RESIDUAL p PROPERTIES OF MAPPING CLASS GROUPS AND SURFACE GROUPS

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ABSTRACT. Let $\mathcal{M}(\Sigma, \mathcal{P})$ be the mapping class group of a punctured oriented surface $(\Sigma, \mathcal{P})$ (where $\mathcal{P}$ may be empty), and let $\mathcal{I}_p(\Sigma, \mathcal{P})$ be the kernel of the action of $\mathcal{M}(\Sigma, \mathcal{P})$ on $H_1(\Sigma \setminus \mathcal{P}, \mathbb{F}_p)$. We prove that $\mathcal{I}_p(\Sigma, \mathcal{P})$ is residually p . In particular, this shows that $\mathcal{M}(\Sigma, \mathcal{P})$ is virtually residually p . For a group G we denote by $\mathcal{I}_p(G)$ the kernel of the natural action of $\text{Out}(G)$ on $H_1(G, \mathbb{F}_p)$. In order to achieve our theorem, we prove that, under certain conditions (G is conjugacy p -separable and has Property A), the group $\mathcal{I}_p(G)$ is residually p . The fact that free groups and surface groups have Property A is due to Grossman. The fact that free groups are conjugacy p -separable is due to Lyndon and Schupp. The fact that surface groups are conjugacy p -separable is, from a technical point of view, the main result of the paper.

1. INTRODUCTION

A group G is said to be *residually nilpotent* if for every $g \in G \setminus \{1\}$ there exists a homomorphism $\varphi : G \rightarrow N$ such that N is a nilpotent group and $\varphi(g) \neq 1$. For $g, h \in G$, we denote by $[g, h] = g^{-1}h^{-1}gh$ the *commutator* of g and h , and, if K, H are two subgroups of G , we denote by $[K, H]$ the *commutator subgroup* of K and H . This is the subgroup generated by $\{[g, h]; g \in K \text{ and } h \in H\}$, and it is normal (resp. characteristic) if both K and H are normal (resp. characteristic). By a *filtration* of a group G we mean an infinite descending chain

$$G = G_1 \triangleright G_2 \triangleright G_3 \triangleright \cdots \triangleright G_n \triangleright \cdots$$

of normal subgroups. We say that $\{G_n\}_{n=1}^{+\infty}$ is a *central filtration* if $[G_n, G_m] \subset G_{m+n}$ for all $m, n \geq 1$, and we say that $\{G_n\}_{n=1}^{+\infty}$ is a *separating filtration* if $\bigcap_{n=1}^{+\infty} G_n = \{1\}$. It is well-known that a group G is residually nilpotent if and only if it admits a separating central filtration. A central filtration $\{G_n\}_{n=1}^{+\infty}$ is called $\mathbb{Z}$ -linear if G_n/G_{n+1} is a free $\mathbb{Z}$ -module for all $n \geq 1$.

Let Σ_ρ be an oriented closed surface of genus $\rho \geq 1$, let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a (possibly empty) finite collection of punctures in Σ_ρ , and let $\text{Homeo}(\Sigma_\rho, \mathcal{P})$ be the group of homeomorphisms $h : \Sigma_\rho \rightarrow \Sigma_\rho$ which preserve the orientation and such that $h(\mathcal{P}) = \mathcal{P}$. The *mapping class group* of $(\Sigma_\rho, \mathcal{P})$ is defined to be the group $\mathcal{M}(\Sigma_\rho, \mathcal{P}) = \pi_0(\text{Homeo}(\Sigma_\rho, \mathcal{P}))$ of isotopy classes of elements of $\text{Homeo}(\Sigma_\rho, \mathcal{P})$. The natural action of $\mathcal{M}(\Sigma_\rho, \mathcal{P})$ on $H_1(\Sigma_\rho \setminus \mathcal{P}, \mathbb{Z})$ leads to a homomorphism $\tau : \mathcal{M}(\Sigma_\rho, \mathcal{P}) \rightarrow GL(H_1(\Sigma_\rho \setminus \mathcal{P}, \mathbb{Z}))$ whose kernel is the *Torelli group* $\mathcal{T}(\Sigma_\rho, \mathcal{P})$.

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The following result is more or less stated in [8] but, as far as I know, its first complete proof is given in [3] (see also [6]).

Theorem 1.1. *The Torelli group $\mathcal{T}(\Sigma_\rho, \mathcal{P})$ has a $\mathbb{Z}$ -linear separating central filtration. In particular, $\mathcal{T}(\Sigma_\rho, \mathcal{P})$ is residually nilpotent.*

The filtration of Theorem 1.1 is actually the well-studied Johnson filtration.

Let $\mathcal{F} = \{G_n\}_{n=1}^{+\infty}$ be a central filtration. Then the group commutator induces on

$$\mathcal{L}_{\mathcal{F}}(G) = \bigoplus_{n=1}^{+\infty} \frac{G_n}{G_{n+1}}$$

a Lie bracket under which it becomes a graded Lie $\mathbb{Z}$ -algebra. If, as in Theorem 1.1, the filtration is $\mathbb{Z}$ -linear, then $\mathcal{L}_{\mathcal{F}}(G)$ is torsion free and one can apply the Poincaré-Birkhoff-Witt theorem to embed $\mathcal{L}_{\mathcal{F}}(G)$ into its enveloping algebra.

Theorem 1.1 also implies that $\mathcal{T}(\Sigma_\rho, \mathcal{P})$ is biorderable (see [19]), which implies many other properties (see [17], [5]).

Let p be a prime number. Our goal in this paper is to prove a result analogous to Theorem 1.1, where the Torelli group is replaced by the congruence subgroup $\mathcal{T}_p(\Sigma_\rho, \mathcal{P})$, and the $\mathbb{Z}$ -linear filtrations by the $\mathbb{F}_p$ -linear filtrations.

We say that a group G is *residually p* if for all $g \in G \setminus \{1\}$ there exists a homomorphism $\varphi : G \rightarrow P$ such that P is a finite p -group and $\varphi(g) \neq 1$. Note that residually p -groups are residually nilpotent as well as residually finite. A central filtration $\mathcal{F} = \{G_n\}_{n=1}^{+\infty}$ of a group G is called *$\mathbb{F}_p$ -linear* if G_n/G_{n+1} is a finite dimensional $\mathbb{F}_p$ -vector space for all $n \geq 1$. Note that, in that case, $\mathcal{L}_{\mathcal{F}}(G)$ becomes a Lie $\mathbb{F}_p$ -algebra, and, therefore, one can apply the Poincaré-Birkhoff-Witt theorem as well.

Let Σ_ρ be a closed surface of genus $\rho \geq 1$, and let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a finite collection of punctures in Σ_ρ . The natural action of $\mathcal{M}(\Sigma_\rho, \mathcal{P})$ on $H_1(\Sigma_\rho \setminus \mathcal{P}, \mathbb{F}_p)$ leads to a homomorphism $\tau_p : \mathcal{M}(\Sigma_\rho, \mathcal{P}) \rightarrow GL(H_1(\Sigma_\rho \setminus \mathcal{P}, \mathbb{F}_p))$ whose kernel is the *congruence subgroup* $\mathcal{T}_p(\Sigma_\rho, \mathcal{P})$. This is a finite index subgroup of $\mathcal{M}(\Sigma_\rho, \mathcal{P})$, and is torsion free if $p \neq 2$ (see [10]).

Our main result is the following.

Theorem 1.2. *$\mathcal{T}_p(\Sigma_\rho, \mathcal{P})$ is residually p .*

Remark. We will show in Section 2 (see Proposition 2.3) that a finitely generated group which is residually p admits an $\mathbb{F}_p$ -linear separating central filtration. On the other hand, $\mathcal{T}_p(\Sigma_\rho, \mathcal{P})$ is a finite index subgroup of $\mathcal{M}(\Sigma_\rho, \mathcal{P})$ which is finitely generated, thus $\mathcal{T}_p(\Sigma_\rho, \mathcal{P})$ is finitely generated. So, by Theorem 1.2, $\mathcal{T}_p(\Sigma_\rho, \mathcal{P})$ admits an $\mathbb{F}_p$ -linear separating central filtration.

The proof of Theorem 1.2 is essentially algebraic and uses the following result known as the Dehn-Nielsen-Baer theorem (see [11], [23]).

Theorem 1.3 (Dehn, Nielsen [18], Baer [2], Magnus [15], Zieschang [22]). *Let Σ_ρ be a closed oriented surface of genus $\rho \geq 1$, and let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a finite collection of punctures in Σ_ρ . Then $\mathcal{M}(\Sigma_\rho, \mathcal{P})$ embeds in $\text{Out}(\pi_1(\Sigma_\rho \setminus \mathcal{P}))$. Moreover, if $\mathcal{P} = \emptyset$, then $\mathcal{M}(\Sigma_\rho) = \mathcal{M}(\Sigma_\rho, \emptyset)$ is an index 2 subgroup of $\text{Out}(\pi_1(\Sigma_\rho))$.*

For a group G , we denote by $I_p(G)$ the kernel of the natural homomorphism $\text{Aut}(G) \rightarrow GL(H_1(G, \mathbb{F}_p))$, and by $\mathcal{I}_p(G)$ the kernel of the natural homomorphism $\text{Out}(G) \rightarrow GL(H_1(G, \mathbb{F}_p))$. Note that $\text{Inn}(G) \subset I_p(G)$ and $\mathcal{I}_p(G) = I_p(G)/\text{Inn}(G)$. Then Theorem 1.2 is a direct consequence of the following.

Theorem 1.4. *Suppose G is either a free group or the fundamental group of a closed oriented surface. Then $I_p(G)$ and $\mathcal{I}_p(G)$ are both residually p .*

The fact that the outer automorphism group of a free group is virtually residually p has been previously proved by Lubotzky [13]. He did not give explicitly any finite index subgroup which is residually p , but I suspect he knew that $\mathcal{I}_p(F)$ is an eligible one. The fact that $I_p(G)$ is residually p is more or less known to experts and follows from the following theorem which is proved in Section 2 (see Theorem 2.4).

Theorem 1.5. *Let G be a finitely generated and residually p group. Then $I_p(G)$ is residually p .*

Now, in order to prove that $\mathcal{I}_p(G)$ is residually p for a given group G , we require some restrictions on G .

For $g, h \in G$, we use the notation $g \sim h$ to mean that g is conjugate to h . We say that a group G is *conjugacy p -separable* if for all $g, h \in G$, either $g \sim h$ or there exists a homomorphism $\varphi : G \rightarrow P$ such that P is a finite p -group and $\varphi(g) \not\sim \varphi(h)$. We say that G has *Property A* if for every automorphism $\alpha \in \text{Aut}(G)$ such that $\alpha(g) \sim g$ for all $g \in G$, we have $\alpha \in \text{Inn}(G)$.

In Section 2 we prove the following.

Theorem 1.6. *Let G be a finitely generated group. If G is conjugacy p -separable and has Property A, then $\mathcal{I}_p(G)$ is residually p .*

The fact that free groups and fundamental groups of oriented closed surfaces have Property A is proved in [7]. The fact that free groups are conjugacy p -separable is stated in [14], Proposition 4.8. However, the proof given in [14] is imprecise and incomplete. We give a complete and detailed proof of this result in Section 3. In Section 4 we prove the following.

Theorem 1.7. *The fundamental group of a closed oriented surface is conjugacy p -separable.*

From a technical point of view, Theorem 1.7 is the main result of the paper.

