DISINTEGRATING TENSOR REPRESENTATIONS OF NILPOTENT LIE GROUPS

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ABSTRACT. Let G be a simply connected nilpotent Lie group and H a closed connected subgroup of G . Given an irreducible unitary representation π of G , we present an explicit disintegration of the restriction $\pi|_H$ of π to H . Such a disintegration relies on the description of the double cosets space $H \backslash G / B$ for an arbitrary closed connected subgroup B of G , and the well-known smooth disintegration of monomial representations of nilpotent Lie groups. As an application we get a concrete disintegration and a criterion of irreducibility for tensor products of a finite number of irreducible representations of G .

1. INTRODUCTION

In this paper, we present a concrete disintegration of the tensor product of two irreducible unitary representations of a simply connected, connected nilpotent Lie group $G = \exp \mathfrak{g}$ into irreducibles. “Concrete” here means that we write down an explicit integral of the spaces of isotopic components involved in the disintegration and an explicit unitary intertwining operator which realizes such a decomposition.

Towards that purpose, we use the well-known fact that a tensor product of two irreducible representations π, π' of G corresponds to the restriction to the diagonal group D of the irreducible representation $\pi \times \pi'$ of the group $G \times G$. We are then led to study the general problem of describing a concrete disintegration of the restriction of an irreducible unitary representation of G to a closed connected subgroup $H = \exp \mathfrak{h}$. The paper is organized as follows: we start by recording some basic results of the representation theory of Lie groups. We especially insist on the structure of nilpotent Lie groups and we introduce in section 2 some notation and conventions useful for the sequel of the paper. Using the orbit method, one realizes any $\pi \in \hat{G}$ as a monomial representation $\pi = \pi_{\ell, \mathfrak{b}} = \text{ind}_B^G \chi_\ell$, where $\ell \in \mathfrak{g}^*$ and where the Lie algebra $\mathfrak{b}$ of B stands for a polarizing subalgebra of $\mathfrak{g}$ at ℓ . It comes out therefore from Mackey ([11]) that the restriction of π to H is disintegrated over the set of double cosets space $H \backslash G / B$ and that any isotopic component is of the form $\text{ind}_{B(x)}^H \chi_{\ell(x)}$, where $B(x) = H \cap \psi(x)B\psi(x)^{-1}$, $x \in H \backslash G / B$, $\ell(x) = \text{Ad}^*(\psi(x))\ell|_{\mathfrak{b}}$ and $\psi : H \backslash G / B \rightarrow G$ is a section for the double cosets space. So,

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the main issue is to provide a comprehensive description of the double cosets space and accurate information on its structural features that the disintegration of the restriction strongly relies on. This will be the aim of section 3, where we provide a description of an open dense subset of $H \backslash G / B$ and of such a section ψ which gives an explicit decomposition of $\pi|_H$ in term of an integral over $H \backslash G / B$ of the representations $\text{ind}_{B(x)\chi_{\ell(x)}}^H$ (see Proposition 3.27). The fourth section is devoted to recalling the results of Baklouti-Ludwig [2] concerning an explicit disintegration of monomial representations. These results will be used as a fundamental genesis to obtain our concrete disintegration of the restriction of π to H . In Section 5, we apply our construction to the disintegration of the tensor product of two irreducible representations π and π' . The disintegration is made over a subvariety of $\mathfrak{g}^*$, which is contained in the sum of the Kirillov orbits of π and π' . We show in section 6 through an example that the formulae describing the disintegration strongly depend on the choice of the polarizations used to realize the representations π and π' . Finally, the last section is devoted to providing necessary and sufficient conditions relying on the representations π and π' for the representation $\pi \otimes \pi'$ to be irreducible.

It is noteworthy to point out in this context that an “abstract” disintegration into irreducibles for restrictions of unitary representations to connected closed subgroups of simply connected nilpotent Lie groups has been given in [5]. Abstract here means that the measure class in $\hat{G}$ for the disintegration of the restriction and the multiplicities of irreducibles appearing in the disintegration is given. In [7] these results are extended to encompass all exponential solvable Lie groups. In [3], another concrete disintegration of the restriction into irreducibles to closed connected subgroups is given.

2. PRELIMINARIES AND NOTATION

Let $\mathfrak{g}$ be a nilpotent Lie algebra, $\mathfrak{b}$ any subalgebra and $G \supset B$ their simply connected Lie groups. Given a flag of ideals

$$\mathfrak{g}_{n+1} := \{0\} \subsetneq \cdots \subsetneq \mathfrak{g}_i \subsetneq \cdots \subsetneq \mathfrak{g}_1 = \mathfrak{g}, \dim(\mathfrak{g}_i / \mathfrak{g}_{i+1}) = 1,$$

we let $G_i := \exp \mathfrak{g}_i$ for every $i = 1, \dots, n$ and we choose a vector $Z_i \in \mathfrak{g}_i \setminus \mathfrak{g}_{i+1}$, so that $\mathfrak{g}_i = \text{span} \langle Z_i, \dots, Z_n \rangle$. We obtain in this way a Jordan-Hölder basis $\mathcal{Z} = (Z_1, \dots, Z_n)$ of $\mathfrak{g}$. Recall that the exponential mapping $\exp : \mathfrak{g} \longrightarrow G$ is a diffeomorphism, so, just for brevity, we denote by $V_1.V_2...V_k := \exp(V_1)\exp(V_2)...\exp(V_k) \in G$, for any k vectors $V_1, \dots, V_k \in \mathfrak{g}$.

Let dg denote the Haar measure on G , that is, for any $f \in L^1(G)$, one has

$$\int_G f(g) dg = \int_{\mathbb{R}^n} f(z_1 Z_1 \dots z_n Z_n) dz.$$

Since G is nilpotent, the quotient space G/B has a G -invariant measure which is unique up to a positive scalar multiple. To describe such a measure, which we shall denote by $d\dot{g}$, we proceed as follows. Let

$$\mathcal{I}^{\mathfrak{g}/\mathfrak{b}} = \{k \in \{1, \dots, n\}, Z_k \notin \mathfrak{b} + \mathfrak{g}_{k+1}\} =: \{k_1 < \dots < k_p\}.$$

We obtain the sequence of subalgebras

$$(2.1) \quad \mathfrak{b}^{p+1} := \mathfrak{b} \subsetneq \cdots \subsetneq \mathfrak{b}^j := \mathbb{R}Z_{k_j} \oplus \mathfrak{b}^{j+1} \subsetneq \cdots \subsetneq \mathfrak{b}^1 = \mathfrak{g}$$

and the *Malcev basis* $\mathcal{M} := (Z_{k_1}, \dots, Z_{k_p})$ of $\mathfrak{g}$ relative to $\mathfrak{b}$. The invariant measure $d\dot{g}$ is then given by

$$(2.2) \quad \mu_{\mathcal{M}}(\varphi) = \mu_{\mathfrak{g}/\mathfrak{b}}(\varphi) = \int_{G/B} \varphi(\dot{g}) d\dot{g} := \int_{\mathbb{R}^p} \varphi(\zeta_1 Z_{k_1} \dots \zeta_p Z_{k_p} \cdot B) d\zeta, \quad \varphi \in \mathcal{C}_c(G/B),$$

where $\mathcal{C}_c(G/B)$ denotes the space of complex-valued continuous functions with compact support on G/B . This is a consequence of the fact that the mapping

$$\begin{aligned} E_{\mathcal{M}} : \quad \mathbb{R}^p &\longrightarrow G/B, \\ \zeta = (\zeta_1, \dots, \zeta_p) &\longmapsto \zeta_1 Z_{k_1} \dots \zeta_p Z_{k_p} \cdot B =: E_{\mathcal{M}}(\zeta) \end{aligned}$$

is a diffeomorphism (for detailed explanations see [6]).

If $\mathfrak{h}$ is another subalgebra of $\mathfrak{g}$, then we denote by $\mathcal{I}^{\mathfrak{h}} \subset \{1, \dots, n\}$ the index set

$$\mathcal{I}^{\mathfrak{h}} := \{i \in \{1, \dots, n\}; \mathfrak{h} + \mathfrak{g}_{i+1} = \mathfrak{h} + \mathfrak{g}_i\} = \{1, \dots, n\} \setminus \mathcal{I}^{\mathfrak{g}/\mathfrak{h}}.$$

We can then assume, without any loss of generalities, that the vectors Z_i , $i \in \mathcal{I}^{\mathfrak{h}}$, are contained in $\mathfrak{h}$, so that

$$(2.3) \quad \mathfrak{h} = \text{span} \{Z_i, i \in \mathcal{I}^{\mathfrak{h}}\}.$$

Definition 2.1. Let $B = \exp \mathfrak{b}$ be a closed connected subgroup of G and let $(\rho, \mathcal{H}_{\rho})$ be a unitary representation of B . We define the induced representation $\tau_{\rho} := \text{ind}_B^G \rho$ of G as follows. Let

$$\begin{aligned} C_c(G/B, \rho) &= \{\xi : G \rightarrow \mathcal{H}_{\rho}, \xi \text{ continuous with compact support modulo } B, \\ &\quad \xi(gb) = \rho(b)^{-1}(\xi(g)) \text{ for all } g \in G, b \in B\}. \end{aligned}$$

We define a scalar product on $C_c(G/B, \rho)$ by

$$\langle \xi, \eta \rangle_{\mathcal{H}} := \int_{G/B} \langle \xi(x), \eta(x) \rangle_{\mathcal{H}_{\rho}} dx, \quad \xi, \eta \in C_c(G/B, \rho)$$

(where $\langle \cdot, \cdot \rangle_{\mathcal{H}_{\rho}}$ stands for the scalar product on $\mathcal{H}_{\rho}$) and therefore a Hilbert norm

$$\|\xi\|_2^2 = \langle \xi, \xi \rangle_{\mathcal{H}} = \int_{G/B} \langle \xi(x), \xi(x) \rangle_{\mathcal{H}_{\rho}} dx.$$

Let us define the left translation $\lambda(g)$ on $C_c(G/B, \rho)$ by

$$\lambda(g)\xi(g') = \xi(g^{-1}g'), \quad \xi \in C_c(G/B, \rho), g, g' \in G.$$

It is easy to see that $\|\lambda(g)\xi\|_2 = \|\xi\|_2$ for every $g \in G$ and $\xi \in C_c(G/B, \rho)$ and so λ defines on the completion $L^2(G/B, \rho)$ of $C_c(G/B, \rho)$ a unitary representation of G , which will be denoted by $\tau_{\rho, B} = \text{ind}_B^G \rho$ or simply by τ_{ρ} . Letting $\ell \in \mathfrak{g}^*$ such that $\langle \ell, [\mathfrak{b}, \mathfrak{b}] \rangle = \{0\}$, we can then define the unitary character $\chi_{\ell}(\exp X) := e^{-i\langle \ell, X \rangle}$, $X \in \mathfrak{b}$, of B and form the representation $\tau_{\ell, \mathfrak{b}} := \tau_{\chi_{\ell}, B}$ of G .

Remarks 2.2.

- (1) We have that the equivalence between $\tau_{\ell, \mathfrak{b}}$ is irreducible and $\mathfrak{b}$ is a polarization of $\mathfrak{g}$ in $\ell \in \mathfrak{g}^*$, i.e. an isotropic subalgebra of $\mathfrak{g}$ with respect to the skew-symmetric bilinear form $B_{\ell} : (X, Y) \mapsto \langle \ell, [X, Y] \rangle$ on $\mathfrak{g}$, with maximal dimension.

- (2) For a given Jordan-Hölder basis $\mathcal{Z} = (Z_1, \dots, Z_n)$ of $\mathfrak{g}$, define the Vergne polarization of ℓ in $\mathfrak{g}$ as the sum

$$\mathfrak{b}(\ell) = \sum_{i=1}^n \mathfrak{g}_i(\ell_i)$$

where $\mathfrak{g}_i(\ell_i) = \{X \in \mathfrak{g}_i, \langle \ell, [X, \mathfrak{g}_i] \rangle = 0\}$ is the stabilizer of $\ell_i := \ell|_{\mathfrak{g}_i}$ in $\mathfrak{g}_i = \text{span}\langle Z_i, \dots, Z_n \rangle$ for $i = 1, \dots, n$.

3. THE SET OF DOUBLE COSETS

Let $G = \exp \mathfrak{g}$ be a connected, simply connected, nilpotent Lie group, $\mathfrak{b}$ and $\mathfrak{h}$ two subalgebras of $\mathfrak{g}$, $B = \exp \mathfrak{b}$ and $H = \exp \mathfrak{h}$ their Lie subgroups. We give a concrete description for $H \backslash G / B$, which will be used to describe later on, in subsection 3.2, an appropriate measure $d\gamma$ which appears in the disintegration of $\pi|_H$ of π to H , where π is an irreducible unitary representation of G which is induced from a character χ_ℓ of B .

3.1. Description of $H \backslash G / B$. For $g \in G$, we denote by $\tilde{g}$ its double coset which is the set $H \cdot g \cdot B = \{hgb, (h, b) \in H \times B\}$. Our purpose is to find an open dense subset of $H \backslash G / B$ which will support the measure $d\gamma(\tilde{g})$ and which is diffeomorphic to a Zariski open subset of $\mathbb{R}^d$ for some $d \in \mathbb{N}^*$. To illustrate this fact, we introduce the following example.

Example 3.1. Let $\mathfrak{g}$ be the 7-dimensional Lie algebra spanned by its Jordan-Hölder basis $\mathcal{Z} = (Z_1, \dots, Z_7)$ and equipped with the nonzero brackets

$$[Z_1, Z_4] = Z_6, \quad [Z_1, Z_5] = Z_7, \quad [Z_2, Z_3] = Z_7.$$

We consider its abelian subalgebras $\mathfrak{h} = \text{span}\langle Z_4, Z_5, Z_7 \rangle$ and $\mathfrak{b} = \text{span}\langle Z_3, Z_4, Z_7 \rangle$. A routine computation shows that any element $g =: z_1 Z_1 \dots z_7 Z_7 \in G$, $(z_1, \dots, z_7) \in \mathbb{R}^7$, can be rewritten as follows:

$$\begin{aligned} g &= (z_1 Z_1 \cdot z_5 Z_5) \cdot z_2 Z_2 \cdot z_6 Z_6 \cdot (z_3 Z_3 \cdot z_4 Z_4 \cdot z_7 Z_7) \\ &= (z_1 z_5 Z_7 \cdot z_5 Z_5 \cdot z_1 Z_1) \cdot z_2 Z_2 \cdot z_6 Z_6 \cdot (z_3 Z_3 \cdot z_4 Z_4 \cdot z_7 Z_7) \\ &\in H \cdot z_1 Z_1 \cdot z_2 Z_2 \cdot z_6 Z_6 \cdot B. \end{aligned}$$

This implies that $\tilde{g} = H \cdot g \cdot B = H \cdot z_1 Z_1 \cdot z_2 Z_2 \cdot z_6 Z_6 \cdot B$.

But if $z_1 \neq 0$, the element $z_1 Z_1 \cdot z_2 Z_2 \cdot z_6 Z_6$ can also be written as $z_1 Z_1 \cdot z_2 Z_2$ conjugated by $\exp(-\frac{z_6}{z_1})Z_4$, which is contained in $H \cap B$. So we deduce that if $z_1 \neq 0$, then $\tilde{g} = H \cdot z_1 Z_1 \cdot z_2 Z_2 \cdot B$. As a conclusion,

$$\begin{aligned} H \backslash G / B &= \{H \cdot z_1 Z_1 \cdot z_2 Z_2 \cdot B, (z_1, z_2) \in \mathcal{V}\} \dot{\cup} \{H \cdot z_2 Z_2 \cdot z_6 Z_6 \cdot B, (z_2, z_6) \in \mathbb{R}^2\} \\ &= \mathbf{p}(G \setminus G_2) \cup \mathbf{p}(G_2), \end{aligned}$$

where $\mathbf{p} : g \mapsto \tilde{g}$ is the canonical projection of G on $H \backslash G / B$, $\mathcal{V} = \mathbb{R}^* \times \mathbb{R}$ is a Zariski open subset of $\mathbb{R}^2$ and G_2 is the closed subgroup of G with Lie algebra $\mathfrak{g}_2 = \text{span}\langle Z_2, \dots, Z_7 \rangle$.

Hence the space $H \backslash G / B$ is the disjoint union of two subsets: the first is the projection of a Zariski open subset of G and the second of a Zariski closed subset. We shall see later that the measure $d\gamma(\tilde{g})$ will be supported on the first set.

We shall prove next that there exists a nonempty Zariski open subset $\mathcal{V}$ of $\mathbb{R}^d$ which is homeomorphic to a subset $\mathcal{U}$ of $H \backslash G / B$ containing the double cosets of almost all (with respect to the Haar measure) elements of G .

- Definitions 3.2.** (1) We say that a subset $\mathcal{O}$ of G
- is symmetric if $\mathcal{O} = \mathcal{O}^{-1} := \{g^{-1}, g \in \mathcal{O}\}$,
 - has the property $(\mathcal{P})$ (with respect to H and B) if $H\mathcal{O}B \subseteq \mathcal{O}$.
- (2) It is easy to see that for any $i \in \{1, \dots, n\}$, the subset

$$\{g \in G, Z_i \notin \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1}\}$$

either contains a Zariski open subset or its complement in G does. We characterize the indices subset $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b})$ of $\{1, \dots, n\}$ as:

$$i \in \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) \iff \begin{cases} \text{the subset } \mathcal{O}_i := \{g \in G, Z_i \notin \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1}\} \\ \quad = \{g \in G, Z_i \notin \text{Ad}(g^{-1})\mathfrak{h} + \mathfrak{b} + \mathfrak{g}_{i+1}\} \\ \text{contains a nonempty Zariski open subset of } G. \end{cases}$$

When this is the case, we denote by $\mathcal{U}_i$ the maximal symmetric Zariski open subset contained in $\mathcal{O}_i$.

