EMBEDDING THEOREMS AND BOUNDARY-VALUE PROBLEMS FOR CUSP DOMAINS

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ABSTRACT. We study the Robin boundary-value problem for bounded domains with isolated singularities. Because trace spaces of space $W_2^1(D)$ on boundaries of such domains are weighted Sobolev spaces $L^{2,\xi}(\partial D)$, existence and uniqueness of corresponding Robin boundary-value problems depends on properties of embedding operators $I_1 : W_2^1(D) \rightarrow L^2(D)$ and $I_2 : W_2^1(D) \rightarrow L^{2,\xi}(\partial D)$ i.e. on types of singularities. We obtain an exact description of weights ξ for bounded domains with ‘outside peaks’ on its boundaries. This result allows us to formulate correctly the corresponding Robin boundary-value problems for elliptic operators. Using compactness of embedding operators I_1, I_2 , we prove also that these Robin boundary-value problems with the spectral parameter are of Fredholm type.

1. INTRODUCTION

An essentially self-contained presentation of a method for a correct formulation and an investigation of the Robin boundary-value problem for second order elliptic equations in domains with ‘outside peaks’ is given in this paper. This study is initiated by the works [12], [6], [8]. Quoted papers are devoted to the Robin boundary-value problem for Lipschitz domains and its straightforward generalizations. It is well known that the embedding operator $I_1 : W_2^1(D) \rightarrow L^2(D)$ is compact for Lipschitz domains [11] and by [8] the operator $I_2 : W_2^1(D) \rightarrow L^2(\partial D)$ is also compact. Therefore the Robin boundary-value problem is of Fredholm type for this class of domains [8].

For domains with isolated singularities the second embedding operator $I_2 : W_2^1(D) \rightarrow L^2(\partial D)$ does not necessarily exist, because traces of functions from $W_2^1(D)$ do not necessarily belong to the space $L^2(\partial D)$. This means that even a correct formulation of the Robin problem for such domains depends on properties of a trace space of $W_2^1(D)$. One of the possible descriptions of trace spaces for bounded C^1 -domains with isolated singularities of an ‘outside peak’ type was proposed in [14], [15] by the second author. For such domains the trace space for $W_2^1(D)$ does not necessarily coincide with $L^2(D)$ and can be described with the help of corresponding weights ξ that depend on singularity types. These results allow us to formulate correctly the Robin boundary-value problem with the help of the weights ξ .

Received by the editors October 9, 2007.

2000 *Mathematics Subject Classification.* Primary 46E35, 35J25.

The first author was supported in part by the Israel Science Foundation grant.

The second author was partially supported by the Russian Foundation for Basic Research (grant 06-01-00735).

We will prove in this paper that embedding operators $I_2 : W_2^1(D) \rightarrow L^{2,\xi}(\partial D)$ are compact for such weights. Because both embedding operators $I_1 : W_2^1(D) \rightarrow L^2(D)$ and $I_2 : W_2^1(D) \rightarrow L^{2,\xi}(\partial D)$ are compact, the Robin boundary-value problem with the spectral parameter is of Fredholm type.

Elliptic boundary-value problems were studied in numerous books and papers. We mention [4] and [10], where many references can be found. In [11] embedding theorems for a variety of domains with nonsmooth boundaries have been studied. The main results of this paper were announced in [16].

Remark. We refer in this paper to domains of class C^1 . We use this condition for simplicity only. All results are also correct for the Lipschitz category.

2. MAIN RESULTS

In this section we introduce necessary definitions and formulate main results.

Recall the classical definition of Sobolev space $W_p^1(D)$. Let D be an open subset of $\mathbb{R}^n$, $n \geq 2$. Define Sobolev space $W_p^1(D)$, $1 \leq p < \infty$ as a normed space of locally summable, weakly differentiable functions $f : D \rightarrow \mathbb{R}$ equipped with the following norm:

$$\|f\|_{W_p^1(D)} = \left[\left(\int_D |f|^p(x) dx \right)^{2/p} + \left(\int_D |\nabla f|^p(x) dx \right)^{2/p} \right]^{1/2}.$$

Here ∇f is the weak gradient of the function f .

2.1. Weak Robin problem for bounded C^1 -domains. Let $G \in \mathbb{R}^n$ be a bounded C^1 -domain i.e. its boundary ∂G is a compact $(n-1)$ -dimensional C^1 -manifold. Recall the classical Robin boundary-value problem

$$(2.1) \quad \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + a(x)u = f(x), \quad x \in G,$$

$$(2.2) \quad \frac{\partial u}{\partial N} + \sigma(x)u = \mu(x), \quad x \in \partial G.$$

Here the functions $a_{i,j}(x) = a_{j,i}, b_i(x), a(x), f(x)$ are C^1 -functions defined on G and $\sigma(x), \mu(x)$ are C^1 -functions defined on ∂G .

As usual, $\frac{\partial u}{\partial N} = \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_j} \cos(\vec{n}, x_i)$, where $\vec{n}$ is the exterior unit normal vector at a point $x \in \partial G$.

Suppose also that the following condition of uniform ellipticity holds:

$$(2.3) \quad C_1 |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq C_2 |\xi|^2$$

for all $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. Here C_1 and C_2 are positive constants.

This formulation is equivalent to the following weak formulation. A function $u \in W_2^1(G)$ is a weak solution of the Robin problem if

$$(2.4) \quad \begin{aligned} \sum_{i,j=1}^n \int_G a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial \bar{\eta}}{\partial x_i} dx - \sum_{i=1}^n \int_G b_i(x) \frac{\partial u}{\partial x_i} \bar{\eta} dx - \int_G a u \bar{\eta} dx \\ = - \int_G f \bar{\eta} dx - \int_{\partial G} \sigma u \bar{\eta} dS_x + \int_{\partial G} \mu \bar{\eta} dS_x \end{aligned}$$

for all $\eta \in W_2^1(G)$. Here dS_x is the standard surface measure.

For the weak formulation, smoothness of functions f, σ, μ is not necessary and can be replaced by the following weak conditions: $a_{i,j}, b_i, a \in L^\infty(G)$, $f \in L^2(G)$, $\sigma \in L^\infty(\partial G)$, $\mu \in L^2(\partial G)$.

2.2. Weak Robin problem for domains with an ‘outside peak’. Suppose domain G is not of class C^1 at one isolated point.

Assumptions on the functions $a_{ij} = a_{ji}$, f and a are the same for any such domain: $f \in L^2(G)$, $a \in L^\infty$ and $a_{ij} = a_{ji} \in L^\infty(G)$. Additional conditions for b_i , σ , μ depend on properties of the boundary and will be formulated later only for domains with ‘outside peaks’.

The next definition is a formal description of domains with an ‘outside peak’.

Definition 2.1. We call a bounded C^1 -domain $G \subset \mathbb{R}^n$ a domain of class OP_φ if

1. There exists such point $O \in \partial G$ that $\partial G \setminus \{O\}$ is an $(n-1)$ -dimensional C^1 -manifold.
2. Let $\Omega \subset \mathbb{R}^{n-1}$ be a bounded domain of the class C^1 and function $\varphi \in C^1([0, 1])$ be such that $\varphi(0) = \varphi'(0) = 0$ and $\varphi'(t) > 0$ for $t \in (0, 1)$. Denote $x' = (x_1, \dots, x_{n-1})$.

