COMPUTATIONAL TOPOLOGY OF EQUIVARIANT MAPS FROM SPHERES TO COMPLEMENTS OF ARRANGEMENTS

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ABSTRACT. The problem of the existence of an equivariant map is a classical topological problem ubiquitous in topology and its applications. Many problems in discrete geometry and combinatorics have been reduced to such a question and many of them resolved by the use of equivariant obstruction theory. A variety of concrete techniques for evaluating equivariant obstruction classes are introduced, discussed and illustrated by explicit calculations. The emphasis is on D_{2n} -equivariant maps from spheres to complements of arrangements, motivated by the problem of finding a 4-fan partition of 2-spherical measures, where D_{2n} is the dihedral group. One of the technical highlights is the determination of the D_{2n} -module structure of the homology of the complement of the appropriate subspace arrangement, based on the geometric interpretation for the generators of the homology groups of arrangements.

1. INTRODUCTION

1.1. Computational equivariant topology. One of the objectives of this paper is to identify the problem of calculating the *topological obstructions for the existence of equivariant maps* as one of the problems paradigmatic for computational topology. Given a group G , a G -equivariant map $f : X \rightarrow Y$ between two G -spaces is a symmetry preserving map, i.e. a map satisfying the condition $f(g \cdot x) = g \cdot f(x)$. In applications in discrete and computational geometry, X is usually a manifold of all “feasible” configurations (the configurations space) while Y is typically a complement of a real, affine subspace arrangement (the test space).

The problem of calculating/evaluating the complexity of equivariant obstructions has many aspects relevant for computational topology, including the following.

- (A₁) The (non)existence of an equivariant map is an essential ingredient in the application of the *configuration space/test map* scheme [36], [19] which has proven to be a very effective tool in solving combinatorial or discrete geometric problems of relevance to computing and analysis of algorithms, [19], [37].
- (A₂) The existence of an equivariant map can often be interpreted as a problem of mapping an object (often a cell of some dimension) to a Euclidean space, subject to some boundary constraints and avoiding some obstacles (arrangement of subspaces). This aspect can be seen as a relative of the

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motion planning problem from robotics (the case of a 1-dimensional cell). The existence of an equivariant map is equivalent to the problem of sectioning a (vector) bundle, and the condition of avoiding obstacles is translated into the question of sectioning a bundle subject to some additional constraints (relations). The latter problem is of utmost importance in many areas of mathematics, including Gromov convex integration theory [16], combinatorial geometry on vector bundles [35], etc.

Conclusion. Both aspects of the problem of existence of an equivariant map are of theoretical interest. If an equivariant map exists, one should be able to describe and evaluate the complexity of an algorithm that constructs such a map. If an equivariant map does not exist, one is interested in the evaluation and complexity of the associated obstruction. One way to achieve both goals is to develop an “effective” obstruction theory. This is of particular interest in the context of (A_2) where such a theory would allow effective placements of objects into an environment with obstacles, subject to some boundary constraints.

1.2. This paper. We use a well-known problem in discrete and computational geometry (Problem 2.1 in the case $k = 4$, $m = 2$) as a motivation, and an associated question from equivariant topology (Problem 2.3) as a point of departure to illustrate many aspects, both theoretical and computational, of the general *existence of an equivariant map* problem. A variety of techniques are introduced and discussed with the emphasis on concrete and explicit calculations. This eventually leads to an almost exhaustive analysis of when such maps do or do not exist in this particular case of interest. The scope and character of these results is illustrated by the following summary of results subsumed by Theorem 4.10.

Summary. Let $G = Q_{4n}$ be the generalized quaternion group and $M(\alpha)$ the complement of the arrangement $\mathcal{A}(\alpha)$ introduced in Section 2.2.

- (A) If $\alpha = (p, p, p, p)$ or $\alpha = (p, p, p, 2p)$, then there are no Q_{4n} -equivariant maps $F : S^3 \rightarrow M(\alpha)$.
- (B) If α is not of the form $(p, q, p + q, p + q)$ or $(p, q, r, p + q)$, $r \notin \{p, p + q\}$, or $(p, p, q, q + r)$, then there exists a Q_{4n} -map $F : S^3 \rightarrow M(\alpha)$.
- (C) If $\alpha = (p, q, p + q, p + q)$ or $\alpha = (p, q, r, p + q)$, $r \notin \{p, p + q\}$, or $\alpha = (p, p, q, q + r)$, the methods of the paper are inconclusive and a further analysis is needed.

The reader is warned however that our analysis of the equivariant problem 2.3 unfortunately does not yield new combinatorial results on the original fan partition problem (problem 2.1). Indeed, all the negative cases covered by part (A) of Theorem 4.10 imply the results already established by the original argument of Bárány and Matoušek [3], [4].

1.3. Obstruction theory in a nutshell. Suppose that X and Y are topological spaces and let G be a group acting on both of them, with the additional assumption that the action on X is **free**. Furthermore, let Z be a closed, G -invariant subspace of Y . We focus our attention in this paper on finite groups G and spaces which are simplicial or CW -complexes; see [19], [37] or [8] for a glossary of basic topological terms. The problem of deciding if there exists a G -equivariant map $f : X \rightarrow Y \setminus Z$ can be approached by two, closely related ways. The first is to build such a map step by step, defining it on the skeletons $X^{(k)}$ one at a time, and attempting to extend

it to the next skeleton $X^{(k+1)}$. The second is to start with a sufficiently generic, equivariant map $f : X \rightarrow Y$ and, in the case that the singularity $S(f) := f^{-1}(Z)$ is nonempty, try to modify f in an attempt to make the set $S(f)$ vanish. In both approaches there may appear *obstructions* which prevent us from completing the process, thus showing that such equivariant maps $f : X \rightarrow Y \setminus Z$ do not exist. In the first approach the obstruction is evaluated in the corresponding *equivariant cohomology group*, while in the second it lies in a “dual” *equivariant homology group*. When a G -action on X is free and the existence of a G -map f depends on the primary obstruction, these two methods coincide. The connection is the equivariant Poincaré duality map [8, Theorem 1.4]. In this paper we combine both approaches in the process of identifying the obstruction; see Section 4.1 for some technical details and references.

FIGURE 1

As emphasized in the introduction, we focus in this paper on “effective evaluation” of obstruction classes. This is achieved by a careful analysis of:

- (a) the singular set $S(f) = f^{-1}(Z)$, which is in this paper just a finite collection $S(f) = \{x_1, \dots, x_m\}$ of points in X ,
- (b) the associated “point classes” $\|f(x_i)\|$, understood as “weights” of points x_i , recording the position of $f(x_i)$ in Z and the way a small neighborhood of $x_i \in X$ is mapped to a small neighborhood of $f(x_i) \in Y$.

Two naive examples of this procedure are depicted in Figure 1, where (A) illustrates a nonequivariant problem where $X = Y = [-1, +1]$ and $Z = \{0\}$, while (B) corresponds to the case where $G = \mathbb{Z}_2$, $X = S^1$, $Y = [-1, +1]$ and $Z = \{0\}$. However simple, these examples capture some of the main features of the general approach, and the reader should keep in mind that all calculations in this paper follow essentially the same pattern. For example the passage from the figure (B) to the figure (A) illustrates the passage from the G -complex X to a “fundamental domain” $A \subset X$, cf. Section 3.1, which allows us to perform the analysis of (a) and (b) in a single, top-dimensional cell $e \subset X$.

2. THE MOTIVATING PROBLEM

2.1. Partition of measures by k -fans. A k -fan $\mathbf{p} = (x; l_1, l_2, \dots, l_k)$ on the sphere S^2 is a point x , called the center of the fan, and k great semicircles $l_1, \dots, l_k$ emanating from x . We always assume counterclockwise enumeration of great semicircles $l_1, \dots, l_k$ of a k -fan. Sometimes we use the notation $\mathbf{p} = (x; \sigma_1, \sigma_2, \dots, \sigma_k)$, where σ_i denotes the open angular sector between l_i and l_{i+1} , $i = 1, \dots, k$.

A measure μ on the sphere S^2 is *proper* if $\mu([a, b]) = 0$ for any circular arc $[a, b] \subset S^2$ and $\mu(U) > 0$ for each nonempty open set $U \subset S^2$.

Let $\mu_1, \mu_2, \dots, \mu_m$ be *proper* Borel probability measures on S^2 . All the results can be extended to more general measures, including the counting measures of finite sets; see [3], [34], [35] for related examples.

Let $(\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathbb{R}_{\geq 0}^k$ be a vector where $\alpha_1 + \alpha_2 + \dots + \alpha_k = 1$. Following [3] we say that a k -fan $(x; l_1, \dots, l_k)$ is an α -*partition* for the collection $\{\mu_j\}_{j=1}^m$ of measures if

$$(\forall i = 1, \dots, k) (\forall j = 1, \dots, m) \mu_j(\sigma_i) = \alpha_i.$$

A vector $\alpha \in \mathbb{R}^k$ is called (m, k) -*admissible* if for any collection of m (proper) measures on S^2 , there exists a simultaneous α -partition. The collection of all (m, k) -admissible vectors is denoted by $\mathcal{A}_{m,k}$. Here is the central problem concerning partitions of measures by k -fans.

Problem 2.1 ([3], [4], [31]). Describe the set $\mathcal{A}_{m,k}$ or, equivalently, find the integers m, k and vectors $\alpha \in \mathbb{R}^k$ such that for any collection $\mathcal{M} = \{\mu_1, \mu_2, \dots, \mu_m\}$ of m (proper) measures, there exists an α -partition for $\mathcal{M}$.

The analysis given in [3] shows that the most interesting cases of the problem are $(3, 2)$, $(2, 3)$ and $(2, 4)$. It was shown in [3] and [4] that $\{(\frac{1}{2}, \frac{1}{2}), (\frac{1}{3}, \frac{2}{3})\} \subseteq \mathcal{A}_{3,2}$, $\{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})\} \cup \{(\frac{a_1}{5}, \frac{a_2}{5}, \frac{a_3}{5})\} \subset \mathcal{A}_{2,3}$ and $\{(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}), (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5})\} \subseteq \mathcal{A}_{2,4}$.

2.2. The configuration space / test map scheme. The configuration space/test map scheme is a fairly general method of translating combinatorial geometric problems into topological problems, more precisely problems involving equivariant maps; see [19], [34], [36], [37]. It was demonstrated in [3] that the problem of α -partitions of spherical measures (Problem 2.1) also admits such a translation. We briefly review this construction focusing on the apparently most challenging case of 4-fan partitions of two measures on the 2-sphere.

Let μ and ν be two Borel, probability measures on S^2 and F_k , the space of all k -fans on the sphere S^2 . The space X_μ of all n -equipartitions of the measure μ is defined by

$$X_\mu = \{(x; l_1, \dots, l_n) \in F_n \mid (\forall i = 1, \dots, n) \mu(\sigma_i) = \frac{1}{n}\}.$$

Every n -fan $(x; l_1, \dots, l_n) \in X_\mu$ is completely determined by the pair (x, l_1) or equivalently the pair (x, y) , where y is the unit tangent vector to l_1 at x . Thus, the space X_μ is the Stiefel manifold $V_2(\mathbb{R}^3)$ of all orthonormal 2-frames in $\mathbb{R}^3$. Recall that $V_2(\mathbb{R}^3) \cong SO(3) \cong \mathbb{R}P^3$.

Let $\mathbb{R}^n$ be a Euclidean space with the standard orthonormal basis $e_1, e_2, \dots, e_n$ and $x_1, x_2, \dots, x_n$ the associated (dual) coordinate functions. Let W_n be the hyperplane $\{x \in \mathbb{R}^n \mid x_1 + x_2 + \dots + x_n = 0\}$ and suppose that the α -vector has the form $\alpha = (\frac{a_1}{n}, \frac{a_2}{n}, \frac{a_3}{n}, \frac{a_4}{n}) \in \frac{1}{n} \mathbb{N}^4 \subset \mathbb{Q}^4$ where $a_1 + a_2 + a_3 + a_4 = n$. Then the test

map $F_\nu : X_\mu \rightarrow W_n \subset \mathbb{R}^n$ for the measure ν is defined by

$$F_\nu(x, y) = F_\nu(x; l_1, \dots, l_n) = (\nu(\sigma_1) - \frac{1}{n}, \nu(\sigma_2) - \frac{1}{n}, \dots, \nu(\sigma_n) - \frac{1}{n}).$$

The natural group of symmetries arising in this problem is the dihedral group D_{2n} . It is interesting and often useful to forget a part of the structure and study the associated problem relative to other subgroups $G \subset D_{2n}$. The associated test subspace is in that case the space $D_G(\alpha) = D(\alpha) := \bigcup \mathcal{A}(\alpha) \subset W_n$ defined as the union of the smallest G -invariant linear subspace arrangement $\mathcal{A}(\alpha)$ in $\mathbb{R}^n$ containing the linear subspace $L(\alpha) \subset W_n$, where

$$(2.1) \quad L(\alpha) := \{x \in \mathbb{R}^n \mid z_1(x) = z_2(x) = z_3(x) = z_4(x) = 0\},$$

and

$$\begin{aligned} z_1(x) &= x_1 + x_2 + \dots + x_{a_1}, & z_2(x) &= x_{a_1+1} + \dots + x_{a_1+a_2}, \\ z_3(x) &= x_{a_1+a_2+1} + \dots + x_{a_1+a_2+a_3}, & z_4(x) &= x_{a_1+a_2+a_3+1} + \dots + x_n. \end{aligned}$$

Obviously $\dim L(\alpha) = n - 4$ and $\text{codim}_{W_n} L(\alpha) = 3$ since for $x \in W_n$, one of the equations $z_i(x) = 0$ is redundant.

The action of $D_{2n} = \langle \omega, \eta \mid \omega^n = \eta^2 = 1, \omega\eta = \eta\omega^{n-1} \rangle$ on the configuration space X_μ and the test space W_n , respectively, is given by

$$\begin{aligned} \omega(x; l_1, \dots, l_n) &= (x; l_2, \dots, l_n, l_1), & \eta(x; l_1, \dots, l_n) &= (-x; l_1, l_n, l_{n-1}, \dots, l_2), \\ \omega(x_1, \dots, x_n) &= (x_2, \dots, x_n, x_1), & \eta(x_1, \dots, x_n) &= (x_n, \dots, x_2, x_1), \end{aligned}$$

where $(x; l_1, \dots, l_n) \in X_\mu$ and $(x_1, \dots, x_n) \in W_n$. D_{2n} acts also on the complement $M(\alpha) = W_n \setminus D(\alpha)$ since $\mathcal{A}(\alpha)$ is the smallest D_{2n} -invariant, linear subspace arrangement which contains the linear subspace $L(\alpha)$. Note that the D_{2n} -action on the configuration space X_μ is **free**. The test map F_ν is a D_{2n} -equivariant. The configuration space X_μ is D_{2n} -homeomorphic to the Stiefel manifold $V_2(\mathbb{R}^3)$, which as a D_{2n} -space has the action described by

$$\omega(x, y) = (x, R_x(\frac{2\pi}{n})(y)), \quad \eta(x, y) = (-x, y),$$

where $R_x(\theta) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the rotation around the axes determined by x through the angle θ . The following lemma is a consequence of this analysis.

Lemma 2.2. *Let $\alpha = (\frac{a_1}{n}, \frac{a_2}{n}, \frac{a_3}{n}, \frac{a_4}{n}) \in \frac{1}{n}\mathbb{N}^4 \subset \mathbb{Q}^4$ be a vector such that $a_1 + a_2 + a_3 + a_4 = n$. Let G be a subgroup of the dihedral group D_{2n} . If there does not exist a G -map $F : V_2(\mathbb{R}^3) \rightarrow M_G(\alpha)$, where $M_G(\alpha) := W_n \setminus D_G(\alpha)$, then for any two measures μ and ν on S^2 , there always exists a 4-fan which simultaneously α -partitions both μ and ν . In other words the nonexistence of such a map implies $\alpha \in \mathcal{A}_{2,4}$.*

In light of Lemma 2.2, it is natural to focus our attention on the following problem as a problem closely related, albeit not equivalent, to the initial α -partition problem.