2. AUTOMORPHISM GROUPS AND OUTER AUTOMORPHISM GROUPS OF RESIDUALLY p GROUPS

We fix a prime number $p \geq 2$. If K and H are two subgroups of a group G , we denote by $[K, H]^p$ the subgroup generated by $\{[g, h]; g \in K \text{ and } h \in H\} \cup \{h^p; h \in H\}$. Note that we do not have $[K, H]^p = [H, K]^p$ in general even if K and H are both normal. The *lower $\mathbb{F}_p$ -linear central filtration* of G is defined to be the filtration $\{\gamma_n^p G\}_{n=1}^{+\infty}$, where $\gamma_1^p G = G$ and $\gamma_{n+1}^p G = [G, \gamma_n^p G]^p$ for $n \geq 1$. Note that $G/\gamma_2^p G = H_1(G, \mathbb{F}_p)$.

The term “central” in the above definition can be used because of the following.

Lemma 2.1. *Let $n, m \geq 1$. Then $[\gamma_m^p G, \gamma_n^p G] \subset \gamma_{n+m}^p G$.*

Proof. We argue by induction on m . If $m = 1$, then

$$[\gamma_m^p G, \gamma_n^p G] = [G, \gamma_n^p G] \subset [G, \gamma_n^p G]^p = \gamma_{n+1}^p G.$$

Assume $m > 1$. We need to prove that $[g, h] \in \gamma_{m+n}^p G$ for every generator g of $\gamma_m^p G$ and every element $h \in \gamma_n^p G$. There are two different cases to consider: (1) $g \in [G, \gamma_{m-1}^p G]$; (2) $g = g_1^p$ for some $g_1 \in \gamma_{m-1}^p G$.

Suppose $g \in [G, \gamma_{m-1}^p G]$. By [16], Theorem 5.2, we have

$$[[G, \gamma_{m-1}^p G], \gamma_n^p G] \subset [[\gamma_{m-1}^p G, \gamma_n^p G], G] \cdot [\gamma_n^p G, G], \gamma_{m-1}^p G].$$

By induction we have

$$\begin{aligned} [[\gamma_{m-1}^p G, \gamma_n^p G], G] &\subset [\gamma_{m+n-1}^p G, G] \subset \gamma_{m+n}^p G, \\ [[\gamma_n^p G, G], \gamma_{m-1}^p G] &\subset [\gamma_{n+1}^p G, \gamma_{m-1}^p G] \subset \gamma_{m+n}^p G, \end{aligned}$$

thus

$$[g, h] \in [[G, \gamma_{m-1}^p G], \gamma_n^p G] \subset \gamma_{m+n}^p G.$$

Now, we assume that $g = g_1^p$ for some $g_1 \in \gamma_{m-1}^p G$. Let g_2 be another element of $\gamma_{m-1}^p G$. We have

$$[g_1 g_2, h] = [g_1, h] \cdot [[g_1, h], g_2] \cdot [g_2, h],$$

and, by induction,

$$[[g_1, h], g_2] \in [[\gamma_{m-1}^p G, \gamma_n^p G], \gamma_{m-1}^p G] \subset [\gamma_{m+n-1}^p G, \gamma_{m-1}^p G] \subset \gamma_{2m+n-2}^p G \subset \gamma_{m+n}^p G,$$

thus

$$[g_1 g_2, h] \equiv [g_1, h] \cdot [g_2, h] \pmod{\gamma_{m+n}^p G}.$$

Iterating this congruence we obtain

$$[g, h] = [g_1^p, h] \equiv ([g_1, h])^p \pmod{\gamma_{m+n}^p G},$$

thus $[g, h] \in \gamma_{m+n}^p G$ since $[g_1, h] \in [\gamma_{m-1}^p G, \gamma_n^p G] \subset \gamma_{m+n-1}^p G$. $\square$

The proof of the following is left to the reader.

Lemma 2.2. *Let P be a finitely generated group. Then P is a finite p -group if and only if there exists some $n \geq 1$ such that $\gamma_n^p P = \{1\}$.*

By a p -filtration of a group G we mean a filtration $\{G_n\}_{n=1}^{+\infty}$ such that G/G_n is a finite p -group for all $n \geq 1$. Note that the p -filtrations that will appear in the proofs of Theorems 2.4 and 2.5 are actually $\mathbb{F}_p$ -linear filtrations, but this is not needed for our purpose. Recall also that a filtration $\{G_n\}_{n=1}^{+\infty}$ is said to be a *separating filtration* if $\bigcap_{n=1}^{+\infty} G_n = \{1\}$.

The main criterion we will use in the proofs of Theorems 2.4 and 2.5 are the following.

Proposition 2.3. (1) *Let G be a (discrete) group. If G has a separating p -filtration, then G is residually p .*

(2) *Let G be a finitely generated group. Then G is residually p if and only if the lower $\mathbb{F}_p$ -linear central filtration $\{\gamma_n^p G\}_{n=1}^{+\infty}$ is a separating filtration.*

Remark. It is easily seen that $\gamma_n^p G$ is finitely generated for all $n \geq 1$ and that $\{\gamma_n^p G\}_{n=1}^{+\infty}$ is a $\mathbb{F}_p$ -linear filtration if G is finitely generated. This is not necessarily true anymore if G is not finitely generated.

Proof. Part (1) is easy to prove and is left to the reader. So, we assume that G is finitely generated and turn to prove part (2). If $\{\gamma_n^p G\}_{n=1}^{+\infty}$ is separating, then G is residually p because $\{\gamma_n^p G\}_{n=1}^{+\infty}$ is a p -filtration (apply part (1)). Suppose that G is residually p . Let $g \in G \setminus \{1\}$. There exists a homomorphism $\varphi : G \rightarrow P$ such that P is a finite p -group and $\varphi(g) \neq 1$. By Lemma 2.2, there exists $n \geq 1$ such that $\gamma_n^p P = \{1\}$. We have $\varphi(\gamma_n^p G) \subset \gamma_n^p P = \{1\}$, thus $g \notin \gamma_n^p G$ since $\varphi(g) \neq 1$. This shows that $\{\gamma_n^p G\}_{n=1}^{+\infty}$ is a separating filtration. $\square$

Theorem 2.4. *Let G be a finitely generated group. If G is residually p , then $I_p(G)$ is residually p .*

Proof. We consider the lower $\mathbb{F}_p$ -linear central filtration $\{\gamma_n^p G\}_{n=1}^{+\infty}$. Since $\gamma_{n+1}^p G$ is a characteristic subgroup of G , the quotient map $G \rightarrow G/\gamma_{n+1}^p G$ determines a homomorphism

$$\pi_n : \text{Aut} G \rightarrow \text{Aut}(G/\gamma_{n+1}^p G).$$

We set $A_n = \text{Ker} \pi_n$ for all $n \geq 1$ and turn to prove that $\{A_n\}_{n=1}^{+\infty}$ is a separating p -filtration of $A_1 = I_p(G)$.

Claim 1. A_n acts trivially on $\gamma_k^p G / \gamma_{n+k}^p G$ for all $k \geq 1$.

Proof of Claim 1. We argue by induction on k . The case $k = 1$ follows from the definition of A_n , so we can assume $k > 1$. We take $\alpha \in A_n$, and turn to prove that $\alpha(g) \equiv g \pmod{\gamma_{n+k}^p G}$ for every generator g of $\gamma_k^p G$.

Suppose first that $g = [g_1, h]$ for some $g_1 \in \gamma_{k-1}^p G$ and $h \in G$. By induction, there exist $x_1 \in \gamma_{k+n-1}^p G$ and $y \in \gamma_{n+1}^p G$ such that $\alpha(g_1) = g_1 x_1$ and $\alpha(h) = hy$. By Lemma 2.1,

$$\begin{aligned} [g_1 x_1, y] &\in [\gamma_{k-1}^p G, \gamma_{n+1}^p G] \subset \gamma_{k+n}^p G, \\ [[g_1 x_1, h], y] &\in [[\gamma_{k-1}^p G, G], \gamma_{n+1}^p G] \subset [\gamma_k^p G, \gamma_{n+1}^p G] \subset \gamma_{k+n+1}^p G \subset \gamma_{k+n}^p G, \end{aligned}$$

thus

$$\alpha(g) = [g_1 x_1, hy] = [g_1 x_1, h] \cdot [g_1 x_1, y] \cdot [[g_1 x_1, h], y] \equiv [g_1 x_1, h] \pmod{\gamma_{k+n}^p G}.$$

Again, by Lemma 2.1,

$$\begin{aligned} [[g_1, h], x_1] &\in [[\gamma_{k-1}^p G, G], \gamma_{k+n-1}^p G] \subset [\gamma_k^p G, \gamma_{k+n-1}^p G] \subset \gamma_{2k+n-1}^p G \subset \gamma_{k+n}^p G, \\ [x_1, h] &\in [\gamma_{k+n-1}^p G, G] \subset \gamma_{k+n}^p G, \end{aligned}$$

thus

$$[g_1 x_1, h] = [g_1, h] \cdot [[g_1, h], x_1] \cdot [x_1, h] \equiv [g_1, h] \pmod{\gamma_{k+n}^p G}.$$

This shows that $\alpha(g) \equiv g \pmod{\gamma_{k+n}^p G}$.

Now, we assume that $g = g_1^p$ for some $g_1 \in \gamma_{k-1}^p G$. Let g_2 be another element of $\gamma_{k-1}^p G$. By induction, there exist $x_1, x_2 \in \gamma_{k+n-1}^p G$ such that $\alpha(g_1) = g_1 x_1$ and $\alpha(g_2) = g_2 x_2$. By Lemma 2.1,

$$[g_2, x_1^{-1}] \in [\gamma_{k-1}^p G, \gamma_{k+n-1}^p G] \subset \gamma_{2k+n-2}^p G \subset \gamma_{k+n}^p G \quad (\text{since } k \geq 2),$$

thus

$$\alpha(g_1 g_2) = g_1 x_1 g_2 x_2 = g_1 g_2 [g_2, x_1^{-1}] x_1 x_2 \equiv g_1 g_2 x_1 x_2 \pmod{\gamma_{k+n}^p G}.$$

Iterating the above congruence we obtain

$$\alpha(g) = \alpha(g_1^p) \equiv g_1^p x_1^p \equiv g_1^p \pmod{\gamma_{k+n}^p G}.$$

Claim 2. The filtration $\{A_n\}_{n=1}^{+\infty}$ is a separating filtration.

Proof of Claim 2. Let $\alpha \in I_p(G)$, $\alpha \neq \text{Id}$. Choose $g \in G$ such that $\alpha(g) \neq g$. Since $\{\gamma_n^p G\}_{n=1}^{+\infty}$ is a separating filtration, there exists $n \geq 1$ such that $g^{-1} \alpha(g) \notin \gamma_{n+1}^p G$; that is, $\alpha(g) \not\equiv g \pmod{\gamma_{n+1}^p G}$. This shows that $\alpha \notin A_n$.

Claim 3. Let $\alpha \in A_n$. Then the map

$$\begin{aligned} u_\alpha : G/\gamma_{n+1}^p G &\rightarrow \gamma_{n+1}^p G/\gamma_{n+2}^p G, \\ [g] &\mapsto [g^{-1}\alpha(g)] \end{aligned}$$

is well-defined.