There exists a neighborhood $U(O)$ of O that can be represented as

$$(2.5) \quad U(O) \cap G = \left\{ x = (x', x_n) \in \mathbb{R}^n : 0 < x_n < 1, \frac{x'}{\varphi(x_n)} \in \Omega \right\}$$

for an appropriate choice of a coordinate system with the origin O in $\mathbb{R}^n$.

The point O is a top of an ‘outside peak’. We will study problem (2.4) for domains of class OP_φ .

Denote by $L^{p,\xi}(\partial G)$ such a space of measurable functions defined on ∂G that

$$\int_{\partial G} |f(x)|^p \xi(x) dS_x \equiv \|f\|_{p,\xi,\partial G}^p < \infty.$$

Here $\xi : \partial G \rightarrow R$ is a fixed nonnegative measurable function (a weight).

We are ready to specify all necessary assumptions for Robin boundary-value problem (2.4) under the uniform ellipticity condition (2.3) for domains of class OP_φ .

Suppose domain G belongs to the class OP_φ . Conditions for functions a_{ij}, b_i, a, f are the same as for the weak Robin problem for bounded C^1 -domains. Assumptions for functions σ, μ are different and depend on function φ .

The functions $\sigma, \mu : \partial G \rightarrow R$ satisfy following conditions:

$$(2.6) \quad \text{ess sup}_{x \in \partial G} \frac{|\sigma(x)|}{\varphi(x_n)} = M_\sigma < \infty, \mu \in L^{2, \frac{1}{\varphi}}(\partial G).$$

Here constant M_σ depends only on function φ that describes the singularity type. The condition for μ is equivalent to $\frac{\mu}{\varphi} \in L^{2,\varphi}(\partial G)$.

New assumptions for σ, μ are essential only in a neighborhood of singularity point $O \in \partial G$. Roughly speaking they correlate with the exact description of the trace space of $W_2^1(G)$ on the boundary ∂G . Reasons for these assumptions will be made clear during the proofs of the main results.

Recall that functions b_i, a, f, u, η (see (2.4)) are complex-valued functions.

2.3. Main results. Let I_1 be the embedding operator of $W_2^1(G)$ into $L^2(G)$, and let I_2 be the embedding operator of $W_2^1(G)$ into $L^{2,\varphi}(\partial G)$. By [14] the space $L^{2,\varphi}(\partial G)$ contains traces of $W_2^1(G)$ on ∂G . Existence, boundedness and compactness of the operator I_1 are well known (see, for example, [11]). Existence and boundedness of the operator I_2 are proved in [14]. Proving compactness of I_2 is the main technical goal of this paper.

Remember that $W_2^1(G)$ with an inner product

$$(2.7) \quad [u, \eta] = \int_G \left[\sum_{i,j=1}^n \frac{\partial u}{\partial x_i} \frac{\partial \bar{\eta}}{\partial x_j} + u \bar{\eta} \right] dx$$

is a Hilbert space.

We adopt a general well-known statement of functional analysis to our study.

Proposition 2.2. *Suppose $G \in OP_\varphi$ and operators I_1 and I_2 are compact. Then the weak Robin problem (2.4)-(2.6) is of Fredholm type; i.e., the problem can be reduced to an operator equation on $W_2^1(G)$,*

$$(2.8) \quad (I + A)u = F,$$

where I is the identity and A is a compact operator.

Compactness of the embedding operator I_2 is the content of the following result:

Theorem 2.3. *If $G \in OP_\varphi$, then the embedding operator I_2 is compact.*

This theorem is a special case, $p = 2$, of the corresponding result for Sobolev spaces $W_p^1(G)$ that will be proved in Section 4. Combining two previous results and using compactness of the embedding operator I_1 [11] we obtain one of the main results of this study.

Theorem 2.4. *If $G \in OP_\varphi$, then the weak Robin problem (2.4)-(2.6) is of Fredholm type.*

Proof of the main results is based on the exact descriptions of the trace spaces of Sobolev spaces $W_p^1(G)$ on boundaries of OP_φ -domains [14]. For the reader's convenience we reproduce this description here.

Furthermore, the relation $A \sim B$ means that a two-sided inequality $C_1 < \frac{A}{B} < C_2$ with constants $0 < C_1 < C_2 < \infty$ depending only on G is correct.

Denote by $TW_p^1(G)$ a normed space of $W_p^1(G)$ -function traces on ∂G with the quotient norm

$$(2.9) \quad \|f\|_{TW_p^1(G)} = \inf \left\{ \|F\|_{W_p^1(G)} : F \in W_p^1(G), F|_{\partial G} = f \right\}.$$

Let $E(x, y) = \max\{\varphi(x_n), \varphi(y_n)\}$ and $\sigma(x, y) = \chi\left(\frac{|x-y|}{E(x,y)}\right)$, where χ is the indicator of $[0; 1]$.

Let $G \in OP_\varphi$, $1 < p < \infty$. By [14],

$$(2.10) \quad \|f\|_{TW_p^1} \sim \|f\|_{p,\varphi,\partial G} + \left\{ \iint_{G \times G} \frac{|f(x) - f(y)|^p}{|x - y|^{n+p-2}} \sigma(x, y) dS_x dS_y \right\}^{1/p}.$$

This means, in particular, that $f/\partial G \in L^{p,\varphi}$.

3. PROOF OF PROPOSITION 2.2

Proof. The space $W_2^1(G)$ is a Hilbert space with the inner product (2.7). Because of the condition (2.3) this inner product is equivalent to the following new one:

$$(3.1) \quad [u, \eta] = \int_G \left[\sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \bar{\eta}}{\partial x_j} + u \bar{\eta} \right] dx.$$

Rewrite the weak Robin boundary problem in terms of this inner product as

$$(3.2) \quad [u, \eta] - \sum_{i=1}^n \int_G b_i(x) \frac{\partial u}{\partial x_i} \bar{\eta} dx - \int_G (a+1) u \bar{\eta} dx + \int_{\partial G} \sigma u \bar{\eta} dS_x = - \int_G f \bar{\eta} dx + \int_{\partial G} \mu \bar{\eta} dS_x.$$

Every integral in (3.2) can be considered as a complex-valued linear functional on $W_2^1(G)$:

$$\begin{aligned} l_{1,u}(\eta) &= - \sum_{i=1}^n \int_G b_i(x) \frac{\partial u}{\partial x_i}(x) \bar{\eta}(x) dx, & l_{2,u}(\eta) &= - \int_G (a+1) u(x) \bar{\eta}(x) dx, \\ l_{3,u}(\eta) &= \int_{\partial G} \sigma(x) u(x) \bar{\eta}(x) dS_x, & l_4(\eta) &= \int_{\partial G} \mu(x) \bar{\eta}(x) dS_x, \\ l_5(\eta) &= - \int_G f(x) \bar{\eta}(x) dx. \end{aligned}$$

Because $a, b_i \in L^\infty(G)$, $f \in L^2(G)$, $u, \frac{\partial u}{\partial x_i} \in L^2(G)$ for any $i = 1, 2, \dots, n$, boundedness of functionals $l_{1,u}, l_{2,u}, l_5$ follows from the Cauchy-Bunyakovski inequality.