Problem 2.3. For a given subgroup G of the dihedral group D_{2n} , and the associated G -spaces $V_2(\mathbb{R}^3)$ and $M_G(\alpha)$, find an explicit description of the set

$$\Lambda_G = \{\alpha \in \frac{1}{n}\mathbb{N}^4 \mid \text{there is a } G\text{-equivariant map } F : V_2(\mathbb{R}^3) \rightarrow M_G(\alpha)\}.$$

2.3. The main problem. The well-known “extension of scalars” equivalence from homological algebra, [10, Section III.3], has a very useful analogue in the category of G -spaces and G -equivariant maps. This equivalence permits us in some cases to change the group and replace the original G -space by a new, more tractable topological space.

Let X be a left G -space and $H \triangleleft G$ be a normal subgroup. Then X/H is a G/H -space with the action given by $gH(Hx) = H(gx)$. A G/H -space Z can also be seen as a G -space via the quotient homomorphism $\pi : G \rightarrow G/H$ where for $g \in G$ and $z \in Z$, $g \cdot z := \pi(g)z$.

Lemma 2.4. *Suppose that X and Z are G -spaces and $H \triangleleft G$ is a normal subgroup of G which acts trivially on Z . Then there exists a G -map $\alpha : X \rightarrow Z$ if and only if there exists a G/H -map $\beta : X/H \rightarrow Z$, where X/G and $Z/H = Z$ are interpreted as G/H -spaces.*

Proof. $\Leftarrow$: Observe that the quotient map $p : X \rightarrow X/H$ is a G -map and that the G/H -space X/H is a G -space via the homomorphism $\pi : G \rightarrow G/H$. If $\beta : X/H \rightarrow Z$ is a G/H -map, it is also a G -map and the composition $\alpha := p \circ \beta : X \rightarrow Z$ is a G -map. $\Rightarrow$: Let $\alpha : X \rightarrow Z$ be a G -map. Then, since H acts trivially on Z , there is a factorization $\alpha = \beta \circ p$ for some $\beta : X/H \rightarrow Z$. We check that β is a G/H -map,

$$\beta(gH \cdot Hx) = \alpha(gx) = g\alpha(x) = g\beta(Hx) = (gH)\beta(Hx). \quad \square$$

As an application of Lemma 2.4 we replace the dihedral group D_{2n} by the generalized quaternion group Q_{4n} and the original configuration space $V_2(\mathbb{R}^3)$ by the sphere S^3 .

Let $S^3 = S(\mathbb{H}) = Sp(1)$ be the group of all unit quaternions and let $\epsilon = \epsilon_{2n} = \cos \frac{\pi}{n} + i \sin \frac{\pi}{n} \in S(\mathbb{H})$ be a root of unity. Group $\langle \epsilon \rangle$ is a subgroup of $S(\mathbb{H})$ of order $2n$. Then, the *generalized quaternion group*, [11, p. 253], is the subgroup

$$Q_{4n} = \{1, \epsilon, \dots, \epsilon^{2n-1}, j, \epsilon j, \dots, \epsilon^{2n-1}j\}$$

of S^3 of order $4n$. Let $H = \{1, \epsilon^n\} = \{1, -1\} \subset Q_{4n}$. The quotient group Q_{4n}/H is isomorphic to the dihedral group D_{2n} of order $2n$.

Lemma 2.5. *There exists a D_{2n} -map $G : V_2(\mathbb{R}^3) \rightarrow M(\alpha)$ if and only if there exists a Q_{4n} -map $F : S^3 \rightarrow M(\alpha)$ where the generalized quaternion group Q_{4n} acts on S^3 as a subgroup of $Sp(1) \cong S^3$, and its action on W_n is described by $\epsilon(x_1, \dots, x_n) = (x_2, \dots, x_n, x_1)$ and $j(x_1, \dots, x_n) = (x_n, \dots, x_2, x_1)$.*

Proof. Note that $\epsilon^n = j^2$ acts trivially on W_n . Moreover, there is an isomorphism $S^3/H \cong \mathbb{R}P^3 \cong SO(3) \cong V_2(\mathbb{R}^3)$. Finally, the $Q_{4n}/H \cong D_{2n}$ action on $S^3/H \cong V_2(\mathbb{R}^3)$ coincides with the D_{2n} -action on $V_2(\mathbb{R}^3)$ described in the preceding section. Hence, this lemma is a direct consequence of Lemma 2.4. $\square$

Remark 2.6. The Q_{4n} -action on S^3 is free. The Q_{4n} -action on W_n is the restriction of the action on $\mathbb{R}^n$ described by

$$\epsilon \cdot e_i = e_{(i+1) \bmod n} \text{ and } j \cdot e_i = e_{n-i+1},$$

where $e_1, \dots, e_n$ is the standard orthonormal basis in $\mathbb{R}^n$.

Lemma 2.5 implies the equivalence of Problem 2.3 with the following problem, which appears to be more tractable from the point of view of equivariant obstruction theory.

Problem 2.7. For a subgroup G of Q_{4n} such that $H = \{1, -1\} \subset G \subseteq Q_{4n}$, describe the set

$$\Gamma_G = \{\alpha \in \frac{1}{n}\mathbb{N}^4 \mid \text{there is a } G\text{-equivariant map } F : S^3 \rightarrow M_G(\alpha)\}.$$

3. TOPOLOGICAL PRELIMINARIES

3.1. The geometry of the group Q_{4n} . In this section two Q_{4n} -invariant CW-structures on S^3 are described. This information is used in the obstruction cocycle computations in Section 4.2.

FIGURE 2. The sphere $S^3 = P_{2n}^{(1)} * P_{2n}^{(2)}$ in the case $n = 3$

Let us represent the sphere S^3 as the join $S^3 = P_{2n}^{(1)} * P_{2n}^{(2)}$ of two $2n$ -gons $P_{2n}^{(1)}$ and $P_{2n}^{(2)}$ with respective vertices v_i and w_j , $0 \leq i, j \leq 2n-1$. The case $n = 3$ is depicted in Figure 2, where $S^3 = S^1 * S^1 = [0, 1] * [0, 1] / \approx$ and “ $\approx$ ” is the equivalence relation on the 3-simplex $\sigma^3 := [0, 1] * [0, 1]$ arising from the identification $S^1 \cong [0, 1] / 0 \sim 1$. The (big) rectangle represents the torus $T^2 := \{\frac{1}{2}x + \frac{1}{2}y \mid x, y \in S^1\}$ subdivided into small rectangles which are the intersections of T^2 with simplices $[v_i, v_{i+1}; w_j, w_{j+1}]$. Since each point $a = \frac{1}{2}x + \frac{1}{2}y$ of this rectangle uniquely determines the line segment $[x, y] \subset S^3$, each subset $A \subset T^2$ determines uniquely a set $\hat{A} \subset S^3$ such that $\hat{A} \cap T^2 = A$. Let $A_1 := \bigcup_{j=0}^{n-1} [v_j, v_{j+1}] \times [w_0, w_1]$ be a rectangle in T^2 and A_2 a closely related “skew” rectangle; see Figure 2. Then both $\hat{A}_1$ and $\hat{A}_2$ are fundamental domains for the Q_{4n} -action on S^3 , both inducing Q_{4n} -invariant CW-structures on S^3 . Recall that a cell A is a fundamental domain for a free G -action of a finite group G on a manifold M if the collection of cells $\{g(A)\}_{g \in G}$ defines a tiling of the manifold in the sense that $M = \bigcup \{g(A)\}_{g \in G}$ and the interiors of cells gA and hA are disjoint for $g \neq h$. Once the fundamental domain A is chosen, its boundary provides all the necessary information about the lower dimensional cells of the associated CW-decomposition.

An analysis of the boundary of the fundamental domain A_2 shows that the associated CW-structure has only one Q_{4n} -cell a in dimension 0, two Q_{4n} -cells b and b' in dimension 1, two Q_{4n} -cells c and c' in dimension 2, and one Q_{4n} -cell e in dimension 3. The reader can easily check that the associated cellular chain complex $\{D_i(S^3, \mathbb{Z})\}_{i=0}^3$ of Q_{4n} -modules coincides with the well-known minimal resolution

of $\mathbb{Z}$ by free Q_{4n} -modules described on page 253 of the classical monograph [11]. More precisely,

$$0 \rightarrow \mathbb{Z}(Q_{4n})e \xrightarrow{\partial} \mathbb{Z}(Q_{4n})c \oplus \mathbb{Z}(Q_{4n})c' \xrightarrow{\partial} \mathbb{Z}(Q_{4n})b \oplus \mathbb{Z}(Q_{4n})b' \xrightarrow{\partial} \mathbb{Z}(Q_{4n})a \rightarrow 0,$$

where

$$\begin{aligned} \partial e &= (\epsilon - 1)c - (\epsilon j - 1)c'; & \partial c &= Nb - (j + 1)b'; & \partial c' &= (\epsilon j + 1)b + (\epsilon - 1)b'; \\ \partial b &= (\epsilon - 1)a; & \partial b' &= (j - 1)a \end{aligned}$$

and $N = 1 + \epsilon + \dots + \epsilon^{n-1} \in \mathbb{Z}(Q_{4n})$. On applying the functor $\text{Hom}_{Q_{4n}}(\cdot, N)$, where $N := H_2(M; \mathbb{Z})$ is a Q_{4n} -module, one obtains the sequence (assuming $\mathbb{Z}$ coefficients),

$$0 \longleftarrow H_2(M) \xleftarrow{\Gamma} H_2(M) \oplus H_2(M) \longleftarrow H_2(M) \oplus H_2(M) \longleftarrow H_2(M) \longleftarrow 0,$$

where $\Gamma(p, q) = (\epsilon - 1)p - (\epsilon j - 1)q$ for $p, q \in H_2(M; \mathbb{Z})$. Now it is not hard to prove

$$H_{Q_{4n}}^3(S^3, H_2(M; \mathbb{Z})) = H_2(M; \mathbb{Z})/\text{Im}\Gamma \cong H_2(M; \mathbb{Z})_{Q_{4n}},$$

where $H_2(M; \mathbb{Z})_{Q_{4n}}$ is the group of coinvariants of the Q_{4n} -module $H_2(M; \mathbb{Z})$, [10, Exercise 1, p. 44]. Alternatively, the last result can be seen as a consequence of the equivariant Poincaré duality, [8, Theorem 1.4].

The complex associated with the fundamental domain A_1 , Figure 2, is not as economical as the complex based on the fundamental domain A_2 . Note however that for our purposes this is not a serious disadvantage. Indeed, all we need to know is the intersection of $e_1 = \text{int}(A_1)$ with the singular set $h^{-1}(\bigcup \mathcal{A}(\alpha))$, for a generic Q_{4n} -equivariant map h , and interpret the answer as an element of the group $H_2(M; \mathbb{Z})_{Q_{4n}}$, cf. Sections 4.2.1 and 4.2.3. On the other hand, the choice of A_1 has an advantage that its top cell $e_1 = A_1$ is a union of simplices from the original simplicial structure on S^3 , Figure 2, which considerably simplifies the computations performed in Section 4.2.3.

3.2. The “point” classes of an arrangement. Assume that orientations on all maximal elements of the arrangement $\mathcal{A}$ and on the ambient space V are given and fixed. The orientations on the orthogonal complements $L_i^\perp$ are chosen in such a way that together with the orientations on L_i , they induce the orientation prescribed on V .

Let $\mathcal{A}$ be an arrangement of linear subspaces in the $(a+b)$ -dimensional Euclidean space V and $\{L_1, \dots, L_k\}$ the set of its elements of maximal dimension. Let us assume that $\dim L_1 = \dots = \dim L_k = a$. Denote the complement of the arrangement by $M = V \setminus \bigcup \mathcal{A}$.

For a given point $x \in L_i \setminus \bigcup_{j \neq i} L_j$, let $D_\epsilon(x) = x + D_\epsilon$ be a small disc around x , where $D_\epsilon = \{y \in L_i^\perp \mid \langle y, y \rangle \leq \epsilon\}$. “Small” means that ϵ is chosen so that $D_\epsilon(x) \cap (\bigcup \mathcal{A}) = \{x\}$. We assume that $D_\epsilon(x)$ is oriented, typically by the orientation inherited from already fixed orientation on $L_i^\perp$. The fundamental class of the pair $(D_\epsilon(x), \partial D_\epsilon(x))$ determines a homology class in $H_b(V, M; \mathbb{Z})$, which we denote by $[x]$ and call the *point class* of x . Note that by the Excision axiom, $[x]$ does not depend on ϵ . Moreover, by the Homotopy axiom, $[x]$ does not change if x is moved inside a connected component of $L_i \setminus \bigcup_{j \neq i} L_j$. Similarly, in light of the isomorphism $H_b(V, M; \mathbb{Z}) \rightarrow H_{b-1}(M; \mathbb{Z})$, the class $\|x\| := \partial[x]$, which is also called the *point class* of x , has all these invariance properties as well.

Let $\hat{V}$ and $\hat{L}_i$ stand for spheres obtained by the one-point compactification. By the definition of the point class $\|x\|$ and the associated orientation, the linking number of $\partial D_\varepsilon(x)$ and $\hat{L}_i$ inside $\hat{V}$ is $+1$.

The point classes have the following properties, which can be deduced from Poincaré-Alexander duality and the Goresky-MacPherson formula.

Proposition 3.1.

- (A) For every maximal element L_i and every $x \in L_i \setminus \bigcup_{j \neq i} L_j$ the class $\|x\| \in H_{b-1}(M; \mathbb{Z})$ is nontrivial.
- (B) If the codimension of $L_{ij} := L_i \cap L_j$ in L_i is greater than 1 for each $j \neq i$ and $x \in L_i \setminus \bigcup_{j \neq i} L_j$, then the point class $\|x\|$ does not depend on $x \in L_i$.
- (C) Let $\text{codim}_{L_i}(L_i \cap L_j) = 1$ for some $j \neq i$ where $L_i = L_i^1 \cup L_i^2$, $L_j = L_j^1 \cup L_j^2$ are decompositions into closed half spaces. If points $x_1 \in \text{int} L_i^1$ and $x_2 \in \text{int} L_i^2$ then $\|x_1\| \neq \|x_2\|$.

FIGURE 3. The point classes

The property of point classes (C) from the previous proposition is illustrated in Figure 3a.

Proposition 3.2. Assume that θ is a b -dimensional simplex in V such that $\theta \cap \bigcup \mathcal{A} = \text{int}(\theta) \cap \bigcup \mathcal{A} = \{x_1, \dots, x_k\}$ is a finite set of points and

$$(\forall x \in \theta \cap \bigcup \mathcal{A}) (\exists \text{ unique } L_x \text{ maximal element of } \mathcal{A}) \{x\} = \theta \cap L_x.$$

The homology class $l \in H_{b-1}(M)$ represented by the boundary $\partial\theta$ of the simplex θ can be represented as the linear combination of point classes (Figure 3b) in the following way:

$$(3.1) \quad l = I(\theta, L_{x_1}) \|x_1\| + \dots + I(\theta, L_{x_k}) \|x_k\|,$$

where $I(\theta, L_{x_i}) \in \{-1, +1\}$ is the intersection number of the oriented simplex θ and L_{x_i} . Here we assume the already fixed orientation on L_{x_i} used in the definition of the point class $\|x_i\|$.

The equality (3.1) is easily deduced from the Excision axiom. It is also implied by the existence of a bordism between $\partial\theta$ and the union of spheres around each x_i ; see Figure 3b.

4. CALCULATION OF THE OBSTRUCTION ELEMENT

In this section we introduce, following [13], the obstruction exact sequence and apply it to the question of the existence of a Q_{4n} -equivariant map $S^3 \rightarrow M_{Q_{4n}}(\alpha)$. The application of the obstruction theory is possible because the Q_{4n} -action on S^3 is **free**.

