$L \log L$ RESULTS FOR THE MAXIMAL OPERATOR IN VARIABLE L^p SPACES

D. CRUZ-URIBE, SFO AND A. FIORENZA

ABSTRACT. We generalize the classical $L \log L$ inequalities of Wiener and Stein for the Hardy-Littlewood maximal operator to variable L^p spaces where the exponent function $p(\cdot)$ approaches 1 in value. We prove a modular inequality with no assumptions on the exponent function, and a strong norm inequality if we assume the exponent function is log-Hölder continuous. As an application of our approach we give another proof of a related endpoint result due to Hästö.

1. INTRODUCTION

The Hardy-Littlewood maximal operator is defined for all $f \in L^1_{loc}(\mathbb{R}^n)$ by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

where the supremum is taken over all balls B containing x . (Equivalently, the supremum may be taken over all balls centered at x , or over all cubes Q containing x .)

It is well known that for $1 < p < \infty$, $M : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$, but that given any $f \in L^1(\mathbb{R}^n)$, $f \neq 0$, $Mf \notin L^1(\mathbb{R}^n)$. In fact, Mf need not even be locally integrable. For instance, if we let

$$f(x) = \frac{1}{x \log(x)^2} \chi_{(0, 1/e)}(x),$$

then on $(0, 1/e)$,

$$Mf(x) \approx \frac{1}{x |\log(x)|},$$

so Mf is not integrable on any interval containing the origin. Wiener [16] (see also [10]) proved that Mf is locally integrable if f is in $L \log L$. More precisely, he showed that given any ball B ,

$$(1.1) \quad \int_B Mf(x) dx \leq 2|B| + C \int_{\mathbb{R}^n} |f(x)| \log(e + |f(x)|) dx.$$

Received by the editors November 30, 2006 and, in revised form, July 23, 2007.

2000 *Mathematics Subject Classification.* Primary 42B25, 42B35.

Key words and phrases. Variable Lebesgue space, maximal operators.

The first author was partially supported by the Stewart-Dorwart faculty development fund of Trinity College. Both authors would like to thank the anonymous referee for the close reading of the original version of this paper.

©2008 American Mathematical Society
 Reverts to public domain 28 years from publication

The L^p boundedness results, $p > 1$, have been generalized to the variable L^p spaces. Given an open set Ω and a measurable function $p(\cdot) : \Omega \rightarrow [1, \infty)$, we define the space $L^{p(\cdot)}(\Omega)$ to be the space of functions such that for some $\lambda > 0$,

$$\rho_\lambda(f) = \int_\Omega \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty.$$

$L^{p(\cdot)}(\Omega)$ becomes a Banach space when equipped with the Musielak-Orlicz norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf \{ \lambda > 0 : \rho_\lambda(f) \leq 1 \}.$$

If the set of λ 's such that $\rho_\lambda(f) \leq 1$ is empty, then we let $\|f\|_{L^{p(\cdot)}(\Omega)} = +\infty$. If $\Omega = \mathbb{R}^n$, we often write simply $L^{p(\cdot)}$ instead of $L^{p(\cdot)}(\mathbb{R}^n)$. Given a set Ω let

$$p_-(\Omega) = \operatorname{ess\,inf}_{x \in \Omega} p(x), \quad p_+(\Omega) = \operatorname{ess\,sup}_{x \in \Omega} p(x).$$

For brevity we write $p_- = p_-(\mathbb{R}^n)$ and $p_+ = p_+(\mathbb{R}^n)$.

Theorem 1.1. *Let $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ be such that $1 < p_- \leq p_+ < \infty$. Suppose further that $p(\cdot)$ satisfies the log-Hölder continuity conditions*

$$(1.2) \quad |p(x) - p(y)| \leq \frac{C}{-\log|x-y|}, \quad x, y \in \mathbb{R}^n, \quad |x-y| < 1/2,$$

and

$$(1.3) \quad |p(x) - p(y)| \leq \frac{C}{\log(e+|x|)}, \quad x, y \in \mathbb{R}^n, \quad |y| \geq |x|.$$

Then the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Theorem 1.1 was first proved by Diening [6] with the stronger hypothesis that $p(\cdot)$ is constant outside of a large ball. The full result was proved independently by Cruz-Uribe, Fiorenza and Neugebauer [4] and Nekvinda [14]. (Nekvinda did so with the second condition replaced by a somewhat more general condition. The relationship between the two conditions is discussed in [2].) More recently, Diening [7] found a complex necessary and sufficient condition on $p(\cdot)$ for the maximal operator to be bounded.

The assumption that $p_- > 1$ is necessary for Theorem 1.1 to be true: in [4] it was shown that if $p_- = 1$ and $p(\cdot)$ is lower semicontinuous, then M cannot be bounded on $L^{p(\cdot)}$. Therefore, in this case we are interested in the local integrability of Mf .

Our first result is a generalization of Wiener's inequality (1.1) with no assumptions on $p(\cdot)$.

Theorem 1.2. *Given $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$, then for any $\epsilon > 0$, there exists a constant C , depending on ϵ and $p(\cdot)$, such that for any ball B ,*

$$(1.4) \quad \int_B Mf(x) dx \leq 2|B| + C \int_{\mathbb{R}^n} |f(x)|^{p(x)} \log(e + |f(x)|)^{q(x)} dx,$$

where $q(x) = \max(\epsilon^{-1}(\epsilon + 1 - p(x)), 0)$.

Remark 1.3. When $p(\cdot) = 1$, (1.4) reduces to (1.1). By a straightforward estimate, we could prove (1.4) with $q(\cdot) = 1$ (and with $2|B|$ replaced by $3|B|$) directly from (1.1): write $f = f_1 + f_2$, where $f_1 = f\chi_{\{f \leq 1\}}$; estimate the integral of Mf_1 by $|B|$ and apply Wiener's result to the integral of Mf_2 (details are left to the reader).

However, the central feature of Theorem 1.2 is that $q(\cdot)$ decreases in size as $p(\cdot)$ increases, and disappears on the set where $p(x) \geq 1 + \epsilon$.

Inequality (1.1) implies an inequality in the scale of Orlicz spaces:

$$(1.5) \quad \|Mf\|_{L^1(B)} \leq C\|f\|_{L \log L(\mathbb{R}^n)},$$

where $L \log L(\mathbb{R}^n)$ is the Orlicz space $L^\Phi(\mathbb{R}^n)$, $\Phi(t) = t \log(e + t)$. As a corollary to Theorem 1.2 we can prove the corresponding inequality in the scale of variable Orlicz spaces (also known as Musielak-Orlicz spaces). Let $\Phi : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow [0, \infty)$ be such that for each $x \in \mathbb{R}^n$, the function $\Phi(x, \cdot)$ is nondecreasing, continuous and convex. Assume that $\Phi(x, 0) = 0$, $\Phi(x, t) > 0$ if $t > 0$, and $\Phi(x, t) \rightarrow \infty$ as $t \rightarrow \infty$. We also assume that for each $t \geq 0$, the function $\Phi(\cdot, t)$ is a measurable function.

