***F*-STABILITY IN FINITE GROUPS**

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ABSTRACT. Let G be a finite group, $S \in \text{Syl}_p(G)$, and $\mathcal{S}$ be the set subgroups containing S . For $M \in \mathcal{S}$ and $V = \Omega_1 Z(O_p(M))$, the paper discusses the action of M on V . Apart from other results, it is shown that for groups of parabolic characteristic p either S is contained in a unique maximal p -local subgroup, or there exists a maximal p -local subgroup in $M \in \mathcal{S}$ such that V is a nearly quadratic 2F-module for M .

1. INTRODUCTION

Let G be a finite group and p a prime. A subgroup P containing a Sylow p -subgroup of G is a p -parabolic subgroup of G , and P is a *local p -parabolic* subgroup if in addition $O_p(P) \neq 1$.

Moreover, G has *characteristic p* if $C_G(O_p(G)) \leq O_p(G)$, and G has *parabolic characteristic p* if every local p -parabolic subgroup has characteristic p .

The standard examples for groups of parabolic characteristic p are the finite simple groups of Lie type in characteristic p . In these examples every proper parabolic subgroup is a local p -parabolic subgroup, and for maximal parabolic subgroups M the normal subgroup $\Omega_1 Z(O_p(M))$, considered as a $GF(p)M$ -module, has a remarkably restricted structure. In this paper we try to understand this phenomena in arbitrary finite groups.

What kind of properties of the module $\Omega_1 Z(O_p(M))$ should one aim at in general? A possible answer arose during our detailed study of the p -local structure of groups of local characteristic p in [MSS], where a group has local characteristic p if each of its p -local subgroups has characteristic p .

Definition 1.1. Let A be an elementary abelian p -group and V a finite dimensional $GF(p)A$ -module. Then A is

- (a) *quadratic* on V if $[V, A, A] = 0$,
- (b) *nearly quadratic* on V if $[V, A, A, A] = 0$ and

$$[V, A] + C_V(A) = [v, A] + C_V(A) \text{ for every } v \in V \setminus [V, A] + C_V(A),$$
- (c) an *offender* on V if $|V/C_V(A)| \leq |A/C_A(V)|$,
- (d) a *2F-offender* on V if $|V/C_V(A)| \leq |A/C_A(V)|^2$,
- (e) *non-trivial* on V if $[V, A] \neq 0$.

A p -subgroup Y of G is called *p -reduced* (for G) if Y is elementary abelian and normal in G , and $O_p(G/C_G(Y)) = 1$. The largest p -reduced subgroup of G is denoted by Y_G ; for the existence of Y_G see Lemma 2.2(a).

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Let M be a subgroup of G . Then M is F -stable (in G) if none of the elementary abelian p -subgroups of $N_G(Y_M)/C_G(Y_M)$ are non-trivial offenders on Y_M . Similarly, M is $2F$ -stable (in G) if none of the elementary abelian p -subgroups of $N_G(Y_M)/C_G(Y_M)$ are non-trivial nearly quadratic $2F$ -offenders on Y_M .

Modules admitting non-trivial $2F$ -offenders have been investigated by Guralnick, Lawther and Malle in [GLM], [GM1], [GM2], and [L]. They have classified all pairs (V, G) , where V is a simple $GF(p)G$ -module and G is a known finite almost quasisimple group containing a non-trivial $2F$ -offender on V .

Their result is a major generalization of earlier results, where G was assumed to contain a non-trivial offender.

For stating our results we need some further definitions.

Definition 1.2. By $\mathcal{S}(X)$ we denote the set of subgroups of G containing X . Let S be Sylow p -subgroup of G . Then:

$$B(S) := C_S(\Omega_1 ZJ(S)),$$

$$C^*(G, S) := \langle C_G(\Omega_1 Z(S)), N_G(C) | 1 \neq C \text{ char } B(S) \rangle,$$

and

$$C^{**}(G, S) = \langle N_G(J(S)), C_G(\Omega_1 Z(S)) \rangle.$$

A *factorization family* for $\mathcal{S}(S)$ is a subset $\mathcal{F}(S) \subseteq \mathcal{S}(S)$ with the following two properties:

- (i) For every $H \in \mathcal{S}(S)$ there exists $M \in \mathcal{F}(S)$ with $H \subseteq C_G(Y_H)M$ and $Y_H \leq Y_M$.
- (ii) If $H \in \mathcal{S}(S)$ and $M \in \mathcal{F}(S)$ with $M \subseteq C_G(Y_M)H$ and $Y_M \leq Y_H$, then $Y_M = Y_H$ and $H \leq M$.

Property (i) implies

$$H/C_H(Y_H) \cong HC_G(Y_H)/C_G(Y_H) \cong (HC_G(Y_H) \cap M)C_G(Y_H)/C_G(Y_H),$$

so the action of H on Y_H is isomorphic to the action of $HC_G(Y_H) \cap M$ on the submodule Y_H of Y_M . In particular, it suffices to identify $M/C_M(Y_M)$ and its action on Y_M to identify $H/C_H(Y_H)$ and Y_H .

Property (ii) is the crucial one for applications since it has strong consequences. For example, if G is of parabolic characteristic p and $S \leq H \leq M \in \mathcal{F}(S)$ such that $M = HC_M(Y_M)$, then M is the unique maximal p -local subgroup of G containing H (see Lemma 3.5).

Of course, it is not clear a priori that factorization families exist. The existence (and uniqueness) will be established in Theorem 3.4.

Theorem 1.3. *Let G be a finite group and $S \in \text{Syl}_p(G)$. There exists a unique factorization family $\mathcal{F}(S)$ for $\mathcal{S}(S)$ in G . Moreover, at most one member of $\mathcal{F}(S)$ is F -stable, and*

$$\Omega_1 Z(S) \leq Y_M \text{ and } M = N_G(Y_M) \text{ for every } M \in \mathcal{F}(S);$$

in particular, the elements of $\mathcal{F}(S)$ are p -local subgroups of G if $S \neq 1$.

In the following results $\mathcal{F}(S)$ is always a factorization family for $\mathcal{S}(S)$. Recall that a finite group H is p -constrained if $H/O_{p'}(H)$ is of characteristic p .

Theorem 1.4. *Let G be a finite group and $S \in \text{Syl}_p(G)$, and let $1 \neq C \text{ char } B(S)$ and $M := N_G(C)$. Suppose that there exists $N \in \mathcal{F}(S)$ that is F -stable.*

- (a) *If $C = B(S)$, then $Y_N = Y_M$ and $N = C_G(Y_M)M = N_G(Y_M)$.*
- (b) *If $Y_N \leq O_p(M)$, then $Y_M = Y_N$ and $M \leq N$.*
- (c) *If M is p -constrained, then $M = O_{p'}(M)(M \cap N)$.*

Theorem 1.5. *Let G be a finite group and $S \in \text{Syl}_p(G)$, and let $M \in \mathcal{S}(S)$ such that $\Omega_1 Z(S) \leq M$ or $M = N_G(C)$ for some $1 \neq C \text{ char } B(S)$. Suppose that there exists $N \in \mathcal{F}(S)$ that is $2F$ -stable.*

- (a) *If $Y_N \leq O_p(M)$, then $M \leq N$.*
- (b) *If M is p -constrained, then $M = O_{p'}(M)(M \cap N)$.*
- (c) *The following hold for any p -constrained $H \in \mathcal{S}(B(S))$ with $H \not\leq O_{p'}(H)N$ (where $\overline{H} = H/O_{p'}(H)$):*
 - (a) $\overline{Y_N} \leq O_p(\overline{H})$.
 - (b) $C_{O_{p'}(\overline{H})}(\overline{Y_N}) \leq \overline{H}$.
 - (c) $Y_{\overline{H}}$ is not F -stable in $\overline{H}$.
 - (d) $C^{**}(\overline{H}, \overline{T}) \leq \overline{H} \cap \overline{N} < \overline{H}$, where $B(S) \leq T \in \text{Syl}_p(H)$.

For groups of parabolic characteristic p more can be said about the members of the factorization family $\mathcal{F}(S)$.

Theorem 1.6. *Let G be a finite group of parabolic characteristic p and let $1 \neq S \in \text{Syl}_p(G)$. Then the members of $\mathcal{F}(S)$ are maximal p -local subgroups of G . Moreover, if $N \in \mathcal{F}(S)$ is $2F$ -stable and $H \in \mathcal{S}(B(S))$ with $B(S) \leq T \in \text{Syl}_p(H)$, then $C^*(H, T) \leq N$.*

Corollary 1.7. *Let G be a finite group of parabolic characteristic p and $S \in \text{Syl}_p(G)$. If S is contained in at least two maximal p -local subgroups of G , then there exists $M \in \mathcal{F}(S)$ such that M is not $2F$ -stable.*

Let G and N be as in Theorem 1.6, and let H be a p -local subgroup containing S such that $H \not\leq N$. Then by Theorem 1.6 $C^*(H, S)$ is a proper subgroup of H . In this case the structure of H can be described precisely by using the Local $C(G, T)$ -Theorem proved in [BHS].