4.1. Obstruction exact sequence. The complement $M = M_{Q_{4n}}(\alpha)$ of the arrangement $\mathcal{A}(\alpha)$ is 1-connected and consequently **2-simple** in the sense that $\pi_1(M)$ acts trivially on $\pi_2(M)$. Then by [13, Section II.3], and by the Hurewicz theorem, there is an isomorphism $\pi_2(M) \cong [S^2, M] \cong H_2(M; \mathbb{Z})$ of Q_{4n} -modules. Thus, the relevant part of the obstruction exact sequence, [13], has the following form:

$$[S^3, M]_{Q_{4n}} \xrightarrow{\theta} \text{Im} \left\{ [S^3_{(2)}, M]_{Q_{4n}} \longrightarrow [S^3_{(1)}, M]_{Q_{4n}} \right\} \xrightarrow{\tau} H^3_{Q_{4n}}(S^3, H_2(M; \mathbb{Z})).$$

Here $S^3_{(1)}$ and $S^3_{(2)}$ are respectively the 1 and 2-skeleton of the sphere S^3 , relative to some Q_{4n} -invariant simplicial or CW-structure. Our initial choice is the simplicial structure arising from the join decomposition $S^3 = P^{(1)}_{2n} * P^{(2)}_{2n}$, where both $P^{(1)}_{2n}$ and $P^{(2)}_{2n}$ are regular $(2n)$ -sided polygons; see Figure 2. More precisely, the vertices of these polygons are, respectively, $v_i := \epsilon^{i-1}a$ and $w_i := \epsilon^i ja$, $i = 0, 1, \dots, 2n-1$, where a is a fixed complex number, $|a| = 1$. The reason we define v_i as $\epsilon^{i-1}a$ is technical and will be explained in Section 4.2.1. Since $[S^3_{(1)}, M]_{Q_{4n}} = \{*\}$ is a one-element set and $[S^3_{(2)}, M]_{Q_{4n}}$ is nonempty, the obstruction exact sequence reduces to

$$[S^3, M]_{Q_{4n}} \longrightarrow \{*\} \xrightarrow{\tau} H^3_{Q_{4n}}(S^3, H_2(M; \mathbb{Z})).$$

The exactness of this sequence means that the set $[S^3, M]_{Q_{4n}} \neq \emptyset$ if and only if the element $\tau(*) \in H^3_{Q_{4n}}(S^3, H_2(M; \mathbb{Z}))$ is equal to zero. The class $\tau(*)$ depends only on M .

The class $\tau(*)$ can be evaluated by studying the “singular set” of a general position equivariant map, [8, Definition 1.5]. A Q_{4n} -simplicial map $h : S^3 \rightarrow W_n$ has a “general position property” if for each simplex σ in S^3 and any linear space U in the arrangement $\mathcal{A}(\alpha)$ which intersects the h -image of σ , the following conditions are satisfied:

$$(4.1) \quad \begin{array}{ll} \text{(A)} & \dim(\sigma) = \dim(h(\sigma)) = 3, \\ \text{(B)} & \dim(U) = n - 3, \\ \text{(C)} & h(\sigma) \cap U = \{pt\} \subset \text{int}(h(\sigma)). \end{array}$$

It is not difficult to check that a “generic map” is in general position and that every equivariant, simplicial map can be put in general position by a small perturbation. For each Q_{4n} -map $h : S^3 \rightarrow W_n$ in general position, there is an associated obstruction cocycle

$$c(h) \in C^3_{Q_{4n}}(S^3, A) = \text{Hom}_{Q_{4n}}(C_3(S^3), A), \quad [c(h)] = \tau(*),$$

where $A = H_2(M; \mathbb{Z})$. If σ is an oriented 3-simplex in S^3 , then $c(h)(\sigma) \in H_2(M; \mathbb{Z})$ is the image $h_*([\partial(\sigma)])$ of the fundamental class of $\partial(\sigma) \cong S^2$ by the map $h_* : H_2(\partial(\sigma); \mathbb{Z}) \rightarrow H_2(M; \mathbb{Z})$. To find an explicit form of the obstruction cocycle $c(h) \in C^3_{Q_{4n}}(S^3, H_2(M; \mathbb{Z})) = \text{Hom}_{Q_{4n}}(C_3(S^3), H_2(M; \mathbb{Z}))$ we recall that by definition

$$c(h)(\theta) \neq 0 \iff h(\theta) \cap \left(\bigcup \mathcal{A}(\alpha) \right) \neq \emptyset$$

and $c(h)(\theta) = h_*([\partial\theta]) \in H_2(M; \mathbb{Z})$.

Before we go further, let us record for future reference an important property of the obstruction cocycle. More details on the restriction and the transfer map can be found in [10, Section III.9].

Lemma 4.1. *The cohomology class of the obstruction cocycle $c(h)$ is a torsion element of the group $H_{Q_{4n}}^3(S^3, H_2(M; \mathbb{Z}))$.*

Proof. Let H be a subgroup of Q_{4n} . There exists a natural “restriction” map $r : H_{Q_{4n}}^3(S^3, H_2(M; \mathbb{Z})) \rightarrow H_H^3(S^3, H_2(M; \mathbb{Z}))$, which on the cochain level is just the “forgetful map” sending a Q_{4n} -cochain $c \in C_{Q_{4n}}^3(S^3, H_2(M; \mathbb{Z}))$ to the same cochain interpreted as an H -cochain. It follows from the definition of the obstruction cocycle that $r(c(h))$ is the obstruction cocycle for the extension of a general position H -map h . Moreover there exists a natural map $\tau : H_H^3(S^3, H_2(M; \mathbb{Z})) \rightarrow H_{Q_{4n}}^3(S^3, H_2(M; \mathbb{Z}))$ in the opposite direction called the transfer map. It is known, [10, Section III.9, Proposition 9.5.(ii)], that the composition of the restriction with the transfer is just multiplication by the index $[Q_{4n} : H]$:

$$\begin{array}{ccccc} H_{Q_{4n}}^3(S^3, H_2(M; \mathbb{Z})) & \rightarrow & H_H^3(S^3, H_2(M; \mathbb{Z})) & \rightarrow & H_{Q_{4n}}^3(S^3, H_2(M; \mathbb{Z})) \\ [c(h)_{Q_{4n}}] & \mapsto & [c(h)_H] & \mapsto & [Q_{4n} : H] \cdot [c(h)_{Q_{4n}}]. \end{array}$$

Note that if H is the trivial group, the cohomology class of the obstruction cocycle $[c(h)_H]$ is zero. This implies that $[G : H] \cdot [c(h)_{Q_{4n}}] = 0$ in $H_{Q_{4n}}^3(S^3, H_2(M; \mathbb{Z}))$; i.e., $[c(h)_{Q_{4n}}]$ is a torsion element of the group $H_{Q_{4n}}^3(S^3, H_2(M; \mathbb{Z}))$. $\square$

Remark 4.2. A consequence of Lemma 4.1 is that special attention should be given to the torsion part of the group $H_{Q_{4n}}^3(S^3, H_2(M; \mathbb{Z}))$.

4.2. The obstruction cocycle.

4.2.1. General position Q_{4n} -equivariant maps. Let us start with the description of a general position, simplicial Q_{4n} -map $h : S^3 \rightarrow W_n$, where the sphere has the simplicial structure $S^3 = P_{2n}^{(1)} * P_{2n}^{(2)}$, described in Section 4.1 and depicted in Figure 2. This will be our standard choice of a general position, Q_{4n} -map, fixed throughout the paper. Let $e_1, \dots, e_n$ be the standard orthonormal basis in $\mathbb{R}^n$ and let $x_1, \dots, x_n$ be the associated dual linear functions. Let $\{u_1, \dots, u_n\}$, where $u_i = e_i - e$ and $e = \frac{1}{n} \sum_{r=1}^n e_r$, be the vertex set of a regular simplex Δ_{n-1} in W_n . If the map h is prescribed in advance on a vertex $t \in P_{2n}^{(1)}$ and if we require that it is a simplicial Q_{4n} -map, then everything else is completely determined. Let $h(t) = u_1$. Then,

$$\begin{aligned} (4.2) \quad h(\epsilon^i t) &= \epsilon^i \cdot h(t) = \epsilon^i \cdot u_1 = \epsilon^i \cdot (e_1 - e) = e_{(i+1) \bmod n} - e = u_{(i+1) \bmod n}, \\ h(jt) &= j \cdot h(t) = j \cdot u_1 = j \cdot (e_1 - e) = e_n - e = u_n, \\ h(\epsilon^i jt) &= \epsilon^i j \cdot h(t) = \epsilon^i \cdot (e_n - e) = e_{(i+n) \bmod n} - e = e_i - e = u_i \bmod n. \end{aligned}$$

If the vertices of $P_{2n}^{(1)}$ and $P_{2n}^{(2)}$ respectively are $v_i := \epsilon^{i-1}t$ and $w_i := \epsilon^i jt$, then the equations (4.2) can be rewritten in a particularly simple form as $h(v_i) = h(w_i) = u_i$ for each i .

4.2.2. The singular set. To decide whether h is a general position map, it is necessary to determine which simplices $\sigma = \sigma_1 * \sigma_2 \subset P_{2n}^{(1)} * P_{2n}^{(2)}$ have the property

that the image $h(\sigma)$ intersects the subspace $L = L(\alpha)$. Note that $h(P_{2n}^{(1)} * P_{2n}^{(2)}) \subseteq \text{sk}_3(\Delta_{n-1})$, where $\text{sk}_j(K)$ is the j -skeleton of a simplicial complex K . Hence, in order to determine $L \cap h(P_{2n}^{(1)} * P_{2n}^{(2)})$, we first find the intersection $L \cap \text{sk}_3(\Delta_{n-1})$. Note that by the defining equations (2.1), for a given increasing sequence of indices $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$, a simplex $\sigma = \text{conv}\{u_{i_1}, u_{i_2}, u_{i_3}, u_{i_4}\} \in \text{sk}_3(\Delta_{n-1})$ intersects $L = L(\alpha)$ if and only if

$$1 \leq i_1 \leq a_1 < a_1 + 1 \leq i_2 \leq a_2 < a_2 + 1 \leq i_3 \leq a_3 < a_3 + 1 \leq i_4 \leq n.$$

Moreover, if these conditions are satisfied, then $y = \lambda_1 u_{i_1} + \dots + \lambda_4 u_{i_4} \in L \cap \text{sk}_3(\Delta_{n-1})$ if and only if

$$\lambda_1 = \frac{a_1}{n}, \quad \lambda_2 = \frac{a_2}{n}, \quad \lambda_3 = \frac{a_3}{n}, \quad \lambda_4 = \frac{a_4}{n}.$$

Suppose that σ is in the image $h(P_{2n}^{(1)} * P_{2n}^{(2)})$. Then, keeping in mind that a typical simplex in $h(P_{2n}^{(1)} * P_{2n}^{(2)})$ has vertices $v_k, v_{k+1}, w_l, w_{l+1}$ and that $h(v_i) = h(w_i) = u_i$, we conclude that either $|i_2 - i_1| = 1$ and $|i_4 - i_3| = 1$ or, in the opposite case, $|i_3 - i_2| = 1$ and $|i_1 - i_4| = 1 \bmod n$. It can be checked that $L \cap h(P_{2n}^{(1)} * P_{2n}^{(2)}) = \{y_1, y_2\}$, where

$$(4.3) \quad \begin{aligned} y_1 &= \frac{a_1}{n} u_{a_1} + \frac{a_2}{n} u_{a_1+1} + \frac{a_3}{n} u_{a_1+a_2+a_3} + \frac{a_4}{n} u_{a_1+a_2+a_3+1}, \\ y_2 &= \frac{a_2}{n} u_{a_1+a_2} + \frac{a_3}{n} u_{a_1+a_2+1} + \frac{a_4}{n} u_n + \frac{a_1}{n} u_1. \end{aligned}$$

This means that there are only two 3-simplices

$$\tau_1 = [u_{a_1}, u_{a_1+1}; u_{a_1+a_2+a_3}, u_{a_1+a_2+a_3+1}] \text{ and } \tau_2 = [u_{a_1+a_2}, u_{a_1+a_2+1}; u_n, u_1]$$

in $h(P_{2n}^{(1)} * P_{2n}^{(2)})$ which intersect L . Note that the point of intersection y_i is in the interior of τ_i ; hence h is a map that always satisfies the first and the third part of the general position condition (4.1) in Section 4.1. The remaining condition that $h(\sigma) \cap U \neq \emptyset$ implies $\dim(U) = n - 3$ will be checked separately in Section 4.2.3, cases **(A)** to **(E)**.

In order to find the singular set $h^{-1}(\bigcup \mathcal{A}(\alpha)) \subseteq S^3$, we have to detect all 3-simplices $\sigma = \sigma_1 * \sigma_2 \subset P_{2n}^{(1)} * P_{2n}^{(2)}$ in the sphere S^3 with the property $\sigma \cap h^{-1}(L) \neq \emptyset$, or in other words the simplices $\sigma = [v_k, v_{k+1}; w_l, w_{l+1}]$ such that either $h(\sigma) = \tau_1$ or $h(\sigma) = \tau_2$. This leads us to the following four systems of equations:

$$(4.4) \quad \begin{array}{ll} h(v_k) = u_{k \bmod n} = u_{a_1} & \text{and} \quad h(w_l) = u_{l \bmod n} = u_{a_1+a_2+a_3}, \\ h(v_k) = u_{k \bmod n} = u_{a_1+a_2+a_3} & \text{and} \quad h(w_l) = u_{l \bmod n} = u_{a_1}, \\ h(v_k) = u_{k \bmod n} = u_{a_1+a_2} & \text{and} \quad h(w_l) = u_{l \bmod n} = u_n, \\ h(v_k) = u_{k \bmod n} = u_n & \text{and} \quad h(w_l) = u_{l \bmod n} = u_{a_1+a_2}. \end{array}$$

It turns out that there are 16, not necessarily different, 3-simplices $\theta_1, \dots, \theta_{16}$ in $P_{2n}^{(1)} * P_{2n}^{(2)}$ which have a nonempty intersection with the singular subset $h^{-1}(L)$, or equivalently which are solutions to one of the four systems of equations (4.4). Here is a complete list where as before v_i and w_j for $i, j \in \{0, \dots, 2n-1\}$ are vertices of

 FIGURE 4. The simplex θ_{11} in the sphere $S^3 = P_{2n}^{(1)} * P_{2n}^{(2)}$

$P_{2n}^{(1)}$ and $P_{2n}^{(2)}$ respectively and $P = a_1$, $Q = a_1 + a_2$, $R = a_1 + a_2 + a_3$. The reader may find Figure 4 useful as a bookkeeping device for generating simplices θ_i :

$$\begin{aligned} \theta_1 &= [v_P, v_{P+1}; w_R, w_{R+1}], & \theta_2 &= [v_{n+P}, v_{n+P+1}; w_R, w_{R+1}], \\ \theta_3 &= [v_P, v_{P+1}; w_{n+R}, w_{n+R+1}], & \theta_4 &= [v_{n+P}, v_{n+P+1}; w_{n+R}, w_{n+R+1}], \\ \theta_5 &= [v_R, v_{R+1}; w_P, w_{P+1}], & \theta_6 &= [v_{n+R}, v_{n+R+1}; w_P, w_{P+1}], \\ \theta_7 &= [v_R, v_{R+1}; w_{n+P}, w_{n+P+1}], & \theta_8 &= [v_{n+R}, v_{n+R+1}; w_{n+P}, w_{n+P+1}], \\ \theta_9 &= [v_Q, v_{Q+1}; w_n, w_{n+1}], & \theta_{10} &= [v_{n+Q}, v_{n+Q+1}; w_n, w_{n+1}], \\ \theta_{11} &= [v_Q, v_{Q+1}; w_0, w_1], & \theta_{12} &= [v_{n+Q}, v_{n+Q+1}; w_0, w_1], \\ \theta_{13} &= [v_n, v_{n+1}; w_Q, w_{Q+1}], & \theta_{14} &= [v_{2n}, v_1; w_Q, w_{Q+1}], \\ \theta_{15} &= [v_n, v_{n+1}; w_{n+Q}, w_{n+Q+1}], & \theta_{16} &= [v_{2n}, v_1; w_{n+Q}, w_{n+Q+1}]. \end{aligned}$$

Then

$$h(\theta_i) = \tau_1 \text{ if } i \in \{1, \dots, 8\} \text{ and } h(\theta_i) = \tau_2 \text{ if } i \in \{9, \dots, 16\}.$$

Simplices θ_1 to θ_4 arise as the solutions of the first system in (4.4), simplices θ_5 to θ_8 are solutions to the second system, etc.

