NONCOMMUTATIVE POISSON STRUCTURES ON ORBIFOLDS

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ABSTRACT. In this paper, we compute the Gerstenhaber bracket on the Hochschild cohomology of $C^\infty(M) \rtimes G$ for a finite group G acting on a compact manifold M . Using this computation, we obtain geometric descriptions for all noncommutative Poisson structures on $C^\infty(M) \rtimes G$ when M is a symplectic manifold. We also discuss examples of deformation quantizations of these noncommutative Poisson structures.

1. INTRODUCTION

It is well known [8] that the deformation theory of an associative algebra A is closely related to the Hochschild cohomology $HH^\bullet(A; A)$ of A . In particular, the infinitesimal deformation of A is governed by $HH^2(A; A)$. Furthermore, if we want the infinitesimal deformation to be integrable, it is necessary (but not sufficient) to require that the 2-cocycle $\Pi \in C^2(A; A)$ associated to the infinitesimal deformation satisfies the equation $[\Pi, \Pi]_G = 0$ in $HH^3(A; A)$, where $[\ , \]_G$ is the Gerstenhaber bracket on $HH^\bullet(A; A)$.

Suppose that A is an algebra of smooth functions on a smooth manifold M . Then, according to the Hochschild-Kostant-Rosenberg theorem, the second Hochschild cohomology classes in $HH^2(A; A)$ satisfying the above integrability conditions are in one-to-one correspondence with Poisson structures on M . Inspired by this relationship between Poisson geometry and deformation theory, Jonathan Block and Ezra Getzler [2] and Ping Xu [18] independently introduced a notion of a noncommutative Poisson structure on an associative algebra in the early 1990s.

Definition 1.1. A noncommutative Poisson structure on an associative algebra A is an element Π in the second Hochschild cohomology group $H^2(A, A)$ of A , whose Gerstenhaber bracket with itself vanishes, i.e. $[\Pi, \Pi]_G = 0$.

In this paper, we want to study noncommutative Poisson structures on orbifolds coming from global quotients. Let M be a compact smooth manifold, and let G be a finite group acting on M . (For simplicity, we will always assume in this paper that the G -action on M is effective.) Our orbifold is the quotient space $X = M/G$. Because X is usually a topological space with quotient singularities, the algebra $C^\infty(M)^G$ of G -invariant smooth functions on M is not regular. Taking a lesson from noncommutative geometry [4], we consider the crossed product algebra $C^\infty(M) \rtimes$

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G as a natural replacement. The crossed product algebra $C^\infty(M) \rtimes G$, though it is noncommutative, has very nice algebraic properties. Our main aim in this paper is to find geometric descriptions for all noncommutative Poisson structures on $C^\infty(M) \rtimes G$ when M is a symplectic manifold with a symplectic action.

As a first step toward understanding noncommutative Poisson structures on $C^\infty(M) \rtimes G$, the second author and his coauthors [13] computed the Hochschild cohomology of $C^\infty(M) \rtimes G$ as a vector space:

$$(1) \quad HH^\bullet(C^\infty(M) \rtimes G; C^\infty(M) \rtimes G) = \Gamma^\infty\left(\bigoplus_{g \in G} \wedge^{\bullet-l(g)} TM^g \otimes \wedge^{l(g)} N^g\right)^G.$$

We explain the notation in the above equation. First, M^g is the fixed point manifold of g , and N^g is the normal bundle of the embedding of M^g in M , where $l(g)$ is the dimension of N^g . The group G acts on the disjoint union $\bigsqcup M^g$ of M^g for all $g \in G$ by the conjugate action. We remark that M^g may have different components with different dimensions, and we take the disjoint union of all the components, and accordingly $l(g)$ is taken as a local constant function on M^g . (Following [13], in this paper we view $C^\infty(M) \rtimes G$ as a bornological algebra with the bornology defined by the Fréchet topology. We take $HH^\bullet$ to be the continuous Hochschild cohomology of a bornological algebra. Accordingly, all computations and constructions in this paper are local with respect to the orbifold M/G . We often work with a vector space (or a G -invariant open subset) with a linear G -action, which we refer to as “local” computation.) We call the stratified space $\bigsqcup M^g/G$ the inertia orbifold $\tilde{X}$ associated to $X = M/G$. In other words, we can interpret Equation (1) as saying that the Hochschild cohomology of $C^\infty(M) \rtimes G$ is equal to the space of “vector fields” on $\tilde{X}$.

The main difficulty in finding noncommutative Poisson structures on $C^\infty(M) \rtimes G$ is to compute the Gerstenhaber bracket on $HH^\bullet(C^\infty(M) \rtimes G; C^\infty(M) \rtimes G)$. To compute the Gerstenhaber bracket, we need to have quasi-isomorphisms between the Hochschild cochain complex

$$C^\bullet(C^\infty(M) \rtimes G; C^\infty(M) \rtimes G)$$

and

$$\Gamma^\infty\left(\bigoplus_{g \in G} \wedge^{\bullet-l(g)} TM^g \otimes \wedge^{l(g)} N^g\right)^G.$$

In [13], we defined a quasi-isomorphism L in the following direction:

$$L : C^\bullet(C^\infty(M) \rtimes G; C^\infty(M) \rtimes G) \longrightarrow \Gamma^\infty\left(\bigoplus_{g \in G} \wedge^{\bullet-l(g)} TM^g \otimes \wedge^{l(g)} N^g\right)^G.$$

The first step of this paper is to define a quasi-isomorphism T in the other direction. This turns out to be much harder to construct than the map L in [13]. We need to construct some nonlocal operators on $C^\infty(M)$, which we call twisted cocycles. These cocycles are closely related to the Lusztig-Demazure operator (cf. [14]). Using the maps T and L , we are able to compute the Gerstenhaber brackets on $HH^\bullet(C^\infty(M) \rtimes G; C^\infty(M) \rtimes G)$. Our results show that the Gerstenhaber bracket on orbifolds is a generalization of the classical Schouten-Nijenhuis bracket on manifolds. We call this bracket the twisted Schouten-Nijenhuis bracket on $\Gamma^\infty\left(\bigoplus_g \wedge^{\bullet-l(g)} TM^g \otimes \wedge^{l(g)} N^g\right)^G$. Using the twisted Schouten-Nijenhuis bracket, we are able to solve the equation $[\Pi, \Pi]_G = 0$ on $HH^2(C^\infty(M) \rtimes G; C^\infty(M) \rtimes G)$.

geometrically. This leads to a full description of noncommutative Poisson structures on $C^\infty(M) \rtimes G$ when M is a symplectic manifold.

If we consider a complex symplectic vector space V with a symplectic G -action, the cocycles used in the definition of symplectic reflection algebras [7] correspond to a special class of noncommutative Poisson structures on $\text{Poly}(V) \rtimes G$, where $\text{Poly}(V)$ is the algebra of polynomials on V . Using the results from [7] we prove in this paper that all these cocycles can be extended to a formal deformation of the algebra $\text{Poly}(V) \rtimes G$. As a generalization, we expect that all the noncommutative Poisson structures discovered in this paper can be extended to formal deformations, which will generalize the symplectic reflection algebras. This question is closely related to the following formality conjecture on orbifolds, which we will discuss in future publications.

Conjecture 1.2. *The Hochschild complex of the algebra $C^\infty(M) \rtimes G$ is a formal differential graded Lie algebra.*

In the last part of this paper, we provide concrete new examples of noncommutative Poisson structures on $\text{Poly}(\mathbb{R}^4) \rtimes (\mathbb{Z}_n \times \mathbb{Z}_m)$ with $\mathbb{Z}_n = \mathbb{Z}/\mathbb{Z}_n$ and $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. These Poisson structures are not symplectic at all, and instead should be viewed as noncommutative quadratic Poisson structures. The connection between these “noncommutative quadratic Poisson structures” and quantum R -matrices will be studied in the near future. In general, there are many interesting examples of noncommutative Poisson structures on orbifolds. We are working with Jean-Michel Oudom in [11] on this material.

Besides the Gerstenhaber bracket, there is also a product on the Hochschild cohomology $HH^\bullet(C^\infty(M) \rtimes G; C^\infty(M) \rtimes G)$. In [15], with Pflaum, Posthuma and Tseng, the second author studied the ring structure on the Hochschild cohomology of the deformed algebras of $C^\infty(M) \rtimes G$, which is closely related to the Chen-Ruan orbifold cohomology [3].

This paper is organized as follows. In Section 2, we briefly recall the basic definitions in Hochschild cohomology and the computations of $HH^\bullet(C^\infty(M) \rtimes G, C^\infty(M) \rtimes G)$ as a vector space in [13]. In Section 3, we will focus on the construction of twisted cocycles and a quasi-isomorphism T ,

$$T : \Gamma^\infty\left(\bigoplus_{g \in G} \wedge^{\bullet-l(g)} TM^g \otimes \wedge^{l(g)} N^g\right)^g \longrightarrow C^\bullet(C^\infty(M) \rtimes G; C^\infty(M) \rtimes G).$$

In Section 4, we will study the Gerstenhaber bracket on the Hochschild cohomology $HH^\bullet(C^\infty(M) \rtimes G; C^\infty(M) \rtimes G)$. In Section 5, we will give a full description of noncommutative Poisson structures on $C^\infty(M) \rtimes G$ when M is a symplectic manifold. Furthermore we will study formal deformations of a special type of noncommutative Poisson structures and compute their second Poisson cohomology group. We end this section by showing two new families of noncommutative quadratic Poisson structures on $\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{Z}_m$.

Remark 1.3. Unless otherwise specified, we work with the field $\mathbb{R}$, real vector spaces, and real manifolds. However, many results in this paper have analogs with the field $\mathbb{C}$, complex vector spaces and affine varieties.

2. HOCHSCHILD COHOMOLOGY OF $C^\infty(M) \rtimes G$

In this section, we briefly review our work in [13] on computing the Hochschild cohomology of the algebra $C^\infty(M) \rtimes G$ as a vector space.

2.1. Hochschild cohomology of an algebra. We review in this subsection the definition of Hochschild cohomology of an algebra. Let A be a unital algebra over $\mathbb{R}$, and let M be a bimodule of A . The degree n Hochschild cochain complex $C^n(A, M)$ of A with coefficients in M consists of $\mathbb{R}$ -linear maps from $A^{\otimes n}$ to M , i.e. $\text{Hom}(A^{\otimes n}, M)$. The coboundary map ∂ on Hochschild cochains $C^\bullet(A, M)$ is defined by

$$\begin{aligned} \partial : C^n(A, M) &\longrightarrow C^{n+1}(A, M) \\ \partial(\xi)(a_0, \dots, a_n) &= a_0 \xi(a_1, \dots, a_n) + \sum_{i=1}^n (-1)^i \xi(\dots, a_{i-1} a_i, \dots) \\ &\quad + (-1)^{n+1} \xi(a_1, \dots, a_{n-1}) a_n. \end{aligned}$$

It is a straightforward check that $\partial^2 = 0$.

Definition 2.1. Define the Hochschild cohomology of A with coefficient in M to be the cohomology of the differential cochain complex $(C^\bullet(A, M), \partial)$.

Remark 2.2. In the case that A is a topological (or bornological) algebra, then we need to consider a topological (bornological) algebra bimodule M , and the topological (bornological) tensor product of A , and continuous (bounded) linear maps from A to M . All the definitions we introduce below naturally extend to the topological (bornological) version and we refer to [15, Appendix A] for details. The above definitions work for arbitrary fields.

The left and right multiplication of A on A makes A a bimodule of A . Gerstenhaber [8] used the Hochschild cohomology $H^\bullet(A, A)$ to study deformation theory of A . On $C^\bullet(A, A)$, besides the coboundary differential, there are two interesting operations:

- (1) the Gerstenhaber bracket, and
- (2) the cup product.

The Gerstenhaber bracket is used in defining a noncommutative Poisson structure in Definition 1.1. We recall its definition.

Definition 2.3. We define a pre-Lie product $\circ$ on $C^\bullet(A; A)$. For $\phi \in C^k(A; A)$, $\psi \in C^l(A; A)$, $\phi \circ \psi \in C^{k+l-1}(A; A)$ is defined by

$$\begin{aligned} &\phi \circ \psi(a_1, \dots, a_{k+l-1}) \\ &= \sum_{i=1}^k (-1)^{(i-1)(l-1)} \phi(a_1, \dots, a_{i-1}, \psi(a_i, \dots, a_{i+l-1}), a_{i+l}, \dots, a_{k+l-1}), \end{aligned}$$

for $a_i \in A$, $i = 1, \dots, k + l - 1$. The Gerstenhaber bracket $[\ , \]$ on $C^\bullet(A; A)$ is defined to be the commutator of the pre-Lie product, i.e.,

$$[\phi, \psi]_G = \phi \circ \psi - (-1)^{(k-1)(l-1)} \psi \circ \phi.$$

The Gerstenhaber bracket is compatible with the differential on $C^\bullet(A, A)$, and therefore defines a Lie bracket on the Hochschild cohomology $HH^\bullet(A, A)$.

2.2. $HH^\bullet(Poly(V) \rtimes G, Poly(V) \rtimes G)$. In this subsection, we consider the algebra $C^\infty(M) \rtimes G$ for a finite group action on a compact manifold M , and explain the computation in [13] of the Hochschild cohomology $HH^\bullet(C^\infty(M) \rtimes G, C^\infty(M) \rtimes G)$. In this subsection we will mainly focus on the local case, i.e., the G -linear action on a

vector space V ; in the next subsection, we explain the extension of the computations to general manifolds.