The proof of the above theorems relies heavily on two elementary results from [PPS] and [Ste], the L -Lemma and the qrc -Lemma. The authors found it remarkable that these results allow us to study finite groups in this context without any $\mathcal{K}$ -group assumption.

In fact, using the L -Lemma another result is proved, which is interesting in its own right and which can be used to improve the qrc -Lemma.

Theorem 1.8. *Let G be a finite group, $S \in \text{Syl}_p(G)$, and V be a finite dimensional faithful $GF(p)G$ -module. Suppose that $O_p(G) = 1$ and S is contained in a unique maximal subgroup of G . Then $|A| = |V/C_V(A)|$ for every offender A of G on V .*

2. ELEMENTARY PROPERTIES

In this section G is a finite group, p is a prime, and $S \in \text{Syl}_p(G)$.

Notation 2.1. Let X be a p -subgroup of G . A subgroup P of G is X -minimal if X is contained in a unique maximal subgroup of P and $X \not\leq O_p(P)$.

Lemma 2.2. *Let L be a subgroup of G and P be a p -parabolic subgroup of L .*

- (a) *There exists a unique largest p -reduced subgroup Y_L of L .*
- (b) *If Y is a p -reduced subgroup of P with $Y \leq O_p(L)$, then $\langle Y^L \rangle$ is p -reduced for L and so $Y \leq Y_L$.*
- (c) *If L is of characteristic p , then $Y_P \leq Y_L$.*

Proof. (a): Let A and B be p -reduced subgroups of L . It suffices to show that AB is also p -reduced. Then Y_L is the product of all p -reduced subgroups of L .

Since A is p -reduced, $B \leq O_p(L) \leq C_L(A)$, and so AB is elementary abelian. Let D be the inverse image of $O_p(L/C_L(AB))$. Since $C_L(AB) \leq C_L(A)$, $DC_L(A)/C_L(A) \leq O_p(L/C_L(A))$ and so $D \leq C_L(A)$. By symmetry, $D \leq C_L(B)$, and thus $D \leq C_L(A) \cap C_L(B) = C_L(AB)$.

(b): Since P is a p -parabolic subgroup of L , $O_p(L) \leq P$. Hence $[Y, O_p(L)] = 1$ since Y is p -reduced in P . By assumption $Y \leq O_p(L)$ and so $Y \leq \Omega_1 Z(O_p(L))$. In particular, $V := \langle Y_P^L \rangle$ is an elementary abelian normal subgroup of L .

Since P contains a Sylow p -subgroup of L , there exists $S_0 \leq P$ such that $S_0 C_L(V)/C_L(V) = O_p(L/C_L(V))$ and $S_0 \in \text{Syl}_p(S_0 C_L(V))$. As $S_0 C_P(V) \trianglelefteq P$ and $C_P(V) \leq C_P(Y_P)$, we get that $S_0 C_P(Y_P) \trianglelefteq P$. Hence $S_0 C_P(V) \leq C_P(Y_P)$ since Y_P is p -reduced in P , and so $[V, S_0 C_P(V)] = 1$ since $S_0 C_P(V) \trianglelefteq P$. Thus V is p -reduced for L , and by (a) $V \leq Y_L$.

(c): As in (b), $[Y_P, O_p(L)] = 1$. Since L is characteristic p , $Y_P \leq O_p(L)$. So (b) implies $Y_P \leq Y_L$. $\square$

Lemma 2.3. *Let $X \leq S \leq P \leq G$. Suppose that P is X -minimal and $N \trianglelefteq P$. Then either $O^p(P) \leq N$ and $P = XN$, or $S \cap N \leq O_p(P)$. In particular, $P = XO^p(P) = \langle X^P \rangle$.*

Proof. Observe that $P = NN_P(S \cap N)$. As P is X -minimal, either $NX = P$ or $NN_P(S \cap N) = P$, and in the second case $S \cap N \leq O_p(P)$.

Since $X \not\leq O_p(P)$, $S \cap XO^p(P) \not\leq O_p(P)$, and so $P = XO^p(P)$. A similar argument gives $P = \langle X^P \rangle$. $\square$

Lemma 2.4. *Let A be an F -stable elementary abelian p -subgroup of G , and let Q be a p -subgroup of G with $A \trianglelefteq Q$. Then the following hold:*

- (a) $A \leq Z(J(Q))$.
- (b) $\langle A^{N_G(Q)} \rangle$ is elementary abelian.

Proof. (a): Let $B \in \mathcal{A}(Q)$. Then B acts on A , and $|B| \geq |C_B(A)A|$ by the maximality of B . Also $C_B(A) \cap A \leq A \cap B \leq C_B(A)$, and so $C_B(A) \cap A = A \cap B$. Hence

$$|C_B(A)||A||C_A(B)|^{-1} \leq |C_B(A)||A||A \cap B|^{-1} = |C_B(A)A| \leq |B|,$$

and $|A/C_A(B)| \leq |B/C_B(A)|$ follows. The F -stability of A gives $[A, B] = 1$, and (a) holds.

(b): This is a direct consequence of (a) since $Z(J(Q)) \trianglelefteq N_G(Q)$. $\square$

Lemma 2.5. *Let Q be a normal p -subgroup of G with $C_G(Q) \leq Q$ and Y be an abelian p -subgroup of G . If $C_Q(Y) \trianglelefteq G$ and Q normalizes Y , then $Y \leq O_p(G)$.*

Proof. Observe that

$$[Q, Y] \leq Q \cap Y \leq C_Q(Y).$$

Since $C_Q(Y) \trianglelefteq G$, this shows that $\langle Y^G \rangle$ centralizes $Q/C_Q(Y)$ and $C_Q(Y)$. Hence $O^p(\langle Y^G \rangle)$ centralizes Q and, since $C_G(Q) \leq Q$, $O^p(\langle Y^G \rangle) = 1$ and $\langle Y^G \rangle$ is a p -group. Thus $Y \leq O_p(G)$. $\square$

Lemma 2.6. *Let A be a finite elementary abelian p -group and V be a finite dimensional $GF(p)A$ -module. Suppose that A is quadratic on V and $[v, A] = [V, A]$ for every $v \in V \setminus C_V(A)$. Then A is a quadratic offender on every A -submodule of V .*

Proof. Since every A -submodule of V satisfies the same hypothesis, it suffices to show that A is an offender on V . Without loss, $[V, A] \neq 1$. Choose $W \leq [V, A]$ with $|[V, A]/W| = p$ and put $\overline{V} = V/W$. Let U be the inverse image of $C_{\overline{V}}(A)$ in V . Then $[U, A] \leq W$, and so $[V, A] \not\leq [U, A]$. Thus $U \leq C_V(A)$ and $C_{\overline{V}}(A) = \overline{C_V(A)}$; in particular, $|V/C_V(A)| = |\overline{V}/C_{\overline{V}}(A)|$. Note that $\overline{V}$ satisfies the hypothesis, so replacing V by $\overline{V}$ we may assume that $|[V, A]| = p$. Let $B < A$ with $|A/B| = p$. Since $[V, B]$ is at most 1 dimensional, B in place of A also satisfies the hypothesis of the lemma. Hence by induction on $|A|$, $|V/C_V(B)| \leq |B|$.

Let $a \in A \setminus B$. Since $|[V, a]| = p$, $|V/C_V(a)| \leq p$, and $|C_V(B)/C_V(B) \cap C_V(a)| \leq p$ also. But $C_V(A) = C_V(B) \cap C_V(a)$ and so

$$|V/C_V(A)| \leq |V/C_V(B)|p \leq |B|p = |A|. \quad \square$$

3. A PARTIAL ORDERING

In this section G is a finite group, p is a prime, and $S \in \text{Syl}_p(G)$.