4.2.3. *The obstruction cocycle $c_{Q_{4n}}(h)$.* By the definition of the obstruction cocycle, for a given 3-simplex θ of S^3 ,

$$c_{Q_{4n}}(h)(\theta) = [\partial h(\theta)],$$

where $[\partial h(\theta)]$ is the homology class determined by the h image of the boundary of θ . Proposition 3.2 implies the formula

$$c_{Q_{4n}}(h)(\theta) = \sum_{y \in h(\theta) \cap \bigcup \mathcal{A}(\alpha)} I(h(\theta), L_y) \|y\|.$$

In order to simplify the computation, it is convenient to replace the simplicial complex structure $P_{2n}^{(1)} * P_{2n}^{(2)}$ on S^3 and its chain complex $\{C_i(S^3, \mathbb{Z})\}_{i=0}^3$ by another, more economical Q_{4n} -invariant, cell complex structure on S^3 . In Section 3.1 two such complexes are introduced. The chain complex $\{D_i(S^3, \mathbb{Z})\}_{i=0}^3$, associated

to the fundamental domain A_2 , is the most economical. However its serious disadvantage is that its top cell e is a “skew cell” in the sense that it is not a union of simplices from the original simplicial decomposition $P_{2n}^{(1)} * P_{2n}^{(2)}$. This is the reason why we work with the cell decomposition associated to the fundamental domain A_1 . As noted in Section 3.1, all that appears in calculations is the top-dimensional cell $e_1 = A_1 = \bigcup_{j=0}^{n-1} [v_j, v_{j+1}; w_0, w_1]$ of this CW-decomposition, so there is no need here to reconstruct the whole CW-complex and the associated chain complex.

The obstruction class computed relative to the cell complex structure based on the cell e_1 , described in Section 3.1, is denoted by $d_{Q_{4n}}(h)$. To evaluate the obstruction cocycle $d_{Q_{4n}}(h)$ on the cell e_1 , we are supposed to determine how many simplices of the form $g \cdot \theta_i$, where $g \in Q_{4n}$ and $i \in \{1, \dots, 16\}$, belong to the cell e_1 . By definition of e_1 , our objective is to determine all elements $g \in Q_{4n}$ and the corresponding simplices θ_i such that $g \cdot \theta_i$ is a simplex of the form $[v_k, v_{k+1}; w_0, w_1]$ for some $0 \leq k \leq n-1$. Note that $jv_k = w_{1-k}$ and $jw_l = v_{n-l+1}$, where the indices in all expressions involving v_k and w_l are evaluated modulo $2n$. It follows that

$$\begin{aligned} \epsilon^i[v_k, v_{k+1}; w_l, w_{l+1}] &= [v_{k+i}, v_{k+i+1}; w_{l+i}, w_{l+i+1}], \\ j[v_k, v_{k+1}; w_l, w_{l+1}] &= [v_{n-l}, v_{n-l+1}; w_{-k}, w_{-k+1}]. \end{aligned}$$

In short, our objective is to inspect the list of all simplices θ_i , $i = 1, \dots, 16$ and determine which group operations turn these simplices into simplices of the form $[v_k, v_{k+1}; w_0, w_1]$ for some $0 \leq k \leq n-1$. By direct calculations we arrive at the following list of simplices and the corresponding group elements:

$$\begin{aligned} \sigma_1 &= [v_{a_1+a_4}, v_{a_1+a_4+1}; w_0, w_1] = \epsilon^{-a_1-a_2-a_3}\theta_2 = j\epsilon^{-a_1}\theta_1, \\ \sigma_2 &= [v_{a_2+a_3}, v_{a_2+a_3+1}; w_0, w_1] = \epsilon^{-a_1}\theta_5 = j\epsilon^{-a_1-a_2-a_3}\theta_7, \\ \sigma_3 &= [v_{a_2+a_1}, v_{a_2+a_1+1}; w_0, w_1] = \theta_{11} = j\epsilon^{-a_1-a_2}\theta_9, \\ \sigma_4 &= [v_{a_3+a_4}, v_{a_3+a_4+1}; w_0, w_1] = \epsilon^{-a_1-a_2}\theta_{13} = j\theta_{14}. \end{aligned}$$

Strictly speaking there are two more representations of the form $g \cdot \theta_j$ for each of the simplices σ_i . Indeed, if $\sigma_i = g \cdot \theta_j$, then $\sigma_i = g' \cdot \theta_{j'}$, where $\theta_{j'} = \epsilon^n \theta_j$ and $g' = g\epsilon^n$.

We conclude that there are four (not necessarily different!) simplices $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ in the cell e_1 such that $h(\sigma_i) \cap D(\alpha) \neq \emptyset$ for each $i \in \{1, 2, 3, 4\}$. Our next observation is that for each of the simplices $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ the intersection $h(\sigma_i) \cap D(\alpha)$ has two (not necessarily different!) points. More explicitly these points are

$$\begin{aligned} x_{11} &= \frac{a_1}{n}v_{a_1+a_4} + \frac{a_2}{n}v_{a_1+a_4+1} + \frac{a_3}{n}w_0 + \frac{a_4}{n}w_1 \in \sigma_1 \cap h^{-1}(\epsilon^{-a_1-a_2-a_3}L), \\ x_{12} &= \frac{a_4}{n}v_{a_1+a_4} + \frac{a_3}{n}v_{a_1+a_4+1} + \frac{a_2}{n}w_0 + \frac{a_1}{n}w_1 \in \sigma_1 \cap h^{-1}(j\epsilon^{-a_1}L), \\ x_{21} &= \frac{a_3}{n}v_{a_2+a_3} + \frac{a_4}{n}v_{a_2+a_3+1} + \frac{a_1}{n}w_0 + \frac{a_2}{n}w_1 \in \sigma_2 \cap h^{-1}(\epsilon^{-a_1}L), \\ x_{22} &= \frac{a_2}{n}v_{a_2+a_3} + \frac{a_1}{n}v_{a_2+a_3+1} + \frac{a_4}{n}w_0 + \frac{a_3}{n}w_1 \in \sigma_2 \cap h^{-1}(j\epsilon^{-a_1-a_2-a_3}L), \\ x_{31} &= \frac{a_2}{n}v_{a_2+a_1} + \frac{a_3}{n}v_{a_2+a_1+1} + \frac{a_4}{n}w_0 + \frac{a_1}{n}w_1 \in \sigma_3 \cap h^{-1}(L), \\ x_{32} &= \frac{a_1}{n}v_{a_2+a_1} + \frac{a_4}{n}v_{a_2+a_1+1} + \frac{a_3}{n}w_0 + \frac{a_2}{n}w_1 \in \sigma_3 \cap h^{-1}(j\epsilon^{-a_1-a_2}L), \\ x_{41} &= \frac{a_4}{n}v_{a_3+a_4} + \frac{a_1}{n}v_{a_3+a_4+1} + \frac{a_2}{n}w_0 + \frac{a_3}{n}w_1 \in \sigma_4 \cap h^{-1}(\epsilon^{-a_1-a_2}L), \\ x_{42} &= \frac{a_3}{n}v_{a_3+a_4} + \frac{a_2}{n}v_{a_3+a_4+1} + \frac{a_1}{n}w_0 + \frac{a_4}{n}w_1 \in \sigma_4 \cap h^{-1}(jL). \end{aligned}$$

From here we deduce that

$$\begin{aligned} h(\sigma_1) \cap D(\alpha) &= (h(\sigma_1) \cap \epsilon^{-a_1-a_2-a_3}L) \cup (h(\sigma_1) \cap j\epsilon^{-a_1}L) = \{h(x_{11}), h(x_{12})\}; \\ h(\sigma_2) \cap D(\alpha) &= (h(\sigma_2) \cap \epsilon^{-a_1}L) \cup (h(\sigma_2) \cap j\epsilon^{-a_1-a_2-a_3}L) = \{h(x_{21}), h(x_{22})\}; \\ h(\sigma_3) \cap D(\alpha) &= (h(\sigma_3) \cap L) \cup (h(\sigma_3) \cap j\epsilon^{-a_1-a_2}L) = \{h(x_{31}), h(x_{32})\}; \\ h(\sigma_4) \cap D(\alpha) &= (h(\sigma_4) \cap \epsilon^{-a_1-a_2}L) \cup (h(\sigma_4) \cap jL) = \{h(x_{41}), h(x_{42})\} \end{aligned}$$

and finally

$$(4.5) \quad h(e_1) \cap D(\alpha) = h(\{x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}, x_{41}, x_{42}\}).$$

We have already remarked that some of the points $h(x_{ij})$ in (4.5) may coincide; i.e., the cardinality of $h(e_1) \cap D(\alpha)$ can be strictly less than 8. It turns out that this cardinality is very sensitive to the choice of integers a_1, a_2, a_3, a_4 . Also, this forces us to check if some one of the intersection points in $h(\sigma) \cap D(\alpha)$ is an element of two or more different maximal elements of the arrangement $\mathcal{A}(\alpha)$. If this would be the case, the assumption $h(\sigma) \cap U \neq \emptyset \Rightarrow \dim(U) = n - 3$ in (4.1) would be violated. Fortunately the inner symmetries of the arrangement $\mathcal{A}(\alpha)$ will provide that the general position assumption for the map h is satisfied.

(A) Let $a_2 = a_4$, $a_1 \neq a_3$. Then $\sigma_1 = \sigma_3$, $\sigma_2 = \sigma_4$ and $x_{11} = x_{32}$, $x_{12} = x_{31}$, $x_{21} = x_{42}$, $x_{22} = x_{41}$. These identities imply that

$$\begin{aligned} h(x_{11}) &= h(x_{32}) \in \epsilon^{-a_1-a_2-a_3}L \cap j\epsilon^{-a_1-a_2}L, & h(x_{12}) &= h(x_{31}) \in j\epsilon^{-a_1}L \cap L, \\ h(x_{22}) &= h(x_{41}) \in j\epsilon^{-a_1-a_2-a_3}L \cap \epsilon^{-a_1-a_2}L, & h(x_{21}) &= h(x_{42}) \in \epsilon^{-a_1}L \cap jL. \end{aligned}$$

The “symmetry” set equality $\epsilon^{-a_1}L = jL$ implies

$$\begin{aligned} j\epsilon^{-a_1-a_2}L &= \epsilon^{(n-1)(-a_1-a_2)}jL = \epsilon^{a_1+a_2}jL = \epsilon^{a_2}L = \epsilon^{a_4}L = \epsilon^{-a_1-a_2-a_3}L, \\ \epsilon^{-a_1}L &= jL, & j\epsilon^{-a_1}L &= \epsilon^{(n-1)(-a_1)}jL = \epsilon^{a_1}jL = L, \\ j\epsilon^{-a_1-a_2-a_3}L &= j\epsilon^{a_4}L = \epsilon^{(n-1)a_4}jL = \epsilon^{-a_4-a_1}L = \epsilon^{-a_2-a_1}L. \end{aligned}$$

Thus, for each point $h(x_{ij})$ there exists a unique, $(n - 3)$ -dimensional subspace $U \subset D(\alpha)$ such that $h(x_{ij}) \in U$.

The case when $a_1 = a_3$, $a_2 \neq a_4$ can be discussed in a similar way with the same conclusion that map h satisfies the general position assumption.

(B) When $a_2 = a_4 \neq a_1 = a_3$ the simplices $\sigma_1, \sigma_3, \sigma_2, \sigma_4$ coincide and $x_{11} = x_{32} = x_{21} = x_{42}$, $x_{12} = x_{31} = x_{22} = x_{41}$. In this case,

$$\begin{aligned} h(x_{11}) &= h(x_{32}) = h(x_{21}) = h(x_{42}) \in \epsilon^{-a_1-a_2-a_3}L \cap j\epsilon^{-a_1-a_2}L \cap \epsilon^{-a_1}L \cap jL, \\ h(x_{12}) &= h(x_{31}) = h(x_{22}) = h(x_{41}) \in j\epsilon^{-a_1}L \cap L \cap j\epsilon^{-a_1-a_2-a_3}L \cap \epsilon^{-a_1-a_2}L. \end{aligned}$$

The set of equalities $\epsilon^{a_1+a_2}L = L$ and $\epsilon^{a_2}L = jL$ imply that h is in general position:

$$\epsilon^{-a_1}L = \epsilon^{a_2}L = jL = j\epsilon^{a_1+a_2}L = j\epsilon^{-a_1-a_2}L = \epsilon^{-a_1-a_2-a_3}L.$$

(C) Assume $a_1 = a_2 = a_3 = a_4$. Then $\sigma_1 = \sigma_3 = \sigma_2 = \sigma_4$ and $x_{11} = x_{32} = x_{21} = x_{42} = x_{12} = x_{31} = x_{22} = x_{41}$. Therefore $h(x_{11}) = \dots = h(x_{41})$ belongs to

$$\epsilon^{-a_1-a_2-a_3}L \cap j\epsilon^{-a_1-a_2}L \cap \epsilon^{-a_1}L \cap jL \cap j\epsilon^{-a_1}L \cap L \cap j\epsilon^{-a_1-a_2-a_3}L \cap \epsilon^{-a_1-a_2}L.$$

Now, with the help of two “symmetry” equalities $\epsilon^{a_1}L = L$ and $jL = L$, we obtain

$$j\epsilon^{-a_1-a_2-a_3}L = j\epsilon^{-a_1-a_2}L = j\epsilon^{-a_1}L = jL = L = \epsilon^{-a_1}L = \epsilon^{-a_1-a_2-a_3}L = \epsilon^{-a_1-a_2}L.$$

(D) Let $a_1 = a_2 \neq a_3 = a_4$. Then $\sigma_1 = \sigma_2$ and $x_{11} = x_{22}$, $x_{12} = x_{21}$. In this case,

$$h(x_{11}) = h(x_{22}) \in \epsilon^{-a_1-a_2-a_3}L \cap j\epsilon^{-a_1-a_2-a_3}L, \quad h(x_{12}) = h(x_{21}) \in j\epsilon^{-a_1}L \cap \epsilon^{-a_1}L.$$

With the equality $jL = \epsilon^{2a_3}L$ we have

$$\begin{aligned} j\epsilon^{-a_1-a_2-a_3}L &= j\epsilon^{a_4}L = \epsilon^{(n-1)a_4}jL = \epsilon^{-a_4+2a_3}L = \epsilon^{a_4}L = \epsilon^{-a_1-a_2-a_3}L, \\ j\epsilon^{-a_1} &= \epsilon^{(n-1)(-a_1)}jL = \epsilon^{a_1+2a_3}L = \epsilon^{-a_1}L, \end{aligned}$$

and conclude that h is in general position.

(E) In all the remaining cases,

$$h(e) \cap D(\alpha) = \{h(x_{11}), h(x_{12}), h(x_{21}), h(x_{22}), h(x_{31}), h(x_{32}), h(x_{41}), h(x_{42})\}.$$

Let H be a G -module and K an additive subgroup of H generated by the elements of the form $g \cdot x - x$, $g \in G$, $x \in H$. The quotient group H/K is the group of coinvariants [10] of the module H and is denoted by H_G .