Let $Poly(V)$ be the algebra of polynomial functions on a vector space V , and let G be a finite group acting linearly on V . We computed [13] the Hochschild cohomology of the crossed product algebra $Poly(V) \rtimes G$. The major step is a construction of a quasi-isomorphism

$$L : C^\bullet(Poly(V) \rtimes G, Poly(V) \rtimes G) \longrightarrow \Gamma^\infty\left(\bigoplus_{g \in G} \wedge^{\bullet-l(g)} TV^g \otimes \wedge^{l(g)} N^g\right)^G.$$

This map was constructed implicitly in the proof of Theorem 3.1 in [13]. We make it explicit in the following.

The map L is a composition of three cochain maps:

Step I.

$$L_1 : C^\bullet(Poly(V) \rtimes G, Poly(V) \rtimes G) \longrightarrow (C^\bullet(Poly(V), Poly(V) \rtimes G))^G,$$

where G acts on $C^\bullet(Poly(V), Poly(V) \rtimes G)$ by

$$g\Psi(a_1, \dots, a_n) = U_{g^{-1}} \cdot \Psi(g(a_1), \dots, g(a_n)) \cdot U_g.$$

Here U_g denotes the element g seen in $Poly(V) \rtimes G$. Given a Hochschild cocycle $\Psi \in C^k(Poly(V) \rtimes G, Poly(V) \rtimes G)$, we define $L_1(\Psi) \in (C^k(Poly(V), Poly(V) \rtimes G))^G$ by

$$L_1(\Psi)(f_1, \dots, f_k) = \frac{1}{|G|} \sum_g (g\Psi)(f_1, \dots, f_k), \quad \forall f_1, \dots, f_k \in Poly(V),$$

where $|G|$ is the order of the group G .

Step II.

$$L_2 : (C^\bullet(Poly(V), Poly(V) \rtimes G))^G \longrightarrow \left(\bigoplus_{g \in G} \Gamma^\infty(\wedge^\bullet TV), \kappa_g \wedge\right)^G.$$

Let A_g be a vector space isomorphic to $Poly(V)$ equipped with the $Poly(V)$ -bimodule structure

$$a \cdot \xi \cdot (b) = a\xi g(b), \text{ for } a, b \in Poly(V), \xi \in Poly(V)_g,$$

where the right hand side is the product of a, ξ , and $g(b)$ as elements in $Poly(V)$. As a $Poly(V)$ - $Poly(V)$ bimodule, $Poly(V) \rtimes G$ has a natural splitting into a direct sum of submodules $\bigoplus_{g \in G} A_g$. Correspondingly, the cochain complex $C^\bullet(Poly(V), Poly(V) \rtimes G)$ has a natural splitting into $\bigoplus_{g \in G} C^\bullet(Poly(V), A_g)$. We define L_2 to be the sum of the maps

$$L_2^g : C^\bullet(Poly(V), A_g) \longrightarrow (\Gamma^\infty(\wedge^\bullet TV), \kappa_g)$$

over all $g \in G$.

On V , we introduce the vector field $X(x) = \sum_i x^i \frac{\partial}{\partial x^i}$, where the x^i are coordinate functions on V . Define the vector field $\kappa_g \in \Gamma^\infty(TV)$ by

$$\kappa_g(x) = X(g(x)) - X(x).$$

We notice that for a permutation σ of k elements fixing $x \in V$, the product $(x_{\sigma(1)}^{i_1} - x^{i_1}) \cdots (x_{\sigma(k)}^{i_k} - x^{i_k})$ is a function on $x_1, \dots, x_k \in V$ by taking the product of the values of the coordinate functions. Given an element $\Psi \in C^k(Poly(V), A_g)$, we

define $L_2^g(\Psi) \in \Gamma^\infty(\wedge^k TV)$, the usual projection to anti-symmetric linear operators, by

$$L_2^g(\Psi)(x) = \sum_{i_1, \dots, i_k} \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma \Psi((x_{\sigma(1)})^{i_1} - x^{i_1}) \cdots (x_{\sigma(k)}^{i_k} - x^{i_k})(x) \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}},$$

where S_k is the permutation group of k elements.

The G -action on $(\bigoplus_{g \in G} \Gamma^\infty(\wedge^\bullet TV), \kappa_g \wedge)$ is defined by

$$h(\sum_g \phi_g)|_{h^{-1}gh} = h_*(\phi_g), \quad \text{for } \sum_g \phi_g \in \bigoplus_g \Gamma^\infty(\wedge^\bullet TV), \text{ and } h \in G.$$

It is straightforward to check that L_2^g is G -equivariant and therefore defines a map

$$L_2 : (C^\bullet(Poly(V), Poly(V) \rtimes G))^G \longrightarrow (\bigoplus_{g \in G} \Gamma^\infty(\wedge^\bullet TV), \kappa_g \wedge)^G.$$

Step III.

$$L_3 : \left(\bigoplus_{g \in G} (\Gamma^\infty(\wedge^\bullet TV), \kappa_g \wedge) \right)^G \longrightarrow \left(\bigoplus_{g \in G} (\Gamma^\infty(\wedge^{\bullet-l(g)} TV^g \otimes \wedge^{l(g)} N^g), 0) \right)^G.$$

Let C_g be the cyclic group generated by g , which has a natural action on V . As C_g is abelian, V is decomposed into a direct sum of C_g irreducible representations. Let V^g be the subspace of all trivial C_g -representations in V , and N^g be the sum of all nontrivial irreducible C_g -representations in V . Therefore, V can be written as $V^g \oplus N^g$.

We define L_3 to be the sum of L_3^g , which is defined to be

$$L_3^g(X) = pr^g(X|_{V^g}),$$

where $X|_{V^g}$ is the restriction of $X \in \wedge^\bullet TV$ to $\wedge^\bullet TV|_{V^g}$, and pr^g projects $\wedge^\bullet TV|_{V^g}$ to $\wedge^{\bullet-l(g)} TV^g \otimes \wedge^{l(g)} N^g$.

The space $\bigoplus_{g \in G} (\Gamma^\infty(\wedge^{\bullet-l(g)} TV^g \otimes \wedge^{l(g)} N^g), 0)$ is closed under G -action on $(\bigoplus_{g \in G} \Gamma^\infty(\wedge^\bullet TV), \kappa_g \wedge)$ and therefore inherits a G -action. Similarly to the computations taken in Step II, we can easily check that L_3^g is G -equivariant and that therefore L_3 defines a map on the G -invariant components.

We proved in Section 3 of [13] that $L = L_3 \circ L_2 \circ L_1$ is a quasi-isomorphism of cochain complexes

$$L : C^\bullet(Poly(V) \rtimes G, Poly(V) \rtimes G) \longrightarrow \Gamma^\infty \left(\bigoplus_{g \in G} \wedge^{\bullet-l(g)} TV^g \otimes \wedge^{l(g)} N^g \right)^G.$$

2.3. The general cases. In the above Steps I-III, we have explained how to compute the Hochschild cohomology of $Poly(V) \rtimes G$ using the quasi-isomorphism L . We explain briefly how to generalize this construction to $C^\infty(M) \rtimes G$ by defining L to be

$$L : C^\bullet(C^\infty(M) \rtimes G, C^\infty(M) \rtimes G) \longrightarrow \Gamma^\infty \left(\bigoplus_{g \in G} \wedge^{\bullet-l(g)} TM^g \otimes \wedge^{l(g)} N^g \right)^G.$$

Firstly, we observe that the map L_1 is a quasi-isomorphism from $C^\bullet(A \rtimes G, A \rtimes G)$ to $(C^\bullet(A, A \rtimes G))^G$ that is true for any algebra A with a finite group action. Therefore, the map L_1 extends to the general case $C^\infty(M) \rtimes G$ naturally:

$$L_1 : C^\bullet(C^\infty(M) \rtimes G, C^\infty(M) \rtimes G) \longrightarrow \left(C^\bullet(C^\infty(M), C^\infty(M) \rtimes G) \right)^G.$$

Secondly, L_3 is generalized to the manifold case, as the map

$$L_3 : \left(\bigoplus_{g \in G} (\Gamma^\infty(\wedge^\bullet TM), \kappa \wedge) \right)^G \longrightarrow \left(\bigoplus_{g \in G} (\Gamma^\infty(\wedge^{\bullet-l(g)} TM^g \otimes \wedge^{l(g)} N^g), 0) \right)^G$$

obtained by composing the projection map $\wedge^\bullet TM|_{M^g} \rightarrow \wedge^{\bullet-l(g)} TM^g \otimes \wedge^{l(g)} N^g$ with the restriction map from $\wedge^\bullet TM$ to $\wedge^\bullet TM|_V$.

Thirdly, to generalize L_2 , we use Connes' map [4, Lemma 44] from the Koszul resolution of $C^\infty(M)$ to its bar resolution for a manifold with an affine structure. We can then define L_2^g from the Hochschild cochain complex $C^\bullet(C^\infty(M), C^\infty(M)_g)$ to $\wedge^\bullet TM$ in a similar way. There is a small issue here that the Connes construction [4, Lemma 44] only works for affine manifolds. Therefore, for a general manifold M , we need to realize the Hochschild cochain complex of $C^\infty(M) \rtimes G$ as a (pre)sheaf over the orbifold M/G , and use Čech techniques to compute the sheaf cohomology of this (pre)sheaf. In this framework, L will be a quasi-isomorphism of (pre)sheaves which is locally defined as the L_g was in Section 2.2, Step II. We refer to [15, Section 3] for details.

At the end of this subsection, we explain the following observation which makes our study easier. Recall Equation (1) concerning the Hochschild cohomology,

$$HH^\bullet(C^\infty(M) \rtimes G; C^\infty(M) \rtimes G) = \Gamma^\infty\left(\bigoplus_{g \in G} \wedge^{\bullet-l(g)} TM^g \otimes \wedge^{l(g)} N^g\right)^G.$$

As g is in the g -centralizer subgroup of G , the g -fixed point component's contribution to $HH^\bullet(C^\infty(M) \rtimes G, C^\infty(M) \rtimes G)$ has to be from g -invariant sections of $\Gamma^\infty(\wedge^{\bullet-l(g)} TV^g \otimes \wedge^{l(g)} N^g)$. As g acts on TV^g trivially, a g -invariant section of $\wedge^{\bullet-l(g)} TV^g \otimes \wedge^{l(g)} N^g$ must have g -invariant component in $\wedge^{l(g)} N^g$. Note that $\wedge^{l(g)} N^g$ is a line bundle over V^g . To have a nonzero g -invariant section, we need that the g -action on $\wedge^{l(g)} N^g$ must be trivial. This implies that $\det(g|_{N^g}) = 1$. Furthermore, we notice that $g|_{N^g}$'s action on N^g is of finite order and therefore can be diagonalized. If N^g is of odd dimension, then by the fact that $\det(g|_{N^g}) = 1$, we conclude that $g|_{N^g}$ has at least one eigenvalue equal to 1. This contradicts the assumption of N^g . Therefore, we conclude that if $\dim(N^g)$ is odd, there is no nonzero contribution to $HH^\bullet(C^\infty(M) \rtimes G, C^\infty(M) \rtimes G)$ from this g -fixed point component. Hence, the Hochschild cohomology of $C^\infty(M) \rtimes G$ has no contribution from g -fixed point submanifolds with odd $l(g)$. Therefore, we conclude

$$(2) \quad HH^\bullet(C^\infty(M) \rtimes G; C^\infty(M) \rtimes G) = \Gamma^\infty\left(\bigoplus_{\substack{g \in G, \\ l(g) \text{ is even}}} \wedge^{\bullet-l(g)} TM^g \otimes \wedge^{l(g)} N^g\right)^G.$$

3. HOCHSCHILD COHOMOLOGY AND QUASI-ISOMORPHISMS

In this section, we construct a quasi-isomorphism

$$T : \Gamma^\infty \left(\bigoplus_{g \in G} \wedge^{\bullet-l(g)} TM^g \otimes \wedge^{l(g)} N^g \right)^G \longrightarrow C^\bullet(C^\infty(M) \rtimes G; C^\infty(M) \rtimes G),$$

which is a quasi-inverse to the map L reviewed in Sections 2.2-2.3. First, we construct a twisted cocycle Ω_g for each element g associated to the determinant line bundle $\wedge^{l(g)} N^g$; second, we use the twisted cocycles to construct the map T . We mainly focus on the local case $\text{Poly}(V) \rtimes G$ and explain at the end how to generalize the construction to general manifolds.

3.1. Twisted cocycle. To construct the map T , we need to understand cocycles in $C^\bullet(\text{Poly}(V), A_g)$, which we call g -twisted cocycles. (Recall that A_g is a bimodule of $\text{Poly}(V)$ isomorphic to $\text{Poly}(V)$ as a vector space but with the g -twisted multiplication from the right (Section 2.2, Step II).) If we trace through the computation Steps I-III in Section 2.2, we see that $H^\bullet(\text{Poly}(V), A_g)$ is isomorphic to $\Gamma^\infty(\wedge^{\bullet-l(g)} TV^g \otimes \wedge^{l(g)} N^g)$. To construct the quasi-inverse of L , we need to associate to each element in $\Gamma^\infty(\wedge^{\bullet-l(g)} TV^g \otimes \wedge^{l(g)} N^g)$ a cocycle in $C^\bullet(\text{Poly}(V), A_g)$. In particular, we need to associate to an element in $\wedge^{l(g)} N^g$ a degree $l(g)$ cocycle in $C^\bullet(\text{Poly}(V), A_g)$. It is well known that an element in $\Gamma^\infty(\wedge^{\bullet-l(g)} TV^g \otimes \wedge^{l(g)} N^g)$ can be viewed as a multi-differential operator on V . A natural guess is that this multi-differential operator might be a cocycle in $C^\bullet(\text{Poly}(V), A_g)$. However, one can easily check that such a multi-differential operator is not a cocycle in $C^\bullet(\text{Poly}(V), A_g)$ except in the case $g = 1$. (For instance, because the right module structure on A_g is twisted by the g -action, a degree 1 cocycle ξ in $C^\bullet(\text{Poly}(V), A_g)$ must satisfy $\xi(ab) = a\xi(b) + \xi(a)g(b)$. One can quickly check that when $g \neq 1$, no nonzero vector fields on V can satisfy such an equation.) Therefore, we have to modify the above natural guess so that the outcome is a cocycle in $C^\bullet(\text{Poly}(V), A_g)$. This leads us to the following construction.