Notation 3.1. Let A and B be subgroups of G . The relation $\ll$ on the subgroups of G is defined by

$$A \ll B : \iff A \subseteq C_G(Y_A)B \text{ and } Y_A \leq Y_B.$$

Furthermore, we define

$$A^\dagger := C_G(Y_A)A \text{ and } \mathcal{S}^\dagger := \{L \leq G \mid L = L^\dagger\}.$$

Lemma 3.2. *Let L and M be subgroups of G .*

- (a) $Y_L \leq Y_{L^\dagger}$, $L \ll L^\dagger$, and $(L^\dagger)^\dagger = L^\dagger$.
- (b) $\mathcal{S}^\dagger = \{L \leq G \mid C_G(Y_L) \leq L\}$.
- (c) $\ll$ is reflexive and transitive.
- (d) $L \subseteq C_G(Y_L)M$ if and only if $L \leq C_G(Y_L)N_M(Y_L)$.
- (e) Suppose that $L \subseteq C_G(Y_L)M$ and $L \cap M$ is a p -parabolic subgroup of L and M . Then Y_L is p -reduced for $N_M(Y_L)$ and $L \ll N_M(Y_L)$.
- (f) If $L = L^\dagger$, then $L \ll M$ if and only if $Y_L \leq Y_M$ and $L = C_G(Y_L)(L \cap M)$.
- (g) Restricted to $\mathcal{S}^\dagger$, $\ll$ is a partial ordering.

Proof. (a): Clearly Y_L is a p -reduced subgroup of $L^\dagger$, so $Y_L \leq Y_{L^\dagger}$. Thus $C_G(Y_{L^\dagger}) \leq C_G(Y_L) \leq L^\dagger$ and $L^\dagger = (L^\dagger)^\dagger$.

(b): This is an immediate consequence of the definition of $L^\dagger$.

(c): Obviously $\ll$ is reflexive. If $A, B, C \leq G$ with $A \ll B$ and $B \ll C$, then $Y_A \leq Y_B \leq Y_C$, and so $Y_A \leq Y_C$. Also $C_G(Y_B) \leq C_G(Y_A)$, and hence

$$A \subseteq C_G(Y_A)B \subseteq C_G(Y_A)C_G(Y_B)C = C_G(Y_A)C.$$

Thus $A \ll C$ and $\ll$ is transitive.

(d): If $L \subseteq C_G(Y_L)M$, then $L \leq N_G(Y_L) \cap C_G(Y_L)M = C_G(Y_L)N_M(Y_L)$. The other direction is obvious.

(e): Since $L \cap M$ is a p -parabolic subgroup of L ,

$$Y_L \leq O_p(L) \leq L \cap M \leq N_M(Y_L),$$

so Y_L is an elementary abelian normal subgroup of $N_M(Y_L)$. Since $L \cap M$ is a p -parabolic subgroup of M , $C_G(Y_L)(L \cap M)$ and thus also $C_G(Y_L)L$ are p -parabolic subgroups of $C_G(Y_L)N_M(Y_L)$.

As Y_L is a p -reduced subgroup of $C_G(Y_L)L$, Lemma 2.2(b) shows that $Y_L = \langle Y_L^{C_G(Y_L)N_M(Y_L)} \rangle$ is p -reduced for $C_G(Y_L)N_M(Y_L)$. Hence Y_L is also a p -reduced subgroup of $N_M(Y_L)$. Thus $Y_L \leq Y_{N_M(Y_L)}$, and so $L \ll N_M(Y_L)$.

(f): Since $L \in \mathcal{S}^\dagger$ we have $C_G(Y_L) \leq L$, and so $L \subseteq C_G(Y_L)M$ implies $L = C_G(Y_L)(L \cap M)$. Now (f) is obvious.

(g): Let $L, M \in \mathcal{S}^\dagger$ with $L \ll M$ and $M \ll L$. Since $Y_L \leq Y_M \leq Y_L$, we have $Y_L = Y_M$. By (f) $L = C_G(Y_L)(L \cap M)$ and $M = C_G(Y_M)(M \cap L)$. Hence $Y_M = Y_L$ gives $L = M$. So the restriction of $\ll$ to $\mathcal{S}^\dagger$ is anti-symmetric. Now (g) follows (c). $\square$

Notation 3.3. Put $\mathcal{S}^\dagger(S) := \{L \in \mathcal{S}^\dagger \mid S \leq L\}$. According to Lemma 3.2(g) $\ll$ restricted to $\mathcal{S}^\dagger(S)$ is a partial ordering on $\mathcal{S}^\dagger(S)$. We denote the set of maximal elements of $\mathcal{S}^\dagger(S)$ with respect to $\ll$ by $\mathcal{F}(S)$.

Theorem 3.4. $\mathcal{F}(S)$ is the unique factorization family for $\mathcal{S}(S)$.

Proof. Let $\mathcal{G}$ be a factorization family for $\mathcal{S}(S)$ and let $M \in \mathcal{G}$. Clearly $M \leq M^\dagger$, and by Lemma 3.2(a), $Y_M \leq Y_{M^\dagger}$. So condition (ii) of Definition 1.2 gives $M = M^\dagger$. Thus $M \in \mathcal{S}^\dagger(S)$ and $\mathcal{G} \subseteq \mathcal{S}^\dagger(S)$.

Now let $\mathcal{G}$ be any subset of $\mathcal{S}^\dagger(S)$. Then condition (i) of Definition 1.2 is fulfilled for $\mathcal{G}$ if and only if for each $L \in \mathcal{S}(S)$ there exists $M \in \mathcal{G}$ with $L \ll M$. Since $L \ll L^\dagger$ and $\ll$ is transitive by Lemma 3.2, we conclude that $\mathcal{G}$ fulfills (i) if and only if $\mathcal{G}$ contains all the maximal elements of $\mathcal{S}^\dagger(S)$ with respect to $\ll$. Also condition (ii) holds if and only if all elements of $\mathcal{G}$ are maximal with respect to $\ll$ in $\mathcal{S}^\dagger(S)$. Thus $\mathcal{F}(S)$ is the unique factorization family for $\mathcal{S}(S)$. $\square$

Lemma 3.5. Let $M \in \mathcal{F}(S)$ and $H \in \mathcal{S}(S)$ with $M = C_M(Y_M)(M \cap H)$. If H is p -constrained, then $H = O_{p'}(H)(H \cap M)$. In particular, if G is of parabolic characteristic p and $S \leq L \leq M$ with $M = C_M(Y_M)L$, then M is the unique maximal p -local subgroup of G containing L .

Proof. Put $\overline{H} = H/O_{p'}(H)$. Since $M = C_M(Y_M)(H \cap M)$, Y_M is p -reduced for $H \cap M$ and $\overline{Y}_M$ is a p -reduced subgroup of $\overline{H} \cap \overline{M}$. So by Lemma 2.2(c), $\overline{Y}_M \leq Y_{\overline{H}}$. Let $Y \leq S$ with $\overline{Y} = Y_{\overline{H}}$ and $K := N_H(Y)$. Then by the Frattini argument, $H = O_{p'}(H)K$. It follows that Y is a p -reduced subgroup of K , so $Y_M \leq Y \leq Y_K$.

As $YO_{p'}(H) \cap M = Y(O_{p'}(H) \cap M)$, we also get, using the Frattini argument one more time,

$$H \cap M = (O_{p'}(H) \cap M)(K \cap M) = O_{p'}(M \cap H)(K \cap M).$$

Thus $M = C_M(Y_M)(H \cap M) \leq C_G(Y_M)K$, since $O_{p'}(M \cap H)$ centralizes Y_M . Now Definition 1.2(ii) implies that $K \leq M$, and so $H = O_{p'}(H)(H \cap M)$. Hence the first statement holds.

To prove the second statement, let H be a p -local subgroup containing L . Then $M = C_M(Y_M)L$ implies $M = C_M(Y_M)(H \cap M)$. On the other hand H is of characteristic p since G has parabolic characteristic p , so H is p -constrained and $O_{p'}(H) = 1$. Hence by the first statement $H = O_{p'}(H)(H \cap M) \leq M$. $\square$

Lemma 3.6. *Let $M \in \mathcal{F}(S)$, $S_0 := C_S(Y_M)$ and $M_0 := N_M(S_0)$.*

- (a) $M = C_M(Y_M)M_0$, $S_0 = O_p(M_0)$ and $C_S(S_0) \leq S_0$.
- (b) $\Omega_1 Z(S) \leq Y_M = Y_{M_0} = \Omega_1 Z(S_0)$.

