MANN PAIRS

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ABSTRACT. Mann proved in the 1960s that for any $n \geq 1$ there is a finite set E of n -tuples $(\eta_1, \dots, \eta_n)$ of complex roots of unity with the following property: if $a_1, \dots, a_n$ are any rational numbers and $\zeta_1, \dots, \zeta_n$ are any complex roots of unity such that $\sum_{i=1}^n a_i \zeta_i = 1$ and $\sum_{i \in I} a_i \zeta_i \neq 0$ for all nonempty $I \subseteq \{1, \dots, n\}$, then $(\zeta_1, \dots, \zeta_n) \in E$. Taking an arbitrary field $\mathbf{k}$ instead of $\mathbb{Q}$ and any multiplicative group in an extension field of $\mathbf{k}$ instead of the group of roots of unity, this property defines what we call a Mann pair $(\mathbf{k}, \Gamma)$. We show that Mann pairs are robust in certain ways, construct various kinds of Mann pairs, and characterize them model-theoretically.

1. INTRODUCTION

Throughout this paper we use the following notation and conventions. We let Ω be an ambient algebraically closed field (taken to be $\mathbb{C}$ if suggested by the context), and let Γ be a subgroup of the multiplicative group $\Omega^\times$. We say that Γ has *finite rank* if Γ has a finitely generated subgroup Γ_0 such that Γ/Γ_0 is a torsion group. We let m, n range over $\mathbb{N} = \{0, 1, 2, \dots\}$. For $r \in \Omega$ and $\vec{s} = (s_1, \dots, s_n) \in \Omega^n$ we put $r\vec{s} := (rs_1, \dots, rs_n) \in \Omega^n$. In particular, if $\alpha \in \Gamma$ and $\vec{\gamma} \in \Gamma^n$, then $\alpha\vec{\gamma} \in \Gamma^n$. We let $\mathbb{U} \subseteq \mathbb{C}^\times$ be the group of complex roots of unity, and $\mathbb{Q}^{\text{ac}} \subseteq \mathbb{C}$ the field of complex algebraic numbers.

Let $a_1, \dots, a_n \in \Omega$, $n \geq 1$, and consider the equation

$$(*) \quad a_1 x_1 + \dots + a_n x_n = 1.$$

A *solution* of $(*)$ is a tuple $(s_1, \dots, s_n) \in \Omega^n$ such that $a_1 s_1 + \dots + a_n s_n = 1$; such a solution is said to be *nondegenerate* if $\sum_{i \in I} a_i s_i \neq 0$ for all nonempty $I \subseteq \{1, \dots, n\}$, and is said to be in Γ if $(s_1, \dots, s_n) \in \Gamma^n$.

In [5] we defined Γ to have the *Mann property* if for all $n \geq 1$ and nonzero $a_1, \dots, a_n$ in the prime field of Ω the equation $(*)$ has only finitely many nondegenerate solutions in Γ ; equivalently, for all $n \geq 1$ and $a_1, \dots, a_n \in \Omega^\times$ the equation $(*)$ has only finitely many non-degenerate solutions in Γ , see [5]. The Mann property is a Mordell-Lang type property. Mann [12] gave an effective proof that $\mathbb{U}$ has the Mann property. More generally, but less effectively, [6], [11], and [14] show that if Ω has characteristic zero and Γ has finite rank, then Γ has the Mann property. In [5] we studied the model theory of structures (Ω, Γ) when Γ has the Mann property. We now define a *uniform* version of the Mann property involving also a subfield of Ω . From now on $\mathbf{k}$ is a subfield of Ω .

Received by the editors November 1, 2007.

2010 *Mathematics Subject Classification*. Primary 03C35, 03C60, 03C98, 11U09.

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Call $(\mathbf{k}, \Gamma)$ a *Mann pair* if for each $n \geq 1$ there is a finite $E \subseteq \Gamma^n$ such that for all $a_1, \dots, a_n \in \mathbf{k}^\times$, all nondegenerate solutions of $(*)$ in Γ lie in E .

Suppose $(\mathbf{k}, \Gamma)$ is a Mann pair. It is obvious that then Γ has the Mann property and that if $\mathbf{k}'$ is a subfield of $\mathbf{k}$ and Γ' a subgroup of Γ , then $(\mathbf{k}', \Gamma')$ is also a Mann pair. Taking $n = 1$ in the definition we see that $\mathbf{k}^\times \cap \Gamma$ is finite, and so all elements of $\mathbf{k}^\times \cap \Gamma$ are roots of unity.

Actually, Theorem 1 of [12] implies that $(\mathbb{Q}, \mathbb{U})$ is a Mann pair; conversely, Corollary 5.1 below says that if $(\mathbb{Q}, \Gamma)$ is a Mann pair and $\Gamma \subseteq (\mathbb{Q}^{\text{ac}})^\times$, then $\Gamma \subseteq \mathbb{U}$. Thus $(\mathbb{Q}, (1 + \sqrt{2})^\mathbb{Z})$ is not a Mann pair, although $(1 + \sqrt{2})^\mathbb{Z} \subseteq (\mathbb{Q}^{\text{ac}})^\times$ has the Mann property and $\mathbb{Q}^\times \cap (1 + \sqrt{2})^\mathbb{Z} = \{1\}$. In Section 2 we indicate simple examples of Mann pairs, such as:

- (1) $(\mathbb{Q}^{\text{ac}}, \exp(\mathbb{Q}^{\text{ac}}))$ is a Mann pair (Lindemann's theorem).
- (2) If $v : K^\times \rightarrow v(K^\times)$ is a valuation on a subfield K of Ω and $\mathbf{k}$ is a subfield of K and Γ a subgroup of $K^\times$ such that v is trivial on $\mathbf{k}$ and injective on Γ , then $(\mathbf{k}, \Gamma)$ is a Mann pair.

A more substantial source of Mann pairs is the following:

Theorem 1.1. *Suppose Ω has characteristic zero, $\mathbf{k}$ is algebraically closed, $\mathbf{k}^\times \cap \Gamma = \{1\}$, and Γ has finite rank. Then $(\mathbf{k}, \Gamma)$ is a Mann pair.*

This is an analogue for Mann pairs of the result from [6], [11], [14] that was mentioned earlier. In Section 3 we first deal with the special case that Γ is finitely generated and $\mathbf{k}(\Gamma)$ has transcendence degree 1 over $\mathbf{k}$; the key here is a height bound by Brownawell and Masser [3]. With this special case as a stepping stone and using various facts established in Sections 4 and 5 we prove Theorem 1.1 in Section 6. This proof is effective, at least in principle, but we do not elaborate upon this aspect in the present paper. Section 5 also describes some curious Mann pairs in positive characteristic as well as some Mann pairs $(\mathbb{Q}_p, \Gamma)$ obtained by Teichmüller lifting.

Next we consider the structure $(\Omega, \mathbf{k}, \Gamma)$, that is, the algebraically closed field Ω with distinguished subsets $\mathbf{k}$ and Γ . In Sections 4 and 7 we show:

Theorem 1.2. *The following are equivalent:*

- (1) $(\mathbf{k}, \Gamma)$ is a Mann pair;
- (2) for each $n \geq 1$ the set $\{(x, y) \in \mathbf{k}^n \times \Gamma^n : x_1 y_1 + \dots + x_n y_n = 0\}$ is a finite union of sets $X \times Y$ where $X \subseteq \mathbf{k}^n$ is definable in the field $\mathbf{k}$ and $Y \subseteq \Gamma^n$ is definable in the group Γ ;
- (3) for all m, n , every subset of $\mathbf{k}^m \times \Gamma^n$ definable in $(\Omega, \mathbf{k}, \Gamma)$ is a finite union of sets $X \times Y$ with $X \subseteq \mathbf{k}^m$ definable in the field $\mathbf{k}$ and $Y \subseteq \Gamma^n$ definable in the group Γ .

Here and below, “definable” means “definable with parameters” unless we specify otherwise. Thus (3) says that the structure induced by $(\Omega, \mathbf{k}, \Gamma)$ on $\mathbf{k}$ and Γ (using parameters from Ω) is just the field structure of $\mathbf{k}$ and the group structure of Γ , respectively, and that these two structures only interact trivially inside $(\Omega, \mathbf{k}, \Gamma)$.

In Section 8 we prove that for Mann pairs $(\mathbf{k}, \Gamma)$ with $[\Omega : \mathbf{k}] > 2$, the complete theory $\text{Th}(\Omega, \mathbf{k}, \Gamma)$ is determined by $\text{Th}(\mathbf{k})$ and $\text{Th}(\Gamma)$ after adding names for enough elements of $\mathbf{k}$ and Γ to witness that $(\mathbf{k}, \Gamma)$ is a Mann pair. Under the additional assumption that $\mathbf{k}$ is algebraically closed we show that $(\Omega, \mathbf{k}, \Gamma)$ is stable, and we make further observations along these lines.

We let $\mathbf{k}^{\text{ac}}$ be the algebraic closure in Ω of any subfield $\mathbf{k}$ of Ω .

2. SOME EXAMPLES OF MANN PAIRS

It will be useful to also consider *homogeneous* linear equations. Let $n \geq 1$ and let $a_1, \dots, a_n \in \Omega$. A *solution* of the homogeneous linear equation

$$a_1x_1 + \dots + a_nx_n = 0$$

is a tuple $(s_1, \dots, s_n) \in \Omega^n$ such that $a_1s_1 + \dots + a_ns_n = 0$; such a solution is said to be *nondegenerate* if $\sum_{i \in I} a_is_i \neq 0$ for all nonempty proper subsets I of $\{1, \dots, n\}$, and is said to be in Γ if $(s_1, \dots, s_n) \in \Gamma^n$.

2.1. Alternative definition of Mann pairs. The equivalence of (1) and (2) in the next lemma expresses being a Mann pair in terms of homogeneous linear equations. Assuming $\mathbf{k}^\times \cap \Gamma$ is finite, the equivalence between (1) and (3) expresses $(\mathbf{k}, \Gamma)$ being a Mann pair purely in terms of the subgroup $G = \mathbf{k}^\times \Gamma$ of $\Omega^\times$ generated by $\mathbf{k}^\times$ and Γ .

Lemma 2.1. *Let $G = \mathbf{k}^\times \Gamma$. Then the following are equivalent:*

- (1) $(\mathbf{k}, \Gamma)$ is a Mann pair;
- (2) for each $n \geq 1$ there is a finite $\Gamma(n) \subseteq \Gamma^n$ such that each nondegenerate solution in Γ of each equation $a_1x_1 + \dots + a_nx_n = 0$ with $a_1, \dots, a_n \in \mathbf{k}^\times$ lies in $\alpha\Gamma(n)$ for some $\alpha \in \Gamma$;
- (3) $\mathbf{k}^\times \cap \Gamma$ is finite, and for each $n \geq 1$ there is a finite $G(n) \subseteq G^n$ such that each nondegenerate solution in G of the equation

$$x_1 + \dots + x_n = 1$$

equals $(c_1g_1, \dots, c_ng_n)$ with $c_1, \dots, c_n \in \mathbf{k}^\times$ and $(g_1, \dots, g_n) \in G(n)$.

Proof. Let $n \geq 1$. First note that if $a_1, \dots, a_n \in \mathbf{k}^\times$ and $(s_1, \dots, s_n) \in \Omega^n$, then $(s_1, \dots, s_n)$ is a nondegenerate solution of $a_1x_1 + \dots + a_nx_n = 1$ iff $(s_1, \dots, s_n, 1)$ is a nondegenerate solution of $a_1x_1 + \dots + a_nx_n - x_{n+1} = 0$.

Therefore, if (1) holds, witnessed by E for our given n as in the definition of *Mann pair* in the Introduction, then (2) holds with

$$\Gamma(n+1) := \{(\gamma_1, \dots, \gamma_n, 1) : (\gamma_1, \dots, \gamma_n) \in E\}.$$

Conversely, if (2) holds, take $\Gamma(n+1)$ such that $\gamma_{n+1} = 1$ for every $(\gamma_1, \dots, \gamma_n, \gamma_{n+1})$ from $\Gamma(n+1)$, and then (1) holds with

$$E := \{(\gamma_1, \dots, \gamma_n) : (\gamma_1, \dots, \gamma_n, 1) \in \Gamma(n+1)\}.$$

Next, observe that if (1) holds, witnessed by E for our given n , then (3) holds with $G(n) := E$. Conversely, if (3) holds, then in (3) we can take $G(n) \subseteq \Gamma^n$, and then (1) holds with

$$E := \{(\alpha_1\gamma_1, \dots, \alpha_n\gamma_n) \in \Gamma^n : \alpha_1, \dots, \alpha_n \in \mathbf{k}^\times \cap \Gamma, (\gamma_1, \dots, \gamma_n) \in G(n)\}.$$

□

Corollary 2.2. *Suppose $(\mathbf{k}, \Gamma)$ is a Mann pair, and Γ' is a subgroup of $\mathbf{k}^\times \Gamma$ such that $\mathbf{k}^\times \cap \Gamma'$ is finite. Then $(\mathbf{k}, \Gamma')$ is a Mann pair.*

2.2. Easy Mann pairs. These are provided by the next lemma.

Lemma 2.3. *Suppose Γ is torsion-free. Then the following are equivalent:*

- (1) *for all $n \geq 1$ and $a_1, \dots, a_n \in \mathbf{k}^\times$, the equation $a_1x_1 + \dots + a_nx_n = 1$ has no nondegenerate solution in Γ that is different from $(1, \dots, 1)$;*
- (2) *whenever $n \geq 1$ and $\gamma_1, \dots, \gamma_n \in \Gamma$ are multiplicatively independent, then they are algebraically independent over $\mathbf{k}$.*

If these conditions are satisfied, then $(\mathbf{k}, \Gamma)$ is a Mann pair.