Recall that $V = V^g \oplus N^g$. We choose coordinates $x^1, \dots, x^{n-l(g)}, y^1, \dots, y^{l(g)}$ on V ($n = \dim(V)$), such that $x^1, \dots, x^{n-l(g)}$ are coordinates on V^g and $y^1, \dots, y^{l(g)}$ are coordinates on N^g . We assume that the g -action on $y^1, \dots, y^{l(g)}$ is diagonalized.¹

For any $\sigma \in S_{l(g)}$, the permutation group of $l(g)$ elements, we introduce the following vectors in N^g . Let $(\tilde{y}^1, \dots, \tilde{y}^{l(g)}) = g(y^1, \dots, y^{l(g)})$. Define

$$\begin{aligned} z_\sigma^0 &= (y^1, \dots, y^{l(g)}), & z_\sigma^1 &= (y^1, \dots, \tilde{y}^{\sigma(1)}, \dots), \\ z_\sigma^2 &= (y^1, \dots, \tilde{y}^{\sigma(1)}, \dots, \tilde{y}^{\sigma(2)}, \dots), \\ z_\sigma^{l(g)-1} &= (\tilde{y}^1, \dots, y^{\sigma(l(g))}, \dots), & z_\sigma^{l(g)} &= (\tilde{y}^1, \dots, \tilde{y}^{l(g)}). \end{aligned}$$

Let Ω_g be an $l(g)$ -cochain in $C^\bullet(A, A_g)$ defined as follows:

$$\begin{aligned} (3) \quad \Omega_g(f_1, \dots, f_{l(g)})(x, y) &:= \frac{1}{l(g)!} \sum_{\sigma \in S_{l(g)}} (-1)^\sigma \\ &\cdot \frac{(f_1(x, z_\sigma^0) - f_1(x, z_\sigma^1))(f_2(x, z_\sigma^1) - f_2(x, z_\sigma^2)) \cdots (f_{l(g)}(x, z_\sigma^{l(g)-1}) - f_{l(g)}(x, z_\sigma^{l(g)}))}{(y^1 - \tilde{y}^1) \cdots (y^{l(g)} - \tilde{y}^{l(g)})}. \end{aligned}$$

¹We can pass to the complex coordinates to achieve such a choice.

We remark that when σ is the identity permutation, we have

$$\begin{aligned} z_{id}^0 &= (y^1, \dots, y^{l(g)}), & z_{id}^1 &= (\tilde{y}^1, \dots, y^{l(g)}), \\ & & & \dots\dots\dots \\ z_{id}^{l(g)-1} &= (\tilde{y}^1, \dots, \tilde{y}^{l(g)-1}, y^{l(g)}), & z_{id}^{l(g)} &= (\tilde{y}^1, \dots, \tilde{y}^{l(g)}). \end{aligned}$$

The corresponding contribution in the summation of expression (3) is

$$\frac{(f_1(x, z_{id}^0) - f_1(x, z_{id}^1))(f_2(x, z_{id}^1) - f_2(x, z_{id}^2)) \cdots (f_{l(g)}(x, z_{id}^{l(g)-1}) - f_{l(g)}(x, z_{id}^{l(g)}))}{(y^1 - \tilde{y}^1) \cdots (y^{l(g)} - \tilde{y}^{l(g)})},$$

which converges to

$$\frac{\partial}{\partial y^1} f_1(x, 0) \cdots \frac{\partial}{\partial y^{l(g)}} f_{l(g)}(x, 0),$$

as $y^1, \dots, y^{l(g)}$ go to 0.

Therefore the identity component in Equation (3) can be viewed as a g analog of the multi-differential operator

$$\frac{\partial}{\partial y^1} \otimes \cdots \otimes \frac{\partial}{\partial y^{l(g)}}.$$

Summing over all permutations, Ω_g can be viewed as a g -analog of the multi-differential operator

$$\Lambda_g = \frac{\partial}{\partial y^1} \wedge \cdots \wedge \frac{\partial}{\partial y^{l(g)}}.$$

It is straightforward to check that Ω_g is a cocycle in $C^{l(g)}(A, A_g)$.

The following is a simple property of Ω_g .

Lemma 3.1. *The twisted cocycle Ω_g satisfies the following properties:*

- (1) $\Omega_g(y^1 \otimes \cdots \otimes y^{l(g)}) = 1;$
- (2) $\Omega_g(y^1 \otimes \cdots \otimes c \otimes \cdots \otimes y^{l(g)}) = 0,$

when c is a constant function.

Proof. A straightforward check. $\square$

We remark that the expression (3) of Ω_g depends on the choices of coordinates $y^1, \dots, y^{l(g)}$. This makes Ω_g in general not invariant with respect to the conjugate action. However, we have the following property.

Proposition 3.2. *Let $C(g)$ be the centralizer subgroup of g , which acts on N^g . If the $C(g)$ -action on N^g is diagonalizable,² there is a natural construction of Ω_g such that*

$$h(\Omega_g) = \det(h|_{N^g})\Omega_g, \quad h \in C(g).$$

Proof. As a $C(g)$ -action on N^g is diagonalizable and g commutes with elements in $C(g)$, g and the $C(g)$ -action on N^g can be diagonalized simultaneously. Therefore, we can find coordinates $y^1, \dots, y^{l(g)}$ on N^g , which are eigenfunctions of g and the $C(g)$ -action. We define Ω_g using the coordinates y^i as in Equation (3). In particular, $\tilde{y}^i = g^i y^i$, and $h(\tilde{y}^i) = h(g^i y^i) = g^i h^i y^i$, where g^i and h^i are eigenvalues

²By a diagonalizable action, we mean that a $C(g)$ -action on N^g splits into a direct sum of 1-dim or 2-dim representations of $C(g)$.

of g and the h -action on y^i . Plugging these expressions in the definition of $h(\Omega_g)$, we obtain the equation

$$h(\Omega_g) = \det(h|_{N^g})\Omega_g, \quad h \in C(g).$$

□

There are two special cases that we know of in which the conditions assumed in Proposition 3.2 are satisfied:

- (1) Group G is abelian.
- (2) The codimension $l(g)$ is 1 or 2. $C(g)$ acts on N^g by isometry. When $l(g) = 1, 2$, the isometry group of N^g is abelian.

At the end of this subsection, we give an example of the twisted cyclic cocycle in a very special case.

Example 3.3. Let V be $\mathbb{R}$, and let $G = \mathbb{Z}/2\mathbb{Z} = \{id, e\}$ act on $\mathbb{R}$ by $e : x \mapsto -x$. In this case, $\Omega_e \in C^1(A, A_e)$ is defined to be

$$\Omega_e(f)(x) = \frac{f(x) - f(-x)}{2x}.$$

The cohomology $HH^\bullet(A, A_e)$ is computed to be

$$HH^\bullet(A, A_e) = \begin{cases} 0 & \bullet \neq 1, \\ \mathbb{R} & \bullet = 1, \end{cases}$$

where $HH^1(A, A_e)$ is generated by Ω_e .

3.2. Inverse map of L . We use the twisted cocycle constructed in the previous step to obtain an inverse map of L :

$$T : \Gamma^\infty\left(\bigoplus_{g \in G} \wedge^{\bullet-l(g)} TV^g \otimes \wedge^{l(g)} N^g\right)^G \longrightarrow C^\bullet(Poly(V) \rtimes G, Poly(V) \rtimes G).$$

We define T as a composition of two maps T_1 and T_2 . The map T_1 is defined as

$$T_1 : \Gamma^\infty\left(\bigoplus_{g \in G} \wedge^{\bullet-l(g)} TV^g \otimes \wedge^{l(g)} N^g\right)^G \longrightarrow C^\bullet(A, Poly(V) \rtimes G)^G.$$

The map T_2 is the standard map constructed in the proof of the Eilenberg-Zilber theorem:

$$T_2 : C^\bullet(A, Poly(V) \rtimes G)^G \subset C^0(G, C^\bullet(A, Poly(V) \rtimes G)) \longrightarrow C^\bullet(A \rtimes G, Poly(V) \rtimes G).$$

Step I. The map T_1 is a sum of the maps

$$T_1^g : \Gamma^\infty(\wedge^{\bullet-l(g)} TV^g \otimes \wedge^{l(g)} N^g) \longrightarrow C^\bullet(A, A_g),$$

which are defined as follows.

Given $\xi \in \Gamma^\infty(\wedge^{\bullet-l(g)} TV^g \otimes \wedge^{l(g)} N^g)$, we write ξ to be $X \otimes \Lambda_g$, where $X \in \Gamma^\infty(\wedge^{\bullet-l(g)} TV^g)$ and $\Lambda_g \in \Gamma^\infty(\wedge^{l(g)} N^g)$ is defined by

$$(4) \quad \frac{\partial}{\partial x^{n-l(g)+1}} \wedge \cdots \wedge \frac{\partial}{\partial x^n}.$$

We define

$$T_1^g(\xi) = X \sharp \Omega_g, \quad \text{for any } \xi \in \Gamma^\infty(\wedge^{k-l(g)} TV^g \otimes \wedge^{l(g)} N^g),$$

where we view X as a multi-differential operator on $\text{Poly}(V)$ for $X \in \wedge^{k-l(g)}TV^g$ and $X\sharp\Omega_g(f_1, \dots, f_k)$ equal to

$$X(f_1, \dots, f_{k-l(g)})\Omega_g(f_{k-l(g)+1}, \dots, f_k).$$

When the $C(g)$ -action on N^g is diagonalizable, as is explained in Proposition 3.2, we have

$$T_1^g(h(\xi)) = h(T_1^{h^{-1}gh}(\xi)), \quad \forall h \in G.$$

We define T_1 to be the sum of T_1^g . The restriction of T_1 to the G -invariant sections gives the desired map

$$T_1 : \Gamma^\infty\left(\bigoplus_{g \in G} \wedge^{\bullet-l(g)}TV^g \otimes \wedge^{l(g)}N^g\right)^G \longrightarrow C^\bullet(A, \text{Poly}(V) \rtimes G)^G.$$

When the $C(g)$ -action on N^g is not diagonalizable, then the image of the above T_1 map is not always G -invariant. Therefore, we need to replace T_1 by

$$\frac{1}{|G|} \sum_h h(T_1(\xi)).$$

Step II. We now explain the construction of T_2 , which is standard in the Eilenberg-Zilber theorem. Given $\Phi \in C^k(A, \text{Poly}(V) \rtimes G)$,

$$(5) \quad T_2(\Phi)(a_1 U_{g_1}, \dots, a_k U_{g_k}) = \Phi(a_1, \dots, g_1 \cdots g_{k-1}(a_k)) U_{g_1 \cdots g_k}.$$

By this Lemma 3.1, we have the following proposition for the map L_2 .

Proposition 3.4. *Given $\xi \in \Gamma^\infty(\bigoplus_g \wedge^{k-l(g)}TV^g \otimes \wedge^{l(g)}N^g)^G$, we write $\xi = \sum_g X_g \otimes \Lambda_g$, where Λ_g is defined as in Equation (4).*

The composition map $L_2 \circ T_1$ satisfies

$$L_2(T_1(\xi)) = \sum_g X_g \otimes \Lambda_g = \xi.$$

Proof. As L_2 is G -equivariant and ξ is G -invariant, we compute $L_2(T_1(\xi))(x)$ as follows: $L_2(T_1(\xi))(x) =$

$$\begin{aligned} &= \sum_{i_1, \dots, i_k} T_1(\xi)((x_1 - x)^{i_1} \cdots (x_k - x)^{i_k}) \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}} \\ &= \sum_g \sum_{i_1, \dots, i_k} T_1(X_g \otimes \Lambda_g)((x_1 - x)^{i_1} \cdots (x_k - x)^{i_k}) \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}} \\ &= \sum_g \sum_{i_1, \dots, i_k} X_g((x_1 - x)^{i_1}, \dots, (x_{k-l(g)} - x)^{i_{k-l(g)}}) \\ &\quad \times \Omega_g((x_{k-l(g)+1} - x)^{i_{k-l(g)+1}}, \dots, (x_k - x)^{i_k}) \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}} \\ &= \sum_g \sum_{i_1, \dots, i_k} X_g((x_1 - x)^{i_1}, \dots, (x_{k-l(g)} - x)^{i_{k-l(g)}}) \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_{k-l(g)}}} \\ &\quad \otimes \Omega_g((x_{k-l(g)+1} - x)^{i_{k-l(g)+1}}, \dots, (x_k - x)^{i_k}) \frac{\partial}{\partial x^{i_{k-l(g)+1}}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}} \\ &= \sum_g X_g \otimes \Lambda_g. \end{aligned}$$

In the last equality, we used Lemma 3.1, which shows Λ_g and Ω_g have the same values on linear functions. $\square$

We define $T = T_2 \circ T_1$, and have the following theorem.