Proof of Claim 3. Let $g_1, g_2 \in G$ such that $g_1 \equiv g_2 \pmod{\gamma_{n+1}^p G}$. Let $x_1, x_2 \in \gamma_{n+1}^p G$ such that $\alpha(g_1) = g_1 x_1$ and $\alpha(g_2) = g_2 x_2$. We have to prove that $x_1 \equiv x_2 \pmod{\gamma_{n+2}^p G}$. Set $a = g_1^{-1} g_2 \in \gamma_{n+1}^p G$. By Claim 1, $\alpha(a)$ is of the form $\alpha(a) = ay$, where $y \in \gamma_{2n+1}^p G \subset \gamma_{n+2}^p G$. We have

$$[a, x_1^{-1}] \in [\gamma_{n+1}^p G, \gamma_{n+1}^p G] \subset \gamma_{2n+2}^p G \subset \gamma_{n+2}^p G$$

and

$$\alpha(g_2) = g_1 x_1 a y = g_1 a [a, x_1^{-1}] x_1 y = g_2 [a, x_1^{-1}] x_1 y,$$

thus

$$x_2 = [a, x_1^{-1}] x_1 y \equiv x_1 \pmod{\gamma_{n+2}^p G}.$$

We denote by H_n the set of set-maps from $G/\gamma_{n+1}^p G$ to $\gamma_{n+1}^p G/\gamma_{n+2}^p G$. The group $\gamma_{n+1}^p G/\gamma_{n+2}^p G$ is a finite dimensional $\mathbb{F}_p$ -vector space and $G/\gamma_{n+1}^p G$ is finite, thus H_n is finite and naturally endowed with a structure of $\mathbb{F}_p$ -vector space. The next claim shows that A_n/A_{n+1} is also a finite dimensional $\mathbb{F}_p$ -vector space for all $n \geq 1$, so it implies that $\{A_n\}_{n=1}^{+\infty}$ is a p -filtration.

Claim 4. The map

$$\begin{aligned} u : A_n &\rightarrow H_n \\ \alpha &\mapsto u_\alpha \end{aligned}$$

is a group homomorphism whose kernel is A_{n+1} .

Proof of Claim 4. Let $\alpha_1, \alpha_2 \in A_n$ and let $g \in G$. We write $\alpha_1(g) = g x_1$, $\alpha_2(g) = g x_2$, and $\alpha_2(x_1) = x_1 y$, where $x_1, x_2 \in \gamma_{n+1}^p G$ and $y \in \gamma_{2n+1}^p G \subset \gamma_{n+2}^p G$. We have $(\alpha_2 \alpha_1)(g) = g x_2 x_1 y$, thus

$$u_{\alpha_2 \alpha_1}([g]) = [x_2 x_1 y] = [x_2 x_1] = u_{\alpha_2}([g]) \cdot u_{\alpha_1}([g]) = (u_{\alpha_2} \cdot u_{\alpha_1})([g]).$$

This shows that u is a homomorphism.

Let $\alpha \in A_{n+1}$. If $g \in G$, then $\alpha(g) \equiv g \pmod{\gamma_{n+2}^p G}$, thus $u_\alpha([g]) = [1]$. This shows that u_α is trivial; that is, α belongs to the kernel of u .

Let α be an element of the kernel of u . Let $g \in G$, and let $x = g^{-1}\alpha(g)$. We have $u_\alpha([g]) = [x] = [1]$, thus $x \in \gamma_{n+2}^p G$, hence $\alpha(g) \equiv g \pmod{\gamma_{n+2}^p G}$. This shows that $\alpha \in A_{n+1}$. $\square$

Theorem 2.5. *Let G be a finitely generated group. If G is conjugacy p -separable and has Property A, then $\mathcal{I}_p(G)$ is residually p .*

Proof. Recall that $\mathcal{I}_p(G) = I_p(G)/\text{Inn}(G)$. So, in order to prove Theorem 2.5, it suffices to find a p -filtration $\{B_n\}_{n=1}^{+\infty}$ of $I_p(G)$ such that $\bigcap_{n=1}^{+\infty} B_n = \text{Inn}(G)$.

We denote by $\pi_n : I_p(G) \rightarrow \text{Aut}(G/\gamma_{n+1}^p G)$ the restriction of $\pi_n : \text{Aut}(G) \rightarrow \text{Aut}(G/\gamma_{n+1}^p G)$ to $I_p(G)$. The kernel of π_n is the group A_n defined in the proof of Theorem 2.4, and $\{A_n\}_{n=1}^{+\infty}$ is a separating p -filtration of $I_p(G)$. Let S_n be the set of conjugacy classes in $G/\gamma_{n+1}^p G$, and let $\text{Sym}(S_n)$ denote the symmetric group of S_n . The group $\text{Aut}(G/\gamma_{n+1}^p G)$ acts naturally on S_n , and this action defines a homomorphism $\theta_n : \text{Aut}(G/\gamma_{n+1}^p G) \rightarrow \text{Sym}(S_n)$. We set $\tilde{\pi}_n = \theta_n \circ \pi_n : I_p(G) \rightarrow \text{Sym}(S_n)$, and $B_n = \text{Ker} \tilde{\pi}_n$.

Claim 1. $\{B_n\}_{n=1}^{+\infty}$ is a p -filtration.

Proof of Claim 1. The fact that $\{A_n\}_{n=1}^{+\infty}$ is a p -filtration means that $I_p(G)/A_n \simeq \text{Im}\pi_n$ is a finite p -group, thus $I_p(G)/B_n \simeq \text{Im}\tilde{\pi}_n = \theta_n(\text{Im}\pi_n)$ is also a finite p -group (it is the image of a finite p -group under a homomorphism).

Claim 2. $\bigcap_{n=1}^{+\infty} B_n = \text{Inn}(G)$.

Proof of Claim 2. The inclusion $\text{Inn}(G) \subset \bigcap_{n=1}^{+\infty} B_n$ is clear, thus we only have to prove the reverse inclusion: $\bigcap_{n=1}^{+\infty} B_n \subset \text{Inn}(G)$. Let $\alpha \in I_p(G)$, $\alpha \notin \text{Inn}(G)$. Since G has Property A, there exists $g \in G$ such that g is not conjugate to $\alpha(g)$. Since G is conjugacy p -separable, there exists a homomorphism $\varphi : G \rightarrow P$ such that P is a finite p -group and $\varphi(g) \not\sim \varphi(\alpha(g))$. By Lemma 2.2, there exists $n \geq 1$ such that $\gamma_{n+1}^p P = \{1\}$. We have $\varphi(\gamma_{n+1}^p G) \subset \gamma_{n+1}^p P = \{1\}$, thus $\gamma_{n+1}^p G \subset \text{Ker}\varphi$. Let $\mu_n : G \rightarrow G/\gamma_{n+1}^p G$ be the quotient map. The above inclusion and the fact that $\varphi(g) \not\sim \varphi(\alpha(g))$ imply that $\mu_n(g) \not\sim \mu_n(\alpha(g))$. This shows that $\tilde{\pi}_n(\alpha)$ does not fix the conjugacy class of $[g] \in G/\gamma_{n+1}^p G$, thus $\tilde{\pi}_n(\alpha) \neq \text{Id}$, therefore $\alpha \notin B_n$. $\square$

3. CONJUGACY p -SEPARABILITY OF FREE GROUPS

We start with an observation that the reader must keep in mind because it will be often used in the remainder of the paper.

Lemma 3.1. *Let G be a group, and let K_1, K_2 be two normal subgroups such that G/K_1 and G/K_2 are finite p -groups. Then $G/(K_1 \cap K_2)$ is a finite p -group.*

Proof. Let $\varphi_1 : G \rightarrow G/K_1$ and $\varphi_2 : G \rightarrow G/K_2$ be the quotient maps. Then $K_1 \cap K_2$ is the kernel of the homomorphism $\varphi : G \rightarrow G/K_1 \times G/K_2$, $g \mapsto (\varphi_1(g), \varphi_2(g))$. $\square$

Now, we prove the following.

Theorem 3.2. *Free groups are conjugacy p -separable.*

Proof. Let $F = F(X)$ be a free group freely generated by a set X . Let $g = x_1^{\varepsilon_1} \cdots x_l^{\varepsilon_l}$ be an element of F written in normal form (i.e. $x_i \in X$, $\varepsilon_i \in \{\pm 1\}$, and $x_{i+1}^{\varepsilon_{i+1}} \neq x_i^{-\varepsilon_i}$ for all i). The number l is called the *word length* of g and is denoted by $|g|_X$, and the set $\{x_1, \dots, x_l\} \subset X$ is called the *support* of g and is denoted by $\text{Sup}(g)$.

We take two elements $g, h \in F$ that are not conjugate, and we prove by induction on $|g|_X + |h|_X$ that there exists a homomorphism $\varphi : F \rightarrow P$ such that P is a finite p -group and $\varphi(g) \not\sim \varphi(h)$.

Step 1. We assume $g = 1$.

We have $h \neq 1$ and F is residually p (see [4], for example); thus there exists a homomorphism $\varphi : F \rightarrow P$ such that P is a finite p -group and $\varphi(h) \neq 1$. We obviously have $\varphi(h) \not\sim \varphi(g) = 1$.

Conclusion of Step 1. We can assume $g \neq 1$ and $h \neq 1$.

Step 2. We assume that $|\text{Sup}(g)| = |\text{Sup}(h)| = 1$.

Then g and h are of the form $g = x^n$ and $h = y^m$, where $x, y \in X$ and $n, m \in \mathbb{Z} \setminus \{0\}$. Moreover, $n \neq m$ if $x = y$. Let $q = p^e$ be a power of p strictly greater than $2|n|$ and than $2|m|$. Set $P = \mathbb{Z}/q\mathbb{Z}$, and let $\varphi : F \rightarrow P$ be the homomorphism defined by $\varphi(x) = 1$, and $\varphi(z) = 0$ for all $z \in X \setminus \{x\}$. We have $\varphi(g) \neq \varphi(h)$, thus $\varphi(g) \not\sim \varphi(h)$ since P is abelian.

Conclusion of Step 2. We can assume that $|\text{Sup}(g)| \geq 2$.

For $f \in F$, we denote by $[f]_1$ the homology class of f in $H_1(F) = \bigoplus_{x \in X} \mathbb{Z}[x]_1$.

Step 3. We assume that $[g]_1 \neq [h]_1$.

Write

$$[g]_1 - [h]_1 = \sum_{i=1}^l a_i [x_i]_1,$$

where $x_1, \dots, x_l \in X$ and $a_i \in \mathbb{Z} \setminus \{0\}$. Let $q = p^e$ be a power of p strictly greater than $2|a_1|$. Set $P = \mathbb{Z}/q\mathbb{Z}$, and let $\varphi : F \rightarrow P$ be the homomorphism defined by $\varphi(x_1) = 1$, and $\varphi(z) = 0$ for all $z \in X \setminus \{x_1\}$. We have $\varphi(g) \neq \varphi(h)$, thus $\varphi(g) \not\sim \varphi(h)$.

Conclusion of Step 3. We can assume that $[g]_1 = [h]_1$.

Step 4. We prove that there exist a homomorphism $\mu : F \rightarrow \mathbb{Z}/p\mathbb{Z}$ and an element $x \in \text{Sup}(g)$ such that $\mu(x) = 1$ and $\mu(g) = \mu(h) = 0$.

Suppose $[g]_1 = [h]_1 = 0$. We choose some $x \in \text{Sup}(g)$ and we define $\mu : F \rightarrow \mathbb{Z}/p\mathbb{Z}$ by $\mu(x) = 1$, and $\mu(z) = 0$ for all $z \in X \setminus \{x\}$. We have $\mu(g) = \mu(h) = 0$.

Suppose $[g]_1 = [h]_1 = a[y]_1$ for some $y \in X$ and $a \in \mathbb{Z} \setminus \{0\}$. Take $x \in \text{Sup}(g) \setminus \{y\}$ (this is possible because $|\text{Sup}(g)| \geq 2$) and define $\mu : F \rightarrow \mathbb{Z}/p\mathbb{Z}$ by $\mu(x) = 1$, and $\mu(z) = 0$ for all $z \in X \setminus \{x\}$. Then $\mu(g) = \mu(h) = 0$.

