SINGULARLY PERTURBED NONLINEAR DIRICHLET PROBLEMS WITH A GENERAL NONLINEARITY

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ABSTRACT. Let Ω be a bounded domain in $\mathbf{R}^n$, $n \geq 3$, with a boundary $\partial\Omega \in C^2$. We consider the following singularly perturbed nonlinear elliptic problem on Ω :

$$\varepsilon^2 \Delta u - u + f(u) = 0, \quad u > 0 \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where the nonlinearity f is of subcritical growth. Under rather strong conditions on f , it has been known that for small $\varepsilon > 0$, there exists a mountain pass solution u_ε of above problem which exhibits a spike layer near a maximum point of the distance function d from $\partial\Omega$ as $\varepsilon \rightarrow 0$. In this paper, we construct a solution u_ε of above problem which exhibits a spike layer near a maximum point of the distance function under certain conditions on f , which we believe to be almost optimal.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbf{R}^n$, $n \geq 3$, with a C^2 boundary $\partial\Omega$. We are interested in the following singularly perturbed nonlinear elliptic problem on Ω :

$$(1) \quad \varepsilon^2 \Delta u - u + f(u) = 0, \quad u > 0 \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where a typical form of $f(u)$ is $|u|^{p-1}u$, $p \in (1, (n+2)/(n-2))$. The most basic solution for (1) is a mountain pass solution. For the existence of a mountain pass solution for (1), the following conditions have been basically assumed in the literature. For a continuous function $f : \mathbf{R} \rightarrow \mathbf{R}$,

(f1) $f(t) = 0$ for $t \leq 0$ and $\lim_{t \rightarrow 0} f(t)/t = 0$;

(f2) there exists $p \in (1, \frac{n+2}{n-2})$ such that $\limsup_{t \rightarrow \infty} f(t)/t^p < \infty$;

(f3-1) there exist $\mu > 2$ and $t_0 > 0$ such that $\mu \int_0^t f(s)ds < f(t)t$ for $t > t_0$.

Furthermore, if we assume additional strong conditions $f \in C^1$, (f1), (f2) and

(f3-2) there exist $\mu > 2$ such that $\mu \int_0^t f(s)ds < f(t)t$ for $t > 0$,

for small $\varepsilon > 0$, there exists a unique maximum point $x_\varepsilon \in \Omega$ of a mountain pass solution u_ε such that a transformed function $v_\varepsilon(x) \equiv u_\varepsilon(\varepsilon(x - x_\varepsilon))$ converges uniformly to a radially symmetric least energy solution of the following *limiting* problem:

$$(2) \quad \Delta u - u + f(u) = 0, \quad u > 0 \text{ in } \mathbf{R}^n \quad \text{and} \quad \lim_{|x| \rightarrow \infty} u(x) = 0.$$

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Thus, u_ε develops a spike layer as $\varepsilon \rightarrow 0$. In their striking paper [23], Ni and Wei proved that the point x_ε of the spike layer converges to the most centered points of Ω , that is, the maximum points of the distance function d from $\partial\Omega$ as $\varepsilon \rightarrow 0$. For the proof, they required the following additional conditions:

- (f4) $f(t)/t$ is nondecreasing on $(0, \infty)$;
- (f5) there exists a unique radially symmetric solution $U \in H^{1,2}(\mathbf{R}^n)$ for $\Delta u - u + f(u) = 0, u > 0$, in $\mathbf{R}^n$ such that if $\Delta V - V + f'(U)V = 0$ and $V \in H^{1,2}(\mathbf{R}^n)$, then $V = \sum_{i=1}^n a_i \frac{\partial U}{\partial x_i}$ for some $a_1, \dots, a_n \in \mathbf{R}$.

(See [13], [21], [24] and [27] for more works on higher energy solutions.) Furthermore, del Pino and Felmer [12] showed the asymptotic behavior of the spike layer by a more simple approach without the nondegeneracy condition (f5). Finally, the author [4] proved the asymptotic behavior without the additional conditions (f4) and (f5) for $n \geq 3$.

There have also been similar works [10], [11], [15], [17], [22] for the problem (1) when the nonlinearity f is typically of the form $f(u) = u - u(u - a)(u - 1)$, $0 < a < \frac{1}{2}$. A culminating result in this direction was obtained by Ni, Takagi and Wei [22]. The result says that if Ω satisfies a geometric condition (C), the same phenomena with the case f satisfying (f1), (f2) and (f3) occur for a mountain pass solution u_ε . Convex sets and annuli satisfy condition (C), but not domains with two or more holes (see [22] for the definition of condition (C)). Their assumptions for $f(t) \equiv t + g(t)$ are the following:

- (g1) $g \in C^{1+\sigma}$ for some $\sigma \in (0, 1)$ and $g(0) = 0, g'(0) < 0$;
- (g2) g has two positive zero z_1 and z_2 , such that $z_1 < z_2$ and g has no other positive zeros;
- (g3) $\int_0^{z_2} g(s)ds > 0$;
- (g4) the function $t \rightarrow f(t)/(t - t_0)$ is decreasing in the interval (t_0, z_2) , where t_0 is the unique number in (z_1, z_2) such that $\int_0^{t_0} f(s)ds = 0$.

In two or three dimensional cases, Dancer [11] proved the same phenomena without the geometric condition (C).

On the other hand, Berestycki and Lions had showed in the classical paper [9] that equation (2) has a radially symmetric least energy solution U satisfying, for some $C, \delta > 0$ and any $|\alpha| \leq 2$,

$$(3) \quad |D^\alpha U(x)| \leq C \exp(-\delta|x|), \quad x \in \mathbf{R}^n,$$

under conditions (f1), (f2) and the following condition:

- (f3) there exists $T > 0$ satisfying $F(T) \equiv \int_0^T f(t)dt > T^2/2$.

Moreover, they showed that for any solution $U \in H^1$ of (2), the following Pohozaev identity holds:

$$(4) \quad \frac{n-2}{2} \int_{\mathbf{R}^n} |\nabla U|^2 dx + \frac{n}{2} \int_{\mathbf{R}^n} U^2 dx - n \int_{\mathbf{R}^n} F(U) dx = 0.$$

We believe that conditions (f1), (f2) and (f3) are almost optimal for an existence of solutions for (2). In fact, it follows from the Pohozaev identity (4) that condition (f3) is necessary and that for $f(u) = u^p, p \geq \frac{n+2}{n-2}$, there exist no solutions of (2) in $H^{1,2}(\mathbf{R}^n)$. When f satisfies (g1), (g2), (g3) and (g4), the mountain pass solution considered in [10], [15], [17] and [22] has its range in $(0, z_2)$. Thus, redefining $g(t) = 0$ for $t \geq z_2$, we see that conditions (g1), (g2), (g3) and (g4) are stronger than

conditions **(f1)**, **(f2)** and **(f3)**. In [18], Jeanjean and Tanaka noticed that a least energy solution of (2) is a mountain pass solution. Since the mountain pass solution is structurally stable, it seems natural to expect that there exists a *corresponding* solution of a singularly perturbed problem (1) for small $\varepsilon > 0$ whenever a limiting problem (2) has a least energy solution. In this paper we will prove that indeed, for $n \geq 3$, there exists a *perturbed* solution and its maximum point converges to a most centered point of Ω as $\varepsilon \rightarrow 0$. Since we assume only **(f1)**, **(f2)**, **(f3)** in this paper, our approach unifies both results under conditions **(f1)**, **(f2)**, **(f3-2)** and under **(g1)**, **(g2)**, **(g3)**, **(g4)**. Furthermore we do not need any geometric condition on domain Ω and any additional smoothness of f . Our approach for an existence of a solution for small $\varepsilon > 0$ is basically variational, but takes advantage of a reduction to a compact set in a finite dimensional reduction method used powerfully when the nondegeneracy condition **(f5)** is satisfied. This approach was successfully carried out in the study of the standing waves for nonlinear Schrödinger equations [5], [6], [7].

We define $\mathcal{S}$ as the set of a solution U for (2) satisfying

$$U(0, \dots, 0, 0) = \max_{x \in \mathbf{R}^n} U(x).$$

Since we assume just that f is continuous, it is not certain whether any solution $u \in \mathcal{S}$ is radially symmetric and monotone decreasing. A result in [9] says that if **(f1)**, **(f2)** and **(f3)** are satisfied, there exists a radially symmetric solution $U \in \mathcal{S}$ which is monotone decreasing. Now we state our main results as follows.

Theorem 1.1. *We assume that $n \geq 3$ and a function $f \in C(\mathbf{R})$ satisfies conditions **(f1)**, **(f2)** and **(f3)**. Then, for sufficiently small $\varepsilon > 0$, there exists a solution u_ε of (1) and a maximum point $x_\varepsilon \in \Omega$ of u_ε such that*

- (i) *for some constant $C, c > 0$, $u_\varepsilon(x) \leq C \exp(-\frac{c}{\varepsilon} \text{dist}(x, x_\varepsilon))$,*
- (ii)

$$\lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon, \partial\Omega) = \max_{x \in \Omega} \text{dist}(x, \partial\Omega),$$

and, for a transformed solution $v_\varepsilon(x) \equiv u_\varepsilon(\varepsilon(x - x_\varepsilon))$, a sequence $\{v_\varepsilon\}_\varepsilon$ converges, up to a subsequence, uniformly to a least energy solution $U \in \mathcal{S}$ of (2).

This paper is organized as follows. In section 2, we introduce some notation and prepare some preliminary results for the proofs of the main result. In section 3, we consider the problem (2) when the domain Ω is a ball. Then, combining the results in sections 2 and 3, we prove our main theorem, Theorem 1.1.

2. PRELIMINARIES

First, we study the related limiting problem (1). The space $H^1 \equiv H^{1,2}(\mathbf{R}^n)$ is a completion of $C_0^\infty(\mathbf{R}^n)$ with respect to the norm

$$\|u\| \equiv \left(\int_{\mathbf{R}^n} |\nabla u|^2 + u^2 dx \right)^{1/2}.$$

Since we are interested in positive solutions, we may assume that $f(t) = 0$ for $t \leq 0$. We define an energy functional Γ on H^1 by

$$\Gamma(u) = \frac{1}{2} \int_{\mathbf{R}^n} |\nabla u|^2 + u^2 dx - \int_{\mathbf{R}^n} F(u) dx,$$

where $F(t) = \int_0^t f(s) ds$.