Let us prove boundedness of the functional $l_{3,u}$. Using the Cauchy-Bunyakovski inequality, condition (2.6), the trace description for domains of class OP_φ [14] and boundedness of I_2 we obtain the following inequality:

$$(3.3) \quad |l_{3,u}(\eta)| = \left| \int_{\partial G} \sigma u \bar{\eta} dS_x \right| = \left| \int_{\partial G} \frac{\sigma}{\varphi} \varphi u \bar{\eta} dS_x \right| \leq M_\sigma \|u\|_{2,\varphi,\partial G} \cdot \|\eta\|_{2,\varphi,\partial G} \\ \leq M_\sigma \|I_2\|^2 \|u\|_{W_2^1(G)} \|\eta\|_{W_2^1(G)}.$$

In (3.3) the $W_2^1(G)$ -norm of functions is induced by the inner product (2.7). Constants M_σ and $\|I_2\|$ depend only on G and function σ . Therefore functional $l_{3,u}$ is bounded. A similar argument is correct for l_4 :

$$(3.4) \quad |l_4(\eta)| = \left| \int_{\partial G} \mu \bar{\eta} dS_x \right| = \left| \int_{\partial G} \frac{\mu}{\varphi} \varphi \bar{\eta} dS_x \right| \leq \left\| \frac{\mu}{\varphi} \right\|_{L^{2,\varphi}(\partial G)} \cdot \|\eta\|_{W_2^1(G)} \\ \leq \|I_2\| \|\mu\|_{L^{2,\frac{1}{\varphi}}(\partial G)} \cdot \|\eta\|_{W_2^1(G)}.$$

We will use the notation $C, C_1, C_2, \dots$ for different positive constants.

By the Riess theorem there exist such bounded operators $B_i : W_2^1(G) \rightarrow W_2^1(G)$, $i = 1, 2, 3$ that $l_{i,u}(\eta) := [B_i u, \eta]$, $l_4(\eta) := [B_4 \mu, \eta]$, $l_5(\eta) := [B_5 f, \eta]$, $\eta \in W_2^1(G)$.

Denote $F := B_4 \mu + B_5 f$. Rewrite (2.4) using B_i and F as

$$[u + (B_1 + B_2 + B_3)u, \eta] = [F, \eta].$$

We will also use a short notation $A := B_1 + B_2 + B_3$.

Let us prove compactness of operator A . The operator A is compact if operators B_1, B_2, B_3 are compact. It is enough to prove that for any operator B_i , $i = 1, 2, 3$, an image of a weakly convergent sequence contains a strongly convergent subsequence.

Let a sequence $\{u_k\}_1^\infty$ weakly converge to u_0 in $W_2^1(G)$. By continuity of B_i the sequence $B_i u_k$ weakly converges to $B_i u_0$. By compactness of operators I_1 and I_2 we can suppose (without loss of generality) that sequences $\{u_k\}_1^\infty$ and $\{B_i u_k\}_1^\infty$ strongly converge in $L^2(G)$ and $L^{2,\varphi}(\partial G)$ correspondingly.

Let us start from B_1 . For simplicity we will use short notations $l_i, i = 1, 2, 3$, instead of $l_{i,u_k - u_m}$. By the definition of B_1 ,

$$\begin{aligned} [B_1(u_k - u_m), B_1(u_k - u_m)] &= |l_1(B_1(u_k - u_m))| \\ &\leq M \|B_1 u_k - B_1 u_m\|_{L^2(G)}. \end{aligned}$$

Here M is a positive constant. Because $[v, v] \sim \|v\|_{W_2^1(G)}^2$ we have

$$\|B_1 u_k - B_1 u_m\|_{W_2^1(G)}^2 \leq C \|B_1 u_k - B_1 u_m\|_{L^2(G)}$$

for a positive constant C .

The last inequality means that $\{B_1 u_k\}_1^\infty$ strongly converges to $B_1 u_0$ in $W_2^1(G)$. Therefore operator B_1 is compact.

A similar argument is correct for operator B_2 :

$$\begin{aligned} \|B_2(u_k - u_m)\|_{W_2^1(G)}^2 &\sim [B_2(u_k - u_m), B_2(u_k - u_m)] \\ &= |l_2(B_2(u_k - u_m))| \leq C \|B_2 u_k - B_2 u_m\|_{L^2(G)} \end{aligned}$$

for a positive constant C . Therefore sequence $\{B_2 u_k\}_1^\infty$ strongly converges to $B_2 u_0$ in $W_2^1(G)$ and operator B_2 is compact.

For operator B_3 we will use compactness of I_2 . Using similar arguments we have

$$\begin{aligned} \|B_3(u_k - u_m)\|_{W_2^1(G)}^2 &\sim [B_3(u_k - u_m), B_3(u_k - u_m)], \\ &= |l_3(B_3(u_k - u_m))| \leq M_\sigma \|u_k - u_m\|_{L^{2,\varphi}(\partial G)} \cdot \|B_3(u_k - u_m)\|_{L^{2,\varphi}(\partial G)}, \end{aligned}$$

proving compactness of B_3 .

Hence operator $A = B_1 + B_2 + B_3$ is compact. Therefore the operator $I + A$ is of the Fredholm type. $\square$

4. COMPACTNESS OF EMBEDDING OPERATOR $I_p : W_p^1(G) \rightarrow L^{p,\varphi}(\partial G)$

Theorem 2.3 is a special case for $p = 2$ of the following result.

Theorem 4.1. *If $G \in OP_\varphi$, then the embedding operator $I_p : W_p^1(G) \rightarrow L^{p,\varphi}(\partial G)$ is compact.*

Fix a domain $G \in OP_\varphi$. Without loss of generality we can suppose that $0 < \varphi'(x_n) < 1$ for any $0 \leq x_n \leq 1$, a top of peak O coincides with the origin of

coordinates and there exists such a neighborhood $V(O)$ of O that $V(O) \cap G$ can be represented as

$$(4.1) \quad \{x = (x', x_n) \in \mathbb{R}^n : 0 < x_n < 1, |x'| < \varphi(x_n)\}.$$

Here notice that $x = (x', x_n) \in \mathbb{R}^n$, $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$.

For the proof of Theorem 4.1 we need the following technical result:

Lemma 4.2. *Any function $F \in W_p^1(G)$, $1 < p < \infty$, has the following representation:*

$$F(x) = \alpha(x_n) + R(x),$$

where $\alpha \in W_p^1(G)$ is a function of one variable, $R \in W_p^1(G)$, and the following inequality is correct:

$$(4.2) \quad \|\varphi^{-1}(x_n)R\|_{L^p(G)} \leq C \|F\|_{W_p^1(G)} < \infty,$$

for some constant C that depends only on G and n .

For the reader's convenience we give here a complete, comparatively simple and independent proof of this result that was proved by the second author in [14].