Suppose $[g]_1 = [h]_1 = a_1[y_1]_1 + \dots + a_l[y_l]_1$, where $y_1, \dots, y_l \in X$, $a_1, \dots, a_l \in \mathbb{Z} \setminus \{0\}$, and $l \geq 2$. Let $m = \gcd(a_1, \dots, a_l)$, $b_i = a_i/m$ for $1 \leq i \leq l$, and $T_1 = b_1[y_1]_1 + \dots + b_l[y_l]_1$. Let $U = \mathbb{Z}[y_1]_1 \oplus \dots \oplus \mathbb{Z}[y_l]_1$ be the $\mathbb{Z}$ -module freely generated by $[y_1]_1, \dots, [y_l]_1$. Since $\gcd(b_1, \dots, b_l) = 1$, T_1 can be completed into a $\mathbb{Z}$ -basis $\{T_1, T_2, \dots, T_l\}$ of U . We define $\bar{\mu} : H_1(F) \rightarrow \mathbb{Z}/p\mathbb{Z}$ by

$$\begin{aligned} \bar{\mu}(T_1) &= 0, & \bar{\mu}(T_2) &= 1, & \bar{\mu}(T_i) &= 0 \text{ for } 3 \leq i \leq l, \\ & & \bar{\mu}(z) &= 0 \text{ for } z \in X \setminus \{y_1, \dots, y_l\}. \end{aligned}$$

The fact that $\bar{\mu}(T_1) = 0$ implies that $\bar{\mu}([g]_1) = \bar{\mu}([h]_1) = 0$, and the fact that $\bar{\mu}(T_2) \neq 0$ implies that there exists $x \in \{y_1, \dots, y_l\} \subset \text{Sup}(g)$ such that $\bar{\mu}([x]_1) \neq 0$. Each element of $(\mathbb{Z}/p\mathbb{Z}) \setminus \{0\}$ generates $\mathbb{Z}/p\mathbb{Z}$, thus one can find an automorphism $\xi : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$ such that $\xi(\bar{\mu}([x]_1)) = 1$. Set $\mu = \xi \circ \bar{\mu} \circ \kappa : F \rightarrow \mathbb{Z}/p\mathbb{Z}$, where $\kappa : F \rightarrow H_1(F)$ is the quotient map. Then $\mu(x) = 1$ and $\mu(g) = \mu(h) = 0$.

Now, for $y \in X \setminus \{x\}$ and $0 \leq i \leq p-1$, we set

$$d(y, i) = x^i y x^{-\mu(x^i y)}.$$

Then $\text{Ker} \mu$ is the free group $F(Y)$ freely generated by

$$Y = \{d(y, i) ; 0 \leq i \leq p-1 \text{ and } y \in X \setminus \{x\}\} \cup \{x^p\}.$$

For $f \in \text{Ker} \mu = F(Y)$ and $0 \leq i \leq p-1$, we set $f_i = x^i f x^{-i} \in F(Y)$.

Step 5. Let $f \in F(Y)$. We show that $|f_i|_Y \leq |f|_X$ for all $0 \leq i \leq p-1$, and that, if $x \in \text{Sup}(f)$, then there exists $i \in \{0, 1, \dots, p-1\}$ such that $|f_i|_Y < |f|_X$.

Set

$$d(x, i) = \begin{cases} 1 & \text{if } 0 \leq i \leq p-2, \\ x^p & \text{if } i = p-1, \end{cases} \quad d(x^{-1}, i) = \begin{cases} x^{-p} & \text{if } i = 0, \\ 1 & \text{if } 1 \leq i \leq p-1. \end{cases}$$

For $y \in X \setminus \{x\}$ and $0 \leq i \leq p-1$ we set

$$d(y^{-1}, i) = x^i y^{-1} x^{-\mu(x^i y^{-1})}.$$

Note that

$$d(y^{-1}, i) = d(y, \mu(x^i y^{-1}))^{-1}$$

for all $y \in X$ (included $y = x$) and all $0 \leq i \leq p-1$.

Let $f = y_1^{\varepsilon_1} \cdots y_l^{\varepsilon_l}$ be the normal form of f with respect to X . For $1 \leq j \leq l$ we set

$$i_j = \mu(x^i y_1^{\varepsilon_1} \cdots y_{j-1}^{\varepsilon_{j-1}}).$$

Then

$$f_i = d(y_1^{\varepsilon_1}, i_1) \cdot d(y_2^{\varepsilon_2}, i_2) \cdots d(y_l^{\varepsilon_l}, i_l).$$

This shows that $|f_i|_Y \leq |f|_X$.

Suppose, moreover, that some y_j is equal to x . We can choose $i \in \{0, 1, \dots, p-1\}$ so that $i_j \neq p-1$ if $\varepsilon_j = 1$, or $i_j \neq 0$ if $\varepsilon_j = -1$. In this case $d(y_j^{\varepsilon_j}, i_j) = 1$ and, therefore, $|f_i|_Y < |f|_X$.

Conclusion of Step 5. We can assume that $|g_{i_0}|_Y < |g|_X$ for some $0 \leq i_0 \leq p-1$, and $|h_i|_Y \leq |h|_X$ for all $0 \leq i \leq p-1$.

By the inductive hypothesis, there exists a homomorphism $\alpha_i : F(Y) \rightarrow A_i$ such that A_i is a finite p -group and $\alpha_i(g_{i_0}) \not\sim \alpha_i(h_i)$ for all $0 \leq i \leq p-1$. Let $L = \bigcap_{i=0}^{p-1} \text{Ker} \alpha_i$, let $B = F(Y)/L$, and let $\beta : F(Y) \rightarrow B$ be the quotient map. Then B is a finite p -group (by Lemma 3.1) and $\beta(g_{i_0}) \not\sim \beta(h_i)$ for all $0 \leq i \leq p-1$. Let $K = \bigcap_{i=0}^{p-1} x^i L x^{-i}$. It is easily seen that K is a normal subgroup of $F = F(X)$ and that $P = F(X)/K$ is a finite p -group. We denote by $\varphi : F \rightarrow P$ the quotient map.

Step 6. We prove that $\varphi(g) \not\sim \varphi(h)$.

Suppose not. Then there exists $f \in F(X)$ such that $\varphi(g_{i_0}) = \varphi(f h f^{-1})$. Write f in the form $f = f' x^i$, where $f' \in F(Y)$ and $0 \leq i \leq p-1$. Then $\varphi(g_{i_0}) = \varphi(f' h_i f'^{-1})$ which implies that $\beta(g_{i_0}) = \beta(f' h_i f'^{-1})$, a contradiction. $\square$

4. CONJUGACY p -SEPARABILITY OF SURFACE GROUPS

In this section we prove the following.

Theorem 4.1. *The fundamental group of a closed oriented surface of genus $\rho \geq 1$ is conjugacy p -separable.*

We fix a genus $\rho \geq 1$, and we denote by Σ_ρ the oriented closed surface of genus ρ . If $\rho = 1$, then $\pi_1(\Sigma_\rho) \simeq \mathbb{Z}^2$, which is obviously conjugacy p -separable; thus we can and will assume $\rho \geq 2$.

We use in our proof the fact that $\pi_1(\Sigma_\rho)$ can be expressed (in many ways) as the amalgamated product of two free groups along a cyclic group. So, we start with some preliminaries on amalgamated products of groups.

Let $G = H_1 *_C H_2$ be the amalgamated product of two groups H_1, H_2 along a subgroup C . A *normal form* of an element $g \in G \setminus C$ is defined to be an expression

$$g = g_1 \cdot g_2 \cdots g_l$$

of g such that

- either $g_i \in H_1 \setminus C$ or $g_i \in H_2 \setminus C$, for all $1 \leq i \leq l$,

- if $g_i \in H_1 \setminus C$, then $g_{i+1} \in H_2 \setminus C$, and if $g_i \in H_2 \setminus C$, then $g_{i+1} \in H_1 \setminus C$, for all $1 \leq i \leq l-1$.

Such an expression always exists but is not unique in general. However, the length l of the normal form is unique. It is called the *Syllable length* of g and is denoted by $\text{Syl}(g)$. A normal form is said to be *cyclically reduced* if its length is either 1 or even. Note that in the latest case (when l is even) we have the equivalences

$$g_l \in H_2 \setminus C \Leftrightarrow g_1 \in H_1 \setminus C, \quad g_l \in H_1 \setminus C \Leftrightarrow g_1 \in H_2 \setminus C.$$

If a normal form of g is cyclically reduced, then all its normal forms are cyclically reduced. We say in that case that g itself is *cyclically reduced*. We also use the convention that the elements of C are cyclically reduced and their syllable lengths are equal to 0. It is easily checked that every element of G is conjugate to a cyclically reduced element.

We refer to [20] for the proof of the following proposition and for a general exposition on amalgamated products of groups.

Proposition 4.2. *Let $G = H_1 *_C H_2$ be the amalgamated product of two groups H_1, H_2 along a subgroup C , and let $g, h \in G$ be two cyclically reduced elements.*

- (1) *If $g \sim h$, then $\text{Syl}(g) = \text{Syl}(h)$.*
- (2) *Suppose $\text{Syl}(g) = \text{Syl}(h) = l \geq 2$, and let $h = h_1 \cdots h_l$ be a normal form of h . We have $g \sim h$ if and only if there exist $u \in C$ and $1 \leq t \leq l$ such that*

$$ugu^{-1} = h_t \cdot h_{t+1} \cdots h_l \cdot h_1 \cdots h_{t-1}.$$

- (3) *Suppose $\text{Syl}(g) = \text{Syl}(h) = l \geq 2$, and let $g = g_1 \cdots g_l$ and $h = h_1 \cdots h_l$ be normal forms of g and h , respectively. There exists $u \in C$ such that $ugu^{-1} = h$ if and only if there exist $l+1$ elements $u_0, u_1, \dots, u_l \in C$ such that $u_0 = u_l$ and $u_{i-1}g_i = h_i u_i$ for all $1 \leq i \leq l$.*

An element g of a group G is said to be *conjugacy p -distinguished* if, for every $h \in G$, either $h \sim g$ or there exists a homomorphism $\varphi : G \rightarrow P$ such that P is a finite p -group and $\varphi(g) \not\sim \varphi(h)$. Note that a group G is conjugacy p -separable if and only if all its elements are conjugacy p -distinguished.

The strategy we use to prove Theorem 4.1 is the following. We start with an element $g \in \pi_1(\Sigma_\rho) \setminus \{1\}$ and show that there exists a decomposition $\pi_1(\Sigma_\rho) = F_1 *_C F_2$ of $\pi_1(\Sigma_\rho)$ as the amalgamated product of two free groups such that g is cyclically reduced of syllable length ≥ 2 (see Proposition 4.4). Afterwards, we take an element $h \in \pi_1(\Sigma_\rho)$ which is not conjugate to g , and we prove that there exists a homomorphism $\varphi : F_1 *_C F_2 \rightarrow P_1 *_C P_2$ such that $P_1 *_C P_2$ is the amalgamated product of two finite p -groups along a cyclic subgroup $\bar{C}$, $\varphi(g)$ is cyclically reduced, $\text{Syl}(\varphi(g)) = \text{Syl}(g) \geq 2$, and $\varphi(h) \not\sim \varphi(g)$ (see Proposition 4.5). We conclude using the following.