Define the space $L^{\Phi(\cdot)}(\mathbb{R}^n)$ to be the set of all functions f such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} \Phi(x, |f(x)|/\lambda) dx < +\infty,$$

equipped with the norm

$$\|f\|_{L^{\Phi(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi(x, |f(x)|\lambda^{-1}) dx \leq 1 \right\}.$$

If $\Phi(x, t) = t^{p(x)}$, then $L^{\Phi(\cdot)}$ reduces to the space $L^{p(\cdot)}$. For complete information on these spaces, see Musielak [13].

A special family of the variable Orlicz spaces is the generalization of the Zygmund spaces $L^p(\log L)^q$. If we let $\Phi(x, t) = t^{p(x)} \log(e + t)^{q(x)}$, then Φ satisfies the above hypotheses and we can define the space $L^{\Phi(\cdot)}$. Hereafter we will denote this space by $L^{p(\cdot)}(\log L)^{q(\cdot)}$.

Corollary 1.4. *Given $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$, there exists a constant C depending on $p(\cdot)$, ϵ and $|B|$ such that*

$$\|Mf\|_{L^1(B)} \leq C\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)}.$$

We can considerably improve the local integrability of Mf if we assume that $p(\cdot)$ satisfies the log-Hölder continuity conditions (1.2) and (1.3).

Theorem 1.5. *Let $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ be a function that satisfies (1.2) and (1.3). Given ϵ , $0 < \epsilon < 1$, then there exist continuous functions $r(\cdot)$ and $q(\cdot)$ such that:*

- (1) $r(\cdot)$ is log-Hölder continuous, $r(x) = p(x)$ whenever $p(\cdot)$ takes on values outside the range $(1, 1 + \epsilon)$, and $1 < r(x) < p(x)$ if $p(\cdot)$ takes values in $(1, 1 + \epsilon)$.
- (2) $0 \leq q(x) \leq 1$, $q(x) = 1$ if $p(x) = 1$, and $q(x) = 0$ if $p(x) \geq 1 + \epsilon$.
- (3) Given a ball B , there is a constant C depending on $|B|$, ϵ and $p(\cdot)$ such that

$$(1.6) \quad \|Mf\|_{L^{r(\cdot)}(B)} \leq C\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)}.$$

Remark 1.6. Theorem 1.5 contains both the classical $L \log L$ inequality (1.5) and the boundedness of the maximal operator on $L^{p(\cdot)}(B)$ if $p_- > 1$ as special cases.

Remark 1.7. In Theorem 1.2, $q(\cdot)$ is essentially a linear function of $p(\cdot)$, but in Theorem 1.5 it is a considerably more complicated function that is roughly a linear function of $1/p(\cdot)$. It would be interesting to determine the optimal exponent function $q(\cdot)$ in each result. (The authors want to thank P. Hästö for suggesting this problem.)

The converse of Wiener's inequality. Stein [15] proved that the converse of (1.1) is also true. More precisely, he showed that given a ball B , if $\text{supp}(f) \subset B$ and $Mf \in L^1(B)$, then

$$(1.7) \quad \int_B |f(x)| \log(e + |f(x)|) dx < \infty.$$

A similar result holds in variable L^p spaces.

Theorem 1.8. *Given a function $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$, let $q(x) = \chi_{\{x: p(x)=1\}}$. Then for any ball B , if $\text{supp}(f) \subset B$ and $\|Mf\|_{L^{p(\cdot)}(B)} < \infty$, then $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(B)} < \infty$.*

Theorem 1.8 seems to be an unsatisfactory converse to Theorems 1.2 and 1.5 since $q(\cdot)$, the exponent on the logarithm, is not continuous. One might conjecture that a sharper result holds, particularly if $p(\cdot)$ and $q(\cdot)$ satisfy some continuity condition. But surprisingly, given very reasonable continuity assumptions on these functions, a sharper inequality cannot hold, as the next example shows.

Example 1.9. Given exponent functions $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ and $q(\cdot) : \mathbb{R}^n \rightarrow [0, 1]$, such that $p_+ < \infty$, $q(x) = 1$ if $p(x) = 1$ and $q(x) < 1$ if $p(x) > 1$, suppose that $p(\cdot)$ satisfies the log-Hölder continuity condition (1.2) and $q(\cdot)$ satisfies the log-log-Hölder condition

$$(1.8) \quad |q(x) - q(y)| \leq \frac{C}{\log \log(|x - y|^{-1})}, \quad |x - y| < e^{-e}.$$

Then there exists a function f supported on a ball B contained in the set where $p(x) > 1$ and $0 < q(x) < 1$ such that $\|Mf\|_{L^{p(\cdot)}(B)} < \infty$ but $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(B)} = \infty$.

Remark 1.10. It is an open question whether a result of this type holds for less continuous $p(\cdot)$ and $q(\cdot)$. Without any evidence we speculate that the growth condition in Theorem 1.11 below may be pertinent.

A theorem of Hästö. The local integrability of Mf when $p_- = 1$ has also been considered by Hästö [11]. He considered the case when the set $\{x : p(x) = 1\}$ has measure 0, and the set where $p(x)$ is close to 1 is small. More precisely, he showed that if Ω is a bounded set, and if for some $\epsilon > 0$ and all $s > 0$ sufficiently small,

$$|\{x \in \Omega : p(x) \leq \lambda_\epsilon(s)\}| \leq Cs, \quad \text{where } \lambda_\epsilon(s) = 1 + (1 + \epsilon) \frac{\log \log(1/s)}{\log(1/s)},$$

then $\|Mf\|_{L^1(\Omega)} \leq \|f\|_{L^{p(\cdot)}(\Omega)}$. The heart of his proof is to show that given these hypotheses, $\|f\|_{L \log L(\Omega)} \leq C\|f\|_{L^{p(\cdot)}(\Omega)}$; the desired conclusion then follows immediately from inequality (1.5).

By combining his ideas with those in the proof of Theorem 1.2 we can give a new proof of his result, one which does not pass through the classical inequality.

Theorem 1.11. *Given $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$, suppose there exist constants ϵ , $0 < \epsilon < 1$, $K > 0$, and δ , $0 < \delta < e^{-e}$, such that for $0 < s \leq \delta$,*

$$|\{x \in \mathbb{R}^n : p(x) \leq \lambda_\epsilon(s)\}| \leq Ks.$$

Then given any ball B , there exists a constant C (depending on $|B|$, $p(\cdot)$, ϵ , δ and K) such that

$$\|Mf\|_{L^1(B)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Theorem 1.11 is modestly stronger than the original result of Hästö since we can take the domain of f to be unbounded. (If we replace f by $f\chi_\Omega$, Ω bounded, we immediately get the same result for any bounded domain.) Further, he assumes that $p_+ < \infty$, whereas we allow unbounded exponents.

Remark 1.12. After we had found our proof of Theorem 1.11 we learned that Futamura and Mizuta [9] had shown that Hästö's result can be sharpened by replacing λ_ϵ , $\epsilon > 0$, with λ_0 . This answers in the affirmative a question asked by Hästö. Their proof again uses the classical result of Wiener.

It seems reasonable to conjecture that we can adapt the proof of Theorem 1.11 to prove a variant of Theorem 1.5: if the set where $p(x) = 1$ has measure zero and the set where $p(\cdot)$ is close to 1 is small, then there exists some exponent function $r(\cdot)$ not identically equal to one such that $\|Mf\|_{L^{r(\cdot)}(B)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. However, a key hypothesis in the proof of Theorem 1.5 was the log-Hölder continuity of $p(\cdot)$, and it is not possible for an exponent function $p(\cdot)$ to satisfy both this and Hästö's growth condition.