Theorem 3.5. *The map T is a quasi-isomorphism. In particular, $L \circ T = id$.*

Proof. We notice that $L_1(T_2) = id$ on $C^\bullet(A; A \rtimes G)^G$, and therefore have

$$L \circ T(\xi) = L_3(L_2(L_1(T_2(T_1(\xi)))) = L_3(L_2(T_1(\xi))),$$

which is equal to ξ by Proposition 3.4. $\square$

3.3. The case of a smooth manifold. In this subsection, we discuss the extension of the construction of T to general manifolds, which is again a composition of T_2 and T_1 .

The map T_2 generalizes to the manifold case directly because it is purely algebraic. The same formula as Equation (5) defines a map

$$T_2 : \left(C^\bullet(C^\infty(M), C^\infty(M) \rtimes G) \right)^G \longrightarrow C^\bullet(C^\infty(M) \rtimes G, C^\infty(M) \rtimes G).$$

To define T_1 , we consider a g -invariant tubular neighborhood $\mathcal{M}^g$ of M^g . The neighborhood $\mathcal{M}^g$ is a fiber bundle over M^g , and we fix a G -invariant Ehresmann connection on $\mathcal{M}^g$. Furthermore, we choose a cutoff function ρ_g on $\mathcal{M}^g$ which is equal to 1 on a g -invariant neighborhood of M^g and vanishes outside $\mathcal{M}^g$. Given a section $\xi_g = X_g \otimes \Lambda_g$ of $\wedge^{\bullet-l(g)} TM^g \otimes \wedge^{l(g)} N^g$ to $\mathcal{M}^g$, we use the Ehresmann connection to extend X_g to a multi-vector field $\tilde{X}_g$ on $\mathcal{M}^g$, and we define Ω_g to be a linear map on $C^\infty(\mathcal{M}^g)$ by the same formula as Equation (3).

We define a cochain map

$$T_1 : \Gamma^\infty\left(\bigoplus_g \wedge^{k-l(g)} TM^g \otimes \wedge^{l(g)} N^g\right)^G \longrightarrow C^k(C^\infty(M); C^\infty(M) \rtimes G)^G$$

by

$$T_1(\xi)(f_1, \dots, f_k) = \sum_g \rho_g \tilde{X}_g(f_1, \dots, f_{k-l(g)}) \Omega_g(f_{k-l(g)+1}, \dots, f_k) U_g,$$

for $\xi = \sum_g X_g \otimes \Lambda_g$.

Again, we can easily compute that $L \circ T = id$. Therefore, since L is a quasi-isomorphism proved in [13], T is also a quasi-isomorphism.

Remark 3.6. The definition of T_1 depends on a choice of the normal bundle N^g , the G -invariant Ehresmann connection, and the cutoff function. Therefore, T is not a canonical map. However, we notice that at any $x \in M^g$, inside $T_x M$, there is a canonical complementary subspace to $T_x M^g \subset T_x M$ determined by the representation of g on $T_x M$ independent of the choices of the metrics. Therefore, it is easy to check (cf. Proposition 3.4) that when restricted to the ∞ -jets of M^g , the map T is independent of all the choices.

4. GERSTENHABER BRACKET

In this section, we compute the Gerstenhaber bracket on the Hochschild cohomology of the algebra $C^\infty(M) \rtimes G$. Since all the computations and constructions are local with respect to the orbifold M/G , it is sufficient to work out everything locally on a vector space. Because the Gerstenhaber bracket on $C^\bullet(Poly(V) \rtimes G, Poly(V) \rtimes G)$ is the commutator of the pre-Lie product, we mainly work on understanding the pre-Lie product, and we state the results for the Gerstenhaber bracket.

4.1. Geometric description. Let $\xi \in \Gamma^\infty(\bigoplus_{\alpha \in G} \wedge^{k-l(\alpha)} TV^\alpha \otimes \wedge^{l(\alpha)} N^\alpha)^G$ and $\eta \in \Gamma^\infty(\bigoplus_{\beta \in G} \wedge^{l-l(\beta)} TV^\beta \otimes \wedge^{l(\beta)} N^\beta)^G$. We compute the pre-Lie product between $T(\xi)$ and $T(\eta)$ by $L(T(\xi) \circ T(\eta))$. We write ξ as the sum of $\xi_\alpha = X_\alpha \otimes \Lambda_\alpha$, and η as the sum of $\eta_\beta = Y_\beta \otimes \Lambda_\beta$, with $X_\alpha \in \Gamma^\infty(\wedge^{k-l(\alpha)} TV^\alpha)$, $Y_\beta \in \Gamma^\infty(\wedge^{l-l(\beta)} TV^\beta)$, $\Lambda_\alpha \in \Gamma^\infty(\wedge^{l(\alpha)} N^\alpha)$, $\Lambda_\beta \in \Gamma^\infty(\wedge^{l(\beta)} N^\beta)$. We compute the Gerstenhaber bracket between $T(\xi)$ and $T(\eta)$ by $L([T(\xi), T(\eta)])$ using the information of the quasi-isomorphisms L and T introduced in Sections 2 and 3.

Step I. We compute $L_1(T(\xi) \circ T(\eta)) \in C^{k+l-1}(Poly(V), Poly(V) \rtimes G)$ first. As ξ and η are both G -invariant, the cocycles $T_1(\xi), T_1(\eta) \in C^\bullet(Poly(V), Poly(V) \rtimes G)$ are also G -invariant, and therefore the averaging in the definition of L_1 is not necessary. Our computations yield:

$$\begin{aligned} & L_1(T(X) \circ T(Y))(f_1, \dots, f_{k+l-1}) \\ &= \sum_s (-1)^{(s-1)(l-1)} T(\xi)(f_1, \dots, f_s, T(\eta)(f_{s+1}, \dots, f_{s+l}), f_{s+l+1}, \dots, f_{k+l-1}) \\ &= \sum_s (-1)^{(s-1)(l-1)} T(\sum_\alpha \xi_\alpha)(f_1, \dots, f_s, T(\sum_\beta \eta_\beta)(f_{s+1}, \dots, f_{s+l}), f_{s+l+1}, f_{k+l-1}) \\ &= \sum_\alpha \sum_\beta \sum_s (-1)^{(s-1)(l-1)} T(\xi_\alpha)(f_1, \dots, f_s, T(\eta_\beta)(f_{s+1}, \dots, f_{s+l}), \\ & \quad f_{s+l+1}, \dots, f_{k+l-1}) \\ &= \sum_g \sum_{\alpha\beta=g} \sum_s (-1)^{(s-1)(l-1)} T_1^\alpha(\xi_\alpha)(f_1, \dots, f_s, T_1^\beta(\eta_\beta)(f_{s+1}, \dots, f_{s+l}), \\ & \quad \beta(f_{s+l+1}), \dots, \beta(f_{k+l-1})) U_g. \end{aligned}$$

Step II. We compute $L_2^g(L_1(T(\xi) \circ T(\eta)))(x)$ by applying L_2^g to the computation in the previous step:

$$\begin{aligned} (6) \quad & \frac{1}{|G|^2} \sum_{i_1, \dots, i_{k+l-1}} \sum_{\alpha\beta=g, h_1, h_2 \in G} \sum_s (-1)^{(s-1)(l-1)} h_1(T_1^\alpha(\xi_\alpha))((x_1 - x)^{i_1}, \dots, (x_s - x)^{i_s}, \\ & \quad h_2(T_1^\beta(\eta_\beta))((x_{s+1} - x)^{i_{s+1}}, \dots, (x_{s+l} - x)^{i_{s+l}}), \beta((x_{s+l+1} - x)^{i_{s+l+1}}), \dots, \\ & \quad \beta((x_{k+l-1} - x)^{i_{k+l-1}})) \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_{k+l-1}}}. \end{aligned}$$

We look at the term $T_1^\beta(\eta_\beta)((x_{s+1} - x)^{i_{s+1}}, \dots, (x_{s+l} - x)^{i_{s+l}})$. Using the expression $\eta_\beta = X_\beta \otimes \Lambda_\beta$, we have

$$\begin{aligned} & T_1^\beta(\eta_\beta)((x_{s+1} - x)^{i_{s+1}}, \dots, (x_{s+l} - x)^{i_{s+l}}) \\ &= X_\beta \sharp \Omega_\beta((x_{s+1} - x)^{i_{s+1}}, \dots, (x_{s+l} - x)^{i_{s+l}}) \\ &= X_\beta((x_{s+1} - x)^{i_{s+1}}, \dots, (x_{s+l-l(\beta)} - x)^{i_{s+l-l(\beta)}}) \\ & \quad \times \Lambda_\beta((x_{s+l-l(\beta)} - x)^{i_{s+l-l(\beta)+1}}, \dots, (x_{s+l} - x)^{i_{s+l}}) \\ &= \eta_\beta((x_{s+1} - x)^{i_{s+1}}, \dots, (x_{s+l} - x)^{i_{s+l}}). \end{aligned}$$

In the second equality of the above equation, we used Lemma 3.1, which stated that Ω_β and Λ_β agree on linear functions.

Substituting the above expression of the $T_1^\beta(\eta_\beta)$ into Equation (6), as η is G -invariant, we have

$$\begin{aligned} & L_2^g(T(\xi_\alpha) \circ T(\eta_\beta)) \\ &= \frac{1}{|G|} \sum_h \sum_{i_1, \dots, i_{k+l-1}} \sum_s (-1)^{(s-1)(l-1)} h(T_1^\alpha(\xi_\alpha)) \left((x_1 - x)^{i_1}, \dots, (x_s - x)^{i_s}, \right. \\ & \quad \eta_\beta((x_{s+1} - x)^{i_{s+1}}, \dots, (x_{s+l} - x)^{i_{s+l}}), \beta((x_{s+l+1} - x)^{i_{s+l+1}}), \dots, \\ & \quad \left. \beta((x_{k+l-1} - x)^{i_{k+l-1}}) \right) \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_{k+l-1}}}. \end{aligned}$$

We discuss several properties of $L_2^g(T(\xi_\alpha) \circ T(\eta_\beta))$.

Lemma 4.1. *The restriction of $L(T(\xi_\alpha) \circ T(\eta_\beta))$ to $V^\alpha \cap V^\beta$ is*

$$\begin{aligned} & \sum_{i_1, \dots, i_{k+l-1}} \sum_s (-1)^{(s-1)(l-1)} \xi_\alpha \left((x_1 - x)^{i_1}, \dots, (x_s - x)^{i_s}, \right. \\ (7) \quad & \eta_\beta((x_{s+1} - x)^{i_{s+1}}, \dots, (x_{s+l} - x)^{i_{s+l}}), \beta((x_{s+l+1} - x)^{i_{s+l+1}}), \dots, \\ & \left. \beta((x_{k+l-1} - x)^{i_{k+l-1}}) \right) \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_{k+l-1}}}. \end{aligned}$$

Proof. Because $T_1(\xi_\alpha) \in C^k(Poly(V); Poly(V)_\alpha)$ and $T_1(\eta_\beta) \in C^l(Poly(V); Poly(V)_\beta)$, the cocycle $T_1(\xi_\alpha) \circ T_1(\eta_\beta)$ is in $C^{k+l-1}(Poly(V); Poly(V)_{\alpha\beta})$. Therefore, the restriction of $L(T(\xi_\alpha) \circ T(\eta_\beta))$ to $V^\alpha \cap V^\beta$ is the same as the restriction of $L_2^{\alpha\beta}(T_1(\xi_\alpha) \circ T_1(\eta_\beta))$ to $V^\alpha \cap V^\beta$.

We observe that the restriction of $\Omega_\alpha(f_1, \dots, f_{l(\alpha)})$ (and $\Omega_\beta(\dots)$) to V^α (and V^β) is the same as $\Lambda_\alpha(f_1, \dots, f_{l(\alpha)})$ (and $\Lambda_\beta(\dots)$) as we set all the variables along the normal direction of V^α equal to 0. Using this property and the G -invariance of ξ , we have that the restriction of

$$\begin{aligned} & T_1(\xi_\alpha) \left((x_1 - x)^{i_1}, \dots, (x_s - x)^{i_s}, T_1(\eta_\beta) \left((x_{s+1} - x)^{i_{s+1}}, \dots, (x_{s+l} - x)^{i_{s+l}} \right), \right. \\ & \left. \beta((x_{s+l+1} - x)^{i_{s+l+1}}), \dots, \beta((x_{k+l-1} - x)^{i_{k+l-1}}) \right) \end{aligned}$$

to $V^\alpha \cap V^\beta$ is the same as

$$\begin{aligned} & \xi_\alpha \left((x_1 - x)^{i_1}, \dots, (x_s - x)^{i_s}, \eta_\beta \left((x_{s+1} - x)^{i_{s+1}}, \dots, (x_{s+l} - x)^{i_{s+l}} \right), \right. \\ & \left. \beta((x_{s+l+1} - x)^{i_{s+l+1}}), \dots, \beta((x_{k+l-1} - x)^{i_{k+l-1}}) \right). \quad \square \end{aligned}$$

Inspired by the results of Lemma 4.1, we introduce the following definition.