Any critical point of Γ in H^1 is a solution of (2). Moreover, any solution $u \in H^1$ (not necessarily positive) of (2) is contained in the following set $\mathcal{P}$:

$$\mathcal{P} \equiv \{u \in H^1 \setminus \{0\} \mid \frac{n-2}{2} \int_{\mathbf{R}^n} |\nabla u|^2 dx + n \int_{\mathbf{R}^n} \frac{u^2}{2} - F(u) dx = 0\},$$

which comes from the Pohozaev identity. The existence of a least energy solution for (2) was proved in [9] when f satisfies **(f1)**, **(f2)**, **(f3)**. It is noticed in [18] that any least energy solution is a mountain pass solution. Here we use a different approach, which is more suitable for our proof of the main result in this paper. We consider the following minimization:

$$(5) \quad I \equiv \inf_{u \in \mathcal{P}} \Gamma(u).$$

Proposition 2.1. *We assume that $n \geq 3$ and that **(f1)**, **(f2)** and **(f3)** hold. Then, there exists a radially symmetric and monotone decreasing minimizer $U \in \mathcal{P} \cap C^2$ of (5) which satisfies equation (2). Any minimizer U of (5) is a least energy solution of (2).*

Proof. From **(f1)** and **(f3)**, we see that $\mathcal{P}$ is not empty. Let $\{u_m\}_{m=1}^\infty$ be a minimizing sequence of (5). Let u_m^* be a radially symmetric rearrangement (Schwartz symmetrization) of u_m , and let $u_m^{*,s}(x) \equiv u_m^*(\frac{x}{s})$. Note that

$$\begin{aligned} \int_{\mathbf{R}^n} |\nabla u_m^{*,s}|^2 dx &= s^{n-2} \int_{\mathbf{R}^n} |\nabla u_m^*|^2 dx \leq s^{n-2} \int_{\mathbf{R}^n} |\nabla u_m|^2 dx, \\ \int_{\mathbf{R}^n} \frac{(u_m^{*,s})^2}{2} - F(u_m^{*,s}) dx &= s^n \int_{\mathbf{R}^n} \frac{(u_m)^2}{2} - F(u_m) dx. \end{aligned}$$

Thus, there exists $s_m \in (0, 1]$ such that $u_m^{*,s_m} \in \mathcal{P}$. Then, we see that

$$\begin{aligned} \Gamma(u_m^{*,s_m}) &= \frac{(s_m)^{n-2}}{2} \int_{\mathbf{R}^n} |\nabla u_m^*|^2 dx - (s_m)^m \frac{n-2}{2n} \int_{\mathbf{R}^n} |\nabla u_m|^2 dx \\ &\leq \left(\frac{(s_m)^{n-2}}{2} - (s_m)^m \frac{n-2}{2n} \right) \int_{\mathbf{R}^n} |\nabla u_m|^2 dx \\ &\leq \left(\frac{1}{2} - \frac{n-2}{2n} \right) \int_{\mathbf{R}^n} |\nabla u_m|^2 dx \\ &= \Gamma(u_m). \end{aligned}$$

Thus, we may assume that u_m is radially symmetric and monotone decreasing for each $m = 1, 2, \dots$. Since u_m is contained in $\mathcal{P}$, it follows that $\Gamma(u_m) = \frac{1}{n} \int_{\mathbf{R}^n} |\nabla u_m|^2 dx$; thus $\{\int_{\mathbf{R}^n} |\nabla u_m|^2 dx\}_m$ is bounded. Moreover, it is easy to see from **(f2)** and Sobolev imbedding that $\{\int_{\mathbf{R}^n} (u_m)^2 dx\}_m$ is bounded. Thus, taking a subsequence if it is necessary, we may assume that u_m converges weakly to some radially symmetric function U in H^1 . We denote $H_r^1 \equiv \{u \in H^1 \mid u(x) = u(|x|)\}$. Then, it follows from [19] that an imbedding $H_r^1 \hookrightarrow L^q(\mathbf{R}^n)$ is compact for $q \in (2, 2n/(n-2))$. Then, it follows that $\lim_{m \rightarrow \infty} \int_{\mathbf{R}^n} F(u_m) dx = \int_{\mathbf{R}^n} F(U) dx$ and

$$\lim_{m \rightarrow \infty} \int_{\mathbf{R}^n} |\nabla u_m|^2 + (u_m)^2 dx \geq \int_{\mathbf{R}^n} |\nabla U|^2 + U^2 dx.$$

Note that

$$\frac{n-2}{2} \int_{\mathbf{R}^n} |\nabla U|^2 dx + n \int_{\mathbf{R}^n} \frac{U^2}{2} - F(U) dx \leq 0.$$

As before, we see that for $U^s(x) \equiv U(\frac{x}{s})$, there exists $t \in (0, 1]$ satisfying $U^t \in \mathcal{P}$. Then, as before, it follows that

$$\Gamma(U^t) \leq \Gamma(U) \leq \lim_{m \rightarrow \infty} \Gamma(u_m).$$

This implies that $U^t \in H_r^1$ is a minimizer of (5). For sake of convenience, let $U \in H_r^1$ be a minimizer of (5). Then, there exists a Lagrange multiplier $\lambda \in \mathbf{R}$ satisfying

$$\Delta U - U + f(U) + \lambda((n-2)\Delta U - nU + nf(U)) = 0 \quad \text{in } \mathbf{R}^n.$$

This implies that

$$(1 + (n-2)\lambda)\Delta U - (1 + n\lambda)U + (1 + n\lambda)f(U) = 0 \quad \text{in } \mathbf{R}^n.$$

From the Pohozaev identity (4), we see that

$$\frac{(1 + (n-2)\lambda)(n-2)}{2} \int_{\mathbf{R}^n} |\nabla U|^2 dx + (1 + n\lambda)n \int_{\mathbf{R}^n} \frac{U^2}{2} - F(U) dx = 0.$$

Since $U \in \mathcal{P}$, it follows that

$$\lambda(n-2) \int_{\mathbf{R}^n} |\nabla U|^2 dx = 0.$$

This implies that $\lambda = 0$. Thus the minimizer U satisfies equation (2). Since U is radially symmetric, it follows that $U \in C^2$. Since any solution of (2) is contained in $\mathcal{P}$, any minimizer of (5) is a least energy solution of (2). This completes the proof. $\square$

Take any $V \in H^1$ satisfying $\Gamma(V) < 0$. Then, we define a mountain pass level $C(V)$ by

$$(6) \quad C(V) \equiv \inf_{\gamma \in \Psi} \max_{t \in [0,1]} \Gamma(\gamma(t)),$$

where $\Psi \equiv \{\gamma \in C([0,1], H^1) \mid \gamma(0) = 0, \gamma(1) = V\}$. Then, we will see that $C(V)$ is independent on V satisfying $\Gamma(V) < 0$ via the following result, which is well known when $f(t)/t$ is monotone increasing.

Proposition 2.2. *We assume that (f1), (f2) and (f3) hold. Suppose that for $V_1, V_2 \in H^1$, $\Gamma(V_i) < 0, i = 1, 2$. Then, there exists a path $g \in C([0,1], H^1)$ such that $g(0) = V_1, g(1) = V_2$ and $\Gamma(g(t)) < 0$ for $t \in [0, 1]$.*

Proof. We find a function $\phi \in C_0^\infty(\mathbf{R}^n)$ such that $\phi(x) = 0$ for $|x| \geq 2$ and $\phi(x) = 1$ for $|x| \leq 1$. For $t > 0$, we define $\phi_t(x) = \phi(tx)$. Then, there exists $t_0 > 0$ such that $\Gamma(sV_i + (1-s)\phi_t V_i) < 0$ for any $s \in [0, 1]$ and $i = 1, 2$. Thus, we may assume that $\text{supp}(V_1) \cup \text{supp}(V_2) \subset B(0, R)$ for some $R > 0$. Define $V_i^t(x) \equiv V_i(\frac{x}{t})$. Then, we see that

$$\Gamma(V_i^t) = \frac{t^{n-2}}{2} \int_{\mathbf{R}^n} |\nabla V_i|^2 dx + t^n \int_{\mathbf{R}^n} \frac{(V_i)^2}{2} - F(V_i) dx.$$

Note that

$$\left. \frac{d\Gamma(V_i^t)}{dt} \right|_{t=1} = n\Gamma(V_i) - \int_{\mathbf{R}^n} |\nabla V_i|^2 < 0.$$

Thus, we see that $\Gamma(V_i^t)$ is strictly decreasing for $t \geq 1$. Moreover, from the facts $\int_{\mathbf{R}^n} \frac{(V_i)^2}{2} - F(V_i) dx < 0$ for each $i = 1, 2$, it follows that $\lim_{t \rightarrow \infty} \Gamma(V_i^t) = -\infty$. Then, we can choose $t_1 > 1$ such that

$$(7) \quad \Gamma(V_1^{t_1}) \leq -1 - \max_{\theta \in [0,1]} |\Gamma(\theta V_2)|.$$

Similarly, we can choose $t_2 > 1$ such that

$$(8) \quad \Gamma(V_2^{t_2}) \leq -1 - \max_{\theta \in [0,1]} |\Gamma(\theta V_1^{t_1})|.$$

Thus, there exists a path connecting V_1 and $V_1^{t_1}$ on which Γ is negative. We define $V_3^s(x) \equiv V_1^{t_1}(x + (2sR(t_1 + t_2), 0, \dots, 0))$, $s \in [0, 1]$. Then, we see that for $s \in [0, 1]$,

$$\Gamma(V_3^s) = \Gamma(V_3^0) = \Gamma(V_1^{t_1}) < 0.$$

We note that for $t \in [1, t_2]$, $\text{supp}(V_3^1) \cap \text{supp}(V_2^t) = \emptyset$. Then, we see that for $\theta \in [0, 1]$,

$$\Gamma(V_3^1 + \theta V_2) = \Gamma(V_3^1) + \Gamma(\theta V_2) < 0.$$

Thus, there exists a path connecting V_1 and $V_3^1 + V_2$ on which Γ is negative. Since $\Gamma(V_3^1 + V_2^t) = \Gamma(V_3^1) + \Gamma(V_2^t)$ for $t \in [1, t_2]$, there exists a path connecting $V_3^1 + V_2$ and $V_3^1 + V_2^{t_2}$ on which Γ is negative. Note that $\Gamma(\theta V_3^1 + V_2^{t_2}) = \Gamma(\theta V_1^{t_1}) + \Gamma(V_2^{t_2})$, $\theta \in [0, 1]$ and $\Gamma(V_2^t) < 0$, $t \in [1, t_2]$. Thus it follows from (8) that there exists a path connecting $V_3^1 + V_2^{t_2}$ and V_2 on which Γ is negative. This completes the proof. $\square$

From Proposition 2.2 above, we see that $C \equiv C(V)$ is independent of V satisfying $\Gamma(V) < 0$.