- (3) Also let for $i \in \{1, \dots, n\} \setminus \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) =: \mathcal{I}(\mathfrak{h}, \mathfrak{b})$,

$$\mathcal{O}_i := \{g \in G, Z_i \in \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1}\}.$$

For $i \in \mathcal{I}(\mathfrak{h}, \mathfrak{b})$, we consider the maximal symmetric Zariski open subset $\mathcal{U}_i$ of $\mathcal{O}_i$, with the property that for every $g \in \mathcal{U}_i$, there exist rational mappings $G \ni g' \mapsto h(g') \in \mathfrak{h}$, $g' \mapsto b(g') \in \mathfrak{b}$, $g' \mapsto b_{i+1}(g') \in \mathfrak{g}_{i+1}$, with no singularities in a Zariski neighbourhood $\mathcal{U}_i(g)$ of g such that $Z_i = h(g') + \text{Ad}(g')b(g) + b_{i+1}(g')$ for all $g' \in \mathcal{U}_i(g)$.

It is easy to see that

Proposition 3.3.

- (1) For $i \in \{1, \dots, n\}$, the following two conditions are equivalent:
- $\mathcal{U}_i \subset \{g \in G, Z_i \in \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1}\}$,
 - $\mathcal{U}_i \subset \{g \in G, Z_i \in \mathfrak{b} + \text{Ad}(g)\mathfrak{h} + \mathfrak{g}_{i+1}\}$.
- (2) The index subset $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b})$ is contained in $\mathcal{I}^{\mathfrak{g}/\mathfrak{h}} \cap \mathcal{I}^{\mathfrak{g}/\mathfrak{b}} =: \mathcal{I}^{\mathfrak{h}, \mathfrak{b}}$.
- (3) Every $\mathcal{O}_i, i \in \{1, \dots, n\}$, has the property $(\mathcal{P})$. Furthermore, the maximal Zariski open subsets $\mathcal{U}_i, i \in \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b})$ of G do.

Proof.

- (1) To find a nonempty Zariski open subset $\mathcal{U}_i$ of G satisfying one of these inclusions, i must be in $\mathcal{I}(\mathfrak{h}, \mathfrak{b})$. When this is the case, suppose that

$$\mathcal{U}_i \subset \{g \in G, Z_i \in \mathfrak{b} + \text{Ad}(g)\mathfrak{h} + \mathfrak{g}_{i+1}\}.$$

Since $\mathcal{U}_i$ is symmetric, for $g \in \mathcal{U}_i$, we have that $Z_i = X + \text{Ad}(g^{-1})Y + S_{i+1}$ for some $X \in \mathfrak{b}, Y \in \mathfrak{h}$ and $S_{i+1} \in \mathfrak{g}_{i+1}$. We obtain that

$$\text{Ad}(g)Z_i = \text{Ad}(g)X + Y + \text{Ad}(g)S_{i+1}$$

and

$$\text{Ad}(g)Z_i = Z_i \pmod{\mathfrak{g}_{i+1}}.$$

By putting $T_{i+1} := (Z_i - \text{Ad}(g)Z_i) + \text{Ad}(g)S_{i+1} \in \mathfrak{g}_{i+1}$, we get $Z_i = \text{Ad}(g)X + Y + T_{i+1}$. Hence $Z_i \in \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1}$, which means that $\mathcal{U}_i \subset \{g \in G, Z_i \in \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1}\}$. Finally, as $\mathfrak{h}$ and $\mathfrak{b}$ play a symmetric role, we are done.

- (2) If $i \in \{1, \dots, n\} \setminus \mathcal{I}^{\mathfrak{g}/\mathfrak{h}} \cap \mathcal{I}^{\mathfrak{g}/\mathfrak{b}} = \mathcal{I}^{\mathfrak{b}} \cup \mathcal{I}^{\mathfrak{h}}$, then $Z_i \in \mathfrak{b} + \mathfrak{g}_{i+1}$ or $Z_i \in \mathfrak{h} + \mathfrak{g}_{i+1}$. Therefore $\text{Ad}(g)(Z_i) \in \mathfrak{b} + \mathfrak{g}_{i+1}$ for every $g \in G$ or $\text{Ad}(g)(Z_i) \in \mathfrak{h} + \mathfrak{g}_{i+1}$ for every $g \in G$ and so $Z_i \in \text{Ad}(g^{-1})\mathfrak{b} + \mathfrak{g}_{i+1} \subset \text{Ad}(g^{-1})\mathfrak{b} + \mathfrak{h} + \mathfrak{g}_{i+1}$ for every $g \in G$ or $Z_i \in \text{Ad}(g^{-1})\mathfrak{h} + \mathfrak{g}_{i+1} \subset \text{Ad}(g^{-1})\mathfrak{h} + \mathfrak{b} + \mathfrak{g}_{i+1}$ for every $g \in G$. Hence $i \in \mathcal{I}(\mathfrak{h}, \mathfrak{b})$.
- (3) If $Z_i = \text{Ad}(g)X + Y + T_{i+1} = X + \text{Ad}(g^{-1})Y + S_{i+1}$ for some $X \in \mathfrak{b}, Y \in \mathfrak{h}$ and $T_{i+1}, S_{i+1} \in \mathfrak{g}_{i+1}$, then for every $(h, b) \in H \times B$, we have that

$$Z_i = \text{Ad}(gb)(\text{Ad}(b^{-1})X) + Y + T_{i+1}$$

and

$$Z_i = X + \text{Ad}((hg)^{-1})(\text{Ad}(h)Y) + S_{i+1}.$$

This shows that the subset of G consisting of all elements g , such that $Z_i \in \text{Ad}(g)\mathfrak{b} + \mathfrak{h} + \mathfrak{g}_{i+1}$ is H -invariant on the right and B -invariant on the left and so is its complement. $\square$

Remark 3.4. Putting $\mathcal{U} := \bigcap_{i \in \mathcal{I}^{\mathfrak{b}, \mathfrak{b}}} \mathcal{U}_i$, we obtain a “canonical” nonempty Zariski open subset of G having the property $(\mathcal{P})$, which allows us to say that

$$(3.1) \quad \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \{1 \leq i \leq n, \forall g \in \mathcal{U}, Z_i \notin \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1}\}.$$

Notation 3.5. We order the set of indices $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \{i_1 < \dots < i_d\}$ and we put

(1)

$$\phi : \mathbb{R}^d \longrightarrow G, \quad (t_1, \dots, t_d) \longmapsto t_1 Z_{i_1} \dots t_d Z_{i_d},$$

(2)

$$\mathbf{p} : G \longrightarrow H \backslash G / B, \quad g \longmapsto \tilde{g} = H \cdot g \cdot B,$$

(3)

$$\mathcal{V} = \phi^{-1}(\mathcal{U}) := \{t \in \mathbb{R}^d, \phi(t) \in \mathcal{U}\}, \quad \tilde{\mathcal{U}} = \mathbf{p}(\mathcal{U}) := \{\tilde{g}, g \in \mathcal{U}\},$$

(4)

$$\tilde{\phi} := \mathbf{p} \circ \phi|_{\mathcal{V}} : \begin{array}{ccc} \mathcal{V} & \longrightarrow & \tilde{\mathcal{U}}, \\ t = (t_1, \dots, t_d) & \longmapsto & H \cdot \phi(t) \cdot B. \end{array}$$

- (5) Finally, we provide the space of the double cosets $H \backslash G / B$ with the quotient topology; i.e. a subset $\tilde{U}$ in $H \backslash G / B$ is open if and only if the subset $U := \mathbf{p}^{-1}(\tilde{U})$ is open in G . The mapping $\mathbf{p}$ is then open, since for every open subset U of G , we have that $\mathbf{p}^{-1}(\mathbf{p}(U)) = HUB = \bigcup_{h \in H, b \in B} hUb$ is open in G .

Lemma 3.6. *In the particular case where $\mathfrak{b}$ is an ideal, we have that $\tilde{\phi}$ is a diffeomorphism between $\mathbb{R}^d$ and $H \backslash G / B$.*

Proof. The closed subgroup $HB = BH$ has $\mathfrak{h} + \mathfrak{b}$ as Lie algebra, and it satisfies $HB \backslash G = H \backslash G / B$. Indeed for every $g \in G$,

$$H \cdot g \cdot B = \{hgb, h \in H, b \in B\} = \{hb'g, h \in H, bgb^{-1} =: b' \in B\} = HB \cdot g.$$

We also notice that $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \mathcal{I}^{\mathfrak{g}/\mathfrak{h}+\mathfrak{b}} =: \{Z_{i_1}, \dots, Z_{i_d}\}$, since the condition $Z_i \notin \mathfrak{h} + \text{Ad } g \mathfrak{b} + \mathfrak{g}_{i+1}$ is now equivalent to the condition $Z_i \notin \mathfrak{h} + \mathfrak{b} + \mathfrak{g}_{i+1}$. This tells us that $\mathcal{U}$ is all of G . Hence $(Z_{i_1}, \dots, Z_{i_d})$ is a Malcev basis for $\mathfrak{g}$ relative to $\mathfrak{h} + \mathfrak{b}$. Finally, we have the diffeomorphism: $t = (t_1, \dots, t_d) \longmapsto HB \cdot (-t_d Z_{i_d}) \dots (-t_1 Z_{i_1})$ between $\mathbb{R}^d$ and the set $HB \backslash G$ of the right classes modulo HB . Composed with the diffeomorphism $HB \backslash G \ni HB \cdot g^{-1} \longmapsto g \cdot HB \in G / HB$, this mapping

becomes $\mathbb{R}^d \ni t = (t_1, \dots, t_d) \mapsto t_1 Z_{i_1} \dots t_d Z_{i_d} \cdot HB \in G/HB$, which is nothing but $\tilde{\phi}$ according to Notation 3.5(4). We deduce then that it is a diffeomorphic mapping. $\square$

Remark 3.7. In Theorem 3.10 we show that the canonical mapping $\tilde{\phi} : \mathcal{V} \longrightarrow \tilde{\mathcal{U}}$ is a homeomorphism. We proceed by induction on the dimension of $\mathfrak{g}/\mathfrak{b}$. We consider then the index

$$l := \max\{i \in \mathcal{I}^{\mathfrak{g}/\mathfrak{b}}\}$$

and we set

$$\mathfrak{a} := \mathfrak{g}_{l+1}, \quad \mathfrak{c} := \mathfrak{b} + \mathfrak{g}_l = \mathfrak{b} \oplus \mathbb{R}Z_l.$$

Since $Z_j \in \mathfrak{b}$ for every $j > l$, it follows that $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{b}$. Furthermore

$$\mathcal{I}^{\mathfrak{g}/\mathfrak{c}} = \mathcal{I}^{\mathfrak{g}/\mathfrak{b}} \setminus \{l\} = \{k_1 < \dots < k_{p-1}\},$$

whence, by intersecting with $\mathcal{I}^{\mathfrak{g}/\mathfrak{h}}$, we obtain that $\mathcal{I}^{\mathfrak{h},\mathfrak{c}} = \mathcal{I}^{\mathfrak{h},\mathfrak{b}} \setminus \{l\}$.

We encounter the following two cases which must be discussed also in the proof of Lemma 3.9.

- a):** $Z_l \notin \mathfrak{h} + \mathfrak{g}_{l+1}$, which means that $l \in \mathcal{I}^{\mathfrak{g}/\mathfrak{h}}$ and so $\mathcal{I}^{\mathfrak{h},\mathfrak{b}} = \mathcal{I}^{\mathfrak{h},\mathfrak{c}} \cup \{l\}$.
- b):** $Z_l \in \mathfrak{h} + \mathfrak{g}_{l+1}$, whence $Z_l \in \mathfrak{h}$ while we assume (2.3). In this case we have $l \notin \mathcal{I}^{\mathfrak{g}/\mathfrak{h}}$ and so $\mathcal{I}^{\mathfrak{h},\mathfrak{b}} = \mathcal{I}^{\mathfrak{h},\mathfrak{c}}$.

Since $\mathfrak{g}_l \subset \mathfrak{g}_{i+1}$ for all $i < l$, it follows that

$$\mathfrak{h} + \text{Ad}(g)\mathfrak{c} + \mathfrak{g}_{i+1} = \mathfrak{h} + \text{Ad}(g)(\mathfrak{b} + \mathfrak{g}_l) + \mathfrak{g}_{i+1} = \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1}, \quad \forall g \in G.$$

This tells us that $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{c}) = \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) \setminus \{l\}$.

We can also deduce from this equality that all the nonempty Zariski open subsets $\mathcal{U}_i^\circ$ ($i \in \mathcal{I}^{\mathfrak{h},\mathfrak{c}}$) of G , defining the canonical Zariski open subset $\mathcal{U}^\circ = \bigcap_{i \in \mathcal{I}^{\mathfrak{h},\mathfrak{c}}} \mathcal{U}_i^\circ$ as in Remark 3.4 for $H \backslash G/C$, coincide with the $\mathcal{U}_i$ ($i \in \mathcal{I}^{\mathfrak{h},\mathfrak{b}} \setminus \{l\}$) for $H \backslash G/B$. In particular, from $\mathcal{I}^{\mathfrak{h},\mathfrak{b}} \supseteq \mathcal{I}^{\mathfrak{h},\mathfrak{c}}$, we obtain that $\mathcal{U} \subseteq \mathcal{U}^\circ$.

Hence in the case **b)**, we have that

$$\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{c}),$$

since $l \notin \mathcal{I}^{\mathfrak{g}/\mathfrak{h}} \supset \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b})$ by Proposition 3.3 (2).

In the first case, we consider the following two eventualities.

- a-1)** $l \in \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b})$: This case is characterized by the existence of a nonempty Zariski open subset $\mathcal{U}_l$ of G , given by Definition 3.2 (2), such that for every $g \in \mathcal{U}_l$, we have $Z_l \notin \mathfrak{h} + \text{Ad}(g)\mathfrak{b}$ (since $\mathfrak{g}_{l+1} \subseteq \text{Ad}(g)\mathfrak{b}$ for $g \in G$).
- a-2)** $l \notin \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b})$: Here we have $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{c}) = \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b})$ as in the case **b)**. Hence for every g in a nonempty Zariski open subset $\mathcal{U}_l$ of G , we have that $Z_l \in \mathfrak{h} + \text{Ad}(g)\mathfrak{b}$.

Remark 3.8. For every $(w, \tilde{g}) \in G_l \times H \backslash G/B$, we have $H \cdot wgw^{-1} \cdot B = \tilde{g}$. This is an immediate consequence of the fact that $wgw^{-1}g^{-1} \in \exp([\mathfrak{g}_l, \mathfrak{g}]) \subset \exp(\mathfrak{g}_{l+1}) \subset B$ ($g \in G$).

Let us first show that

Lemma 3.9. *The mapping*

$$\tilde{\phi} : \mathcal{V} \ni (t_1, \dots, t_d) \mapsto H \cdot t_1 Z_{i_1} \dots t_d Z_{i_d} \cdot B \in \tilde{\mathcal{U}}$$

is bijective.

Proof. We will proceed by induction on the codimension p of B . There is nothing to prove if $p = 0$ while $H \backslash G / B = \{\tilde{e} = G\}$ when $B = G$. If $p = 1$, then the claim is trivial in view of Lemma 3.6; in fact, a subalgebra of codimension 1 in $\mathfrak{g}$ is an ideal of $\mathfrak{g}$. Let $l = \max \mathcal{I}^{\mathfrak{g}/\mathfrak{b}} = \{k_1 < \dots < k_p = l\}$ and $\mathfrak{c} := \mathbb{R}Z_l + \mathfrak{b}$ as in Remark 3.7. We apply the induction hypothesis to $\mathfrak{g}, \mathfrak{h}$ and $\mathfrak{c}$, giving us the bijection $\tilde{\phi}^\circ : \mathcal{V}^\circ \ni s \mapsto H \cdot \phi^\circ(s) \cdot C \in \tilde{\mathcal{U}}^\circ$. Let us introduce the projection

$$\begin{aligned} \Lambda^\circ : \quad H \backslash G / B &\longrightarrow H \backslash G / C, \\ H \cdot g \cdot B = \tilde{g} &\longmapsto \tilde{g} = H \cdot g \cdot C. \end{aligned}$$

We notice that the surjectivity of $\tilde{\phi}$ is equivalent to $H\phi(\mathcal{V})B = \mathcal{U}$. Since we have already the first inclusion “ $\subseteq$ ” thanks to the property $(\mathcal{P})$, it’s sufficient to prove that

$$\{h\phi(t)b, h \in H, t \in \mathcal{V}, b \in B\} \supseteq \mathcal{U}.$$

We now discuss the three cases of Remark 3.7.

Case a-1) $l \in \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \{i_1, \dots, i_d = l\}$.