Definition 4.2. Let V be a vector space with a linear endomorphism γ . For all $\xi \in \wedge^k TV$, $\eta \in \wedge^l TV$, the γ -twisted pre-Lie product $\xi \circ_\gamma \eta$ is defined to be

$$\begin{aligned} & \sum_{i_1, \dots, i_{k+l-1}} \sum_s (-1)^{(s-1)(l-1)} \xi \left((x_1 - x)^{i_1}, \dots, (x_s - x)^{i_s}, \right. \\ & \eta \left((x_{s+1} - x)^{i_{s+1}}, \dots, (x_{s+l} - x)^{i_{s+l}} \right), \gamma \left((x_{s+l+1} - x)^{i_{s+l+1}} \right), \dots, \\ & \left. \gamma \left((x_{k+l-1} - x)^{i_{k+l-1}} \right) \right) (x) \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_{k+l-1}}}. \end{aligned}$$

Step III. In the following, we look at the $\alpha\beta$ component of the Gerstenhaber bracket in the case when $V^\alpha \cap V^\beta = V^{\alpha\beta}$.

Lemma 4.3. *Assume $V^\alpha \cap V^\beta = V^{\alpha\beta}$. If $\xi_\alpha = X_\alpha \otimes \Lambda_\alpha$ contains directions along N^β , then the contribution of ξ_α and η_β to the $\alpha\beta$ component of $L(T(\xi_\alpha) \circ T(\eta_\beta))$ vanishes.*

Proof. Firstly, as $V^{\alpha\beta} = V^\alpha \cap V^\beta$, we can apply Lemma 3.1 to compute $L(T(\xi_\alpha) \circ T(\eta_\beta))$ using the twisted pre-Lie product between ξ_α and η_β .

Secondly, we look at the term $\eta_\beta(\cdots)$ in $X_\alpha \circ_\alpha \eta_\beta$ at $g = \alpha\beta$. It contains derivations along all the directions of N^β .

Thirdly, by Lemma 3.1, if the component of $L(T(\xi_\alpha) \circ T(\eta_\beta))$ at $\frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_{k+l-1}}}$ does not vanish, we must have at least one of $(x_1 - x)^{i_1}, \dots, (x_s - x)^{i_s}, \eta_\beta((x_{s+1} - x)^{i_{s+1}}, \dots, (x_{s+l} - x)^{i_{s+l}}), \beta((x_{s+l+1} - x)^{i_{s+l+1}}), \dots, \beta((x_{k+l-1} - x)^{i_{k+l-1}})$ which is supported along the N^β direction because $X_\alpha \otimes \Lambda_\alpha$ contains derivations along N^β . Furthermore, we notice that $\eta_\beta((x_{s+1} - x)^{i_{s+1}}, \dots, (x_{s+l} - x)^{i_{s+l}})$ lies along the V^β direction. This implies that at least one of $(x_1 - x)^{i_1}, \dots, (x_s - x)^{i_s}, \beta((x_{s+l+1} - x)^{i_{s+l+1}}), \dots, \beta((x_{k+l-1} - x)^{i_{k+l-1}})$ must be along the N^β direction. Noticing that the action of β on V preserves the decomposition $V = V^\beta \oplus N^\beta$, we conclude that at least one of $(x_1 - x)^{i_1}, \dots, (x_s - x)^{i_s}, (x_{s+l+1} - x)^{i_{s+l+1}}, \dots, (x_{k+l-1} - x)^{i_{k+l-1}}$ is along N^β .

Summarizing the above observations, we see that $T(X_\alpha \otimes \Lambda_\alpha) \circ T(Y_\beta \otimes \Lambda_\beta)$ contains too many derivations along N^β since in the above two parts we have a total of more than $l(\beta)$ many derivations along the N^β direction, which is of dimension $l(\beta)$. We conclude that

$$L(T(X_\alpha \otimes \Lambda_\alpha) \circ T(Y_\beta \otimes \Lambda_\beta)) = L(T(\xi_\alpha) \circ T(\eta_\beta)) = 0.$$

□

Lemma 4.3 shows that to compute the $\alpha\beta$ component of $L(T(\xi_\alpha) \circ T(\eta_\beta))$ we can assume that ξ_α is contained in the V^β direction. Therefore, there is no need to consider the β -action on $(x_{s+l+1} - x)^{i_{s+l+1}}, \dots, (x_{k+l-1} - x)^{i_{k+l-1}}$ in Equation (7), which concerns $L(T(\xi_\alpha) \circ T(\eta_\beta))$. In this case, the twisted pre-Lie product between ξ_α and η_β is reduced to

$$\sum_{i_1, \dots, i_{k+l-1}} \sum_s (-1)^{(s-1)(l-1)} \xi_\alpha((x_1 - x)^{i_1}, \dots, (x_s - x)^{i_s}, \eta_\beta((x_{s+1} - x)^{i_{s+1}}, \dots, (x_{s+l} - x)^{i_{s+l}}, (x_{s+l+1} - x)^{i_{s+l+1}}, \dots, (x_{k+l-1} - x)^{i_{k+l-1}})) \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_{k+l-1}}}.$$

This is the standard pre-Lie product $\xi_\alpha \circ \eta_\beta$ between ξ_α and η_β .

We summarize the above computation into the following theorem.³

Theorem 4.4. *Consider*

$$\begin{aligned} \xi &= \sum_\alpha \xi_\alpha \in \Gamma^\infty(\bigoplus_\alpha \wedge^{k-l(\alpha)} TV^\alpha \otimes \wedge^{l(\alpha)} N^\alpha)^G, \\ \eta &= \sum_\beta \eta_\beta \in \Gamma^\infty(\bigoplus_\beta \wedge^{l-l(\beta)} TV^\beta \otimes \wedge^{l(\beta)} N^\beta)^G. \end{aligned}$$

³A theorem similar to the one that follows was stated in the first version of [1], but the proof there contained a crucial gap.

If $V^\alpha \cap V^\beta = V^{\alpha\beta}$, then the contribution of ξ_α and η_β to the $\alpha\beta$ component of $L(T(\xi_\alpha) \circ T(\eta_\beta))$ is

$$pr^{\alpha\beta}(\xi_\alpha \circ \eta_\beta|_{V^{\alpha\beta}}),$$

where $\circ$ in the above formula is the standard pre-Lie product on V .

Proof. The proof is straightforward from Lemma 4.3. $\square$

We discuss a special but important case that $V^\alpha \cap V^\beta = V^{\alpha\beta}$ holds true. The following lemma is well known, and we include its proof for the reader's convenience.

Lemma 4.5. For $\alpha, \beta \in G$, the condition $V^\alpha + V^\beta = V$ is equivalent to the equality $l(\alpha) + l(\beta) = l(\alpha\beta)$, and when $V^\alpha + V^\beta = V$, $V^\alpha \cap V^\beta = V^{\alpha\beta}$.

Proof. As $\dim(V^\alpha) + \dim(V^\beta) = \dim(V^\alpha + V^\beta) + \dim(V^\alpha \cap V^\beta)$, we have that

$$\begin{aligned} l(\alpha) + l(\beta) &= \dim(V) - \dim(V^\alpha) + \dim(V) - \dim(V^\beta) \\ &= \dim(V) - \dim(V^\alpha + V^\beta) + \dim(V) - \dim(V^\alpha \cap V^\beta) \\ &\geq \dim(V) - \dim(V^\alpha + V^\beta) + \dim(V) - \dim(V^{\alpha\beta}) \\ &= \dim(V) - \dim(V^\alpha + V^\beta) + l(\alpha\beta), \end{aligned}$$

where we have used the fact that $V^\alpha \cap V^\beta \subset V^{\alpha\beta}$. Therefore, $l(\alpha) + l(\beta) = l(\alpha\beta)$ implies that $V = V^\alpha + V^\beta$.

On the other hand, assume that $V^\alpha + V^\beta = V$ and let $\langle \cdot, \cdot \rangle$ be a G -invariant metric on V . For any $v \in V^{\alpha\beta}$, we have $\alpha\beta(v) = v$, and accordingly $\beta(v) = \alpha^{-1}(v)$, and $\beta(v) - v = \alpha^{-1}(v) - v$. Furthermore, as the metric $\langle \cdot, \cdot \rangle$ is G -invariant, we see that $\beta(v) - v$ is orthogonal to V^β with respect to the metric $\langle \cdot, \cdot \rangle$ and $\alpha^{-1}(v) - v$ is orthogonal to V^α . Therefore $\beta(v) - v = \alpha^{-1}(v) - v$ is orthogonal to $V^\alpha + V^\beta$, which is equal to V by the assumption. This implies that v must belong to $V^\alpha \cap V^\beta$, and we have $V^\alpha \cap V^\beta = V^{\alpha\beta}$. This together with the above equations implies that

$$l(\alpha) + l(\beta) = \dim(V) - \dim(V^\alpha + V^\beta) + \dim(V) - \dim(V^{\alpha\beta}) = l(\alpha\beta).$$

$\square$

Example 4.6. We consider the set S of all $g \in G$ such that $l(g) = 2$. The set S is invariant under conjugation as $l(g)$ is invariant under conjugate action. We consider $\pi = \sum_{g \in S} \pi_g \in \Gamma^\infty(\bigoplus_{g \in S} \wedge^2 N^g)^G$ and $\phi_{id} \in \Gamma^\infty(\wedge^\bullet V)^G$, which is supported at the identity component.

For π and ϕ_{id} , the conditions of Theorem 4.4 are satisfied. Therefore, we have

$$L(T(\pi) \circ T(\phi_{id})) = \sum_{g \in S} pr^g(\pi_g \circ \phi_{id}|_{V^g})$$

and

$$L([T(\pi), T(\phi_{id})]) = \sum_{g \in S} pr^g([\pi_g, \phi_{id}]|_{V^g}).$$

4.2. Abelian group action. In the following, we discuss the special case when G is abelian. Under this assumption, we obtain a more explicit description of the twisted Schouten-Nijenhuis bracket. As G is abelian, the G -action on V is decomposed into a direct sum of 1 and 2 dimensional irreducible representations of G , and we have global well-defined coordinate functions, which are eigenvectors for all g -actions. In particular, the conditions of Proposition 3.2 are satisfied and the map T_1 in Section 3.2 without averaging is already G -equivariant.

Remark 4.7. Discussions in this subsection can be extended to the case where all elements in (α) (the set of elements in the same conjugacy class as α) commute with all elements in (β) .

Lemma 4.8. *Let*

$$\xi = \sum_{\alpha} X_{\alpha} \otimes \Lambda_{\alpha} \in \Gamma^{\infty}(\bigoplus_{\alpha} \wedge^{k-l(\alpha)} TV^{\alpha} \otimes \wedge^{l(\alpha)} N^{\alpha})^G$$

and

$$\eta = \sum_{\beta} Y_{\beta} \otimes \Lambda_{\beta} \in \Gamma^{\infty}(\bigoplus_{\beta} \wedge^{l-l(\beta)} TV^{\beta} \otimes \wedge^{l(\beta)} N^{\beta})^G.$$

If $V^{\alpha} + V^{\beta} \neq V$ for all α, β , then

$$L(T(\xi) \circ T(\eta)) = 0.$$

Proof. Following the computations similar to those in Equation (6), we have that $L(T(\xi_{(\alpha)}) \circ T(\eta_{(\beta)}))$ is equal to

$$\begin{aligned} & \sum_{i_1, \dots, i_{k+l-1}} \sum_{\alpha \beta = g} \sum_s (-1)^{(s-1)(l-1)} T_1^{\alpha}(\xi_{\alpha}) \left((x_1 - x)^{i_1}, \dots, (x_s - x)^{i_s}, \right. \\ & \quad \left. T_1^{\beta}(\eta_{\beta}) \left((x_{s+1} - x)^{i_{s+1}}, \dots, (x_{s+l} - x)^{i_{s+l}} \right), \right. \\ & \quad \left. \beta((x_{s+l+1} - x)^{i_{s+l+1}}), \dots, \beta((x_{k+l-1} - x)^{i_{k+l-1}}) \right) \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_{k+l-1}}}. \end{aligned}$$

Since $V^{\alpha} + V^{\beta} \neq V$, then its normal directions $N^{\perp} = N^{\alpha} \cap N^{\beta}$ are nontrivial. We observe that $T(\xi_{\alpha})$ contains all the derivations⁴ along N^{α} , and $T(\eta_{\beta})$ contains all the derivations along N^{β} . As η_{β} is a section of $\wedge^{l-l(\beta)} TV^{\beta} \otimes \wedge^{l(\beta)} N^{\beta}$, $T_1^{\beta}(\eta_{\beta})((x_{s+1} - x)^{i_{s+1}}, \dots, (x_{s+l} - x)^{i_{s+l}})$ is independent of $N^{\perp}$. Therefore, we see that to have a nonzero contribution in $L(T(\xi_{\alpha}) \circ T(\eta_{\beta}))$, the coordinates $x^{i_1}, \dots, x^{i_{s+l}}, \beta(x^{i_{s+l+1}}), \dots, \beta(x^{i_{k+l-1}})$ must contain two copies of the variables along $N^{\perp}$ and one copy of the variables along $N^{\beta}/N^{\perp}$. However, because $i_1, \dots, i_{k+l-1}$ are distinguished and N^{β} is β -invariant, the coordinates $x^{i_1}, \dots, x^{s+k-1}, \beta(x^{s+k}), \dots, \beta(x^{i_{k+l-1}})$ have at most one copy of the variables along the N^{β} direction. There are not enough variables along the $N^{\perp}$ direction. This implies that $L(T(\xi_{(\alpha)}) \circ T(\eta_{(\beta)}))$ vanishes. $\square$

By Lemma 4.8, we are reduced to considering the Gerstenhaber bracket in the case when $V^{\alpha} + V^{\beta} = V$, which by Lemma 4.5 implies $V^{\alpha} \cap V^{\beta} = V^{\alpha\beta}$. Therefore, we can use Theorem 4.4 to compute the Gerstenhaber bracket.