We define $H_r^1 \equiv \{u \in H^{1,2}(\mathbf{R}^n) \mid u(x) = u(|x|)\}$. Then, we have further the following pathwise connectedness in H_r^1 .

Proposition 2.3. *Let U, V be radially symmetric functions in H_r^1 such that $\Gamma(U), \Gamma(V) < 0$. Then, there exists a path $g \in C([0, 1], H_r^1)$ such that $g(0) = U, g(1) = V$ and $\Gamma(g(t)) < 0$ for $t \in [0, 1]$.*

Proof. We can take $a > 0$ such that $\Gamma(u) < 0$ if $\|U - u\| \leq a$ or $\|V - u\| \leq a$. Since $C_0^\infty(\mathbf{R}^n) \cap H_r^1$ is dense in H_r^1 , we can take $U_1, V_1 \in C_0^\infty(\mathbf{R}^n) \cap H_r^1$ such that $\|U - U_1\| \leq a$ and $\|V - V_1\| \leq a$. Thus, it suffices to find a path $g \in C([0, 1], H_r^1)$ such that $g(0) = U_1, g(1) = V_1$ and $\Gamma(g(t)) < 0$ for $t \in [0, 1]$. From Proposition 2.2, there exists a path $h \in C([0, 1], H^1)$ such that $h(0) = U_1, h(1) = V_1$ and $\Gamma(h(t)) < 0$ for $t \in [0, 1]$. Note that for each $t \in [0, 1]$, the function $h(t)$ in the proof of Proposition 2.2 is of the form $tU_1(sx + x_0) + t'V_1(s'x + x'_0)$ for some $t, t', s, s' \in \mathbf{R}$ and $x_0, x'_0 \in \mathbf{R}^n$. Thus, for each $t \in [0, 1]$, the function $h(t)$ is in $C^\infty \cap H^1$. We let $g(t)$ be the Schwartz symmetrization of $h(t)$. Then, we see that $g(t) \in H_r^1$ and $\Gamma(g(t)) < 0$. Since $h(t) \in C^\infty(\mathbf{R}^n)$, the function $h(t)$ is co-area regular (see [1, Definition 1.2.6] for a definition of co-area regularity). Then, by [1, Theorem 1.4], it follows that $g \in C([0, 1], H_r^1)$. This completes the proof. $\square$

Let U be a radially symmetric least energy solution of (2). Then, denoting $U_t(x) = U(\frac{x}{t})$, we see that

$$\Gamma(U_t) = \left(\frac{t^{n-2}}{2} - \frac{(n-2)t^n}{2n}\right) \int_{\mathbf{R}^n} |\nabla U|^2 dx.$$

This implies that $I \geq C$. For $u \in H^1$, we define

$$P(u) = \frac{n-2}{2} \int_{\mathbf{R}^n} |\nabla u|^2 dx + n \int_{\mathbf{R}^n} \frac{u^2}{2} - F(u) dx.$$

For any $\gamma \in \Psi$, $P(\gamma(t))dx > 0$ if $t > 0$ is small, and

$$P(\gamma(1)) = n\Gamma(\gamma(1)) - \int_{\mathbf{R}^n} |\nabla \gamma(1)|^2 dx < 0.$$

Thus, there exists a $t_0 \in [0, 1]$ such that $P(\gamma(t_0)) = 0$. This implies that $I \leq C$. Thus we conclude the following result.

Proposition 2.4.

$$I = C.$$

Let $\mathcal{S}$ be the set of a least energy solution U of (2) satisfying $U(0) = \max_{x \in \mathbf{R}^n} U(x)$. Then, we see the following compactness and decay estimate which was proved in [5].

Proposition 2.5. *For $N \geq 3$, $\mathcal{S}$ is compact in $H^1(\mathbf{R}^n)$. Moreover, there exist $C, c > 0$, independent of $U \in \mathcal{S}$, such that*

$$U(x) + |\nabla U(x)| \leq C \exp(-c|x|).$$

For $t > 0$, the map $x \mapsto \frac{x}{t}$ is a conformal map on $\mathbf{R}^n$, which had been used for obtaining the Pohozaev identity (4). For each $t > 0$, we define a corresponding map on a ball $B(0, R)$ as follows:

$$(9) \quad y(x, t) = y_R(x, t) = \frac{x}{t + \frac{|x|}{R}(1-t)}.$$

This map was used in [4]. For each $t \in (0, \infty)$, $y(\cdot, t)$ is a diffeomorphism from $B(0, R)$ onto $B(0, R)$. Moreover, it holds that

$$(10) \quad \lim_{t \rightarrow 0} y(x, t) = R \frac{x}{|x|} \text{ locally uniformly on } \overline{B(0, R)} \setminus \{0\},$$

$$(11) \quad y(x, 1) = x$$

and

$$(12) \quad \lim_{t \rightarrow \infty} y(x, t) = 0 \text{ uniformly on each compact subset of } B(0, R).$$

For any $u \in H_0^{1,2}(B(0, R))$, we define

$$u^t(x) = u(y(x, t)), x \in B(0, R), t \in (0, \infty).$$

We denote

$$T = T(t, y, R) = (t-1) \frac{|y|}{R} + 1, \quad a_{ij} = a_{ij}(t, y, R) = \frac{T-1}{T} \frac{y_i y_j}{|y|^2}.$$

The identity $n \times n$ -matrix is denoted by I . Then, the following estimates were shown in [4].

Proposition 2.6. *It follows that for $A = (a_{ij})_{i,j=1}^n$,*

$$(13) \quad dx = \left(\frac{t}{T}\right)^n |\det(I - A)| dy,$$

$$(14) \quad \left| \frac{dT}{dt} \right| \leq \frac{|y|}{R}, \quad \frac{d^2 T}{dt^2} = 0$$

and

$$(15) \quad |\nabla u^t(x)|^2 = \frac{1}{t^2 T^4} \sum_{i=1}^n \left(T^2 + \frac{2(T-1)y_i^2}{|y|^2} + \frac{(T-1)^2 y_i^2}{T^2 |y|^2} \right) \left(\frac{\partial u}{\partial y_i} \right)^2.$$

Moreover, for each $a > 1$ and $b \in (1, R]$, there exists some $C > 0$ such that

$$(16) \quad \max_{|t| \leq a, |y| \leq b} |T(t, y, R) - 1| \leq Cab/R, \quad \min_{1/a \leq t, |y| \leq R} T(t, y, R) \geq 1/a,$$

$$(17) \quad \max_{|y| \leq b} |\det(I - A) - 1| \leq Ca^n |t - 1|,$$

$$(18) \quad \max_{1/a \leq t \leq a, |y| \leq b} \left| \frac{d}{dt} \det(I - A) \right| \leq Ca^{n+1} \frac{b}{R},$$

$$(19) \quad \max_{1/a \leq t \leq a, |y| \leq b} \left| \frac{d^2}{dt^2} \det(I - A) \right| \leq Ca^{n+2} \frac{b^2}{R^2}$$

and

$$(20) \quad \max_{1/a \leq t \leq a, |y| \leq b} \left| T^2 + \frac{2(T-1)y_i^2}{|y|^2} + \frac{(T-1)^2 y_i^2}{T^2 |y|^2} - 1 \right| \leq Ca^4 b^2 / R^2.$$

For $R, D > 0$, we consider the following exterior boundary value problem:

$$(21) \quad \begin{aligned} \Delta u - u + f(u) &= 0, \quad u > 0 \quad \text{in } \mathbf{R}^n \setminus B(0, R-1), \\ \lim_{|x| \rightarrow \infty} u(x) &= 0, \quad u = D \quad \text{on } \partial B(0, R-1). \end{aligned}$$

Proposition 2.7. *We assume that f satisfies (f1) and (f2). Fix $R_0 > 0$ and assume that $R > R_0$. Then, if $D^2 R^{N-1} > 0$ is sufficiently small, problem (21) has a radially symmetric solution $u \in C^2$ satisfying $\frac{du(r)}{dr} \leq 0$ for $r \geq R-1$.*

The proof of Proposition 2.7 above is given in [4, Proposition 3.2].

3. THE SIMPLEST DOMAIN CASE

In this section we will consider the problem (1) when a domain Ω is a ball. Thus we consider the following problem:

$$(22) \quad \Delta u - u + f(u) = 0, \quad u > 0 \text{ in } B(0, R), \quad u = 0 \text{ on } \partial B(0, R).$$

We define

$$H_r^{1,2}(B(0, R)) \equiv \{u \in H_0^{1,2}(B(0, R)) \mid u(x) = u(|x|)\}$$

and

$$H_r^{1,2}(\mathbf{R}^n) \equiv \{u \in H_0^{1,2}(\mathbf{R}^n) \mid u(x) = u(|x|)\}.$$

We define $\mathcal{S}$ as the set of a least energy solution U of (2) satisfying $U(0) = \max_{x \in \mathbf{R}^n} U(x)$. For $u \in H_r^{1,2}(B(0, R))$, we define a functional Γ_R by

$$\Gamma_R(u) = \frac{1}{2} \int_{B(0, R)} |\nabla u|^2 + u^2 dx - \int_{B(0, R)} F(u) dx.$$

For any nonnegative $u \in H^{1,2}(\mathbf{R}^n)$, we let u^* be the Schwartz symmetrization of u , and define

$$\mathcal{S}^{*,\delta} \equiv \{|U|^* \in H_r^{1,2}(\mathbf{R}^n) \mid U \in H^{1,2}(\mathbf{R}^n), \|U - \mathcal{S}\| \leq \delta\}.$$

By Proposition 2.5, it follows that for each $\delta \geq 0$, $\mathcal{S}^{*,\delta}$ is bounded. We find a radially symmetric function $\phi_R \in C_0^\infty(\mathbf{R}^n)$ such that $\phi_R(x) = 1$ for $|x| \leq R-2$, $\phi_R(x) = 0$ for $|x| \geq R$ and $|\nabla \phi_R| \leq 1$. We define

$$\mathcal{S}_R^{*,\delta} \equiv \{\phi_R U^* \mid U^* \in \mathcal{S}^\delta\} \subset H_r^{1,2}(B(0, R))$$

and

$$\mathcal{S}_R^\delta \equiv \{u \in H_r^{1,2}(B(0, R)) \mid \|U - \mathcal{S}_R^{*,\delta}\| \leq \delta\}.$$

Here we show the existence of an *almost* mountain pass solution in $\mathcal{S}_R^\delta$ for (22) for large $R > 0$ when **(f1)**, **(f2)** and **(f3)** are satisfied.