Proof. Choose nonnegative functions $h \in C_0^\infty(\mathbb{R}^{n-1})$ and $K \in C_0^\infty(\mathbb{R})$ such that $\text{supp } h \subset B_1(0') := \{x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} : |x'| < 1\}$, $\text{supp } K \subset [\frac{1}{2}, 1]$,

$$\int_{\mathbb{R}^{n-1}} h(x') dx' = 1, \quad \int_{-\infty}^{\infty} K(t) dt = 1. \quad \text{Suppose}$$

$$\Omega(x, r, \theta', y) = \varphi^{-n}(x_n) h\left(\frac{x' + r\theta'}{\varphi(x_n)}\right) r^{n-2} K\left(\frac{y}{\varphi(x_n)}\right),$$

where $r \geq 0$, $y \in \mathbb{R}^1$, and $\theta' \in S_1(0') = \partial B_1(0')$. We will use a family of kernels

$$\Omega_\varepsilon := \Omega\left(x, \frac{r}{\varepsilon}, \theta', \frac{y}{\varepsilon}\right).$$

Denote

$$F_\varepsilon(x) = \int_{S_1(0')} dS_{\theta'} \int_0^\infty dr \int_{-\infty}^\infty F(x' + r\theta', x_n + y) \Omega_\varepsilon\left(x, \frac{r}{\varepsilon}, \theta', \frac{y}{\varepsilon}\right) \frac{dy}{\varepsilon^2}.$$

The proof of this lemma can be divided onto four parts.

1. Let us demonstrate that $F_\varepsilon \rightarrow F$ in $L^{p,loc}(G)$ for $\varepsilon \rightarrow 0$. Using the change of variables $r\theta' = z'$, $y = z_n$ we obtain that

$$F_\varepsilon(x) = \int_{\mathbb{R}^n} F(x + z) h\left(\frac{\varepsilon x' + z'}{\varepsilon \varphi(x_n)}\right) K\left(\frac{z_n}{\varepsilon \varphi(x_n)}\right) \frac{dz}{\varepsilon^n \varphi^n(x_n)}.$$

Let U be a compact subset of G . Denote $x_n^0 = \inf\{x_n : x = (x', x_n) \in U\}$. Then $x_n^0 > 0$. Using monotonicity of φ and compactness of U we have that $\varphi(x_n^0) < \varphi(x_n) < C_0 \varphi(x_n^0)$, where $x = (x', x_n) \in U$ and the constant C_0 depends only on U and φ . By construction of K we have $\frac{1}{2} \leq \frac{z_n}{\varepsilon \varphi(x_n)} \leq 1$. Hence

$$K\left(\frac{z_n}{\varepsilon \varphi(x_n)}\right) \leq C_1 \chi\left(\frac{z_n}{C_0 \varepsilon \varphi(x_n^0)}\right).$$

Because $h\left(\frac{\varepsilon x' + z'}{\varepsilon \varphi(x_n)}\right) \neq 0$ for $\left|\frac{\varepsilon x' + z'}{\varepsilon \varphi(x_n)}\right| < 1$, then

$$\left|h\left(\frac{\varepsilon x' + z'}{\varepsilon \varphi(x_n)}\right)\right| \leq C_2 \chi\left(\frac{|z'|}{2C_0 \varepsilon \varphi(x_n^0)}\right).$$

In these inequalities, $C_1 = \sup\{K(\tau) : \tau \in \mathbf{R}\}$, $C_2 = \sup\{|h(u')| : u' \in \mathbf{R}^{n-1}\}$ and χ is the indicator of $[0; 1]$. The second inequality is based on the inequality $|z'| \leq \varepsilon|x'| + \varepsilon\varphi(x_n) \leq 2\varepsilon\varphi(x_n) \leq 2C_0\varepsilon\varphi(x_n^0)$ that is a direct consequence of an obvious inequality $|\varepsilon x' + z'| \leq \varepsilon\varphi(x_n)$.

Because

$$\int_{\mathbf{R}^n} h\left(\frac{\varepsilon x' + z'}{\varepsilon \varphi(x_n)}\right) K\left(\frac{z_n}{\varepsilon \varphi(x_n)}\right) \frac{dz}{\varepsilon^n \varphi^n(x_n)} = 1,$$

we get

$$|F_\varepsilon(x) - F(x)| \leq C \int_{\mathbf{R}^n} |F(x+z) - F(x)| \chi\left(\frac{z_n}{C_0 \varepsilon \varphi(x_n^0)}\right) \chi\left(\frac{|z'|}{2C_0 \varepsilon \varphi(x_n^0)}\right) \frac{dz}{\varepsilon^n \varphi^n(x_n^0)}.$$

Integrating both parts of the previous inequality in degree p on U and using the Minkovski inequality

$$\begin{aligned} \|F_\varepsilon - F\|_{L^p(U)} &\leq \\ C \int_{\mathbf{R}^n} \left(\int_U |F(x+z) - F(x)|^p dx \right)^{\frac{1}{p}} &\chi\left(\frac{z_n}{C_0 \varepsilon \varphi(x_n^0)}\right) \chi\left(\frac{|z'|}{2C_0 \varepsilon \varphi(x_n^0)}\right) \frac{dz}{\varepsilon^n \varphi^n(x_n^0)} \\ &\leq C \sup\{\|F(x+z) - F(x)\|_{L^p(U)} : |z| \leq 2C_0 \varepsilon \varphi(x_n^0)\}. \end{aligned}$$

Using continuity of function $F \in L^p(G)$ in the sense of L^p (see [13], ch.1) the proof can be finished.

Therefore we obtained the following integral representation:

$$F(x) = F_1(x) - \int_0^1 \frac{\partial}{\partial \varepsilon} F_\varepsilon(x) d\varepsilon$$

for any $F \in L^p(G)$.

We will use the short notation

$$R(x) = - \int_0^1 \frac{\partial}{\partial \varepsilon} F_\varepsilon(x) d\varepsilon.$$

Recall that

$$F_1(x) = \int_{-\infty}^{\infty} dy \int_S dS'_\theta \int_0^{\infty} F(x' + r\theta', x_n + y) \Omega(x, r, \theta', y) dr.$$

Using the following change of variables $x' + r\theta' = z'$, $x_n + y = z_n$, $r^{n-2} dr d\theta' = dz'$, $dy = dz_n$ we obtain

$$(4.3) \quad \alpha(x_n) := F_1(x) = \int_{\mathbf{R}^n} F(z) h\left(\frac{z'}{\varphi(x_n)}\right) K\left(\frac{z_n - x_n}{\varphi(x_n)}\right) \frac{dz}{\varphi^n(x_n)}.$$

Here we used a new notation $\alpha(x_n) := F_1(x)$ emphasizing that function $F_1(x)$ depends only on x_n .

2. Let us demonstrate that $\alpha \in W_p^1(G)$ for $1 < p < \infty$. By construction of α we have

$$(4.4) \quad \|\alpha\|_{L^p(G)}^p = \int_G |\alpha(x_n)|^p dx = \int_0^1 dx_n \int_{\{|x'| < \varphi(x_n)\}} |\alpha(x_n)|^p dx' \\ \leq \int_0^1 \int_{\{|x'| < \varphi(x_n)\}} \left| \int_{R^n} |F(z)| \cdot h\left(\frac{z'}{\varphi(x_n)}\right) \cdot K\left(\frac{z_n - x_n}{\varphi(x_n)}\right) \frac{dz}{\varphi^n(x_n)} \right|^p dx' dx_n.$$

Recall that $\text{supp } h \subset B_1(0')$ and $K \subset [1/2; 1]$. Therefore the domain of integration for z is the following set: $U := \{z : |z'| < \varphi(x_n), x_n + \varphi(x_n)/2 < z_n < x_n + \varphi(x_n)\}$, which belongs to G .