Theorem 4.3. *Let $d_{Q_{4n}}(h) \in C_{Q_{4n}}^3(S^3, H_2(M; \mathbb{Z}))$ denote the obstruction cocycle for the map h defined above.*

(A) *Let $\alpha = (p, p, p, p)$. Then*

$$d_{Q_{4n}}(h)(e) = \|y_1\| + K.$$

(B) *Let $\alpha = (p, q, r, q)$ or $\alpha = (q, p, q, r)$, $p \neq r$. Then*

$$[d_{Q_{4n}}(h)(e)] = 4\|y_1\| + K.$$

(C) *Let $\alpha = (p, q, p, q)$, $p \neq q$. Then*

$$[d_{Q_{4n}}(h)(e)] = 2\|y_1\| + K.$$

(D) *Let $\alpha = (p, p, q, q)$ or $\alpha = (p, q, q, p)$, $p \neq q$. Then*

$$[d_{Q_{4n}}(h)(e)] = 2\|y_1\| - 2\|y_2\| + K.$$

(E) *In all the remaining cases,*

$$[d_{Q_{4n}}(h)(e)] = 4\|y_1\| - 4\|y_2\| + K.$$

Proof. Before we calculate obstruction cocycles in all these cases remember that

$$c_{Q_{4n}}(h)(\theta) = \sum_{y \in h(\theta) \cap D(\alpha)} I(h(\theta), L_y) \|y\|.$$

Also, for every $g \in Q_{4n}$ there is an identity [14, Proposition 4.5, p. 198]

$$I(h(g\theta), L_{g \cdot y}) = I(h(\theta), L_y).$$

The elementary linear algebra calculation implies that

$$I(h(\theta_i), L) = \begin{cases} 1, & i \in \{1, \dots, 8\} \\ -1, & i \in \{9, \dots, 16\}. \end{cases} \quad h(\theta_i) \cap L = h(x_{\theta_i}) = \begin{cases} y_1, & i \in \{1, \dots, 8\} \\ y_2, & i \in \{9, \dots, 16\} \end{cases}.$$

We assume that the orientation on the subspace $\epsilon^r jL$ is induced by the orientation on L . If there is a set equality $\epsilon^r jL = \epsilon^k L$, then we prefer the orientation induced by $\epsilon^k L$. From the defining equality for the obstruction cocycle we get

$$d_{Q_{4n}}(h)(e) = \sum_{\theta \subset e: h(\theta) \cap D(\alpha) \neq \emptyset} [\partial h(\theta)] = \sum_{\theta \subset e} \sum_{y \in h(\theta) \cap D(\alpha)} I(h(\theta), L_y) \|y\|.$$

Here $[\partial h(\theta)]$ denotes the homology class determined by the h image of the boundary of the simplex θ . Also, observe that

$$\begin{aligned} \|h(x_{11})\| &= \epsilon^{-a_1-a_2-a_3} \|y_1\|, & \|h(x_{12})\| &= j\epsilon^{-a_1} \|y_1\|, \\ \|h(x_{21})\| &= \epsilon^{-a_1} \|y_1\|, & \|h(x_{22})\| &= j\epsilon^{-a_1-a_2-a_3} \|y_1\|, \\ \|h(x_{31})\| &= \|y_2\|, & \|h(x_{32})\| &= j\epsilon^{-a_1-a_2} \|y_2\|, \\ \|h(x_{41})\| &= \epsilon^{-a_1-a_2} \|y_2\|, & \|h(x_{42})\| &= j\|y_2\|. \end{aligned}$$

Now we are ready to calculate the obstruction cocycle $d_{Q_{4n}}(h)(e)$. (A) In the case $a_1 = a_2 = a_3 = a_4$, according to previous discussion,

$$d_{Q_{4n}}(h)(e) = \|h(x_{31})\| = \|y_1\|.$$

(B) If $a_2 = a_4$ and $a_1 \neq a_3$, or $a_1 = a_3$ and $a_2 \neq a_4$, then

$$d_{Q_{4n}}(h)(e) = (\epsilon^{-a_1-a_2-a_3} + j\epsilon^{-a_1} + \epsilon^{-a_1} + j\epsilon^{-a_1-a_2-a_3}) \|y_1\|.$$

(C) If $a_2 = a_4 \neq a_1 = a_3$, then

$$d_{Q_{4n}}(h)(e) = (\epsilon^{-a_1-a_2-a_3} + j\epsilon^{-a_1}) \|y_1\|.$$

(D) If $a_1 = a_2 \neq a_3 = a_4$, or $a_2 = a_3 \neq a_1 = a_4$, then

$$d_{Q_{4n}}(h)(e) = (\epsilon^{-a_1-a_2-a_3} + j\epsilon^{-a_1}) \|y_1\| - (1 + \epsilon^{-a_1-a_2}) \|y_2\|.$$

(F) In all the remaining cases, we have

$$\begin{aligned} d_{Q_{4n}}(h)(e) &= (\epsilon^{-a_1-a_2-a_3} + j\epsilon^{-a_1} + \epsilon^{-a_1} + j\epsilon^{-a_1-a_2-a_3}) \|y_1\| \\ &\quad - (1 + j\epsilon^{-a_1-a_2} + \epsilon^{-a_1-a_2} + j) \|y_2\|. \end{aligned} \quad \square$$

4.3. The cohomology group $H_{Q_{4n}}^3(S^3, H_2(M; \mathbb{Z}))$. The isomorphism

$$H_{Q_{4n}}^3(S^3, H_2(M; \mathbb{Z})) \cong H_2(M; \mathbb{Z})_{Q_{4n}}$$

motivates the study of the Q_{4n} -module structure on $H_2(M(\alpha); \mathbb{Z})$.

4.3.1. The Homology of $H_2(M(\alpha); \mathbb{Z})$. The basic idea for computing the homology $H_2(M; \mathbb{Z})$ as an Abelian group is to rely on the Poincaré-Alexander duality and work with the arrangement $\mathcal{A}(\alpha)$ instead of the complement M . The complement $M = W_n \setminus \bigcup \mathcal{A}(\alpha)$ can be written in the form $S^{n-1} \setminus \bigcup \hat{\mathcal{A}}(\alpha)$, where $\hat{\mathcal{A}}(\alpha)$ is the one-point compactification of the arrangement $\mathcal{A}(\alpha)$. There is a sequence of isomorphisms

$$\begin{aligned} (4.6) \quad H_2(M, \mathbb{Z}) &= H_2(S^{n-1} \setminus \bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}) \quad \text{Poincaré-Alexander duality} \\ &\cong H^{(n-1)-2-1}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}) \quad \text{Universal Coefficient theorem} \\ &\cong \text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}(H_{n-5}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z}). \end{aligned}$$

The following observation allows us to simplify the computation of the group $\text{Ext}(H_{n-5}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z})$ in all cases we are interested in.

Lemma 4.4. *If X is a simplicial complex which is homotopy equivalent to a k -dimensional complex Y , then $H_k(X; \mathbb{Z})$ has no torsion and consequently the group $\text{Ext}(H_k(X; \mathbb{Z}); \mathbb{Z})$ is a trivial group.*

In the process of computing the homology $H_2(M, \mathbb{Z})$ via the isomorphism (4.8), when the Ext-group is trivial, we use the Goresky-MacPherson decomposition (with $\mathbb{Z}$ coefficients)

$$\begin{aligned} H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha)) &\cong \bigoplus_{V \in P(\alpha)} H_{n-4}(\Delta(P(\alpha)_{<V}) * S^{\dim V}) \\ &\cong \bigoplus_{d=0}^{n-4} \bigoplus_{V \in P(\alpha): \dim V=d} \tilde{H}_{n-5-d}(\Delta(P(\alpha)_{<V})), \end{aligned}$$

where $P(\alpha)$ is the intersection poset of the arrangement $\mathcal{A}(\alpha)$. Hence, we have to determine

$$\tilde{H}_{n-5-\dim p}(\Delta(P(\alpha)_{<V}))$$

for each $V \in P(\alpha)$, where by convention $\tilde{H}_{-1}(\emptyset) = \mathbb{Z}$.

Theorem 4.5. *Let $M = W_n \setminus \bigcup \mathcal{A}(\alpha)$ be the complement of the minimal D_{2n} -invariant arrangement $\mathcal{A}(\alpha)$ containing the subspace $L(\alpha)$ defined by (2.1) (or equivalently the Q_{4n} -arrangement, where j^2 acts trivially). Let $\alpha = (a_1, a_2, a_3, a_4) \in \mathbb{N}^4$ such that $a_1 + a_2 + a_3 + a_4 = n$.*

- (A) *If $\alpha = (p, p, p, p)$, then $H_2(M; \mathbb{Z}) \cong \mathbb{Z}^{n/4}$.*
- (B) *If $\alpha = (p, q, p, q)$ and $p \neq q$, then $H_2(M; \mathbb{Z}) \cong \mathbb{Z}^{n/2}$.*
- (C) *If $\alpha = (p, p, p, 2p)$, then $H_2(M; \mathbb{Z}) \cong \mathbb{Z}^{n + \frac{4p}{5}}$.*
- (D) *If $\alpha = (1, 1, 1, 3)$, then $H_2(M; \mathbb{Z}) \cong \mathbb{Z}^{13}$.*
- (E) *If $\alpha = (q, q, q, p)$, $p \notin \{q, 2q\}$ and $3q + p > 6$, then $H_2(M; \mathbb{Z}) \cong \mathbb{Z}^{2n}$.*
- (F) *If $\alpha = (p, q, p, p+q)$ and $p \neq q$, then $H_2(M; \mathbb{Z}) \cong \mathbb{Z}^{2n}$.*
- (G) *If $\alpha = (p, p, 2p, 2p)$, then $H_2(M; \mathbb{Z}) \cong \mathbb{Z}^{2n + \frac{n}{3}}$.*
- (H) *If $\alpha = (p, q, p+q, p+q)$ and $p \neq q$, then $H_2(M; \mathbb{Z}) \cong ?$.*
- (I) *If $\alpha = (p, p, q, q)$ and $p \neq q$, $p \neq 2q$, $q \neq 2p$, then $H_2(M; \mathbb{Z}) \cong \mathbb{Z}^n$.*
- (J) *If $\alpha = (p, q, r, q)$ and $r \notin \{q, p, p+q\}$, then $H_2(M; \mathbb{Z}) \cong \mathbb{Z}^n$.*
- (K) *If $\alpha = (p, q, r, p+q)$ and $r \notin \{p, p+q\}$, then $H_2(M; \mathbb{Z}) \cong ?$.*
- (L) *If $\alpha = (p, p, q, q+r)$, and $p \neq q$, $r \neq 0$, then $H_2(M; \mathbb{Z}) \cong ?$.*
- (M) *In all the remaining cases $H_2(M; \mathbb{Z}) \cong \mathbb{Z}^{2n}$.*

Remark 4.6. The computation of $H_2(M; \mathbb{Z})$ in the cases (H), (K) and (L) is probably within reach of the existing methods; however, we leave it as an open problem.

The computation of $H_2(M; \mathbb{Z})$ relies on the isomorphism (4.6) and the fact that the homology group $H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$ is free. This means that we actually compute $H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$; i.e., with the help of the Goresky-MacPherson formula we compute $\tilde{H}_{\dim L - \dim p - 1}(\Delta(P(\alpha)_{<V}; \mathbb{Z}))$ for each $V \in P(\alpha)$.

Proof. (A). In this case there are $\frac{n}{4}$ maximal elements $L, \epsilon L, \dots, \epsilon^{p-1}L$ of the arrangement $\mathcal{A}(\alpha)$. Since for every $i, j \in \{0, \dots, p-1\}$, $i \neq j$,

$$(4.7) \quad \dim L - \dim(\epsilon^i L \cap \epsilon^j L) > 1,$$

the Goresky-MacPherson formula implies that only maximal elements contribute to $H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$. The equation (4.7) also guarantees that the assumption of Lemma 4.4 is satisfied, so the corresponding Ext-group is trivial.

(B), (I), (J) and (M). The arrangements for these cases have the same property as in the case (A),

$$(\forall g \in D_{2n}) \quad L \neq gL \Rightarrow \dim L - \dim(L \cap gL) > 1.$$

Only the number of maximal elements differs. Thus the calculations follow in the footsteps of the case (A).

(C). The arrangement has n maximal elements. The four among them $\epsilon^p L$, $\epsilon^{2p} L$, $\epsilon^{3p} L$ and $\epsilon^{4p} L$ have the property $\dim(L \cap \epsilon^{k \cdot p} L) = \dim L - 1$, $k = 1, \dots, 4$. In addition,

$$L \cap \epsilon^p L = L \cap \epsilon^{2p} L = L \cap \epsilon^{3p} L = L \cap \epsilon^{4p} L.$$

Denote the intersection $L \cap \epsilon^q L \cap \epsilon^{2q} L \cap \epsilon^{3q} L \cap \epsilon^{4q} L$ by V . Then the Hasse diagram of the subposet $\{p \in P(\alpha) : \dim p \geq n - 5\}$ is depicted in Figure 5. Before using

FIGURE 5. The Hasse diagram for the case (C)

the Goresky-MacPherson decomposition observe that for $n \geq 5$,

$$(\forall V \in P(\alpha)) \dim V \leq n - 6 \implies \tilde{H}_{n-5-\dim V}(\Delta(P(\alpha)_{<V})) = 0.$$

Indeed, for each element $W \in P(\alpha)_{<V}$ such that $\dim W = n - 4$ there exists a unique element $V_W \in P(\alpha)_{<V}$ with the property $\dim V_W = n - 5$ and $W < V_W$. There is a monotone map $f : P(\alpha)_{<V} \rightarrow P(\alpha)_{<V} - \{W \mid \dim W = n - 4\}$ defined by

$$W \longmapsto \begin{cases} W & \text{for } \dim W \leq n - 5, \\ V_W & \text{for } \dim W = n - 4, \end{cases}$$

which satisfies the conditions of the Quillen fiber lemma. This implies that f induces a homotopy equivalence; hence

$$\tilde{H}_{n-5-\dim V}(\Delta(P(\alpha)_{<V})) = \tilde{H}_{n-5-\dim V}(\Delta(Q)) = 0$$

since $\dim \Delta(Q) < n - 5 - \dim V$. So, the only relevant part of the poset $P(\alpha)$ for the computation of $H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$ is the part in Figure 5. Similarly, this analysis guarantees that the conditions of Lemma 4.4 are satisfied, which again implies that the associated Ext-group is trivial.

(D). The proof follows from the Goresky-MacPherson decomposition and the Hasse diagram of the intersection poset, shown in Figure 6.

FIGURE 6. The Hasse diagram for the case (D)

(E). When $\alpha = (a_1, a_2, a_3, a_4) = (q, q, q, p)$, $p \neq 2q$, the arrangement has n maximal elements and exactly two, $\epsilon^q L$ and $\epsilon^{2q+p} L$, have maximal L intersection, $\dim(L \cap \epsilon^q L) = \dim(L \cap \epsilon^{2q+p} L) = \dim L - 1$. It is not hard to describe the first

three levels of the intersection poset $P(\alpha)$. There are n elements $L, \epsilon L, \dots, \epsilon^{n-1}L$ of dimension $n-4$, n elements

$$\begin{aligned} & L \cap \epsilon^q L, \epsilon^q L \cap \epsilon^{2q} L, \dots, \epsilon^{2q+p} L \cap L; \\ & \epsilon L \cap \epsilon^{q+1} L, \epsilon^{q+1} L \cap \epsilon^{2q+1} L, \dots, \epsilon^{2q+p+1} L \cap \epsilon L; \\ & \epsilon^{d-1} L \cap \epsilon^{q+d-1} L, \epsilon^{q+d-1} L \cap \epsilon^{2q+d-1} L, \dots, \epsilon^{2q+p+d-1} L \cap \epsilon^{d-1} L \end{aligned}$$

of dimension $n-5$ (where $d = (q, n)$) and finally n elements

$$L \cap \epsilon^q L \cap \epsilon^{2q+p} L, \epsilon(L \cap \epsilon^q L \cap \epsilon^{2q+p} L), \dots, \epsilon^{n-1}(L \cap \epsilon^q L \cap \epsilon^{2q+p} L)$$

of dimension $n-6$. Thus, the Hasse diagram of the subposet $\{V \in P(\alpha) : \dim V \geq n-5\}$ is of the form depicted in Figure 7. We prove that only this part of

FIGURE 7. The first Hasse diagram for the case (E)

the Hasse diagram of the intersection poset $P(\alpha)$ contributes to the homology $H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$. Indeed, for $n > 6$,

$$(\forall V \in P(\alpha)) \dim V \leq n-6 \implies \tilde{H}_{n-5-\dim V}(\Delta(P(\alpha)_{<V})) = 0.$$

Observe that $\dim \Delta(P(\alpha)_{<V}) \leq n-5-\dim V$. If $\dim \Delta(P(\alpha)_{<V}) < n-5-\dim V$ we have nothing to do, so we assume that $\dim \Delta(P(\alpha)_{<V}) = n-5-\dim V$. Let $K = \{W \in P(\alpha)_{<V} \mid \dim W \neq n-6\}$. Let us show that the inclusion $i : K \hookrightarrow P(\alpha)_{<V}$ satisfies the assumptions of the Quillen fiber lemma. It suffices to check that for every $W = \epsilon^i(L \cap \epsilon^q L \cap \epsilon^{2q+p} L) \in P(\alpha)_{<V} - K$ the order complex $\Delta(i^{-1}((P(\alpha)_{<V})_{\leq W}))$ is contractible. For such a $W = \epsilon^i(L \cap \epsilon^q L \cap \epsilon^{2q+p} L)$, we observe that $\Delta(i^{-1}((P(\alpha)_{<V})_{\leq W}))$ is contractible. Hence, $\Delta(P(\alpha)_{<V}) \simeq \Delta(K)$. Since $\dim \Delta(P(\alpha)_{<V}) < n-5-\dim V$, we conclude that $\tilde{H}_{n-5-\dim V}(K) = 0$. As before, from the Goresky-MacPherson formula we deduce the following decomposition:

$$H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha); \mathbb{Z}) \cong \bigoplus_{d=n-5}^{n-4} \bigoplus_{V \in P(\alpha) : \dim V = d} \tilde{H}_{n-5-d}(\Delta(P(\alpha)_{<V}); \mathbb{Z}) \cong \mathbb{Z}^n \oplus \mathbb{Z}^n.$$

The case of the group Ext is handled as in previous cases.