Proposition 3.1. *We assume that **(f1)**, **(f2)** and **(f3)** hold. Suppose that there exists a path $\gamma_R \in C([0, 1], H_r^{1,2}(B(0, R)))$ and small $\delta > 0$ satisfying*

- (i) $\gamma_R(0) = 0$, $\Gamma_R(\gamma_R(1)) < 0$;
- (ii) for some $t_R \in (0, 1)$,

$$\Gamma_R(\gamma_R(t_R)) = \max_{t \in [0, 1]} \Gamma_R(\gamma_R(t)) \equiv C_R,$$

$$\gamma_R(t_R) \in \mathcal{S}_R^\delta \quad \text{and} \quad \lim_{R \rightarrow \infty} C_R = I;$$

- (iii) *there exists $\mu > 0$ such that for sufficiently large $R > 0$, $\Gamma_R(\gamma_R(t)) \leq I - \mu$ if $\gamma_R(t) \notin \mathcal{S}_R^\delta$.*

Then, for sufficiently large $R > 0$, there exists a radially symmetric positive solution $u_R \in C^2$ of (22) such that $\Gamma_R(u_R) \leq C_R$ and $\lim_{R \rightarrow \infty} \|u_R - \mathcal{S}\| = 0$. Moreover, for any $\mu > 0$, there exist constants $c, C > 0$ such that

$$C_R \geq I + c \exp(-2(1 + \mu)R).$$

Remark 3.1. For a radially symmetric solution U of (2), there exists $t_0 > 0$ such that $\Gamma(U^{t_0}) < 0$. We define $\gamma_R(t) = \phi_R U^{t t_0}$. Then, the path γ_R satisfies (i), (ii) and (iii) of Proposition 3.1 (see Proposition 4.1). This implies the existence of a solution u_R satisfying the conclusion of Proposition 3.1. An important fact in Proposition 3.1 is that we find a solution u_R whose energy $\Gamma_R(u_R)$ is less than or equal to the maximum of Γ_R over any path γ_R satisfying (i), (ii) and (iii). This fact will be important in proving our main result in the next section.

For any $c \in \mathbf{R}$, we define $\Gamma_R^c \equiv \{u \in H_r^{1,2}(B(0, R)) \mid \Gamma_R(u) \leq c\}$. We will show that for sufficiently large $R > 0$, there exists a solution $u_R \in \mathcal{S}_R^\delta \cap \Gamma_R^{C_R}$ of problem (22). To show this, we need the following lemma.

Lemma 3.1. *For sufficiently small $\delta > 0$, there exists $\nu > 0$ such that for sufficiently large $R > 0$,*

$$|\Gamma'_R(u)| \geq \nu \text{ for } u \in \Gamma_R^{C_R} \cap (\mathcal{S}_R^{2\delta} - \mathcal{S}_R^\delta).$$

Proof. We regard a member u of $H_0^{1,2}(B(0, R))$ as that of $H^{1,2}(\mathbf{R}^n)$ by defining $u(x) = 0$ for $|x| \geq R$.

To the contrary, we assume that there exists $u_R \in \Gamma_R^{C_R} \cap (\mathcal{S}_R^{2\delta} - \mathcal{S}_R^\delta)$ satisfying $\liminf_{R \rightarrow \infty} \Gamma'_R(u_R) = 0$. Note that $\{\mathcal{S}_R^\delta\}_R$ is bounded in $H_r^{1,2}(\mathbf{R}^n)$. Then, u_R converges weakly to some $U \in H_r^{1,2}(\mathbf{R}^n)$ along a subsequence as $R \rightarrow \infty$. Without loss of generality, we may assume that u_R converges weakly to U in $H_r^{1,2}(\mathbf{R}^n)$ as $R \rightarrow \infty$, and $\lim_{R \rightarrow \infty} \Gamma'(u_R) = 0$. Then, U is a solution of (22). We claim that

$U \in \mathcal{S} \cap H_r^{1,2}(\mathbf{R}^n)$ and u_R converges strongly to U in $H_r^{1,2}(\mathbf{R}^n)$ as $R \rightarrow \infty$. This contradicts the fact that $u_R \notin \mathcal{S}_R^\delta$, and completes the proof.

Since $\{u_R\}_R$ is bounded in $H_r^{1,2}(\mathbf{R}^n)$, it follows that for any $D > 0$,

$$\lim_{|z| \rightarrow \infty} \int_{B(z,D)} (u_R)^2 dx = 0 \quad \text{uniformly in } R.$$

Then, from the result of Lions (see [20, Lemma I.1]), we see that

$$\lim_{A \rightarrow \infty} \int_{\mathbf{R}^n \setminus B(0,A)} (u_R)^{p+1} dx = 0 \quad \text{uniformly in } R.$$

This implies that $\lim_{R \rightarrow \infty} \int_{\mathbf{R}^n} F(u_R) dx = \int_{\mathbf{R}^n} F(U) dx$. Then, since $\|U\|^2 \leq \liminf_{R \rightarrow \infty} \int_{\mathbf{R}^n} |\nabla u_R|^2 + (u_R)^2 dx$, it follows that $\Gamma(U) \leq C$. By Proposition 2.5, we see that $U \in \mathcal{S} \cap H_r^{1,2}(\mathbf{R}^n)$. Since $\Gamma(U) = C$, we deduce that

$$\|U\|^2 = \lim_{R \rightarrow \infty} \int_{\mathbf{R}^n} |\nabla u_R|^2 + (u_R)^2 dx.$$

This implies that u_R converges strongly to U as $R \rightarrow \infty$. This completes the proof. $\square$

Proof of Proposition 3.1. Suppose that for large $R > 0$, there does not exist a solution $u_R \in H_r^{1,2}(B(0,R))$ of (22) such that $\Gamma_R(u_R) \leq C_R$ and $u_R \in \mathcal{S}_R^\delta$. Then, there exists $\nu' = \nu'(R) > 0$ such that $|\Gamma'(u)| \geq \nu'$ for $u \in \mathcal{S}_R^\delta$. Then, by a deformation lemma (see [26]) and Lemma 3.1, we get a path γ'_R such that for large $R > 0$, $\max_{t \in [0,1]} \Gamma_R(\gamma'_R(t)) < I = C$, $\gamma'_R(0) = 0$ and $\Gamma_R(\gamma'_R(1)) < 0$. This contradicts Proposition 2.1, Proposition 2.3 and Proposition 2.4. Thus, there exists a solution $u_R \in H_r^{1,2}(B(0,R))$ of (22) such that $\Gamma_R(u_R) \leq C_R$ and $u_R \in \mathcal{S}_R^\delta$. Then, for some $U \in \mathcal{S}$, u_R converges strongly to U in $H_r^{1,2}(\mathbf{R}^n)$ as $R \rightarrow \infty$.

Since we have shown the existence of a solution u_R of (22) with $\Gamma_R(u_R) \leq C_R$, we can follow the proof [4, Proposition 4.3] for the lower energy estimate $C_R \geq I + c \exp(-2(1+\mu)R)$. For completeness, we give a detailed proof in the Appendix. $\square$

4. GENERAL DOMAIN CASE

For any set $A \subset \mathbf{R}^n$ and $\varepsilon > 0$, we define $A_\varepsilon \equiv \{x \in \mathbf{R}^n \mid \varepsilon x \in A\}$. By defining $v(x) = u(\varepsilon x)$, problem (1) is transformed to

$$(23) \quad \Delta v - v + f(v) = 0, \quad u > 0 \quad \text{on} \quad \Omega_\varepsilon, \quad v = 0 \quad \text{on} \quad \partial\Omega_\varepsilon.$$

We denote $H_\varepsilon \equiv H_0^{1,2}(\Omega_\varepsilon)$. By a translation, we may assume that

$$\text{dist}(0, \partial\Omega) = \max_{x \in \Omega} \text{dist}(x, \partial\Omega).$$

We define $D \equiv \text{dist}(0, \partial\Omega)$. For $u \in H_\varepsilon$, we define

$$\Gamma_\varepsilon(u) = \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla u|^2 + u^2 dx - \int_{\Omega_\varepsilon} F(u) dx,$$

where $F(t) \equiv \int_0^t f(s) ds$. Then, a critical point of Γ_ε corresponds to a solution of (23). For $u \in H^{1,2}(\mathbf{R}^n)$, we define

$$\Gamma(u) = \frac{1}{2} \int_{\mathbf{R}^n} |\nabla u|^2 + u^2 dx - \int_{\mathbf{R}^n} F(u) dx.$$

For small $\varepsilon > 0$, we find a function $\chi_\varepsilon \in C^1(\mathbf{R}^n, [0, 1])$ such that $\chi_\varepsilon(x) = 0$ for $x \notin \Omega_\varepsilon$, $\chi_\varepsilon(x) = 1$ for $x \in \Omega_\varepsilon$ with $\text{dist}(x, \partial\Omega_\varepsilon) \geq 2$ and $|\nabla\chi_\varepsilon| \leq 1$. Let $\mathcal{S}$ be the set of a least energy solution U for (2) satisfying $U(0) = \max_{x \in \mathbf{R}^n} U(x)$. We define

$$\mathcal{S}_\varepsilon \equiv \{U_\varepsilon^y \equiv \chi_\varepsilon U(\cdot - y) \mid y \in \Omega_\varepsilon, U \in \mathcal{S}\}.$$

We fix a radially symmetric monotone decreasing function $U \in \mathcal{S}$, and we find a radially symmetric function $\phi_\varepsilon \in C_0^\infty(B(0, D/\varepsilon), [0, 1])$ satisfying $\phi_\varepsilon(x) = 1$ for $|x| \leq D/\varepsilon - 1$ and $|\nabla\phi_\varepsilon| \leq 2$. For $t > 0$, we define

$$U_\varepsilon^t(x) = \phi_\varepsilon(x)U\left(\frac{x}{t}\right), \quad x \in \Omega_\varepsilon.$$

Then, there exists $t_0 > 0$ such that $\Gamma_\varepsilon(U_\varepsilon^{t_0}) < -1$ for sufficiently small $\varepsilon > 0$. Now we see the following upper estimate.