Let’s recall that $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{c}) = \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) \setminus \{l\} = \{i_1, \dots, i_{d-1}\}$, the map ϕ° goes from $\mathbb{R}^{d-1}$ to G and that we have the inclusion

$$\bigcap_{i \in \mathcal{I}^{\mathfrak{h}, \mathfrak{c}}} \mathcal{U}_i = \mathcal{U}^\circ \supseteq \mathcal{U} = \mathcal{U}^\circ \cap \mathcal{U}_l.$$

By the induction hypothesis, we know that

$$\tilde{\phi}^\circ : \mathcal{V}^\circ \ni s = (t_1, \dots, t_{d-1}) \longmapsto H \cdot \phi^\circ(s) \cdot C \in \tilde{\mathcal{U}}^\circ$$

is bijective. Finally, we get as in Notation 3.5 (3), the nonempty Zariski open subset $\mathcal{V}^\circ = \phi^{\circ-1}(\mathcal{U}^\circ)$ of $\mathbb{R}^{d-1}$ and the open dense subset $\tilde{\mathcal{U}}^\circ = \{\tilde{g} = H \cdot g \cdot C, g \in \mathcal{U}^\circ\}$ of $H \backslash G / C$.

Surjectivity of $\tilde{\phi}$:

Let $u \in \mathcal{U} = \mathcal{U}^\circ \cap \mathcal{U}_l$. As $\mathcal{U}^\circ = H\phi^\circ(\mathcal{V}^\circ)C$, we can write $u = h\phi^\circ(s)c$ for certain $h \in H, s = (s_1, \dots, s_{d-1}) \in \mathcal{V}^\circ$ and $c \in C$. We have $c = (\exp \gamma Z_l)b$ where $b \in B$ and $\gamma \in \mathbb{R}$. Denoting by $t = (s_1, \dots, s_{d-1}, t_d := \gamma) \in \mathbb{R}^d$, we obtain that $u = h\phi(t)b$.

But $t \in \mathcal{V}$ since $\phi(t) = h^{-1}ub^{-1}$ is in $\mathcal{U}$ thanks to $(\mathcal{P})$. Finally $u \in H\phi(\mathcal{V})B$.

Injectivity of $\tilde{\phi}$:

At first, we remark that if $t = (t_1, \dots, t_d) \in \mathbb{R}^d$ is in $\mathcal{V}$, then $s := (t_1, \dots, t_{d-1}) \in \mathbb{R}^{d-1}$ must be in $\mathcal{V}^\circ$. Indeed, $\phi(t) \in \mathcal{U} = \mathcal{U}^\circ \cap \mathcal{U}_l \subset \mathcal{U}^\circ$ and so, thanks to the property $(\mathcal{P})$, the stability of $\mathcal{U}^\circ$ by right multiplication with elements of C gives $\phi^\circ(s) = \phi(t) \exp(-t_d Z_l) \in \mathcal{U}^\circ$; hence $s \in \mathcal{V}^\circ$.

Now let $t = (t_1, \dots, t_d), t' = (t'_1, \dots, t'_d) \in \mathcal{V}$ such that $H \cdot \phi(t) \cdot B = H \cdot \phi(t') \cdot B$, whence $H \cdot \phi(s) \cdot C = H \cdot \phi(s') \cdot C$, where $s = (t_1, \dots, t_{d-1})$ and $s' = (t'_1, \dots, t'_{d-1})$ belong to $\mathcal{V}^\circ$. Then $s = s'$ while $\tilde{\phi}^\circ$ is injective. Finally, to show that $t_d = t'_d$, we make the following calculation. Let $h = \exp Y \in H$ and $b = \exp X \in B$, such that $\phi(t) = h\phi(t')b$. As a consequence of the previous discussion, we can consider the other relation $\phi(t') = \phi(t) \exp((t'_d - t_d)Z_l) =: u$ between these two elements of $\mathcal{U}$,

to obtain

$$\begin{aligned}
\phi(t') \exp((t_d - t'_d)Z_l) &= \exp(Y)\phi(t') \exp(X) \\
&\Leftrightarrow \\
\exp(-Y)(u \exp((t_d - t'_d)Z_l)u^{-1}) &= u \exp(X)u^{-1} \\
&\Leftrightarrow \\
\exp(-Y) \exp(\operatorname{Ad}(u)(t_d - t'_d)Z_l) &= \exp(\operatorname{Ad}(u)X) \\
&\Rightarrow \\
-Y + (t_d - t'_d)Z_l &= \operatorname{Ad}(u)X \pmod{\mathfrak{g}_{l+1}}.
\end{aligned}$$

In fact,

$$\begin{aligned}
\exp(\operatorname{Ad}(u)(t_d - t'_d)Z_l) &= \exp((t_d - t'_d)Z_l) \pmod{G_{l+1}}, \\
&\Rightarrow \\
\exp(-Y) \exp(\operatorname{Ad}(u)(t_d - t'_d)Z_l) &= \exp(-Y + (t_d - t'_d)Z_l) \pmod{G_{l+1}} \\
&\quad (\text{by the Campbell-Baker-Hausdorff formula}).
\end{aligned}$$

But when $(t_d - t'_d)Z_l = Y + \operatorname{Ad}(u)X \pmod{\mathfrak{g}_{l+1}}$, and if we suppose $t_d \neq t'_d$, then $Z_l \in \mathfrak{h} + \operatorname{Ad}(u)\mathfrak{b}$, which contradicts the fact that $u \in \mathcal{U}$. Hence $t_d = t'_d$ and $\tilde{\phi}$ is injective.

Case a-2) $\mathcal{I}^{\mathfrak{b}, \mathfrak{b}} = \mathcal{I}^{\mathfrak{b}, \mathfrak{c}} \cup \{l\}$ but $l \notin \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \{i_1, \dots, i_d\} = \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{c})$.

Here, we have $\phi^\circ = \phi : \mathbb{R}^d \rightarrow G$, but the nonempty Zariski subsets of $\mathbb{R}^d$ defined by $\mathcal{V}^\circ = \phi^{-1}(\mathcal{U}^\circ)$ and $\mathcal{V} = \phi^{-1}(\mathcal{U})$ do not coincide in general. In fact, we know that $\mathcal{U}^\circ = \bigcap_{i \in \mathcal{I}^{\mathfrak{b}, \mathfrak{c}}} \mathcal{U}_i \supseteq \mathcal{U} = \mathcal{U}^\circ \cap \mathcal{U}_l$, where the Zariski open subset $\mathcal{U}_l$ of G satisfies $\forall g \in \mathcal{U}_l, Z_l \in \mathfrak{h} + \operatorname{Ad}(g)\mathfrak{b}$. By the induction hypothesis, $\tilde{\mathcal{U}}^\circ = \{\tilde{g} = H \cdot g \cdot C, g \in \mathcal{U}^\circ\}$ is in bijection with $\mathcal{V}^\circ$ thanks to $\tilde{\phi}^\circ$.

Surjectivity of $\tilde{\phi}$:

Let $u \in \mathcal{U} = \mathcal{U}^\circ \cap \mathcal{U}_l \subset \mathcal{U}^\circ$. Then u can be written as $h\phi^\circ(t)c = h\phi(t)c$, where $h \in H, c = \exp(\gamma Z_l)b \in C$ and $t \in \mathcal{V}^\circ$. Then also $v := \phi(t) \exp(\gamma Z_l) \in \mathcal{U}$, because of property (P). We shall show that

$$v = h'\phi(t)b'$$

for some $h' \in H, b' \in B$. This then shows that $t \in \mathcal{V} = \phi^{-1}(\mathcal{U})$ and so we see that $\tilde{\phi}$ is surjective.

Now since $v \in \mathcal{U}_l$, we have $Z_l \in \operatorname{Ad}(v^{-1})\mathfrak{h} + \mathfrak{b}$. Hence there exist $Y =: \log h' \in \mathfrak{h}$ and $X =: \log x \in \mathfrak{b}$, such that

$$\begin{aligned}
\gamma Z_l &= \operatorname{Ad}(v^{-1})Y + X \\
&\Leftrightarrow -X + \gamma Z_l = \operatorname{Ad}(v^{-1})Y \\
&\Rightarrow \exp(\gamma Z_l)x^{-1}q = v^{-1}h'v \text{ (for some } q \in G_{l+1} \subset B) \\
&\Rightarrow \exp(\gamma Z_l) = v^{-1}h'vb'' \text{ (for some } b'' \in B) \\
&\Rightarrow v = \phi(t) \exp(\gamma Z_l) = \phi(t)(\exp(-\gamma Z_l)\phi(t)^{-1}h'\phi(t) \exp(\gamma Z_l)b'') \\
&\Rightarrow v = \phi(t)(\phi(t)^{-1}h'\phi(t)b') = h'\phi(t)b' \text{ (for some } b' \in B).
\end{aligned}$$

Injectivity of $\tilde{\phi}$:

This is an immediate consequence from the injectivity of $\tilde{\phi}^\circ$, since $\mathcal{V} \subset \mathcal{V}^\circ$ and $\tilde{\phi} = \tilde{\phi}|_{\mathcal{V}}$.

Case b) $l \notin \mathcal{I}^{\mathfrak{g}/\mathfrak{h}}$.

This means that $Z_l \in \mathfrak{h}$. Now we have $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{c}) = \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \{i_1, \dots, i_d\}$, $\phi^\circ = \phi$, $\mathcal{U}^\circ = \mathcal{U} = \bigcap_{i \in \mathcal{I}^{\mathfrak{h}, \mathfrak{b}}} \mathcal{U}_i$ and $\mathcal{V}^\circ = \phi^{\circ^{-1}}(\mathcal{U}^\circ) = \phi^{-1}(\mathcal{U}) = \mathcal{V}$. But of course the spaces $H \backslash G/B$ and $H \backslash G/C$ are different, and $\tilde{\phi}^\circ$ and $\tilde{\phi}$ are related by the formula

$$\begin{aligned} \tilde{\phi}^\circ = \Lambda^\circ \circ \tilde{\phi}: \quad \mathcal{V} &\longrightarrow \tilde{\mathcal{U}}^\circ \\ t = (t_1, \dots, t_d) &\longmapsto H \cdot \phi(t) \cdot C. \end{aligned}$$

Surjectivity of $\tilde{\phi}$:

As we have already $\mathcal{U} = \mathcal{U}^\circ = H\phi^\circ(\mathcal{V}^\circ)C = H\phi(\mathcal{V})C$, it suffices to prove the inclusion $H\phi(\mathcal{V})C \subseteq H\phi(\mathcal{V})B$. Now if we take $u = h\phi(t)c$ where the elements $h, t = (t_1, \dots, t_d), c = \exp(\gamma Z_l)b, \gamma$ and b are respectively in $H, \mathcal{V}, C, \mathbb{R}$ and B , then

$$u = h\phi(t)c = h\phi(t)\exp(\gamma Z_l)b = (h\exp(\gamma Z_l))\exp(-\gamma Z_l)\phi(t)\exp(\gamma Z_l)b,$$

which gives

$$\begin{aligned} H \cdot u \cdot B &= H \cdot h\exp(\gamma Z_l)\exp(-\gamma Z_l)\phi(t)\exp(\gamma Z_l)b \cdot B \\ &= H \cdot \exp(-\gamma Z_l)\phi(t)\exp(\gamma Z_l) \cdot B = H\phi(t)B, \end{aligned}$$

by Remark 3.8.

Injectivity of $\tilde{\phi}$:

This is exactly as in the case a-2); in fact, $\tilde{\phi}^\circ = \Lambda^\circ \circ \tilde{\phi}$ is injective, and so is $\tilde{\phi}$. $\square$

Theorem 3.10. *The mapping $\tilde{\phi}: \mathcal{V} \ni t = (t_1, \dots, t_d) \longmapsto H \cdot t_1 Z_{i_1} \dots t_d Z_{i_d} \cdot B \in \tilde{\mathcal{U}}$ is a homeomorphism.*

Proof. The mapping $\tilde{\phi} = \mathbf{p} \circ \phi|_{\mathcal{V}}$ is evidently continuous since this is the case for ϕ and $\mathbf{p}$.

Hence it is enough to prove that the mapping

$$H \times \mathbb{R}^d \times B \longrightarrow G, (h, t, b) \longmapsto h\phi(t)b$$

is open.

We do this again by induction on the dimension of G/B , using the notation of the proof of Lemma 3.9.

If the dimension of $G/B = 1$, then B is normal and we apply Lemma 3.6.

In **Case a-1)**, it suffices to consider subsets $W_H \subset H$, which are open in H , $W = W_0 \times]a, b[\subset \mathcal{V}$ which are open in $\mathbb{R}^{d-1} \times \mathbb{R}$ and subsets $W_B \subset B$, which are open in B and to prove that $W_H \phi(W) W_B$ is open in G . But by the induction hypothesis for the groups H and C , we know that

$$W_H \phi(W) W_B = W_B \phi^\circ(W_0) \exp(]a, b[Z_l) W_B$$

is open in G , since $\exp(]a, b[Z_l) W_B$ is open in B .

In **Case a-2)** consider open subsets W_H of H , $W = W_0 \times]a, b[$ open in $\mathcal{V} \subset \mathbb{R}^{d-1} \times \mathbb{R}$ and W_B open in B and prove that $W_H \phi(W) W_B$ is open in G . Let $b \in W_B, h \in W_H, t = (t_1, \dots, t_d) \in W$. We know that $\phi = \phi|_{\mathcal{V}}$ and that W is open in $\mathcal{V}^\circ$. Since $l \in \mathcal{I}^{\mathfrak{g}/\mathfrak{h}} \cap \mathcal{I}^{\mathfrak{g}/\mathfrak{b}} \setminus \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b})$, there exist rational mappings

$$g \mapsto Y(g) \in \mathfrak{h}, \quad g \mapsto X(g) \in \mathfrak{b},$$

defined on a Zariski open neighborhood of $\phi(t)$, which we can assume to contain W , such that for every g in this neighborhood we have that

$$\begin{aligned} Z_l &= \text{Ad}(g^{-1})Y(g) + X(g) \\ \Rightarrow \exp(t_d Z_l - t_d X(g)) &= g^{-1} \exp(t_d Y(g))g, \quad t_d \in \mathbb{R}, \\ \Rightarrow \exp(t_d Z_l) b(g, t_d)^{-1} &= g^{-1} \exp(t_d Y(g))g \\ &\quad (\text{where } (g, t_d) \mapsto b(g, t_d) \in B \text{ is rational}), \\ \Rightarrow \exp(t_d Z_l) &= g^{-1} h(g, t_d) g b(g, t_d) \quad (\text{where } h(g, t_d) = \exp(t_d Y(g)) \in H). \end{aligned}$$

Hence, for $b' \in B, h' \in H, t' \in W, t_d \in \mathbb{R}$, we get

$$(3.2) \quad h' \phi(t') \exp(t_d Z_l) b' = h' h(\phi(t'), t_d) \phi(t') b(g, t_d) b'.$$

Since the mapping

$$\begin{aligned} H \times B \times \mathbb{R} \times W &\longrightarrow H \times B \times \mathbb{R} \times W \\ (h', b', t_d, t') &\longmapsto (h' h(\phi(t'), t_d), b(g, t_d) b', t_d, t') \end{aligned}$$

is a diffeomorphism, we can find a small neighbourhood $W'_H \subset W_H$ of h , $W'_B \subset W_B$ of b and $W' \subset W$ of t , such that for every $t_d \in]-\varepsilon, \varepsilon[$ for some $\varepsilon > 0$, we have that

$$(h' h(\phi(t'), t_d), b(g, t_d) b') \subset W_H \times W_B.$$

This implies by (3.2) that

$$h\phi(t)b \in W'_B \phi(W') \exp[] - \varepsilon, \varepsilon[Z_l] W'_B \subset W_H \phi(W) W_B.$$

By the induction hypothesis, we know that $W'_B \phi(W') \exp[] - \varepsilon, \varepsilon[Z_l] W'_B$ is an open neighborhood of $h\phi(t)b$ and thus the set $W_H \phi(W) W_H$ is open in G .

In Case b), we have that Z_l is an element of $\mathfrak{c}$. We consider again open subsets W_H of H , $W = W_0 \times]a, b[$ open in $\mathcal{V} \subset \mathbb{R}^d$ and W_B open in B and prove that $W_H \phi(W) W_B$ is open in G . Let $b \in W_B, h \in W_H, t = (t_1, \dots, t_d) \in W$. For $h' \in H, b' \in B, s \in \mathbb{R}$ and $t \in W$, we have that

$$\begin{aligned} (3.3) \quad h' \phi(t) \exp(s Z_l) b' &= h' \exp(s Z_l) (\exp(-s Z_l) \phi(t) \exp(s Z_l)) b' \\ &= h' \exp(s Z_l) \phi(t) b(s, t) b' \\ &\quad (\text{where } b(s, t) \in B \text{ is polynomial in } (s, t)). \end{aligned}$$

This implies as in the case a-2) that there exist small neighbourhoods $W'_H \subset W_H$ of h , $W'_B \subset W_B$ of b and $W' \subset W$ of t , such that for every $s \in]-\varepsilon, \varepsilon[$ (for some $\varepsilon > 0$), $h' \in W'_H, b' \in W'_B$, we have that

$$(3.4) \quad (h' h, b(s, t) b') \subset W_H \times W_B.$$

Therefore, by (3.3) and (3.4),

$$h\phi(t)b \in W'_B \phi(W') \exp[] - \varepsilon, \varepsilon[Z_l] W'_B \subset W_H \phi(W) W_B$$

and we conclude as in the case a-2). $\square$

Notation 3.11. We define the function $\theta : H \backslash G / B \longrightarrow \mathbb{R}^d$ such that for every $x \in \tilde{\mathcal{U}}$, we have

$$\theta(x) = t = (t_1, \dots, t_d) \in \mathcal{V} \iff x = \tilde{\phi}(t) = H \cdot t_1 Z_{i_1} \dots t_d Z_{i_d} \cdot B$$

and we denote by

$$\psi : H \backslash G / B \longrightarrow G$$

the inverse of the mapping $\phi(\mathcal{V}) \longrightarrow H \backslash G / B$. Then $\psi = \phi \circ \theta$ is a continuous section of the open dense subset $\tilde{\mathcal{U}}$ of $H \backslash G / B$ into G .