Lemma 4.9. *Let G be an abelian group. If $l - l(\beta) \geq 2$, then for $Y_{\beta} \in \Gamma^{\infty}(\wedge^{l-l(\beta)} TN^{\alpha})^{C(\beta)}$,*

$$L(T(\xi_{\alpha}) \circ T(\eta_{\beta})) = 0.$$

Proof. Because of Lemma 4.8 and Theorem 4.4, we can drop the β twist in Equation (7). At the component $g = \alpha\beta$, we look at the number of derivations along the direction of N^{α} . The component $T(\xi_{\alpha})$ contributes $l(\alpha)$ and $T(Y_{\beta} \otimes \Lambda_{\beta})$ contributes $l - l(\beta)$ number of derivations. Therefore, $T(\xi_{\alpha}) \circ T(Y_{\beta} \otimes \Lambda_{\beta})$ contains at least the following number of derivations along N^{α} :

$$l(\alpha) + (l - l(\beta)) - 1 \geq l(\alpha) + 2 - 1 \geq l(\alpha) + 1.$$

This implies the statement of this lemma, because $\dim(N^{\alpha}) = l(\alpha)$. $\square$

⁴Rigorously speaking, $\Omega_{\alpha}, \Omega_{\beta}$ are not derivations. Here, we use the word “derivation” loosely, because they behave like derivations on linear functions.

In the case when G is abelian, the expressions of the pre-Lie product and Gerstenhaber bracket in Theorem 4.4 can be simplified.

Theorem 4.10. *Let*

$$\xi = \sum_{\alpha \in (\alpha)} X_\alpha \otimes \Lambda_\alpha \in \Gamma^\infty(\wedge^{k-l(\alpha)} TV^\alpha \otimes \wedge^{l(\alpha)} N^\alpha)^{C(\alpha)}$$

and

$$\eta = \sum_{\beta \in (\beta)} Y_\beta \otimes \Lambda_\beta \in \Gamma^\infty(\wedge^{l-l(\beta)} TV^\beta \otimes \wedge^{l(\beta)} N^\beta)^{C(\beta)}.$$

Then the component of $L(T(\xi) \circ T(\eta))$ in

$$\Gamma^\infty(\wedge^{k+l-l(\alpha)-1} TV^{\alpha\beta} \otimes \wedge^{l(\alpha)} N^{\alpha\beta})^{C(\alpha\beta)}$$

is computed as follows.

- (1) When $V^\alpha + V^\beta \neq V$ for all α, β , then $L(T(\xi) \circ T(\eta)) = 0$.
- (2) When $V^\alpha + V^\beta = V$, as G is abelian, we write $V = V^{\alpha\beta} \oplus N^\alpha \oplus N^\beta$, where $V^{\alpha\beta}$ is the invariant subspace of $\alpha\beta$, N^α is the subspace orthogonal to V^α , and N^β is the subspace orthogonal to V^β . In this case, $V^\alpha = V^{\alpha\beta} \oplus N^\beta$ and $V^\beta = V^{\alpha\beta} \oplus N^\alpha$. According to this decomposition, we write

$$\begin{aligned} X_\alpha &= \sum X_\alpha^{i_1 \dots i_s, p_1 \dots p_{k-l(\alpha)-s}} \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_s}} \wedge \frac{\partial}{\partial x^{p_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{p_{k-l(\alpha)-s}}} \\ &\in \Gamma^\infty(\wedge^s TV^{\alpha\beta}) \otimes \Gamma^\infty(\wedge^{k-l(\alpha)-s} TN^\beta), \\ Y_\beta &= \sum Y_\beta^{j_1 \dots j_t, q_1 \dots q_{l-l(\beta)-t}} \frac{\partial}{\partial x^{j_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{j_t}} \wedge \frac{\partial}{\partial x^{q_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{q_{l-l(\beta)-t}}} \\ &\in \Gamma^\infty(\wedge^t TV^{\alpha\beta}) \otimes \Gamma^\infty(\wedge^{k-l(\beta)-t} TN^\alpha). \end{aligned}$$

The component of $L(T(\xi_{(\alpha)}) \circ T(\eta_{(\beta)}))$ in

$$\Gamma^\infty(\wedge^{k+l-l(g)-1} TV^g \otimes \wedge^{l(g)} (N^\alpha \oplus N^\beta))^{C(g)}$$

is computed to be

$$\begin{aligned} &\sum_{\substack{g=\lambda\mu \\ \lambda \in (\alpha), \mu \in (\beta) \\ l(g)=l(\lambda)+l(\mu)}} \sum_{i_1, \dots, i_{k-l(\lambda)}, j_1, \dots, j_{l-l(\mu)}} (-1)^{(z-1)(l-1)+(k-z)l+(l-l(\mu))l(\lambda)} X_\lambda^{i_1 \dots \widehat{i_z} \dots i_{k-l(\lambda)}} \\ &\frac{\partial}{\partial x^{i_z}} Y_\mu^{j_1 \dots j_{l-l(\mu)}} \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \widehat{\frac{\partial}{\partial x^{i_z}}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_{k-l(\lambda)}}} \wedge \frac{\partial}{\partial x^{j_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{j_{l-l(\mu)}}} \otimes \Lambda_\lambda \wedge \Lambda_\mu \\ &+ \sum_{i_1, \dots, i_{k-l(\lambda)}, j_1, \dots, j_{l-l(\mu)-1}, q_z} (-1)^{(k-l(\lambda))(l-1)+(k-l(\mu))(l(\lambda)-1)} X_\lambda^{i_1 \dots i_{k-l(\lambda)}} \\ &\frac{\partial}{\partial x^{q_z}} Y_\mu^{j_1, \dots, j_{l-l(\mu)-1}, q_z} \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_{k-l(\lambda)}}} \wedge \frac{\partial}{\partial x^{j_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{j_{l-l(\mu)-1}}} \otimes \Lambda_\lambda \wedge \Lambda_\mu. \end{aligned}$$

Proof. The first statement is a corollary of Lemma 4.8. It remains to show the second statement.

As G is abelian, we can simultaneously diagonalize the α, β action on V . Using the fact that $V^\alpha + V^\beta = V$, we can write $V = V^{\alpha\beta} \oplus N^\alpha \oplus N^\beta$ such that $V^\alpha = V^{\alpha\beta} \oplus N^\beta$ and $V^\beta = V^{\alpha\beta} \oplus N^\alpha$.

By Lemma 4.3, we conclude that X_α must be from $\Gamma^\infty(\wedge^{k-l(\alpha)} TV^{\alpha\beta})$ to have a nontrivial contribution in $L([T(\xi_{(\alpha)}), T(\eta_{(\beta)})])$.

Similarly, by Lemma 4.9, we conclude that to have a nontrivial contribution in $L([T(\xi_\alpha), T(\eta_\beta)])$, the direction Y_β has to be from one of the following spaces:

$$(i) Y_\beta \in \Gamma^\infty(\wedge^{l-l(\beta)} TV^{\alpha\beta}), \quad (ii) Y_\beta \in \Gamma^\infty(\wedge^{l-l(\beta)-1} TV^{\alpha\beta}) \otimes \Gamma^\infty(TN^\alpha).$$

We apply Theorem 4.4 to compute the pre-Lie product. Because both $V^{\alpha\beta}$ and N^α are subspaces of V^β , which is the fixed point set of β , we can drop the β twist of the pre-Lie product. Therefore, we are left with the standard pre-Lie product.

When $Y_\beta \in \Gamma^\infty(\wedge^{l-l(\beta)} TV^{\alpha\beta})$, we have $L([T(\xi_\alpha), T(\eta_\beta)]) =$

$$\sum_{i_1, \dots, i_{k-l(\alpha)}, j_1, \dots, j_{l-l(\beta)}} (-1)^{(z-1)(l-1)+(k-z)l+(l-l(\beta))l(\alpha)} X_\alpha^{i_1 \dots \widehat{i_z} \dots i_{k-l(\alpha)}} \frac{\partial}{\partial x^{i_z}} Y_\beta^{j_1 \dots j_{l-l(\beta)}} \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \widehat{\frac{\partial}{\partial x^{i_z}}} \dots \wedge \frac{\partial}{\partial x^{i_{k-l(\alpha)}}} \wedge \frac{\partial}{\partial x^{j_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{j_{l-l(\beta)}}}.$$

When $Y_\beta \in \Gamma^\infty(\wedge^{l-l(\beta)-1} TV^{\alpha\beta}) \otimes \Gamma^\infty(TN^\alpha)$, we compute $L([T(\xi_\alpha), T(\eta_\beta)]) =$

$$\sum_{i_1, \dots, i_{k-l(\alpha)}, j_1, \dots, j_{l-l(\beta)-1}, q_z} (-1)^{(k-l(\alpha))(l-1)+(k-l(\beta))(l(\alpha)-1)} X_\alpha^{i_1 \dots i_{k-l(\alpha)}} \frac{\partial}{\partial q_z} Y_\beta^{j_1, \dots, j_{l-l(\beta)-1}, q_z} \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_{k-l(\alpha)}}} \wedge \frac{\partial}{\partial x^{j_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{j_{l-l(\beta)-1}}}.$$

□

Corollary 4.11. *Let G be an abelian group acting a manifold M . The (g) component of the twisted Schouten-Nijenhuis bracket is*

$$L([T(\xi), T(\eta)]) = pr^g \left(\sum_{\substack{g=\lambda\mu \\ \lambda \in (\alpha), \mu \in (\beta) \\ l(g)=l(\lambda)+l(\mu)}} [\tilde{\xi}_\lambda, \tilde{\eta}_\mu]_{M^{\lambda\mu}} \right).$$

5. NONCOMMUTATIVE POISSON STRUCTURE AND SYMPLECTIC REFLECTION ALGEBRAS

In this section, we want to find geometric expressions of all possible noncommutative Poisson structures on $C^\infty(M) \rtimes G$.

5.1. Noncommutative Poisson structure. We assume that the manifold M has a G -invariant symplectic structure. Because the group G is finite, there always exists a G -invariant compatible almost complex structure J on M . Accordingly, for any g in G , the fixed point manifold M^g is again a symplectic manifold with a compatible almost complex structure [9]. Therefore, all the fixed point manifolds $M^g, g \in G$ are of even dimension. Furthermore, the restriction of the $l(g)$ -th wedge power of the corresponding Poisson structure defines a global section on $\wedge^{l(g)} N^g$. Therefore, we can choose Λ_g to be constant for any $g \in G$.

Since all the fixed point manifolds have at least codimension equal to 2, there is no contribution of $\Gamma^\infty(\bigoplus_{g; l(g)=1} TM^g \otimes N^g)^G$ in the Hochschild cohomology of $C^\infty(M) \rtimes G$.

For a noncommutative Poisson structure on $C^\infty(M) \rtimes G$, we need to consider degree 2 Hochschild cohomology classes. Since there is no contribution from g -fixed point submanifolds with $l(g) = 1$ (Equation (2)), we have the following description of the degree 2 Hochschild cohomology of $C^\infty(M) \rtimes G$:

$$HH^2(C^\infty(M) \rtimes G, C^\infty(M) \rtimes G) = \Gamma^\infty(\wedge^2 TM)^G \oplus \Gamma^\infty\left(\bigoplus_{g \in G, l(g)=2} \wedge^2 N^g\right)^G,$$

where the G -action on the second component is by conjugation. Inspired by the above equation, we define the set S of elements $g \in G$ such that the fixed point subspace of g is of codimension 2. It is easy to see that S is closed under the conjugate action of G .

With the above discussion, we are ready to state the following geometric description of noncommutative Poisson structures on $C^\infty(M) \rtimes G$.

We assume that the manifold M has a G -invariant symplectic structure. As G is finite, there always exists a G -invariant compatible almost complex structure J on M . Therefore, for any g in G , the fixed point manifold M^g is a symplectic submanifold with a compatible almost complex structure [9] and accordingly is of even dimension. Furthermore, we have that the restriction of the $l(g)$ -th wedge power of the corresponding Poisson structure defines a global section on $\wedge^{l(g)} N^g$, which is G -invariant.

Theorem 5.1. *Assume that M is a symplectic manifold,⁵ with a G symplectic action. Let π be an element in $\Gamma^\infty(\wedge^2 TM)^G$, and $\sum_{g \in S} \Lambda_g$ be an element in $(\bigoplus_{g \in S} \wedge^2 N^g)^G$. Then $\pi + \sum_g \Lambda_g$ is a noncommutative Poisson structure on $C^\infty(M) \rtimes G$ if and only if*

- (1) on M , $[\pi, \pi] = 0$;
- (2) for any $g \in S$, $pr^g([\pi, \tilde{\Lambda}_g]|_{M^g}) = 0$.

In the above, we have assumed that there is no group element of G except the identity that is a stabilizer of an open subset of M . In this case, we call the G -action reduced.

Proof. We compute $L([T(\pi + \sum_{g \in S} \Lambda_g), T(\pi + \sum_{g \in S} \Lambda_g)])$. It decomposes into the sum of four terms:

$$\begin{aligned} & i) L([T(\pi), T(\pi)]), \quad ii) \sum_{\alpha \in S} L([T(\pi), T(\Lambda_\alpha)]), \\ & iii) \sum_{\alpha \in S} L([T(\Lambda_\alpha), T(\pi)]), \quad iv) \sum_{\alpha, \beta \in S} L([T(\Lambda_\alpha), T(\Lambda_\beta)]). \end{aligned}$$

We compute the above i)- iv) terms separately.