Proof. (a): The Frattini argument gives $M = C_M(Y_M)M_0$. Hence $O_p(M_0) = S_0$, since Y_M is p -reduced. Clearly $Y_M \leq \Omega_1 Z(S_0)$, and so

$$C_S(S_0) \leq C_S(\Omega_1 Z(S_0)) \leq C_S(Y_M) \leq S_0.$$

(b): Let $S_0 \leq S_1 \leq S$ with

$$S_1 C_{M_0}(\Omega_1 Z(S_0)) / C_{M_0}(\Omega_1 Z(S_0)) = O_p(M_0 / C_{M_0}(\Omega_1 Z(S_0))).$$

Then $S_1 C_M(Y_M) / C_M(Y_M)$ is normalized by $M_0 C_M(Y_M) = M$. Since $O_p(M / C_M(Y_M)) = 1$ we get $S_1 \leq C_M(Y_M)$, so $S_1 = S_0 = O_p(M_0)$ by (a), and $\Omega_1 Z(S_0)$ is p -reduced for M_0 . Together with Lemma 3.2(a) this gives

$$Y_M \leq \Omega_1 Z(S_0) \leq Y_{M_0} \leq Y_{M_0^\dagger}.$$

In particular, $M \ll M_0^\dagger$, and the maximality of M yields $Y_M = Y_{M_0^\dagger}$. Now (b) follows, since also $\Omega_1 Z(S) \leq \Omega_1 Z(S_0)$. $\square$

Proof of Theorems 1.3 and 1.4. By Theorem 3.4 $\mathcal{F}(S)$ is the unique factorization family for $\mathcal{S}(S)$. Let $M \in \mathcal{F}(S)$. By Lemma 3.6(b) $\Omega_1 Z(S) \leq Y_M$ and by Lemma 3.2(e) $M \ll N_G(Y_M)$. Hence the maximality of M gives $M = N_G(Y_M)$.

Assume that there exists $N \in \mathcal{F}(S)$ that is F -stable, i.e. $Y := Y_N$ is F -stable in $N_G(Y_N) = N$. Then by Lemma 2.4 $B(S) \leq C_G(Y)$ and

$$N = C_G(Y)N_N(B(S)) \subseteq C_G(Y)L, \text{ where } L := N_G(B(S));$$

in particular, $Y \leq \Omega_1 Z(B(S)) \leq O_p(N_G(B(S)))$. Now Lemma 2.2(b) implies that $Y \leq Y_L$, and so by Definition 1.2(ii) $Y = Y_L$. It follows that $N = N_G(Y_L)$. In particular, N is the unique F -stable member of $\mathcal{F}(S)$. This finishes the proof of Theorem 1.3 and also shows Theorem 1.4(a).

Now let $1 \neq C \text{ char } B(S)$ and put $M := N_G(C)$. Then $N_N(B(S)) \leq M$ and thus also $N = C_G(Y)(M \cap N)$. Suppose that $Y_N \leq O_p(M)$. Then as above Lemma 2.2(b) implies that $Y_N \leq Y_M$, and by Definition 1.2(ii) $Y_N = Y_M$ and $M \leq N$. So Theorem 1.4(b) holds.

Suppose next that M is p -constrained. From $N = C_G(Y)(N \cap M)$ and Lemma 3.5 we get that $M = O_{p'}(M)(M \cap N)$. Hence Theorem 1.4(c) holds. $\square$

4. THE L -LEMMA AND THE qrc -LEMMA

In this chapter we will work with the following hypothesis.

Hypothesis 4.1. Let P be a finite group of characteristic p , $T \in \text{Syl}_p(P)$, $Y \trianglelefteq T$, and $R := C_T(Y)$. Suppose that P is $RO_p(P)$ -minimal with M being the unique maximal subgroup of P containing $RO_p(P)$.

Notation 4.2. Let X be a finite group and V a finite dimensional $GF(p)X$ -module. By $c(V, X)$ we denote the number of non-central chief factors of X in V (in a given chief series). We define $q(V, X) := 0$ if every quadratically acting subgroup of X already centralizes V , and

$$q(V, X) := \min\{\log_{|A/C_A(V)|} |V/C_V(A)| \mid A \leq X, [V, A, A] = 1 \neq [V, A]\}$$

otherwise. Moreover, $r(V, X) := 0$ if V does not possess non-central X -chief-factors, and

$$r(V, X) := \min\{q(C, X) \mid C \text{ non-central } X\text{-chief-factor on } V\}$$

otherwise.

Lemma 4.3 (*L-Lemma*). *Assume Hypothesis 4.1. Let A be a subgroup of T such that $A \not\leq O_p(P)$. Then there exists a subgroup $L \leq P$ with $AO_p(P) \leq L$ satisfying:*

- (a) $AO_p(L)$ is contained in a unique maximal subgroup L_0 of L , and $L_0 = L \cap M^g$ for some $g \in P$.
- (b) $L = \langle A, A^x \rangle O_p(L)$ for every $x \in L \setminus L_0$.
- (c) L is not contained in any P -conjugate of M .

Proof. See [PPS]. □

The next lemma is very similar to [Ste, 3.3].

Lemma 4.4. *Assume Hypothesis 4.1. Suppose $V := \langle Y^P \rangle$ is elementary abelian, $C_{O_p(P)} \not\trianglelefteq P$ and $c(V, P) = 1$. Then $[O_p(P), O^p(P)]$ is a non-trivial quadratic offender on Y .*

Proof. Since P is $RO_p(P)$ -minimal, we get from Lemma 2.3 that $P = RO^p(P)O_p(P)$. Put

$$Q := [O_p(P), O^p(P)], \quad W := [V, O^p(P)] \quad \text{and} \quad D := C_V(O^p(P)).$$

Since $c(V, P) = 1$, $W/W \cap D$ is a chief factor for P on V . Hence $[W, O_p(P)] \leq D$. Note that $P = TO^p(P)$ normalizes YW , and so $V = YW$. Thus $[Y, Q]D \leq P$. Observe that R centralizes $[Y, Q]D/D$. Since $O^p(P) \leq \langle R^P \rangle$ we conclude that $[Y, Q, O^p(P)] \leq D$; hence $O^p(P)$ centralizes $[Y, Q]$. So $P = TO^p(P)$ normalizes $[Y, Q]$ and $[V, Q] = [Y, Q] \leq D$. It follows that

$$[V, O^p(P), O_p(P)] = [W, O_p(P)] \leq D \quad \text{and} \quad [O_p(P), O^p(P), V] = [Q, V] \leq D.$$

Hence the Three Subgroup Lemma implies $[V, O_p(P), O^p(P)] \leq D$, and so

$$(*) \quad [V, O_p(P)] \leq D.$$

Pick $x \in Y \setminus D$. Since R centralizes x we conclude from (*) that $P = RO^p(P)O_p(P)$ normalizes $\langle x^{O^p(P)} \rangle D$, and so $W \leq \langle x^{O^p(P)} \rangle D$. Put $X := [x, Q]$. Since $X \leq D$ it follows that

$$[W, Q] \leq [\langle x^{O^p(P)} \rangle D, Q] = [x, Q] = X.$$

As $[V, Q, O^p(P)] = 1 \leq X$ and $[V, O^p(P), Q] = [W, Q] \leq X$, the Three Subgroup Lemma implies $[Q, O^p(P), V] \leq X$. Since $[Q, O^p(P)] = Q$ we get $[V, Q] = X$. In particular,

$$[y, Q] = X \quad \text{for every } y \in Y \setminus C_Y(Q).$$

Now Lemma 2.6 shows that Q is a quadratic offender on Y .

If Q acts trivially on Y , then $Q \leq C_{O_p(P)}(Y)$, and so $C_{O_p(P)}(Y) \trianglelefteq TO^p(P) = P$, a contradiction. □

Lemma 4.5. *Let L be a finite group acting on a p -group E , and let A and B be p -subgroups of L and X and Z subgroups of E . Suppose that*

- (i) $B \not\leq O_p(L)$,
- (ii) $[E, A] \leq X \leq C_E(A)$ and $[E, B] \leq Z \leq C_E(B)$,
- (iii) L is $AO_p(L)$ -minimal and $[E, O^p(L)] \neq 1$,
- (iv) X is normalized by E and $O_p(L)$, X is abelian, and $E = \langle X^L \rangle$.

Then

- (a) $C_B(E) \leq B \cap O_p(L)$,
- (b) $E = X^g Z = X^g C_E(B)$ for some $g \in L$,
- (c) $Z = [E, B]C_Z(E) = [E, b]C_Z(E)$ for all $b \in B \setminus O_p(L)$,
- (d) $[B, E, E, E] = 1$,
- (e) $|Z/C_Z(E)| \leq |ZD/D| \leq |E/C_E(B)|$,
- (f) $|BO_p(L)/O_p(L)| \leq |E/C_E(B)|$.

Proof. By (iii) there exists a unique maximal subgroup L_0 of L containing $AO(L)$, and by Lemma 2.3 $(\bigcap_{g \in L} L_0^g)/O_p(L)$ is a p' -group.

Pick $b \in B \setminus O_p(L)$. Then there exists $g \in L$ with $b \notin L_0^g$. Put $H := \langle A^g, b \rangle$. Then $L = HO_p(L)$ since $H \not\leq L_0^g$. Furthermore, we put $D := \bigcap_{g \in L} X^g$.