Proof. Apply Lemma 8.2 in [5] (but Γ , E , G in that lemma are $\{1\}$, $\mathbf{k}$, Γ in the present lemma). $\square$

Since $\pi \notin \mathbb{Q}^{\text{ac}}$, the group $\exp(\mathbb{Q}^{\text{ac}}) \subseteq \mathbb{C}^\times$ is torsion-free, and by Lindemann's theorem on exponentials ([9], Appendix 1), condition (2) of the lemma is satisfied with $\mathbf{k} = \mathbb{Q}^{\text{ac}} \subseteq \mathbb{C}$ and $\Gamma = \exp(\mathbb{Q}^{\text{ac}})$. Thus $(\mathbb{Q}^{\text{ac}}, \exp(\mathbb{Q}^{\text{ac}}))$ is a Mann pair. Here is another application of the lemma, which applies for example to Hahn fields $K = \mathbf{k}((\Gamma))$.

Corollary 2.4. *Let $v : K^\times \rightarrow v(K^\times)$ be a valuation on a subfield K of Ω . Suppose $\mathbf{k}$ is a subfield of K and Γ a subgroup of $K^\times$ such that v is trivial on $\mathbf{k}$ and injective on Γ . Then condition (2) of Lemma 2.3 is satisfied, and so $(\mathbf{k}, \Gamma)$ is a Mann pair.*

A group with the Mann property can produce a Mann pair in an elementary extension as follows:

Corollary 2.5. *Suppose G is a subgroup of $\mathbf{k}^\times$ with the Mann property, $(\mathbf{k}^*, G^*)$ is an elementary extension of $(\mathbf{k}, G)$, and Γ is a subgroup of G^* such that $\mathbf{k}^\times \cap \Gamma = \{1\}$. Then $(\mathbf{k}, \Gamma)$ is a Mann pair.*

Proof. Note that Γ is torsion-free. Let $a_1, \dots, a_n \in \mathbf{k}^\times$, $n \geq 1$. Then the equation $a_1x_1 + \dots + a_nx_n = 1$ has the same finite number of nondegenerate solutions in G as in G^* , so cannot have nondegenerate solutions in Γ different from $(1, \dots, 1)$, the unique element of $G^n \cap \Gamma^n$. Hence condition (1) of Lemma 2.3 is satisfied, and so $(\mathbf{k}, \Gamma)$ is a Mann pair. $\square$

If $\mathbf{k}$ has positive characteristic, then Lemma 2.3 is the only source of Mann pairs $(\mathbf{k}, \Gamma)$ with torsion-free Γ :

Corollary 2.6. *Suppose $\mathbf{k}$ has positive characteristic, Γ is torsion-free, and $(\mathbf{k}, \Gamma)$ is a Mann pair. Then conditions (1) and (2) of Lemma 2.3 are satisfied.*

Proof. Let $a_1, \dots, a_n \in \mathbf{k}^\times$, $n \geq 1$, and suppose $(\gamma_1, \dots, \gamma_n) \in \Gamma^n$ is a nondegenerate solution of $a_1x_1 + \dots + a_nx_n = 1$. Let ϕ be the Frobenius map on $\mathbf{k}$. Then for each m we have a nondegenerate solution

$$(\phi^m(\gamma_1), \dots, \phi^m(\gamma_n)) \in \Gamma^n$$

of the equation $\phi^m(a_1)x_1 + \dots + \phi^m(a_n)x_n = 1$. By the equivalence (1) $\Leftrightarrow$ (2) of Lemma 2.1 this yields that all γ_i are roots of unity, so equal to 1 by torsion-freeness of Γ . $\square$

3. MANN PAIRS IN FUNCTION FIELDS OF ONE VARIABLE

Let $\mathbf{k}$ be algebraically closed, and let $F \subseteq \Omega$ be a function field of one variable over $\mathbf{k}$; that is, F is a field extension of finite degree of $\mathbf{k}(t)$ for some $t \in \Omega \setminus \mathbf{k}$. In particular, F has transcendence degree 1 over $\mathbf{k}$. Below we use some standard facts about such function fields; for proofs of these facts, see Chapter I, §2 of [10].

Let $\mathcal{R}(F|\mathbf{k})$, the *Riemann space* of F over $\mathbf{k}$, be the set of all valuations $v : F^\times \rightarrow \mathbb{Z}$ on F with value group $v(F^\times) = \mathbb{Z}$ that are trivial on $\mathbf{k}$. We let v range over $\mathcal{R}(F|\mathbf{k})$. For each $f \in F^\times$ we have $v(f) \neq 0$ for only finitely many v , and $\sum_v v(f) = 0$. Let $\mathcal{D}(F|\mathbf{k})$ be the group of *divisors* of F over $\mathbf{k}$; that is,

$$\mathcal{D}(F|\mathbf{k}) := \bigoplus_v \mathbb{Z}v$$

is the free abelian group on the Riemann space of F over $\mathbf{k}$. To $f \in F^\times$ we assign its *principal divisor* $(f) := \sum_v v(f)v \in \mathcal{D}(F|\mathbf{k})$. The group morphism

$$F^\times \rightarrow \mathcal{D}(F|\mathbf{k}), \quad f \mapsto (f)$$

has kernel $\mathbf{k}^\times$. In particular, if Γ is a subgroup of $F^\times$ with $\mathbf{k}^\times \cap \Gamma = \{1\}$, then this morphism is injective on Γ , and so the image of Γ under this morphism is an isomorphic copy of Γ inside the free abelian group $\mathcal{D}(F|\mathbf{k})$. It follows that each such Γ is free as an abelian group.

Given finite $S \subseteq \mathcal{R}(F|\mathbf{k})$, an *S-unit* is an element $u \in F^\times$ such that $v(u) = 0$ for all $v \notin S$.

Lemma 3.1. *Let L be a finite-dimensional $\mathbf{k}$ -linear subspace of F and let $\Gamma \subseteq F^\times$ be finitely generated with $\mathbf{k}^\times \cap \Gamma = \{1\}$. Then $L \cap \Gamma$ is finite.*

Proof. Let $b_1, \dots, b_m$ be a basis of the $\mathbf{k}$ -linear space L , and let $\gamma_1, \dots, \gamma_n$ generate the group Γ . Take a finite $S \subseteq \mathcal{R}(F|\mathbf{k})$ such that all b_i and γ_j are *S*-units. Take a natural number d such that $v(b_i) \geq -d$ for all $v \in S$ and $i = 1, \dots, m$. Then $v(f) \geq 0$ for all $f \in L$ and v outside S , and $v(f) \geq -d$ for all $f \in L$ and $v \in S$.

Suppose now that $\gamma \in L \cap \Gamma$. Then $v(\gamma) = 0$ for v outside S and $v(\gamma) \geq -d$ for $v \in S$. In view of $\sum_v v(\gamma) = 0$, this gives $v(\gamma) \leq |S|d$ for all $v \in S$. It follows that the image of $L \cap \Gamma$ in $\mathcal{D}(F|\mathbf{k})$ is finite, and thus $L \cap \Gamma$ is finite. $\square$

Let $u_1, \dots, u_n \in F$ not be all zero. We define their *height* by

$$H(u_1, \dots, u_n) := - \sum_v \min\{v(u_1), \dots, v(u_n)\}.$$

This height is projective: $H(fu_1, \dots, fu_n) = H(u_1, \dots, u_n)$ for $f \in F^\times$.

Example. Let $F = \mathbf{k}(t)$ with t transcendental over $\mathbf{k}$. Suppose that the polynomials $u_1, \dots, u_n \in \mathbf{k}[t]$ have no common zero in $\mathbf{k}$, $n \geq 1$. It is easy to check that then $H(u_1, \dots, u_n) = \max\{\deg_t u_1, \dots, \deg_t u_n\}$.

The following important bound is from [3]:

*Let Ω have characteristic zero, let g be the genus of the function field $F|\mathbf{k}$, and let S be a finite subset of $\mathcal{R}(F|\mathbf{k})$ and $n \geq 2$. Suppose $u_1, \dots, u_n$ are *S*-units and $(u_1, \dots, u_n)$ is a nondegenerate solution of $x_1 + \dots + x_n = 0$. Then*

$$H(u_1, \dots, u_n) \leq \frac{1}{2}(n-1)(n-2)\{|S| + \max(0, 2g-2)\}.$$

In combination with the previous lemma this has the following consequence.

Corollary 3.2. *Suppose Ω has characteristic zero and $\Gamma \subseteq F^\times$ is finitely generated with $\mathbf{k}^\times \cap \Gamma = \{1\}$. Then $(\mathbf{k}, \Gamma)$ is a Mann pair.*

Proof. Take a finite $S \subseteq \mathcal{R}(F|\mathbf{k})$ such that all $\gamma \in \Gamma$ are S -units. Let $n \geq 2$ and let $a_1, \dots, a_n \in \mathbf{k}^\times$ and $\gamma_1, \dots, \gamma_n \in \Gamma$ be such that $(\gamma_1, \dots, \gamma_n)$ is a nondegenerate solution of $a_1x_1 + \dots + a_nx_n = 0$. Dividing by γ_n we arrange $\gamma_n = 1$, and we need to show that this leaves only finitely many possibilities for $(\gamma_1, \dots, \gamma_{n-1})$. Now by the bound above we have $v(\gamma_i) \geq -N$ for all $v \in S$ and $i = 1, \dots, n-1$, where $N := \frac{1}{2}(n-1)(n-2)\{|S| + \max(0, 2g-2)\}$, so each γ_i lies in the $\mathbf{k}$ -linear subspace

$$L := \{f \in F : v(f) \geq 0 \text{ for all } v \notin S, v(f) \geq -N \text{ for all } v \in S\}$$

of F . Now L is finite-dimensional by [10], p. 7. In view of Lemma 3.1 this gives the desired finiteness. $\square$

Example. To illustrate the effective nature of this proof, assume that Ω has characteristic zero, and consider the case $F = \mathbf{k}(t)$ of a rational function field (so $g = 0$), where Γ is generated as a group by

$$t - c_1, \dots, t - c_M, \quad \text{with distinct } c_1, \dots, c_M \in \mathbf{k}.$$

Let $n \geq 2$ be given. Call a tuple $(\gamma_1, \dots, \gamma_n) \in \Gamma^n$ *reduced* if all γ_i lie in $\mathbf{k}[t]$ and $\gamma_1, \dots, \gamma_n$ have no common zero in $\mathbf{k}$ when viewed as polynomials in t over $\mathbf{k}$. Let $a_1, \dots, a_n \in \mathbf{k}^\times$. Any nondegenerate solution in Γ of the equation

$$a_1x_1 + \dots + a_nx_n = 0$$

can be multiplied by an element of Γ to give a nondegenerate reduced solution $(\gamma_1, \dots, \gamma_n)$. In this situation the inequality from [3] yields

$$\max\{\deg_t \gamma_1, \dots, \deg_t \gamma_n\} \leq \frac{1}{2}(n-1)(n-2)(M+1),$$

which is satisfied by only finitely many reduced tuples $(\gamma_1, \dots, \gamma_n)$, which we can list explicitly. Given a reduced tuple $(\gamma_1, \dots, \gamma_n)$ satisfying the inequality, the existence of $a_1, \dots, a_n \in \mathbf{k}^\times$ such that $(\gamma_1, \dots, \gamma_n)$ is a nondegenerate solution of $a_1x_1 + \dots + a_nx_n = 0$ is equivalent (effectively) to the existence of a solution to a certain finite system of linear equations and inequations with coefficients in the field $\mathbb{Q}(c_1, \dots, c_M)$.

4. ALLOWING COEFFICIENTS FROM Ω

In this section we consider Γ as acting on itself by multiplication, making Γ into a Γ -set as defined in [5], §4. In particular, any subset of Γ^n definable in the Γ -set Γ is definable in the group Γ . Our aim is to prove the following result and some related facts.