Example 3.12. Returning to the previous example, we can see that $\mathcal{I}^{\mathfrak{h}, \mathfrak{b}} = \mathcal{I}^{\mathfrak{g}/\mathfrak{h}} \cap \mathcal{I}^{\mathfrak{g}/\mathfrak{b}} = \{1, 2, 6\}$ and $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \{1, 2\}$. In fact, if $g = z_1 Z_1 \dots z_7 Z_7 \in G$ (with $(z_1, \dots, z_7) \in \mathbb{R}^7$), then $\text{Ad}(g)\mathfrak{b} = \text{span}\langle Z_3, Z_4 + z_1 Z_6, Z_7 \rangle$, so in the Zariski open subset $\mathcal{U}_6 := G \setminus G_2$ of G ($z_1 \neq 0$), we have that $\forall g \in \mathcal{U}_6$, $Z_6 \in \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_7 = \text{span}\langle Z_3, Z_4, Z_5, z_1 Z_6, Z_7 \rangle$, but for $i = 1, 2$, it's clear that $Z_i \notin \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1}$, $\forall g \in G =: \mathcal{U}_i$. In conclusion,

$$\begin{aligned} \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) &= \{i \in \mathcal{I} = \{1, 2, 6\}, Z_i \notin \mathfrak{h} + \text{Ad}(g)\mathfrak{b} + \mathfrak{g}_{i+1} \ \forall g \in \mathcal{U} = G \setminus G_2\} \\ &= \{1, 2\}. \end{aligned}$$

Now for $H \setminus G/B$, thanks to $\mathcal{V} = \phi^{-1}(\mathcal{U}) = \{t \in \mathbb{R}^2, t_1 Z_1 \cdot t_2 Z_2 \in G \setminus G_2\} = \mathbb{R}^* \times \mathbb{R}$ and $\tilde{\mathcal{U}} = \mathbf{p}(\mathcal{U}) = \{H \cdot z_1 Z_1 \dots z_7 Z_7 \cdot B, z = (z_1, \dots, z_7) \in \mathbb{R}^* \times \mathbb{R}^6\}$ where these maps are given by

$$\begin{aligned} \phi: \mathbb{R}^2 &\longrightarrow G \\ (t_1, t_2) &\longmapsto t_1 Z_1 \cdot t_2 Z_2 \end{aligned} \quad \text{and} \quad \begin{aligned} \mathbf{p}: G &\longrightarrow H \setminus G/B, \\ g &\longmapsto H \cdot g \cdot B \end{aligned}$$

we have that

$$\begin{aligned} \tilde{\phi}: \mathbb{R}^* \times \mathbb{R} &\longrightarrow \tilde{\mathcal{U}} \\ (t_1, t_2) &\longmapsto H \cdot t_1 Z_1 \cdot t_2 Z_2 \cdot B, \end{aligned}$$

which is a homeomorphism thanks to the previous theorem. In particular, this confirms the relation $\mathbf{p}(G \setminus G_2) = \tilde{\phi}(\mathbb{R}^* \times \mathbb{R})$ of the first example's calculation.

3.2. A measure on $H \setminus G/B$. Our aim is to disintegrate the restriction to a closed subgroup $H = \exp(\mathfrak{h})$ of an irreducible representation $\pi_{\ell, \mathfrak{b}}$ of G ($\mathfrak{b} = \text{Lie}(B)$) is a polarization of $\mathfrak{g}$ at $\ell \in \mathfrak{g}^*$). The first step is to realize the Hilbert space $L^2(G/B, \chi_\ell)$ as a direct integral. Therefore we need a measure $d\gamma$ on the space $H \setminus G/B$ such that

$$L^2(G/B, \chi_\ell) \simeq \int_{H \setminus G/B}^{\oplus} L^2(H/B(x), \chi_{\ell(x)}) d\gamma(x),$$

where the linear functional

$$(3.5) \quad \ell(x) = \text{Ad}^*(\psi(x))\ell \quad (x \in H \setminus G/B)$$

defines a character of the closed subgroup

$$B(x) := H \cap \psi(x)B\psi(x)^{-1} = \exp(\mathfrak{b}(x) := \mathfrak{h} \cap \text{Ad}(\psi(x))\mathfrak{b}).$$

Notation 3.13. More generally, for any $g \in G$ denote by $\mathfrak{b}(g) = \mathfrak{h} \cap \text{Ad}(g)\mathfrak{b}$ the Lie algebra of $B(g) = H \cap gBg^{-1}$. We may also use the notation $B(x)$ or $B(t)$ (respectively $\mathfrak{b}(x)$ and $\mathfrak{b}(t)$) when $x = \tilde{\phi}(t) \in H \setminus G/B$, instead of $B(\phi(t))$ (respectively $\mathfrak{b}(\phi(t))$).

Let $\mathcal{Y} = \{Y_1, \dots, Y_m\}$ be a Jordan-Hölder basis of $\mathfrak{h}$. As in section 2, let

$$E_{\mathcal{Y}}: \mathbb{R}^m \longrightarrow H, \quad (y_1, \dots, y_m) \mapsto \exp(y_1 Y_1) \cdots \exp(y_m Y_m)$$

be the diffeomorphism used to define the Haar measure $d_{\mathcal{Y}}$ on H by

$$\int_H f(h) d_{\mathcal{Y}}(h) := \int_{\mathbb{R}^m} f \circ E_{\mathcal{Y}}(y) dy, \quad f \in C_c(H).$$

We know that for any subalgebra $\mathfrak{k} \subset \mathfrak{h}$, if we define the index set $\mathcal{I}^{\mathfrak{b}/\mathfrak{k}}$ by

$$\mathcal{I}^{\mathfrak{b}/\mathfrak{k}} := \{i \in \{1, \dots, m\}, Y_i \notin \mathfrak{k} + \mathfrak{h}_{i+1}\} = \{j_1 < \dots < j_r\},$$

then the subset

$$(3.6) \quad \mathcal{N} := \{Y_j, j \in \mathcal{I}^{\mathfrak{h}/\mathfrak{k}}\}$$

is a Malcev basis of $\mathfrak{h}$ relative to $\mathfrak{k}$ and the mapping

$$E_{\mathcal{N}} : \mathbb{R}^r \longrightarrow H, \quad (s_1, \dots, s_r) \mapsto \exp(s_1 Y_{j_1}) \cdots \exp(s_r Y_{j_r}),$$

composed with the mapping $: H \ni h \mapsto \check{h} = h \cdot K \in H/K$, gives the diffeomorphism

$$\check{E}_{\mathcal{N}} : \mathbb{R}^r \longrightarrow H/K, \quad (s_1, \dots, s_r) \mapsto \exp(s_1 Y_{j_1}) \cdots \exp(s_r Y_{j_r}) \cdot K.$$

In particular, similarly to (2.2), we obtain an H -invariant measure on H/K denoted $d_{\mathcal{N}}(h)$ or simply $d\check{h}$, by putting

$$(3.7) \quad \int_{H/K} \varphi(\check{h}) d_{\mathcal{N}}(h) := \int_{\mathbb{R}^r} \varphi \circ \check{E}_{\mathcal{N}}(s) ds, \quad \varphi \in C_c(H/K).$$

Proposition 3.14. *There exists a Zariski open subset $\mathcal{U}''$ of G with property (P), and an index set $\mathcal{J}^{\mathfrak{h}/\mathfrak{b}(G)} \subset \mathcal{I}^{\mathfrak{h}}$, such that*

$$\mathcal{J}^{\mathfrak{h}/\mathfrak{b}(G)} = \mathcal{I}^{\mathfrak{h}/\mathfrak{h} \cap \text{Ad}(g)\mathfrak{b}} \quad \text{for all } g \in \mathcal{U}''.$$

Proof. For every $i \in \mathcal{I}^{\mathfrak{h}}$, the set

$$\mathcal{S}'_i = \{g \in G, Z_i \notin \mathfrak{h} \cap \text{Ad}(g)\mathfrak{b} + \mathfrak{h} \cap \mathfrak{g}_{i+1}\}$$

has property (P) and is either contained in a Zariski closed subset of G or it contains a Zariski open subset $\mathcal{S}_i$ with property (P).

The collection of all the i 's verifying the second eventuality gives us the subset $\mathcal{J}^{\mathfrak{h}/\mathfrak{b}(G)}$ of $\mathcal{I}^{\mathfrak{h}}$ and $\mathcal{U}'' = \bigcap_{i \in \mathcal{J}^{\mathfrak{h}/\mathfrak{b}(G)}} \mathcal{S}_i$. $\square$

Notation 3.15. Let $\{j_1 < \dots < j_r\}$ be the elements of $\mathcal{J}^{\mathfrak{h}/\mathfrak{b}(G)}$. Then $\mathcal{N} := \{Y_{j_1}, \dots, Y_{j_r}\}$ is a Malcev basis of $\mathfrak{h}$ relative to $\mathfrak{b}(g)$ for all $g \in \mathcal{U}''$. Now we put

$$\mathcal{U}' := \mathcal{U} \cap \mathcal{U}'' \subset G, \quad \mathcal{V}' := \phi^{-1}(\mathcal{U}') \subset \mathbb{R}^d, \quad \tilde{\mathcal{U}}' := \mathbf{p}(\mathcal{U}') \subset H \backslash G/B.$$

The subsets $\mathcal{U}'$ and $\mathcal{V}'$ respectively of G and $\mathbb{R}^d$ are Zariski open subsets and $\tilde{\mathcal{U}}'$ is open and dense in $H \backslash G/B$. To simplify the notation, these subsets will be denoted respectively by $\mathcal{U}, \mathcal{V}$ and $\tilde{\mathcal{U}}$.

Remark 3.16. Since the representations $\tau_{\ell, \mathfrak{b}}$ and $\tau_{\text{Ad}^*(g)\ell, \text{Ad}(g)\mathfrak{b}}$, $g \in G$, are equivalent, we can replace the subalgebra $\mathfrak{b}$ by a conjugated algebra $\text{Ad}(g)\mathfrak{b}$, with $g \in \mathcal{U}$. This allows us to assume, if we want, that e is contained in the Zariski open subset $\mathcal{U}$. This means that we can suppose that 0 is an element of $\mathcal{V}$.

Theorem 3.17. *The mapping*

$$\Phi : \mathbb{R}^r \times \mathcal{V} \longrightarrow G/B : (s, t) \mapsto E_{\mathcal{N}}(s)\phi(t) \cdot B$$

is injective and polynomial, its image is equal to $\mathcal{U} \bmod B$ and it is regular on a Zariski open subset of $\mathbb{R}^r \times \mathcal{V}$.

Proof. Let $g \in \mathcal{U}$. Then $g = h\phi(t)b$ with $t \in \mathcal{V}, h \in H$ and $b \in B$. We can write $h = E_{\mathcal{N}}(s)h_0$, where $h_0 = \phi(t)b_0\phi(t)^{-1}$ (with $b_0 \in B$), is an element of $H \cap \phi(t)B\phi(t)^{-1}$. Hence,

$$\Phi(s, t) = E_{\mathcal{N}}(s)\phi(t) \cdot B = E_{\mathcal{N}}(s)\phi(t)b_0 \cdot B = E_{\mathcal{N}}(s)\phi(t)b_0\phi(t)^{-1}\phi(t) \cdot B = g \cdot B.$$

If $\Phi(s, t) = \Phi(s', t')$ for two couples (s, t) and (s', t') in $\mathbb{R}^r \times \mathcal{V}$, then $H \cdot \phi(t) \cdot B = H \cdot \phi(t') \cdot B$ and since $t \in \mathcal{V}$, it follows from the injectivity of $\tilde{\phi}$, that $t = t'$. Hence $E_{\mathcal{N}}(s) = E_{\mathcal{N}}(s')$ and the injectivity of the mapping $E_{\mathcal{N}}$ implies that $s = s'$.

Now, if we choose for instance the Malcev basis $\mathcal{M} = \{Z_{k_1}, \dots, Z_{k_p}\}$ of $\mathfrak{g}$ relative to $\mathfrak{b}$ defined in (2.1), then for

$$z = \exp\left(\sum_{j=1}^n z_j Z_j\right) \in G,$$

we have that

$$z \mod B = E_{\mathcal{U}}(Q_1(z_1, \dots, z_n), \dots, Q_p(z_1, \dots, z_n)) \mod B,$$

where $Q_1, \dots, Q_k : \mathbb{R}^n \rightarrow \mathbb{R}$ are real-valued polynomials. This tells us that Φ is a polynomial mapping, since the mapping $(s, t) \rightarrow E_{\mathcal{N}}(s)\phi(t)$ is clearly polynomial. The differential of the mapping Φ in zero is given by the family of vectors

$$\{Y_{j_1}, \dots, Y_{j_r}, Z_{i_1}, \dots, Z_{i_d}\} \mod \mathfrak{b}.$$

These vectors form a basis of $\mathfrak{g}$ modulo $\mathfrak{b}$ (but not necessarily a Malcev basis). Hence Φ is regular in $(0, 0)$ and since Φ is polynomial, Φ is regular on a Zariski open subset of $\mathbb{R}^r \times \mathbb{R}^d$. $\square$

Remark 3.18. As a consequence of the previous theorem, we have the following condition on the cardinals:

$$\sharp \mathcal{I}^{\mathfrak{g}/\mathfrak{b}} = \sharp \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) + \sharp \mathcal{J}^{\mathfrak{h}/\mathfrak{b}(G)},$$

but in general,

$$\mathcal{I}^{\mathfrak{g}/\mathfrak{b}} \neq \mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) \cup \mathcal{J}^{\mathfrak{h}/\mathfrak{b}(G)}.$$

This is shown by the following example.

Example 3.19. We use again Example 3.1. Clearly we have that $\mathcal{I}^{\mathfrak{g}/\mathfrak{b}} = \{1, 2, 5, 6\}$ and we have seen that $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \{1, 2\}$ and it follows easily that $\mathcal{J}^{\mathfrak{h}/\mathfrak{b}(G)} = \{4, 5\}$.

Corollary 3.20. *We recall the definitions of the sets of indices.*

- $\mathcal{I}(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{b}) = \{i_1, \dots, i_d\}$ giving the directions $(Z_{i_1}, \dots, Z_{i_d})$ which characterize almost all elements of $H \backslash G/B$ by Theorem 3.10.
- $\mathcal{J}^{\mathfrak{h}/\mathfrak{b}(G)} = J = \{j_1, \dots, j_r\}$ giving the Malcev basis $\mathcal{N} = (Y_{j_1}, \dots, Y_{j_r})$ of $\mathfrak{h}$ relative to $\mathfrak{b}(g)$ for every $g \in \mathcal{U}''$ defined in Notation 3.15.
- $\mathcal{I}^{\mathfrak{g}/\mathfrak{b}} = \{k_1, \dots, k_p\}$ ($p = r + d$), giving the Malcev basis $\mathcal{M} = (Z_{k_1}, \dots, Z_{k_p})$ of $\mathfrak{g}$ relative to $\mathfrak{b}$ defined in (2.1).

As a consequence of Theorem 3.17, we have that the mapping

$$\begin{aligned} \Psi = E_{\mathcal{M}}^{-1} \circ \Phi : \quad & \mathbb{R}^r \times \mathcal{V} \longrightarrow \mathbb{R}^p \\ (s = (s_1, \dots, s_r), t = (t_1, \dots, t_d)) & \longmapsto \zeta = (\zeta_1, \dots, \zeta_p) \quad / \\ E_{\mathcal{N}}(s)\phi(t) \cdot B = s_1 Y_{j_1} \dots s_r Y_{j_r} \cdot t_1 Z_{i_1} \dots t_d Z_{i_d} \cdot B &= \zeta_1 Z_{k_1} \dots \zeta_p Z_{k_p} \cdot B = E_{\mathcal{M}}(\zeta) \end{aligned}$$

is a polynomial diffeomorphism into its image $\mathcal{O}$, which is a Zariski open subset of $\mathbb{R}^p$.