Theorem 4.3 (Ivanova, [12]). *Let $H = P_1 *_C P_2$ be the amalgamated product of two finite p -groups P_1, P_2 along a cyclic group $\bar{C}$. Then every cyclically reduced element of H of syllable length ≥ 2 is conjugacy p -distinguished.*

Remark. The exact result which can be found in [12] is that: if the amalgamated product $P_1 *_C P_2$ of two finite p -groups is residually p , then any element of $P_1 *_C P_2$ of infinite order is conjugacy p -distinguished. The fact that g is of infinite order if it is cyclically reduced and of syllable length ≥ 2 is a classical fact that can be found for instance in [20]. The fact that $P_1 *_C P_2$ is residually p if $\bar{C}$ is cyclic is proved

in [9]. Note that the amalgamated product of two finite p -groups is not residually p in general (see [9]).

We now turn to the first step of our proof of Theorem 4.1.

Proposition 4.4. *Let $g \in \pi_1(\Sigma_\rho) \setminus \{1\}$. Then there exists a decomposition $\pi_1(\Sigma_\rho) = F_1 *_C F_2$ of $\pi_1(\Sigma_\rho)$ as an amalgamated product such that $F_1 = F(x_1, \dots, x_n, y_1, \dots, y_n)$ is a free group of rank $2n$, $F_2 = F(x'_1, \dots, x'_m, y'_1, \dots, y'_m)$ is a free group of rank $2m$, C is the cyclic subgroup generated by*

$$\gamma = [x_1, y_1] \cdots [x_n, y_n] = [x'_1, y'_1] \cdots [x'_m, y'_m],$$

and the element $g \in F_1 *_C F_2$ is cyclically reduced of syllable length ≥ 2 .

Proof. Let $a : \mathbb{S}^1 \hookrightarrow \Sigma_\rho$ be a separating simple closed curve of Σ_ρ (see Figure 4.1). The curve a separates Σ_ρ into two subsurfaces Σ_1 and Σ_2 , each of them with a unique boundary component. Let $a_1 : \mathbb{S}^1 \hookrightarrow \partial\Sigma_1$ (resp. $a_2 : \mathbb{S}^1 \hookrightarrow \partial\Sigma_2$) be the boundary curve of Σ_1 (resp. of Σ_2). Then $\Sigma_\rho = (\Sigma_1 \sqcup \Sigma_2) / \sim$, where $\sim$ is the equivalence relation which identifies $a_1(z)$ with $a_2(\bar{z})$ for all $z \in \mathbb{S}^1$. Set $B_0 = a(1)$, let $\bar{a} : [0, 1] \rightarrow \Sigma_\rho$ be the loop based at B_0 defined by $\bar{a}(t) = a(e^{2i\pi t})$, and let $\gamma \in \pi_1(\Sigma_\rho, B_0)$ be the class of $\bar{a}$. Then $\pi_1(\Sigma_\rho)$ has the amalgamated decomposition $\pi_1(\Sigma_\rho, B_0) = \pi_1(\Sigma_1, B_0) *_C \pi_1(\Sigma_2, B_0)$, where C is the infinite cyclic subgroup generated by γ . Let n (resp. m) be the genus of Σ_1 (resp. Σ_2). Then $\pi_1(\Sigma_1, B_0) = F_1 = F(x_1, \dots, x_n, y_1, \dots, y_n)$ is a free group of rank $2n$, $\pi_1(\Sigma_2, B_0) = F_2 = F(x'_1, \dots, x'_m, y'_1, \dots, y'_m)$ is a free group of rank $2m$, and the generators can be chosen so that

$$\gamma = [x_1, y_1] \cdots [x_n, y_n] = [x'_1, y'_1] \cdots [x'_m, y'_m].$$

FIGURE 4.1. Amalgamated decomposition of $\pi_1(\Sigma_\rho)$.

Let $g \in \pi_1(\Sigma_\rho, B_0) \setminus \{1\}$. Let $\bar{c}_g : [0, 1] \rightarrow \Sigma_\rho$ be a loop based at B_0 which represents g , and let $c_g : \mathbb{S}^1 \rightarrow \Sigma_\rho$ be the closed curve defined by $c_g(e^{2i\pi t}) = \bar{c}_g(t)$. It is easily shown that an element $h \in \pi_1(\Sigma_\rho, B_0)$ is conjugate to g if and only if c_h is homotopic to c_g (here the homotopies are free with no fixed point).

Let $c : \mathbb{S}^1 \rightarrow \Sigma_\rho$ be a closed curve homotopic to c_g and which intersects a transversely. Let $\bar{c} : [0, 1] \rightarrow \Sigma_\rho$ be the loop defined by $\bar{c}(t) = c(e^{2i\pi t})$, and let $0 \leq t_1 < t_2 < \cdots < t_l < 1$ such that $\bar{c}(t_i) \in a(\mathbb{S}^1)$ for all $1 \leq i \leq l$, and $\bar{c}(t) \notin a(\mathbb{S}^1)$ if $t \notin \{t_1, \dots, t_l\}$. In other words, we have $c(\mathbb{S}^1) \cap a(\mathbb{S}^1) = \{\bar{c}(t_1), \dots, \bar{c}(t_l)\}$, where the points are counted with multiplicity. For $1 \leq i \leq l$, let $s_i \in [0, 1[$ such that $\bar{c}(t_i) = \bar{a}(s_i)$, and let

$$h_i = \left[\bar{a}|_{[0, s_i]} \cdot \bar{c}|_{[t_i, t_{i+1}]} \cdot (\bar{a}|_{[0, s_{i+1}]})^{-1} \right] \in \pi_1(\Sigma_\rho, B_0)$$

(here we assume $t_{l+1} = t_1$, $s_{l+1} = s_1$, and $[t_l, t_1] = [t_l, 1] \cup [0, t_1]$). We have either $\bar{c}([t_i, t_{i+1}]) \subset \Sigma_1$ or $\bar{c}([t_i, t_{i+1}]) \subset \Sigma_2$. Moreover, if $\bar{c}([t_i, t_{i+1}]) \subset \Sigma_1$ (resp. $\bar{c}([t_i, t_{i+1}]) \subset \Sigma_2$), then $h_i \in \pi_1(\Sigma_1, B_0) = F_1$ (resp. $h_i \in \pi_1(\Sigma_2, B_0) = F_2$).

If $h_i \in \langle \gamma \rangle = C$, then there is a closed curve c' homotopic to c such that $\bar{c}'$ follows $\bar{c}$ between 0 and $t_i - \varepsilon$, $\bar{c}'$ follows $\bar{c}$ between $t_{i+1} + \varepsilon$ and 1, and $\bar{c}'$ remains in the same component of $\Sigma_\rho \setminus a(\mathbb{S}^1)$ between $t_i - \varepsilon$ and $t_{i+1} + \varepsilon$. Hence, we can reduce the number l in this manner. So, if l is minimal, then

$$h = h_1 \cdot h_2 \cdots h_l$$

is a normal form. It is also cyclically reduced because l is even (since a is separating) and h is conjugate to g . We set $l = I(a, g)$. By Proposition 4.2, this number is well-defined.

Now, our goal is to prove that we can choose a so that $I(a, g) \neq 0$. We assume that $I(a, g) = 0$ for every separating simple closed curve $a : \mathbb{S}^1 \hookrightarrow \Sigma_\rho$, and we look for a contradiction.

We take $2\rho - 3$ separating simple closed curves $a_1, \dots, a_{2\rho-3} : \mathbb{S}^1 \hookrightarrow \Sigma_\rho$ that are pairwise disjoint and such that the components of the natural compactification of $\Sigma_\rho \setminus (\bigcup_{i=1}^{2\rho-3} a_i(\mathbb{S}^1))$ are all pantalons (spheres with 3 holes) or tori with one hole (see Figure 4.2). Since $I(a_i, g) = 0$ for all $1 \leq i \leq 2\rho - 3$, we can assume that c is included in a component of $\Sigma_\rho \setminus (\bigcup_{i=1}^{2\rho-3} a_i(\mathbb{S}^1))$.

FIGURE 4.2. A decomposition of Σ_ρ into pantalons and one-hole tori.

Let S_0 be a tubular neighborhood of c , and let $d_1, \dots, d_r : \mathbb{S}^1 \hookrightarrow \partial S_0$ be the boundary curves of S_0 . Up to permutation, we can assume that d_i bounds a disk $\mathbb{D}_i$ in $\Sigma_\rho \setminus S_0$ for all $k+1 \leq i \leq r$, and d_i does not bound any disk for all $1 \leq i \leq k$. Set $S = S_0 \cup (\bigcup_{i=k+1}^r \mathbb{D}_i)$. Then S is an essential subsurface of Σ_ρ (i.e. $\pi_1(S)$ embeds in $\pi_1(\Sigma_\rho)$) which can be assumed to be included in some component of $\Sigma_\rho \setminus (\bigcup_{i=1}^{2\rho-3} a_i(\mathbb{S}^1))$. In particular, S is either an annulus, a pantalon, or a one-hole torus.

We denote by $I(x, y)$ the geometric intersection number of two simple closed curves x, y (see [1]). Let $d : \mathbb{S}^1 \hookrightarrow \Sigma_\rho$ be a boundary curve of S . Let $a : \mathbb{S}^1 \hookrightarrow \Sigma_\rho$ be a separating simple closed curve of Σ_ρ such that $I(a, d) \geq 2$ (the proof of the existence of such a curve is left to the reader). Let a' be an arc of a whose extremities, P and Q , belong to S , and let x be a path in S which joins P to Q . We consider $a' \cup x$ as a closed curve. Then one can prove (with some effort) that: if $a' \cup x$ is homotopic to a constant curve, then a' is included in S . This fact and the equality $I(a, g) = 0$ imply that c is homotopic to a curve c' included in S and such that $c' \cap a = \emptyset$. So, c' is included in a connected component of $S \setminus a(\mathbb{S}^1)$. Such a component is either a disk or an annulus if S is either a pantalon or a one-hole

torus, and is a disk if S is an annulus. We conclude that c_g is homotopic to a constant curve, a contradiction. $\square$

The second step of our proof of Theorem 4.1 is given by the following.

Proposition 4.5. *Let $\pi_1(\Sigma_\rho) = F_1 *_C F_2$ be an amalgamated decomposition of $\pi_1(\Sigma_\rho)$ such that $F_1 = F(x_1, \dots, x_n, y_1, \dots, y_n)$ is a free group of rank $2n$, $F_2 = F(x'_1, \dots, x'_m, y'_1, \dots, y'_m)$ is a free group of rank $2m$, and C is the cyclic group generated by*

$$\gamma = [x_1, y_1] \cdots [x_n, y_n] = [x'_1, y'_1] \cdots [x'_m, y'_m].$$

Let $g \in \pi_1(\Sigma_\rho)$ be a cyclically reduced element of syllable length $2l \geq 2$, and let $h \in \pi_1(\Sigma_\rho)$ be another element not conjugate to g . Then there exists a homomorphism $\varphi : F_1 *_C F_2 \rightarrow P_1 *_C P_2$ such that $P_1 *_C P_2$ is the amalgamated product of two finite p -groups along a cyclic group $\bar{C}$, $\varphi(g)$ is cyclically reduced of syllable length equal to $2l = \text{Syl}(g)$, and $\varphi(h)$ is not conjugate to $\varphi(g)$.

The following Lemmas 4.6 to 4.9 are preliminaries to the proof of Proposition 4.5. Lemmas 4.6 and 4.9 are due to Stebe (see [21], Lemma 1 and Lemma 6). The proof of Lemma 4.8 is long and technical, so we put it separately in the last section.