Proposition 4.1. *Let $(\mathbf{k}, \Gamma)$ be a Mann pair and $r_1, \dots, r_n \in \Omega$. Then*

$$\{(x, y) \in \mathbf{k}^n \times \Gamma^n : r_1x_1y_1 + \dots + r_nx_ny_n = 0\}$$

is a finite union of sets $X \times Y$ with X a $\mathbf{k}$ -linear subspace of $\mathbf{k}^n$ and $Y \subseteq \Gamma^n$ defined in the Γ -set Γ by a conjunction of atoms in the language of Γ -sets.

We first show this for $r_1, \dots, r_n \in \mathbf{k}$, next for $r_1, \dots, r_n \in \mathbf{k}(\Gamma)$, and then in general.

4.1. Notation. In this section we let $\vec{a} = (a_1, \dots, a_n)$ and $\vec{b} = (b_1, \dots, b_n)$ range over $\mathbf{k}^n$, and α, β, γ over Γ , and

$$\vec{\alpha} = (\alpha_1, \dots, \alpha_n), \quad \vec{\beta} = (\beta_1, \dots, \beta_n), \quad \vec{\gamma} = (\gamma_1, \dots, \gamma_n)$$

over Γ^n . Let $n \geq 1$ and put

$$\begin{aligned} \Sigma_n(\mathbf{k}, \Gamma) &:= \{(\vec{a}, \vec{\gamma}) \in \mathbf{k}^n \times \Gamma^n : a_1\gamma_1 + \dots + a_n\gamma_n = 0\}, \\ \Sigma_n(\mathbf{k}, \Gamma; \vec{\gamma}) &:= \{\vec{a} \in \mathbf{k}^n : (\vec{a}, \vec{\gamma}) \in \Sigma_n(\mathbf{k}, \Gamma)\}. \end{aligned}$$

Imposing nondegeneracy yields the set $\Sigma_n^{\text{nd}}(\mathbf{k}, \Gamma)$ of all $(\vec{a}, \vec{\gamma}) \in (\mathbf{k}^\times)^n \times \Gamma^n$ such that $\vec{\gamma}$ is a nondegenerate solution of $a_1x_1 + \dots + a_nx_n = 0$. We also introduce for each $\vec{\gamma}$ the corresponding section

$$\Sigma_n^{\text{nd}}(\mathbf{k}, \Gamma; \vec{\gamma}) := \{\vec{a} \in (\mathbf{k}^\times)^n : (\vec{a}, \vec{\gamma}) \in \Sigma_n^{\text{nd}}(\mathbf{k}, \Gamma)\}.$$

If $(\vec{a}, \vec{\gamma}) \in \Sigma_n^{\text{nd}}(\mathbf{k}, \Gamma)$, then $(\vec{a}, \alpha\vec{\gamma}) \in \Sigma_n^{\text{nd}}(\mathbf{k}, \Gamma)$, so $\Sigma_n^{\text{nd}}(\mathbf{k}, \Gamma)$ is a union of sets of the form $\Sigma_n^{\text{nd}}(\mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma\vec{\gamma}$. Being a Mann pair can now be expressed as follows:

$(\mathbf{k}, \Gamma)$ is a Mann pair iff for each $n \geq 1$ there is a finite $\Gamma(n) \subseteq \Gamma^n$ such that

$$\Sigma_n^{\text{nd}}(\mathbf{k}, \Gamma) = \bigcup_{\vec{\gamma} \in \Gamma(n)} \Sigma_n^{\text{nd}}(\mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma\vec{\gamma}.$$

Whenever $(\mathbf{k}, \Gamma)$ is a Mann pair we let $\Gamma(n)$ for $n \geq 1$ be as above.

We also want to allow coefficients outside $\mathbf{k}$ and accordingly, given $\vec{r} = (r_1, \dots, r_n) \in \Omega^n$, we let $\Sigma(\vec{r}, \mathbf{k}, \Gamma)$ be the set of all

$$(\vec{a}, \vec{\gamma}) = (a_1, \dots, a_n, \gamma_1, \dots, \gamma_n) \in \mathbf{k}^n \times \Gamma^n$$

such that $\vec{\gamma}$ is a solution of $r_1a_1x_1 + \dots + r_na_nx_n = 0$, and let $\Sigma^{\text{nd}}(\vec{r}, \mathbf{k}, \Gamma)$ be the set of all $(\vec{a}, \vec{\gamma}) \in (\mathbf{k}^\times)^n \times \Gamma^n$ such that $\vec{\gamma}$ is a nondegenerate solution of $r_1a_1x_1 + \dots + r_na_nx_n = 0$. In particular, $\Sigma^{\text{nd}}(\vec{r}, \mathbf{k}, \Gamma) = \emptyset$ if $n \geq 2$ and $r_i = 0$ for some $i \in \{1, \dots, n\}$. Also, let $\Sigma^{\text{nd}}(\vec{r}, \mathbf{k}, \Gamma; \vec{\gamma})$ be the set of $\vec{a} \in (\mathbf{k}^\times)^n$ such that $(\vec{a}, \vec{\gamma}) \in \Sigma^{\text{nd}}(\vec{r}, \mathbf{k}, \Gamma)$.

4.2. Mann pairs over F . Besides $\mathbf{k}$ we also let F denote a subfield of Ω . Usually, $\mathbf{k}$ serves as the base field, and F will be an extension of $\mathbf{k}$.

We say that $(\mathbf{k}, \Gamma)$ is a Mann pair over F if for every tuple $\vec{r} = (r_1, \dots, r_n)$ from $F^\times$, $n \geq 2$, there is a finite subset $\Gamma(\vec{r})$ of Γ^n , such that

$$\Sigma^{\text{nd}}(\vec{r}, \mathbf{k}, \Gamma) = \bigcup_{\vec{\gamma} \in \Gamma(\vec{r})} \Sigma^{\text{nd}}(\vec{r}, \mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma\vec{\gamma}.$$

It is clear that $(\mathbf{k}, \Gamma)$ is a Mann pair iff it is a Mann pair over $\mathbf{k}$. We proceed to show that if $(\mathbf{k}, \Gamma)$ is a Mann pair, then it is a Mann pair over Ω .

Let $\vec{r} = (r_1, \dots, r_n) \in F^n$, $n \geq 1$. Let $\mathcal{P}$ be a partition of $\{1, \dots, n\}$ into distinct sets. Then we define $\Sigma_{\mathcal{P}}(\vec{r}, \mathbf{k}, \Gamma)$ to be the set of all

$$(a_1, \dots, a_n, \beta_1, \dots, \beta_n) \in \mathbf{k}^n \times \Gamma^n$$

such that for $I \in \mathcal{P}$ the tuple $(\beta_i)_{i \in I}$ is a nondegenerate solution of

$$\sum_{i \in I} r_i a_i x_i = 0.$$

With $\vec{r}_I := (r_i)_{i \in I}$ for $I \in \mathcal{P}$, this means

$$\Sigma_{\mathcal{P}}(\vec{r}, \mathbf{k}, \Gamma) = \prod_{I \in \mathcal{P}} \Sigma^{\text{nd}}(\vec{r}_I, \mathbf{k}, \Gamma).$$

Suppose now that $(\mathbf{k}, \Gamma)$ is a Mann pair over F . Then we have for $I \in \mathcal{P}$ a finite $\Gamma(\vec{r}_I) \subseteq \Gamma^I$ such that

$$\Sigma^{\text{nd}}(\vec{r}_I, \mathbf{k}, \Gamma) = \bigcup_{\vec{\gamma} \in \Gamma(\vec{r}_I)} \Sigma^{\text{nd}}(\vec{r}_I, \mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma \vec{\gamma}.$$

It follows that

$$\Sigma_{\mathcal{P}}(\vec{r}, \mathbf{k}, \Gamma) = \prod_{I \in \mathcal{P}} \bigcup_{\vec{\gamma} \in \Gamma(\vec{r}_I)} \Sigma^{\text{nd}}(\vec{r}_I, \mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma \vec{\gamma}.$$

Thus, with $\vec{\gamma}_I := (\gamma_i)_{i \in I}$ and $\Gamma(\vec{r}) = \Gamma(\vec{r}, \mathcal{P}) := \prod_{I \in \mathcal{P}} \Gamma(\vec{r}_I) \subseteq \Gamma^n$,

$$\begin{aligned} \Sigma_{\mathcal{P}}(\vec{r}, \mathbf{k}, \Gamma) &= \bigcup_{\vec{\gamma} \in \Gamma(\vec{r})} \prod_{I \in \mathcal{P}} [\Sigma^{\text{nd}}(\vec{r}_I, \mathbf{k}, \Gamma; \vec{\gamma}_I) \times \Gamma \vec{\gamma}_I] \\ &\subseteq \bigcup_{\vec{\gamma} \in \Gamma(\vec{r})} \prod_{I \in \mathcal{P}} [\Sigma(\vec{r}_I, \mathbf{k}, \Gamma; \vec{\gamma}_I) \times \Gamma \vec{\gamma}_I]. \end{aligned}$$

The last product set is contained in $\Sigma(\vec{r}, \mathbf{k}, \Gamma)$ under an obvious identification, so with $\mathcal{P}$ ranging over the partitions of $\{1, \dots, n\}$, we get

$$\Sigma(\vec{r}, \mathbf{k}, \Gamma) = \bigcup_{\mathcal{P}} \Sigma_{\mathcal{P}}(\vec{r}, \mathbf{k}, \Gamma) = \bigcup_{\mathcal{P}} \bigcup_{\vec{\gamma} \in \Gamma(\vec{r}, \mathcal{P})} \prod_{I \in \mathcal{P}} [\Sigma(\vec{r}_I, \mathbf{k}, \Gamma; \vec{\gamma}_I) \times \Gamma \vec{\gamma}_I].$$

This yields the following result.

Lemma 4.2. *Suppose $(\mathbf{k}, \Gamma)$ is a Mann pair over F , and $\vec{r} \in F^n$, $n \geq 1$. Then $\Sigma(\vec{r}, \mathbf{k}, \Gamma)$ is a finite union of sets $P \times Q$, where P is a $\mathbf{k}$ -linear subspace of $\mathbf{k}^n$, and $Q \subseteq \Gamma^n$ is defined in Γ by a conjunction of atoms in the language of Γ -sets.*

Note that the intersection of two sets of the form $P \times Q$, where P, Q are as in Lemma 4.2, is again of the same form. Hence, if $(\mathbf{k}, \Gamma)$ is a Mann pair over F , and $\vec{r}, \vec{s} \in F^n$, $n \geq 1$, then $\Sigma(\vec{r}, \mathbf{k}, \Gamma) \cap \Sigma(\vec{s}, \mathbf{k}, \Gamma)$ is a finite union of sets $P \times Q$, where P, Q are as in Lemma 4.2. Next we prove a converse of Lemma 4.2.

Lemma 4.3. *Let $\vec{r} \in \Omega^n$. Suppose that*

$$\Sigma(\vec{r}, \mathbf{k}, \Gamma) = \bigcup_{j=1}^k P_j \times Q_j,$$

where $P_1, \dots, P_k$ are subsets of $\mathbf{k}^n$, and each Q_j is a subset of Γ^n defined in Γ by a conjunction of atoms in the language of Γ -sets. Then there is a finite subset $\Gamma(\vec{r})$ of Γ^n such that

$$\Sigma^{\text{nd}}(\vec{r}, \mathbf{k}, \Gamma) = \bigcup_{\vec{\gamma} \in \Gamma(\vec{r})} \Sigma^{\text{nd}}(\vec{r}, \mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma \vec{\gamma}.$$

Proof. We can assume $\Gamma \neq \{1\}$. Let $j \in \{1, \dots, k\}$ be such that $Q_j \neq \emptyset$. By the discussion in [5] just after Corollary 4.2, we can take $\vec{\gamma} \in \Gamma^n$ and a partition of $\{1, \dots, n\}$ into distinct sets $I(1), \dots, I(l)$ such that

$$Q_j = \Gamma \vec{\gamma}(1) \times \dots \times \Gamma \vec{\gamma}(l) \subseteq \prod_{\lambda=1}^l \Gamma^{I(\lambda)} = \Gamma^n,$$

where $\vec{\gamma}(\lambda) := \{(\gamma_i)\}_{i \in I(\lambda)} \in \Gamma^{I(\lambda)}$ for $\lambda = 1, \dots, l$.

Claim. Let $l > 1$, $\vec{a} \in P_j$ and $\vec{\beta} \in Q_j$. Then $(a_1\beta_1, \dots, a_n\beta_n)$ is a degenerate solution of $r_1x_1 + \dots + r_nx_n = 0$.

Proof of the claim. For each $\alpha \in \Gamma$,

$$\sum_{i \in I(1)} r_i a_i \beta_i + \sum_{i \notin I(1)} r_i a_i \beta_i = 0 = \sum_{i \in I(1)} r_i a_i \beta_i + \sum_{i \notin I(1)} r_i a_i \alpha \beta_i.$$

So $\sum_{i \notin I(1)} r_i a_i \beta_i = \alpha \sum_{i \notin I(1)} r_i a_i \beta_i$ for all $\alpha \in \Gamma$. Hence $\sum_{i \notin I(1)} r_i a_i \beta_i = 0$, and thus $(a_1 \beta_1, \dots, a_n \beta_n)$ is a degenerate solution, proving the claim.