(a): Again by (iii) $O^p(L) \not\leq C_L(E)$, so Lemma 2.3 shows that $C_L(E)/C_{O_p(L)}(E)$ is a p' -group. Now (a) follows.

(b): By (iv) $O_p(L)$ normalizes X , so $X^L = X^H$. It follows that $D = \bigcap_{h \in H} X^h$ and $C_{X^g}(b) \leq C_{X^g}(H) \leq D$. From $E = \langle X^L \rangle = \langle X^H \rangle$ and (ii) we conclude that

$$(*) \quad E = X^g[E, H] = X^g[E, A^g][E, b] = X^g[E, b] = X^g Z.$$

Since $Z \leq C_E(B)$ by (ii), (b) holds.

(c): Since $X^g \cap Z \leq C_{X^g}(H) \leq D$, we get $X^g \cap Z = D \cap Z$, and so by (b) and (*)

$$|E/X| = |E/X^g| = |ZX^g/X^g| = |Z/Z \cap X^g| = |Z/Z \cap D|.$$

Moreover, using (*)

$$Z = (X^g \cap Z)[E, b] \leq (Z \cap D)[E, b] \leq C_Z(E)[E, b] \leq C_Z(E)[E, B] \leq Z,$$

and so (c) holds.

(d): Since L is $AO_p(L)$ -minimal, $A \not\leq O_p(L)$, and so (b) can be applied with A and A^g in place of A and B . Then $E = XX^t$ for some $t \in L$; in particular $[X^t, X] \leq X \cap X^t \leq D \leq Z(E)$. Thus $E' \leq D \leq Z(E)$ and $[B, E, E, E] \leq [E', E] = 1$. So (d) holds.

(e): By (ii) and (b)

$$C_E(B) \leq C_{X^g}(B)Z = C_{X^g}(H)Z = DZ.$$

Hence

$$|E/C_E(B)| \geq |E/ZD| = |X^g ZD/ZD| = |X^g/X^g \cap ZD| = |X^g/D|.$$

On the other hand, by (b) $|E/X| = |Z/Z \cap X^g| = |Z/Z \cap D|$, while the same result applied to A in place of B gives $|E/X| = |X/D| = |X^g/D|$. Since $D \leq Z(E)$ this gives

$$|E/C_E(B)| \geq |Z/Z \cap D| \geq |Z/C_Z(E)|.$$

(f): Let $x \in X^g \setminus D$ and suppose that $[x, b] \in D$. Then $\langle x \rangle D$ is normalized by $\langle X^g, b \rangle = H$, and so $x \in D$, a contradiction. This shows that $[x, c] \notin D$ for every

$c \in B \setminus O_p(L)$. Since B acts quadratically on the abelian group E/D we conclude

$$|[x, B]D/D| = |\{[x, c]D \mid c \in B\}| \geq |BO_p(L)/O_p(L)|.$$

Note that by (ii), $[x, B]D \leq ZD$, and so (f) now follows from (e). $\square$

Theorem 4.6. *Assume Hypothesis 4.1. Let V be a finite dimensional $GF(p)P$ -module such that $[V, O_p(P)] = 0$ and $[V, O^p(P)] \neq 0$. Then $q(V, P) = 0$ or $q(V, P) \geq 1$.*

Proof. Let $A \leq T$ be quadratic on V with $[V, A] \neq 0$. We need to show that $|V/C_V(A)| \geq |A/C_A(V)|$. The proof is by induction on $|A|$.

Let Y be a non-central P -chief factor in V . By Lemma 2.3 $C_T(Y) \leq O_p(P) \leq C_T(V)$. It follows that

$$|Y/C_Y(A)| \leq |V/C_V(A)| \text{ and } |A/C_A(Y)| = |A/C_A(V)|$$

for every $A \leq T$. Hence we may assume that

(1°) V is a non-trivial simple P -module.

We now apply Lemma 4.3. Then there exists $A \leq L$ such that L has the properties given in Lemma 4.3. In particular, there exists $g \in P$ such that $A \leq T^g \cap L \in \text{Syl}_p(L)$ and that $L \cap M^g$ is the unique maximal subgroup of L containing $AO_p(L)$. Put $U := \langle C_V(T^g)^L \rangle$.

(2°) $[U, O^p(L)] \neq 0$ and $[U, A] \neq 0$.

By (1°) $C_V(T^g)$ is not P -invariant, so $N_P(C_V(T^g)) \leq M^g$. Since $L \not\leq M^g$, we get that $[U, O^p(L)] \neq 0$ and thus also $[U, A] \neq 0$.

(3°) Put $D := C_A(U)$. Then $|A/D| \leq |U/C_U(A)|$.

Observe that by the definition of U , $[U, O_p(L)] = 0$. Thus, for $E := U$, $B := A$, and $X := Z := C_U(A)$, L satisfies the hypothesis of Lemma 4.5. By Lemma 4.5(f)

$$|A/D| = |A/C_A(U)| \leq |A/A \cap O_p(L)| \leq |U/C_U(A)|.$$

So (3°) holds.

(4°) $|D/C_D(V)| \leq |V/C_V(D)|$.

Since $[U, A] \neq 0$, $D < A$ and (4°) follows by induction on $|A|$.

Using (3°) and (4°) we compute

$$\begin{aligned} |A/C_A(V)| &= |A/D| |D/C_D(V)| \leq |U/C_U(A)| |V/C_V(D)| \\ &\leq |C_V(D)/C_V(A)| |V/C_V(D)| = |V/C_V(A)|. \end{aligned} \quad \square$$

The next lemma is a variation of [Ste, 3.2].

Lemma 4.7 (*grc-Lemma*). *Assume Hypothesis 4.1. Let $V := \langle Y^P \rangle$. Suppose that*

(i) $Y \leq \Omega_1 Z(J(O_p(P)))$,

(ii) $C_{O_p(P)}(Y) \not\leq P$,

(iii) $J(R) \not\leq O_p(P)$.

Then $V \leq \Omega_1 Z(J(O_p(P)))$, $V \neq Y$, $N_L(Y) \leq M$, $[V, O^p(P)] \neq 1$, $C_T(V) \leq O_p(P)$, and there exists $A \in \mathcal{A}(R)$ with

$$[V, A, A] = 1 \neq [V, A] \text{ and } A \not\leq O_p(P).$$

Moreover, one of the following holds, where $q := q(Y, O_p(P))$, $r := r(V, P)$ and $c := c(V, P)$:

- (a) $0 \neq q \leq 1$.
- (b) $2 \leq c$, $1 \leq r$, and $(q-1)(rc-1) \leq 1$. In particular, $0 \neq q \leq 2$.

Proof. By (i) $V \leq ZJ(O_p(P))$. If $V = Y$, then $C_{O_p(P)}(Y) \trianglelefteq P$, a contradiction to (ii). Hence $V \neq Y$ and $RO_p(P) \leq T \leq N_P(Y) < P$. Since P is $RO_p(P)$ -minimal, we conclude that $N_P(Y) \leq M$. So $[V, O^p(P)] \neq 1$. Hence Lemma 2.3 gives $C_T(V) \leq O_p(P)$. In particular, (P, Y) satisfies Hypothesis III of [Ste].

As $C_T(V) \leq O_p(P)$, (iii) shows that there exists $A \in \mathcal{A}(R)$ such that $[V, A] \neq 1$. By the Timmesfeld Replacement Theorem [KS] we may assume that $[V, A, A] = 1$. Moreover, (i) implies that $A \not\leq O_p(P)$.

Suppose that $c = 1$. Then Lemma 4.4 shows that (a) holds. Thus, we may assume from now on that $c \geq 2$.

Suppose that $[A \cap O_p(P), V] = 1$. Again by Lemma 2.3 $O_p(P) \cap A = C_A(V) = C_A(U)$ for every non-central P -chief factor U of V . On the other hand, by the maximality of A , $|V/C_V(A)| \leq |A/C_A(V)|$, and thus also $|U/C_U(A)| \leq |A/C_A(U)|$. Hence Theorem 4.6 implies that $c = 1$, which contradicts our assumption. We have shown that $[A \cap O_p(P), V] \neq 1$; in particular $q \neq 0$. Now [Ste, 3.2(c)] and Theorem 4.6 yield (b). $\square$

Lemma 4.8. *Assume Hypothesis 4.1. Suppose that*

- (i) $C_{O_p(P)}(Y) \not\trianglelefteq P$,
- (ii) $Y \leq Z(J(O_p(P))) \cap Z(J(T))$,
- (iii) $J(T) \not\leq O_p(P)$.