Notation 3.21. For $t \in \mathcal{V}$, we denote by E_t the diffeomorphism between H and $\mathbb{R}^r \times B(t)$ such that for $h \in H$, and $(s, b) \in \mathbb{R}^r \times B(t)$,

$$E_t(h) = (s, b) \iff h = E_{\mathcal{N}}(s)b.$$

Write also for $t \in \mathcal{V}$ and $h \in H$, the element $s(h, t)$ of $\mathbb{R}^r$ satisfying

$$h = E_{\mathcal{N}}(s(h, t)) \mod B(t).$$

We now fix $t \in \mathcal{V}$ and $h \in H$ in order to define the polynomial mapping $q_{h,t} : \mathbb{R}^r \longrightarrow \mathbb{R}^r$ by

$$q_{h,t}(s) = \ddot{E}_{\mathcal{N}}^{-1}(hE_{\mathcal{N}}(s)) = s(hE_{\mathcal{N}}(s), t), s \in \mathbb{R}^r.$$

It follows then that

$$q_{hh',t} = q_{h,t} \circ q_{h',t}, \quad h, h' \in H, t \in \mathcal{V}.$$

Proposition 3.22. *The Jacobian $Jac(s, t)$ of the mapping Φ of Theorem 3.17 is constant in $s \in \mathbb{R}^r$.*

Proof. Indeed, if $f \in C_c(\mathcal{U}/B)$, $t \in \mathcal{V}$, and $h' \in H$, using the left-invariant measure $d\dot{g}$ on G/B , we get

$$\begin{aligned} \int_{\mathbb{R}^d} \int_{\mathbb{R}^r} f(\Phi(s, t)) |Jac(s, t)| ds dt &= \int_{G/B} f(g) d\dot{g} = \int_{G/B} f(h'g) d\dot{g} \\ &= \int_{\mathbb{R}^d} \int_{H/B(t)} f(h'h\phi(t)) |Jac(s((h, t), t))| d_{\mathcal{N}}(h) dt \\ &= \int_{\mathbb{R}^d} \int_{H/B(t)} f(h\phi(t)) |Jac(s(h'^{-1}h, t), t)| d\ddot{h} dt \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^r} f(E_{\mathcal{N}}(s)\phi(t)) |Jac(q_{h'^{-1},t}(s), t)| ds dt \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^r} f(\Phi(s, t)) |Jac(q_{h'^{-1},t}(s), t)| ds dt. \end{aligned}$$

This shows that $|Jac(q_{h'^{-1},t}(s), t)| = |Jac(s, t)|$ for every $h' \in H, s \in \mathbb{R}^r$ and $t \in \mathcal{V}$. Therefore $Jac(s, t) = Jac(0, t)$ for every $s \in \mathbb{R}^r, t \in \mathcal{V}$, since Φ is a real-valued polynomial function. $\square$

Definition 3.23. Let

$$F(t) := Jac(0, t), t \in \mathcal{V}.$$

Then $F : \mathcal{V} \longrightarrow \mathbb{R}$ is a function on $\mathcal{V}$ which admits a polynomial extension on $\mathbb{R}^d$.

Definition 3.24. We can define now our measure on the topological space $H \setminus G/B$. We have the Zariski dense open subset $\mathcal{U}$ of G , which has property $(\mathcal{P})$ and therefore its complement $\mathcal{C}$ in G is Zariski closed and has also property $(\mathcal{P})$. The projection $\mathbf{p} : G \longrightarrow H \setminus G/B$ maps $\mathcal{U}$ to the open dense subset $\tilde{\mathcal{U}}$ and $\mathcal{C}$ to the closed subset $\tilde{\mathcal{C}}$. We define a measure $d\gamma$, which is supported on $\tilde{\mathcal{U}}$, by the rule

$$\int_{H \setminus G/B} f(x) d\gamma(x) := \int_{\mathcal{V}} f(\phi(t)) |F(t)| dt.$$

Proposition 3.25. *Let $f \in C_c(G/B)$. Then*

$$\int_{G/B} f(g) d\dot{g} = \int_{H \setminus G/B} \left(\int_{H/B(x)} f(h\psi(x)) d\ddot{h} \right) d\gamma(x).$$

Proof. We can make the following computations:

$$\begin{aligned}
\int_{G/B} f(g) d\dot{g} &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^r} f(\Phi(s, t)) |Jac(s, t)| ds dt \\
&= \int_{\mathcal{V}} \left(\int_{\mathbb{R}^r} f(E_{\mathcal{N}}(s) \phi(t)) ds \right) |F(t)| dt \\
&= \int_{\mathcal{V}} \left(\int_{H/B(t)} f(h \phi(t)) d_{\mathcal{N}}(h) \right) |F(t)| dt
\end{aligned}$$

and justify them by

- Theorem 3.17 for the first equality,
- Proposition 3.22 and Definition 3.23 for the second one,
- Reference (3.7) for the last statement; in fact $\mathcal{N}$ is a Malcev basis of $\mathfrak{h}$ relative to all the $\mathfrak{b}(t), t \in \mathcal{V}$.

We finally conclude by Definition 3.24. $\square$

Notation 3.26. Let $H = \exp(\mathfrak{h})$ be a closed subgroup of the simply connected nilpotent Lie group $G = \exp(\mathfrak{g})$. Let $\ell \in \mathfrak{g}^*$, let $B = \exp(\mathfrak{b})$ be a polarization at ℓ and $\pi_{\ell} = \pi_{\ell, \mathfrak{b}} = \text{ind}_B^G \chi_{\ell}$ the irreducible representation associated to $(\ell, \mathfrak{b})$. Let for $g \in \mathcal{U}$,

$$\ell(g) = \text{Ad}^*(g)\ell|_{\mathfrak{b}} \quad \text{and} \quad \tau_{\ell(g), \mathfrak{b}(g)} = \text{ind}_{B(g)}^H \chi_{\ell(g)} =: \sigma(g).$$

Then we can show that the representations $\sigma(g')$ and $\sigma(g)$ of H are equivalent whenever g' and g are “equivalent”, i.e. whenever g and g' are in the same double coset. Hence, when $\tilde{g} = \tilde{\phi}(t) \in \tilde{\mathcal{U}}$, we define $\sigma(\tilde{g}) = \sigma(t) := [\sigma(g)]$ as the equivalence class of the unitary representation $\sigma(g)$.

Proposition 3.27. *The restriction of $\pi_{\ell, \mathfrak{b}}$ to H is a direct integral over the double cosets space $H \backslash G/B$ (with respect to the measure $d\gamma$) and the contribution of $x \in \tilde{\mathcal{U}} \subset H \backslash G/B$ may be computed as follows. Recall (3.5) where $\ell(x) \in \mathfrak{h}^*$ is the character $\ell(\psi(x))$ on the closed subgroup $B(x) = \exp(\mathfrak{b}(x))$ of H . Hence, the isotopic components in the disintegration of $\pi_{\ell, \mathfrak{b}}$ to H are the representations $\sigma(x) = \tau_{\ell(x), \mathfrak{b}(x)}$ whose Hilbert spaces are $\mathcal{H}_{\sigma(x)} = L^2(H/B(x), \chi_{\ell(x)})$. We have*

$$L^2(G/B, \chi_{\ell}) \simeq \int_{H \backslash G/B}^{\oplus} \mathcal{H}_{\sigma(x)} d\gamma(x)$$

and

$$\pi_{\ell, \mathfrak{b}|_H} \simeq \int_{H \backslash G/B}^{\oplus} \sigma(x) d\gamma(x).$$

Proof. Let L_0 be the space of all the vector fields $F = (f_x \in C_c(H/B(x)))_{x \in \tilde{\mathcal{U}}}$, such that the function $\mathbb{R}^r \times \tilde{\mathcal{U}} \ni (s, x) \mapsto f_x(E_{\mathcal{N}}(s))$ is continuous with compact support. On L_0 we have the Hilbert space norm

$$\|F\|_2^2 := \int_{H \backslash G/B} \int_{\mathbb{R}^r} |F_x(E_{\mathcal{N}}(s))|^2 ds d\gamma(x) = \int_{H \backslash G/B} \|F_x\|_{L^2(H/B(x), \chi_{\ell(x)})}^2 d\gamma(x).$$

We construct now an isometric isomorphism S between $L^2(G/B, \chi_{\ell})$ and the direct integral

$$\int_{H \backslash G/B}^{\oplus} L^2(H/B(x), \chi_{\ell(x)}) d\gamma(x),$$

which can be defined as the completion of L_0 for the norm $\|\cdot\|_2$ of L_0 .

For $\xi \in C_c(G/B, \chi_\ell)$, define the function

$$(3.8) \quad S\xi(x)(h) := \xi(h\psi(x)), x \in \tilde{\mathcal{U}}, h \in H.$$

For $x \in \tilde{\mathcal{U}}$, the function $S(\xi)(x)$ is then an element of $C_c(H/B(x), \chi_{\ell(x)})$, since it satisfies the covariance relation

$$S(\xi)(x)(hb) = \xi(hb\phi(t)) = \xi(h\phi(t)\phi(t)^{-1}b\phi(t)) = \chi_{\ell(x)}\xi(h\phi(t)) = \chi_{\ell(x)}S(\xi)(x)(h),$$

where $\phi(t) = \psi(x)$, $h \in H$, and $b \in B(x)$. It is also continuous with compact support modulo $B(x)$ for every $x \in \tilde{\mathcal{U}}$, since the mapping $\Phi : \mathbb{R}^r \times \mathcal{V} \rightarrow \mathcal{U}$ is a homeomorphism.

We obtain in this way a vector field $(S(\xi)(x) \in C_c(H/B(x), \chi_{\ell(x)}))_{x \in \tilde{\mathcal{U}}}$, which is contained in L_0 . The mapping $S : C_c(G/B, \chi_\ell) \rightarrow L_0$ is isometric by Proposition 3.25, it is a linear isomorphism and it obviously intertwines the representation $\pi_{\ell|H}$ and the direct integral representation

$$\int_{H \backslash G/B}^{\oplus} \tau_{\chi_{\ell(x)}, B(x)} d\gamma(x).$$

□

4. DISINTEGRATION OF INDUCED REPRESENTATIONS

We have seen in the preceding section that the disintegration of the restriction of the irreducible representation $\pi_{\ell, \mathfrak{b}}$ to H reduces now to the disintegration of the monomial representations $\sigma(x) = \tau_{\chi_{\ell(x)}, B(x)}$ of H into irreducibles. We shall use the techniques developed in [2].

In Notation 3.15, we introduced the Malcev basis $\mathcal{N} = \{Y_{j_1}, \dots, Y_{j_r}\}$ of $\mathfrak{h}$ relative to all the $\mathfrak{b}(g)$, $g \in \mathcal{U}$. Let us use also the following notation.

Notation 4.1. For every $t \in \mathcal{V}$,

- we have similarly to (2.1) a flag of subalgebras

$$(4.1) \quad \mathfrak{b}^{r+1}(t) := \mathfrak{b}(t) \subset \dots \subset \mathfrak{b}^c(t) = \text{span}\langle Y_{j_c}, \dots, Y_{j_r}, \mathfrak{b}(t) \rangle \subset \dots \subset \mathfrak{b}^1(t) = \mathfrak{h}.$$

We denote their Lie groups by $B^c(t) := \exp \mathfrak{b}^c(t)$, $c = 1, \dots, r$.

- Let $\mathcal{X}(t)$ be a Jordan-Hölder basis of $\mathfrak{b}(t)$ generated by rationally varied vectors $X_i(t)$, $i = r+1, \dots, m$, with no singularities in $t \in \mathcal{V}$. The corresponding Malcev basis of $\mathfrak{h}$ is then

$$\mathcal{Y}(t) = \{X_1(t) := Y_{j_1}, \dots, X_r(t) := Y_{j_r}, X_{r+1}(t), \dots, X_m(t)\},$$

and has the dual basis in $\mathfrak{h}^*$

$$\mathcal{Y}(t)^* = \{X_1(t)^*, \dots, X_r(t)^*, X_{r+1}(t)^*, \dots, X_m(t)^*\}, \quad t \in \mathcal{V}.$$

- Let

$$(4.2) \quad \Gamma(t) = \ell(t) + \mathfrak{b}(t)^\perp \quad \text{where} \quad \ell(t) := \text{Ad}^*(\phi(t))\ell|_{\mathfrak{h}}.$$

Also let $d_c(t) := \max\{\dim(\text{Ad}^*(B^c(t))(f)); f \in \Gamma(t)\}$ for any $1 \leq c \leq r$ and put by convention $d_{r+1}(t) := 0$. We obtain then an index set

$$(4.3) \quad I(t) := \{c_1 < \dots < c_{q(t)}\} := \{c \in \{1, \dots, r\}, \quad d_c(t) = d_{c+1}(t)\}$$

and an affine space, on which we take the Lebesgue measure dv ,

$$(4.4) \quad \mathcal{R}(t) := \{\ell(t) + \sum_{c_i \in I(t)} v_i X_{c_i}(t)^*, \quad v = (v_1, \dots, v_{q(t)}) \in \mathbb{R}^{q(t)}\}.$$

We can again assume, if necessary by shrinking $\mathcal{V}$, that there exists a fixed index set I such that for every $t \in \mathcal{V}$ we have that

$$(4.5) \quad I(t) = I = \{c_1, \dots, c_q\} \text{ and so } q(t) = q = |I|.$$

- Denote now by

$$(4.6) \quad \ell(t, v) := \ell(t) + \sum_{c \in I(t)} v_i Y_{j_c}(t)^* \in \Gamma(t), v \in \mathbb{R}^q$$

and by $\mathfrak{k}(t, v)$ the Vergne polarization of $\mathfrak{h}$ at $\ell(t, v)$ relative to the Jordan-Hölder basis

$$\mathcal{Y} = \{Y_1, \dots, Y_m\} = \{Z_j, j \in \mathcal{I}^{\mathfrak{h}}\}.$$

We denote by $\rho_{(t,v)} = \rho_w$ the unitary irreducible representation of H ,

$$(4.7) \quad \rho_f = \text{ind}_{K(t,v)}^H \chi_f \in \hat{H},$$

for any $f = f_w = \ell(t, v) \in \mathcal{R}(t)$ where $w = (t, v), v \in \mathbb{R}^q$ and $K(t, v) = \exp(\mathfrak{k}(t, v))$.

- Finally, we identify any $x \in \tilde{\mathcal{U}} \subset H \backslash G / B$ with $\theta(x) \in \mathbb{R}^d$; in particular when $t \in \mathcal{V}$ and $x = \tilde{\phi}(t)$, then we put $\mathcal{R}(x) := \mathcal{R}(t)$. It follows that if $v \in \mathcal{R}(x)$, then $(x, v) := (t, v), \rho_{(x,v)} := \rho_{(t,v)}, K(x, v) := K(t, v)$, etc...

It follows from the proofs used in the paper [2] that there exists a Zariski open subset $\mathcal{W}$ of $\mathbb{R}^d \times \mathbb{R}^q$ contained in $\mathcal{V} \times \mathbb{R}^q$ such that for every $w = (t, v) \in \mathcal{W}$, we have a Malcev basis

$$\mathcal{X}(w) = \{X_1(w), \dots, X_d(w)\}$$

of $\mathfrak{h}$ relative to $\mathfrak{k}(t, v)$ and a Malcev basis

$$(4.8) \quad \mathcal{Y}(w) = \{Y_1(w), \dots, Y_e(w)\}$$

of $\mathfrak{k}(t, v)$ relative to the subalgebra $\mathfrak{k}(t, v) \cap \mathfrak{b}(t)$ such that all the vectors $X_j(w)$ and $Y_i(w)$ vary rationally without singularities in $w = (t, v) \in \mathcal{W}$. Define now the operators

$$T_w : L^2(H/B(t), \chi_{\ell(t)}) \longrightarrow L^2(H/K(t, v), \chi_{\ell(w)}), \quad w = (t, v) \in \mathcal{W}$$

such that

$$T_w(\eta)(h) = \int_{K(t,v)/K(t,v) \cap B(t)} \eta(hb) \chi_{\ell(w)}(b) d_{\mathcal{Y}(w)}(b), \eta \in L^2(H/B(t), \chi_{\ell(t)}), h \in H$$

and where the measure $d_{\mathcal{Y}(w)}$ on $K(w)/(K(w) \cap B(t))$ is built using the Malcev basis $\mathcal{Y}(w)$.

Thus, for a fixed $t \in \mathcal{V}$, we obtain an isometric intertwining operator

$$\mathcal{T}^t : L^2(H/B(t), \chi_{\ell(t)}) \longrightarrow \int_{\mathcal{R}(t)}^{\oplus} L^2(H/K(t, v), \chi_{\ell(t,v)}) dv$$

characterized by the following properties.

- (1) It takes any $\eta \in L^2(H/B(t), \chi_{\ell(t)})$ to

$$\mathcal{T}^t(\eta) : \mathcal{R}(t) \rightarrow \bigcup_{\mathcal{R}(t)} L^2(H/K(t, v), \chi_{\ell(t,v)})$$

such that for $f_w \in \mathcal{R}(t)$ with $w = (t, v) \in \mathcal{W}$, $(\mathcal{T}^t \eta)(f_w) := T_w(\eta) \in L^2(H/K(w), \chi_{\ell(w)})$. Notice that the norm on this space is defined via the measure $d_{\mathcal{X}(w)}$ on $H/K(w)$.