As a result of this claim, we get a subset J of $\{1, \dots, k\}$, and a tuple $\vec{\gamma}_j \in \Gamma^n$ for each $j \in J$ such that

$$\Sigma^{\text{nd}}(\vec{r}, \mathbf{k}, \Gamma) \subseteq \bigcup_{j \in J} P_j \times \Gamma \vec{\gamma}_j.$$

Since obviously $\bigcup_{j \in J} \Sigma^{\text{nd}}(\vec{r}, \mathbf{k}, \Gamma; \vec{\gamma}_j) \times \Gamma \vec{\gamma}_j \subseteq \Sigma^{\text{nd}}(\vec{r}, \mathbf{k}, \Gamma)$, this yields

$$\Sigma^{\text{nd}}(\vec{r}, \mathbf{k}, \Gamma) = \bigcup_{j \in J} \Sigma^{\text{nd}}(\vec{r}, \mathbf{k}, \Gamma; \vec{\gamma}_j) \times \Gamma \vec{\gamma}_j.$$

□

Note: $P_1, \dots, P_k$ in this lemma are not assumed to be linear subspaces of $\mathbf{k}^n$. Applying the lemma with $\vec{r} = (1, \dots, 1)$, we obtain

Lemma 4.4. *Suppose for each $n \geq 1$ the set $\Sigma_n(\mathbf{k}, \Gamma)$ is a finite union of sets $P \times Q$ with $P \subseteq \mathbf{k}^n$ and Q a subset of Γ^n defined in Γ by a positive quantifier-free formula in the language of Γ -sets. Then $(\mathbf{k}, \Gamma)$ is a Mann pair.*

Next we improve Lemma 4.4 and obtain a strong version of the implication (2) $\implies$ (1) of Theorem 1.2. We shall use facts from [5], but we alert the reader that in the statement and proof of Proposition 5.11 of [5], “subgroups of G^n ” should be “cosets of subgroups of G^n ”, and “ B_n ” should be “ B_m ”.

Proposition 4.5. *Suppose for each $n \geq 1$ the set $\Sigma_n(\mathbf{k}, \Gamma)$ is a finite union of sets $X \times Y$ where $X \subseteq \mathbf{k}^n$ and where $Y \subseteq \Gamma^n$ is a boolean combination of cosets of subgroups of Γ^n . Then $(\mathbf{k}, \Gamma)$ is a Mann pair.*

Proof. Let $n \geq 1$. For $(x, y) \in \mathbf{k}^n \times \Gamma^n$ we have

$$x_1 y_1 + \dots + x_n y_n = 1 \iff ((x_1, \dots, x_n, -1), (y_1, \dots, y_n, 1)) \in \Sigma_{n+1}(\mathbf{k}, \Gamma).$$

Using the hypothesis with $n+1$ instead of n it follows easily that the set

$$\{(x, y) \in \mathbf{k}^n \times \Gamma^n : x_1 y_1 + \dots + x_n y_n = 1\}$$

is a finite union of sets $X \times Y$, where $X \subseteq \mathbf{k}^n$ and where $Y \subseteq \Gamma^n$ is a boolean combination of cosets of subgroups of Γ^n . Then Proposition 5.11 of [5] yields that Γ has the Mann property. Next,

$$\Sigma_n(\mathbf{k}, \Gamma) = \bigcup_{i \in I} X_i \times Y_i$$

with finite I and each $X_i \subseteq \mathbf{k}^n$, and where each $Y_i \subseteq \Gamma^n$ is a boolean combination of cosets of subgroups of Γ^n . We can also arrange that $X_i \neq \emptyset$ for all i and $X_i \cap X_j = \emptyset$ for all distinct $i, j \in I$.

By Corollary 5.1 and Lemma 5.5 in [5], it follows that each set Y_i is definable in Γ by a positive quantifier-free formula in the language of Γ -sets. Then Lemma 4.4 yields that $(\mathbf{k}, \Gamma)$ is a Mann pair. □

Lemma 4.6. *If $(\mathbf{k}, \Gamma)$ is a Mann pair, then it is a Mann pair over $\mathbf{k}(\Gamma)$.*

Proof. Assume $(\mathbf{k}, \Gamma)$ is a Mann pair, and let $r_1, \dots, r_n \in \mathbf{k}(\Gamma)^\times$. By Lemma 4.3, it suffices to prove that $\Sigma(\vec{r}, \mathbf{k}, \Gamma)$ is a finite union of sets $P \times Q$, where $P \subseteq \mathbf{k}^n$, and $Q \subseteq \Gamma^n$ is defined by a finite conjunction of atoms in the language of Γ -sets. We may assume that $r_1, \dots, r_n \in \mathbf{k}[\Gamma]$. Then $r_i = \sum_{j=1}^k a_{ij} \beta_j$ with $a_{ij} \in \mathbf{k}$, and $\beta_j \in \Gamma$ for $i = 1, \dots, n$ and $j = 1, \dots, k$.

Let $\vec{a} = (a_{11}, \dots, a_{1k}, \dots, a_{n1}, \dots, a_{nk}) \in \mathbf{k}^{nk}$. Then

$$(c_1, \dots, c_n, \gamma_1, \dots, \gamma_n) \in \Sigma(\vec{r}, \mathbf{k}, \Gamma)$$

$$\iff$$

$$(c_1, \dots, c_1, \dots, c_n, \dots, c_n, \beta_1 \gamma_1, \dots, \beta_k \gamma_1, \dots, \beta_1 \gamma_n, \dots, \beta_k \gamma_n) \in \Sigma(\vec{a}, \mathbf{k}, \Gamma).$$

By Lemma 4.2, $\Sigma(\vec{a}, \mathbf{k}, \Gamma)$ is a finite union of sets $P \times Q$, where $P \subseteq \mathbf{k}^{nk}$, and $Q \subseteq \Gamma^{nk}$ is defined in Γ by a conjunction of atoms in the language of Γ -sets. Fix such P, Q , and define

$$P' := \{(c_1, \dots, c_n) \in \mathbf{k}^n : (c_1, \dots, c_1, \dots, c_n, \dots, c_n) \in P\},$$

$$Q' := \{(\gamma_1, \dots, \gamma_n) \in \Gamma^n : (\beta_1 \gamma_1, \dots, \beta_k \gamma_1, \dots, \beta_1 \gamma_n, \dots, \beta_k \gamma_n) \in Q\}.$$

Then for $(c_1, \dots, c_n, \gamma_1, \dots, \gamma_n) \in \mathbf{k}^n \times \Gamma^n$,

$$(c_1, \dots, c_n, \gamma_1, \dots, \gamma_n) \in P' \times Q'$$

$$\iff$$

$$(c_1, \dots, c_1, \dots, c_n, \dots, c_n, \beta_1 \gamma_1, \dots, \beta_k \gamma_n) \in P \times Q.$$

Hence, $\Sigma(\vec{a}, \mathbf{k}, \Gamma)$ being a finite union of sets $P \times Q$ with P and Q as above, $\Sigma(\vec{r}, \mathbf{k}, \Gamma)$ is the union of the corresponding sets $P' \times Q'$. This finishes the proof of the lemma. $\square$

Proposition 4.7. *If $(\mathbf{k}, \Gamma)$ is a Mann pair, then it is a Mann pair over Ω .*

Proof. Let $\vec{r} = (r_1, \dots, r_n) \in \Omega^n$. Take a basis $b_1, \dots, b_m$ of the $\mathbf{k}(\Gamma)$ -linear space $\mathbf{k}(\Gamma)r_1 + \dots + \mathbf{k}(\Gamma)r_n$. Then $r_j = \sum_{i=1}^m r_{ij} b_i$ with $r_{ij} \in \mathbf{k}(\Gamma)$, so that for all $(\vec{c}, \vec{\gamma}) \in \mathbf{k}^n \times \Gamma^n$ we have: $(\vec{c}, \vec{\gamma}) \in \Sigma(\vec{r}, \mathbf{k}, \Gamma)$ if and only if $(c_1 \gamma_1, \dots, c_n \gamma_n)$ is a solution of the system

$$r_{11}x_1 + \dots + r_{1n}x_n = 0,$$

$$r_{21}x_1 + \dots + r_{2n}x_n = 0,$$

$$\vdots$$

$$r_{m1}x_1 + \dots + r_{mn}x_n = 0.$$

Suppose now that $(\mathbf{k}, \Gamma)$ is a Mann pair. Then it is a Mann pair over $\mathbf{k}(\Gamma)$ by the previous lemma. It remains to use Lemmas 4.2 and 4.3. $\square$

Clearly, this proposition and Lemma 4.2 yield Proposition 4.1.

For any n -tuple $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, consider the character

$$\chi_k : (\Omega^\times)^n \rightarrow \Omega^\times, \quad \chi_k(y_1, \dots, y_n) := y_1^{k_1} \cdots y_n^{k_n}.$$

This is a multiplicative group homomorphism. For any $e \in \mathbb{N}$, let $\mathcal{D}(n, e)$ be the finite collection of subgroups of $(\Omega^\times)^n$ that are intersections of kernels of characters χ_k with $|k| = |k_1| + \dots + |k_n| \leq e$.

Proposition 4.8. *Let $(\mathbf{k}, \Gamma)$ be a Mann pair, let $X_1, \dots, X_m, Y_1, \dots, Y_n$ be distinct indeterminates, and let the polynomials $f_1, \dots, f_M \in \Omega[X, Y]$ have degree $\leq d$ in $X = (X_1, \dots, X_m)$ and degree $\leq e$ in $Y = (Y_1, \dots, Y_n)$, where $M, d, e \in \mathbb{N}$, and put*

$$Z := \{(x, y) \in \mathbf{k}^m \times \Gamma^n : f_1(x, y) = \dots = f_M(x, y) = 0\}.$$

Then Z is a finite union of sets $P \times Q$, where

$$P = \{x \in \mathbf{k}^m : g_1(x) = \dots = g_N(x) = 0\}$$

for suitable $N \in \mathbb{N}$ and polynomials $g_1, \dots, g_N \in \mathbf{k}[X]$ of degree $\leq d$, and where $Q \subseteq \Gamma^n$ is a coset of a subgroup $D \cap \Gamma^n$ of Γ^n with $D \in \mathcal{D}(n, e)$.

Proof. The intersection of finitely many cosets of such subgroups is either empty or again a coset of such a subgroup. Hence we may (and shall) assume that $M = 1$. Put $f := f_1$. Then $f = \sum_{(i,j) \in I \times J} a_{ij} X^i Y^j$ where all $a_{ij} \in \Omega$ and I is the set of multi-indices $i = (i_1, \dots, i_m) \in \mathbb{N}^m$ with $|i| = i_1 + \dots + i_m \leq d$ and J is the set of multi-indices $j = (j_1, \dots, j_n) \in \mathbb{N}^n$ with $|j| = j_1 + \dots + j_n \leq e$. By Proposition 4.7 and Lemma 4.2 the set

$$\{((x_i)_{i \in I}, (y_j)_{j \in J}) \in \mathbf{k}^I \times \Gamma^J : \sum_{(i,j) \in I \times J} a_{ij} x_i y_j = 0\}$$

is a finite union of subsets of $\mathbf{k}^I \times \Gamma^J$ of the form

$$V \times \{y \in \Gamma^J : \gamma_1 y_{i(1)} = y_{j(1)}, \dots, \gamma_k y_{i(k)} = y_{j(k)}\}$$

with V a $\mathbf{k}$ -linear subspace of $\mathbf{k}^I$, $\gamma_1, \dots, \gamma_k \in \Gamma$, and with $k \in \mathbb{N}$ and indices $i(1), j(1), \dots, i(k), j(k)$ in J . It remains to observe that for such $\gamma_1, \dots, \gamma_k$ and $i(1), j(1), \dots, i(k), j(k)$ the set

$$\{y \in \Gamma^J : \gamma_1 \chi_{i(1)}(y) = \chi_{j(1)}(y), \dots, \gamma_k \chi_{i(k)}(y) = \chi_{j(k)}(y)\}$$

is a coset of the subgroup $D \cap \Gamma^n$ of Γ^n where D is the intersection of the kernels of $\chi_{i(1)-j(1)}, \dots, \chi_{i(k)-j(k)}$. $\square$

5. ROBUSTNESS OF MANN PAIRS

In this section we assume that $(\mathbf{k}, \Gamma)$ is a Mann pair. In addition, $K \supseteq \mathbf{k}$ is a subfield of Ω , and Δ, Γ' are subgroups of $\Omega^\times$ with $\Gamma' \supseteq \Gamma$. We claim:

- (1) $[K : \mathbf{k}] < \infty \Rightarrow (K, \Gamma)$ is a Mann pair;
- (2) $[\Gamma' : \Gamma] < \infty \Rightarrow (\mathbf{k}, \Gamma')$ is a Mann pair;
- (3) $(K \supseteq \mathbf{k}(\Gamma))$ and (K, Δ) is a Mann pair $\Rightarrow (\mathbf{k}, \Gamma\Delta)$ is a Mann pair;
- (4) $(K \text{ is linearly disjoint from } \mathbf{k}(\Gamma) \text{ over } \mathbf{k}) \Rightarrow (K, \Gamma)$ is a Mann pair.