There exist $\delta > 0$ and $o < c < x_n$ such that

$$\varphi(x_n + \delta\varphi(x_n)) = \varphi(x_n) + \varphi'(c)\delta\varphi(x_n) = \varphi(x_n)(1 + \delta\varphi'(c)).$$

By monotonicity of φ we have for $\delta = 1/2$ the following inequality:

$$(4.5) \quad \frac{\varphi(x_n)}{2} \leq \varphi(z_n) \leq \frac{3\varphi(x_n)}{2}.$$

The integrand in (4.4) can be rewritten in a more convenient way as

$$\varphi^{-n} |F| \cdot |h| \cdot |K| = |F| \cdot (|h|^{1/p} |K|^{1/p} \varphi^{-\frac{n}{p}}) \cdot (|h|^{p-1/p} |K|^{p-1/p} \varphi^{-\frac{(p-1)n}{p}}).$$

Using the Hölder inequality and changing the order of integration we obtain

$$(4.6) \quad \|\alpha\|_{L^p(G)}^p = C \int_G |F(z)|^p \Psi(z, x_n) dz,$$

where

$$C = \left\{ \int_G \Psi(z, x_n) dz \right\}^{p-1}, \\ \Psi(z, x_n) = \int_0^1 dx_n \int_{\{|x'| < \varphi(x_n)\}} h\left(\frac{z'}{\varphi(x_n)}\right) K\left(\frac{z_n - x_n}{\varphi(x_n)}\right) \frac{dx'}{\varphi^n(x_n)}.$$

We will estimate $\Psi(z, x_n)$.

Let us first change x_n into $\tau := \frac{z_n - x_n}{\varphi(x_n)}$. Hence

$$\frac{dx_n}{\varphi(x_n)} = - \frac{d\tau}{1 + \frac{z_n - x_n(\tau)}{\varphi(x_n(\tau))} \varphi'(x_n)}.$$

Therefore

$$\int_0^1 K\left(\frac{z_n - x_n}{\varphi(x_n)}\right) \frac{dx_n}{\varphi(x_n)} \leq C \int_{-\infty}^{\infty} K(\tau) \frac{d\tau}{1 - \varphi'(x_n(\tau))} < \infty,$$

because of the boundedness of function K with compact support. Using the same argument for (4.6) we obtain finally the following estimate for the L^p -norm of α :

$$\|\alpha\|_{L^p(G)}^p \leq C \int_G |F(z)|^p dz.$$

Let us prove that $\alpha \in W_p^1(G)$.

Using the following change of variables in (4.3),

$$z' = \varphi(x_n)s', z_n = x_n + \varphi(x_n)s_n,$$

we get

$$\alpha(x_n) = \int_{R^n} F(\varphi(x_n)s', x_n + \varphi(x_n)s_n) h(s') K(s_n) ds.$$

After differentiation we obtain

$$\begin{aligned} \alpha'(x_n) = & \int_{R^n} (\varphi'(\nabla' F(\varphi(x_n)s', x_n + \varphi(x_n)s_n), s')) \\ & + D_n F(\varphi(x_n)s', x_n + \varphi(x_n)s_n)(1 + \varphi'(x_n)s_n)) h(s') K(s_n) ds. \end{aligned}$$

Returning to the original variable we finally get

$$\begin{aligned} |\alpha'(x_n)| \leq & C \int_{R^n} |\nabla' F(z)| \cdot h\left(\frac{z'}{\varphi(x_n)}\right) \cdot K\left(\frac{z_n - x_n}{\varphi(x_n)}\right) \frac{dz}{\varphi^n(x_n)} \\ & + C \int_{R^n} |D_n F(z)| \cdot h\left(\frac{z'}{\varphi(x_n)}\right) \cdot K\left(\frac{z_n - x_n}{\varphi(x_n)}\right) \frac{dz}{\varphi^n(x_n)}. \end{aligned}$$

This inequality permits us to estimate $\|\alpha'\|_{L^p(G)}$ by the same way as $\|\alpha\|_{L^p(G)}$. Therefore $\alpha \in W_p^1(G)$.

3. Function R can be represented as $R = R_1 + R_2$, where

$$\begin{aligned} R_1(x) = & - \int_0^1 \frac{d\varepsilon}{\varepsilon^{n+1}} \int_{\mathbb{R}^n} (\nabla' F(z), z' - x') h\left(\frac{z' - (1 - \varepsilon)x'}{\varepsilon\varphi(x_n)}\right) K\left(\frac{z_n - x_n}{\varepsilon\varphi(x_n)}\right) \frac{dz}{\varphi^n(x_n)}, \\ R_2(x) = & - \int_0^1 \frac{d\varepsilon}{\varepsilon^{n+1}} \int_{\mathbb{R}^n} D_n F(z) \cdot (z_n - x_n) K\left(\frac{z_n - x_n}{\varepsilon\varphi(x_n)}\right) h\left(\frac{z' - (1 - \varepsilon)x'}{\varepsilon\varphi(x_n)}\right) \frac{dz}{\varphi^n(x_n)}. \end{aligned}$$

Here $\nabla' = \left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_{n-1}}\right)$, $z = (z', z_n)$, $D_n = \frac{\partial}{\partial z_n}$.

Let us verify this representation. Remember first that

$$\Omega\left(x, \frac{r}{\varepsilon}, \theta', \frac{y}{\varepsilon}\right) = \frac{r^{n-2}}{\varphi^n(x_n)\varepsilon^{n-2}} h\left(\frac{\varepsilon x' + r\theta'}{\varepsilon\varphi(x_n)}\right) K\left(\frac{y}{\varepsilon\varphi(x_n)}\right).$$

Hence

$$\begin{aligned} (4.7) \quad & \frac{\partial}{\partial \varepsilon} \left[\varepsilon^{-2} \Omega\left(x, \frac{r}{\varepsilon}, \theta', \frac{y}{\varepsilon}\right) \right] \\ & = - \frac{nr^{n-2}}{\varepsilon^{n+1}\varphi^n(x_n)} hK - \frac{r^{n-1}}{\varepsilon^{n+2}\varphi^{n+1}(x_n)} \sum_{j=1}^{n-1} D_j h \theta_j - \frac{yr^{n-2}}{\varepsilon^{n+2}\varphi^{n+1}(x_n)} hK' \\ & \quad - \frac{K}{\varepsilon^{n+1}\varphi^n(x_n)} \frac{\partial}{\partial r} [r^{n-1}h] - \frac{r^{n-2}}{\varepsilon^{n+1}\varphi^n(x_n)} h \frac{\partial}{\partial y} \left[yK\left(\frac{y}{\varepsilon\varphi(x_n)}\right) \right]. \end{aligned}$$