Proposition 1.13. *Let $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ be a bounded measurable function and let $\delta < e^{-e}$, ϵ , C , and K be positive constants. If $p_- = 1$, then it is not possible for $p(\cdot)$ to satisfy the log-Hölder continuity condition (1.2) and satisfy*

$$(1.9) \quad |\{x \in \mathbb{R}^n : p(x) \leq \lambda_\epsilon(s)\}| \leq Ks, \quad 0 < s < \delta.$$

A result of the type conjectured above may still be true, but we have no insight on how to prove it without log-Hölder continuity.

Organization of the paper. The rest of the paper is organized as follows: in Section 2 we prove Theorem 1.2 and Corollary 1.4. In Section 3 we prove Theorem 1.5. In Section 4 we prove Theorem 1.8 and construct Example 1.9. Finally, in Section 5 we prove Theorem 1.11 and Proposition 1.13.

Throughout the paper, the notation will be standard or defined as needed. In order to emphasize that we are dealing with variable exponents, we will always write $p(\cdot)$ instead of p to denote an exponent function. Unless otherwise specified, C and c will denote positive constants which will depend only on the dimension n , any underlying sets (such as a ball B), and the exponent function $p(\cdot)$.

2. PROOF OF THEOREM 1.2 AND COROLLARY 1.4

The proof of Theorem 1.2 requires two lemmas. The first is a generalization due to Aguilar Cañestro and Ortega Salvador [1] of a modular inequality in [4]. For completeness we include the short proof.

Lemma 2.1. *Given $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$, then for all $t > 0$,*

$$|\{x \in \mathbb{R}^n : Mf(x) > t\}| \leq C \int_{\{x: |f(x)| > t/2\}} \left(\frac{|f(x)|}{t/2} \right)^{p(x)} dx.$$

Proof. Fix $t > 0$, and write $f = f_1 + f_2$, where $f_1(x) = f(x)\chi_{\{x: |f(x)| > t/2\}}(x)$. Then for all x , $Mf_2(x) \leq t/2$, so

$$Mf(x) \leq Mf_1(x) + Mf_2(x) \leq Mf_1(x) + t/2.$$

Therefore, by the weak $(1, 1)$ inequality for the maximal operator (see [10]), and since $p(x) \geq 1$,

$$\begin{aligned} |\{x \in \mathbb{R}^n : Mf(x) > t\}| &\leq |\{x \in \mathbb{R}^n : Mf_1(x) > t/2\}| \\ &\leq C \int_{\{x: |f(x)| > t/2\}} \frac{|f(x)|}{t/2} dx \leq C \int_{\{x: |f(x)| > t/2\}} \left(\frac{|f(x)|}{t/2} \right)^{p(x)} dx. \end{aligned}$$

□

To state the second lemma, recall that a function A is log-convex if $\log A$ is convex.

Lemma 2.2. *For any $a > 1$, the function*

$$A(x) = \begin{cases} \frac{a^x - 1}{x} & 0 < x \leq 1 \\ \log(a) & x = 0 \end{cases}$$

is log-convex. In particular, given ϵ , $0 < \epsilon \leq 1$, for all x , $0 \leq x \leq \epsilon$,

$$A(x) \leq \epsilon^{-1} a^x \log(a)^{1-x/\epsilon}.$$

Proof. We first use the power series expansion of e^x to show that

$$(2.1) \quad \frac{a^x - 1}{\log(a)} \geq xa^{x/2}.$$

Since $2^n \geq n + 1$, for all $n \geq 0$,

$$xa^{x/2} = \sum_{n=0}^{\infty} \frac{x^{n+1} \log(a)^n}{2^n n!} \leq \sum_{n=0}^{\infty} \frac{x^{n+1} \log(a)^n}{(n+1)!} = \frac{1}{\log(a)} \sum_{n=1}^{\infty} \frac{x^n \log(a)^n}{n!} = \frac{a^x - 1}{\log(a)}.$$

We now show that $C(x) = \log(A(x))$ is convex by showing that $C''(x) \geq 0$:

$$C''(x) = \frac{1}{x^2} - \frac{a^x \log(a)^2}{(a^x - 1)^2} \geq \frac{1}{x^2} - a^x \left(\frac{1}{xa^{x/2}} \right)^2 = 0.$$

(The middle inequality follows from (2.1).)

Finally, to establish the desired inequality, we first let $\epsilon = 1$. Then by log-convexity we have that for all $x \leq 1$,

$$A(x) = A(1 \cdot x + 0 \cdot (1 - x)) \leq A(1)^x A(0)^{1-x} \leq a^x \log(a)^{1-x}.$$

Now fix $\epsilon < 1$ and $0 < x \leq \epsilon$. Then

$$A(x/\epsilon) = \frac{a^{x/\epsilon} - 1}{x/\epsilon} \leq a^{x/\epsilon} \log(a)^{1-x/\epsilon}.$$

Since this inequality is true for any $a > 1$, replace a by $\bar{a} = a^{1/\epsilon} > 1$; then we get that

$$\frac{\bar{a}^x - 1}{x/\epsilon} \leq \bar{a}^x \log(\bar{a})^{1-x/\epsilon},$$

which in turn implies

$$\frac{\bar{a}^x - 1}{x} \leq \epsilon^{-x/\epsilon} \bar{a}^x \log(\bar{a})^{1-x/\epsilon} \leq \epsilon^{-1} \bar{a}^x \log(\bar{a})^{1-x/\epsilon}.$$

□

Proof of Theorem 1.2. Given the exponent function $p(\cdot)$, fix a function f such that the right-hand side of (1.4) is finite. Without loss of generality we may assume that f is non-negative.

Fix $\epsilon > 0$ and define the function $\bar{p}$ by

$$\bar{p}(x) = \begin{cases} \frac{p(x)+1}{2} & \text{if } 1 \leq p(x) < 1 + \epsilon \\ p(x) & \text{if } p(x) \geq 1 + \epsilon. \end{cases}$$

For all x , $p(x)/2 \leq \bar{p}(x) \leq p(x)$. Fix a ball B ; then

$$\begin{aligned} \int_B Mf(x) dx &\leq 2|B| + \int_{\{x \in B : Mf(x) > 2\}} Mf(x) dx \\ &= 2|B| + \int_2^\infty |\{x \in B : Mf(x) > t\}| dt; \end{aligned}$$

by Lemma 2.1,

$$\leq 2|B| + C \int_2^\infty \int_{\{x : f(x) > t/2\}} \left(\frac{f(x)}{t/2} \right)^{\bar{p}(x)} dx dt;$$

by the change of variables $s = t/2$ this becomes

$$\leq 2|B| + C \int_1^\infty \int_{\{x : f(x) > s\}} \left(\frac{f(x)}{s} \right)^{\bar{p}(x)} dx ds;$$

by Fubini's theorem,

$$= 2|B| + C \int_{\{x : f(x) > 1\}} \int_1^{f(x)} s^{-\bar{p}(x)} ds f(x)^{\bar{p}(x)} dx.$$