Proof of (1). Here K is an extension field of finite degree m over $\mathbf{k}$. Let $b_1, \dots, b_m$ be a basis of the $\mathbf{k}$ -linear space K . Let $n \geq 1$, and consider $(\vec{a}, \vec{\gamma}) \in K^n \times \Gamma^n$. Then with $a_j = \sum_{i=1}^m a_{ij} b_i$ ($1 \leq j \leq n$, all $a_{ij} \in \mathbf{k}$),

$$\begin{aligned} (\vec{a}, \vec{\gamma}) \in \Sigma_n(K, \Gamma) &\iff \sum_{j=1}^n (a_{1j} b_1 + \dots + a_{mj} b_m) \gamma_j = 0 \\ &\iff F(a_{11}, \dots, a_{mn}, \gamma_1, \dots, \gamma_n) = 0 \end{aligned}$$

$$\text{with } F := \sum_{j=1}^n (X_{1j} b_1 + \dots + X_{mj} b_m) Y_j \in K[X_{11}, \dots, X_{mn}, Y_1, \dots, Y_n].$$

By Proposition 4.8, the set

$$\{((c_{ij}), \gamma) \in \mathbf{k}^{mn} \times \Gamma^n : F(c_{11}, \dots, c_{mn}, \gamma_1, \dots, \gamma_n) = 0\}$$

is a union $\bigcup_{i \in I} P_i \times Q_i$ with finite I , where each $P_i \subseteq \mathbf{k}^{mn}$, and each $Q_i \subseteq \Gamma^n$ is a coset of a subgroup of Γ^n . Therefore we have $\Sigma_n(K, \Gamma) = \bigcup_{i \in I} P'_i \times Q_i$ with $P'_i = \{\vec{a} \in K^n : (a_{ij}) \in P_i\}$, where $\vec{a}$ determines (a_{ij}) as above.

Corollary 5.1. *If $\gamma \in \Gamma$ is algebraic over $\mathbf{k}$, then γ is a root of unity. In particular, if $(\mathbb{Q}, \Gamma)$ is a Mann pair with $\Gamma \subseteq A^\times$, then $\Gamma \subseteq \mathbb{U}$.*

Proof of (2). Assume $\Gamma' = \bigcup_{\alpha \in C} \alpha\Gamma$ where $C \subseteq \Gamma'$ is finite. Then

$$\Sigma_n(\mathbf{k}, \Gamma') = \bigcup_{\vec{\alpha} \in C^n} \{(\vec{a}, \vec{\alpha}\vec{\gamma}) : (\vec{a}, \vec{\gamma}) \in \Sigma(\vec{\alpha}, \mathbf{k}, \Gamma),$$

and it remains to use Proposition 4.7, Lemma 4.2 and Proposition 4.5.

Proof of (3). Assume $K \supseteq \mathbf{k}(\Gamma)$ and (K, Δ) is a Mann pair. Let $n \geq 1$ and take a finite $\Delta(n) \subseteq \Delta^n$ such that

$$\Sigma_n^{\text{nd}}(K, \Delta) = \bigcup_{\vec{\delta} \in \Delta(n)} \Sigma_n^{\text{nd}}(K, \Delta; \vec{\delta}) \times \Delta\vec{\delta}.$$

For each $\vec{\delta} \in \Delta(n)$ we have a finite $\Gamma(\vec{\delta}) \subseteq \Gamma^n$ such that

$$\Sigma^{\text{nd}}(\vec{\delta}, \mathbf{k}, \Gamma) = \bigcup_{\vec{\gamma} \in \Gamma(\vec{\delta})} \Sigma^{\text{nd}}(\vec{\delta}, \mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma\vec{\gamma}.$$

It follows that, with $\vec{\gamma}\vec{\delta} := (\gamma_1\delta_1, \dots, \gamma_n\delta_n)$, we have

$$\Sigma_n^{\text{nd}}(\mathbf{k}, \Gamma\Delta) = \bigcup_{\vec{\delta} \in \Delta(n)} \bigcup_{\vec{\gamma} \in \Gamma(\vec{\delta})} \Sigma_n^{\text{nd}}(\mathbf{k}, \Gamma\Delta; \vec{\gamma}\vec{\delta}) \times \Gamma\Delta\vec{\gamma}\vec{\delta}.$$

Proof of (4). Assume K is linearly disjoint from $\mathbf{k}(\Gamma)$ over $\mathbf{k}$. Let $n \geq 1$; it is enough to obtain $\Sigma_n(K, \Gamma)$ as a finite union of sets $P \times Q$ with $P \subseteq K^n$, and $Q \subseteq \Gamma^n$ defined in Γ by a conjunction of atoms in the language of Γ -sets.

Suppose $(\vec{r}, \vec{\gamma}) \in \Sigma_n(K, \Gamma)$. Then take a basis $b_1, \dots, b_m$ of the $\mathbf{k}$ -linear space $\mathbf{k}r_1 + \dots + \mathbf{k}r_n$. Then $r_j = \sum_{i=1}^m a_{ij}b_i$ with $a_{ij} \in \mathbf{k}$, $j = 1, \dots, n$; hence

$$(a_{i1}, \dots, a_{in}, \vec{\gamma}) \in \Sigma_n(\mathbf{k}, \Gamma) \text{ for } i = 1, \dots, m.$$

Conversely, given any $a_{ij} \in \mathbf{k}$ ($1 \leq i, j \leq n$) such that

$$(a_{i1}, \dots, a_{in}, \vec{\gamma}) \in \Sigma_n(\mathbf{k}, \Gamma) \text{ for } i = 1, \dots, n$$

and any $b_1, \dots, b_n \in K$ (not necessarily linearly independent over $\mathbf{k}$), we have $(\vec{r}, \vec{\gamma}) \in \Sigma_n(K, \Gamma)$, where $r_j = \sum_{i=1}^n a_{ij}b_i$ for $j = 1, \dots, n$. It follows that for all $(\vec{r}, \vec{\gamma}) \in K^n \times \Gamma^n$ we have: $(\vec{r}, \vec{\gamma}) \in \Sigma_n(K, \Gamma)$ if and only if there is an $n \times n$ matrix $A = (a_{ij}) \in \mathbf{k}^{n^2}$ and a vector $\vec{b} \in K^n$ such that

$$(a_{i1}, \dots, a_{in}, \vec{\gamma}) \in \Sigma_n(\mathbf{k}, \Gamma) \text{ for } i = 1, \dots, n, \text{ and } A\vec{b} = \vec{r},$$

with $\vec{b}$ and $\vec{r}$ viewed as column vectors. By Lemma 4.2 the set of all $(A, \vec{\gamma}) \in \mathbf{k}^{n^2} \times \Gamma^n$, with $A = (a_{ij})$, such that the displayed condition holds, is a union $\bigcup_{\lambda \in \Lambda} P_\lambda \times Q_\lambda$ with finite Λ where each $P_\lambda \subseteq \mathbf{k}^{n^2}$, and each $Q_\lambda \subseteq \Gamma^n$ is defined in Γ by a conjunction of atoms in the language of Γ -sets.

For $\lambda \in \Lambda$, let P'_λ be the set of all $\vec{r} \in K^n$ for which there is a matrix $A \in P_\lambda$ and a vector $\vec{b} \in K^n$ such that $A\vec{b} = \vec{r}$. Then by the above, $\Sigma_n(K, \Gamma) = \bigcup_{\lambda \in \Lambda} P'_\lambda \times Q_\lambda$.

5.1. Curious Mann pairs in positive characteristic. Let p be a prime number, and let $\Omega \supseteq \mathbb{F}_p$ be of characteristic p so that every $\mathbb{F}_{p^e}$ with e a positive integer is a subfield of Ω . Let P be a finite nonempty set of prime numbers different from p , let $S(P)$ be the set of all positive integers all of whose prime factors are in P . Put

$$G := \{a \in \Omega^\times : a^N = 1 \text{ for some } N \in S(P)\},$$

an infinite subgroup of $\Omega^\times$ isomorphic to the direct product of the Prüfer groups $\mathbb{Z}(l^\infty)$ with $l \in P$.

Theorem 8.9 in [5] says that G has the Mann property, but this is also a consequence of earlier results in [2] and [15]. We now strengthen this as follows. The proof in [5] starts with a certain large enough finite field $\mathbb{F}_{p^e}$ where e is a positive integer. Put

$$f := e \cdot \prod_{l \notin P} l^\infty, \quad g := e \cdot \prod_{l \in P} l^\infty \quad (\text{products of supernatural numbers}).$$

Then $\mathbb{F}_{p^e}(G) = \mathbb{F}_{p^g} \subseteq \Omega$, and so the infinite field $\mathbb{F}_{p^f}$ is linearly disjoint from $\mathbb{F}_{p^e}(G)$ over $\mathbb{F}_{p^e}$, so by (4) above we have a Mann pair $(\mathbb{F}_{p^f}, G)$ with the curious property that $\mathbb{F}_{p^f}(G) = \mathbb{F}_p^{\text{ac}}$.

5.2. Some Mann pairs in mixed characteristic. Let p be a prime number and $E \supseteq \mathbb{F}_p$ a perfect field of characteristic p . Let $W[E] \supseteq W[\mathbb{F}_p] = \mathbb{Z}_p$ be the ring of Witt vectors over E , let $W(E) \supseteq W(\mathbb{F}_p) = \mathbb{Q}_p$ be its fraction field, and let $\tau : E^\times \rightarrow W(E)^\times$ be the Teichmüller lifting. It was shown in [5], 8.4, that $\tau(E^\times)$ satisfies a strong form of the Mann property. In fact:

Proposition 5.2. $(\mathbb{Q}_p, \tau(E^\times))$ is a Mann pair.

Proof. Consider first the case $E = \mathbb{F}_p^{\text{ac}}$. Then

$$\tau(E^\times) = \mathbb{U}[p'] := \{x \in W(E) : x^n = 1 \text{ for some } n \text{ not divisible by } p\}.$$

By arguments as in the proof of Theorem 1 in [12] one shows that $(\mathbb{Q}_p, \mathbb{U}[p'])$ is a Mann pair. (Alternatively, $(\mathbb{Q}, \mathbb{U}[p'])$ is a Mann pair by Theorem 1 in [12], and it remains to use (4) and the fact that $\mathbb{Q}_p$ and $\mathbb{Q}(\mathbb{U}[p'])$ are linearly disjoint over $\mathbb{Q}$.)

In the general case, we first arrange that E is algebraically closed so that $E \supseteq \mathbb{F}_p^{\text{ac}}$ and $W(E) \supseteq W(\mathbb{F}_p^{\text{ac}})$. Then $E^\times = (\mathbb{F}_p^{\text{ac}})^\times G$, where the subgroup G of $E^\times$ is torsion-free. Then $\tau(E^\times) = \mathbb{U}[p']\tau(G)$. By the proof of 8.4 in [5], whenever $\gamma_1, \dots, \gamma_n \in \tau(G)$ are multiplicatively independent, they are algebraically independent over $W(\mathbb{F}_p^{\text{ac}})$. Then Lemma 2.3 yields that $(W(\mathbb{F}_p^{\text{ac}}), \tau(G))$ is a Mann pair. Since $(\mathbb{Q}_p, \mathbb{U}[p'])$ is a Mann pair, we obtain from (3) that $(\mathbb{Q}_p, \tau(E^\times))$ is a Mann pair. $\square$

6. PROOF OF THEOREM 1.1

In this section we assume that Ω has characteristic zero, and, besides $\mathbf{k}$, we also let E, F, K be subfields of Ω . The following contains Proposition 5.16 of [5] as a special case, with almost the same proof.

Lemma 6.1. *Let E be a subfield of K such that all p^{th} roots of unity in K are in E , for every prime p . Let G be a pure subgroup of $E^\times$, and put*

$$H := \{h \in K^\times : h^d \in G \text{ for some positive integer } d\}.$$

Then for $a_1, \dots, a_n \in E^\times$, the equation $a_1x_1 + \dots + a_nx_n = 1$ has the same non-degenerate solutions in G as in H .

Proof. By Lemma 5.15 of [5], it is enough to show the following.

Claim. Let p be a prime number, and let $h \in K^\times$ be such that $h^p = g \in G$ and $h \notin G$. Then $X^p - g \in E[X]$ is irreducible, and $Gh^\mathbb{Z}$, the subgroup of $K^\times$ generated by G and h , is pure in $E(h)^\times$.