Using (4.7) we get the following expression for $R(x)$:

$$\begin{aligned}
 R(x) &= - \int_0^1 \frac{\partial F_\epsilon}{\partial \epsilon} d\epsilon \\
 &= - \int_0^1 d\epsilon \int_{-\infty}^{\infty} dy \int_S dS'_\theta \int_0^\infty F(x' + r\theta', x_n + y) \frac{\partial}{\partial \epsilon} \left[\Omega(x', \frac{r}{\epsilon}, \theta', \frac{y}{\epsilon}) \epsilon^{-2} \right] dr \\
 &= - \int_0^1 d\epsilon \int_S dS_{\theta'} \int_0^\infty \left(\sum_{j=1}^{n-1} D_j F \theta_j \right) r^{n-1} h \left(\frac{\epsilon x' + r\theta'}{\epsilon \varphi(x_n)} \right) dr \\
 &= \int_0^1 d\epsilon \int_S dS_{\theta'} \int_0^\infty h \left(\frac{\epsilon x' + r\theta'}{\epsilon \varphi(x_n)} \right) \frac{r^{n-2}}{\epsilon^{n+1} \varphi^n(x_n)} dr \int_{-\infty}^\infty y D_n F K \left(\frac{y}{\epsilon \varphi(x_n)} \right) dy.
 \end{aligned}$$

Using the following change of variables,

$$z' = x' + r\theta', dz' = r^{n-2} dr dS'_\theta, x_n + y = z_n, dz_n = dy,$$

we obtain the above-claimed decomposition $R = R_1 + R_2$:

$$\begin{aligned}
 R(x) &= - \int_0^1 \frac{d\epsilon}{\epsilon^{n+1} \varphi^n(x_n)} \int_{R^n} (\nabla' F(z), z' - x') K \left(\frac{z_n - x_n}{\epsilon \varphi(x_n)} \right) h \left(\frac{z' - (1 - \epsilon)x'}{\epsilon \varphi(x_n)} \right) dz \\
 &\quad - \int_0^1 \frac{d\epsilon}{\epsilon^{n+1} \varphi^n(x_n)} \int_{R^n} D_n F(z) \cdot (z_n - x_n) K \left(\frac{z_n - x_n}{\epsilon \varphi(x_n)} \right) h \left(\frac{z' - (1 - \epsilon)x'}{\epsilon \varphi(x_n)} \right) dz \\
 &= R_1 + R_2.
 \end{aligned}$$

4. Similarly to the plane case [15] and to the proof of (4.2) we will demonstrate that $\varphi^{-1}R_i \in L^p(G)$, $i = 1, 2$. We will prove this fact only for R_1 . For R_2 the proof is similar.

We have

$$\begin{aligned}
 (4.8) \quad &|\varphi^{-1}(x_n)R_1(x)| \\
 &\leq \int_0^1 d\epsilon \int_{\mathbb{R}^n} |\nabla' F(z)| \cdot \left| \frac{z' - x'}{\epsilon \varphi(x_n)} \right| h \left(\frac{z' - (1 - \epsilon)x'}{\epsilon \varphi(x_n)} \right) K \left(\frac{z_n - x_n}{\epsilon \varphi(x_n)} \right) \frac{dz}{\epsilon^n \varphi^n(x_n)}.
 \end{aligned}$$

Denote by $X(x)$ the extension of $|\varphi^{-1}(x_n)R_1(x)|$ on $\mathbb{R}^n \setminus G$ by zero and by $Y(z)$ the extension of $|\nabla' F(z)|$ by zero on $\mathbb{R}^n \setminus G$. Recall the generalized Minkovski inequality. Let $\psi(x, y)$ be a measurable nonnegative function defined on $A \times B$, where $A \in R^n$, $B \in R^m$ are measurable sets. Then for any $1 \leq p < \infty$,

$$\left\{ \int_A \left(\int_B \psi(x, y) dy \right)^p dx \right\}^{1/p} \leq \int_B \left(\int_A \psi^p(x, y) dx \right)^{1/p} dy.$$

Using the generalized Minkovski inequality we get

$$(4.9) \quad \|X\|_{L_p(\mathbf{R}^n)} \leq \int_0^1 \left\{ \int_{\mathbf{R}^n} \left[\frac{1}{\varepsilon^n \varphi^n(x_n)} \int_{\mathbf{R}} \Phi(z, x) \left(\int_{\mathbf{R}^{n-1}} h \left(\frac{z' - (1-\varepsilon)x'}{\varepsilon \varphi(x_n)} \right) dz' \right)^{\frac{p-1}{p}} K dz_n \right]^p dx \right\}^{1/p} d\varepsilon.$$

Here

$$\Phi(z, x) = \left\{ \int_{\mathbf{R}^{n-1}} Y^p(z) h \left(\frac{z' - (1-\varepsilon)x'}{\varepsilon \varphi(x_n)} \right) dz' \right\}^{1/p}.$$

Because

$$\left(\int_{\mathbf{R}^{n-1}} h \left(\frac{z' - (1-\varepsilon)x'}{\varepsilon \varphi(x_n)} \right) dz' \right)^{\frac{p-1}{p}} \leq C [\varepsilon \varphi(x_n)]^{\frac{(n-1)(p-1)}{p}}$$

we finally obtain

$$(4.10) \quad \|X\|_{L_p(\mathbf{R}^n)} \leq C \int_0^1 \left\{ \int_{\mathbf{R}^n} \left[\int_{\mathbf{R}} \Phi(z, x) K \left(\frac{z_n - x_n}{\varepsilon \varphi(x_n)} \right) dz_n \right]^p \frac{dx}{(\varepsilon \varphi(x_n))^{n+p-1}} \right\}^{1/p} d\varepsilon.$$

Using for the term in square brackets from the right hand side of (4.10) the Hölder inequality for the variable z_n and taking into account the inequality

$$\left\{ \int_{-\infty}^{\infty} K \left(\frac{z_n - x_n}{\varepsilon \varphi(x_n)} \right) dz_n \right\}^{p-1} \leq C (\varepsilon \varphi(x_n))^{p-1},$$

we obtain the following L^p -estimate:

$$(4.11) \quad \|X\|_{L^p(G)} \leq C \int_0^1 \left\{ \int_{\mathbb{R}^n} Y^p(z) \left[\int_{\mathbb{R}^n} h \left(\frac{z' - (1-\varepsilon)x'}{\varepsilon \varphi(x_n)} \right) K \left(\frac{z_n - x_n}{\varepsilon \varphi(x_n)} \right) \frac{dx}{(\varepsilon \varphi(x_n))^n} \right] dz \right\}^{\frac{1}{p}} d\varepsilon.$$

Denote by J the term in square brackets on the right hand side of (4.11). By direct calculations we have

$$\begin{aligned} J &= \int_{\mathbb{R}^n} h \left(\frac{z' - (1-\varepsilon)x'}{\varepsilon \varphi(x_n)} \right) K \left(\frac{z_n - x_n}{\varepsilon \varphi(x_n)} \right) \frac{dx}{(\varepsilon \varphi(x_n))^n} \\ &= \int_{-\infty}^{\infty} K \left(\frac{z_n - x_n}{\varepsilon \varphi(x_n)} \right) \left(\int_{\mathbb{R}^{n-1}} h \left(\frac{z' - (1-\varepsilon)x'}{\varepsilon \varphi(x_n)} \right) dx' \right) \frac{dx_n}{(\varepsilon \varphi(x_n))^n} \\ &\leq C \int_{-\infty}^{\infty} K \left(\frac{z_n - x_n}{\varepsilon \varphi(x_n)} \right) \frac{dx_n}{\varepsilon \varphi(x_n)}. \end{aligned}$$