Proposition 4.1. *It follows that*

$$\lim_{\varepsilon \rightarrow 0} \Gamma_\varepsilon(U_\varepsilon^t) = \frac{1}{4}(t^{n-2} - \frac{n-2}{n}t^n) \int_{\mathbf{R}^n} |\nabla U|^2 dx \quad \text{uniformly for } t \in (0, t_0).$$

Moreover, for any $\mu \in (0, 1)$, there exists $C > 0$ such that for small $\varepsilon > 0$,

$$c_\varepsilon \equiv \max_{t \in [0, t_0]} \Gamma_\varepsilon(U_\varepsilon^t) \leq I + C \exp(-2(1 - \mu)D/\varepsilon).$$

Proof. Here, we will not give a proof. For the proof, see [4, Propositions 4.1, 4.2]. $\square$

For $\delta > 0$ and $\mathcal{A} \subset H_\varepsilon$, we define

$$\mathcal{A}^\delta = \{v \in H_\varepsilon \mid \min_{u \in \mathcal{A}} \|v - u\|_\varepsilon \leq \delta\}.$$

We define

$$\Gamma_\varepsilon^c \equiv \{u \in H_\varepsilon \mid \Gamma_\varepsilon(u) \leq c\}.$$

We will find a critical point $u_\varepsilon \in \mathcal{S}_\varepsilon^\delta \cap \Gamma_\varepsilon^{c_\varepsilon}$ for small $\delta, \varepsilon > 0$. We prepare some propositions.

Proposition 4.2. *For small $\delta > 0$, there exist $\alpha > 0$ and $\varepsilon_0 > 0$ such that for all $u \in (\mathcal{S}_\varepsilon^\delta - \mathcal{S}_\varepsilon^{\delta/2}) \cap \Gamma_\varepsilon^{c_\varepsilon}$ and $\varepsilon \in (0, \varepsilon_0)$,*

$$|\Gamma'_\varepsilon(u)| \geq \alpha.$$

Proof. To the contrary, suppose that there exists a small $\delta > 0$ and $u_\varepsilon \in (\mathcal{S}_\varepsilon^\delta - \mathcal{S}_\varepsilon^{\delta/2}) \cap \Gamma_\varepsilon^{c_\varepsilon}$ satisfying $\liminf_{\varepsilon \rightarrow 0} |\Gamma'_\varepsilon(u_\varepsilon)| = 0$. For convenience's sake, we may assume that $\lim_{\varepsilon \rightarrow 0} |\Gamma'_\varepsilon(u_\varepsilon)| = 0$. Then, there exist $\{x_\varepsilon\} \subset \Omega_\varepsilon$ and $\{U_\varepsilon\} \subset \mathcal{S}_\varepsilon$ such that $\|u_\varepsilon - \chi_\varepsilon U_\varepsilon(\cdot - x_\varepsilon)\|_\varepsilon \leq \delta$. From now on, we regard $u \in H_\varepsilon$ as a member in H^1 by defining $u \equiv 0$ on $\mathbf{R}^n \setminus \Omega_\varepsilon$.

Suppose that there exist $r > 0$ and $y_\varepsilon \in \Omega_\varepsilon$ satisfying

$$\lim_{\varepsilon \rightarrow 0} |x_\varepsilon - y_\varepsilon| = \infty \quad \text{and} \quad \limsup_{\varepsilon \rightarrow 0} \int_{B(y_\varepsilon, r)} (u_\varepsilon)^2 dx > 0.$$

We define $w_\varepsilon(x) \equiv u_\varepsilon(x + y_\varepsilon)$. Taking a subsequence if it is necessary, we may assume that w_ε converges weakly to a nonnegative function W in $H^{1,2}(\mathbf{R}^n)$ as $\varepsilon \rightarrow 0$ and $\lim_{\varepsilon \rightarrow 0} \int_{B(y_\varepsilon, r)} (u_\varepsilon)^2 dx > 0$. Then, denoting $H_\xi^d \equiv \{x \in \mathbf{R}^n \mid \langle x, \xi \rangle > d\}$ for $\xi \in S^{n-1}$ and $d \in \mathbf{R} \cup \{-\infty\}$, we see that for some $\xi \in S^{n-1}$ and $d \in \mathbf{R} \cup \{-\infty\}$, $W \in H_0^{1,2}(H_\xi^d)$ satisfies the following equation:

$$(24) \quad \Delta W - W + f(W) = 0, \quad W > 0 \text{ in } H_\xi^d \text{ and } W(x) = 0 \text{ on } \partial H_\xi^d.$$

We may assume that $W \in H_0^{1,2}(\mathbf{R}^n)$ by defining $W \equiv 0$ on $\mathbf{R}^n \setminus H_\xi^d$. We take $x_0 \in \partial H_\xi^d$ if $d \neq -\infty$, and $x_0 = 0$ if $d = -\infty$. Then, since $\nabla W \cdot (x - x_0) \in H_0^{1,2}(H_\xi^d)$, it follows that

$$\left. \frac{d\Gamma(W(t(\cdot - x_0) + x_0))}{dt} \right|_{t=1} = 0.$$

Then, through a simple calculation, we see that

$$(25) \quad \frac{n-2}{2} \int_{\mathbf{R}^n} |\nabla W|^2 dx + n \int_{\mathbf{R}^n} \frac{W^2}{2} - F(W) dx = 0.$$

By Proposition 2.1, we see that $\Gamma(W) \geq I$. Note from (25) that $\Gamma(W) = \frac{1}{n} \int_{\mathbf{R}^n} |\nabla W|^2 dx$. This implies that for some large $r > 0$,

$$\lim_{\varepsilon \rightarrow 0} \int_{B(y_\varepsilon, r)} |\nabla u_\varepsilon|^2 + (u_\varepsilon)^2 dx > \frac{1}{2n} \int_{\mathbf{R}^n} |\nabla W|^2 dx \geq \frac{I}{2}.$$

If $\delta \in (0, \sqrt{\frac{I}{2}})$ and $\varepsilon > 0$ is sufficiently small, this contradicts the fact that $u_\varepsilon \in \mathcal{S}_R^\delta$. This shows that for any $r > 0$ and $y_\varepsilon \in \Omega_\varepsilon$ with $\lim_{\varepsilon \rightarrow 0} |x_\varepsilon - y_\varepsilon| = \infty$,

$$\limsup_{\varepsilon \rightarrow 0} \int_{B(y_\varepsilon, r)} (u_\varepsilon)^2 dx = 0.$$

Then, by [20, Lemma I.1], we see that $\lim_{r \rightarrow \infty} \int_{\mathbf{R}^n \setminus B(x_\varepsilon, r)} (u_\varepsilon)^{p+1} dx = 0$ uniformly small for $\varepsilon > 0$. Now we may assume that $u_\varepsilon(\cdot + x_\varepsilon)$ converges weakly to some U in $H^{1,2}(\mathbf{R}^n)$. This implies that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} F(u_\varepsilon) dx = \int_{\mathbf{R}^n} F(U) dx.$$

Since $\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_\varepsilon \geq \|U\|$, it follows that

$$(26) \quad \liminf_{\varepsilon \rightarrow 0} \Gamma_\varepsilon(u_\varepsilon) \geq \Gamma(U).$$

Then, for some $\xi \in S^{n-1}$ and $d \in \mathbf{R} \cup \{-\infty\}$, $U \in H_0^{1,2}(H_\xi^d)$ satisfies the following equation:

$$(27) \quad \Delta U - U + f(U) = 0, \quad U > 0 \text{ in } H_\xi^d \text{ and } W(x) = 0 \text{ on } \partial H_\xi^d.$$

As before, we see that

$$(28) \quad \frac{n-2}{2} \int_{\mathbf{R}^n} |\nabla U|^2 dx + n \int_{\mathbf{R}^n} \frac{U^2}{2} - F(U) dx = 0.$$

By Proposition 2.1, we see that $\Gamma(U) \geq C$. Moreover, we see from the maximum principles and Proposition 2.1 that $\Gamma(U) > I$ if $d \neq -\infty$. Then, we see from (26) that $\Gamma(U) = I$. This implies that for some $y_0 \in \mathbf{R}^n$, $U(\cdot + y_0) \in \mathcal{S}$, $\lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon, \partial \Omega_\varepsilon) = \infty$, and that $\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_\varepsilon = \|U\|$. Thus, $u_\varepsilon(\cdot + x_\varepsilon)$ converges strongly to U in $H^{1,2}(\mathbf{R}^n)$; this implies that

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - W(\cdot - x_\varepsilon - y_0)\|_\varepsilon = 0 \quad \text{for some } W \in \mathcal{S}.$$

Note that $\lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon, \partial \Omega_\varepsilon) = \infty$; thus $x_\varepsilon + y_0 \in \Omega_\varepsilon$ for small $\varepsilon > 0$ and $\lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon + y_0, \partial \Omega_\varepsilon) = \infty$. This contradicts the fact that $u_\varepsilon \notin \mathcal{S}_\varepsilon^{\delta/2}$. This completes the proof. $\square$

Proposition 4.3. *For sufficiently small $\varepsilon > 0$, there exists a critical point of Γ_ε in $\mathcal{S}_\varepsilon^\delta \cap \Gamma_\varepsilon^{c_\varepsilon}$.*

Proof. Suppose that there exist no critical points of Γ_ε in $\mathcal{S}_\varepsilon^\delta \cap \Gamma_\varepsilon^{c_\varepsilon}$. Then, by Proposition 4.1 and Proposition 4.2, we deduce via a gradient flow (see [26]) that for small $\sigma > 0$, independent of small $\varepsilon > 0$, there exists a path $p_\varepsilon : [0, t] \rightarrow H_\varepsilon$ and $c > 0$ satisfying

- (i) $p_\varepsilon(t) = U_\varepsilon^t$ for $t \in [0, T] \setminus (1 - \sigma, 1 + \sigma)$;
- (ii) $p_\varepsilon(t) \in \mathcal{S}_\varepsilon^\delta$ for $t \in (1 - \sigma, 1 + \sigma)$;
- (iii) for small $\varepsilon > 0$, $\Gamma_\varepsilon(p_\varepsilon(t)) \leq \frac{1}{2}\Gamma(U) - c\delta$.