- (i) $L([T(\pi), T(\pi)])$. On the identity component, the Gerstenhaber bracket corresponds to the standard Schouten-Nijenhuis bracket. Therefore, we have

$$L([T(\pi), T(\pi)]) = [\pi, \pi],$$

which is again on the identity component.

⁵From the proof, we see that all we really need is a G -invariant almost complex structure on M .

- (ii) $\sum_{\alpha \in S} L([T(\pi), T(\Lambda_\alpha)])$. The identity component contributes π and the α component contributes Λ_α . We use Example 4.6 to compute these terms,

$$\sum_{\alpha \in S} L([T(\pi), T(\Lambda_\alpha)]) = \sum_{\alpha \in S} pr^\alpha([\pi, \tilde{\Lambda}_\alpha]_{M^\alpha}),$$

where $[\pi, \tilde{\Lambda}_\alpha]_{M^\alpha}$ is on the α component.

- (iii) $\sum_{\alpha \in S} L([T(\Lambda_\alpha), T(\pi)])$. This case is similar to the previous case. We apply Example 4.6 to compute the terms

$$\sum_{\alpha \in S} L([T(\Lambda_\alpha), T(\pi)]) = \sum_{\alpha \in S} pr^\alpha([\tilde{\Lambda}_\alpha, \pi]_{M^\alpha}),$$

where $[\tilde{\Lambda}_\alpha, \pi]_{M^\alpha}$ is on the α component.

- (iv) $\sum_{\alpha, \beta \in S} L([T(\Lambda_\alpha), T(\Lambda_\beta)])$. We need to use the assumption that M is a symplectic manifold. Accordingly, M^α is a symplectic submanifold of codimension 2. Furthermore, if we fix a G -invariant compatible almost complex structure on M , M^α is an almost complex submanifold. Hence, for each $x \in M^\alpha \cap M^\beta$ with $\alpha, \beta \in S$, $T_x M^\alpha$ and $T_x M^\beta$ are almost complex subspaces of $T_x M$. The sum $T_x M^\alpha + T_x M^\beta$ is again an almost complex subspace of $T_x M$ and satisfies one of the following properties:

- (a) $T_x M^\alpha + T_x M^\beta = T_x M$,
- (b) $T_x M^\alpha = T_x M^\beta = T_x M^{\alpha\beta}$,
- (c) $T_x M^\alpha = T_x M^\beta$ with $T_x M^{\alpha\beta} = T_x M$.

We discuss these three cases separately.

(a) If $T_x M^\alpha + T_x M^\beta = T_x M$, then by Lemma 4.5, $T_x M^{\alpha\beta} = T_x M^\alpha \cap T_x M^\beta$. Therefore, $M^{\alpha\beta}$ is a codimension 4 submanifold near x . We notice that $[T(\Lambda_\alpha), T(\Lambda_\beta)]$ is a 3-cocycle. By Equation (2) with $\bullet = 3$, we see that any contribution to the degree 3 Hochschild cohomology of $C^\infty(M) \rtimes G$ from a g -fixed point submanifold with $l(g) = 4$ is trivial. Hence, the $\alpha\beta$ component of $\sum_{\alpha, \beta \in S} L([T(\Lambda_\alpha), T(\Lambda_\beta)])(x)$ vanishes.

(b) If $T_x M^\alpha = T_x M^\beta = T_x M^{\alpha\beta}$, then $M^\alpha = M^\beta = M^{\alpha\beta} = M^\alpha \cap M^\beta$ near x . Therefore, we apply Lemma 4.1 to compute $L(T(\Lambda_\alpha) \circ T(\Lambda_\beta))$ and $L(T(\Lambda_\beta) \circ T(\Lambda_\alpha))$ by the twisted pre-Lie product (Definition 4.2). We observe that the normal bundles N^α and N^β are both symplectic orthogonal to $M^\alpha = M^\beta$. Therefore, N^α is the same as N^β . In Definition 4.2, since the β -action preserves $N^\beta = N^\alpha$, to have a nontrivial outcome we need all 3 terms $(x_s - x)^{i_s}$ for $s = 1, \dots, 3$ to be along the $N^\alpha = N^\beta$ direction.

This is because Λ_α and Λ_β both contains 2 derivations along $N^\alpha = N^\beta$. On the other hand, $N^\alpha = N^\beta$ is only of 2 dimensions. Any wedge product of 3 vectors along N^α vanishes. Therefore, the $\alpha\beta$ component of $\sum_{\alpha, \beta \in S} L([T(\Lambda_\alpha), T(\Lambda_\beta)])(x)$ vanishes.

(c) If $T_x M^\alpha = T_x M^\beta$ but $T_x M^{\alpha\beta} = T_x M$, then this shows that $\alpha\beta$ acts on $T_x M$ trivially and therefore there is a neighborhood of x which is totally fixed by $\alpha\beta$. Hence, by the assumption that the G -action on M is reduced, we know $\alpha\beta = 1$. (As $N^\alpha = N^\beta$ is of 2 dimensions, the action of the centralizer subgroups $C(\alpha)$ and $C(\beta)$ on N^α , which are subgroups of $U(1)$, can be diagonalized simultaneously. We choose the eigenfunctions of this action to be the coordinate functions. See Lemma 3.2.) We notice that

$T(\Lambda_\alpha)$ and $T(\Lambda_\beta)$ both contains 2 derivations along the $N^\alpha = N^\beta$ direction. It is not difficult to check that $L([T(\Lambda_\alpha), T(\Lambda_\beta)])$ contains the wedge product of 3 vector fields along the $N^\alpha = N^\beta$ direction. Therefore, the identity component of $L([T(\Lambda_\alpha), T(\Lambda_\beta)])(x)$ vanishes for the same reason as in part (b).

In summary, we have that

$$\begin{aligned} & L([T(\pi + \sum_{g \in S} \Lambda_g), T(\pi + \sum_g \Lambda_g)]) \\ &= [\pi, \pi] + \sum_{\alpha \in S} pr^\alpha([\pi, \tilde{\Lambda}_\alpha]|_{M^\alpha}) + \sum_{\alpha \in S} pr^\alpha([\tilde{\Lambda}_\alpha, \pi]|_{M^\alpha}) \\ &= [\pi, \pi] + 2 \sum_{\alpha \in S} pr^\alpha([\pi, \tilde{\Lambda}_\alpha]|_{M^\alpha}). \end{aligned}$$

Therefore, if $\pi + \sum_{g \in S} \Lambda_g$ is a noncommutative Poisson structure, then $[\pi, \pi] = 0$ and $pr^\alpha([\pi, \tilde{\Lambda}_\alpha]|_{M^\alpha}) = 0$ for any $\alpha \in S$. $\square$

Corollary 5.2. *Let V be a real symplectic vector space with a G -invariant linear symplectic form ω . Assume that the G -action on V is reduced as in Theorem 5.1. Let π be the corresponding Poisson structure of ω . Then $\kappa = \pi + \sum_{\alpha \in S} \Lambda_\alpha$ is a noncommutative Poisson structure on $\text{Poly}(V) \rtimes G$ if and only if Λ_α is constant on V^α .*

Proof. Under the assumption of the corollary, the restriction of the symplectic form ω to N^α for $\alpha \in S$ is a symplectic two-form. We denote the corresponding Poisson structure on N^α by π_α . Accordingly, we can write $\Lambda_\alpha = f_\alpha \pi_\alpha$, where f_α is a polynomial function on V^α .

By Theorem 5.1, $\pi + \sum_{\alpha \in S} f_\alpha \pi_\alpha$ is a noncommutative Poisson structure if and only if

- (1) $[\pi, \pi] = 0$,
- (2) $[\pi, f_\alpha]|_{V^\alpha} = 0$.

Equation (1) is automatically satisfied because π is Poisson. Because $[\pi, \pi_\alpha] = 0$, Equation (2) is reduced to

$$pr^\alpha([\pi, f_\alpha \pi_\alpha]|_{V^\alpha}) = pr^\alpha([\pi, f_\alpha] \wedge \pi_\alpha|_{V^\alpha}) = [\pi, f_\alpha]|_{V^\alpha} \wedge \pi_\alpha = 0.$$

In the second equality, we have used the fact that as f_α is a function on V^α , the bracket $[\pi, f_\alpha]$ is a vector field along V^α . Therefore, $\pi + \sum_{\alpha \in S} f_\alpha \pi_\alpha$ is a noncommutative Poisson structure if and only if $[\pi, f_\alpha]|_{V^\alpha} = [\pi, f_\alpha] = 0$, for all $\alpha \in S$. Because ω is symplectic, $[\pi, f_\alpha] = 0$ forces f_α to be a constant. Therefore, Λ_α is also a constant on V^α for all $\alpha \in S$. $\square$

5.2. Remarks on deformation quantizations. It is known that the set of noncommutative Poisson structures on an algebra A is in one-to-one correspondence to the set of infinitesimal deformations of A . It is natural to ask whether one can integrate the infinitesimal deformation associated to a noncommutative Poisson structure to a real one. This question relates to the idea of deformation quantization in mathematical physics. In [16], the second author introduced a notion of deformation quantization of a noncommutative Poisson structure, which we now recall.

Definition 5.3. A deformation quantization of a noncommutative Poisson structure Π on an associative algebra A is an associative product $\star_{\hbar}$ on $A[[\hbar]]$, such that $f \star_{\hbar} g = \sum_i \hbar^i C_i(f, g)$ for all $f, g \in A$ satisfying the following conditions:

- (1) $C_0(f, g) = fg$;
- (2) the Hochschild cohomology class $[C_1]$ is equal to Π .

It is natural to ask whether all the noncommutative Poisson structures defined in Theorem 5.1 can be deformation quantized. One special case has already been studied extensively, namely, when the noncommutative Poisson structure comes from a G -invariant Poisson structure on M (i.e., when there are no components supported on fixed point submanifolds of codimension 2). The deformation quantizations of these types of noncommutative Poisson structures on $C^\infty(M) \rtimes G$ were studied in [16], [17], [5], [13], etc. Another well-studied and well-known special case is the following proposition, which is essentially due to Etingof and Ginzburg [7, Theorem 1.3].

Proposition 5.4. *The noncommutative Poisson structure on a symplectic vector space obtained in Corollary 5.2 can be deformation quantized.*

Proof. In the following, we work with the field $\mathbb{C}$, because we will use the construction of the symplectic reflection algebras in [7] and Theorem 1.3 therein. Everything extends to the field $\mathbb{R}$, because Theorem 1.3 in [7] still holds in the real case. (The real group algebra of a finite group is semisimple.)

In [7], a symplectic reflection algebra $H_{t,c}$ is introduced as

$$TV \rtimes G / \langle x \otimes y - y \otimes x - \kappa(x, y) \in T^2V \oplus \mathbb{C}G \rangle_{x, y \in V},$$

where (V, ω) is a finite dimensional complex symplectic vector space over $\mathbb{C}$, and TV is its tensor algebra, and T^2V is $V \otimes V$, and κ is defined to be

$$\kappa(x, y) = t\pi(x, y) + \sum_{\alpha \in S} c_\alpha \pi_\alpha U_\alpha,$$

a G -invariant section of $\wedge^2 V + \bigoplus_{\alpha \in S} \wedge^2 N^\alpha$.

We assign V degree 1, and $\mathbb{C}G$ degree 0. This defines an increasing filtration $F_\bullet$ on $H_{t,c}$. It was proved by Etingof and Ginzburg in [7, Theorem 1.3] that $H_{t,c}$ satisfies the Poincaré-Birkhoff-Witt property, i.e. that tautological embedding $V \hookrightarrow gr(H_{t,c})$ extends to an isomorphism $Q : \text{Poly}(V) \rtimes G \rightarrow gr(H_{t,c})$ of vector spaces. We define gr_i to be the projection from $gr(H_{t,c})$ to its i th degree component.