For the last inequality we used the following estimate:

$$\int_{\mathbf{R}^{n-1}} h \left(\frac{z' - (1-\varepsilon)x'}{\varepsilon \varphi(x_n)} \right) dx' \leq C [\varepsilon \varphi(x_n)]^{n-1}.$$

This estimate can be proved by the following way. Let

$$A = \left\{ x' \in \mathbf{R}^{n-1} : \left| \frac{z' - x'}{\varepsilon \varphi(x_n)} \right| < 1 \right\}, \quad B = \{ x' \in \mathbf{R}^{n-1} : |z' - x'| < 2\varepsilon \varphi(x_n) \}.$$

Using the definition of class OP_φ and the inequalities

$$\left| \frac{z' - x'}{\varepsilon \varphi(x_n)} \right| - 1 \leq \left| \frac{z' - x'}{\varepsilon \varphi(x_n)} \right| - \frac{|x'|}{\varphi(x_n)} \leq \left| \frac{z' - (1 - \varepsilon)x'}{\varepsilon \varphi(x_n)} \right| < 1,$$

it follows that $A \subset B$. Therefore

$$\int_{\mathbf{R}^{n-1}} h \left(\frac{z' - (1 - \varepsilon)x'}{\varepsilon \varphi(x_n)} \right) dx' \leq \int_A h dx' \leq \int_B h dx'.$$

Let us change variable x_n to $\tau := \frac{z_n - x_n}{\varepsilon \varphi(x_n)} = \tau$ ($x_n \rightarrow \tau$). Hence

$$\frac{dx}{\varepsilon \varphi(x_n)} = - \frac{d\tau}{1 + \frac{z_n - x_n(\tau)}{\varepsilon \varphi(x_n(\tau))} \varphi'(x_n)}.$$

Therefore

$$\int_{-\infty}^{\infty} K \left(\frac{z_n - x_n}{\varepsilon \varphi(x_n)} \right) \frac{dx}{\varepsilon \varphi(x_n)} \leq C \int_{-\infty}^{\infty} K(\tau) \frac{d\tau}{1 - \varepsilon \varphi'(x_n(\tau))} \leq C,$$

where constant C depends only on $F(x)$. Using the last inequality (4.11) we obtain finally

$$(4.12) \quad \|X\|_{L^p(\mathbb{R}^n)} \leq C \|Y\|_{L^p(\mathbb{R}^n)} \leq C \|F\|_{W_p^1(G)}.$$

□

Using the previous Lemma 4.2 we will prove Theorem 4.1.

Proof of Theorem 4.1. Let $\mathfrak{M} \subset W_p^1(G)$ be a bounded set and $\Lambda = I_p(\mathfrak{M}) \subset L^{p,\varphi}(\partial G)$. To prove compactness of $I_2 : W_p^1(G) \rightarrow L^{p,\varphi}(\partial G)$ we need to construct for any $\varepsilon > 0$ a finite ε -network of Λ .

By Lemma 4.2 any function $F \in \mathfrak{M}$ can be represented as $F = \alpha + R$, $\alpha = \alpha(x_n)$. Denote by Λ_α the set of all functions $\alpha = F - R$ such that $F \in \mathfrak{M}$, and denote by Λ_R the set of all functions R that correspond to $F \in \mathfrak{M}$. Hence $\Lambda = \Lambda_\alpha + \Lambda_R$.

Because the embedding operator $I_1 : W_p^1(G) \rightarrow L^p(G)$ is compact, the set Λ_α is relatively compact in $L^p(G)$. Therefore for any $\varepsilon_1 > 0$ there exists an ε_1 -network $\beta_1, \dots, \beta_N$ of Λ_α in $L^p(G)$. Denote $G_\delta = B_\delta(0) \cap G$, where $B_\delta(0) = \{x \in \mathbb{R}^n : |x| < \delta\}$. Further we suppose that $0 < \delta < 1$. Using continuity of function $F \in L^p(G)$ in the sense of L^p (see [13], ch.1), we can find $\delta > 0$ such that for any $j = 1, 2, \dots, N$ the following inequality is correct:

$$\|\beta_j\|_{p, G_\delta} < \varepsilon_1.$$

Hence for any $\alpha \in \Lambda_\alpha$ there exists such a β_j that

$$(4.13) \quad \|\alpha\|_{p, G_\delta} \leq \|\alpha - \beta_j\|_{p, G_\delta} + \|\beta_j\|_{p, G_\delta} < 2\varepsilon_1.$$

By construction of the domain G in a neighborhood of its peak with the top O (see (2.5)) we have by direct calculation

$$\int_{G_\delta} |\alpha(x_n)|^p dx = \omega_{n-1} \int_0^\delta |\alpha(x_n)|^p \varphi^{n-1}(x_n) dx_n,$$

where ω_{n-1} is the volume of the unit $(n-1)$ -dimensional ball.

On the other hand we will demonstrate (using (2.5)) that

$$\int_{\partial G_\delta} |\alpha(x_n)|^p \varphi(x_n) dS_x \sim \int_0^\delta |\alpha(x_n)|^p \varphi^{n-1}(x_n) dx_n.$$

The argument is simple. The manifold $\partial G_\delta \setminus \{0\}$ is a finite union of charts of the following type:

$$(4.14) \quad \partial G_{\delta,\sigma}^{(n-1)} = \left\{ x = (x'', x_{n-1}, x_n) : 0 < x_n < \delta, \right. \\ \left. x_{n-1} = \sqrt{\varphi^2(x_n) - |x''|^2}, |x''| = \left(\sum_{i=1}^{n-2} x_i^2 \right)^{1/2} < \sigma \varphi(x_n) \right\},$$

where $\sigma \in (0, 1)$. By direct calculations,

$$dS_x = \varphi(x_n) \sqrt{\frac{1 + (\varphi'(x_n))^2}{\varphi^2(x_n) - |x''|^2}} dx'' dx_n,$$

and by the inequality $|x''| < \sigma \varphi(x_n)$,

$$\varphi(x_n) \sqrt{\frac{1 + (\varphi'(x_n))^2}{\varphi^2(x_n) - |x''|^2}} \sim 1.$$

Therefore

$$(4.15) \quad \int_{\partial G_\delta^{(n-1)}} |\alpha(x_n)|^p \varphi(x_n) dS_x \sim \int_0^\delta dx_n \int_{\{|x''| < \sigma \varphi(x_n)\}} |\alpha(x_n)|^p \varphi(x_n) dx'' \\ \sim \int_0^\delta |\alpha(x_n)|^p \varphi^{n-1}(x_n) dx_n.$$

Combining this estimate and (4.13) we can conclude that integrals

$$\int_{\partial G_\delta} |\alpha(x_n)|^p \varphi(x_n) dS_x, \quad \alpha \in \Lambda_\alpha,$$

are uniformly small for all $\alpha \in \Lambda_\alpha$ if $\delta > 0$ is small enough.