(F). There are exactly two maximal elements $\epsilon^{p+q}L$ and $\epsilon^{2p+q}L$ with the property

$$\dim(L \cap \epsilon^{p+q}L) = \dim(L \cap \epsilon^{2p+q}L) = \dim L - 1.$$

Since the subposet $\{V \in P(\alpha) : \dim V \geq n-5\}$ of $P(\alpha)$ has the same shape as the appropriate subposet in the proof of part (E), it is not hard to see that the computation for this case goes completely along the lines of the preceding proof.

(G). In this case the arrangement has n maximal elements and two maximal elements, $\epsilon^{2p}L$ and $\epsilon^{4p}L$, with the maximal L intersection,

$$\dim(L \cap \epsilon^{2p}L) = \dim(L \cap \epsilon^{4p}L) = \dim L - 1.$$

The elements of $P(\alpha)$ of dimension $n-5$ are: $L \cap \epsilon^{2p}L, \epsilon(L \cap \epsilon^{2p}L), \dots, \epsilon^{n-1}(L \cap \epsilon^{2p}L)$. There are $2p$ elements of dimension $n-6$ for $n > 6$: $L \cap \epsilon^{2p}L \cap \epsilon^{4p}L, \epsilon(L \cap \epsilon^{2p}L \cap \epsilon^{4p}L), \dots, \epsilon^{2p}(L \cap \epsilon^{2p}L \cap \epsilon^{4p}L)$ and only one for $n = 6$. We use this information to

draw the Hasse diagram (Figure 8) for the subposet $\{V \in P(\alpha) : \dim V \geq n - 6\}$. Let us convince ourselves that for $n > 6$

FIGURE 8. The Hasse diagram for the case (G)

$$(\forall V \in P(\alpha)) \dim V \leq n - 7 \implies \tilde{H}_{n-5-\dim V}(\Delta(P(\alpha)_{<V})) = 0.$$

Let $R \stackrel{\text{def}}{=} P(\alpha)_{<V} \cap \{W \in P(\alpha) \mid \dim W = n - 6\} = \{V_1, \dots, V_k\}$ where $k \leq 2p$ and $Q = P(\alpha)_{<V} \cap \{W \in P(\alpha) \mid \dim W \leq n - 6\}$. Since for every element $W \in P(\alpha)_{<V} \cap \{W \in P(\alpha) \mid \dim W > n - 6\}$ there exists a unique element $V_W \in R$ such that $W < V_W$, we can define a monotone map $f : P(\alpha)_{<V} \rightarrow Q$ by the formula

$$f(W) = \begin{cases} W, & \dim W \leq n - 6, \\ V_W, & \dim W > n - 6. \end{cases}$$

Again the Quillen fiber lemma implies that f induces a homotopy equivalence $\Delta(P(\alpha)_{<V}) \simeq \Delta(Q)$. Indeed, for each $W \in Q$ the poset $f^{-1}((P(\alpha)_{<V})_{\leq W})$ has a maximum; hence $\Delta(f^{-1}((P(\alpha)_{<V})_{\leq W}))$ is contractible. It follows from $\dim \Delta(Q) < n - 5 - \dim V$ that

$$\tilde{H}_{n-5-\dim V}(\Delta(P(\alpha)_{<V})) = \tilde{H}_{n-5-\dim V}(\Delta(Q)) = 0.$$

From here we infer that the Goresky-MacPherson formula decomposition reduces to the first three levels of the Hasse diagram of $P(\alpha)$ and

$$H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}) \cong \mathbb{Z}^n \oplus \mathbb{Z}^n \oplus \mathbb{Z}^{\frac{n}{3}}.$$

□

4.3.2. *The Q_{4n} -module structure on $H_2(M(\alpha); \mathbb{Z})$.* To detect the Q_{4n} -module structure on $H_2(M(\alpha); \mathbb{Z})$ we have to understand how the isomorphisms

$$(4.8) \quad H_2(M, \mathbb{Z}) \xrightarrow{\phi} \text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z}) \xrightarrow{\psi} H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}),$$

valid if $\text{Ext}(H_{n-5}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z}) = 0$, respect the action of the group. In essence the isomorphism ϕ is given by the linking number of homology classes in the sphere $\hat{W}_n$. The Poincaré-Alexander duality map is a G -map up to an “orientation character”. The duality map is defined as the map $X \mapsto X \cap O$, where O is a fundamental class of S^{n-1} and “ $\cap$ ” the usual “cap”-pairing between homology and cohomology. So in order to describe the G -action via the Poincaré-Alexander duality (4.6), one has to take into account the associated “orientation character”, i.e. the way O transforms by the action of G . Since the action of G on $W_n \subset S^{n-1}$ is linear, $g(O) = \det(g) \cdot O$, for each g in G . The isomorphism of the Universal Coefficient Theorem (4.6) is a

G -map when the action on $\text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z})$ is given by

$$(g \cdot \xi)(x) = \xi(g^{-1} \cdot x)$$

for $g \in Q_{4n}$, $\xi \in \text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z})$ and $x \in H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$. The map ϕ ,

$$(4.9) \quad H_2(M, \mathbb{Z}) \rightarrow \text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z}),$$

is a Q_{4n} isomorphism if the “ $*$ ”-action on $\text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z})$ is given by

$$(4.10) \quad (g * \xi)(x) := \det(g) \xi(g^{-1} \cdot x) = \det(g^{-1}) \xi(g^{-1} \cdot x).$$

Therefore,

$$(4.11) \quad H_2(M, \mathbb{Z})_{Q_{4n}} \cong \text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z})_{Q_{4n}}.$$

Remark 4.7. As we have seen in Theorem 4.5, the homology $H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$ is free. Therefore, there is an **isomorphism of abelian groups**

$$(4.12) \quad \text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z}) \cong H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}).$$

The isomorphism (4.12) **does not** induce an isomorphism of Q_{4n} -module structures given by ambient or modified actions. This fact is an extremely important detail [26], [24]. Therefore, we are forced to work inside the Q_{4n} -module $\text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z})$ or directly inside $H_2(M, \mathbb{Z})$. Nevertheless, there are situations when these two module structures coincide. For example, let G be a finite group and H a subgroup. Let the G -module structure on $\xi \in \text{Hom}(\mathbb{Z}[G/H], \mathbb{Z})$ be defined by $(g \cdot \xi)(x) := \xi(g^{-1} \cdot x)$, for $x \in \mathbb{Z}[G/H]$. Then there is a G -module isomorphism

$$\text{Hom}(\mathbb{Z}[G/H], \mathbb{Z}) \cong \mathbb{Z}[G/H].$$

Theorem 4.8. *Let $M = W_n \setminus \bigcup \mathcal{A}(\alpha)$ be the complement of the minimal D_{2n} -invariant arrangement $\mathcal{A}(\alpha)$ containing the subspace $L(\alpha)$ defined by (2.1) (or equivalently the Q_{4n} -arrangement, where j^2 acts trivially). Let $\alpha = (a_1, a_2, a_3, a_4) \in \mathbb{N}^4$ such that $a_1 + a_2 + a_3 + a_4 = n$.*

(A) *If $\alpha = (p, p, p, p)$, then $H_2(M; \mathbb{Z}) \cong \mathbb{Z}[Q_{4n}]\xi / \langle (\epsilon^p + 1)\xi, (j - 1)\xi \rangle$.*

(B) *If $\alpha = (p, q, p, q)$ and $p \neq q$, then*

$$H_2(M; \mathbb{Z}) \cong \mathbb{Z}[Q_{4n}]\xi / \langle (\epsilon^{p+q} - 1)\xi, (j^2 - 1)\xi, (\epsilon^{-q}j + 1)\xi \rangle.$$

(C) *If $\alpha = (p, p, p, 2p)$ and*

$$K = \langle (1 + \epsilon^p + \epsilon^{2p} + \epsilon^{3p} + \epsilon^{4p} + j + \epsilon^p j + \epsilon^{2p} j + \epsilon^{3p} j + \epsilon^{4p} j)\xi, (j^2 - 1)\xi, (\epsilon^n - 1)\xi \rangle,$$

then

$$H_2(M; \mathbb{Z}) \cong \mathbb{Z}[Q_{4n}]\xi / K.$$

(D) *If $\alpha = (1, 1, 1, 3)$ and*

$$K = \langle (\epsilon^{-3}j + 1)\xi, (\epsilon^6 - 1)\xi, (1 + \epsilon^{-2}j)\eta, (\epsilon^6 - 1)\eta, (1 - j)\zeta, (1 + \epsilon)\zeta \rangle,$$

then

$$H_2(M; \mathbb{Z}) \cong \mathbb{Z}[Q_{4n}]\xi \oplus \mathbb{Z}[Q_{4n}]\eta \oplus \mathbb{Z}[Q_{4n}]\zeta / K.$$

(E) *If $\alpha = (q, q, q, p)$, $p \notin \{q, 2q\}$ and $3q + p > 6$, then*

$$H_2(M; \mathbb{Z}) \cong \mathbb{Z}[Q_{4n}]\xi \oplus \mathbb{Z}[Q_{4n}]\eta / \langle (1 + \epsilon^{-p}j)\xi, (\epsilon^n - 1)\xi, (j^2 - 1)\eta, (1 + \epsilon^{-2p-2q}j)\eta \rangle.$$

(F) *If $\alpha = (p, q, p, p + q)$ and $p \neq q$, then*

$$H_2(M; \mathbb{Z}) \cong \mathbb{Z}[Q_{4n}]\xi \oplus \mathbb{Z}[Q_{4n}]\eta / \langle (1 + \epsilon^{-p-q}j)\xi, (\epsilon^n - 1)\xi, (j^2 - 1)\eta, (1 + j)\eta \rangle.$$

- (G) If $\alpha = (p, p, 2p, 2p)$, then $H_2(M; \mathbb{Z}) \cong ?$.
 (H) If $\alpha = (p, q, p+q, p+q)$ and $p \neq q$, then $H_2(M; \mathbb{Z}) \cong ?$.
 (I) If $\alpha = (p, p, q, q)$ and $p \neq q$, $p \neq 2q$, $q \neq 2p$, then

$$H_2(M; \mathbb{Z}) \cong \mathbb{Z}[Q_{4n}]\xi / \langle (e^n - 1)\xi, (\epsilon^{-2q}j - 1)\xi \rangle.$$

- (J) If $\alpha = (p, q, r, q)$ and $r \notin \{q, p, p+q\}$, then

$$H_2(M; \mathbb{Z}) \cong \mathbb{Z}[Q_{4n}]\xi / \langle (e^n - 1)\xi, (\epsilon^p j + 1)\xi \rangle.$$

- (K) If $\alpha = (p, q, r, p+q)$ and $r \notin \{p, p+q\}$, then $H_2(M; \mathbb{Z}) \cong ?$.
 (L) If $\alpha = (p, p, q, q+r)$, and $p \neq q$, $r \neq 0$, then $H_2(M; \mathbb{Z}) \cong ?$.
 (M) In all the remaining cases, $H_2(M; \mathbb{Z}) \cong \mathbb{Z}[Q_{4n}]\xi / \langle (e^n - 1)\xi \rangle$.

Proof. The proof of every instance of the theorem contains three steps:

- study of the Q_{4n} -module structure on $H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$ with the ambient action,
- study of the Q_{4n} -module structure on $\text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z})$ given by (4.10) using a concrete basis of $H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$,
- description of the Q_{4n} -module structure of $\text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z})$.

The fact that j^2 and ϵ^n are acting trivially on W_n is affecting all module structures we study.

(A) Let $x \in H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$ be the homology class induced by the subspace L . The set equality $L = \epsilon^p L$ translates to the equality $x = (-1)^{p-1} \epsilon^p \cdot x$ in homology, because ϵ^p acts on the orientation of the geometric representative of x by $(-1)^{p-1}$. Indeed, the decomposition $L \oplus L^\perp = \mathbb{R}^n$ implies the identity

$$\det(\epsilon^p) = \det(\epsilon^p|_L) \det(\epsilon^p|_{L^\perp}).$$

To detect the value $\det(\epsilon^p|_{L^\perp})$ let us fix

$$\mathbf{e} = [e_1 + \cdots + e_p, e_{p+1} + \cdots + e_{2p}, e_{2p+1} + \cdots + e_{3p}, e_{3p+1} + \cdots + e_n]$$

as the basis of $L^\perp$. Then ϵ^p acts on $\mathbf{e}$ by permuting elements of the basis in the following way:

$$\epsilon^p \mathbf{e} = [e_{p+1} + \cdots + e_{2p}, e_{2p+1} + \cdots + e_{3p}, e_{3p+1} + \cdots + e_n, e_1 + \cdots + e_p].$$

Since the permutation of the basis elements is odd, $\det(\epsilon^p|_{L^\perp}) = -1$. Thus, $\det(\epsilon^p) = (-1)^p$ implies that $\det(\epsilon^p|_L) = (-1)^{p-1}$.

The set equality $L = jL$ in a similar fashion induces the relation $x = (-1)^{\binom{n}{2}} j \cdot x$ in $H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$. Indeed, j acts on the basis $\mathbf{e}$ of $L^\perp$ as the even permutation $\begin{pmatrix} 1234 \\ 4321 \end{pmatrix}$. We also inherit relations $\epsilon^n \cdot x = x$ and $j^2 \cdot x = x$.

The set of elements $\Lambda = \{x, \epsilon \cdot x, \dots, \epsilon^{p-1} \cdot x\}$ is a basis of the free abelian group $H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$. Let $\xi \in \text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z})$ be an element given on the basis Λ by $\xi(x) = 1$, $\xi(\epsilon^i \cdot x) = 0$, for $i \in \{1, \dots, p-1\}$. Then the set $\{\xi, \epsilon * \xi, \dots, \epsilon^{p-1} * \xi\}$ is a basis of the free abelian group $\text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z})$. From the definition of the “ $*$ ”-action (4.10), it follows that for $i \in \{0, 1, \dots, p-1\}$,

$$\begin{aligned} \epsilon^p * \xi(\epsilon^i \cdot x) &= \det(\epsilon^p) \xi(\epsilon^{i-p} \cdot x) = (-1)^p \xi(\epsilon^{i+3p} \cdot x) = (-1)^p \xi(\epsilon^i \cdot (\epsilon^{3p} \cdot x)) \\ &= (-1)^p \xi(\epsilon^i \cdot ((-1)^{3p-3} x)) = (-1)^{4p-3} \xi(\epsilon^i \cdot x) = -\xi(\epsilon^i \cdot x). \end{aligned}$$

Thus $\epsilon^p * \xi = -\xi$. In the same way, for $i \in \{0, 1, \dots, p-1\}$,

$$\begin{aligned} j * \xi(\epsilon^i \cdot x) &= \det(j) \xi(j^{-1} \epsilon^i \cdot x) = (-1)^{\binom{n}{2}} \xi(j^3 \epsilon^i \cdot x) = (-1)^{\binom{n}{2}} \xi(\epsilon^n j \epsilon^i \cdot x) \\ &= (-1)^{\binom{n}{2}} \xi(\epsilon^n \epsilon^{-i} j \cdot x) = \xi(\epsilon^{4p-i} \cdot x) = \xi(\epsilon^{p-i} \cdot (\epsilon^{3p} \cdot x)) \\ &= -\xi(\epsilon^{p-i} \cdot x) = \xi(\epsilon^i \cdot x) \end{aligned}$$

and consequently $j * \xi = \xi$.