We evaluate the inner integral depending on the size of $\bar{p}(x)$. If $\bar{p}(x) = 1$, then

$$\int_1^{f(x)} s^{-\bar{p}(x)} ds = \int_1^{f(x)} s^{-1} ds = \log(f(x)).$$

If $\bar{p}(x) \geq 1 + \epsilon$, then

$$\int_1^{f(x)} s^{-\bar{p}(x)} ds = \frac{1}{\bar{p}(x) - 1} \left(1 - f(x)^{1-\bar{p}(x)} \right) \leq 1/\epsilon.$$

If $1 < \bar{p}(x) < 1 + \epsilon$, then we actually have that $1 < \bar{p}(x) < 1 + \epsilon/2$. Thus, since x is such that $f(x) > 1$,

$$\begin{aligned} \int_1^{f(x)} s^{-\bar{p}(x)} ds &= \frac{f(x)^{1-\bar{p}(x)}}{\bar{p}(x) - 1} \left(f(x)^{\bar{p}(x)-1} - 1 \right) \\ &\leq \left(\frac{f(x)^{\bar{p}(x)-1} - 1}{\bar{p}(x) - 1} \right); \end{aligned}$$

by Lemma 2.2 with $a = f(x)$, with x replaced by $\bar{p}(x) - 1$, and with ϵ replaced by $\epsilon/2$, we have that

$$\leq 2\epsilon^{-1} [\log(f(x))]^{1-(\bar{p}(x)-1)/(\epsilon/2)} f(x)^{\bar{p}(x)-1}.$$

Returning now to our original estimate, if we define the sets

$$\begin{aligned} E_1 &= \{x : f(x) > 1\} \cap \{x : \bar{p}(x) = 1\}, \\ E_2 &= \{x : f(x) > 1\} \cap \{x : \bar{p}(x) \geq 1 + \epsilon\}, \\ E_3 &= \{x : f(x) > 1\} \cap \{x : 1 < \bar{p}(x) < 1 + \epsilon\}, \end{aligned}$$

then

$$\begin{aligned} 2|B| + C \int_{\{x: f(x) > 1\}} \int_1^{f(x)} s^{-\bar{p}(x)} ds f(x)^{\bar{p}(x)} dx \\ = 2|B| + C \int_{E_1} f(x)^{\bar{p}(x)} \log(f(x)) dx \\ + C\epsilon^{-1} \int_{E_2} f(x)^{\bar{p}(x)} dx \\ + C\epsilon^{-1} \int_{E_3} f(x)^{2\bar{p}(x)-1} [\log(f(x))]^{1-(\bar{p}(x)-1)/(\epsilon/2)} dx \\ \leq 2|B| + C \int_{\{x: f(x) > 1\}} f(x)^{p(x)} [\log(e + f(x))]^{q(x)} dx \\ \leq 2|B| + C \int_{\mathbb{R}^n} f(x)^{p(x)} [\log(e + f(x))]^{q(x)} dx. \end{aligned}$$

This completes the proof. $\square$

Proof of Corollary 1.4. Fix a function f and a ball B ; then we can apply Theorem 1.2 to the family of functions f/λ , $\lambda > 0$, to get

$$\begin{aligned} (2|B| + C)^{-1} \|Mf\|_{L^1(B)} \\ = \inf \left\{ \lambda > 0 : \int_B M(f/\lambda)(x) dx \leq 2|B| + C \right\} \\ \leq \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)} [\log(e + |f(x)/\lambda|)]^{q(x)} dx \leq 1 \right\} \\ = \|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)}. \end{aligned} \quad \square$$

3. PROOF OF THEOREM 1.5

In our proof we need a few basic facts about variable L^p and Orlicz spaces. For proofs, see Kováčik and Rákosník [12] and Musielak [13, Section 8, p. 43].

Lemma 3.1. *Given the exponent function $p(\cdot) : \Omega \rightarrow [1, \infty)$, if $\|f\|_{L^{p(\cdot)}(\Omega)} \leq 1$, then*

$$\int_{\Omega} |f(x)|^{p(x)} dx \leq \|f\|_{L^{p(\cdot)}(\Omega)}.$$

Conversely, if

$$\int_{\Omega} |f(x)|^{p(x)} dx < \infty,$$

then $\|f\|_{L^{p(\cdot)}(\Omega)} < \infty$.

Lemma 3.2. *Given variable Orlicz spaces $L^{\Phi_1(\cdot)}$ and $L^{\Phi_2(\cdot)}$, if $\Phi_1(x, t) \leq \Phi_2(x, t)$, $x \geq 0$, $t > 0$, then for all f , $\|f\|_{L^{\Phi_1(\cdot)}(\mathbb{R}^n)} \leq \|f\|_{L^{\Phi_2(\cdot)}(\mathbb{R}^n)}$.*

Similarly, given any ball B , if for some $t_0 > 0$, $\Phi_1(x, t) \leq \Phi_2(x, t)$, $x \geq 0$, $t > t_0$, then there exists a constant C depending on $|B|$ such that $\|f\|_{L^{\Phi_1(\cdot)}(B)} \leq C\|f\|_{L^{\Phi_2(\cdot)}(B)}$.

The proof also depends on the following lemma due to Capone, Cruz-Uribe and Fiorenza [2], who used it to give a new proof of the boundedness of the maximal operator on variable L^p spaces.

Lemma 3.3. *Given $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ such that $1 \leq p_- \leq p_+ < \infty$, and such that (1.2) and (1.3) hold, then there exists a bounded function S such that if $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 1$, then*

$$Mf(x)^{p(x)} \leq CM(f(\cdot)^{p(\cdot)})(x) + S(x).$$

Remark 3.4. The statement of Lemma 3.3 in [2] (see Theorem 4.1) only asserts that $S \in L^1$ if $p_- > 1$; however, it is immediate from the proof that we can take $p_- = 1$, and that S is bounded since it is some (variable) power of $(e + |x|)^{-n}$. Furthermore, the statement has the hypothesis that $|f|_{L^{p(\cdot)}} = \rho_1(f) \leq 1$, but this follows from our assumption that $\|f\|_{L^{p(\cdot)}} \leq 1$ by Lemma 3.1.

Finally, we need a property of log-Hölder continuous functions. The proof is straightforward; see, for example, the proof of Lemma 4.3 in [5].

Lemma 3.5. *Let f and g be two functions that satisfy the log-Hölder conditions (1.2) and (1.3). Then $\max(f, g)$ and $\min(f, g)$ also satisfy these conditions.*

Proof of Theorem 1.5. Fix ϵ , $0 < \epsilon < 1$. We first define $r(x)$. Let

$$R(x) = p(x) + (p(x) - 1)(p(x) - (1 + \epsilon)) = (p(x) - 1)(p(x) - \epsilon) + 1.$$

Since $p(\cdot)$ is log-Hölder continuous, $p_+ < \infty$, so

$$|p(x)^2 - p(y)^2| \leq 2p_+|p(x) - p(y)|;$$

hence, $p(x)^2$ satisfies (1.2) and (1.3), and so $R(\cdot)$ does as well. Now let $r(x) = \min(p(x), R(x))$. Then by Lemma 3.5, $r(\cdot)$ satisfies (1.2) and (1.3). If $p(x) = 1$ or if $p(x) \geq 1 + \epsilon$, then $r(x) = p(x)$. If x is such that $1 < p(x) < 1 + \epsilon$, then $1 < r(x) < p(x)$.