Lemma 4.6 (Stebe, [21]). *Let $F = F(X)$ be a free group, let $g \in F \setminus \{1\}$ be a non-trivial element, and let $q = p^e$ be a fixed power of p . Then there exists a homomorphism $\varphi : F \rightarrow P$ such that P is a finite p -group and the order of $\varphi(g)$ is exactly q .*

Lemma 4.7. *Let $F = F(x_1, \dots, x_n, y_1, \dots, y_n)$ be a free group of rank $2n$, let $\gamma = [x_1, y_1] \cdots [x_n, y_n]$, and let C be the cyclic subgroup generated by γ . Let $g \in F \setminus C$. Then there exists a homomorphism $\varphi : F \rightarrow P$ such that P is a finite p -group and $\varphi(g) \notin \varphi(C)$.*

Proof. Since γ is primitive, the centralizer of γ in F is equal to C . Now, $g \notin C$, thus $[g, \gamma] \neq \{1\}$. The group F is residually p , thus there exists a homomorphism $\varphi : F \rightarrow P$ such that P is a finite p -group and $\varphi([g, \gamma]) = [\varphi(g), \varphi(\gamma)] \neq 1$. It follows that $\varphi(g) \notin \varphi(C) = \langle \varphi(\gamma) \rangle$. $\square$

As pointed out before, the proof of the following is the object of Section 5.

Lemma 4.8. *Let $F = F(x_1, \dots, x_n, y_1, \dots, y_n)$ be a free group of rank $2n$, and let $\gamma = [x_1, y_1] \cdots [x_n, y_n]$. Let $g, h \in F$. If $\gamma^a g \neq h \gamma^b$ for all $a, b \in \mathbb{Z}$, then there exists a homomorphism $\varphi : F \rightarrow P$ such that P is a finite p -group and $\varphi(\gamma)^a \varphi(g) \neq \varphi(h) \varphi(\gamma)^b$ for all $a, b \in \mathbb{Z}$.*

Lemma 4.9 (Stebe, [21]). *Let P be a finite p -group. Let $\xi, \omega \in P$ such that $[\omega, \xi \omega \xi^{-1}] \neq 1$, and let $a, b \in \mathbb{Z}$. If $\omega^a = \xi \omega^b \xi^{-1}$, then p divides a and b .*

Corollary 4.10. *Let P be a finite p -group. Let $e \geq 0$, let $\xi, \omega \in P$ such that $[\omega^{p^r}, \xi \omega^{p^r} \xi^{-1}] \neq 1$ for all $0 \leq r \leq e$, and let $a, b \in \mathbb{Z}$. If $\omega^a = \xi \omega^b \xi^{-1}$, then p^{e+1} divides a and b .*

Proof. We show by induction on r that p^{r+1} divides a and b for all $0 \leq r \leq e$. The case $r = 0$ is a direct consequence of Lemma 4.9. We assume $r \geq 1$. By induction, p^r divides a and b . We set $a = p^r a'$ and $b = p^r b'$. We have

$$(\omega^{p^r})^{a'} = \xi (\omega^{p^r})^{b'} \xi^{-1},$$

thus, by Lemma 4.9, p divides a' and b' , therefore p^{r+1} divides a and b . $\square$

Proof of Proposition 4.5. We can assume that

$$g = g_1 \cdot g'_1 \cdots g_l \cdot g'_l$$

is a normal form of g , where $g_i \in F_1 \setminus C$ and $g'_i \in F_2 \setminus C$ for all $1 \leq i \leq l$.

Step 1. We assume that $h \in F_1$.

By Lemma 4.7, there exists a homomorphism $\alpha_{1i} : F_1 \rightarrow A_{1i}$ such that A_{1i} is a finite p -group and $\alpha_{1i}(g_i) \notin \alpha_{1i}(C)$ for all $1 \leq i \leq l$. Let $L_1 = \bigcap_{i=1}^l \text{Ker} \alpha_{1i}$, let $B_1 = F_1/L_1$, and let $\beta_1 : F_1 \rightarrow B_1$ be the quotient map. Then B_1 is a finite p -group (see Lemma 3.1) and $\beta_1(g_i) \notin \beta_1(C)$ for all $1 \leq i \leq l$. Similarly, there exists a homomorphism $\beta_2 : F_2 \rightarrow B_2$ such that B_2 is a finite p -group and $\beta_2(g'_i) \notin \beta_2(C)$ for all $1 \leq i \leq l$. Let $q_1 = p^{e_1}$ be the order of $\beta_1(\gamma)$, let $q_2 = p^{e_2}$ be the order of $\beta_2(\gamma)$, and let $q = q_1 q_2 = p^{e_1+e_2}$. By Lemma 4.6, there exists a homomorphism $\psi_j : F_j \rightarrow Q_j$ such that Q_j is a finite p -group and the order of $\psi_j(\gamma)$ is equal to q , for $j = 1, 2$. Let $K_j = \text{Ker} \beta_j \cap \text{Ker} \psi_j$, let $P_j = F_j/K_j$, and let $\varphi_j : F_j \rightarrow P_j$ be the quotient map, for $j = 1, 2$. Then P_1 is a finite p -group, $\varphi_1(g_i) \notin \varphi_1(C)$ for all $1 \leq i \leq l$, and the order of $\varphi_1(\gamma)$ is equal to q . Similarly, P_2 is a finite p -group, $\varphi_2(g'_i) \notin \varphi_2(C)$ for all $1 \leq i \leq l$, and the order of $\varphi_2(\gamma)$ is equal to q . Let $H = P_1 *_{\varphi_1(\gamma)=\varphi_2(\gamma)} P_2 = P_1 *_{\bar{C}} P_2$, where $\bar{C}$ is the cyclic group generated by $\varphi_1(\gamma) = \varphi_2(\gamma)$. Then the homomorphisms φ_1, φ_2 induce a homomorphism $\varphi : F_1 *_C F_2 \rightarrow P_1 *_{\bar{C}} P_2$.

By construction,

$$\varphi(g) = \varphi_1(g_1) \cdot \varphi_2(g'_1) \cdots \varphi_1(g_l) \cdot \varphi_2(g'_l)$$

is a normal form which is cyclically reduced. Furthermore, we have $\varphi(h) \in P_1$, thus, by Proposition 4.2, $\varphi(h)$ is not conjugate to $\varphi(g)$.

Conclusion of Step 1. We can assume that h is not conjugate to an element of F_1 and is not conjugate to an element of F_2 . So, we can assume that h is cyclically reduced of syllable length $2k \geq 2$ and has a normal form

$$h = h_1 \cdot h'_1 \cdots h_k \cdot h'_k$$

with $h_j \in F_1 \setminus C$ and $h'_j \in F_2 \setminus C$ for all $1 \leq j \leq k$.

Step 2. We assume that $k \neq l$.

It is easily proved using the same arguments as in Step 1 that there exist homomorphisms $\varphi_1 : F_1 \rightarrow P_1$, $\varphi_2 : F_2 \rightarrow P_2$ such that P_1 and P_2 are finite p -groups, $\varphi_1(g_i), \varphi_1(h_j) \notin \varphi_1(C)$ for all $1 \leq i \leq l$ and all $1 \leq j \leq k$, $\varphi_2(g'_i), \varphi_2(h'_j) \notin \varphi_2(C)$ for all $1 \leq i \leq l$ and all $1 \leq j \leq k$, and the order of $\varphi_1(\gamma)$ is equal to the order of $\varphi_2(\gamma)$. Let $H = P_1 *_{\varphi_1(\gamma)=\varphi_2(\gamma)} P_2 = P_1 *_{\bar{C}} P_2$, where $\bar{C}$ is the cyclic group generated by $\varphi_1(\gamma) = \varphi_2(\gamma)$. Then φ_1 and φ_2 induce a homomorphism $\varphi : F_1 *_C F_2 \rightarrow P_1 *_{\bar{C}} P_2$.

By construction

$$\varphi(g) = \varphi_1(g_1) \cdot \varphi_2(g'_1) \cdots \varphi_1(g_l) \cdot \varphi_2(g'_l) \quad \text{and} \quad \varphi(h) = \varphi_1(h_1) \cdot \varphi_2(h'_1) \cdots \varphi_1(h_k) \cdot \varphi_2(h'_k)$$

are cyclically reduced normal forms. By Proposition 4.2, we conclude that $\varphi(g) \not\sim \varphi(h)$.

Conclusion of Step 2. We can assume $k = l$.

For the next step we need the following.

Claim 1. Let $f \in F_1 \setminus C$ and $n \in \mathbb{Z} \setminus \{0\}$. Then $[\gamma^n, f\gamma^n f^{-1}] \neq 1$.

Proof of Claim 1. Suppose that $[\gamma^n, f\gamma^n f^{-1}] = 1$. Let H be the subgroup of F_1 generated by $\{\gamma, f\}$. Either H is free of rank 2 freely generated by $\{\gamma, f\}$, or H is cyclic. Since $[\gamma^n, f\gamma^n f^{-1}] = 1$, the group H is not freely generated by $\{\gamma, f\}$, thus H is a cyclic group. The element γ is primitive, thus H is equal to $\langle \gamma \rangle = C$, therefore $f \in C$, a contradiction.

Step 3. We construct a homomorphism $\varphi : F_1 *_C F_2 \rightarrow P_1 *_C P_2$, where P_1, P_2 are finite p -groups, C is the cyclic group generated by $\varphi(\gamma)$, and $\text{Syl}(\varphi(g)) = \text{Syl}(\varphi(h)) = 2l$.

For $1 \leq t \leq l$, we set

$$h^{(t)} = h_{t+1} \cdot h'_{t+1} \cdots h_{t+l} \cdot h'_{t+l},$$

where the indices are considered mod l . By Proposition 4.2, the fact that g is not conjugate to h is equivalent to

$$\gamma^a h^{(t)} \gamma^{-a} \neq g \quad \text{for all } a \in \mathbb{Z} \text{ and all } 1 \leq t \leq l.$$

It is easily proved using the same arguments as in Step 1 that there exist homomorphisms $\alpha_1 : F_1 \rightarrow A_1$, $\alpha_2 : F_2 \rightarrow A_2$ such that A_1, A_2 are finite p -groups, $\alpha_1(g_i), \alpha_1(h_i) \notin \alpha_1(C)$ for all $1 \leq i \leq l$, and $\alpha_2(g'_i), \alpha_2(h'_i) \notin \alpha_2(C)$ for all $1 \leq i \leq l$.

Now, for a given $1 \leq t \leq l$, we construct homomorphisms $\beta_1^{(t)} : F_1 \rightarrow B_1^{(t)}$, $\beta_2^{(t)} : F_2 \rightarrow B_2^{(t)}$, where $B_1^{(t)}, B_2^{(t)}$ are finite p -groups.

Case 1. There exists $1 \leq i \leq l$ such that $\gamma^a h_{t+i} \neq g_i \gamma^b$ for all $a, b \in \mathbb{Z}$.

By Lemma 4.8, there exists a homomorphism $\beta_1^{(t)} : F_1 \rightarrow B_1^{(t)}$ such that $B_1^{(t)}$ is a finite p -group and $\beta_1^{(t)}(\gamma^a h_{t+i}) \neq \beta_1^{(t)}(g_i \gamma^b)$ for all $a, b \in \mathbb{Z}$. On the other hand, we set $B_2^{(t)} = \{1\}$, and $\beta_2^{(t)} : F_2 \rightarrow B_2^{(t)}$ is the trivial map.

Case 2. Not in Case 1, and there exists $1 \leq i \leq l$ such that $\gamma^a h'_{t+i} \neq g'_i \gamma^b$ for all $a, b \in \mathbb{Z}$.