(B) Again $x \in H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$ denotes the homology class induced by L . There are two set equalities, $L = \epsilon^{p+q} L$ and $L = \epsilon^{-q} j L$, besides two usual $L = \epsilon^n L$, $L = j^2 L$. Elements ϵ^{p+q} and $\epsilon^{-q} j$ act on $L^\perp$ as permutations $\begin{pmatrix} 1234 \\ 3412 \end{pmatrix}$ and $\begin{pmatrix} 1234 \\ 3214 \end{pmatrix}$, respectively. Therefore, in homology,

$$\epsilon^{p+q} \cdot x = \det(\epsilon^{p+q}) x, \epsilon^{-q} j \cdot x = -\det(\epsilon^{-q} j) x, \epsilon^n \cdot x = x, j^2 \cdot x = x.$$

The set of elements $\Lambda = \{x, \epsilon \cdot x, \dots, \epsilon^{p+q-1} \cdot x\}$ is a basis of the free abelian group $H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$. Let $\xi \in \text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z})$ be an element given on the basis Λ by $\xi(x) = 1$, $\xi(\epsilon^i \cdot x) = 0$, for $i \in \{1, \dots, p+q-1\}$. Then the set $\{\xi, \epsilon * \xi, \dots, \epsilon^{p+q-1} * \xi\}$ is a basis of the free abelian group $\text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z})$. From the definition of the “ $*$ ”-action (4.10), it follows that for $i \in \{0, 1, \dots, p+q-1\}$,

$$\begin{aligned} \epsilon^{p+q} * \xi(\epsilon^i \cdot x) &= \det(\epsilon^{p+q}) \xi(\epsilon^{i-p-q} \cdot x) = \det(\epsilon^{p+q}) \xi(\epsilon^{i+p+q} \cdot x) \\ &= \det(\epsilon^{p+q}) \xi(\epsilon^i \cdot (\epsilon^{p+q} \cdot x)) = \xi(\epsilon^i \cdot x) \end{aligned}$$

and

$$\begin{aligned} \epsilon^{-q} j * \xi(\epsilon^i \cdot x) &= \det(\epsilon^{-q} j) \xi(j^3 \epsilon^{i+q} \cdot x) = \det(\epsilon^{-q} j) \xi(\epsilon^n j \epsilon^{i+q} \cdot x) \\ &= \det(\epsilon^{-q} j) \xi(\epsilon^{n-i-q} j \cdot x) = -\xi(\epsilon^{n-i} \cdot x) \\ &= -\xi(\epsilon^{p+q-i} \cdot (\epsilon^{p+q} \cdot x)) = -\xi(\epsilon^{p+q-i} \cdot x) = -\xi(\epsilon^i \cdot x). \end{aligned}$$

Thus, $\epsilon^{p+q} * \xi = \xi$, $\epsilon^{-q} j * \xi = -\xi$ and $j^2 * \xi = \xi$.

(C) Let I be the intersection $L \cap \epsilon^p L = L \cap \epsilon^p L \cap \epsilon^{2p} L \cap \epsilon^{3p} L \cap \epsilon^4 L$ and L^+ be a closed subspace defined by

$$L^+ = \{x \in L \mid x_{3p+1} + \dots + x_{4p} \geq 0\}.$$

We make this choice because $y_1 \in \text{int} L^+$ (y_1 is given by 4.3). The “opposite” closed half-space will be denoted by L^- .

Let $x, y \in H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$ denote the homology classes induced by L and by $L^+ \cup \epsilon^p L^+$. Since $\epsilon^{-2p} j$ acts on the basis of $L^\perp$ as the odd permutation $\begin{pmatrix} 1234 \\ 3214 \end{pmatrix}$, the set equality $L = \epsilon^{-2p} j L$ implies the relation

$$(4.13) \quad (\epsilon^{-2p} j) \cdot x = -\det(\epsilon^{-2p} j) x.$$

With $j^2 \cdot x = \epsilon^n \cdot x = x$ this completes the set of relations for element x .

To describe the action of Q_{4n} on y let us first choose an orientation on $L^+ \cup \epsilon^p L^+$. Let $\mathfrak{e}$ be a basis,

$$\begin{aligned} \mathfrak{e} = [e_1 + \dots + e_p, e_{p+1} + \dots + e_{2p}, e_{2p+1} + \dots + e_{3p}, \\ e_{3p+1} + \dots + e_{4p}, e_{4p+1} + \dots + e_n], \end{aligned}$$

of $I^\perp$ and $\mathfrak{f}$ a basis of I such that $[\mathfrak{f}, \mathfrak{e}]$ coincides with the prescribed orientation on W_n . Let $[\mathfrak{e}, v]$ be a basis of L such that vector v points inside L^+ . We assume that $[\mathfrak{e}, v]$ is the prescribed orientation on L . Then the geometric class $L^+ \cup \epsilon^p L^+$ is oriented in the following way: $(L^+, [\mathfrak{f}, v]; \epsilon^p L^+, [-\mathfrak{f}, \epsilon^p v])$. The $-\mathfrak{f}$ appears because

the orientations on the boundary of the half-spaces have to be opposite. The element ϵ^p acts on the basis $\mathfrak{e}$ of $I^\perp$ by the even permutation $\begin{pmatrix} 12345 \\ 23451 \end{pmatrix}$. This implies that ϵ^p acts on the orientation of I by the multiplication with $\det(\epsilon^p)$. The geometric representatives of classes y , $\epsilon^p \cdot y$, $\epsilon^{2p} \cdot y$, $\epsilon^{3p} \cdot y$, $\epsilon^{4p} \cdot y$ and their orientations are given by

$$\begin{array}{lll} y & \rightarrow & (L^+, [\mathfrak{f}, v]); \quad \epsilon^p L^+, [-\mathfrak{f}, \epsilon^p v]) \\ \epsilon^p \cdot y & \rightarrow & (\epsilon^p L^+, [\det(\epsilon^p)\mathfrak{f}, \epsilon^p \cdot v]); \quad \epsilon^{2p} L^+, [-\det(\epsilon^p)\mathfrak{f}, \epsilon^{2p} v]) \\ \epsilon^{2p} \cdot y & \rightarrow & (\epsilon^{2p} L^+, [\det(\epsilon^{2p})\mathfrak{f}, \epsilon^{2p} \cdot v]); \quad \epsilon^{3p} L^+, [-\det(\epsilon^{2p})\mathfrak{f}, \epsilon^{3p} v]) \\ \epsilon^{3p} \cdot y & \rightarrow & (\epsilon^{3p} L^+, [\det(\epsilon^{3p})\mathfrak{f}, \epsilon^{3p} \cdot v]); \quad \epsilon^{4p} L^+, [-\det(\epsilon^{3p})\mathfrak{f}, \epsilon^{4p} v]) \\ \epsilon^{4p} \cdot y & \rightarrow & (\epsilon^{3p} L^+, [\det(\epsilon^{4p})\mathfrak{f}, \epsilon^{4p} \cdot v]); \quad L^+, [-\det(\epsilon^{4p})\mathfrak{f}, v]), \end{array}$$

where vector $v \in L$ points inside L^+ . Summing up with appropriate signs and taking care of orientations, we get the relation

$$(4.14) \quad y + \det(\epsilon^p)\epsilon^p \cdot y + \det(\epsilon^{2p})\epsilon^{2p} \cdot y + \det(\epsilon^{3p})\epsilon^{3p} \cdot y + \det(\epsilon^{4p})\epsilon^{4p} \cdot y = 0.$$

The element $\epsilon^{-2p}j$ acts on the basis $\mathfrak{e}$ of $I^\perp$ by the even permutation $\begin{pmatrix} 12345 \\ 32154 \end{pmatrix}$ and consequently $\epsilon^{-2p}j$ acts on the orientation of I by $\det(\epsilon^{-2p}j)$. As we have seen, $\epsilon^{-2p}j$ acts on the orientation of L by $-\det(\epsilon^{-2p}j)$. Also observe that $\epsilon^{-2p}jL^+ = L^-$. Thus the geometric representative of the class $\epsilon^{-2p}j \cdot y$ has orientation given by

$$\begin{array}{ll} (\epsilon^{-2p}jL^+, [\det(\epsilon^{-2p}j)\mathfrak{f}, \epsilon^{-2p}j \cdot v]); & \epsilon^{-2p}j\epsilon^p L^+, [-\det(\epsilon^{-2p}j)\mathfrak{f}, \epsilon^{-2p}j\epsilon^p v]) \\ (L^-, [\det(\epsilon^{-2p}j)\mathfrak{f}, \epsilon^{-2p}j \cdot v]); & \epsilon^p L^-, [-\det(\epsilon^{-2p}j)\mathfrak{f}, \epsilon^{-2p}j\epsilon^p v]). \end{array} =$$

This implies the relation

$$(4.15) \quad y - \det(\epsilon^{-2p}j)\epsilon^{-2p}j \cdot y = x - \det(\epsilon^p)\epsilon^p \cdot x.$$

The reader may find Figure 9 useful as a guide for these calculations. Thus we have

FIGURE 9. Moving of the element y along the action

an isomorphism of Q_{4n} -modules

$$H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}) \cong \mathbb{Z}[Q_{4n}]x \oplus \mathbb{Z}[Q_{4n}]y/A,$$

where

$$A = \left\langle \begin{array}{l} (\det(\epsilon^{-2p}j)\epsilon^{-2p}j + 1)x, (\epsilon^n - 1)x, (j^2 - 1)y, \\ (1 + \det(\epsilon^p)\epsilon^p + \det(\epsilon^{2p})\epsilon^{2p} + \det(\epsilon^{3p})\epsilon^{3p} + \det(\epsilon^{4p})\epsilon^{4p})y \\ (1 - \det(\epsilon^{-2p}j)\epsilon^{-2p}j)y + (\det(\epsilon^p)\epsilon^p - 1)x \end{array} \right\rangle.$$

Unfortunately, to describe the Q_{4n} -module structure on $\text{Hom}(H_{n-4}(\bigcup \widehat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z})$ we have to work with the concrete basis of $H_{n-4}(\bigcup \widehat{\mathcal{A}}(\alpha), \mathbb{Z})$. Let $\Lambda = \{x, \epsilon \cdot x, \dots, \epsilon^{n-1} \cdot x, y, \epsilon \cdot y, \dots, \epsilon^{4p-1} \cdot y\}$ be a basis of the free abelian group $H_{n-4}(\bigcup \widehat{\mathcal{A}}(\alpha), \mathbb{Z})$. Let $\xi \in \text{Hom}(H_{n-4}(\bigcup \widehat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z})$ be an element given on the basis Λ by

$$\xi(x) = 1, \xi(y) = 1, (\forall z \in \Lambda \setminus \{x, y\}) \xi(z) = 0.$$

Let $\Pi = 1 + \epsilon^p + \epsilon^{2p} + \epsilon^{3p} + \epsilon^{4p} + j + \epsilon^p j + \epsilon^{2p} j + \epsilon^{3p} j + \epsilon^{4p} j$. Using the definition of the “ $\ast$ ”-action (4.10), we compute $\Pi \ast \xi$ on elements of the basis Λ . It can be checked that $\Pi \ast \xi = 0$. For example, for the basis element x we have

$$\begin{aligned} (\Pi \ast \xi)(x) &= \sum_{i=0}^4 \epsilon^{ip} \ast \xi(x) + \sum_{i=0}^4 \epsilon^{ip} j \ast \xi(x) \\ &= \xi \left(\sum_{i=0}^4 \det(\epsilon^{ip}) \epsilon^{ip} \cdot x + \sum_{i=0}^4 \det(\epsilon^{ip} j) \epsilon^{ip} j \cdot x \right) \\ &= \sum_{i=0}^4 \det(\epsilon^{ip}) \xi(\epsilon^{ip} \cdot x) + \sum_{i=0}^4 \det(\epsilon^{ip} j) \xi(\epsilon^{ip} j \cdot x) \\ &= \xi(x) + \det(\epsilon^{3p} j) \xi(\epsilon^{3p} j \cdot x). \end{aligned}$$

Since $\epsilon^{3p} j \cdot x$ is not a basis element, ξ is evaluated via the relation (4.13):

$$(\Pi \ast \xi)(x) = \xi(x) + \det(\epsilon^{3p} j) \xi(\epsilon^{3p} j \cdot x) = \xi(x) + \det(\epsilon^{3p} j) \xi(-\det(\epsilon^{3p} j)x) = 0.$$

For the basis element y , applying relation (4.14) we obtain

$$(\Pi \ast \xi)(y) = \xi \left(\sum_{i=0}^4 \det(\epsilon^{ip}) \epsilon^{ip} \cdot y + \sum_{i=0}^4 \det(\epsilon^{ip} j) \epsilon^{ip} j \cdot y \right) = 0.$$

Now it is not hard to deduce that there is a Q_{4n} -module isomorphism

$$\text{Hom}(H_{n-4}(\bigcup \widehat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}[Q_{4n}]\xi/K,$$

where

$$K = \langle 1 + \epsilon^p + \epsilon^{2p} + \epsilon^{3p} + \epsilon^{4p} + j + \epsilon^p j + \epsilon^{2p} j + \epsilon^{3p} j + \epsilon^{4p} j \rangle \xi, (j^2 - 1)\xi, (\epsilon^n - 1)\xi \rangle.$$

(D) Let $x \in H_{n-4}(\bigcup \widehat{\mathcal{A}}(\alpha), \mathbb{Z})$ be the homology class induced by L . We analyze the set equality $\epsilon^{-3}j \cdot L = L$. Let $\mathfrak{e} = [e_1, e_2, e_3, e_4 + e_5 + e_6]$ be the fixed basis of $L^\perp$. Then $\epsilon^{-3}j$ acts on $\mathfrak{e}$ as the odd permutation $\begin{pmatrix} 1234 \\ 3214 \end{pmatrix}$. This implies the equality $x = -\det(\epsilon^{-3}j)\epsilon^{-3}j \cdot x$. Also, $j^2 \cdot x = \epsilon^6 \cdot x = x$.

Let us denote by L^+ and $(\epsilon L)^+$ the following half-spaces:

$$L^+ = \{x \in L \mid x_4 \geq 0\} \text{ and } (\epsilon L)^+ = \{x \in \epsilon L \mid x_1 \geq 0\}.$$

In contrast to the previous situation (C), here we have $\epsilon \cdot L^+ \neq (\epsilon L)^+$. Let $y \in H_{n-4}(\bigcup \widehat{\mathcal{A}}(\alpha), \mathbb{Z})$ be the homology class induced by $L^+ \cup (\epsilon L)^+$. Again the set equality $\epsilon^{-2}j \cdot (L^+ \cup (\epsilon L)^+) = L^+ \cup (\epsilon L)^+$ is the starting point of our calculation. The element $\epsilon^{-2}j$ acts on the basis $[e_1, e_2, e_3, e_4, e_5 + e_6]$ of $(L \cap \epsilon L)^\perp$ as the even permutation $\begin{pmatrix} 12345 \\ 43215 \end{pmatrix}$. Thus $\epsilon^{-2}j$ acts on the orientation of $L \cap \epsilon L$ as the multiplication with $\det(\epsilon^{-2}j)$. Let $\mathfrak{f}$ be the basis of $L \cap \epsilon L$. We choose the following orientation of the geometric representative of the element y :

$$(L^+, [\mathfrak{f}, v] ; (\epsilon L)^+, [-\mathfrak{f}, \epsilon^{-2}j \cdot v]),$$

where $v \in L$ points inside L^+ . The vector $\epsilon^{-2}j \cdot v$ points inside $(\epsilon L)^+$. The action of $\epsilon^{-2}j$ on fixed orientations provides

$$\left((\epsilon L)^+, [\det(\epsilon^{-2}j)\mathfrak{f}, \epsilon^{-2}j \cdot v]; L^+, [-\det(\epsilon^{-2}j)\mathfrak{f}, v] \right)$$

because $\epsilon^{-2}jL^+ = (\epsilon L)^+$ and $\epsilon^{-2}j(\epsilon L)^+ = L^+$. Therefore,

$$y + \det(\epsilon^{-2}j) \epsilon^{-2}j \cdot y = 0.$$

Again, $j^2 \cdot y = \epsilon^6 \cdot y = y$.