To define $q(\cdot)$ we first modify $r(\cdot)$. Let $F = \{x : p(x) \leq 1 + \epsilon/3\}$ and let

$$r^* = \sup_{x \in F} r(x).$$

(If F is empty, let $r^* = 1 + \epsilon/3$.) Define $\tilde{r}(x) = \min(r(x), r^*)$, and let

$$q(x) = \max\left(\frac{3}{\epsilon} \left(1 + \frac{\epsilon}{3} - \frac{p(x)}{\tilde{r}(x)}\right), 0\right).$$

By Lemma 3.5, $\tilde{r}(\cdot)$ is log-Hölder continuous, so $q(\cdot)$ is continuous. Since $\tilde{r}(x) \leq r(x) \leq p(x)$, we have $0 \leq q(x) \leq 1$. Furthermore, if $x \in F$, $r(x) < p(x) \leq 1 + \epsilon/3$, so $r^* \leq 1 + \epsilon/3$. Thus for all $x \in \mathbb{R}^n$, $\tilde{r}(x) \leq r^* \leq 1 + \epsilon/3$. Therefore, if x is such that $p(x) \geq 1 + \epsilon$, then

$$\frac{p(x)}{\tilde{r}(x)} \geq \frac{1 + \epsilon}{1 + \epsilon/3} \geq 1 + \frac{\epsilon}{3}.$$

Hence, if $p(x) \geq 1 + \epsilon$, $q(x) = 0$. Similarly, if $p(x) = 1$, $\tilde{r}(x) = r(x) = 1$, so $q(x) = 1$.

Now fix B and $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$; by homogeneity we may assume without loss of generality that $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}} = 1$. Then by Lemma 3.2, $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 1$ and by Lemma 3.1,

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx \leq 1.$$

To complete the proof, again by Lemma 3.1 it will suffice to show that

$$(3.1) \quad \int_B Mf(x)^{r(x)} dx \leq C,$$

where C depends only on $|B|$, ϵ , and $p(\cdot)$.

We define the following sets:

$$\begin{aligned} B_1 &= \{x \in B : Mf(x) \leq 1, p(x) > 1 + \epsilon/3\}, \\ B_2 &= \{x \in B : Mf(x) > 1, p(x) > 1 + \epsilon/3\}, \\ B_3 &= \{x \in B : p(x) \leq 1 + \epsilon/3\}. \end{aligned}$$

Now divide the integral in (3.1) into three pieces:

$$\int_B Mf(x)^{r(x)} dx = \int_{B_1} Mf(x)^{r(x)} dx + \int_{B_2} Mf(x)^{r(x)} dx + \int_{B_3} Mf(x)^{r(x)} dx.$$

We estimate each integral in turn. The first is straightforward:

$$\int_{B_1} Mf(x)^{r(x)} dx \leq |B_1| \leq |B| < \infty.$$

To estimate the second, first note that since $r(x) \leq p(x)$,

$$\int_{B_2} Mf(x)^{r(x)} dx \leq \int_{B_2} Mf(x)^{p(x)} dx.$$

If $x \in B_2$, $p(x)/(1 + \epsilon/3) \geq 1$ and $p(\cdot)/(1 + \epsilon/3)$ is log-Hölder continuous, so by Lemma 3.3,

$$\leq C \int_{B_2} M(|f(\cdot)|^{p(\cdot)/(1+\epsilon/3)})^{1+\epsilon/3} dx + C \int_{B_2} S(x)^{1+\epsilon/3} dx.$$

Since the maximal operator is bounded on $L^{1+\epsilon/3}$ and S is bounded,

$$\begin{aligned} &\leq C \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx + C \\ &\leq C. \end{aligned}$$

Finally, to estimate the third integral, since $\tilde{r}(\cdot)$ satisfies the hypotheses of Lemma 3.3 we have that

$$\begin{aligned} \int_{B_3} Mf(x)^{r(x)} dx &= \int_{B_3} Mf(x)^{\tilde{r}(x)} dx \\ &\leq C \int_{B_3} M(|f(\cdot)|^{\tilde{r}(\cdot)}) dx + \int_{B_3} S(x) dx \\ &\leq C \int_B M(|f(\cdot)|^{\tilde{r}(\cdot)}) dx + C. \end{aligned}$$

Now apply Theorem 1.2 with exponent $p(\cdot)/\tilde{r}(\cdot)$ and with ϵ replaced by $\epsilon/3$ to get

$$\begin{aligned} &\leq C|B| + C \int_{\mathbb{R}^n} |f(x)|^{p(x)} \log(e + |f(x)|^{\tilde{r}(x)})^{q(x)} dx + C \\ &\leq C|B| + C \int_{\mathbb{R}^n} |f(x)|^{p(x)} \log(e + |f(x)|)^{q(x)} dx + C \\ &\leq C. \end{aligned}$$

This completes the proof. $\square$

Remark 3.6. It can also be shown that the exponent function $q(\cdot)$ is log-Hölder continuous. The details are left to the reader.

4. PROOF OF THEOREM 1.8 AND CONSTRUCTION OF EXAMPLE 1.9

Proof of Theorem 1.8. Fix a ball B and f with $\text{supp}(f) \subset B$. Without loss of generality we may assume $\|Mf\|_{L^{p(\cdot)}(B)} = 1$, so by Lemma 3.2, $\|Mf\|_{L^1(B)} \leq 1$. Hence, by (1.7), Stein's converse of Wiener's inequality,

$$\int_B |f(x)| \log(e + |f(x)|) dx \leq C,$$

where C depends on $|B|$. Therefore,

$$\begin{aligned} &\int_B |f(x)|^{p(x)} \log(e + |f(x)|)^{q(x)} dx \\ &= \int_{\{x:p(x)=1\}} |f(x)| \log(e + |f(x)|) dx + \int_{\{x:p(x)>1\}} |f(x)|^{p(x)} dx \\ &\leq \int_{\{x:p(x)=1\}} |f(x)| \log(e + |f(x)|) dx + \int_{\{x:p(x)>1\}} Mf(x)^{p(x)} dx \\ &\leq C, \end{aligned}$$

so by Lemma 3.1, $\|f\|_{L^{p(\cdot)}(\log)^{q(\cdot)}(B)}$ is finite. $\square$

The construction of Example 1.9 requires one lemma whose proof will be given after the construction itself.

Lemma 4.1. *Let $a(\cdot), b(\cdot) : [0, e^{-e}] \rightarrow (0, \infty)$ be such that $a(0) < 1$, $a(\cdot)$ satisfies (1.2) and $b(\cdot)$ satisfies (1.8). If we define*

$$f(x) = x^{-a(x)} \log(1/x)^{-b(x)} \chi_{(0, e^{-e})},$$

then there exists a constant C such that for all $x \in (0, e^{-e})$,

$$Mf(x) \leq Cf(x).$$

Remark 4.2. It is part of the “folklore” of weighted norm inequalities that $f(x) = x^{-a} \log(1/x)^{-b}$, $0 < a < 1$, satisfies the Muckenhoupt A_1 condition on $(0, e^{-e})$. Lemma 4.1 is a generalization of this result.

Construction of Example 1.9. To avoid complications, we will construct the example on the real line; the same ideas can be extended to higher dimensions. We will also assume that there exists a point x such that $p(x) > 1$, since otherwise the example is vacuous.