Proof of the Claim. Since $h \notin G$ and K and E have the same p^{th} roots of unity, g is not a p^{th} power in $E^\times$. Thus by Theorem 9.1 from [9] the polynomial $X^p - g$ is irreducible in $E[X]$. To show that $Gh^\mathbb{Z} = \bigcup_{i=0}^{p-1} Gh^i$ is pure in $E(h)^\times$, suppose towards a contradiction that $f \in E(h)^\times$ and $f^d \in Gh^\mathbb{Z}$, where d is an integer greater than 1, but $f \notin Gh^\mathbb{Z}$. We can reduce to the case that d is prime and $f^m \notin G$ for $1 \leq m < d$. So by Theorem 9.1 of [9] again, $X^d - f^d$ is irreducible in $E[X]$. As $E(f) \subseteq E(h)$ and $[E(h) : E] = p$, we get $d = p$. Let ζ be a primitive p^{th} root of unity in the algebraic closure of K . Then $E(h) \cap E(\zeta) = E$ and $E(h, \zeta)$ is a cyclic extension of degree p of $E(\zeta)$. Let $\sigma \in \text{Gal}(E(h, \zeta) | E(\zeta))$ be given by $\sigma(h) = \zeta h$. Then $\sigma(f) = \zeta^k f$ with $0 < k < p$. With $f = c_0 + c_1h + \dots + c_{p-1}h^{p-1}$ and all $c_i \in E$ this gives

$$\begin{aligned} \sigma(f) &= c_0 + \zeta c_1h + \dots + \zeta^{p-1}c_{p-1}h^{p-1} \\ &= \zeta^k c_0 + \zeta^k c_1h + \dots + \zeta^k c_{p-1}h^{p-1}. \end{aligned}$$

This forces $c_i = 0$ for all $i \neq k$, so $f = c_k h^k$, as desired. $\square$

Lemma 6.2. *Suppose $\mathbf{k}$ is algebraically closed, $F \supseteq \mathbf{k}$, $\text{trdeg}(F | \mathbf{k}) = 1$, $\mathbf{k}^\times \cap \Gamma = \{1\}$, and $\Gamma \subseteq F^\times$ has finite rank. Then $(\mathbf{k}, \Gamma)$ is a Mann pair.*

Proof. We can assume that $\Gamma \neq \{1\}$. Consider the divisible closure

$$H := \{f \in F^\times : f^d \in \mathbf{k}^\times \Gamma \text{ for some positive integer } d\},$$

of $\mathbf{k}^\times \Gamma$ in $F^\times$. Now, as an abelian group, $\mathbf{k}^\times$ is divisible and hence injective, so we can take a subgroup $\Gamma' \supseteq \Gamma$ of H such that $\mathbf{k}^\times \cap \Gamma' = \{1\}$ and $\mathbf{k}^\times \Gamma' = H$. It follows that Γ' is pure in $F^\times$ and Γ'/Γ is a torsion group. Replacing Γ by Γ' we arrange that Γ is pure in $F^\times$. Take a finitely generated subgroup Γ_0 of Γ such that Γ/Γ_0 is a torsion group. Then $E := \mathbf{k}(\Gamma_0)$ is a function field over $\mathbf{k}$ of one variable. Take a finite subset S of $\mathcal{R}(E | \mathbf{k})$ such that all elements of Γ_0 are S -units, and put $\Gamma_1 := E^\times \cap \Gamma$. Since Γ_1/Γ_0 is a torsion group, all elements of Γ_1 are also S -units. Hence Γ_1 is isomorphic to a subgroup of $\bigoplus_{v \in S} \mathbb{Z}v$, and is thus finitely generated. Then by Corollary 3.2, $(\mathbf{k}, \Gamma_1)$ is a Mann pair. Since Γ is pure in $F^\times$ and $\mathbf{k}$ contains all roots of unity, Γ_1 is pure in $E^\times$. Now apply Lemmas 6.1 and 2.1 to get that $(\mathbf{k}, \Gamma)$ is a Mann pair. $\square$

Theorem 6.3. *Suppose $\mathbf{k}$ is algebraically closed, $\mathbf{k}^\times \cap \Gamma = \{1\}$, and Γ has finite rank. Then $(\mathbf{k}, \Gamma)$ is a Mann pair.*

Proof. Let F be the algebraic closure of $\mathbf{k}(\Gamma)$ in Ω , and put

$$H := \{f \in F^\times : f^d \in \mathbf{k}^\times \Gamma \text{ for some positive integer } d\},$$

the divisible closure of $\mathbf{k}^\times \Gamma$ in $F^\times$. Now, as an abelian group, $\mathbf{k}^\times$ is divisible and hence injective, so we can take a subgroup $\Gamma' \supseteq \Gamma$ of H such that $\mathbf{k}^\times \cap \Gamma' = \{1\}$ and $\mathbf{k}^\times \Gamma' = H$. It follows that Γ' is divisible and Γ'/Γ is a torsion group. So replacing Γ by Γ' if necessary we can assume in the rest of the proof that Γ is divisible. Let $m := \text{trdeg}(\mathbf{k}(\Gamma)|\mathbf{k})$, and take a chain

$$\mathbf{k} = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_m = F$$

of algebraically closed subfields of Ω such that for $i = 0, \dots, m-1$ we have $\text{trdeg}(K_{i+1}|K_i) = 1$. Put

$$\Gamma_i := \Gamma \cap K_i^\times, \quad i = 1, \dots, m.$$

We claim that $(\mathbf{k}, \Gamma_i)$ is a Mann pair for $i = 1, \dots, m$. For $i = 1$ this holds by Lemma 6.2, since $\mathbf{k}(\Gamma_1) \subseteq K_1$ and K_1 has transcendence degree 1 over $\mathbf{k}$. Suppose $1 \leq i < m$ and $(\mathbf{k}, \Gamma_i)$ is a Mann pair. Since Γ_i is divisible we have a subgroup Δ of Γ_{i+1} such that $\Gamma_{i+1} = \Gamma_i \Delta$ and $\Gamma_i \cap \Delta = \{1\}$. Then Δ has finite rank and $K_i^\times \cap \Delta = \{1\}$, and $K_i(\Delta) \subseteq K_{i+1}$, so (K_i, Δ) is a Mann pair. It follows from (3) in Section 5 that then $(\mathbf{k}, \Gamma_i \Delta) = (\mathbf{k}, \Gamma_{i+1})$ is a Mann pair.

This finishes the inductive proof of the claim. For $i = m$ this claim yields the desired conclusion. $\square$

In this theorem we cannot replace the finite rank assumption by the weaker condition that Γ has the Mann property, as the following example shows. Take distinct indeterminates $a_1, x_1, a_2, x_2, a_3, x_3, \dots$ and put

$$\mathbf{k} := \mathbb{Q}(a_1, a_2, a_3, \dots)^{\text{ac}}, \quad K := \mathbf{k}(x_1, x_2, x_3, \dots).$$

Let Γ be the subgroup of $K^\times$ generated by $x_1, y_1, x_2, y_2, x_3, y_3, \dots$, where $y_i := 1 - a_i x_i$. Then $\mathbf{k}$ is algebraically closed, $\mathbf{k}^\times \cap \Gamma = \{1\}$, but $(\mathbf{k}, \Gamma)$ is not a Mann pair, since $a_i x_i + y_i = 1$ for all i . On the other hand, Γ has the Mann property because $x_1, y_1, x_2, y_2, \dots$ are algebraically independent over $\mathbb{Q}$ (but not over $\mathbf{k}$).

The condition $\mathbf{k}^\times \cap \Gamma = \{1\}$ in the theorem can be relaxed:

Corollary 6.4. *Suppose $\mathbf{k}$ is algebraically closed, $\mathbf{k}^\times \cap \Gamma$ is finite and Γ has finite rank. Then $(\mathbf{k}, \Gamma)$ is a Mann pair.*

Proof. Put $d := |\mathbf{k}^\times \cap \Gamma|$. The hypothesis implies that then the subgroup $\Delta := \{\gamma^d : \gamma \in \Gamma\}$ of Γ is of finite rank and has finite index in Γ , and that $\mathbf{k}^\times \cap \Delta = \{1\}$. Hence by the theorem, $(\mathbf{k}^{\text{ac}}, \Delta)$ is a Mann pair, and thus $(\mathbf{k}, \Gamma)$ is a Mann pair by item (2) of the previous section. $\square$

Another proof. The referee mentioned that Theorem 6.3 can be obtained from [7]. Here is a sketch of how. With the assumptions of Theorem 6.3 we can arrange in addition that Ω is equipped with a derivation making it a differentially closed field with $\mathbf{k}$ as its field of constants. Below, “definable”, “Morley rank”, and “Morley degree” are with respect to Ω^{eq} , with Ω viewed as a differentially closed field. The derivation of Ω yields the logarithmic derivative map ld as a definable group morphism

$$x \mapsto x'/x : \mathbb{G}_{\text{m}} = \Omega^\times \rightarrow \mathbb{G}_{\text{a}} = \Omega,$$

with kernel $\mathbf{k}^\times$. Let $\gamma_1, \dots, \gamma_r \in \Gamma$ with $r \in \mathbb{N}$ generate a subgroup Γ_0 such that Γ/Γ_0 is a torsion group. Then

$$\text{ld}(\Gamma) \subseteq \mathbb{Q} \cdot \text{ld}(\gamma_1) + \dots + \mathbb{Q} \cdot \text{ld}(\gamma_r) \subseteq \mathbf{k} \cdot \text{ld}(\gamma_1) + \dots + \mathbf{k} \cdot \text{ld}(\gamma_r) =: V.$$

Put $G := \text{ld}^{-1}(V)$, a definable subgroup of $\mathbb{G}_m$ of finite Morley rank with $\mathbf{k}^\times \subseteq G$ and $\Gamma \subseteq G$. Let $n \geq 2$ and let $X \subseteq G^n$ be the set of nondegenerate solutions of the equation $x_1 + \dots + x_n = 0$ in G ; by Lemma 2.1 it is enough to obtain a finite set $F(n) \subseteq G^n$ such that

$$X \subseteq \{(c_1 g_1, \dots, c_n g_n) : c_1, \dots, c_n \in \mathbf{k}^\times, (g_1, \dots, g_n) \in F(n)\}.$$

The semiabelian variety $\mathbb{G}_m^n$ and its diagonal algebraic subgroup $\Delta := \{(g, \dots, g) : g \in \mathbb{G}_m\}$ are defined over $\mathbf{k}$, which gives the semiabelian variety $A := \mathbb{G}_m^n / \Delta$ over $\mathbf{k}$. Let $\bar{G} \subseteq A$ be the image of G^n under the natural map $\mathbb{G}_m^n \rightarrow A$, so $\bar{G}$ is a definable subgroup of A of finite Morley rank containing $A(\mathbf{k})$. For $x \in G^n$, let $\bar{x}$ be its image in $\bar{G}$, and let $\bar{X} := \{\bar{x} : x \in X\} \subseteq \bar{G}$. It is now an exercise to show that each definable subset of $\bar{X}$ of Morley degree 1 has trivial model-theoretic stabilizer in $\bar{G}$, as defined in [7]. This allows us to apply Corollary 2.7 of [7] inductively to obtain that $\bar{X}$ is covered by finitely many cosets of $A(\mathbf{k})$; this gives the finite set $F(n) \subseteq G^n$, as desired.

This proof does not seem to have the same kind of effectiveness as the proof based on the Brownawell-Masser height bound.

7. DEFINABLE SETS IN MANN PAIRS

Let $\mathcal{L}$ be the language of rings augmented by two distinct unary relation symbols. Let T be the $\mathcal{L}$ -theory whose models are the structures $(\Omega, \mathbf{k}, \Gamma)$ where Ω is an algebraically closed field with a subfield $\mathbf{k}$, and a subgroup Γ of $\Omega^\times$. Let $\mathcal{L}_{\Sigma}^{\text{f}, \text{g}}$ be the 2-sorted language, with sorts f, g (the *field sort* and the *group sort*), and with the following nonlogical symbols:

- constant symbols 0 and 1 of sort f ,
- a unary function symbol $-$ of sort $(\text{f}; \text{f})$,
- binary function symbols $+$ and $\cdot$ of sort $(\text{f}, \text{f}; \text{f})$,
- a constant symbol 1 of sort g ,
- a unary function symbol $^{-1}$ of sort $(\text{g}; \text{g})$,
- a binary function symbol $\cdot$ of sort $(\text{g}, \text{g}; \text{g})$,
- for each $n \geq 1$, a $2n$ -ary relation symbol Σ_n of sort $(\text{f}, \dots, \text{f}, \text{g}, \dots, \text{g})$ (n places of sort f and n places of sort g).