This contradicts Propositions 2.1, 2.2 and 2.4. This completes the proof. $\square$

Proof. Completion of the Main Theorem. Let u_ε be a solution of (23) with

$$\Gamma_\varepsilon(u_\varepsilon) \leq c_\varepsilon$$

and $x_\varepsilon \in \Omega_\varepsilon$ be a maximum point of u_ε . It is standard to see that

$$0 < \liminf_{\varepsilon \rightarrow 0} u_\varepsilon(x_\varepsilon) < \limsup_{\varepsilon \rightarrow 0} u_\varepsilon(x_\varepsilon) < \infty.$$

Moreover, as in the proof of Proposition 4.2, taking $\delta \in (0, \sqrt{I/2})$, we see that $\lim_{|y-x_\varepsilon| \rightarrow \infty} u_\varepsilon(x) = 0$. Then, it follows from a maximum principle that for any $\delta \in (0, 1)$, there exists $C > 0$ satisfying $|\nabla u_\varepsilon(x)| + u_\varepsilon(x) \leq C \exp(-\delta|x - x_\varepsilon|)$. Then we deduce from Proposition 2.1 that $u_\varepsilon(\cdot + x_\varepsilon)$ converges, up to a subsequence, uniformly to a least energy solution $U \in \mathcal{S}$. Thus, denoting $d_\varepsilon \equiv \text{dist}(\varepsilon x_\varepsilon, \partial\Omega)$, we see that $d_\varepsilon/\varepsilon = \lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon, \partial\Omega_\varepsilon) \rightarrow \infty$ as $\varepsilon \rightarrow 0$. We will show that $\lim_{\varepsilon \rightarrow 0} d_\varepsilon = D$.

From now on, any $u \in H_0^{1,2}(\Omega'), \Omega \subset \mathbf{R}^n$, is regarded as a member of $H^{1,2}(\mathbf{R}^n)$ by defining $u = 0$ on $\mathbf{R}^n \setminus \Omega'$. By a translation, we may assume that $x_\varepsilon = 0$. For any $\beta_1 > \beta_2 > 0$ and $C > 2(\beta_1 - \beta_2)$, we find that $\psi_\varepsilon \in C_0^\infty(\mathbf{R}^n)$ such that

$$\psi_\varepsilon(x) = \begin{cases} 1 & \text{if } |x| \leq (d_\varepsilon + \beta_2)/\varepsilon, \\ 0 & \text{if } |x| \geq (d_\varepsilon + \beta_1)/\varepsilon, \end{cases}$$

$0 \leq \psi_\varepsilon(x) \leq 1$, $|\nabla \psi_\varepsilon| \leq C\varepsilon$ and $|\nabla^2 \psi_\varepsilon| \leq C\varepsilon^2$. Let $\tilde{\Omega}_\varepsilon = B(0, \frac{d_\varepsilon + \beta_1}{\varepsilon}) \cap \Omega_\varepsilon$. Then, there exists $\beta_3 = \beta_3(\varepsilon) \in (0, \beta_1)$ satisfying $|\tilde{\Omega}_\varepsilon| = |B(0, \frac{d_\varepsilon + \beta_3}{\varepsilon})|$. Since $\partial\Omega \in C^2$, it follows that $\{\beta_3(\varepsilon)\}_\varepsilon$ is bounded away from 0 and β_1 . We take $\beta_2 > 0$ so that $\beta_3 < \beta_2 < \beta_1$. From the decay estimate of u_ε , there exists some $C' > 0$ such that

$$(29) \quad |\Gamma_\varepsilon(u_\varepsilon) - \Gamma_\varepsilon(u_\varepsilon \psi_\varepsilon)| \leq C' \exp(-\frac{2\delta}{\varepsilon}(d_\varepsilon + \beta_2)).$$

We define $z_\varepsilon = u_\varepsilon \psi_\varepsilon$ and $z_\varepsilon^t(x) = z_\varepsilon(y(x, t))$ with

$$y(x, t) = x / (t + \frac{\varepsilon|x|}{d_\varepsilon + \beta_1}(1 - t)).$$

Since $\Delta z_\varepsilon - z_\varepsilon + f(z_\varepsilon) = 0$ in $\Omega_\varepsilon \cap B(0, \frac{d_\varepsilon + \beta_2}{\varepsilon})$, it follows that

$$\begin{aligned}
& \left. \frac{d\Gamma(z_\varepsilon^t)}{dt} \right|_{t=1} \\
&= \int_{B(0, \frac{d_\varepsilon + \beta_1}{\varepsilon}) \cap \Omega_\varepsilon} \nabla z_\varepsilon \cdot \nabla \left(x \cdot \nabla z_\varepsilon \frac{\varepsilon|x| - d_\varepsilon - \beta_1}{d_\varepsilon + \beta_1} \right) dx \\
&\quad + \int_{B(0, \frac{d_\varepsilon + \beta_1}{\varepsilon}) \cap \Omega_\varepsilon} (z_\varepsilon - f(z_\varepsilon)) \left(x \cdot \nabla z_\varepsilon \frac{\varepsilon|x| - d_\varepsilon - \beta_1}{d_\varepsilon + \beta_1} \right) dx \\
&= \int_{\partial(B(0, \frac{d_\varepsilon + \beta_2}{\varepsilon}) \cap \Omega_\varepsilon)} \frac{\partial z_\varepsilon}{\partial \nu} x \cdot \nabla z_\varepsilon \frac{\varepsilon|x| - d_\varepsilon - \beta_1}{d_\varepsilon + \beta_1} d\sigma \\
&\quad + \int_{B(0, \frac{d_\varepsilon + \beta_1}{\varepsilon}) \cap \Omega_\varepsilon \setminus B(0, \frac{d_\varepsilon + \beta_2}{\varepsilon})} \nabla z_\varepsilon \cdot \nabla \left(x \cdot \nabla z_\varepsilon \frac{\varepsilon|x| - d_\varepsilon - \beta_1}{d_\varepsilon + \beta_1} \right) dx \\
&\quad + \int_{B(0, \frac{d_\varepsilon + \beta_1}{\varepsilon}) \cap \Omega_\varepsilon \setminus B(0, \frac{d_\varepsilon + \beta_2}{\varepsilon})} (z_\varepsilon - f(z_\varepsilon)) \left(x \cdot \nabla z_\varepsilon \frac{\varepsilon|x| - d_\varepsilon - \beta_1}{d_\varepsilon + \beta_1} \right) dx,
\end{aligned}$$

where ν and $d\sigma$ are the outward normal and the volume element on $\partial(B(0, \frac{d_\varepsilon + \beta_2}{\varepsilon}) \cap \Omega_\varepsilon)$, respectively. From the $W^{2,p}$ estimates, [16, Theorem 9.13] and the decay estimates of u_ε , we deduce that for any $\delta \in (0, 1)$, there exists some $C > 0$ satisfying

$$(30) \quad \left| \frac{d\Gamma(z_\varepsilon^t)}{dt} \right|_{t=1} \leq C \exp\left(-\frac{2\delta}{\varepsilon}(d_\varepsilon + \beta_3)\right).$$

Denoting

$$T_\varepsilon = T_\varepsilon(y) \equiv (t-1) \frac{\varepsilon|y|}{d_\varepsilon + \beta_1} + 1 \quad \text{and} \quad A = A_\varepsilon \equiv \left(\frac{T-1}{T} \frac{y_i y_j}{|y|^2} \right),$$

we deduce from Proposition 2.6 that

$$\begin{aligned}
(31) \quad & \int_{\mathbf{R}^n} |\nabla z_\varepsilon^t|^2 dx \\
&= t^{n-2} \sum_{i=1}^n \int_{\mathbf{R}^n} \left| \frac{\partial z_\varepsilon}{\partial y_i} \right|^2 T^2 + \frac{2(T-1)y_i^2}{|y|^2} + \frac{(T-1)^2 y_i^2}{T^2 |y|^2} \left| \frac{\det(I-A)}{T^{n+4}} \right| dy,
\end{aligned}$$

$$(32) \quad \int_{\mathbf{R}^n} |z_\varepsilon^t|^2 dx = t^n \int_{\mathbf{R}^n} |z_\varepsilon|^2 \frac{|\det(I-A)|}{T^n} dy$$

and

$$(33) \quad \int_{\mathbf{R}^n} F(z_\varepsilon^t) dx = t^n \int_{\mathbf{R}^n} F(z_\varepsilon) \frac{|\det(I-A)|}{T^n} dy.$$

Note that for fixed $a, b > 1$, $\lim_{\varepsilon \rightarrow 0} T_\varepsilon = 1$ and $\lim_{\varepsilon \rightarrow 0} A_\varepsilon = 0$ uniformly for $t \in (1/a, a)$ and $|y| \leq b$. From the exponential decays of z_ε and U , we deduce that if $a > 1$ is fixed,

$$(34) \quad \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^n} |\nabla z_\varepsilon^t|^2 dx = t^{n-2} \int_{\mathbf{R}^n} |\nabla U|^2 dx,$$

$$(35) \quad \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^n} |z_\varepsilon^t|^2 dx = t^n \int_{\mathbf{R}^n} U^2 dx,$$

$$(36) \quad \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^n} F(z_\varepsilon^t) dx = t^n \int_{\mathbf{R}^n} F(U) dx$$

uniformly for $t \in (1/a, a)$. Then, it follows from the Pohozaev identity (4) that

$$(37) \quad \lim_{\varepsilon \rightarrow 0} \Gamma(z_\varepsilon^t) = \left(\frac{t^{n-2}}{2} - \frac{n-2}{2n} t^n \right) \int_{\mathbf{R}^n} |\nabla U|^2 dx$$

uniformly for $t \in (1/a, a)$. Moreover, differentiating both sides of (31), (32), (33) and using Proposition 2.6, we see that for sufficiently small $\gamma > 0$,

$$\lim_{\varepsilon \rightarrow 0} \frac{d^2 \Gamma(z_\varepsilon^t)}{dt^2} \leq -\frac{n-2}{2} \int_{\mathbf{R}^n} |\nabla U|^2 dx \quad \text{for } t \in (1-\gamma, 1+\gamma).$$