Since $p(\cdot)$ and $q(\cdot)$ are continuous and $q(x) < 1$ if $p(x) > 1$, there exists $\delta > 0$ and an interval on which $p(x) \geq 1 + \delta$ and $\delta < q(x) < 1 - \delta$. By translation we

may assume the left endpoint of this interval is 0; without loss of generality we will assume that it is the interval $[0, e^{-e})$. (If the interval is shorter than this in length we can take an appropriate subinterval.)

Define $f(x) = x^{-a(x)} \log(1/x)^{-b(x)}$, where

$$a(x) = \frac{1}{p(x)}, \quad b(x) = \frac{1+q(x)}{p(x)}.$$

Since

$$\left| \frac{1}{p(x)} - \frac{1}{p(y)} \right| \leq \frac{|p(x) - p(y)|}{p_-^2},$$

$a(\cdot)$ satisfies the log-Hölder condition (1.2). Further, $a(0) \leq a_+ \leq (1+\delta)^{-1} < 1$. Similarly,

$$\begin{aligned} |b(x) - b(y)| &\leq p_-^{-2} (|p(x) - p(y)| + |p(x)q(y) - p(x)q(x)| + |p(x)q(x) - p(y)q(x)|) \\ &\leq p_-^{-2} (|p(x) - p(y)| + p_+|q(y) - q(x)| + q_+|p(x) - p(y)|), \end{aligned}$$

and so $b(\cdot)$ satisfies the log-log-Hölder condition (1.8). Therefore, by Lemma 4.1, $Mf(x) \leq Cf(x)$ for $x \in [0, e^{-e})$.

Hence, since $q(x) > \delta$ for all x ,

$$\begin{aligned} \int_0^{e^{-e}} Mf(x)^{p(x)} dx &\leq C \int_0^{e^{-e}} f(x)^{p(x)} dx \\ &= C \int_0^{e^{-e}} x^{-1} \log(1/x)^{-1-q(x)} dx \\ &\leq C \int_0^{e^{-e}} x^{-1} \log(1/x)^{-1-\delta} dx \\ &< +\infty. \end{aligned}$$

Thus by Lemma 3.1, $Mf \in L^{p(\cdot)}(0, e^{-e})$.

On the other hand, to see that $f \notin L^{p(\cdot)}(\log L)^{q(\cdot)}$, first note that there exists $0 < \gamma < e^{-e}$ such that if $0 < x < \gamma$,

$$\begin{aligned} \log(e + f(x)) &\geq \log(f(x)) = a(x) \log(1/x) - b(x) \log \log(1/x) \\ &\geq \frac{\log(1/x)}{p_+} - 2 \log \log(1/x) \geq \frac{\log(1/x)}{2p_+}. \end{aligned}$$

Hence,

$$\begin{aligned} \int_0^{e^{-e}} f(x)^{p(x)} \log(e + f(x))^{q(x)} dx &\geq c \int_0^\gamma x^{-1} \log(1/x)^{-1-q(x)} \log(1/x)^{q(x)} dx \\ &= c \int_0^\gamma x^{-1} \log(1/x)^{-1} dx = +\infty. \end{aligned}$$

Since $p_+ < \infty$ and $q_+ < \infty$, it follows from the definition of the variable Orlicz norm that $f \notin L^{p(\cdot)}(\log L)^{q(\cdot)}(0, e^{-e})$. $\square$

Proof of Lemma 4.1. Fix $x \in (0, e^{-e})$; since $\text{supp}(f)$ is contained in this interval, there exists an interval $I \subset (0, e^{-e})$ containing x such that

$$Mf(x) \leq \frac{2}{|I|} \int_I f(t) dt.$$

Therefore,

$$\frac{Mf(x)}{f(x)} \leq \frac{2}{|I|} x^{a(x)} \log(1/x)^{b(x)} \int_I t^{-a(t)} \log(1/t)^{-b(t)} dt.$$

We now apply the continuity assumptions on $a(\cdot)$ and $b(\cdot)$. First, from (1.8) we have that

$$\begin{aligned} \log(1/x)^{b(x)} &= \log(1/x)^{b(0)} \log(1/x)^{b(x)-b(0)} \\ &\leq \log(1/x)^{b(0)} \exp(|b(x) - b(0)| \log \log(1/x)) \leq C \log(1/x)^{b(0)}. \end{aligned}$$

Similarly, we have for each $t \in I$ that

$$\log(1/t)^{-b(t)} \leq C \log(1/t)^{-b(0)}.$$

Exactly the same argument using (1.2) shows that $x^{a(x)} \leq Cx^{a(0)}$ and $t^{-a(t)} \leq Ct^{-a(0)}$.

Hence,

$$\frac{Mf(x)}{f(x)} \leq Cx^{a(0)} \log(1/x)^{b(0)} \frac{1}{|I|} \int_I t^{-a(0)} \log(1/t)^{-b(0)} dt.$$

Since the function $t^{-a(0)} \log(1/t)^{-b(0)}$ is decreasing, we can increase its average by taking the average over the interval $(0, x)$. Thus, since $a(0) < 1$,

$$\frac{Mf(x)}{f(x)} \leq Cx^{a(0)-1} \log(1/x)^{b(0)} \int_0^x t^{-a(0)} \log(1/t)^{-b(0)} dt \leq Cx^{a(0)-1} \int_0^x t^{-a(0)} dt \leq C.$$

This completes the proof. $\square$

5. PROOF OF THEOREM 1.11 AND PROPOSITION 1.13

Proof of Theorem 1.11. The proof is initially very similar to the proof of Theorem 1.2. Fix a function $f \in L^{p(\cdot)}(\mathbb{R}^n)$. Without loss of generality we may assume that f is non-negative. We may also assume that $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = 1$, so by Lemma 3.1,

$$\int_{\mathbb{R}^n} f(x)^{p(x)} dx \leq 1.$$

Fix a ball B ; then again by Lemma 3.1 it will suffice to prove

$$(5.1) \quad \int_B Mf(x) dx \leq C,$$

where C depends only on $|B|$, ϵ , δ , K , and $p(\cdot)$. Fix $\gamma > 1/\epsilon$ and define the function

$$\bar{p}(x) = \begin{cases} \frac{\gamma+p(x)}{1+\gamma} & 1 < p(x) < \Lambda \\ p(x) & p(x) \geq \Lambda, \end{cases}$$

where the value Λ will be chosen below and will depend only on δ . Note that $p(x)/(\gamma+1) \leq \bar{p}(x) \leq p(x)$ for all x .

We now argue exactly as we did in the proof of Theorem 1.2 to get

$$(5.2) \quad \int_B Mf(x) dx \leq 2|B| + C \int_{\{x: f(x) > 1\}} \frac{1 - f(x)^{1-\bar{p}(x)}}{\bar{p}(x) - 1} f(x)^{\bar{p}(x)} dx.$$

To estimate this integral, we decompose the set $E = \{x \in \mathbb{R}^n : f(x) > 1\}$. Choose $k > 1$ so that $e^{-k} \leq \delta$. For $j \geq k$, let $\alpha_j = (1 + \epsilon) \log(j)/j$. Define the sets

$$A_0 = \{x \in E : p(x) > 1 + \alpha_k\},$$

$$A_j = \{x \in E : 1 + \alpha_{j+1} < p(x) \leq 1 + \alpha_j\}, \quad j \geq k.$$

Then $E = A_0 \cup \bigcup_{j \geq k} A_j$ and by hypothesis, if $j \geq k$, $|A_j| \leq C e^{-j}$.