To estimate the function $R \in \Lambda_R$ that corresponds to $F \in \mathfrak{M}$ we will use Lemma 4.2. Denote $\partial_0 G = \{x \in \partial G : |x'| = \varphi(x_n), 0 < x_n < 1\}$. First we will estimate

the integral

$$\int_{\partial_0 G_\delta} |R(x)|^p \varphi(x_n) dS_x,$$

for some $\delta > 0$.

The boundary $\partial_0 G_\delta$ can be covered by a finite number of charts $\partial G_{\delta,\sigma}^{(k)}, k = 1, 2, \dots, n-1$ (see (4.14)). Let us prove estimates for R for the chart $\partial G_{\delta,\sigma}^{(n-1)}$. We will use the notation $x = (x'', x_{n-1}, x_n) \in \partial_0 G_{\delta,\sigma}^{(n-1)}$. Then

$$x_{n-1} = \sqrt{\varphi^2(x_n) - |x''|^2}, |x''| < \sigma\varphi(x_n).$$

Choose $s : 0 < s < \sqrt{\varphi^2(x_n) - |x''|^2}$. Then

$$R(x) = R(x'', s, x_n) + \int_s^{x_{n-1}} D_{n-1} R(x'', \tau, x_n) d\tau,$$

where $D_{n-1} = \frac{\partial}{\partial x_{n-1}}$.

Integrating in s we obtain

$$(4.16) \quad |R(x)|^p \varphi(x_n) \leq C_1 \int_0^{\sqrt{\varphi^2(x_n) - |x''|^2}} |R(x'', s, x_n)|^p ds \\ + C_2 \varphi(x_n) \left(\int_0^{\sqrt{\varphi^2(x_n) - |x''|^2}} |D_{n-1} R(x'', \tau, x_n)|^p d\tau \right)^p.$$

For the estimate (4.16) we used that $\sqrt{\varphi^2(x_n) - |x''|^2} \sim \varphi(x_n)$ if $|x''| < \sigma\varphi(x_n)$. Integrating (4.16) on $\partial_0 G_{\delta,\sigma}^{(n-1)}$, taking into account $dS_x \sim dx'' dx_n$ and using Hölder's inequality for variable τ , we get the following inequality:

$$\int_{\partial_0 G_{\delta,\sigma}^{(n-1)}} |R(x)|^p \varphi(x_n) dS_x \\ \leq C_1 \int_0^\delta dx_n \int_{\{|x''| < \sigma\varphi(x_n)\}} dx'' \int_0^{\sqrt{\varphi^2(x_n) - |x''|^2}} |R(x'', s, x_n)|^p ds \\ + C_2 \int_0^\delta \varphi^p(x_n) dx_n \int_{\{|x''| < \sigma\varphi(x_n)\}} dx'' \int_0^{\sqrt{\varphi^2(x_n) - |x''|^2}} |D_{n-1} R(x'', \tau, x_n)|^p d\tau \\ \leq C_1 \int_{G_\delta} |R(x)|^p dx + C_2 \varphi^p(\delta) \int_{G_\delta} |D_{n-1} R(x)|^p dx.$$

Combining with (4.12) we obtain

$$(4.17) \quad \int_{\partial_0 G_{\delta, \sigma}^{(n-1)}} |R(x)|^p \varphi(x_n) dS_x \leq C \varphi^p(\delta) \left[\int_G \left| \frac{R(x)}{\varphi(x_n)} \right|^p dx + \int_G |D_{n-1} R(x)|^p dx \right].$$

Using this estimate and Lemma 4.2 we get finally

$$(4.18) \quad \|R\|_{p, \varphi, \partial_0 G_\delta} \leq C \varphi(\delta) \|F\|_{W_p^1(G)}.$$

This estimate and corresponding estimates for the integrals of $|\alpha(x_n)|^p$ on $\partial_0 G_\delta$ for $\alpha \in \Lambda_\alpha$ allow us to conclude that for any $\epsilon_2 > 0$ there exists such a $\delta_0 > 0$ that for all $\delta \in (0, \delta_0)$ and for all $F \in \mathfrak{M}$ the following inequality is correct:

$$(4.19) \quad \|F\|_{p, \varphi, \partial_0 G_\delta} < \epsilon_2.$$

Let δ be an arbitrary number from $(0, \delta_0)$. Denote by k_δ a function of the class $C^1([0, 1])$ with the following properties: $0 \leq k_\delta \leq 1$ on $[0, 1]$, $k_\delta(t) = 1$ when $\delta < t \leq 1$, $k_\delta(t) = 0$ when $0 \leq t \leq \delta/2$. Denote by $\mathfrak{M}(\delta)$ traces of functions F from $\mathfrak{M}$ on $G \setminus \overline{G_{\delta/2}}$. The set $\mathfrak{M}(\delta)$ is bounded in $W_p^1(G \setminus \overline{G_{\delta/2}})$ and therefore [13] is compactly embedded into $L^{p, \varphi}(\partial_0 G \setminus \partial_0 G_{\delta/2}) \sim L^p(\partial_0 G \setminus \partial_0 G_{\delta/2})$ (for fixed δ). Let $\epsilon > 0$ be an arbitrary number. Choose a finite $\epsilon/2$ -network for $\Lambda(\delta) = I_2(\mathfrak{M}(\delta)) \subset L^p(\partial_0 G \setminus \partial_0 G_{\delta/2})$ that is a finite set of functions $\tilde{\nu}_1, \dots, \tilde{\nu}_r$. Then the set of functions $\nu_j = k_\delta \tilde{\nu}_j$, $j = 1, 2, \dots, r$ represents a finite $\epsilon/2$ -network for the set $k_\delta \Lambda(\delta) = \{k_\delta F : F \in \mathfrak{M}(\delta)\}$ into $L^{p, \varphi}(\partial_0 G)$. It is possible to suppose that for the same δ the inequality (4.19) is correct with $\epsilon_2 = \epsilon/2$.

Finally we will prove that the set of functions $\nu_1, \dots, \nu_r$ represents an ϵ -network of Λ into $L^{p, \varphi}(\partial_0 G)$. The argument is standard. Let $F \in \mathfrak{M}$. Then $F = (1 - k_\delta)F + k_\delta F$. Because $k_\delta F \in k_\delta \Lambda(\delta)$ there exists such a ν_i that $\|k_\delta F - \nu_i\|_{p, \varphi, \partial_0 G} < \epsilon/2$. Then by (4.19),

$$\|F - \nu_i\|_{p, \varphi, \partial_0 G} \leq \|(1 - k_\delta)F\|_{p, \varphi, \partial_0 G} + \|k_\delta F - \nu_i\|_{p, \varphi, \partial_0 G} < \epsilon/2 + \epsilon/2 = \epsilon.$$

Therefore $\nu_1, \dots, \nu_r$ is the desired ϵ -network for $\Lambda = I_2(\mathfrak{M})$ in $L^{p, \varphi}(\partial_0 G)$. $\square$

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