Then there exist subgroups $A \in \mathcal{A}(T)$ and L of P such that the following hold:

- (a) L is $AO_p(L)$ -minimal.
- (b) $O_p(P)A \leq T \cap L \in \text{Syl}_p(L)$, and $M \cap L$ is the unique maximal subgroup of L containing $AO_p(L)$.
- (c) $Y \not\trianglelefteq L$, and $V_0 := \langle Y^L \rangle$ is abelian.
- (d) If $Y \leq Z(J(O_p(L)))$, then L and Y satisfy the hypothesis of Lemma 4.7 with L in place of P .

Proof. From (ii) $[Y, J(T)] = 1$, and so $J(T) \leq R$ and $J(R) = J(T)$. So the assumptions of Lemma 4.7 are fulfilled. In particular, V is elementary abelian and there exists $A \in \mathcal{A}(T)$ with $[V, A, A] = 1 \neq [V, A]$ and $A \not\leq O_p(P)$.

Hence we are allowed to apply the L -Lemma, Lemma 4.3. This gives a subgroup L having the properties (a)–(c) given in Lemma 4.3. By Lemma 4.3(c) L is not a p -group, and so L is $AO_p(L)$ -minimal. This is (a).

According to Lemma 4.3(a) there exists $g \in P$ such that $AO_p(L) \leq T^g \cap L \in \text{Syl}_p(L)$, and $L \cap M^g$ is the unique maximal subgroup containing $AO_p(L)$. Hence replacing A by $A^{g^{-1}}$ and L by $L^{g^{-1}}$ we may assume that (b) holds.

Clearly, $Y \not\trianglelefteq L$ since $L \not\leq M$ but $N_P(Y) \leq M$. Since V is abelian, V_0 is abelian and (c) holds.

From $A \not\leq O_p(P)$ and (ii) we get that $J(C_{T \cap L}(Y)) \not\leq O_p(L)$ and L is $C_{T \cap L}(Y)O_p(L)$ -minimal. Hence Lemma 4.7(iii) holds for L and Y . Assume that $C_{O_p(L)}(Y) \trianglelefteq L$. Then also $C_{O_p(P)}(Y) \trianglelefteq L$ and thus $P = \langle T, L \rangle \leq N_P(C_{O_p(P)}(Y))$. This contradicts (i). Hence also Lemma 4.7(ii) holds for L and Y , and (d) follows. $\square$

5. F -STABILITY

In this section we explore the following hypothesis:

Hypothesis 5.1. Let p be a prime and H a finite group. Suppose that Y is an elementary abelian p -subgroup of H such that for $T \in \text{Syl}_p(N_H(Y))$ and $R := C_T(Y)$ the following hold:

- (i) $Y \trianglelefteq N_H(J(R))$.
- (ii) Y is F -stable in H .
- (iii) Either $Y \leq O_p(H)$ or H is of characteristic p .

This hypothesis is motivated by the following observation:

Lemma 5.2. Let G be a finite group, $S \in \text{Syl}_p(G)$ and $J(S) \leq H \leq G$, and let $\mathcal{F}(S)$ be a factorization family for $\mathcal{S}(S)$. Suppose that $N \in \mathcal{F}(S)$ is F -stable.

- (a) If $Y_N \leq O_p(H)$, then $Y := Y_H$ and H satisfy Hypothesis 5.1.
- (b) If H is p -constrained and $\overline{H} := H/O_{p'}(H)$, then $\overline{Y_N}$ and $\overline{H}$ satisfy Hypothesis 5.1 in place of Y and H .

Proof. Let $T \in \text{Syl}_p(H)$ with $J(S) \leq T$. Put $Y := Y_N$ and $R := C_T(Y)$. Since $Y \trianglelefteq S$ and Y is F -stable, Lemma 2.4(a) implies that $Y \leq \Omega_1 Z(J(S)) \leq H$ and $J(S) = J(T) = J(R)$. Observe that $Y \leq O_p(N_G(J(S)))$, and so by Theorem 1.4(b), $N_G(J(S)) \leq N$. In particular, $T \leq N_G(Y)$, and so $T \in \text{Syl}_p(N_H(Y))$. Now (a) follows.

Assume that H is p -constrained. Then $\overline{H} = H/O_{p'}(H)$ is of characteristic p . By the Frattini argument, $N_{\overline{H}}(\overline{Y}) = \overline{N_H(Y)}$ and $N_{\overline{H}}(J(\overline{R})) = \overline{N_H(J(R))}$. Moreover since Y is F -stable in G , $\overline{Y}$ is F -stable in $\overline{H}$. Thus Hypothesis 5.1 holds for $\overline{Y}$ and $\overline{H}$. $\square$

Lemma 5.3. Assume Hypothesis 5.1. Then $Y \trianglelefteq T$, $Y \leq Z(J(T))$, $J(R) = J(T)$, $N_H(T) \leq N_H(Y)$ and $T \in \text{Syl}_p(H)$.

Proof. Clearly $Y \trianglelefteq T$. Thus by Lemma 2.4(a), $[Y, J(T)] = 1$. So $J(T) \leq R$ and $J(T) = J(R)$. Therefore $N_H(T) \leq N_H(J(R))$, and so by Hypothesis 5.1(i) $N_H(T) \leq N_H(Y)$. Hence $T \in \text{Syl}_p(H)$. $\square$

Theorem 5.4. Assume Hypothesis 5.1 and suppose $C_{O_p(H)}(Y) \not\trianglelefteq H$. Then Y is not $2F$ -stable in H .

Proof. If any subgroup of H satisfies the conclusion of Theorem 5.4 with respect to Y , then H does also. Thus we may assume

- (1°) No proper subgroup of H satisfies the hypothesis of Theorem 5.4 with respect to Y .

Put

$$H_0 = N_H(C_{O_p(H)}(Y)).$$

From Lemma 5.3 we conclude

- (2°) $N_H(T) \leq N_H(Y) \leq H_0$, $J(R) = J(T)$, $Y \leq ZJ(R)$ and $T \in \text{Syl}_p(H_0)$.

Next we show:

- (3°) Let $J(R)O_p(H) \leq \tilde{H} < H$. Then $C_{O_p(\tilde{H})}(Y) \trianglelefteq \tilde{H}$ and $\tilde{H} \leq H_0$.

By (2°) there exists $\tilde{R} \in \text{Syl}_p(C_{\tilde{H}}(Y))$ with $J(R) \leq \tilde{R}$ and $J(R) = J(T) = J(\tilde{R})$, and so $Y \leq N_{\tilde{H}}(J(\tilde{R}))$. Since $O_p(H) \leq O_p(\tilde{H})$, $Y \leq O_p(\tilde{H})$ or $\tilde{H}$ is of characteristic p . Moreover, Y is F -stable in $\tilde{H}$, and so Y and $\tilde{H}$ satisfy Hypothesis 5.1. Now (1°) shows that $C_{O_p(\tilde{H})}(Y) \leq \tilde{H}$. Hence $C_{O_p(H)}(Y) = O_p(H) \cap C_{O_p(\tilde{H})}(Y)$ is normal in $\tilde{H}$ and $\tilde{H} \leq H_0$.

(4°) H_0 is the unique maximal subgroup of H containing $J(R)O_p(H)$, and H is $J(R)O_p(H)$ -minimal.

The first statement follows from (3°). If $J(R) \leq O_p(H)$, then by Lemma 5.3, $J(R) = J(O_p(H)) \leq H$, and so by Hypothesis 5.1(i), $H \leq N_H(Y) \leq H_0$, a contradiction. Hence $J(R)O_p(H) \not\leq O_p(H)$, and H is $J(R)O_p(H)$ -minimal.

(5°) Put $W := \langle Y^{H_0} \rangle$. Then W is elementary abelian.

We will first show that $Y \leq O_p(H_0)$. If $Y \leq O_p(H)$, this is obvious. Otherwise H is of characteristic p , and by Lemma 5.3, $O_p(H)$ normalizes Y . So by Lemma 2.5 and (2°) $Y \leq O_p(H_0) \leq T$. Now (5°) follows from Hypothesis 5.1(ii) and Lemma 2.4(b).

Let $\mathcal{W}$ be the set of all p -subgroups D of H satisfying:

- (a) $WO_p(H) \leq N_H(D) \not\leq H_0$.
- (b) $D = J(D) \leq H_0$.