By Lemma 4.8, there exists a homomorphism $\beta_2^{(t)} : F_2 \rightarrow B_2^{(t)}$ such that $B_2^{(t)}$ is a finite p -group and $\beta_2^{(t)}(\gamma^a h'_{t+i}) \neq \beta_2^{(t)}(g'_i \gamma^b)$ for all $a, b \in \mathbb{Z}$. On the other hand, we set $B_1^{(t)} = \{1\}$, and $\beta_1^{(t)} : F_1 \rightarrow B_1^{(t)}$ is the trivial map.

Case 3. For all $1 \leq i \leq l$ there exist $a_i, a'_i, b_i, b'_i \in \mathbb{Z}$ such that $\gamma^{a_i} h_{t+i} = g_i \gamma^{b_i}$ and $\gamma^{a'_i} h'_{t+i} = g'_i \gamma^{b'_i}$.

Set $c_i = b_i - a'_i$ for $1 \leq i \leq l$, $c'_i = b'_i - a_{i+1}$ for $1 \leq i \leq l-1$, and $c'_l = b'_l - a_1$. Then

$$\gamma^{a_1} h^{(t)} \gamma^{-a_1} = g_1 \gamma^{c_1} g'_1 \gamma^{c'_1} \cdots g_l \gamma^{c_l} g'_l \gamma^{c'_l}.$$

Moreover, since h is not conjugate to g , not all the c_i 's or the c'_i 's are zero. Let

$$u = \gcd(c_1, \dots, c_l, c'_1, \dots, c'_l),$$

and write $u = p^e v$, where p does not divide v . Since F_1 is residually p , by Claim 1 there exists a homomorphism $\beta_1^{(t)} : F_1 \rightarrow B_1^{(t)}$ such that $B_1^{(t)}$ is a finite p -group and

$$\beta_1^{(t)}([\gamma^{p^r}, g_i^{-1} \gamma^{p^r} g_i]) \neq 1$$

for all $1 \leq i \leq l$ and all $0 \leq r \leq e$. Similarly, there exists a homomorphism $\beta_2^{(t)} : F_2 \rightarrow B_2^{(t)}$ such that $B_2^{(t)}$ is a finite p -group and

$$\beta_2^{(t)}([\gamma^{p^r}, g_i'^{-1} \gamma^{p^r} g_i']) \neq 1$$

for all $1 \leq i \leq l$ and all $0 \leq r \leq e$.

Now, let $L_j = \text{Ker} \alpha_j \cap (\bigcap_{i=1}^l \text{Ker} \beta_j^{(t)})$, let $N_j = F_j/L_j$, and let $\psi_j : F_j \rightarrow N_j$ be the quotient map, for $j = 1, 2$. Let $q_1 = p^{e_1}$ be the order of $\psi_1(\gamma)$, let $q_2 = p^{e_2}$ be the order of $\psi_2(\gamma)$, and let $q = q_1 q_2 = p^{e_1+e_2}$. By Lemma 4.6, there exist homomorphisms $\psi'_1 : F_1 \rightarrow N'_1$, $\psi'_2 : F_2 \rightarrow N'_2$ such that N'_1 and N'_2 are finite p -groups, and the order of $\psi'_j(\gamma)$ is equal to q for $j = 1, 2$.

Let $K_j = \text{Ker} \psi_j \cap \text{Ker} \psi'_j$, let $P_j = F_j/K_j$, and let $\varphi_j : F_j \rightarrow P_j$ be the quotient map, for $j = 1, 2$. The groups P_1, P_2 are finite p -groups, and the orders of $\varphi_1(\gamma)$ and $\varphi_2(\gamma)$ are both equal to $q = p^{e_1+e_2}$. Let $H = P_1 *_{\varphi_1(\gamma)=\varphi_2(\gamma)} P_2 = P_1 *_{\bar{C}} P_2$, where $\bar{C}$ is the cyclic group generated by $\varphi_1(\gamma) = \varphi_2(\gamma)$. Then φ_1 and φ_2 induce a homomorphism $\varphi : F_1 *_C F_2 \rightarrow P_1 *_{\bar{C}} P_2$.

Step 4. We prove that $\varphi(h) \not\sim \varphi(g)$.

By construction,

$\varphi(g) = \varphi_1(g_1) \cdot \varphi_2(g'_1) \cdots \varphi_1(g_l) \cdot \varphi_2(g'_l)$ and $\varphi(h) = \varphi_1(h_1) \cdot \varphi_2(h'_1) \cdots \varphi_1(h_l) \cdot \varphi_2(h'_l)$ are cyclically reduced normal forms. So, by Proposition 4.2, it suffices to show that

$$\varphi(g) \neq \varphi(\gamma^a h^{(t)} \gamma^{-a})$$

for all $a \in \mathbb{Z}$ and all $1 \leq t \leq l$.

Case 1. There exists $1 \leq i \leq l$ such that $\gamma^a h_{t+i} \neq g_i \gamma^b$ for all $a, b \in \mathbb{Z}$.

By construction of $\beta_1^{(t)}$, we have

$$\varphi_1(\gamma^a h_{t+i}) \neq \varphi_1(g_i \gamma^b)$$

for all $a, b \in \mathbb{Z}$. By Proposition 4.2, it follows that $\varphi(\gamma^a h^{(t)} \gamma^{-a}) \neq \varphi(g)$ for all $a \in \mathbb{Z}$.

Case 2. Not in Case 1, and there exists $1 \leq i \leq l$ such that $\gamma^a h'_{t+i} \neq g'_i \gamma^b$ for all $a, b \in \mathbb{Z}$.

Then it is easily proved using the same arguments as in Case 1 that $\varphi(\gamma^a h^{(t)} \gamma^{-a}) \neq \varphi(g)$ for all $a \in \mathbb{Z}$.

Case 3. For all $1 \leq i \leq l$ there exist $a_i, a'_i, b_i, b'_i \in \mathbb{Z}$ such that $\gamma^{a_i} h_{t+i} = g_i \gamma^{b_i}$ and $\gamma^{a'_i} h'_{t+i} = g'_i \gamma^{b'_i}$.

Following the construction of the homomorphisms $\beta_1^{(t)}$ and $\beta_2^{(t)}$ we set

$$\gamma^{a_1} h^{(t)} \gamma^{-a_1} = g_1 \gamma^{c_1} g'_1 \gamma^{c'_1} \cdots g_l \gamma^{c_l} g'_l \gamma^{c'_l},$$

and $u = p^e v = \gcd(c_1, \dots, c_l, c'_1, \dots, c'_l)$, where p does not divide v . Suppose there exists $d \in \mathbb{Z}$ such that $\varphi(\gamma^d h^{(t)} \gamma^{-d}) = \varphi(g)$. By Proposition 4.2, there exist $d_1, \dots, d_l, d'_1, \dots, d'_l \in \mathbb{Z}$ such that

$$\varphi(\gamma^{d'_i-1} g_i \gamma^{c_i}) = \varphi(g_i \gamma^{d_i}) \quad \text{and} \quad \varphi(\gamma^{d_i} g'_i \gamma^{c'_i}) = \varphi(g'_i \gamma^{d'_i})$$

for all $1 \leq i \leq l$, where $d'_0 = d'_l$. Let $1 \leq i \leq l$. The above equality implies that

$$\varphi_1(g_i^{-1} \gamma^{d'_i-1} g_i) = \varphi_1(\gamma^{d_i-c_i}).$$

Moreover, by construction of $\beta_1^{(t)}$, $\varphi_1([\gamma^{p^r}, g_i^{-1} \gamma^{p^r} g_i]) \neq 1$ for all $0 \leq r \leq e$; thus, by Corollary 4.10, p^{e+1} divides d'_{i-1} and $d_i - c_i$. Similarly, p^{e+1} divides d_i and $d'_i - c'_i$ for all $1 \leq i \leq l$. It follows that p^{e+1} divides all the c_i 's and all the c'_i 's, a contradiction. So, $\varphi(\gamma^d h^{(t)} \gamma^{-d}) \neq \varphi(g)$ for all $d \in \mathbb{Z}$.

Proof of Theorem 4.1. Let $g, h \in \pi_1(\Sigma_\rho)$ such that $g \not\sim h$. We can assume $g \neq 1$. By Proposition 4.4, the group $\pi_1(\Sigma_\rho)$ has an amalgamated decomposition $\pi_1(\Sigma_\rho) = F_1 *_C F_2$, where $F_1 = F(x_1, \dots, x_n, y_1, \dots, y_n)$ is a free group of rank $2n$, $F_2 = F(x'_1, \dots, x'_m, y'_1, \dots, y'_m)$ is a free group of rank $2m$, and C is the cyclic group generated by

$$\gamma = [x_1, y_1] \cdots [x_n, y_n] = [x'_1, y'_1] \cdots [x'_m, y'_m],$$

and such that g is cyclically reduced of syllable length ≥ 2 . By Proposition 4.5, there exists a homomorphism $\alpha : F_1 *_C F_2 \rightarrow P_1 *_C P_2$ such that P_1, P_2 are finite p -groups, $\bar{C}$ is the cyclic group generated by $\alpha(\gamma)$, $\alpha(g)$ is cyclically reduced of syllable length ≥ 2 , and $\alpha(g) \not\sim \alpha(h)$. By Theorem 4.3, there exists a homomorphism $\beta : P_1 *_C P_2 \rightarrow P$ such that P is a finite p -group and $\beta(\alpha(g)) \not\sim \beta(\alpha(h))$. Let $\varphi = \beta \circ \alpha : \pi_1(\Sigma_\rho) \rightarrow P$. Then $\varphi(g) \not\sim \varphi(h)$. $\square$

5. A TECHNICAL LEMMA

Let $F = F(x_1, \dots, x_n, y_1, \dots, y_n)$ a free group of rank $2n$, and let $\gamma = [x_1, y_1] \cdots [x_n, y_n]$. In this section we prove the following.

Lemma 4.8. *Let $g, h \in F$. If $\gamma^a g \neq h \gamma^b$ for all $a, b \in \mathbb{Z}$, then there exists a homomorphism $\varphi : F \rightarrow P$ such that P is a finite p -group and $\varphi(\gamma^a g) \neq \varphi(h \gamma^b)$ for all $a, b \in \mathbb{Z}$.*

The following Lemmas 5.1 and 5.2 are preliminaries to the proof of Lemma 4.8.

Lemma 5.1. *Let $g \in F$. Then there exist a homomorphism $\varphi : F \rightarrow \mathbb{Z}/p^2\mathbb{Z}$ and a free generating set Z for $\text{Ker}\varphi$, such that $\gamma \in Z$, and $g\gamma g^{-1}$ is of the form $g\gamma g^{-1} = wzw^{-1}$, with $w \in \text{Ker}\varphi$ and $z \in Z$.*

Proof. Let Σ be a surface of genus n with one hole, and let $c : \mathbb{S}^1 \hookrightarrow \partial\Sigma$ be the boundary curve of Σ . Let $B_0 = c(1)$. Then we identify F with $\pi_1(\Sigma, B_0)$ and γ with the homotopy class of the loop $\bar{c} : [0, 1] \rightarrow \Sigma$, $t \mapsto c(e^{2i\pi t})$.

Let $\tilde{\Sigma}$ be the surface of genus $p^2(n-1) + 1$ with p^2 holes which we represent as follows.

