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