(2) It intertwines the monomial representation

$$\sigma(t) = \text{ind}_{B(t)}^H \chi_{\ell(t)}$$

with its disintegration into irreducibles of H given by

$$\int_{\mathcal{R}(t)}^{\oplus} \rho_{(t,v)} dv = \int_{\mathcal{R}(t)}^{\oplus} \text{ind}_{K(t,v)}^H \chi_{\ell(t,v)} dv.$$

Combining these manipulations with Theorem 3.3 of [2] and with Proposition 3.27, we obtain Theorem 4.2 giving an isometric intertwining operator

$$\mathcal{T} : L^2(G/B, \chi_{\ell}) \longrightarrow \int_{\mathcal{V}}^{\oplus} \left(\int_{\mathcal{R}(t)}^{\oplus} L^2(H/K(t, v), \chi_{\ell(t, v)}) dv \right) |F(t)| dt$$

between $\pi_{\ell}|_H$ and its decomposition into irreducibles of H ,

$$(4.9) \quad \pi_{\ell, \mathfrak{b}|_H} \simeq \int_{H \setminus G/B}^{\oplus} \left(\int_{\mathcal{R}(x)}^{\oplus} \rho_{(x,v)} dv \right) d\gamma(x) \simeq \int_{\mathcal{W}}^{\oplus} \rho_w dw,$$

where the measure dw that we take on $\mathcal{W}$ is $|F(t)| dt \otimes dv$ on $\mathbb{R}^d \times \mathbb{R}^q$. Let's recall the operator S introduced in the proof of Proposition 3.27 whose expression

$$S(\xi) \in \int_{H \setminus G/B}^{\oplus} L^2(H/B(x), \chi_{\ell(x)}) d\gamma(x) :$$

$$H \setminus G/B \ni x \mapsto S\xi(x) \in L^2(H/B(x), \chi_{\ell(x)})$$

is given by (3.8) $\forall h \in H$, $S\xi(x)(h) = \xi(h\psi(x))$ for every $\xi \in \mathcal{S}(G/B, \chi_{\ell})$. So define

$$\mathcal{T}(\xi) : \mathcal{V} \ni t \mapsto \mathcal{T}\xi(t) = \mathcal{T}^t(S\xi(\tilde{\phi}(t))) \in \int_{\mathcal{R}(t)}^{\oplus} L^2(H/K(t, v), \chi_{\ell(t, v)}) dv,$$

$$\mathcal{R}(t) \ni f_{(t,v)} \mapsto T_{(t,v)}(S\xi(\tilde{\phi}(t))) \in L^2(H/K(t, v), \chi_{\ell(t, v)}).$$

Then, for every $f_w = \ell(t, v) \in \mathcal{R}(x)$ where $w = (t, v) \in \mathcal{W}$, we see that

$$\mathcal{T}\xi(w) := \mathcal{T}\xi(x, f_w) = T_w(S\xi(x)) \in L^2(H/K(w), \chi_{\ell(w)})$$

$$(4.10) \quad H \ni h \mapsto \int_{K(t,v)/(K(t,v) \cap B(t))} \xi(hb\phi(t)) \chi_{\ell(w)}(b) d_{\mathcal{Y}(w)}(b) \in \mathbb{C}.$$

Theorem 4.2. *Let $G = \exp \mathfrak{g}$ be a connected simply connected nilpotent Lie group and H a closed connected subgroup of G . Let $\mathcal{Z}$ be a Jordan-Hölder basis of $\mathfrak{g}$, ℓ in $\mathfrak{g}^*$ and $\mathfrak{b}$ the Vergne polarization at ℓ with respect to $\mathcal{Z}$. Also let π be an irreducible representation of G realized as $\pi_{\ell, \mathfrak{b}} = \text{ind}_B^G \chi_{\ell}$. Then the operator $\mathcal{T}$ defined on $\mathcal{S}(G/B, \chi_{\ell})$ as in Reference (4.10) uniquely extends to a unitary intertwining operator which decomposes the restriction of π to H as in Reference (4.9).*

Remark 4.3. Let's remark that for any $\xi \in \mathcal{S}(G/B, \chi_{\ell})$, one has

$$\|\xi\|_{L^2(G/B, \chi_{\ell})}^2 = \int_{H \setminus G/B} \left(\int_{\mathcal{R}(x)} \|T_{(x,v)} S\xi(x)\|_{L^2(H/K(x,v); \chi_{\ell(x,v)})}^2 dv \right) d\gamma(x)$$

$$= \int_{\mathcal{W}} \|\mathcal{T}\xi(w)\|_{L^2(H/K(w); \chi_{\ell(w)})}^2 dw.$$

5. DISINTEGRATING TENSOR PRODUCTS OF IRREDUCIBLE REPRESENTATIONS

Let $G = \exp \mathfrak{g}$ be a simply connected, connected Lie group and let $\pi = \pi_\ell, \pi' = \pi_{\ell'}$ be two irreducible unitary representations of G . We take a Jordan-Hölder basis $\mathcal{Z} = \{Z_1, \dots, Z_n\}$ of $\mathfrak{g}$ and we obtain the corresponding Vergne polarizations $\mathfrak{b}$ at ℓ and $\mathfrak{b}'$ at ℓ' . We then realize π as $\pi_{\ell, \mathfrak{b}}$ and π' as $\pi_{\ell', \mathfrak{b}'}$.

Let $\mathbb{G} := G \times G$ and $\mathbb{B} := B \times B'$ be the direct products respectively of G with G and B with B' . We denote by $\mathcal{G} := \mathfrak{g} \times \mathfrak{g}$ and $\mathcal{B} := \mathfrak{b} \times \mathfrak{b}'$ their respective Lie algebras. Also let $\Pi = \pi \times \pi'$ be the outer tensor product of π and π' . We can realize Π as an induced representation. It acts by left translation on the space

$$\mathcal{H}_\Pi := L^2(\mathbb{G}/\mathbb{B}, \chi_{(\ell, \ell')}) \simeq \mathcal{H}_\pi \hat{\otimes} \mathcal{H}_{\pi'},$$

where (ℓ, ℓ') is the linear functional defined on $\mathfrak{g} \times \mathfrak{g}$ by

$$(\ell, \ell')(Z, Z') := \ell(Z) + \ell'(Z'), \quad Z, Z' \in \mathfrak{g},$$

and so $\mathcal{B}$ is a polarization at (ℓ, ℓ') . Hence $\Pi = \pi_{(\ell, \ell'), \mathcal{B}}$.

Let $\mathbb{D} = \{(g, g); g \in G\}$ be the diagonal group of $\mathbb{G}$. This group is of course isomorphic to G and the tensor product $\pi \otimes \pi'$ is the representation of G acting on the space of Π through the rule

$$\pi \otimes \pi'(g) := \Pi(g, g), \quad g \in G.$$

We are led to investigate the structure of the space of double cosets $\mathbb{D} \backslash \mathbb{G} / \mathbb{B}$.

Proposition 5.1. *The mapping*

$$\mathbb{M} : B \backslash G / B' \longrightarrow \mathbb{D} \backslash \mathbb{G} / \mathbb{B}, \quad B \cdot g \cdot B' \mapsto \mathbb{D} \cdot (e, g) \cdot \mathbb{B},$$

is well defined and is a bijection.

Proof. Indeed, if $g \in G$ and $g' \in B \cdot g \cdot B'$, then we have that $g' = bgb'$ for some $b \in B$ and $b' \in B'$ and so

$$\begin{aligned} \mathbb{D} \cdot (e, g') \cdot \mathbb{B} &= \mathbb{D} \cdot (e, bgb') \cdot \mathbb{B} = \{(ub_2, ubgb'b'_2), u \in G, b_2 \in B, b'_2 \in B'\} \\ &= \{(ub^{-1}b_2, ugb'_2), u \in G, b_2 \in B, b'_2 \in B'\} \\ &= \{(ub_2, ugb'_2), u \in G, b_2 \in B, b'_2 \in B'\} \\ &= \mathbb{D} \cdot (e, g) \cdot \mathbb{B}. \end{aligned}$$

Hence our mapping is well defined. Since

$$\mathbb{D} \cdot (g, g') \cdot \mathbb{B} = \mathbb{D} \cdot (g, g) \cdot (e, g^{-1}g') \cdot \mathbb{B} = \mathbb{D} \cdot (e, g^{-1}g') \cdot \mathbb{B} = \mathbb{M}(B \cdot (g^{-1}g') \cdot B'),$$

it is also surjective. If $\mathbb{M}(B \cdot g \cdot B') = \mathbb{M}(B \cdot g' \cdot B')$ for two elements g, g' in G , then $(e, g') = (u, u)(e, g)(b, b')$ for some $u \in G, b \in B$ and $b' \in B'$. Hence, $ub = e$ and $ugb' = g'$, which means that $g' = b^{-1}gb'$ is an element of $B \cdot g \cdot B'$. Therefore $\mathbb{M}$ is injective and so it is a bijection. $\square$

According to the two preceding sections, we first need a Jordan-Hölder basis of $\mathfrak{g} \times \mathfrak{g}$ in order to perform our computations.

We make the following choice. Let

$$\mathcal{Z}_i := \begin{cases} (Z_j, Z_j) & \text{if } i = 2j - 1 \text{ is odd,} \\ (Z_j, -Z_j) & \text{if } i = 2j \text{ is even} \end{cases} \quad (j \in \{1, \dots, n\}).$$

The basis $\mathcal{A} = \{A_1, \dots, A_n\}$ of the Lie algebra $\mathfrak{d} = \{(Z, Z); Z \in \mathfrak{g}\}$ of the diagonal group $\mathbb{D}$ is then given by the vectors

$$A_j := (Z_j, Z_j) = \mathcal{Z}_{2j-1}, \quad j = 1, \dots, n.$$

Hence

$$(5.1) \quad \mathcal{I}^{\mathfrak{d}} = \{2j-1, j \in \{1, \dots, n\}\}, \quad \mathcal{I}^{\mathcal{G}/\mathfrak{d}} = \{2j, j \in \{1, \dots, n\}\}.$$

We must determine the index sets $\mathcal{I}(\mathcal{G}/\mathfrak{d}, \mathcal{G}/\mathcal{B}) \subset \mathcal{I}^{\mathcal{G}/\mathfrak{d}} \cap \mathcal{I}^{\mathcal{G}/\mathcal{B}}$. By (5.1) we have that

$$\mathcal{I}(\mathcal{G}/\mathfrak{d}, \mathcal{G}/\mathcal{B}) \subset \{2j, j \in \{1, \dots, n\}\}.$$

Proposition 5.2.

$$\mathcal{I}(\mathcal{G}/\mathfrak{d}, \mathcal{G}/\mathcal{B}) = \{2j, j \in \mathcal{I}(\mathfrak{g}/\mathfrak{b}, \mathfrak{g}/\mathfrak{b}')\}.$$

Proof. Let $\mathcal{U}$ be a Zariski open subset of G with property $(\mathcal{P})$, such that for every $j \in \mathcal{I}(\mathfrak{g}/\mathfrak{b}, \mathfrak{g}/\mathfrak{b}')$,

$$Z_j \notin \mathfrak{b} + \text{Ad}(g)\mathfrak{b}' + \mathfrak{g}_{j+1} \text{ for every } g \in \mathcal{U}.$$

Let

$$\mathbb{U} := \{(g, g') \in \mathbb{G}, g^{-1}g' \in \mathcal{U}\}.$$

Then $\mathbb{U}$ is a Zariski open subset of $\mathbb{G}$. Let us show that for every $j \in \mathcal{I}(\mathfrak{g}/\mathfrak{b}, \mathfrak{g}/\mathfrak{b}')$, we have

$$(5.2) \quad Z_{2j} = (Z_j, -Z_j) \notin \mathfrak{d} + \text{Ad}(g, g')(\mathcal{B}) + (\mathfrak{g} \times \mathfrak{g}')_{2j+1}, \forall (g, g') \in \mathbb{U}.$$

Then necessarily the index $2j$ is in $\mathcal{I}(\mathcal{G}/\mathfrak{d}, \mathcal{G}/\mathcal{B})$. Suppose that for some elements j and (g, g') respectively of $\mathcal{I}(\mathfrak{g}/\mathfrak{b}, \mathfrak{g}/\mathfrak{b}')$ and $\mathbb{U}$, the relation (5.2) is false. Then we have that

$$(Z_j, -Z_j) = (Y, Y) + (\text{Ad}(g)(X), \text{Ad}(g')(X')) + (U, U')$$

for certain vectors Y, X, X' in $\mathfrak{g}, \mathfrak{b}, \mathfrak{b}'$ respectively and U and $U' \in \mathfrak{g}_{j+1}$, since $(\mathfrak{g} \times \mathfrak{g}')_{2j+1} = \mathfrak{g}_{j+1} \times \mathfrak{g}_{j+1}$.

Hence

$$\begin{cases} Z_j &= Y + \text{Ad}(g)(X) + U, \\ -Z_j &= Y + \text{Ad}(g')(X') + U', \end{cases}$$

which gives us the identity

$$Z_j = \text{Ad}(g)\left(\frac{X}{2}\right) + \text{Ad}(g')\left(\frac{-X'}{2}\right) + \frac{U - U'}{2}.$$

This implies that

$$\begin{aligned} Z_j &\equiv \frac{X}{2} + \text{Ad}(g^{-1}g')\left(\frac{-X'}{2}\right) \pmod{\mathfrak{g}_{j+1}} \\ &\in \mathfrak{b} + \text{Ad}(g^{-1}g')(\mathfrak{b}') \pmod{\mathfrak{g}_{j+1}}, \end{aligned}$$

where $g^{-1}g' \in \mathcal{U}$ (by the definition of $\mathbb{U}$). This relation contradicts the fact that $j \in \mathcal{I}(\mathfrak{g}/\mathfrak{b}, \mathfrak{g}/\mathfrak{b}')$.

Now let $2j \in \mathcal{I}(\mathcal{G}/\mathfrak{d}, \mathcal{G}/\mathcal{B})$. Then for every (g, g') in a Zariski open subset of $\mathbb{G}$, we have that

$$(Z_j, -Z_j) \notin \mathfrak{d} + \text{Ad}(g)(\mathfrak{b}) \times \text{Ad}(g')(\mathfrak{b}') + \mathfrak{g}_{j+1} \times \mathfrak{g}_{j+1}.$$

Suppose now that $j \notin \mathcal{I}(\mathfrak{g}/\mathfrak{b}, \mathfrak{g}/\mathfrak{b}')$. Then in the Zariski open subset $\mathcal{U}$ of G , we have that

$$Z_j \in \mathfrak{b} + \text{Ad}(g)\mathfrak{b}' + \mathfrak{g}_{j+1}, \quad \forall g \in \mathcal{U}.$$

Let (g, g') be one element of the Zariski open subset $\mathbb{U}$ of $\mathbb{G}$. This means that $g'' := g^{-1}g' \in \mathcal{U}$. There exists $X \in \mathfrak{b}$ and $X' \in \mathfrak{b}'$, such that

$$Z_j = X + \text{Ad}(g'')(X') \mod \mathfrak{g}_{j+1}.$$

Hence

$$\text{Ad}(g)Z_j = \text{Ad}(g)X + \text{Ad}(g')X' \mod \mathfrak{g}_{j+1}$$

and Z_j can then be written as

$$Z_j = \text{Ad}(g)X + \text{Ad}(g')X' \mod \mathfrak{g}_{j+1}.$$

Let

$$Y = -\text{Ad}(g)X + \text{Ad}(g')X'.$$

It is easy to verify then that

$$\begin{aligned} (Z_j, -Z_j) &= (Y, Y) + (\text{Ad}(g)(2X), \text{Ad}(g')(-2X')) \mod \mathfrak{g}_{j+1} \times \mathfrak{g}_{j+1} \\ &\subset \mathfrak{d} + \text{Ad}(g, g')(\mathfrak{g} \times \mathfrak{g}') \mod \mathfrak{g}_{j+1} \times \mathfrak{g}_{j+1}. \end{aligned}$$

This is impossible since $(g, g') \in \mathbb{U}$. Hence $j \in \mathcal{I}(\mathfrak{g}/\mathfrak{b}, \mathfrak{g}/\mathfrak{b}')$. $\square$

Corollary 5.3. *Let $\mathcal{U}$ be the nonempty Zariski open subset of G defined in Remark 3.4 for the subgroups B and B' and let $\mathbb{U} = \{(g, g') \in \mathbb{G}; g^{-1}g' \in \mathcal{U}\}$. Let $\tilde{\mathcal{U}}$ be the projection of $\mathcal{U}$ into $B \backslash G / B'$ and $\tilde{\mathbb{U}}$ the projection of $\mathbb{U}$ into $\mathbb{D} \backslash \mathbb{G} / \mathbb{B}$. Then the mapping $\mathbb{M}$ of Proposition 5.1 maps $\tilde{\mathcal{U}}$ homeomorphically onto $\tilde{\mathbb{U}}$.*

Now let $\mathcal{U}$ be the Zariski open subset of G with property $(\mathcal{P})$ such that

$$j \in \mathcal{I}(\mathfrak{g}/\mathfrak{b}, \mathfrak{g}/\mathfrak{b}') \iff Z_j \notin \mathfrak{b} + \text{Ad}(g)\mathfrak{b}' + \mathfrak{g}_{j+1}, \quad \text{for all } g \in \mathcal{U}.$$

Let us write

$$\mathcal{I}(\mathfrak{g}/\mathfrak{b}, \mathfrak{g}/\mathfrak{b}') = \{i_1, \dots, i_d\}.$$

Then we have the mapping defined on $\mathbb{R}^d$ for the set of double cosets $B \backslash G / B'$ as we introduced in Notation 3.5,

$$\phi(t) = \exp(t_1 Z_{i_1}) \cdots \exp(t_d Z_{i_d}),$$

and the equivalent one corresponding to the set of double cosets $\mathbb{D} \backslash \mathbb{G} / \mathbb{B}$ which is $(\phi, \check{\phi})$. In fact,

$$\exp(t_1(Z_{i_1}, -Z_{i_1})) \cdots \exp(t_d(Z_{i_d}, -Z_{i_d})) = (\phi(t), \phi(-t)).$$

Then the linear functionals $(\ell, \ell')(t) = \text{Ad}^*((\phi, \check{\phi})(t))(\ell, \ell')|_{\mathfrak{d}}$ of $\mathfrak{d}$ that we need as in (4.2), are identified to those on $\mathfrak{g}$ given by

$$(\ell, \ell')(t) := \text{Ad}^*(\phi(t))\ell + \text{Ad}^*(\phi(-t))\ell' \in \text{Ad}^*(G)\ell + \text{Ad}^*(G)\ell', \quad t \in \mathbb{R}^d,$$

while for $Z \in \mathfrak{g}$, we have

$$\begin{aligned} \langle (\ell, \ell')(t), (Z, Z) \rangle &= \langle \text{Ad}^*(\phi(t), \phi(-t))(\ell, \ell')|_{\mathfrak{d}}, (Z, Z) \rangle \\ &= \langle \text{Ad}^*(\phi(t))\ell + \text{Ad}^*(\phi(-t))\ell', Z \rangle. \end{aligned}$$

We must determine now the index set I defined in (4.5) for the algebras $\mathfrak{b}$ and $\mathfrak{b}'$ and the elements $\ell(t, v)$ in $(\ell, \ell')(t) + \mathcal{B}(t)^\perp$ defined in (4.6).