Choose a small $\varepsilon > 0$. For $1 \leq k \leq n-1$ and $0 \leq l \leq p^2 - 1$ we set

$$O_{kl} = ke^{2i\pi l/p^2}, \quad V_{kl} = (k-1+\varepsilon)e^{2i\pi l/p^2}, \quad U_{kl} = (k-\varepsilon)e^{2i\pi l/p^2}.$$

These are complex numbers. We denote by $\mathcal{C}_{kl}$ the circle of radius ε centered at O_{kl} , and by $\mathcal{D}_{kl}$ the interval $\mathcal{D}_{kl} = [V_{kl}, U_{kl}]$, for $1 \leq k \leq n-1$ and $0 \leq l \leq p^2 - 1$. We denote by $\mathcal{C}_0$ the circle centered at 0 of radius ε . We consider the connected graph

$$\Gamma = \mathcal{C}_0 \cup \left(\bigcup_{k,l} \mathcal{C}_{kl} \right) \cup \left(\bigcup_{k,l} \mathcal{D}_{kl} \right)$$

embedded in $\mathbb{C}$ (see Figure 5.1).

We embed $\mathbb{C}$ in $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ by $\xi \mapsto (\xi, 0)$, and we denote by $\tilde{\Sigma}_0$ the boundary of a regular neighborhood of Γ in $\mathbb{R}^3$. Then $\tilde{\Sigma}_0$ is a closed oriented surface of genus $p^2(n-1) + 1$. Let H_l be the plane orthogonal to $\mathbb{R}e^{2i\pi l/p^2}$ and which contains

FIGURE 5.1. The graph Γ for $n = 3$ and $p = 2$.

$(n - 1 + \varepsilon - \varepsilon^2)e^{2i\pi l/p^2}$. It is easily seen that this construction can be made so that $H_l \cap \tilde{\Sigma}_0$ is a circle, $\tilde{c}_l$, which bounds a disk $\mathbb{D}_l$ embedded in $\tilde{\Sigma}_0$. We set $\tilde{\Sigma} = \tilde{\Sigma}_0 \setminus (\bigcup_{l=0}^{p^2-1} \mathbb{D}_l)$. Then $\tilde{\Sigma}$ is an oriented surface of genus $p^2(n - 1) + 1$ with p^2 holes, and its boundary curves are $\tilde{c}_0, \tilde{c}_1, \dots, \tilde{c}_{p^2-1}$.

Let $D = \{0\} \times \mathbb{R}$ be the central vertical line, and let $\theta : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the rotation of angle $2\pi/p^2$ around D . Clearly, we can assume that $\theta(\tilde{\Sigma}) = \tilde{\Sigma}$, and $\theta \circ \tilde{c}_l = \tilde{c}_{l+1}$ for all $0 \leq l \leq p^2 - 1$ (where the indices are considered mod p^2). Set $G = \langle \theta \rangle \simeq \mathbb{Z}/p^2\mathbb{Z}$. By the above, G acts freely on $\tilde{\Sigma}$ and $\tilde{\Sigma}/G$ is a one-hole surface of genus n which can be identified with Σ . Set $\tilde{B}_l = \tilde{c}_l(1)$ for all $0 \leq l \leq p^2 - 1$. So, we have a regular covering $\pi : \tilde{\Sigma} \rightarrow \Sigma$ (see Figure 5.2) which gives rise to an exact sequence

$$1 \rightarrow \pi_1(\tilde{\Sigma}, \tilde{B}_0) \longrightarrow \pi_1(\Sigma, B_0) \longrightarrow G \simeq \mathbb{Z}/p^2\mathbb{Z} \rightarrow 1.$$

FIGURE 5.2. The regular covering $\tilde{\Sigma} \rightarrow \Sigma$ (for $n = 2$ and $p = 2$).

For $0 \leq l \leq p^2 - 1$, we denote by $\hat{c}_l : [0, 1] \rightarrow \tilde{\Sigma}$ the loop based at $\tilde{B}_l$ defined by $\hat{c}_l(t) = \tilde{c}_l(e^{2i\pi t})$, we choose a path $\hat{d}_l : [0, 1] \rightarrow \tilde{\Sigma}$ from $\tilde{B}_0$ to $\tilde{B}_l$, and we set $z_l = [\hat{d}_l \hat{c}_l \hat{d}_l^{-1}] \in \pi_1(\tilde{\Sigma}, \tilde{B}_0)$. For a good choice of the $\hat{d}_l$'s, the set $\{z_0, z_1, \dots, z_{p^2-1}\}$ can be completed into a generating system

$$\hat{Z} = \{u_1, \dots, u_{p^2(n-1)+1}, v_1, \dots, v_{p^2(n-1)+1}, z_0, z_1, \dots, z_{p^2-1}\}$$

for $\pi_1(\tilde{\Sigma}, \tilde{B}_0)$ so that $\pi_1(\tilde{\Sigma}, \tilde{B}_0)$ has a presentation with generating set $\hat{Z}$ and unique relation

$$[u_1, v_1] \cdots [u_{p^2(n-1)+1}, v_{p^2(n-1)+1}] = z_0 z_1 \cdots z_{p^2-1}.$$

In particular, for a chosen $k_0 \in \{0, 1, \dots, p^2 - 1\}$, $\pi_1(\tilde{\Sigma}, \tilde{B}_0)$ is the free group freely generated by $Z = \hat{Z} \setminus \{z_{k_0}\}$. Also, the path $\hat{d}_0$ can and will be assumed to be the constant path.

Let $\omega : [0, 1] \rightarrow \Sigma$ be a loop based at B_0 which represents g , let $\hat{\omega} : [0, 1] \rightarrow \tilde{\Sigma}$ be the lift of ω such that $\hat{\omega}(0) = \tilde{B}_0$, and let $l_0 \in \{0, 1, \dots, p^2 - 1\}$ such that $\hat{\omega}(1) = \tilde{B}_{l_0}$. Since $p^2 \geq 3$, we can choose k_0 such that $k_0 \neq 0$ and $k_0 \neq l_0$. Then $\gamma = [c] = [\hat{c}_0] = z_0 \in Z$ and

$$g\gamma g^{-1} = [\hat{\omega} \hat{c}_{l_0} \hat{\omega}^{-1}] = w z_{l_0} w^{-1},$$

where $w = [\hat{\omega} \hat{d}_{l_0}^{-1}] \in \pi_1(\tilde{\Sigma}, \tilde{B}_0)$. $\square$

Remark. In the above proof we only need the condition $p^2 \geq 3$, so we can replace p^2 by p if $p \neq 2$.

Lemma 5.2. *Let $g, h \in F$. If $\gamma^a g \gamma^b g^{-1} \neq h$ for all $a, b \in \mathbb{Z}$, then there exists a homomorphism $\varphi : F \rightarrow P$ such that P is a finite p -group and $\varphi(\gamma^a g \gamma^b g^{-1}) \neq \varphi(h)$ for all $a, b \in \mathbb{Z}$.*

Proof. By Lemma 5.1, there exist a homomorphism $\pi : F \rightarrow \mathbb{Z}/p^2\mathbb{Z}$ and a free generating set Z for $\text{Ker}\pi$ such that $\gamma \in Z$ and $g\gamma g^{-1}$ is of the form uz_0u^{-1} , where $u \in \text{Ker}\pi$ and $z_0 \in Z$.

Suppose $\pi(h) \neq 1$. Set $P = \mathbb{Z}/p^2\mathbb{Z}$ and $\varphi = \pi : F \rightarrow P$. Then $\varphi(\gamma^a g \gamma^b g^{-1}) = 1 \neq \varphi(h)$ for all $a, b \in \mathbb{Z}$. So, we can assume that $h \in \text{Ker}\pi$.

Let $u = \omega_1^{c_1} \omega_2^{c_2} \cdots \omega_l^{c_l}$ be the normal form of u with respect to Z , where $c_i \in \mathbb{Z} \setminus \{0\}$, $\omega_i \in Z$, and $\omega_i \neq \omega_{i+1}$ for all i . For convenience, we assume that $\omega_1 = \gamma$ and that c_1 may be equal to 0. Moreover, we can also assume that $\omega_l \neq z_0$. We set $v = \omega_2^{c_2} \cdots \omega_l^{c_l}$. For $a, b \in \mathbb{Z}$, the normal form of $\gamma^{a+c_1} v z_0^b v^{-1}$ is

$$\begin{array}{ll} \gamma^{a+c_1} & \text{if } b = 0, \\ \gamma^{a+b+c_1} & \text{if } v = 1 \text{ and } z_0 = \gamma, \\ \gamma^{a+c_1} \omega_2^{c_2} \cdots \omega_l^{c_l} z_0^b \omega_l^{-c_l} \cdots \omega_2^{-c_2} & \text{otherwise.} \end{array}$$

Let $h\gamma^{c_1} = z_1^{d_1} \cdots z_k^{d_k}$ be the normal form of $h\gamma^{c_1}$ with respect to Z . Let $q = p^e$ be a power of p strictly greater than $2|c_i|$ and than $2|d_j|$ for all $2 \leq i \leq l$ and all $1 \leq j \leq k$.

Let

$$G = \langle Z \mid z^q = 1 \text{ for } z \in Z \rangle.$$

The group G is the free product of $|Z|$ copies of $\mathbb{Z}/q\mathbb{Z}$, and there is a natural homomorphism $\alpha : \text{Ker}\pi = F(Z) \rightarrow G$ which sends z to z for all $z \in Z$. Let

$$U = \{\gamma^{a+c_1} v z_0^b v^{-1}; a, b \in \mathbb{Z}\}.$$

Note that the condition $\gamma^a g \gamma^b g^{-1} \neq h$ for all $a, b \in \mathbb{Z}$ is equivalent to $h\gamma^{c_1} \notin U$. Now, observe that $\alpha(h\gamma^{c_1}) = z_1^{b_1} \cdots z_k^{b_k}$ is the normal form of $\alpha(h\gamma^{c_1})$ which, by construction, does not coincide with the normal form of any element of $\alpha(U)$, thus $\alpha(h\gamma^{c_1}) \notin \alpha(U)$.

The set $\alpha(U)$ is finite and the group G is residually p , thus there exists a homomorphism $\beta : G \rightarrow B$ such that B is a finite p -group and $(\beta \circ \alpha)(h\gamma^{c_1}) \notin (\beta \circ \alpha)(U)$. Set $\psi = \beta \circ \alpha : \text{Ker} \pi = F(Z) \rightarrow B$. Then $\psi(h) \neq \psi(\gamma^a u z_0^b u^{-1})$ for all $a, b \in \mathbb{Z}$.

Let $f \in F$ such that $\pi(f) = 1$. Let $K = \bigcap_{j=0}^{p^2-1} (f^j \text{Ker} \psi f^{-j})$. The group K is a normal subgroup of F and the quotient $P = F/K$ is a finite p -group. Let $\varphi : F \rightarrow P$ be the quotient map. Then $\varphi(\gamma^a g \gamma^b g^{-1}) \neq \varphi(h)$ for all $a, b \in \mathbb{Z}$. $\square$

Proof of Lemma 4.8. The condition $\gamma^a g \neq h\gamma^b$ for all $a, b \in \mathbb{Z}$ is equivalent to $\gamma^a g \gamma^{-b} g^{-1} \neq h g^{-1}$ for all $a, b \in \mathbb{Z}$. By Lemma 5.2, there exists a homomorphism $\varphi : F \rightarrow P$ such that P is a finite p -group and $\varphi(\gamma^a g \gamma^{-b} g^{-1}) \neq \varphi(h g^{-1})$ for all $a, b \in \mathbb{Z}$. Then $\varphi(\gamma^a g) \neq \varphi(h\gamma^b)$ for all $a, b \in \mathbb{Z}$. $\square$

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