Clearly, $1 \in \mathcal{W}$ and so $\mathcal{W} \neq \emptyset$. Pick $D \in \mathcal{W}$ such that first $|A|$ is maximal for $A \in \mathcal{A}(D)$ and then $|D|$ is maximal. Put $N := N_H(D)$ and $T_0 := DO_p(H)$ and let $T_1 \in \text{Syl}_p(N \cap H_0)$. Since $T_0 \leq O_p(N \cap H_0)$, $T_0 \leq T_1$. As W is H_0 -invariant and by (2°) $T \in \text{Syl}_p(H_0)$, there exists $g \in H_0$ with $W^g = W$ and $T_1^g \leq T$; in particular $D^g \in \mathcal{W}$. Thus, after replacing D by D^g we may assume that $T_1 \leq T$.

(6°) $Y \leq Z(J(T_1))$, and if $Y \leq T_0$, then $Y \leq Z(J(T_0))$.

Since $T_1 \leq T$, T_1 normalizes Y . So (6°) follows from Lemma 2.4(a).

(7°) Let U be a p -subgroup of H_0 containing D . Suppose that $W \leq N_H(U)$ and $N_H(U) \not\leq H_0$. Then $J(U) = D$, and if $Y \leq U$, then $Y \leq Z(D)$.

Observe that $W \leq N_H(U) \leq N_H(UO_p(H))$. Hence $N_H(UO_p(H)) \not\leq H_0$ and $J(UO_p(H)) \in \mathcal{W}$. Since $D \leq U \leq UO_p(H)$, the maximal choice of D gives $D = J(UO_p(H)) = J(U)$.

Suppose that $Y \leq U$. Then $J(U) = D \leq T_1 \leq N_H(Y)$, and so by Lemma 2.4(a), $Y \leq Z(J(U)) = Z(D)$.

(8°) $J(T) \neq D$ and $J(T_1) \neq D$.

Suppose $J(T) = D$. Then by Lemma 5.3 and Hypothesis 5.1(i), $N \leq N_G(J(T)) \leq N_G(Y)$, and so $N \leq N_H(C_{O_p(H)}(Y)) \leq H_0$, contrary to the choice of D .

Suppose $J(T_1) = D$. Then $N_T(T_1) \leq N_H(J(T_1)) = N$, so $N_T(T_1) \leq T_1$, $T = T_1$ and $J(T) = J(T_1) = D$, a contradiction.

(9°) Let U be a p -subgroup of H_0 containing WD . Suppose that $J(U) \neq D$ or $Y \not\leq Z(D)$. Then U is not contained in any H -conjugate of H_0 other than H_0 .

Let $g \in H$ with $U \leq H_0^g$ and $U \leq T_2 \in \text{Syl}_p(H_0 \cap H_0^g)$. If $J(T_2) = D$, then also $D = J(U)$ and thus $Y \leq Z(D)$. Thus either $J(T_2) \neq D$ or $Y \not\leq Z(D)$. So (7°) gives $N_H(T_2) \leq H_0$. This implies $N_{H_0^g}(T_2) \leq H_0 \cap H_0^g$, and so $T_2 \in \text{Syl}_p(H_0^g)$. By (4°), H_0^g is the unique maximal subgroup of H containing T_2 . Since $T_2 \leq H_0$ we get $H_0 = H_0^g$.

(10°) $T_1 \in \text{Syl}_p(N)$, $J(T_1) \not\leq O_p(N)$, and $WJ(T_1)D$ is not contained in any other H -conjugate of H_0 .

By (8°) and (7°) $N_H(J(T_1)D) \leq H_0$, so $N_N(T_1) \leq N \cap H_0$ and $T_1 \in \text{Syl}_p(N)$. If $J(T_1) \leq O_p(N)$, then $J(T_1) = J(O_p(N))$ and $N \leq N_H(J(T_1)D) \leq H_0$, a contradiction.

Put $U := WJ(T_1)D$. By (8°), $J(U) \neq D$, and so the last statement in (10°) follows from (9°).

(11°) There exists a $WJ(T_1)T_0$ -minimal subgroup $H_1 \leq N$ such that $H_1 \cap H_0$ is a maximal subgroup of H_1 and $J(O_p(H_1)) = D$.

By definition of $\mathcal{W}$, $N \not\leq H_0$. Choose $WJ(T_1)T_0 \leq H_1 \leq N$ such that H_1 is minimal with $H_1 \not\leq H_0$. Since $H_1 \not\leq H_0$, $N_H(O_p(H_1)) \not\leq H_0$. Also $WO_p(H)$ normalizes $O_p(H_1)$, and $D \leq O_p(H_1)$. So by (7°) $J(O_p(H_1)) = D$. Since $J(T_1) \neq D$ by (8°) we conclude $J(T_1) \not\leq O_p(H_1)$. Hence also $WJ(T_1)T_0 \not\leq O_p(H_1)$, and H_1 is $WJ(T_1)T_0$ -minimal.

In the following let H_1 be as in (11°). Pick $WJ(T_1)T_0 \leq T_3 \in \text{Syl}_p(H_1 \cap H_0)$. Then H_1 is T_3 -minimal, and so $T_3 \in \text{Syl}_p(H_1)$. Since $T_3 \leq N \cap H_0$ and $T_1 \in \text{Syl}_p(H_0 \cap N)$, there exists $g \in N \cap H_0$ with $J(T_1) \leq T_3 \leq T_1^g$. Hence g normalizes $J(T_1)$, D and W , and thus also $WJ(T_1)T_0$. So replacing H_1 by H_1^g and T_3 by T_3^g we may assume that $T_3 \leq T_1 \leq T$.

Case 1. The case $Y \leq O_p(H_1)$.

(12°) Let $\overline{H}_1 := H_1/O_{p'}(H_1)$. Then $\overline{Y}$ and $\overline{H}_1$ satisfy the hypotheses of Lemma 4.8.

Since $T_3 \leq T_1$, $Y \trianglelefteq T_3$. On the other hand, $Y \not\leq H_1$, and by (2°), $C_{H_1}(O_p(H_1)) \leq C_{H_1}(Y) \leq H_0$. Hence Lemma 2.3 shows that $\overline{H}_1$ is of characteristic p . By (5°) and (6°) $Y \leq Z(WJ(T_1))$, so $\overline{H}_1$ satisfies Hypothesis 4.1. Hence (8°) and (11°) give Hypothesis 4.8(iii), while Hypothesis 2.4(a) gives Hypothesis 4.8(ii).

Assume that $C_{O_p(\overline{H}_1)}(\overline{Y}) \trianglelefteq \overline{H}_1$. Then $C_{O_p(H_1)}(Y) \trianglelefteq H_1$, and since $O_p(H) \leq O_p(H_1)$ also $C_{O_p(H)}(Y) \trianglelefteq H_1$, which contradicts $H_1 \not\leq H_0$. Hence also Hypothesis 4.8(i) holds.

According to (12°) we are allowed to apply Lemma 4.8 to $\overline{Y}$ and $\overline{H}_1$. Let $\overline{L}$ be with the properties given there. Since $Y \leq O_p(L) \leq T_1$, we get from Lemma 2.4(a) that $Y \leq Z(J(O_p(L)))$. Thus, by Lemma 4.8(d) $\overline{L}$ and $\overline{Y}$ satisfy the hypothesis of Lemma 4.7.

Since Y and thus also $\overline{Y}$ is $2F$ -stable, we are in case Lemma 4.7(b), so $0 \neq q(\overline{Y}, O_p(\overline{H}_1)) \leq 2$. Hence there exists a non-trivial quadratic $2F$ -offender on Y , and the lemma is proved in Case 1.

Case 2. The case $Y \not\leq O_p(H_1)$.

By our assumption on H , in this case H has characteristic p . Hence also H_1 has characteristic p since $O_p(H) \leq H_1$. We now apply the L -Lemma, Lemma 4.3, with W and H_1 in place of A and P . Then there exists $WO_p(H_1) \leq L$ such that

- (i) L is $WO_p(L)$ -minimal, and
- (ii) there exists $g \in H_1$ such that $L_0 := H_0^g \cap L$ is the unique maximal subgroup of L containing $WO_p(L)$.

(13°) $L_0 = L \cap H_0$, $Y \not\leq Z(D)$ and $J(O_p(L)) = D$.

Let g be as in (ii). Then $WD \leq H_0 \cap H_0^g$ and $Y \not\leq Z(D)$ since $D \leq O_p(H_1)$. Hence (9°) implies $H_0 = H_0^g$; in particular $L_0 = L \cap H_0$ and $L \not\leq H_0$. Now (7°) also gives $J(O_p(L)) = D$.

According to (13°) we may assume, after conjugation by a suitable element of $H_0 \cap H_1$, that

(14°) $WO_p(L) \leq T \cap L \in \text{Syl}_p(L_0)$. In particular, $O_p(L) \leq T$ and $O_p(L)$ normalizes Y .