Remarks 5.4.

- (1) Let us remark that the subalgebra $\mathcal{B}(t)$ of $\mathcal{G}$ defined as in Notation 3.13 can be identified to a subalgebra of $\mathfrak{g}$ denoted by $\mathfrak{k}(t)$, while

$$(5.3) \quad \mathcal{B}(t) = \mathfrak{d} \cap \text{Ad}(\phi(t))(\mathcal{B}) \simeq \text{Ad}(\phi(t))\mathfrak{b} \cap \text{Ad}(\phi(-t))\mathfrak{b}' =: \mathfrak{k}(t).$$

- (2) Since $\text{Ad}(\phi(t))\mathfrak{b}$ is a polarization at $\text{Ad}^*(\phi(t))\ell$ and similarly for $\text{Ad}(\phi(-t))\mathfrak{b}'$, we have that

$$\begin{aligned} (\ell, \ell')(t) + \mathfrak{k}(t)^\perp &= \text{Ad}^*(\phi(t))\ell + \text{Ad}^*(\phi(-t))\ell' + (\text{Ad}(\phi(t))\mathfrak{b} \cap \text{Ad}(\phi(-t))\mathfrak{b}')^\perp \\ &= \text{Ad}^*(\phi(t))\ell + \text{Ad}^*(\phi(-t))\ell' + \text{Ad}(\phi(t))\mathfrak{b}^\perp + \text{Ad}(\phi(-t))\mathfrak{b}'^\perp \\ &= \text{Ad}^*(\phi(t)B\phi(t)^{-1})\ell + \text{Ad}^*(\phi(-t)B'\phi(-t)^{-1})\ell' \\ &\subset \text{Ad}^*(G)\ell + \text{Ad}^*(G)\ell'. \end{aligned}$$

Hence, the affine subsets $\mathcal{R}(t)$ defined in (4.4) are all contained in the sum of the orbits $\Omega_\pi + \Omega_{\pi'}$ of ℓ and of ℓ' .

Notation 5.5. Let us also denote for $t \in \mathcal{V}$,

$$\mathfrak{p}(t) := \text{Ad}(\phi(t))\mathfrak{b}, \quad \mathfrak{p}'(t) := \text{Ad}(\phi(-t))\mathfrak{b}'.$$

Then $\mathfrak{p}(t)$ is a polarization at $\text{Ad}^*(\phi(t))\ell$ and $\mathfrak{p}'(t)$ at $\text{Ad}^*(\phi(-t))\ell'$. When we put $P(t) = \exp \mathfrak{p}(t)$ and $P'(t) = \exp \mathfrak{p}'(t)$ for $t \in \mathcal{V}$, then the subalgebra of their intersection $P(t) \cap P'(t)$ is not in general a polarization of $\mathfrak{g}$ at $(\ell, \ell')(t)$. That's why we have to disintegrate the unitary representation

$$\sigma(t) = \text{ind}_{P(t) \cap P'(t)}^G \chi_{(\ell, \ell')(t)} := \sigma(x) \quad (\text{where } x = \tilde{\phi}(t) \in B \backslash G / B')$$

into irreducibles of G . In fact, we view it as the unitary representation of $\mathbb{G}$ given by

$$\begin{aligned} \sigma(\tilde{x}) &:= \text{ind}_{P(t) \times P'(t)}^{G \times G} \chi_{(\ell, \ell')(t)} \\ &(\text{where } \tilde{x} = \mathbb{M}(x) \in \mathbb{D} \backslash \mathbb{G} / \mathbb{B} \text{ for } x = \tilde{\phi}(t) \in B \backslash G / B'). \end{aligned}$$

From Proposition 3.27, we deduce that

$$(5.4) \quad (\pi \times \pi')|_{\mathbb{D}} \simeq \int_{\mathbb{D} \backslash \mathbb{G} / \mathbb{B}}^\oplus \sigma(\tilde{x}) d\gamma(\tilde{x}).$$

Thus, we obtain the first writing of the disintegration of $\pi \otimes \pi'$:

$$\begin{aligned} (5.5) \quad \pi \otimes \pi' &\simeq \int_{B \backslash G / B'}^\oplus \sigma(x) d\gamma(x) \\ &\simeq \int_{\mathcal{V}}^\oplus \sigma(t) |F(t)| dt, \end{aligned}$$

where F is the function defined as in Definition 3.23 (for the subgroups B and B').

To obtain the final decomposition, we have just to repeat the process developed in the fourth section. So, we consider the Zariski open subset $\mathcal{W} \subset \mathbb{R}^d \times \mathbb{R}^q$

contained in $\bigcup_{t \in \mathcal{V}} \mathcal{R}(t)$ and we put for $w = (t, v) \in \mathcal{W}$ with $t \in \mathcal{V}, v \in \mathbb{R}^q$ and $x = \tilde{\phi}(t) \in B \backslash G / B'$,

- the linear functional $f_w = f_{(x,v)} \in \mathcal{R}(x) = \mathcal{R}(t)$ as in (4.6),

$$\begin{aligned} f_w &= \ell(t, v) = \ell(t) + \sum_{c \in I(t)} v_i Y_{j_c}(t)^* \\ &\in \Omega_\pi + \Omega_{\pi'} \quad (\text{by Remark 5.4.2}), \end{aligned}$$

- the unitary irreducible representation of G as in (4.7),

$$\rho_w = \text{ind}_{K(w)}^G \chi_{f_w} =: \rho_{(x,v)}$$

where $\mathfrak{k}(w) = \text{Lie}(K(w))$ is here the Vergne polarization of $\mathfrak{g}$ at f_w ,

- and we define the measure $d\omega$ on $\Omega_\pi + \Omega_{\pi'}$ as the image of $dw = |F(t)| dt \otimes dv$ on $\mathbb{R}^d \times \mathbb{R}^q$ through the mapping $\mathcal{W} \ni w \mapsto \omega := f_w \in \Omega_\pi + \Omega_{\pi'}$.

We can now apply Theorem 4.2 for the special case of Π and $\mathbb{D}$. We obtain an operator

$$\mathcal{T} : L^2(\mathbb{G}/\mathbb{B}, \chi_{(\ell, \ell')}) \longrightarrow \int_{\mathcal{W}}^{\oplus} L^2(G/K(w), \chi_{f_w}) dw$$

between the tensor product and its disintegration into irreducibles of G ,

$$\pi \otimes \pi' \simeq \int_{B \backslash G / B'}^{\oplus} \left(\int_{\mathcal{R}(x)}^{\oplus} \rho_{(x,v)} dv \right) d\gamma(x).$$

By considering the previous computations, this decomposition can be expressed differently; in fact

$$(5.6) \quad \pi \otimes \pi' \simeq \int_{\Omega_\pi + \Omega_{\pi'}}^{\oplus} \rho(\omega) d\omega \simeq \int_{\mathcal{W}}^{\oplus} \rho_w dw$$

and the operator $\mathcal{T}$ is at first defined on $\mathcal{S}(\mathbb{G}/\mathbb{B}, \chi_{(\ell, \ell')})$ as follows. For every $\xi \in \mathcal{S}(\mathbb{G}/\mathbb{B}, \chi_{(\ell, \ell')})$, we put $\mathcal{T}(\xi) : \mathcal{W} \ni w \mapsto \mathcal{T}\xi(w) \in L^2(G/K(w), \chi_{f_w})$, such that for any $w = (t, v) \in \mathcal{W}$,

$$(5.7) \quad \mathcal{T}\xi(w) : G \ni g \longmapsto \int_{K(w)/(K(w) \cap P(t) \cap P'(t))} \xi(gb\phi(t), gb\phi(-t)) \chi_{f_w}(b) d_{\mathcal{Y}(w)}(b) \in \mathbb{C}$$

where $\mathcal{Y}(w)$ is as in (4.8). Then $\mathcal{T}$ extends to a unitary intertwining operator and one has

$$\int_{\mathcal{W}} \|\mathcal{T}\xi(w)\|_{L^2(G/K(w), \chi_{f_w})}^2 dw = \|\xi\|_{L^2(\mathbb{G}/\mathbb{B}, \chi_{(\ell, \ell')})}^2.$$

Theorem 5.6. *Let $G = \exp \mathfrak{g}$ be a connected simply connected nilpotent Lie group and $\mathcal{Z}$ a Jordan-Hölder basis of $\mathfrak{g}$. Let $\ell \in \mathfrak{g}^*$ (respectively $\ell' \in \mathfrak{g}^*$) and $\mathfrak{b}$ (respectively $\mathfrak{b}'$) be the Vergne polarization at ℓ (respectively at ℓ'), with respect to $\mathcal{Z}$. We realize $\pi \in \hat{G}$ as $\pi_{\ell, \mathfrak{b}} = \text{ind}_B^G \chi_\ell$ where $B = \exp(\mathfrak{b})$, and similarly for $\pi' \in \hat{G}$. Then their tensor product is a direct integral over the sum of orbits $\Omega_\pi + \Omega_{\pi'}$ (with respect to the measure $d\omega$). Finally, an explicit unitary intertwining operator $\mathcal{T}$ which decomposes $\pi \otimes \pi'$ as in Reference (5.6) is provided by the formula (5.7).*

6. AN EXAMPLE

We show in the following example that the choice of a polarization for a given linear functional has a strong influence on the concrete form of the disintegration of the representation $\pi \otimes \pi'$.

Let $\mathfrak{g}$ be the Lie algebra spanned by the vectors $\langle Z_1, \dots, Z_7 \rangle$ and equipped with the nontrivial brackets

$$\begin{aligned} [Z_1, Z_4] &= Z_7, & [Z_2, Z_4] &= Z_5, & [Z_2, Z_6] &= Z_7, \\ [Z_3, Z_4] &= Z_6, & [Z_3, Z_5] &= Z_7. \end{aligned}$$

Then $\mathcal{Z} = \{Z_1, \dots, Z_7\}$ is a Jordan-Hölder basis of $\mathfrak{g}$ and one can easily verify that the coadjoint orbit $G \cdot \ell$ of any $\ell \in \mathfrak{g}^*$ with $\ell(Z_7) \neq 0$ is equal to $\ell + Z_7^\perp$. Let $\mathcal{Z}^* = \{Z_1^*, \dots, Z_7^*\}$ be the dual basis of $\mathcal{Z}$.

Now choose $\ell = Z_7^* + Z_6^* \in \mathfrak{g}^*$. Its stabilizer in $\mathfrak{g}$ is reduced to the center $\mathbb{R}Z_7$ of $\mathfrak{g}$. There are several possibilities for polarizations at ℓ . We take 4 such polarizations:

$$\begin{aligned} \mathfrak{b}_1 &:= \text{span}\langle Z_7, Z_6, Z_5, Z_4 \rangle, \\ \mathfrak{b}_2 &:= \text{span}\langle Z_7, Z_3, Z_2, Z_1 \rangle, \\ \mathfrak{b}_3 &:= \text{span}\langle Z_7, Z_6, Z_4 - Z_5, Z_3 \rangle, \\ \mathfrak{b}_4 &:= \text{span}\langle Z_7, Z_6, Z_4, Z_1 \rangle. \end{aligned}$$

Then $\mathfrak{b}_1$ is the Vergne polarization associated to the Jordan-Hölder basis $\mathcal{Z}$. It is also an abelian ideal of $\mathfrak{g}$. $\mathfrak{b}_2$ and $\mathfrak{b}_4$ are also abelian subalgebras but not $\mathfrak{b}_3$.

Now let

$$\pi_j := \text{ind}_{B_j}^G \chi_\ell, \quad j = 1, 2, 3, 4.$$

6.1. Disintegration of $\pi_1 \otimes \pi_2$. We have that $\mathfrak{b}_1$ is an ideal of $\mathfrak{g}$ and $\mathfrak{b}_1 + \mathfrak{b}_2 = \mathfrak{g}$. Hence $\mathcal{I}(\mathfrak{g}/\mathfrak{b}_1, \mathfrak{g}/\mathfrak{b}_2) = \emptyset$. Hence by (5.5), we have that

$$\pi_1 \otimes \pi_2 \simeq \sigma(0) = \text{ind}_{B_1 \cap B_2}^G \chi_{2\ell}.$$

We must determine the index set I of (4.3). We have now that $2\ell + (\mathfrak{b}_1 \cap \mathfrak{b}_2)^\perp = 2\ell + Z_7^\perp$ and the family $\{Z_1, \dots, Z_6\}$ is a Malcev basis of $\mathfrak{g}$ relative to $\mathfrak{b}_1 \cap \mathfrak{b}_2$. We must determine the indices set I . We have for a linear form $f = 2Z_7^* + \sum_{j=2}^6 \beta_j Z_j^* \in 2\ell + Z_7^\perp$ that

$$\begin{array}{llll} \mathfrak{g}_6(f_6) & = & \mathfrak{g}_6 & \not\subset \mathfrak{g}_7 \implies 6 \in I, \\ \mathfrak{g}_5(f_5) & = & \mathfrak{g}_5 & \not\subset \mathfrak{g}_6 \implies 5 \in I, \\ \mathfrak{g}_4(f_4) & = & \mathfrak{g}_4 & \not\subset \mathfrak{g}_5 \implies 4 \in I, \\ \mathfrak{g}_3(f_3) & = & \text{span}\langle Z_7, Z_6, Z_4 - \frac{\beta_6}{2} Z_5 \rangle & \subset \mathfrak{g}_4 \implies 3 \notin I, \\ \mathfrak{g}_2(f_2) & = & \text{span}\langle Z_7, Z_4 - \frac{\beta_6}{2} Z_5 + \frac{\beta_6}{2} Z_6 \rangle & \subset \mathfrak{g}_3 \implies 2 \notin I, \\ \mathfrak{g}(f) & = & \mathfrak{g}_7 & \subset \mathfrak{g}_2 \implies 1 \notin I, \end{array}$$

where $f_j := f|_{\mathfrak{g}_j}$. This implies that $I = \{k_1 = 4, k_2 = 5, k_3 = 6\}$ and the space $\mathcal{R}$ of disintegration of $\pi_1 \otimes \pi_2$ is the space of $\sigma(0)$ given by $\mathcal{R}(0) = 2Z_7^* + \mathbb{R}Z_6^* + \mathbb{R}Z_5^* + \mathbb{R}Z_4^*$. We can take for every $w = (v_4, v_5, v_6) \in \mathcal{W} = \mathbb{R}^3$ the Vergne polarization $\mathfrak{b} = \mathfrak{b}_1$ associated to the Jordan-Hölder basis $\mathcal{Z}$ at $f_w \in \mathcal{R}$. The mapping $T_w : L^2(\mathbb{G}/B_1 \times B_2, \chi_{\ell, \ell})$ of Theorem 5.6 is then given for $g \in G$ by

$$\begin{aligned} & T_w(\xi)(g) \\ &= \int_{\mathbb{R}^3} \xi(g(u_4 Z_4 \cdot u_5 Z_5 \cdot u_6 Z_6), g(u_4 Z_4 \cdot u_5 Z_5 \cdot u_6 Z_6)) e^{-2\pi i(v_4 u_4 + v_5 u_5 + (v_6 + 1)u_6 + 2Z_7^*)} du, \end{aligned}$$

and

$$\mathcal{H}_{\pi_1} \hat{\otimes} \mathcal{H}_{\pi_2} \simeq \int_{\mathbb{R}^3}^{\oplus} L^2(G/B, \chi_{v_4 Z_4^* + v_5 Z_5^* + (v_6+1)Z_6^* + 2Z_7^*}) du_4 du_5 du_6.$$

6.2. Disintegration of $\pi_3 \otimes \pi_4$. We have here that

$$\mathcal{I}^{\mathfrak{g}/\mathfrak{b}_3} = \{1, 2, 5\}, \quad \mathcal{I}^{\mathfrak{g}/\mathfrak{b}_4} = \{2, 3, 5\}$$

and so $I^{\mathfrak{g}/\mathfrak{b}_3} \cap I^{\mathfrak{g}/\mathfrak{b}_4} = \{2, 5\}$.