Of course, “1 used as a symbol of sort f ” is different from “1 used as a symbol of sort g ” and likewise with the multiplication symbol. For a model $(\Omega, \mathbf{k}, \Gamma)$ of T we construe $(\mathbf{k}, \Gamma)$ as an $\mathcal{L}_{\Sigma}^{\text{f}, \text{g}}$ -structure by interpreting the symbols in the obvious way; in particular, each Σ_n is interpreted as the previously defined $\Sigma_n(\mathbf{k}, \Gamma) \subseteq \mathbf{k}^n \times \Gamma^n$.

In the next result we do *not* assume that $(\mathbf{k}, \Gamma)$ is a Mann pair.

Lemma 7.1. *Let $(\Omega, \mathbf{k}, \Gamma) \models T$. Then every subset of $\mathbf{k}^m \times \Gamma^n$ definable in $(\Omega, \mathbf{k}, \Gamma)$ is definable in the $\mathcal{L}_{\Sigma}^{\text{f}, \text{g}}$ -structure $(\mathbf{k}, \Gamma)$.*

The diagram shows a sequence of vectors in a 2D space. At the bottom left is vector $\mathbf{k}$. An arrow points up to $\mathbf{k}'$. From $\mathbf{k}'$, an arrow points up and to the right to $\mathbf{k}'(\Gamma')$. From $\mathbf{k}'(\Gamma')$, an arrow points up and to the right to Ω . From $\mathbf{k}$, an arrow points up and to the right to Ω . From $\mathbf{k}$, an arrow points up and to the right to $\mathbf{k}(\Gamma)$. From $\mathbf{k}(\Gamma)$, an arrow points up to $\mathbf{k}'(\Gamma')$. From $\mathbf{k}(\Gamma)$, an arrow points up and to the right to Ω . A vertical arrow also points from Ω to Ω' .

$$\mathrm{tp}_{(\mathbf{k}', \Gamma')} ((\vec{a}, \vec{\alpha}) | (\mathbf{k}, \Gamma)) = \mathrm{tp}_{(\mathbf{k}', \Gamma')} ((\vec{b}, \vec{\beta}) | (\mathbf{k}, \Gamma)).$$
$$\mathrm{tp}_{(\Omega', \mathbf{k}', \Gamma')} ((\vec{a}, \vec{\alpha}) | (\Omega, \mathbf{k}, \Gamma)) = \mathrm{tp}_{(\Omega', \mathbf{k}', \Gamma')} ((\vec{b}, \vec{\beta}) | (\Omega, \mathbf{k}, \Gamma)).$$
$$\mathrm{tp}_{\mathbf{k}'}(\vec{a}|\mathbf{k}) = \mathrm{tp}_{\mathbf{k}'}(\vec{b}|\mathbf{k}) \text{ and } \mathrm{tp}_{\Gamma'}(\vec{\alpha}|\Gamma) = \mathrm{tp}_{\Gamma'}(\vec{\beta}|\Gamma).$$
$$\mathrm{tp}_{(\mathbf{k}', \Gamma')}((\vec{a}, \vec{\alpha}) | (\mathbf{k}, \Gamma)) = \mathrm{tp}_{(\mathbf{k}', \Gamma')}((\vec{b}, \vec{\beta}) | (\mathbf{k}, \Gamma)).$$
$$(\mathbf{k}', \Gamma') \models \Sigma_N(\vec{c}', \vec{\gamma}') \iff (\mathbf{k}', \Gamma') \models \Sigma_N(\sigma(\vec{c}'), \phi(\vec{\gamma}')).$$

We prove the forward implication. (The backward implication follows in the same way.) Assume $(\mathbf{k}', \Gamma') \models \Sigma_N(\vec{c}', \vec{\gamma}')$. We have $\Sigma_N(\mathbf{k}, \Gamma) = \bigcup_{i \in I} P_i \times Q_i$, where I is

finite, each $P_i \subseteq \mathbf{k}^N$ is definable in the field $\mathbf{k}$ and each $Q_i \subseteq \Gamma^N$ is definable in the group Γ . Then $\Sigma_N(\mathbf{k}', \Gamma') = \bigcup_{i \in I} P'_i \times Q'_i$, where $P'_i \subseteq (\mathbf{k}')^N$ is defined in $\mathbf{k}'$ by any formula with parameters from $\mathbf{k}$ that defines P_i in the field $\mathbf{k}$, and $Q'_i \subseteq (\Gamma')^N$ is defined in Γ' by any formula with parameters from Γ that defines Q_i in the group Γ . Take $i \in I$ such that $\vec{c}' \in P'_i$ and $\vec{\gamma}' \in Q'_i$. It is clear that then $\sigma(\vec{c}') \in P_i$ and $\phi(\vec{\gamma}') \in Q_i$, so $(\mathbf{k}', \Gamma') \models \Sigma_N(\sigma(\vec{c}'), \phi(\vec{\gamma}'))$, as desired. $\square$

8. THE ELEMENTARY THEORY OF Ω WITH A MANN PAIR

In this section we discuss various model-theoretic properties of models $(\Omega, \mathbf{k}, \Gamma)$ of T where $(\mathbf{k}, \Gamma)$ is a Mann pair.

8.1. Elementary equivalence and smallness. We first prove that the theory of a model $(\Omega, \mathbf{k}, \Gamma)$ of T is determined by the $\mathcal{L}_{\Sigma}^{\text{f,g}}$ -theory of $(\mathbf{k}, \Gamma)$ whenever $\mathbf{k} \cup \Gamma$ is small in Ω . For this we do not need $(\mathbf{k}, \Gamma)$ to be a Mann pair. (A definition and basic properties of *small* are in [5], Section 2.)

Lemma 8.1. *Let $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ be models of T such that $\mathbf{k}_i \cup \Gamma_i$ is small in Ω_i for $i = 1, 2$, and $(\mathbf{k}_1, \Gamma_1) \equiv (\mathbf{k}_2, \Gamma_2)$ as $\mathcal{L}_{\Sigma}^{\text{f,g}}$ -structures. Then $(\Omega_1, \mathbf{k}_1, \Gamma_1) \equiv (\Omega_2, \mathbf{k}_2, \Gamma_2)$.*

Proof. It is harmless to assume CH (the continuum hypothesis), so we can reduce to the case that $(\Omega_i, \mathbf{k}_i, \Gamma_i)$ is saturated of cardinality $\aleph_1$ for $i = 1, 2$. Then we have an $\mathcal{L}_{\Sigma}^{\text{f,g}}$ -isomorphism

$$(\iota_f, \iota_g) : (\mathbf{k}_1, \Gamma_1) \rightarrow (\mathbf{k}_2, \Gamma_2).$$

This yields a ring isomorphism $\mathbf{k}_1[\Gamma_1] \rightarrow \mathbf{k}_2[\Gamma_2]$ extending both ι_f and ι_g , and hence a field isomorphism

$$\iota : \mathbf{k}_1(\Gamma_1)^{\text{ac}} \rightarrow \mathbf{k}_2(\Gamma_2)^{\text{ac}}.$$

By the smallness assumption the transcendence degree of Ω_i over $\mathbf{k}_i(\Gamma_i)^{\text{ac}}$ is $\aleph_1$ for $i = 1, 2$. Thus we can extend ι to a field isomorphism $\Omega_1 \rightarrow \Omega_2$, which is then an $\mathcal{L}$ -isomorphism. $\square$

Let $\mathbf{k}$ be a subfield of the algebraically closed field Ω . A theorem of E. Artin says that if $1 < [\Omega : \mathbf{k}] < \infty$, then $\mathbf{k}$ is real closed with $\Omega = \mathbf{k}(\sqrt{-1})$. As noted at the end of Section 2 in [5] it follows that by work of Keisler [8] the following are equivalent:

- (1) $\mathbf{k}$ is small in Ω ;
- (2) $[\Omega : \mathbf{k}] > 2$;
- (3) $[\Omega : \mathbf{k}] = \infty$.

We use this equivalence in the proof of the next lemma, which is analogous to Lemma 6.1 in [5]. (The proofs are not at all similar.)

Lemma 8.2. *Assume $(\Omega, \mathbf{k}, \Gamma) \models T$, where $[\Omega : \mathbf{k}] > 2$ and $(\mathbf{k}, \Gamma)$ is a Mann pair. Then $\mathbf{k} \cup \Gamma$ is small in Ω .*

Proof. We can assume that Γ is infinite. Take a proper elementary extension $(\Omega^*, \mathbf{k}^*, \Gamma^*)$ of $(\Omega, \mathbf{k}, \Gamma)$. Using that $(\mathbf{k}, \Gamma)$ is a Mann pair, it follows easily that

the subfields $\mathbf{k}^*$ and $\mathbf{k}(\Gamma^*)$ of Ω^* are linearly disjoint over $\mathbf{k}$. See the diagram below where all arrows are inclusions:

It follows that $\mathbf{k}^*(\Gamma)$ and $\mathbf{k}(\Gamma^*)$ are linearly disjoint over $\mathbf{k}(\Gamma)$. Among the subsets of Γ^* that are multiplicatively independent over Γ , take one, say B , that is maximal. It follows from Lemma 5.12 in [5] that B is algebraically independent over $\mathbf{k}(\Gamma)$. Hence B is algebraically independent over $\mathbf{k}^*(\Gamma)$. The maximality property of B guarantees that $B \neq \emptyset$ and for every $\gamma \in \Gamma^*$ there is a positive integer d with $\gamma^d \in \mathbf{k}^*(\Gamma)(B)$. Since $\mathbf{k}^*(\Gamma)(B)$ is a nontrivial purely transcendental extension of $\mathbf{k}^*(\Gamma)$ it follows that $\mathbf{k}^*(\Gamma^*, \sqrt{-1})$ is not algebraically closed, and so $\mathbf{k}^*(\Gamma^*, \sqrt{-1}) \neq \Omega^*$. Hence by the remark preceding this lemma, $\mathbf{k}^* \cup \Gamma^*$ is small in Ω^* , and thus $\mathbf{k} \cup \Gamma$ is small in Ω ; see Section 2 in [5]. $\square$

8.2. Stability.

Proposition 8.3. *Let $(\Omega, \mathbf{k}, \Gamma) \models T$ be such that $\mathbf{k}$ is algebraically closed and $(\mathbf{k}, \Gamma)$ is a Mann pair. Then $(\Omega, \mathbf{k}, \Gamma)$ is stable, and if Γ is divisible, then $(\Omega, \mathbf{k}, \Gamma)$ is ω -stable.*

Proof. Take an infinite cardinal κ such that the abelian group Γ is κ -stable. We show that then $(\Omega, \mathbf{k}, \Gamma)$ is κ -stable. We can assume $\mathbf{k} \neq \Omega$, and that $|\Omega| = \kappa$. Take a κ^+ -saturated elementary extension $(\Omega', \mathbf{k}', \Gamma')$ of $(\Omega, \mathbf{k}, \Gamma)$.

By the proofs of Lemma 7.1 and Proposition 7.2, the type of an element of $\mathbf{k}'$ over Ω in $(\Omega', \mathbf{k}', \Gamma')$ is determined by its type over $\mathbf{k}$ in the field $\mathbf{k}'$. Likewise, the type of an element of Γ' over Ω in $(\Omega', \mathbf{k}', \Gamma')$ is determined by its type over Γ in the group Γ' .

Now let $t \in \Omega(\mathbf{k}' \cup \Gamma')^{\text{ac}}$, say $t \in \Omega(\vec{a}, \vec{\gamma})^{\text{ac}}$ with $\vec{a} \in (\mathbf{k}')^m$ and $\vec{\gamma} \in (\Gamma')^n$. Then $\text{tp}_{(\Omega', \mathbf{k}', \Gamma')}(t|\Omega)$ is determined by $\text{tp}_{\mathbf{k}'}(\vec{a}|\mathbf{k})$, $\text{tp}_{\Gamma'}(\vec{\gamma}|\Gamma)$ and the specification of a polynomial $P(X, Y, T) \in \Omega[X, Y, T]$ where $X = (X_1, \dots, X_m)$, $Y = (Y_1, \dots, Y_n)$ and T is a single indeterminate such that $P(\vec{a}, \vec{\gamma}, T) \in \Omega(\vec{a}, \vec{\gamma})[T]$ is irreducible and $P(\vec{a}, \vec{\gamma}, t) = 0$.

Finally, by the last argument of the proof of Lemma 8.1, all elements of Ω' outside $\Omega(\mathbf{k}' \cup \Gamma')^{\text{ac}}$ realize the same type in $(\Omega', \mathbf{k}', \Gamma')$ over Ω .

Hence we have at most κ many different 1-types in $(\Omega', \mathbf{k}', \Gamma')$ over Ω . $\square$

Remark. One can show that if $(\Omega, \mathbf{k}, \Gamma)$ is a model of T and $(\mathbf{k}, \Gamma)$ is a Mann pair, then the subsets $\mathbf{k}$ and Γ of Ω are definable in the structure $(\Omega, \mathbf{k} \cup \Gamma)$. Using this fact, Proposition 8.3 also follows from Fact 2.1 and Theorem 4.8 in [1].