We further subdivide the sets A_j , $j \geq k$: define

$$B_j = \left\{ x \in A_j : f(x) \geq \left(\frac{\gamma+1}{\alpha_{j+1}} \right)^{\frac{\gamma+1}{\gamma\alpha_{j+1}}} \right\},$$

$$C_j = \left\{ x \in A_j : f(x) < \left(\frac{\gamma+1}{\alpha_{j+1}} \right)^{\frac{\gamma+1}{\gamma\alpha_{j+1}}} \right\}.$$

Then the right-hand side of (5.2) is bounded by

$$\begin{aligned} & C \int_{A_0} \frac{1 - f(x)^{1-\bar{p}(x)}}{\bar{p}(x) - 1} f(x)^{\bar{p}(x)} dx + C \sum_{j \geq k} \int_{B_j} \frac{1 - f(x)^{1-\bar{p}(x)}}{\bar{p}(x) - 1} f(x)^{\bar{p}(x)} dx \\ & + C \sum_{j \geq k} \int_{C_j} \frac{1 - f(x)^{1-\bar{p}(x)}}{\bar{p}(x) - 1} f(x)^{\bar{p}(x)} dx \\ & = I_1 + I_2 + I_3. \end{aligned}$$

We will estimate each term separately. To estimate I_1 , we now fix $\Lambda = 1 + \alpha_k$. Then for all $x \in A_0$, $\bar{p}(x) = p(x) \geq \Lambda$, so

$$I_1 = C \int_{A_0} \frac{1 - f(x)^{1-p(x)}}{p(x) - 1} f(x)^{p(x)} dx \leq C(\Lambda - 1)^{-1} \int_{A_0} f(x)^{p(x)} dx \leq C.$$

(Here the constant C depends on $p(\cdot)$, ϵ and δ .)

To estimate I_2 , first note that the integrand in each term is bounded by

$$\frac{f(x)^{\gamma(1-\bar{p}(x))}}{\bar{p}(x) - 1} f(x)^{\bar{p}(x) + \gamma(\bar{p}(x) - 1)} = \frac{f(x)^{\gamma(1-\bar{p}(x))}}{\bar{p}(x) - 1} f(x)^{p(x)}.$$

We will now show that the fraction on the right-hand side is bounded by 1. On A_j ,

$$1 + \frac{\alpha_{j+1}}{\gamma+1} < \bar{p}(x) \leq 1 + \frac{\alpha_j}{\gamma+1}.$$

Therefore, on B_j ,

$$\frac{f(x)^{\gamma(1-\bar{p}(x))}}{\bar{p}(x) - 1} \leq \left(\frac{\alpha_{j+1}}{\gamma+1} \right)^{\gamma \cdot \frac{\gamma+1}{\gamma\alpha_{j+1}} \cdot \frac{\alpha_{j+1}}{\gamma+1}} \left(\frac{\alpha_{j+1}}{\gamma+1} \right)^{-1} = 1,$$

so we have that

$$I_2 \leq C \sum_{j \geq k} \int_{B_j} f(x)^{p(x)} dx \leq C \int_{\mathbb{R}^n} f(x)^{p(x)} dx \leq C.$$

Finally, we estimate I_3 . By the definition of C_j ,

$$I_3 \leq C \sum_{j \geq k} \int_{C_j} \frac{f(x)^{\bar{p}(x)}}{\bar{p}(x) - 1} dx \leq C \sum_{j \geq k} \left(\frac{\gamma+1}{\alpha_{j+1}} \right)^{\frac{\gamma+1}{\gamma\alpha_{j+1}}(1 + \frac{\alpha_j}{\gamma+1})} \left(\frac{\alpha_{j+1}}{\gamma+1} \right)^{-1} |C_j|.$$

We estimate the j -th term:

$$\begin{aligned} & \left(\frac{\gamma+1}{\alpha_{j+1}} \right)^{\frac{\gamma+1}{\gamma\alpha_{j+1}}(1+\frac{\alpha_j}{\gamma+1})} \left(\frac{\alpha_{j+1}}{\gamma+1} \right)^{-1} |C_j| \\ & \leq C \left(\frac{\gamma+1}{\alpha_{j+1}} \right)^{\frac{\gamma+1}{\gamma\alpha_{j+1}}(1+\frac{\alpha_j}{\gamma+1})+1} e^{-j} \\ & \leq C \exp \left(\left(\frac{\gamma+1}{\gamma\alpha_{j+1}} \left(1 + \frac{\alpha_j}{\gamma+1} \right) + 1 \right) \log \left(\frac{\gamma+1}{\alpha_{j+1}} \right) - j \right) \\ & = C \exp \left(\left(\frac{\gamma+1}{\gamma\alpha_{j+1}} + \frac{\alpha_j}{\gamma\alpha_{j+1}} + 1 \right) \log \left(\frac{\gamma+1}{\alpha_{j+1}} \right) - j \right); \end{aligned}$$

since $\alpha_j/\alpha_{j+1} \leq 2$,

$$\leq C \exp \left(\left(\frac{\gamma+1}{\gamma\alpha_{j+1}} + \frac{3}{\gamma} \right) \log \left(\frac{\gamma+1}{\alpha_{j+1}} \right) - j \right);$$

by the definition of α_{j+1} ,

$$= \exp \left(\left(\log \left(\frac{\gamma+1}{1+\epsilon} \right) + \log \left(\frac{j+1}{\log(j+1)} \right) \right) \left(\frac{\gamma+1}{\gamma(1+\epsilon)} \frac{j+1}{\log(j+1)} + \frac{3}{\gamma} \right) - j \right).$$

Since $\gamma > 1/\epsilon$, $\gamma+1 < \gamma(1+\epsilon)$. Hence, for all j sufficiently large, there exists $\beta > 0$ such that the exponent is dominated by $-\beta j$. Therefore, we have that for some large constant C ,

$$I_3 \leq C \sum_{j \geq k} e^{-\beta j} < \infty.$$

Combining the previous three estimates, we see that (5.1) holds; this completes our proof. $\square$

Proof of Proposition 1.13. Assume to the contrary that there exists $p(\cdot)$ that simultaneously satisfies (1.2) and (1.9). We will derive a contradiction.

There are two cases.

Case I. If there exists a ball B such that $p_-(B) = 1$, then, since (1.2) implies that $p(\cdot)$ is continuous, there exists $x_0 \in \mathbb{R}^n$ such that $p(x_0) = 1$.

In this case, we begin by observing that $\lambda_\epsilon : (0, e^{-e}) \rightarrow (1, 1 + (1+\epsilon)/e)$ is strictly increasing and so invertible. Therefore, we can rewrite (1.9) as

$$|\{x \in \mathbb{R}^n : p(x) \leq t\}| \leq K \lambda_\epsilon^{-1}(t)$$

for all $t > 1$ sufficiently close to 1.