This and (30) implies that there exists $t_\varepsilon > 0$ satisfying

$$\Gamma(z_\varepsilon^{t_\varepsilon}) = \max_{t \in (1-\gamma, 1+\gamma)} \Gamma(z_\varepsilon^t) \quad \text{and} \quad |t_\varepsilon - 1| \leq C \exp\left(-\frac{2\delta}{\varepsilon}(d_\varepsilon + \beta_3)\right)$$

for some $C > 0$. Then, through the Taylor expansion as in (46), we deduce that for small $\gamma > 0$,

$$|\Gamma_\varepsilon(z_\varepsilon) - \max_{t \in [1-\gamma, 1+\gamma]} \Gamma_\varepsilon(z_\varepsilon^t)| \leq C \exp\left(-\frac{4\delta}{\varepsilon}(d_\varepsilon + \beta_3)\right).$$

By (f1), (34) and (35), we see that for some small $t_0 > 0$,

$$\frac{d\Gamma(s z_\varepsilon^{t_0})}{ds} \geq 0 \quad \text{for } s \in [0, 1] \quad \text{and small } \varepsilon > 0.$$

Moreover, by (34), (35) and (36), we see that for some $t_1 > 0$,

$$\Gamma(z_\varepsilon^{t_1}) < -1 \quad \text{for small } \varepsilon > 0.$$

Then, we define a continuous curve $\alpha_\varepsilon(t) : [0, 1] \rightarrow H_0^{1,2}(\mathbf{R}^n)$ by

$$\alpha_\varepsilon(t) = \begin{cases} 2t z_\varepsilon^{t_0}, & t \in [0, 1/2], \\ z_\varepsilon^{(2-2t)t_0 + (2t-1)t_1}, & t \in [1/2, 1]. \end{cases}$$

From (37), we see that for sufficiently small $\varepsilon > 0$,

$$\max_{t \in [0, 1]} \Gamma_\varepsilon(\alpha_\varepsilon(t)) = \max_{t \in [1/2, 1]} \Gamma_\varepsilon(\alpha_\varepsilon(t)) = \max_{t \in [t_0, t_1]} \Gamma_\varepsilon(z_\varepsilon^t) = \max_{t \in [1-\gamma, 1+\gamma]} \Gamma_\varepsilon(z_\varepsilon^t).$$

Thus, it follows that

$$\max_{t \in [0, 1]} \Gamma_\varepsilon(\alpha_\varepsilon(t)) \leq \Gamma_\varepsilon(z_\varepsilon) + C \exp\left(-\frac{4\delta}{\varepsilon}(d_\varepsilon + \beta_3)\right).$$

Since $\lim_{\varepsilon \rightarrow 0} c_\varepsilon = I = C$, it follows that for any small $\alpha > 0$, there exists a constant $\beta > 0$ such that

$$\limsup_{\varepsilon \rightarrow 0} \max_{t \in [0, t_1] \setminus (1-\beta, 1+\beta)} \Gamma(z_\varepsilon^t) < I - \alpha.$$

We let S_ε^t be the Schwartz symmetrization of $\alpha_\varepsilon(t)$ for each $t > 0$. Then,

$$S_\varepsilon^t \in H_0^{1,2}\left(B\left(0, \frac{d_\varepsilon + \beta_3}{\varepsilon}\right)\right) \quad \text{and} \quad S_\varepsilon^0 = 0, \quad \Gamma(S_\varepsilon^{t_1}) < 0.$$

By a rearrangement argument, we see that

$$\max_{t \in [0, t_1]} \Gamma(\alpha_\varepsilon(t)) \geq \max_{t \in [0, t_1]} \Gamma(S_\varepsilon^t) \quad \text{and} \quad \limsup_{\varepsilon \rightarrow 0} \max_{t \in [0, t_1] \setminus (1-\beta, 1+\beta)} \Gamma(S_\varepsilon^t) < I - \alpha.$$

Moreover, for any small $\delta > 0$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$,

$$S_\varepsilon^1 = (z_\varepsilon)^* \in \mathcal{S}_{(d_\varepsilon + \beta_3)/\varepsilon}^{\delta/2}.$$

It is easy to see that S_ε^t is continuous with respect to $t \in [0, t_1]$. Thus, $S_\varepsilon^t \in \mathcal{S}_{(d_\varepsilon + \beta_3)/\varepsilon}^\delta$ if $|t - 1|$ is sufficiently small. Then, by Proposition 3.1, there exists a solution w_ε of (22) with $R = (d_\varepsilon + \beta_3)/\varepsilon$ such that

$$\Gamma(w_\varepsilon) \leq \max_{t \in [0, t_1]} \Gamma(S_\varepsilon^t),$$

and that for some $C > 0$,

$$\Gamma(w_\varepsilon) \geq I + \exp\left(-\frac{2\delta_1}{\varepsilon}(d_\varepsilon + \beta_3)\right).$$

Thus we conclude that for any $\delta_1 > 1 > \delta_2$, there exist $C_1, C_2, C_3 > 0$ satisfying

$$\begin{aligned} \Gamma_\varepsilon(u_\varepsilon) &\geq \Gamma_\varepsilon(z_\varepsilon) - C_1 \exp\left(-\frac{2\delta}{\varepsilon}(d_\varepsilon + \beta_2)\right) \\ &\geq \Gamma(U) + C_2 \exp\left(-\frac{2\delta_1}{\varepsilon}(d_\varepsilon + \beta_3)\right) \\ &\quad - C_1 \exp\left(-\frac{2\delta}{\varepsilon}(d_\varepsilon + \beta_2)\right) - C_3 \exp\left(-\frac{4\delta_2}{\varepsilon}(d_\varepsilon + \beta_3)\right). \end{aligned}$$

Thus, it follows that for any $\beta > 0$, there exists $C > 0$ satisfying

$$c_\varepsilon \geq \Gamma_\varepsilon(u_\varepsilon) \geq I + C \exp\left(-\frac{2}{\varepsilon}(d_\varepsilon + \beta)\right).$$

Then, by Proposition 4.1, we conclude that $\lim_{\varepsilon \rightarrow 0} d_\varepsilon = D$. It is standard to see other properties of u_ε . This completes the proof. $\square$

APPENDIX: A LOWER ENERGY ESTIMATE IN PROPOSITION 3.1

Now we show the lower estimate: for any $\mu \in (0, 1)$, there exists $c > 0$ such that $C_R \geq \Gamma(U) + c \exp(-2(1 + \mu)R)$.

Since $\{\mathcal{S}_R^\delta\}_{R>1}$ is bounded, we see that

$$\left\{ \int_{B(0, R)} |\nabla u_R|^2 + (u_R)^2 dx \right\}_{R>1}$$

is bounded. Then, since u_R is radially symmetric, it follows that $\lim_{x \rightarrow \infty} u_R(x) = 0$ uniformly for $R > 1$. Then, by the comparison principle, we see that for any $\gamma \in (0, 1)$, there exists a constant $C > 0$, independent of $R > 1$, satisfying

$$(38) \quad u_R(x) \leq C \exp(-\gamma|x|), \quad x \in B(0, R).$$

Then, we see from the elliptic estimates (refer to [16]) that for any $\gamma \in (0, 1)$, there exists $C' > 0$ satisfying

$$(39) \quad \left| \frac{du_R}{dr} \right| \leq C' \exp(-\gamma E).$$

We may assume that u_R converges weakly to some U in $H^{1,2}(\mathbf{R}^n)$ as $R \rightarrow \infty$. It is easy to see that U is a solution of (2). Then, we deduce from Proposition 2.4 and the above estimates that U is a least energy solution of (2) and u_R converges uniformly to U as $R \rightarrow \infty$. For $\gamma \in (0, 1)$ and $C > 0$, we define

$$\psi_R(x) = C(\exp(-|x|(1 + \gamma)) - \exp(-R(1 + \gamma))).$$

Then we see that

$$\begin{aligned} & \Delta\psi_R - \psi_R + \frac{f(u_R)}{u_R}\psi_R \\ &= \left((1+\gamma)^2 - \frac{(n-1)(1+\gamma)}{r} - 1 + \frac{f(u_R)}{u_R} \right) C \exp(-(1+\gamma)|x|) \\ & \quad + \left(1 - \frac{f(u_R)}{u_R} \right) C \exp(-R(1+\gamma)). \end{aligned}$$

By (38), there exists $R_0 > 0$, independent of large R , such that

$$\Delta\psi_R - \psi_R + \frac{f(u_R)}{u_R}\psi_R \geq 0 \text{ for } R_0 \leq |x| < R.$$

We take a constant $c' > 0$ such that $u_R(R_0) \geq c'\psi_R(R_0)$ for large R . Then, by the comparison principle, we see that $u_R(r) \geq c'\psi_R(r)$ for $r \in (r_0, R)$. In particular, we see that

$$u_R(R-1) \geq c'\psi_R(R-1) = c' \exp(-R(1+\gamma))(\exp(1+\gamma) - 1).$$

Thus, for any $\gamma \in (0, 1)$, there exist $c, C > 0$ such that

$$(40) \quad c \exp(-R(1+\gamma)) \leq u_R(R-1) \leq C \exp(-R(1-\gamma)).$$

Then, from Proposition 2.7, we see that for sufficiently large $R > 1$, there exists a radially symmetric decreasing solution V_R of

$$\begin{aligned} \Delta u - u + f(u) &= 0 \quad \text{on } \mathbf{R}^n \setminus \overline{B(0, R-1)}, \\ u(R-1) &= u_R(R-1) \quad \text{and} \quad \lim_{x \rightarrow \infty} u(x) = 0. \end{aligned}$$

Then, from the comparison principle and the decay property (40), we deduce that for any $\gamma \in (0, 1)$, there exists $C > 0$ satisfying

$$(41) \quad |V_R(r)| + \left| \frac{dV_R(r)}{dr} \right| \leq C \exp(-\gamma r) \quad \text{for } r \geq R-1.$$