By (13°), $Y \not\leq ZJ(O_p(L))$. Since $O_p(L)$ normalizes Y , we get from Lemma 2.4 that

(15°) $Y \not\leq O_p(L)$.

Put

$$A := W, \quad B := Y, \quad X := O_p(L) \cap A, \quad E := \langle X^L \rangle, \quad Z := O_p(L) \cap B.$$

By (5°) A is abelian, and by (14°) $O_p(L)$ normalizes A and B . Moreover, since $O_p(H_1) \leq O_p(L)$ and H_1 has characteristic p , $[E, O^p(L)] \neq 1$. It follows that the hypotheses of Lemma 4.5 are satisfied.

By Lemma 4.5(a), $C_Y(E) \leq Y \cap O_p(L)$, and so by Lemma 4.5(c), $[b, Y]C_Y(E) = [E, Y]C_Y(E) = Y \cap O_p(L)$ for all $b \in Y \setminus O_p(L)$. Moreover, by Lemma 4.5(d) $[Y, E, E, E] = 1$ and by Lemma 4.5(e), (f) we have $|Y/C_Y(E)| \leq |E/C_E(Y)|^2$. So E is a nearly quadratic $2F$ -offender on Y . Hence the lemma also holds in Case 2. $\square$

Lemma 5.5. *Assume Hypothesis 5.1. Suppose that $Y \not\leq H$ and Y is $2F$ -stable. Then $\Omega_1 Z(T) \not\leq H$.*

Proof. Let $T \leq P \leq H$ and P be minimal with $Y \not\leq P$. By Lemma 5.3, $N_H(T) \leq N_H(Y)$, so $T \not\leq P$ and P is T -minimal. Put

$$Q := C_{O_p(P)}(Y), \quad V_0 := \Omega_1(Z(Q)), \quad V := C_{V_0}(O_p(P)), \quad \overline{P} := P/C_P(V).$$

If $Z(T) \not\leq O_p(P)$, $Z(T) \not\leq P$. So we may assume $\Omega_1 Z(T) \leq O_p(P)$ and thus $\Omega_1 Z(T) = C_V(T)$. By Theorem 5.4, $Q \leq P$. Since either $Y \leq O_p(H) \leq O^p(P)$ or P is of characteristic p , Lemma 2.5 implies $Y \leq O_p(P)$. Thus $Y \leq V_0$. Since $Y \not\leq P$, we get that $[V_0, O^p(P)] \neq 1$. By Lemma 5.3, $J(R) = J(T)$, and so by Hypothesis 5.1(i), $J(R) \not\leq O_p(P)$. Hence Lemma 2.3 shows that $[O^p(P), J(R)] = O^p(P)$. Since $J(R)$ centralizes Y , $[O_p(P), J(R)] \leq O_p(P) \cap J(T) \leq Q$, and so $[O_p(P), O^p(P)] \leq Q$. The $P \times Q$ -Lemma yields $[V, O^p(P)] \neq 1$.

Again Lemma 2.3 gives $C_T(V) = O_p(P)$ and $O_p(\overline{P}) = 1$. Moreover $\overline{J(T)} \neq 1$ since $J(R) \not\leq O_p(P)$. Hence $\overline{P}$ and V satisfy the hypothesis of [BHS, 5.6]. It follows that $[C_V(T), P] \neq 1$. Since $C_V(T) = \Omega_1 Z(T)$ and $P = \langle T^P \rangle$ we conclude that $\Omega_1 Z(T) \not\leq P$, and so also $Z(T) \not\leq H$. $\square$

6. THE PROOF OF THEOREMS 1.5, 1.6, 1.8 AND COROLLARY 1.7

Recall that Theorems 1.3 and 1.4 have been proved in Section 3.

Proof of Theorem 1.5. (a): Observe that $N = N_G(Y_N)$ by Theorem 1.3. Suppose $Y_N \leq O_p(M)$. If $M = N_G(C)$ for $1 \neq C \in \text{char } B(S)$, then Theorem 1.5(a) follows from Theorem 1.4(b). If $\Omega_1 Z(S) \leq M$, then Lemma 5.2(a) shows that Y_N and M satisfy Hypothesis 5.1. Hence Lemma 5.5 gives $Y_N \leq M$, and so $M \leq N$.

(b): Put $\overline{M} := M/O_{p'}(M)$. Then Lemma 5.2(b) shows that $\overline{Y_N}$ and $\overline{M}$ satisfy Hypothesis 5.1. Thus Lemma 5.5 gives $\overline{Y_N} \trianglelefteq \overline{M}$. By the Frattini argument $M = O_{p'}(M)N_M(Y_N) = O_{p'}(M)(M \cap N)$.

(c): Let $B(S) \leq H \leq G$ and H be p -constrained with $H \neq O_{p'}(H)(H \cap N)$, and let $B(S) \leq T \in \text{Syl}_p(N_H(Y_N))$. Put $\overline{H} := H/O_{p'}(H)$. Then again Lemma 5.2(b) shows that $\overline{Y_N}$ and $\overline{H}$ satisfy Hypothesis 5.1. Hence by Theorem 5.4, $C_{O_p(\overline{H})}(\overline{Y_N}) \trianglelefteq H$, and by Lemma 2.5, $\overline{Y_N} \leq O_p(\overline{H})$. From Lemma 5.5 applied to $N_{\overline{H}}(\Omega_1 Z(\overline{T}))$ we get $\overline{Y_N} \trianglelefteq N_{\overline{H}}(\Omega_1 Z(\overline{T}))$. Recall that $Y_N \leq \Omega_1 Z(J(S)) \leq O_p(N_G(J(S)))$. Thus by Theorem 1.4(b), $N_G(J(S)) \leq N$. Since $B(T) = B(S)$ we have $J(T) = J(S)$, and so $\overline{Y_N} \trianglelefteq C^{**}(\overline{H}, \overline{T})$. By the Frattini argument $N_{\overline{H}}(\overline{Y_N}) = \overline{N_H(Y_N)} = \overline{H \cap N}$. Hence also $C^{**}(\overline{H}, \overline{T}) \leq \overline{H \cap N}$.

Proof of Theorem 1.6. Let $P \in \mathcal{F}(S)$. By Theorem 1.3 P is a p -local subgroup of G . Let L be a maximal p -local subgroup containing P . By Lemma 2.2(c) $Y_P \leq Y_L$, and so $P \ll L$. Hence by Lemma 3.4 $P = L$.

Suppose that $N \in \mathcal{F}(S)$ is $2F$ -stable. Let $M = N_G(C)$ for $1 \neq C \in \text{char } B(S)$ or $M = N_G(\Omega_1 Z(S))$. Then $S \leq M$, so M has characteristic p since G is of parabolic characteristic p . Hence Theorem 1.5(b) implies $M \leq N$.

Let $H \in \mathcal{S}(B(S))$ and $B(S) \leq T \in \text{Syl}_p(H)$. Then $B(S) = B(T)$, and so $N_H(C) \leq N$ for $1 \neq C \in \text{char } B(T)$. Also $T \leq S^g$ for some $g \in N_G(B(S)) \leq N$ and $\Omega_1 Z(S^g) \leq J(S) \leq T$, so $\Omega_1 Z(S^g) \leq Z(T)$ and $C_H(\Omega_1 Z(T)) \leq C_G(\Omega_1 Z(S^g)) \leq N^g = N$. Thus $C^*(H, T) \leq H \cap N$.

Proof of Corollary 1.7. By Theorem 1.6 the members of $\mathcal{F}(S)$ are maximal p -local subgroups. We may assume that there exists a $2F$ -stable $N \in \mathcal{F}(S)$.

Let L be a maximal p -local subgroup containing S with $L \not\leq N$ and choose $M \in \mathcal{F}(S)$ with $L \ll M$. Then $L \leq C_G(Y_L)M$. On the other hand, by Lemma 2.2(c), $\Omega_1 Z(S) \leq Y_L$, and so by Theorem 1.6 $C_G(Y_M) \leq C_G(\Omega_1 Z(S)) \leq N$. Since $L \not\leq N$ we conclude that $M \not\leq N$ and $M \neq N$. By Theorem 1.4 N is the only member of $\mathcal{F}(S)$ which is F -stable. Hence M is not F -stable.

Proof of Theorem 1.8. Let P be the semi-direct product of G and V . Then $O_p(P) = V$ and $[V, O^p(P)] \neq 1$. Let A be an offender on V such that $|A| |C_V(A)|$ is maximal. Because of [KS, 9.2.3] we may assume that A is quadratic on V . Hence Theorem 4.6 implies $|A/C_A(V)| = |V|$, and Theorem 1.8 follows.

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