Similarly, if we apply the log-Hölder condition (1.2) in a neighborhood of x_0 , we get that for all x sufficiently close to x_0 ,

$$|p(x) - 1| \leq \frac{C}{-\log(|x - x_0|)}.$$

Hence, for all s sufficiently small,

$$\{x \in \mathbb{R}^n : |x - x_0| < s\} \subset \left\{ x \in \mathbb{R}^n : p(x) \leq 1 + \frac{C}{\log(1/s)} \right\},$$

and, in particular,

$$(5.3) \quad \omega_n s^n \leq \left| \left\{ x \in \mathbb{R}^n : p(x) \leq 1 + \frac{C}{\log(1/s)} \right\} \right|,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Now define $\psi : (0, 1) \rightarrow (1, \infty)$ by

$$\psi(s) = 1 + \frac{C}{\log(1/s)}.$$

Then ψ is also strictly increasing and invertible. Thus we can rewrite (5.3) as

$$\omega_n (\psi^{-1}(t))^n \leq |\{x \in \mathbb{R}^n : p(x) \leq t\}|$$

for all $t > 1$ sufficiently close to 1.

Therefore, we will get the desired contradiction if we can show that for all such t ,

$$K\lambda_\epsilon^{-1}(t) < \omega_n (\psi^{-1}(t))^n,$$

or equivalently that

$$t < \lambda_\epsilon (\omega_n K^{-1} (\psi^{-1}(t))^n).$$

To show this, note that if we take t sufficiently close to 1, then we have that

$$\begin{aligned} \log \left(\log(\omega_n K^{-1}) + \frac{nC}{t-1} \right) &> \frac{2(nC+1)}{1+\epsilon}, \\ (t-1) \log(\omega_n^{-1} K) &< 1. \end{aligned}$$

Thus, for all such t ,

$$\begin{aligned} \lambda_\epsilon (\omega_n K^{-1} (\psi^{-1}(t))^n) &= 1 + (1+\epsilon) \frac{\log(\log(\omega_n^{-1} K \psi^{-1}(t)^{-n}))}{\log(\omega_n^{-1} K \psi^{-1}(t)^{-n})} \\ &> 1 + (1+\epsilon) \frac{\frac{2(nC+1)}{1+\epsilon}}{\log(\omega_n^{-1} K) + \frac{nC}{t-1}} \\ &= 1 + (nC+1) \frac{2(t-1)}{(t-1) \log(\omega_n^{-1} K) + nC} \\ &> 1 + 2(t-1) \\ &> t. \end{aligned}$$

This is the desired contradiction, which completes the proof of Case I.

Case II. Suppose that $p_- = 1$ but $p(x) > 1$ for all $x \in \mathbb{R}^n$. Then there exists a sequence $\{x_n\}$ such that $|x_n| \rightarrow \infty$ and $p(x_n) \rightarrow 1$ as $n \rightarrow \infty$. Furthermore, by passing to a subsequence we may assume that for all n and m , $|x_n - x_m| \geq 2$.

Now fix any $s > 0$. Then there exists $N > 0$ such that if $n \geq N$, $p(x_n) < \lambda_\epsilon(s)$. But then by condition (1.2), there exists σ , $0 < \sigma < 1$, such that if $|x - x_n| < \sigma$ for some $n \geq N$, then $p(x) < \lambda_\epsilon(s)$. The balls $B_\sigma(x_n)$ are disjoint, so their union has infinite measure. Therefore, the set $\{x \in \mathbb{R}^n : p(x) \leq \lambda_\epsilon(s)\}$ has infinite measure. This contradicts the assumption that (1.9) holds, and our proof is complete. $\square$

REFERENCES

- [1] M.I. Aguilar Cañestro and P. Ortega Salvador, *Weighted weak type inequalities with variable exponents for Hardy and maximal operators*, Proc. Japan Acad. 82, Ser. A Math. Sci. (2006), 126–130. MR2279278 (2007i:42018)
- [2] C. Capone, D. Cruz-Urbe, SFO, and A. Fiorenza, *The fractional maximal operator on variable L^p spaces*, Rev. Mat. Iberoamericana, 23 (2007), no. 3, 743–770. MR2414490
- [3] D. Cruz-Urbe, SFO, and A. Fiorenza *Approximate identities in variable L^p spaces*, Math. Nach. 280 (2007), 256–270 MR2292148
- [4] D. Cruz-Urbe, SFO, A. Fiorenza and C.J. Neugebauer, *The maximal function on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. 28 (2003), 223–238, and 29 (2004), 247–249. MR1976842 (2004c:42039) MR2041952 (2004m:42018)
- [5] D. Cruz-Urbe, SFO, A. Fiorenza, J.M. Martell and C. Pérez, *The boundedness of classical operators on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. 31 (2006), 239–264. MR2210118 (2006m:42029)
- [6] L. Diening, *Maximal functions on generalized Lebesgue spaces $L^{p(x)}$* , Math. Inequal. Appl. 7(2004), 245–253. MR2057643 (2005k:42048)
- [7] L. Diening, *Maximal Function on Musielak-Orlicz Spaces and Generalized Lebesgue Spaces*, Bull. Sci. Math. 129 (2005), no. 8, 657–700. MR2166733 (2006e:46032)
- [8] L. Diening, P. Hästö, and A. Nekvinda, *Open problems in variable exponent Lebesgue and Sobolev spaces*, FSDONA04 Proceedings (Drabek and Rakosnik (eds.); Milovy, Czech Republic, Academy of Sciences of the Czech Republic, Prague, 2005, 38–58.
- [9] T. Futamura and Y. Mizuta, *Maximal functions for Lebesgue spaces with variable exponent approaching 1*, Hiroshima Math. J. 36 (2006), 23–28. MR2213640 (2006k:42034)
- [10] J. García-Cuerva and J.L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Math. Studies 116, North-Holland, Amsterdam, 1985. MR807149 (87d:42023)
- [11] P. Hästö, *The maximal operator in Lebesgue spaces with variable exponents approaching 1*, Math. Nach., 280 (2007), 74–82. MR2290383 (2007j:42009)
- [12] O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$* , Czechoslovak Math. J. 41(116) (1991), 4, 592–618. MR1134951 (92m:46047)
- [13] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, 1034, Springer-Verlag, Berlin, 1983. MR724434 (85m:46028)
- [14] A. Nekvinda, *Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$* , Math. Ineq. Appl. 7 (2004), 255–265. MR2057644 (2005f:42045)
- [15] E.M. Stein, *Note on the class $L \log L$* , Studia Math. 32 (1969), 305–310. MR0247534 (40:799)
- [16] N. Wiener, *The ergodic theorem*, Duke Math. J. 5 (1939), 1–18. MR1546100

DEPARTMENT OF MATHEMATICS, TRINITY COLLEGE, HARTFORD, CONNECTICUT 06106-3100
E-mail address: david.cruzuribe@trincoll.edu

DIPARTIMENTO DI COSTRUZIONI E METODI MATEMATICI IN ARCHITETTURA, UNIVERSITÀ DI NAPOLI, VIA MONTEOLIVETO, 3, I-80134 NAPOLI, ITALY – AND – ISTITUTO PER LE APPLICAZIONI DEL CALCOLO “MAURO PICONE”, SEZIONE DI NAPOLI, CONSIGLIO NAZIONALE DELLE RICERCHE, VIA PIETRO CASTELLINO, 111, I-80131 NAPOLI, ITALY
E-mail address: fiorenza@unina.it