8.3. Axiomatizing $(\Omega, \mathbf{k}, \Gamma)$. Our goal here is to show that if $(\Omega, \mathbf{k}, \Gamma)$ is a model of T and $(\mathbf{k}, \Gamma)$ is a Mann pair with $\mathbf{k}$ small in Ω , then the elementary theory of $(\Omega, \mathbf{k}, \Gamma)$ is completely determined by the elementary theories of the field $\mathbf{k}$ and of the group Γ . We achieve this after adding names for enough elements of $\mathbf{k}$ and Γ to witness that $(\mathbf{k}, \Gamma)$ is a Mann pair.

Fix a model $(\Omega_0, \mathbf{k}_0, \Gamma_0)$ of T such that $(\mathbf{k}_0, \Gamma_0)$ is a Mann pair. For each $n \geq 2$, fix a finite subset $\Gamma_0(n)$ of Γ_0^n such that

$$\Sigma_n^{\text{nd}}(\mathbf{k}_0, \Gamma_0) = \bigcup_{\vec{\gamma} \in \Gamma_0(n)} \Sigma_n^{\text{nd}}(\mathbf{k}_0, \Gamma_0; \vec{\gamma}) \times \Gamma_0 \vec{\gamma},$$

and for each $\vec{\gamma} \in \Gamma_0(n)$, fix a basis $B(\vec{\gamma}) \subseteq \mathbf{k}_0^n$ of the $\mathbf{k}_0$ -linear subspace $\Sigma_n(\mathbf{k}_0, \Gamma_0; \vec{\gamma})$ of $\mathbf{k}_0^n$. Let $\mathcal{L}(\mathbf{k}_0, \Gamma_0)$ be the language $\mathcal{L}$ augmented by names for the elements of $\mathbf{k}_0 \cup \Gamma_0$.

In the following definition and subsequent remarks, $(\Omega, \mathbf{k}, \Gamma)$ ranges over models of T that contain $(\Omega_0, \mathbf{k}_0, \Gamma_0)$ as a substructure. We construe such $(\Omega, \mathbf{k}, \Gamma)$ as an $\mathcal{L}(\mathbf{k}_0, \Gamma_0)$ -structure in the obvious way, and we say that $(\Omega, \mathbf{k}, \Gamma)$ *satisfies the Mann-axioms of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$* if for each $n \geq 2$:

- (1) $\Sigma_n^{\text{nd}}(\mathbf{k}, \Gamma) = \bigcup_{\vec{\gamma} \in \Gamma_0(n)} \Sigma_n^{\text{nd}}(\mathbf{k}, \Gamma; \vec{\gamma}) \times \Gamma \vec{\gamma}$;
- (2) for each $\vec{\gamma} \in \Gamma_0(n)$, the $\mathbf{k}$ -linear subspace $\Sigma_n(\mathbf{k}, \Gamma; \vec{\gamma})$ of $\mathbf{k}^n$ is generated by $B(\vec{\gamma})$.

Remarks. The reason for this terminology is that we have a set $\text{Mann}(\mathbf{k}_0, \Gamma_0)$ of sentences in the language $\mathcal{L}(\mathbf{k}_0, \Gamma_0)$ such that for all $(\Omega, \mathbf{k}, \Gamma)$ as above,

$$(\Omega, \mathbf{k}, \Gamma) \models \text{Mann}(\mathbf{k}_0, \Gamma_0)$$

$$\iff$$

$$(\Omega, \mathbf{k}, \Gamma) \text{ satisfies the Mann axioms of } (\Omega_0, \mathbf{k}_0, \Gamma_0).$$

In particular, if $(\Omega, \mathbf{k}, \Gamma)$ is an elementary extension of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$, then $(\Omega, \mathbf{k}, \Gamma)$ satisfies the Mann axioms of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$.

Let $\mathcal{L}^f := \{0, 1, -, +, \cdot\}$ be the one-sorted language whose symbols are those of $\mathcal{L}_{\Sigma}^{f, g}$ involving only sort f , and let $\mathcal{L}^f(\mathbf{k}_0)$ be $\mathcal{L}^f$ augmented by names for the elements of $\mathbf{k}_0$. Let $\mathcal{L}^g := \{1, ^{-1}, \cdot\}$ be the one-sorted language whose symbols are those of $\mathcal{L}_{\Sigma}^{f, g}$ involving only sort g , and let $\mathcal{L}^g(\Gamma_0)$ be $\mathcal{L}^g$ augmented by names for the elements of Γ_0 . Let $n \geq 1$, let $x = (x_1, \dots, x_n)$ be a tuple of distinct f -variables, and let $y = (y_1, \dots, y_n)$ be a tuple of distinct g -variables. A careful look at the proof of Lemma 4.2 for $\vec{r} = (1, \dots, 1)$ yields the following: there are quantifier-free $\mathcal{L}^f(\mathbf{k}_0)$ -formulas $\phi_1(x), \dots, \phi_m(x)$ and quantifier-free $\mathcal{L}^g(\Gamma_0)$ -formulas $\psi_1(y), \dots, \psi_m(y)$ such that for all $(\Omega, \mathbf{k}, \Gamma) \models \text{Mann}(\mathbf{k}_0, \Gamma_0)$,

$$\Sigma_n(\mathbf{k}^n, \Gamma^n) = \bigcup_{i=1}^m \phi_i(\mathbf{k}^n) \times \psi_i(\Gamma^n).$$

This uniformity is crucial in the proof of the next result.

Theorem 8.4. *Let $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ be models of T such that*

- (1) $[\Omega_1 : \mathbf{k}_1] > 2$ and $[\Omega_2 : \mathbf{k}_2] > 2$;
- (2) $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_2, \mathbf{k}_2, \Gamma_2)$;
- (3) $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ satisfy the Mann-axioms of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$.

Then: $(\Omega_1, \mathbf{k}_1, \Gamma_1) \equiv_{\mathbf{k}_0 \cup \Gamma_0} (\Omega_2, \mathbf{k}_2, \Gamma_2) \iff \mathbf{k}_1 \equiv_{\mathbf{k}_0} \mathbf{k}_2$ and $\Gamma_1 \equiv_{\Gamma_0} \Gamma_2$.

Proof. The forward direction is clear. For the converse, assume $\mathbf{k}_1 \equiv_{\mathbf{k}_0} \mathbf{k}_2$ and $\Gamma_1 \equiv_{\Gamma_0} \Gamma_2$. Without loss we can also assume that $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ are κ -saturated, where $\kappa > |\mathbf{k}_0| + |\Gamma_0|$.

We construct a back-and-forth system between the $\mathcal{L}_{\Sigma}^{f,g}$ -structures $(\mathbf{k}_1, \Gamma_1)$ and $(\mathbf{k}_2, \Gamma_2)$. Let $\mathcal{S}_1$ be the set of substructures $(\mathbf{k}'_1, \Gamma'_1)$ of $(\mathbf{k}_1, \Gamma_1)$ that extend $(\mathbf{k}_0, \Gamma_0)$ and have cardinality $< \kappa$ (so $\mathbf{k}'_1$ is a subring, not necessarily a subfield, of $\mathbf{k}_1$, and Γ'_1 is a subgroup of Γ_1). Define $\mathcal{S}_2$ likewise, with $(\mathbf{k}_2, \Gamma_2)$ in place of $(\mathbf{k}_1, \Gamma_1)$, and let $\mathcal{I}$ be the set of isomorphisms

$$\iota = (\iota_f, \iota_g) : (\mathbf{k}'_1, \Gamma'_1) \rightarrow (\mathbf{k}'_2, \Gamma'_2),$$

where $(\mathbf{k}'_i, \Gamma'_i) \in \mathcal{S}_i$ for $i = 1, 2$, such that ι is the identity on $(\mathbf{k}_0, \Gamma_0)$, $\iota_f : \mathbf{k}'_1 \rightarrow \mathbf{k}'_2$ is a partial elementary map from $\mathbf{k}_1$ to $\mathbf{k}_2$, and $\iota_g : \Gamma'_1 \rightarrow \Gamma'_2$ is a partial elementary map from Γ_1 to Γ_2 . It is clear that the identity map of $(\mathbf{k}_0, \Gamma_0)$ belongs to $\mathcal{I}$.

Next we show that $\mathcal{I}$ is a back-and-forth system. By symmetry it is enough to show that we can go forth. Let

$$\iota = (\iota_f, \iota_g) : (\mathbf{k}'_1, \Gamma'_1) \rightarrow (\mathbf{k}'_2, \Gamma'_2)$$

be in $\mathcal{I}$. Replacing $\mathbf{k}'_1$ and $\mathbf{k}'_2$ by their fraction fields inside $\mathbf{k}_1$ and $\mathbf{k}_2$ and extending ι_f accordingly, without changing ι_g , we arrange that $\mathbf{k}'_1$ and $\mathbf{k}'_2$ are subfields of $\mathbf{k}_1$ and $\mathbf{k}_2$.

Let $a \in \mathbf{k}_1 \setminus \mathbf{k}'_1$ be given. Using saturation we can take $b \in \mathbf{k}_2 \setminus \mathbf{k}'_2$ and a field isomorphism $j : \mathbf{k}'_1(a) \rightarrow \mathbf{k}'_2(b)$ that extends ι_f , sends a to b , and is a partial elementary map from $\mathbf{k}_1$ to $\mathbf{k}_2$. It is obvious that $(\mathbf{k}'_1(a), \Gamma'_1) \in \mathcal{S}_1$ and $(\mathbf{k}'_2(b), \Gamma'_2) \in \mathcal{S}_2$, and it follows easily from the remark just before the lemma that $(j, \iota_g) \in \mathcal{I}$.

Now let $\alpha \in \Gamma_1 \setminus \Gamma'_1$. By saturation we can find $\beta \in \Gamma_2 \setminus \Gamma'_2$ and a group isomorphism $h : \alpha^{\mathbb{Z}}\Gamma'_1 \rightarrow \beta^{\mathbb{Z}}\Gamma'_2$ from Γ_1 to Γ_2 that extends ι_g , sends α to β , and is a partial elementary map from Γ_1 to Γ_2 . It is obvious that $(\mathbf{k}'_1, \alpha^{\mathbb{Z}}\Gamma'_1) \in \mathcal{S}_1$ and $(\mathbf{k}'_2, \beta^{\mathbb{Z}}\Gamma'_2) \in \mathcal{S}_2$. It follows easily from the remark just before the lemma that $(\iota_f, h) : (\mathbf{k}'_1, \alpha^{\mathbb{Z}}\Gamma'_1) \rightarrow (\mathbf{k}'_2, \beta^{\mathbb{Z}}\Gamma'_2)$ belongs to $\mathcal{I}$.

We have now shown that $\mathcal{I}$ is a nonempty back-and-forth system. It follows that $(\mathbf{k}_1, \Gamma_1) \equiv_{(\mathbf{k}_0, \Gamma_0)} (\mathbf{k}_2, \Gamma_2)$. From (1), (3), and Lemma 8.2 we obtain that $\mathbf{k}_i \cup \Gamma_i$ is small in Ω_i for $i = 1, 2$. Then a proof such as that of Lemma 8.1 yields

$$(\Omega_1, \mathbf{k}_1, \Gamma_1) \equiv_{\mathbf{k}_0 \cup \Gamma_0} (\Omega_2, \mathbf{k}_2, \Gamma_2) \iff \mathbf{k}_1 \equiv_{\mathbf{k}_0} \mathbf{k}_2 \text{ and } \Gamma_1 \equiv_{\Gamma_0} \Gamma_2,$$

as desired. $\square$

For algebraically closed $\mathbf{k}_i$, this result takes the following form:

Corollary 8.5. *Let $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ be models of T such that*

- (1) $\mathbf{k}_1$ and $\mathbf{k}_2$ are algebraically closed, $\mathbf{k}_1 \neq \Omega_1$, $\mathbf{k}_2 \neq \Omega_2$;
- (2) $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_0, \mathbf{k}_0, \Gamma_0) \subseteq (\Omega_2, \mathbf{k}_2, \Gamma_2)$;
- (3) $(\Omega_1, \mathbf{k}_1, \Gamma_1)$ and $(\Omega_2, \mathbf{k}_2, \Gamma_2)$ satisfy the Mann axioms of $(\Omega_0, \mathbf{k}_0, \Gamma_0)$.

Then: $(\Omega_1, \mathbf{k}_1, \Gamma_1) \equiv_{\mathbf{k}_0 \cup \Gamma_0} (\Omega_2, \mathbf{k}_2, \Gamma_2) \iff \Gamma_1 \equiv_{\Gamma_0} \Gamma_2$.

ACKNOWLEDGEMENT

The referee has pointed out that Theorem 1.1 also follows from Hrushovski and Pillay [7]. At the end of Section 6 we indicate how. The referee also mentioned that Sections 7 and 8 are related to earlier work by Casanovas and Ziegler [4] and by Pillay [13].

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