Let us denote by $F_1, \dots, F_6$ the following convex cones:

$$\begin{aligned} F_1 &= \{x \in L \mid x_4 \geq 0, x_6 \geq 0\}, & F_2 &= \{x \in \epsilon L \mid x_5 \geq 0, x_1 \geq 0\}, \\ F_3 &= \{x \in \epsilon^2 L \mid x_6 \geq 0, x_2 \geq 0\}, & F_4 &= \{x \in \epsilon^3 L \mid x_1 \geq 0, x_3 \geq 0\}, \\ F_5 &= \{x \in \epsilon^4 L \mid x_2 \geq 0, x_4 \geq 0\}, & F_6 &= \{x \in \epsilon^5 L \mid x_3 \geq 0, x_5 \geq 0\}. \end{aligned}$$

Let $z \in H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$ be the element geometrically represented by the union $S = F_1 \cup \dots \cup F_6$. Then $g \cdot S = S$ for every $g \in Q_{4n}$. Let us fix a basis for each of the convex cones by

$$\begin{aligned} F_1 &\rightarrow [e_4, e_6], & F_2 &\rightarrow [e_5, e_1], & F_3 &\rightarrow [e_6, e_2], \\ F_4 &\rightarrow [e_1, e_3], & F_5 &\rightarrow [e_2, e_4], & F_6 &\rightarrow [e_3, e_5]. \end{aligned}$$

Since the element ϵ does not change the orientation of S , we have $\epsilon \cdot z - z = 0$. On the other hand, the element j does change the orientation of S and so $j \cdot z + z = 0$. Figure 10 may be used as a guide for visualizing geometric representatives of classes x, y and z . Let $\Lambda = \{x, \epsilon \cdot x, \dots, \epsilon^5 \cdot x, y, \epsilon \cdot y, \dots, \epsilon^5 \cdot y, z\}$ be a basis of the free abelian

FIGURE 10. The geometric representatives in the case $(1, 1, 1, 3)$

group $H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$. Let $\xi, \eta, \zeta \in \text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z})$ be the elements given on the basis Λ by

$$\begin{aligned} \xi(x) &= 1, (\forall t \in \Lambda \setminus \{x\}) \xi(t) = 0, \\ \eta(y) &= 1, (\forall t \in \Lambda \setminus \{y\}) \eta(t) = 0, \\ \zeta(z) &= 1, (\forall t \in \Lambda \setminus \{z\}) \zeta(t) = 0. \end{aligned}$$

By direct computation on the basis Λ one can prove that

$$\epsilon^{-3}j \cdot \xi + \xi = 0, \eta + \epsilon^{-2}j \cdot \eta = 0, \epsilon \cdot \zeta + \zeta = 0, \epsilon \cdot \zeta - \zeta = 0,$$

because $\det(\epsilon) = -1$ and $\det(j) = 1$.

(E) The proof follows in the footsteps of the previous cases. Let $x \in H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$ be the homology class induced by L . The set equality $L = \epsilon^{-p}jL$

implies the relation $(1 + \det(\epsilon^{-p}j)\epsilon^{-p}j) \cdot x = 0$. Let $y \in H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$ be the homology class induced by the union of the following half-spaces:

$$L^+ = \{x \in L \mid x_{3q+1} + \dots + x_{4p} \geq 0\}, \quad (\epsilon^q L)^+ = \{x \in \epsilon^q L \mid x_1 + \dots + x_p \geq 0\}.$$

Again $\epsilon^q L^+ \neq (\epsilon^q \cdot L)^+$. The set equality $\epsilon^{-2p-2q}j \left(L^+ \cup (\epsilon^q L)^+ \right) = L^+ \cup (\epsilon^q L)^+$, with careful orientation calculation, implies the relation $(1 + \det(\epsilon^{-2p-2q}j)\epsilon^{-2p-2q}j) \cdot y = 0$.

If we use the basis $\Lambda = \{x, \epsilon \cdot x, \dots, \epsilon^{n-1} \cdot x, y, \epsilon \cdot y, \dots, \epsilon^{n-1} \cdot y\}$ for $H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$, and define $\xi, \eta \in \text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z})$ by

$$\begin{aligned} \xi(x) &= 1, \quad (\forall t \in \Lambda \setminus \{x\}) \quad \xi(t) = 0, \\ \eta(y) &= 1, \quad (\forall t \in \Lambda \setminus \{y\}) \quad \eta(t) = 0, \end{aligned}$$

the result follows.

(F) The statement follows as in the case (E) using the set equalities $L = \epsilon^{-(p+q)}jL$ and $L \cap \epsilon^{p+q}L = j(L \cap \epsilon^{p+q}L)$.

(I) Let $x \in H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$ be the homology class induced by L . The element $\epsilon^{-2q}j$ acts on the basis of $L^\perp$ by an even permutation. Thus the set equality $\epsilon^{-2q}jL = L$ transforms into the relation $((\det \epsilon^{-2q}j)\epsilon^{-2q}j - 1) \cdot x = 0$. For a basis $\Lambda = \{x, \epsilon \cdot x, \dots, \epsilon^{n-1} \cdot x\}$ of $H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$, we define $\xi \in \text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z})$ on Λ by

$$\xi(x) = 1, \quad (\forall t \in \Lambda \setminus \{x\}) \quad \xi(t) = 0.$$

Then

$$\begin{aligned} \epsilon^{-2q}j * \xi(\epsilon^i \cdot x) &= \det(\epsilon^{-2q}j)\xi(j\epsilon^{2q}\epsilon^i \cdot x) = \det(\epsilon^{-2q}j)\xi(\epsilon^{-2q-i}j \cdot x) \\ &= \det(\epsilon^{-2q}j)\xi(\epsilon^{-i} \cdot (\epsilon^{-2q}j \cdot x)) = \xi(\epsilon^{-i} \cdot x) \\ &= \xi(\epsilon^i \cdot x). \end{aligned}$$

The last equality follows from the definition of ξ . Thus, $\epsilon^{-2q}j * \xi = \xi$.

(J) Let $x \in H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$ be the homology class induced by L . The element $\epsilon^p j$ acts on $L^\perp$ by an odd permutation $\begin{pmatrix} 1234 \\ 3214 \end{pmatrix}$. Thus the set equality $\epsilon^{-2q}jL = L$ transforms to the relation $(\det(\epsilon^{-2q}j)\epsilon^{-2q}j + 1) \cdot x = 0$ in $H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z})$. Like in the previous cases for the particular choice of an element $\xi \in \text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z})$ we obtain the relation $(\epsilon^{-2q}j + 1) * \xi = 0$.

(M) In all the remaining cases there are no nontrivial relations except

$$\epsilon^n = j^2 = 1.$$

□

The knowledge of the Q_{4n} -module structure on $H_2(M; \mathbb{Z})$ allows us to read off the group of coinvariants.

Corollary 4.9. *With the assumptions of Theorem 4.8 the corresponding groups of coinvariants are the following:*

- (A) If $\alpha = (p, p, p, p)$, then $H_2(M; \mathbb{Z})_{Q_{4n}} \cong \mathbb{Z}_2$.
- (B) If $\alpha = (p, q, p, q)$ and $p \neq q$, then $H_2(M; \mathbb{Z})_{Q_{4n}} \cong \mathbb{Z}_2$.
- (C) If $\alpha = (p, p, p, 2p)$, then $H_2(M; \mathbb{Z})_{Q_{4n}} \cong \mathbb{Z}_{10}$.
- (D) If $\alpha = (1, 1, 1, 3)$, then $H_2(M; \mathbb{Z})_{Q_{4n}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
- (E) If $\alpha = (q, q, q, p)$, $p \notin \{q, 2q\}$ and $3q + p > 6$, then $H_2(M; \mathbb{Z})_{Q_{4n}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
- (F) If $\alpha = (p, q, p, p + q)$ and $p \neq q$, then $H_2(M; \mathbb{Z})_{Q_{4n}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

- (G) If $\alpha = (p, p, 2p, 2p)$, then $H_2(M; \mathbb{Z})_{Q_{4n}} \cong ?$.
- (H) If $\alpha = (p, q, p + q, p + q)$ and $p \neq q$, then $H_2(M; \mathbb{Z})_{Q_{4n}} \cong ?$.
- (I) If $\alpha = (p, p, q, q)$ and $p \neq q$, $p \neq 2q$, $q \neq 2p$, then $H_2(M; \mathbb{Z})_{Q_{4n}} \cong \mathbb{Z}$.
- (J) If $\alpha = (p, q, r, q)$ and $r \notin \{q, p, p + q\}$, then $H_2(M; \mathbb{Z})_{Q_{4n}} \cong \mathbb{Z}_2$.
- (K) If $\alpha = (p, q, r, p + q)$ and $r \notin \{p, p + q\}$, then $H_2(M; \mathbb{Z})_{Q_{4n}} \cong ?$.
- (L) If $\alpha = (p, p, q, q + r)$, and $p \neq q$, $r \neq 0$, then $H_2(M; \mathbb{Z})_{Q_{4n}} \cong ?$.
- (M) In all the remaining cases $H_2(M; \mathbb{Z})_{Q_{4n}} \cong \mathbb{Z}$.

4.4. The main theorem. This section is central to the paper. We gather together all the information collected in previous sections and use it to obtain a reasonably complete answer to Problem 2.3.

Theorem 4.10. *Suppose that Q_{4n} is a generalized quaternion group and $M(\alpha)$ the complement of the arrangement $\mathcal{A}(\alpha)$ introduced in Section 2.2. Then our knowledge about the existence of Q_{4n} -equivariant maps $F : S^3 \rightarrow M(\alpha)$ (Problem 2.3) can be summarized as follows.*

- (A) If $\alpha = (p, p, p, p)$ or $\alpha = (p, p, p, 2p)$, then there are no Q_{4n} -equivariant maps $F : S^3 \rightarrow M(\alpha)$.
- (B) If α is not of the form $(p, q, p + q, p + q)$, or $(p, q, r, p + q)$, $r \notin \{p, p + q\}$, or $(p, p, q, q + r)$, then there exists a Q_{4n} -map $F : S^3 \rightarrow M(\alpha)$.
- (C) If $\alpha = (p, q, p + q, p + q)$, or $\alpha = (p, q, r, p + q)$, $r \notin \{p, p + q\}$, or $\alpha = (p, p, q, q + r)$, the methods of the paper are inconclusive and a further analysis is needed.

Proof. **(A.1)** Let $\alpha = (p, p, p, p)$. In order to prove that $[d_{Q_{4n}}(h)(e)] = [\|y_1\|] \neq 0$ we rely on the isomorphism (4.8),

$$\phi : H_2(M(\mathcal{A}(\alpha)), \mathbb{Z}) \rightarrow \text{Hom}(H_{n-4}(\bigcup \widehat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z}).$$

Using the fact that this isomorphism is related to the linking number and that $y_1 \in L \setminus \bigcup_{g \in Q_{4n}: gL \neq L} gL$, we have that $(y_1 + L^\perp)$ links only with $\widehat{L}$, and therefore on the basis Λ ,

$$\phi(\|y_1\|)(\epsilon^i \cdot x) = \text{link}(\widehat{\epsilon^i L}, (\widehat{y_1 + L^\perp})) = \begin{cases} 1, & i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Here $\widehat{\epsilon^i L}$ and $(\widehat{y_1 + L^\perp})$ are respectively one-point compactifications of the linear subspace $\epsilon^i L$ and the affine subspace $y_1 + L^\perp$ in the sphere $\widehat{W}_n \approx S^{n-1}$. Thus $\phi(\|y_1\|) = \xi$.

The obstruction element $d_{Q_{4n}}(h)(e) = \|y_1\|$ is mapped to the generator of the group $H_2(M; \mathbb{Z})_{Q_{4n}} \cong \mathbb{Z}_2$ by the quotient map $H_2(M; \mathbb{Z}) \rightarrow H_2(M; \mathbb{Z})_{Q_{4n}}$. Therefore, there is no Q_{4n} -map $F : S^3 \rightarrow M(\alpha)$.

(A.2) Let $\alpha = (p, p, p, 2p)$. Then $d_{Q_{4n}}(h)(e) = 4\|y_1\|$, by Theorem 4.3, (B). Like in (A.1) we compute the ϕ image of the point class $\|y_1\|$. Since $y_1 \in L^+$ (see (4.3) and Theorem 4.8 (C)) we have on the basis Λ ,

$$\begin{aligned} \phi(\|y_1\|)(\epsilon^i \cdot x) &= \text{link}(\widehat{\epsilon^i L}, (\widehat{y_1 + L^\perp})) = \begin{cases} 1, & i = 0, \\ 0, & \text{otherwise,} \end{cases} \\ \phi(\|y_1\|)(\epsilon^j \cdot y) &= \text{link}(\epsilon^j(L^+ \cup \epsilon^p L^+), (\widehat{y_1 + L^\perp})) = \begin{cases} 1, & j = 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Consequently, $\phi(\|y_1\|) = \xi \in \text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z})$. Thus, $[d_{Q_{4n}}(h)(e)] = 4 \cdot \|y_1\| \in H_2(M; \mathbb{Z})_{Q_{4n}} \cong \mathbb{Z}_{10}$ is a nonzero element $4 * (\text{generator})$. Therefore, there are no Q_{4n} -maps $F : S^3 \rightarrow M(\alpha)$.

(B) Assume that α is not of the form (p, p, p, p) , $(p, p, p, 2p)$, $(p, q, p + q, p + q)$, $(p, q, r, p + q)$, $r \notin \{p, p + q\}$ or $(p, p, q, q + r)$. Then the cohomology class of the obstruction cocycle $[d_{Q_{4n}}(h)(e)]$ is divisible by 2 (Theorem 4.3). On the other hand this cohomology class is a torsion element of $H_2(M; \mathbb{Z})_{Q_{4n}}$ (Lemma 4.1). Since Corollary 4.9 implies that the only torsion summands which can appear are copies of $\mathbb{Z}_2$, we conclude that $[d_{Q_{4n}}(h)(e)] = 0$. Thus a Q_{4n} -map $F : S^3 \rightarrow M(\alpha)$ must exist in all these cases. $\square$

4.5. Concluding remarks. Perhaps one of the main general problems of the emerging field of Computational Topology is to determine the scope of the field and major lines of prospective research. This includes the identification of classes of model problems where topology and computational mathematics interact in an essential way.

One of our objectives in this paper was to demonstrate that the calculation of the equivariant obstruction for the existence of an equivariant map is one such problem. The algorithms used in these calculations are based on important results of topology and geometric combinatorics, including the Goresky-MacPherson formula for the homology of the complement of an affine arrangement, equivariant obstruction exact sequence, etc. However, the crux of the computation is in the introduction and manipulation of the so-called “point” and “broken point” homology classes, which are not only of theoretical interest but allow quite efficient calculations, one step away from an effective computer implementation.

The paper focuses on the currently active problem involving G -equivariant maps where G is a subgroup of the dihedral group D_{2n} , alternatively the generalized quaternion group Q_{4n} , but the technique is quite general and may be adapted to other problems of a similar nature.

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