Since for every $g = \prod_{j=1}^7 (g_j Z_j)$, $g_2 \neq 0$,

$$\begin{aligned} Z_5 &= Z_5 - Z_4 + \text{Ad}((g_1 Z_1)(g_2 Z_2)(g_3 Z_3)(g_4 Z_4))\left(\frac{1}{g_2}(-Z_4)\right) \\ &\quad \text{mod } \mathfrak{g}_6 \subset \mathfrak{b}_3 + \text{Ad}(g)(\mathfrak{b}_4) \quad \text{mod } \mathfrak{g}_6, \end{aligned}$$

it follows that $5 \notin \mathcal{I}(\mathfrak{g}/\mathfrak{b}_3, \mathfrak{g}/\mathfrak{b}_4)$. Hence, since $[\mathfrak{g}, \mathfrak{g}] = \text{span}\langle Z_5, Z_6, Z_7 \rangle$, we have that

$$\mathcal{I}(\mathfrak{g}/\mathfrak{b}_3, \mathfrak{g}/\mathfrak{b}_4) = \{2\}.$$

We must compute the function F of Reference (3.23). Let $\Phi : \mathbb{R}^2 \times \mathbb{R} \longrightarrow G/B_4$ be the mapping such that

$$\Phi(s_1, s_2, t) = (s_1 Z_1 \cdot s_2 (Z_4 - Z_5) \cdot t Z_2) \cdot B_4 = (s_1 Z_1 \cdot t Z_2 \cdot (s_2 t + 1) Z_5) \cdot B_4.$$

Therefore

$$\Phi(s_1, s_2, t) \equiv (s_1, t, 1 + s_2 t)$$

and so

$$F(t) = t.$$

We deduce from (5.5) the first step of the disintegration

$$\pi_3 \otimes \pi_4 \simeq \int_{\mathbb{R}}^{\oplus} \sigma(t) |t| dt,$$

where $\sigma(t) = \sigma(B_3 \cdot \exp(t Z_2) \cdot B_4) = \text{ind}_{P(t) \cap P'(t)}^G \chi_{(\ell, \ell')(t)}$ has $\mathcal{R}(t) = 2Z_7^* + \mathbb{R}Z_5^* + \mathbb{R}Z_4^*$ as the space of disintegration for every $t \in \mathbb{R}$. In fact, we have that

- $\text{Ad}^*(t Z_2)\ell = \ell - t Z_6^* \Rightarrow \ell(t) = \text{Ad}^*(t Z_2)\ell + \text{Ad}^*(-t Z_2)\ell = 2\ell, t \in \mathbb{R}$,
- the Lie algebra $(\mathfrak{p}(t) \cap \mathfrak{p}'(t))$ of Reference (5.3) is given for $t \in \mathbb{R}$ by

$$\text{Ad}(t Z_2)\mathfrak{b}_3 \cap \text{Ad}(-t Z_2)\mathfrak{b}_4 = \text{span}\{Z_6, Z_7\}.$$

Hence the space of disintegration is this time $\mathcal{W} = \mathbb{R} \times \mathbb{R}^2$ and for $w = (t, (v_4, v_5)) \in \mathcal{W}$, the mapping T_w is defined by

$$T_w \xi(g) = \int_{\mathbb{R}^2} \xi(g(u_4 Z_4 \cdot u_5 Z_5 \cdot t Z_2), g(u_4 Z_4 \cdot u_5 Z_5 \cdot (-t Z_2))) e^{-2\pi i(u_4 v_4 + u_5 v_5)} du_4 du_5$$

and

$$\mathcal{H}_{\pi_3} \hat{\otimes} \mathcal{H}_{\pi_4} \simeq \int_{\mathbb{R}}^{\oplus} \int_{\mathbb{R}^2}^{\oplus} L^2(G/B, \chi_{v_4 Z_4^* + v_5 Z_5^* + 2Z_7^*}) |t| dv_4 dv_5 dt.$$

7. CRITERIA FOR IRREDUCIBILITY

In this section we shall determine a necessary and sufficient condition for the irreducibility of the tensor representation $\pi \otimes \pi'$. We shall prove the following theorem.

Theorem 7.1. *Let $G = \exp \mathfrak{g}$ be a simply connected nilpotent Lie group, $\pi = \pi_\ell$ and $\pi' = \pi_{\ell'}$ two irreducible unitary representations of G . The following assertions are equivalent.*

- (1) $\pi \otimes \pi' \in \hat{G}$.
- (2) For any polarizations $\mathfrak{b}$ at ℓ and $\mathfrak{b}'$ at ℓ' , we have that

$$\begin{cases} \mathfrak{b} + \mathfrak{b}' = \mathfrak{g} & \text{and} \\ \mathfrak{b} \cap \mathfrak{b}' \text{ is a polarization at } \ell + \ell'. \end{cases}$$

- (3) There exist some polarizations $\mathfrak{b}$ at ℓ and $\mathfrak{b}'$ at ℓ' such that

$$\begin{cases} \mathfrak{b} + \mathfrak{b}' = \mathfrak{g} & \text{and} \\ \mathfrak{b} \cap \mathfrak{b}' \text{ is a polarization at } \ell + \ell', \end{cases}$$

- (4)

$$\text{Ad}^*(G)\ell + \text{Ad}^*(G)\ell' = \text{Ad}^*(G)(\ell + \ell')$$

and

$$\mathfrak{g}(\ell) + \mathfrak{g}(\ell') = \mathfrak{g}.$$

Proof. 1) $\implies$ 2)

By (5.4), the space of double cosets $\mathbb{D} \backslash \mathbb{G} / \mathbb{B}$ must be reduced to one point. This is then the case of $B \backslash G / B'$ while they are in bijection as we mentioned in Proposition 5.1. This implies by Lemma 3.9 that $\mathcal{I}(\mathfrak{g}/\mathfrak{b}, \mathfrak{g}/\mathfrak{b}') = \emptyset$ and $G = BB'$. In particular this tells us that $L^2(\mathbb{G}/\mathbb{B}, \chi_{(\ell, \ell')})$ is isometrically isomorphic to $L^2(G/B \cap B', \chi_{\ell + \ell'})$. In fact, we have constructed in the proof of Proposition 3.27 the isometric operator S which, in our case here, maps $L^2(\mathbb{G}/\mathbb{B}, \chi_{(\ell, \ell')})$ to $L^2(\Delta G/B \times B', \chi_{(\ell, \ell')})$, but from Remarks 5.4(1), we see that this last space is isometrically isomorphic to $L^2(G/B \cap B', \chi_{\ell + \ell'})$, and so $\pi \otimes \pi' \simeq \text{ind}_{B \cap B'}^G \chi_{\ell + \ell'}$. Hence the representation $\text{ind}_{B \cap B'}^G \chi_{\ell + \ell'}$ is irreducible, which forces $\mathfrak{b} \cap \mathfrak{b}'$ to be a polarization at $\ell + \ell'$ by Kirillov's theorem.

2) $\implies$ 3) is trivial.

3) $\implies$ 4)

Let us first show that $\text{Ad}^*(G)\ell + \text{Ad}^*(G)\ell' = \text{Ad}^*(G)(\ell + \ell')$. We do this by an induction on $\dim(\mathfrak{g})$. If this dimension is 1, the implication is evident, since we work then with characters. If ℓ' is a character, then we always have that $\text{Ad}^*(G)\ell + \text{Ad}^*(G)\ell' = \text{Ad}^*(G)\ell + \ell' = \text{Ad}^*(G)(\ell + \ell')$. We can thus assume that $\mathfrak{b}'$ is a proper subalgebra of $\mathfrak{g}$. Let $\mathfrak{h}$ be again an ideal of codimension 1 in $\mathfrak{g}$ containing $\mathfrak{b}'$. Then $\mathfrak{b} \not\subset \mathfrak{h}$ by the condition $\mathfrak{b} + \mathfrak{b}' = \mathfrak{g}$. Hence $\mathfrak{b} \cap \mathfrak{h}$ is a polarization at $\ell|_{\mathfrak{h}}$ and $\mathfrak{b}'$ is a polarization at $\ell'|_{\mathfrak{h}}$. Furthermore, we can also see easily that their sum $\mathfrak{b} \cap \mathfrak{h} + \mathfrak{b}'$ is $\mathfrak{h}$ entire, and that their intersection $(\mathfrak{b} \cap \mathfrak{h}) \cap \mathfrak{b}' = \mathfrak{b} \cap \mathfrak{b}'$ is a polarization of $\mathfrak{h}$ at $\ell|_{\mathfrak{h}} + \ell'|_{\mathfrak{h}}$, but also a polarization of $\mathfrak{g}$ at $\ell + \ell'$ too, which means that the G -orbit $\text{Ad}^*(G)(\ell + \ell')$ is saturated with respect to $\mathfrak{h}$. It follows from the induction hypothesis that

$$\text{Ad}^*(H)\ell|_{\mathfrak{h}} + \text{Ad}^*(H)\ell'|_{\mathfrak{h}} = \text{Ad}^*(H)(\ell|_{\mathfrak{h}} + \ell'|_{\mathfrak{h}}).$$

Therefore

$$\mathrm{Ad}^*(H)\ell + \mathrm{Ad}^*(H)\ell' + \mathfrak{h}^\perp = \mathrm{Ad}^*(H)(\ell + \ell') + \mathfrak{h}^\perp.$$

Now, since the G -orbit $\mathrm{Ad}^*(G)(\ell)$ is nonsaturated with respect to $\mathfrak{h}$ contrary to $\mathrm{Ad}^*(G)(\ell')$ and $\mathrm{Ad}^*(G)(\ell + \ell')$, they replace respectively $\mathrm{Ad}^*(H)\ell$, $\mathrm{Ad}^*(H)\ell' + \mathfrak{h}^\perp$ and $\mathrm{Ad}^*(H)(\ell + \ell') + \mathfrak{h}^\perp$. Finally we obtain

$$\mathrm{Ad}^*(G)\ell + \mathrm{Ad}^*(G)\ell' = \mathrm{Ad}^*(G)(\ell + \ell').$$

Let us show now that $\mathfrak{g}(\ell) + \mathfrak{g}(\ell') = \mathfrak{g}$.

We have for any polarization $\mathfrak{b}$ at ℓ that

$$\dim(\mathfrak{b}) = \frac{\dim(\mathfrak{g}) + \dim(\mathfrak{g}(\ell))}{2}.$$

Hence, since $\mathfrak{b} + \mathfrak{b}' = \mathfrak{g}$ and $\mathfrak{b} \cap \mathfrak{b}'$ is a polarization at $\ell + \ell'$ and since

$$\dim(\mathfrak{b}) + \dim(\mathfrak{b}') = \dim(\mathfrak{g}) + \dim(\mathfrak{b} \cap \mathfrak{b}'),$$

we have that

$$\frac{2\dim(\mathfrak{g}) + \dim(\mathfrak{g}(\ell)) + \dim(\mathfrak{g}(\ell'))}{2} = \dim(\mathfrak{g}) + \frac{\dim(\mathfrak{g}) + \dim(\mathfrak{g}(\ell + \ell'))}{2}.$$

Therefore, since $\mathfrak{g}(\ell) \cap \mathfrak{g}(\ell') \subset \mathfrak{g}(\ell + \ell')$,

$$\begin{aligned} \dim(\mathfrak{g}(\ell) + \mathfrak{g}(\ell')) + \dim(\mathfrak{g}(\ell + \ell')) &\geq \dim(\mathfrak{g}(\ell) + \mathfrak{g}(\ell')) + \dim(\mathfrak{g}(\ell) \cap \mathfrak{g}(\ell')) \\ &= \dim(\mathfrak{g}(\ell)) + \dim(\mathfrak{g}(\ell')) \\ &= \dim(\mathfrak{g}(\ell + \ell')) + \dim(\mathfrak{g}), \end{aligned}$$

and so $\dim(\mathfrak{g}(\ell) + \mathfrak{g}(\ell')) \geq \dim(\mathfrak{g})$, which implies that $\mathfrak{g} = \mathfrak{g}(\ell) + \mathfrak{g}(\ell')$.

4) $\implies$ 1)

Let $\mathfrak{b}$ (resp. $\mathfrak{b}'$) be a polarization at ℓ (resp. at ℓ'). Then $\mathfrak{g}(\ell) \subset \mathfrak{b}$, $\mathfrak{g}(\ell') \subset \mathfrak{b}'$ and hence $\mathfrak{b} + \mathfrak{b}' = \mathfrak{g}$. Therefore BB' contains an open subset of G , whence it is itself open. Since the product of two closed connected subgroups of G is always closed, it follows that BB' is open and closed, which implies that $BB' = G$. Therefore $\pi_{\ell, \mathfrak{b}} \otimes \pi_{\ell', \mathfrak{b}'} \simeq \mathrm{ind}_{B \cap B'}^G \chi_{\ell + \ell'}$. If we can show that $\mathfrak{b} \cap \mathfrak{b}'$ is a polarization at $\ell + \ell'$, then we know that $\pi \otimes \pi'$ is irreducible. The condition $\mathrm{Ad}^*(G)l + \mathrm{Ad}^*(G)l' = \mathrm{Ad}^*(G)(l + l')$ implies that the sum of the tangent spaces $\mathrm{ad}^*(\mathfrak{g})l$ and $\mathrm{ad}^*(\mathfrak{g})l'$ of the two orbits $\mathrm{Ad}^*(G)l$ and $\mathrm{Ad}^*(G)l'$ is contained in the tangent space $\mathrm{ad}^*(\mathfrak{g})(l + l')$ of the orbit $\mathrm{Ad}^*(G)(l + l')$. Furthermore if for some $X \in \mathfrak{g}(l)$ and some $X' \in \mathfrak{g}(l')$ we have that $\mathrm{ad}^*(X)l' = \mathrm{ad}^*(-X')l = 0$, then $\mathrm{ad}^*(X')l \in \mathfrak{g}(l)^\perp \cap \mathfrak{g}(l')^\perp = (\mathfrak{g}(l) + \mathfrak{g}(l'))^\perp = \mathfrak{g}^\perp = \{0\}$. Hence the mapping

(7.1)

$$\mathrm{ad}^*(\mathfrak{g})l' \times \mathrm{ad}^*(\mathfrak{g})l \rightarrow \mathrm{ad}^*(\mathfrak{g})(l + l'), (\mathrm{ad}^*(X')l, \mathrm{ad}^*(X)l') \mapsto \mathrm{ad}^*(X')l + \mathrm{ad}^*(X)l'$$

is injective. Therefore

$$\dim(\mathfrak{g}/\mathfrak{g}(l)) + \dim(\mathfrak{g}/\mathfrak{g}(l')) \leq \dim(\mathfrak{g}/\mathfrak{g}(l + l')).$$

Hence

$$\begin{aligned} 2\dim(\mathfrak{g}) - \dim(\mathfrak{g}(l)) - \dim(\mathfrak{g}(l')) &\leq \dim(\mathfrak{g}) - \dim(\mathfrak{g}(l + l')) \\ \implies \\ \dim(\mathfrak{g}(l) + \mathfrak{g}(l')) - \dim(\mathfrak{g}(l)) - \dim(\mathfrak{g}(l')) &\leq -\dim(\mathfrak{g}) - \dim(\mathfrak{g}(l + l')) \\ \implies \\ \dim(\mathfrak{g}(l + l')) &\leq \dim(\mathfrak{g}(l) \cap \mathfrak{g}(l')). \end{aligned}$$

Therefore $\mathfrak{g}(\ell) \cap \mathfrak{g}(\ell') = \mathfrak{g}(\ell + \ell')$, since $\mathfrak{g}(\ell) \cap \mathfrak{g}(\ell') \subset \mathfrak{g}(\ell + \ell')$.
This tells us then that

$$\begin{aligned}
 \dim(\mathfrak{b} \cap \mathfrak{b}') &= \dim(\mathfrak{b}) + \dim(\mathfrak{b}') - \dim(\mathfrak{b} + \mathfrak{b}') \\
 &= \dim(\mathfrak{b}) + \dim(\mathfrak{b}') - \dim(\mathfrak{g}) \\
 &= \frac{2\dim(\mathfrak{g}) + \dim(\mathfrak{g}(\ell)) + \dim(\mathfrak{g}(\ell'))}{2} - \dim(\mathfrak{g}) \\
 &= \frac{\dim(\mathfrak{g}(\ell)) + \dim(\mathfrak{g}(\ell'))}{2} \\
 &= \frac{\dim(\mathfrak{g}) + \dim(\mathfrak{g}(\ell) \cap \mathfrak{g}(\ell'))}{2} \\
 &= \frac{\dim(\mathfrak{g}) + \dim(\mathfrak{g}(\ell + \ell'))}{2}.
 \end{aligned}$$

Hence $\mathfrak{b} \cap \mathfrak{b}'$ is a polarization at $\ell + \ell'$ while $\langle \ell + \ell', [\mathfrak{b} \cap \mathfrak{b}', \mathfrak{b} \cap \mathfrak{b}'] \rangle = 0$. $\square$

Example 7.2. Let $\mathfrak{g}$ be the Lie algebra spanned by the basis $\langle Z_1, \dots, Z_6 \rangle$ and equipped with the brackets

$$[Z_2, Z_3] = Z_5, \quad [Z_1, Z_4] = Z_6.$$

Let $\ell = Z_5^*, \ell' = Z_6^* \in \mathfrak{g}^*$. Take the polarizations

$$\mathfrak{b} := \text{span}\langle Z_6, Z_5, Z_4, Z_3, Z_1 \rangle$$

at ℓ and

$$\mathfrak{b}' := \text{span}\langle Z_6, Z_5, Z_4, Z_3, Z_2 \rangle$$

at ℓ' .

Then $\mathfrak{b} + \mathfrak{b}' = \mathfrak{g}$ and

$$\mathfrak{b} \cap \mathfrak{b}' = \text{span}\langle Z_6, Z_5, Z_4, Z_3 \rangle,$$

which is a polarization at $\ell + \ell'$. Hence $\pi_\ell \otimes \pi_{\ell'} \in \hat{G}$.

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