Define

$$w_R(x) \equiv \begin{cases} u_R(x) & \text{for } |x| \leq R-1, \\ V_R(x) & \text{for } |x| \geq R-1, \end{cases}$$

and $w_R^t(x) \equiv w_R(\frac{x}{t})$. Since the least energy solution U is a mountain pass solution of (2) (see [18]) and w_R converges uniformly to U , it follows that

$$\max_{t \in (0, \infty)} \Gamma(w_R^t) \geq \Gamma(U).$$

We define

$$u_R^t(x) = \begin{cases} u_R(\frac{x}{t}) & \text{for } t > 0, x \in B(0, tR), \\ 0 & \text{for } t > 0, x \in \mathbf{R}^n \setminus B(0, tR). \end{cases}$$

From (4) and a change of variables, we see that as $R \rightarrow \infty$,

$$\begin{aligned} & \Gamma(u_R^t) \\ &= \frac{t^{n-2}}{2} \int_{B(0, R)} |\nabla u_R|^2 dx + \frac{t^n}{2} \int_{B(0, R)} |u_R|^2 - 2F(u_R) dx \\ (42) \quad & \rightarrow \frac{t^{n-2}}{2} \int_{\mathbf{R}^n} |\nabla U|^2 dx + \frac{t^n}{2} \int_{\mathbf{R}^n} U^2 - 2F(U) dx \\ &= \left(\frac{t^{n-2}}{2} - \frac{(n-2)t^n}{2n} \right) \int_{\mathbf{R}^n} |\nabla U|^2 dx \end{aligned}$$

uniformly for $t \in (0, \infty)$. We take $s_R > 0$ satisfying

$$\Gamma(u_R^{s_R}) = \max_{t \in (0, \infty)} \Gamma(u_R^t).$$

Then, it is easy to see that $\lim_{R \rightarrow \infty} s_R = 1$. Then, by the Pohozaev identity (4), we see that as $R \rightarrow \infty$,

$$\begin{aligned} & \frac{d^2 \Gamma(u_R^t)}{dt^2} \\ &= (n-2)(n-3) \frac{t^{n-4}}{2} \int_{B(0,R)} |\nabla u_R|^2 dx \\ & \quad + n(n-1) \frac{t^{n-2}}{2} \int_{B(0,R)} |u_R|^2 dx - n(n-1) t^{n-2} \int_{B(0,R)} F(u_R) dx \\ & \rightarrow (n-2)(n-3) \frac{t^{n-4}}{2} \int_{\mathbf{R}^n} |\nabla U|^2 dx + n(n-1) \frac{t^{n-2}}{2} \int_{\mathbf{R}^n} U^2 - 2F(U) dx \\ &= \frac{n-2}{2} \int_{\mathbf{R}^n} |\nabla U|^2 dx \left((n-3)t^{n-4} - (n-1)t^{n-2} \right). \end{aligned}$$

Thus, there exist $C > 0, t' \in (0, 1)$ and $R_0 > 0$ such that

$$(43) \quad \frac{d^2 \Gamma(u_R^t)}{dt^2} < -C \quad \text{for } R \geq R_0 \text{ and } t \in (1-t', 1+t').$$

Let ω_{n-1} be the volume of S^{n-1} . Then we see that

$$\begin{aligned} & \left. \frac{d\Gamma(u_R^t)}{dt} \right|_{t=1} \\ &= - \int_{B(0,R)} \nabla u_R \nabla (x \cdot \nabla u_R) + u_R (x \cdot \nabla u_R) - f(u_R) (x \cdot \nabla u_R) dx \\ &= -\omega_{n-1}(R)^N \left(\frac{du_R(R)}{dr} \right)^2. \end{aligned}$$

Then, by (39), we see that for any $\gamma \in (0, 1)$, there exists $C > 0$ satisfying

$$(44) \quad \left| \frac{d\Gamma(u_R^t)}{dt} \right|_{t=1} \leq C \exp(-2\gamma R).$$

From (42), (43) and (44), we deduce that there exists $D > 0$ satisfying

$$(45) \quad |1 - s_R| \leq D \exp(-2\gamma R).$$

Note that

$$(46) \quad \Gamma(u_R^{s_R}) = \Gamma(u_R^1) + \left. \frac{d\Gamma(u_R^t)}{dt} \right|_{t=1} (s_R - 1) + O(|s_R - 1|^2).$$

Thus, we see that for some $C' > 0$,

$$(47) \quad \Gamma(u_R^t) \leq \Gamma(u_R) + C' \exp(-3R), \quad t \in (0, \infty).$$

We define $V_R^t(x) \equiv V_R(\frac{x}{t})$ for $t > 0$, and $e(v) \equiv \frac{1}{2} |\nabla v|^2 + \frac{1}{2} v^2 - F(v)$. Then we see that

$$\begin{aligned} & \Gamma(u_R^t) \\ &= \Gamma(w_R^t) + \int_{B(0,tR) \setminus B(0,t(R-1))} e(u_R^t) dx - \int_{\mathbf{R}^n \setminus B(0,t(R-1))} e(V_R^t) dx \\ &\equiv A(t) + B(t) - C(t). \end{aligned}$$

We take $t(R) > 0$ so that $A(t(R)) = \max_{t \in (0, \infty)} A(t)$. Since U is a mountain pass solution (see ([18])), it follows that

$$(48) \quad A(t(R)) \geq \Gamma(U).$$

Since w_R converges to a least energy solution U of (2), it follows that $\lim_{R \rightarrow \infty} t(R) = 1$. By (41) and a similar manner as in (45), we deduce that for any $\gamma \in (0, 1)$, there exists $C > 0$ satisfying $|t(R) - 1| \leq C \exp(-2\gamma R)$. Moreover, from (40) and (41), we deduce that for any $\gamma \in (0, 1)$, there exists $C > 0$ satisfying

$$|B'(1) - C'(1)| \leq C \exp(-2\gamma R).$$

This implies that for some $C > 0$,

$$(49) \quad \left| (B(t(R)) - C(t(R))) - (B(1) - C(1)) \right| \leq C \exp(-3R).$$

Note that

$$\int_{B(0,R) \setminus B(0,R-1)} u_R \Delta u_R - (u_R)^2 + f(u_R) u_R dx = 0$$

and

$$\int_{\mathbf{R}^n \setminus B(0,R-1)} V_R \Delta V_R - (V_R)^2 + f(V_R) V_R dx = 0.$$

Since $\lim_{t \rightarrow 0} F(t)/t^2 = \lim_{t \rightarrow 0} f(t)/t = 0$, it follows from integration by parts that

$$\begin{aligned} B(1) &= \frac{1}{2} \int_{B(0,R) \setminus B(0,R-1)} |\nabla u_R|^2 + (u_R)^2 - f(u_R) u_R dx \\ &\quad + \int_{B(0,R) \setminus B(0,R-1)} \frac{1}{2} f(u_R) u_R - F(u_R) dx \\ &= -\frac{1}{2} \omega_{n-1} (R-1)^{n-1} u_R (R-1) \frac{du_R(R-1)}{dr} \\ &\quad + \omega_{n-1} (R-1)^{n-1} o((u_R(R-1))^2). \end{aligned}$$

By a similar manner, we deduce that

$$C(1) = -\frac{1}{2} (R-1)^{n-1} V_R (R-1) \frac{dV_R(R-1)}{dr} + \omega_{N-1} (R-1)^{n-1} o((V_R(R-1))^2).$$

Thus, we see that

$$\begin{aligned} (50) \quad & B(1) - C(1) \\ & \geq \omega_{N-1} (R-1)^{n-1} u_R (R-1) \left(\frac{1}{2} \frac{dV_R(R-1)}{dr} - \frac{1}{2} \frac{du_R(R-1)}{dr} \right) \\ & \quad + \omega_{N-1} (R-1)^{n-1} o((u_R(R-1))^2). \end{aligned}$$

We define

$$\tilde{V}_R(r) \equiv V_R(r + R - 1)/u_R(R - 1)$$

and

$$\tilde{u}_R(r) \equiv u_R(r + R - 1)/u_R(R - 1).$$

Then we see that

$$\frac{d^2 \tilde{V}_R}{dr^2} + \frac{n-1}{r+R-1} \frac{d\tilde{V}_R}{dr} - \tilde{V}_R + \frac{1}{u_R(R-1)} f(u_R(R-1) \tilde{V}_R) = 0, \quad r > 0,$$

and

$$\frac{d^2 \tilde{u}_R}{dr^2} + \frac{n-1}{r+R-1} \frac{d\tilde{u}_R}{dr} - \tilde{u}_R + \frac{1}{u_R(R-1)} f(u_R(R-1) \tilde{u}_R) = 0, \quad r \in (0, 1).$$

Since $\lim_{s \rightarrow 0} \frac{f(s)}{s} = 0$, it is standard to show from Proposition 2.7 that $\tilde{V}_R$ converges locally uniformly on $[0, \infty)$ in C^1 -sense to a unique positive function V_1 satisfying

$$\frac{d^2 V_1}{dr^2}(r) - V_1(r) = 0, \quad r \in (0, \infty), \quad V_1(0) = 1, V_1(r) \leq 1.$$

Likewise, $\tilde{u}_R$ converges uniformly on $[0, 1]$ in C^1 -sense to a unique positive function V_2 satisfying

$$\frac{d^2 V_2}{dr^2}(r) - V_2(r) = 0, \quad r \in (0, 1), \quad V_2(0) = 1, V_2(1) = 0.$$

It is obvious that

$$\frac{dV_1(0)}{dt} - \frac{dV_2(0)}{dt} > 0.$$

This implies that for some $C > 0$,

$$\frac{dU_R(R-1)}{dr} - \frac{du_R(R-1)}{dr} \geq Cu_R(R-1)$$

if $R > 0$ is sufficiently large. Then, we see from (50) that there exists $C > 0$ satisfying

$$B(1) - C(1) \geq C(R-1)^{n-1}(u_R(R-1))^2 \quad \text{for large } R > 0.$$

Then we see from (40) and (49) that for any $\delta > 1$, there exists a constant $C > 0$ satisfying

$$B(t(R)) - C(t(R)) \geq C \exp(-2\delta R).$$

Combining this with (47) and (48), we deduce that for any $\mu > 0$, there exists a constant $C > 0$ satisfying

$$\Gamma(u_R) \geq \Gamma(U) + C \exp(-2(1+\mu)R).$$

This completes the proof.

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