We define a formal deformation quantization of $\text{Poly}(V) \rtimes G$ as follows. For $fU_\alpha, gU_\beta \in \text{Poly}(V) \rtimes G$,

$$fU_\alpha \star gU_\beta = \sum_{i, j, k=0}^{\infty} \hbar^{i+j-k} Q^{-1} \left(gr_k (gr_i (Q(fU_\alpha)) gr_j (Q(gU_\beta))) \right).$$

In particular, $C_i(fU_\alpha, gU_\beta)$ is defined to be

$$C_i(fU_\alpha, gU_\beta) = \sum_{p+q-r=i} Q^{-1} \left(gr_r (gr_p (Q(fU_\alpha)) gr_q (Q(gU_\beta))) \right).$$

Because $Q(fU_\alpha)$ and $Q(gU_\beta)$ are of finite degrees, p, q in the summation are both finite. Therefore, the sum in the definition of C_i is finite and C_i is well defined.

$$\begin{aligned}
& \text{We check that } \star \text{ is associative. } (fU_\alpha \star gU_\beta) \star hU_g \\
&= \sum_{i,j,k=0}^{\infty} \hbar^{i+j-k} Q^{-1} \left(gr_k (gr_i (Q(fU_\alpha)) gr_j (Q(gU_\beta))) \right) \star hU_g \\
&= \sum_{i,j,k,p,q,r=0}^{\infty} \hbar^{i+j-k} \hbar^{p+q-r} Q^{-1} \left(gr_r \left(gr_p \left(Q(Q^{-1} (gr_k (gr_i (Q(fU_\alpha)) gr_j (Q(gU_\beta)))) \right) \right) \right. \\
&\quad \left. gr_q (Q(hU_g)) \right) \\
&= \sum_{i,j,k=0}^{\infty} \sum_{p,q,r=0}^{\infty} \hbar^{i+j+p+q-k-r} Q^{-1} \left(gr_r \left(gr_p \left(gr_k (gr_i (Q(fU_\alpha)) gr_j (Q(gU_\beta))) \right) gr_q (Q(hU_g)) \right) \right) \\
&= \sum_{i,j,k=p,q,r}^{\infty} \hbar^{i+j+q-r} Q^{-1} \left(gr_r \left(gr_k \left(gr_i (Q(fU_\alpha)) gr_j (Q(gU_\beta)) \right) gr_q (Q(hU_g)) \right) \right) \\
&= \sum_{i,j,q=0}^{\infty} \hbar^{i+j+q} \sum_{r=0}^{\infty} \hbar^{i+j+q-r} Q^{-1} \left(gr_r \left(\sum_{k=0}^{\infty} gr_k \left(gr_i (Q(fU_\alpha)) gr_j (Q(gU_\beta)) \right) gr_q (Q(hU_g)) \right) \right) \\
&= \sum_{i,j,k,r=0}^{\infty} \hbar^{i+j+q-r} Q^{-1} \left(gr_r \left(gr_i (Q(fU_\alpha)) gr_j (Q(gU_\beta)) gr_q (Q(hU_g)) \right) \right),
\end{aligned}$$

which by a similar computation is equal to

$$fU_\alpha \star (gU_\beta \star hU_g).$$

We look at $C_1(fU_\alpha, gU_\beta) = \sum_{i+j-k=1} Q^{-1} \left(gr_k (gr_i (Q(fU_\alpha)) gr_j (Q(gU_\beta))) \right)$. To check that C_1 is cohomologous to κ , we compute $L(C_1)$ as follows using the definition of L :

$$\begin{aligned}
&= L_3 \left(\sum_{i_1, i_2} C_1(x_1^{i_1} - x^{i_1}, x_2^{i_2} - x^{i_2}) \frac{\partial}{\partial x^{i_1}} \wedge \frac{\partial}{\partial x^{i_2}} \right) \\
&= L_3 \left(\sum_{i_1, i_2} gr_0((x_1 - x)^{i_1} (x_2 - x)^{i_2}) \frac{\partial}{\partial x^{i_1}} \wedge \frac{\partial}{\partial x^{i_2}} \right) \\
&= L_3 \left(\sum_{i_2 < i_1} (x^{i_1} x^{i_2} - x^{i_2} x^{i_1}) \frac{\partial}{\partial x^{i_1}} \wedge \frac{\partial}{\partial x^{i_2}} \right) \\
&= \sum_{i_2 < i_1} \left(\omega(x^{i_1}, x^{i_2}) + \sum_{\alpha \in S} c_\alpha \omega_\alpha(x^{i_1}, x^{i_2}) U_\alpha \right) \frac{\partial}{\partial x^{i_1}} \wedge \frac{\partial}{\partial x^{i_2}} \\
&= \frac{1}{2} \sum_{i_1, i_2} \left(\omega(x^{i_1}, x^{i_2}) + \sum_{\alpha \in S} c_\alpha \omega_\alpha(x^{i_1}, x^{i_2}) U_\alpha \right) \frac{\partial}{\partial x^{i_1}} \wedge \frac{\partial}{\partial x^{i_2}}.
\end{aligned}$$

In the third equality of the above equation, we have used the definition of C_1 and the product structure in $H_{t,c}$. When $i_1 < i_2$, the term $x^{i_1} x^{i_2}$ has no degree 0 term. When $i_1 > i_2$, the degree 0 term of $x^{i_1} x^{i_2}$ is $x^{i_1} x^{i_2} - x^{i_2} x^{i_1}$.

In conclusion, $\star$ is a deformation quantization of $A \rtimes G$ with the noncommutative Poisson structure equal to $\frac{1}{2}\kappa$. $\square$

Remark 5.5. The generalization of the deformation quantization defined in Proposition 5.4 to affine varieties was studied by Etingof [6].

It is natural to ask whether the deformation quantization constructed in Theorem 5.4 is unique up to isomorphism. From Poisson geometry, we know that

the isomorphism classes of a deformation quantization of a Poisson structure are determined by its second Poisson cohomology, introduced in [18].

We briefly recall the definition of Poisson cohomology here. Let π be a noncommutative Poisson structure on an A . As $[\pi, \pi]_G = 0$ in $HH^\bullet(A, A)$, the operator $d^\pi : HH^\bullet(A, A) \rightarrow HH^{\bullet+1}(A, A)$, defined by

$$d^\pi(\phi) = [\pi, \phi], \quad \phi \in H^\bullet(A, A),$$

satisfies $d^\pi \circ d^\pi = 0$. The cohomology of $HH^\bullet(A, A)$ with respect to d^π is called the Poisson cohomology of π .

We denote the Poisson cohomology group of $Poly(V) \rtimes G$ associated to κ by $H_\kappa^\bullet(Poly(V) \rtimes G)$. In the following we compute $H_\kappa^2(Poly(V) \rtimes G)$.

Proposition 5.6. *Let (V, ω) be a real symplectic vector space with a finite group G symplectic action. Let κ be defined as in Corollary 5.2. Then the second Poisson cohomology of κ is isomorphic to*

$$\left\{ \sum_{g \in S} c_g \pi_g \in \Gamma^\infty\left(\bigoplus_{g \in S} \wedge^2 N^g\right)^G \mid \text{for all } g \in S, c_g \text{ is a constant on } V^g \right\}.$$

Proof. By [9], the fixed point subspace V^g of g is always a symplectic vector space, which implies that V^g is of even dimension. According to Equation (1) and the fact that the codimensions of all V^α are even, the second Hochschild cohomology of $Poly(V) \rtimes G$ is

$$\Gamma^\infty(\wedge^2 TV)^G \oplus \left(\sum_{g \in S} \wedge^0 V^g \otimes \wedge^2 N^g \right)^G.$$

Let $\Xi + \sum_{g \in S} f_g \pi_g$ be an element of $\Gamma^\infty(\wedge^2 TV)^G \oplus \left(\sum_{g \in S} \wedge^0 V^g \otimes \wedge^2 N^g \right)^G$. We compute $L([T(\Xi + \sum_{g \in S} f_g \pi_g), T(\kappa)])$ as follows. The computation is similar to computations in the proof of Theorem 5.1. There are four terms, which we deal with separately:

$$\begin{aligned} & 1) \ L([T(\Xi), T(\pi)]), \quad 2) \ L([T(\Xi), T(\sum_{g \in S} c_g \pi_g)]), \\ & 3) \ L([T(\sum_{g \in S} f_g \pi_g), T(\pi)]), \quad 4) \ L([T(\sum_{g \in S} f_g \pi_g), T(\sum_{g \in S} c_g \pi_g)]). \end{aligned}$$

The following computation follows similar reasoning to that found in the proof of Theorem 5.1:

$$\begin{aligned} & 1) \ L([T(\Xi), T(\pi)]) = [\Xi, \pi]; \\ & 2) \ L([T(\Xi), T(\sum_{g \in S} c_g \pi_g)]) = \sum_{g \in S} pr^g(c_g [\Xi, \pi_g]|_{V^g}); \\ & 3) \ L([T(\sum_{g \in S} f_g \pi_g), T(\pi)]) = \sum_{g \in S} pr^g([f_g \pi_g, \pi]|_{V^g}) = \sum_{g \in S} pr^g([f_g, \pi] \wedge \pi_g)|_{V^g}; \\ & 4) \ L([T(\sum_{g \in S} f_g \pi_g), T(\sum_{g \in S} c_g \pi_g)]) = 0. \end{aligned}$$

According to the above computation, $\Xi + \sum_{g \in S} f_g \pi_g$ is κ closed if and only if it satisfies the following equations:

$$(8) \quad \begin{aligned} 1) \quad & [\Xi, \pi] = 0, \\ 2) \quad & pr^g(\{c_g[\Xi, \pi_g] + [f_g, \pi] \wedge \pi_g\}|_{V^g}) = 0, \text{ for all } g \in G. \end{aligned}$$

We denote the space of solutions to the above equations by Z_κ^2 .

Next we compute the Poisson coboundary in $(\Gamma^\infty(\wedge^2 TV) \oplus_{\alpha \in S} \wedge^2 N^g)^G$.

According to Equation (1), the cohomology $HH^1(Poly(V) \rtimes G, Poly(V) \rtimes G)$ is a G -invariant vector field on V , as V^g for $g \neq id$ has at least codimension 2 for all g . Let $X \in \Gamma^\infty(TV)^G$. We compute $L([T(\kappa), T(X)])$ using Example 4.6:

$$L([T(\kappa), T(X)]) = [\pi, X] + \sum_{g \in S} pr^g(c_g[\pi_g, X]|_{V^g}).$$

We denote the space of elements in $\Gamma^\infty(\wedge^2 TV)^G \oplus (\sum_{g \in S} \wedge^0 V^g)^G$ of the above form by B_κ^2 .

We want to find the quotient Z_κ^2/B_κ^2 . Given $\Xi \in \Gamma^\infty(\wedge^2 TV)^G$ with $[\Xi, \pi] = 0$, we use the fact that π is from a symplectic form to find a G -invariant vector field $X \in \Gamma^\infty(TV)$ such that $[\pi, X] = \Xi$. Because $L([T(\kappa), T(X)]) =: \Xi + \sum_g h_g \pi_g = [\pi, X] + \sum_{g \in S} c_g pr^g([\pi_g, X]|_{V^g})$ is κ -closed we conclude that

$$\Xi + \sum_{g \in S} h_g \pi_g - (\Xi + \sum_g f_g \pi_g) = \sum_{g \in S} (h_g - f_g) \pi_g$$

is also κ -closed. Substituting this expression into the second equation of (8) with $\Xi = 0$, we have that for all $g \in S$,

$$pr^g([\pi, (h_g - f_g)] \wedge \pi_g|_{V^g}) = [\pi, (h_g - f_g)]|_{V^g} \wedge \pi_g|_{V^g} = 0.$$

In the above equality, we have used the fact that $h_g - f_g$ is supported on V^g and therefore $[\pi, h_g - f_g]$ is an element in TV^g also supported on V^g . Accordingly, we have

$$[\pi, h_g - f_g]|_{V^g} = 0 \Leftrightarrow [\pi, h_g - f_g] = 0.$$

Because π is symplectic, this implies that $h_g - f_g$ has to be a constant. It is obvious that $X + \sum_{g \in S} (h_g + c_g) \pi_g$ is κ -closed.

We are left to show that nonzero elements of the form $\sum_{g \in S} a_g \pi_g$ are not coboundaries for any constant $a_g, g \in S$. If $X \in \Gamma^\infty(TV)^G$ such that $L(T(\kappa), T(X)) = \sum_{g \in S} a_g \pi_g$, then by similar computations to those above, we have that

$$[\pi, X] = 0, \quad pr^g(c_g[\pi_g, X]|_{V^g}) = a_g \pi_g$$

for any $g \in S$.

As π is from a symplectic form, $[\pi, X] = 0$ implies that there is a function f such that $X = [\pi, f]$. As g acts on V preserving the symplectic form, we are allowed to write π as a sum of π_g and π^g , where π_g is a bivector supported along N^g , and π^g is a bivector supported along V^g . Therefore, we have

$$[\pi_g, X] = [\pi_g, [\pi, f]] = [\pi_g, [\pi_g, f]] + [\pi_g, [\pi^g, f]].$$

We easily compute that $[\pi_g, [\pi_g, f]] \equiv 0$ and $pr^g([\pi_g, [\pi^g, f]]) \equiv 0$ using the fact N^g is of dimension 2. This shows that $[\pi_g, X]$ contains no component proportional to π_g . Accordingly, $pr^g(c_g[\pi_g, X]) = 0 = a_g \pi_g$.

In conclusion, we see that the quotient space Z_κ^2/B_κ^2 is

$$\left\{ \sum_{g \in S} c_g \pi_g \in \Gamma^\infty \left(\bigoplus_{g \in S} \wedge^2 N^g \right)^G \mid \text{for all } g \in S, c_g \text{ is a constant on } V^g \right\}.$$

□

Remark 5.7. Proposition 5.6 shows that the dimension of all infinitesimal deformations of a noncommutative Poisson structure κ is equal to the size of the set S . Furthermore, it is easy to check that all the infinitesimal deformations actually correspond to Poisson structures. This gives another explanation of Corollary 5.2.

Propositions 5.4 and 5.6 inspire a series of interesting questions. The cocycle κ is a very special type of noncommutative Poisson structure on $A \rtimes G$ defined in Theorem 5.1. In [11], with Jean-Michel Oudom, we proved that, with a mild assumption, all linear Poisson structures on $\text{Poly}(V) \rtimes G$ can be quantized. This inspires the question of whether all the noncommutative Poisson structures defined in Theorem 5.1 can be deformation quantized. If their deformation quantizations exist, how many are there? All these problems have a general version on $C^\infty(M) \rtimes G$. It is closely related to the Conjecture 1 in [5] by Dolgushev and Etingof. We plan to address these questions in the near future.

5.3. Noncommutative quadratic Poisson structures. In this subsection, we provide some new examples of noncommutative Poisson structures other than those in Corollary 5.2. We can easily see that these Poisson structures are not symplectic at all, and they should be viewed as generalized quadratic Poisson structures.

We consider the space of $\mathbb{R}^4 = \mathbb{C} \times \mathbb{C}$ with the following $\mathbb{Z}_n \times \mathbb{Z}_m$ action, where $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. Let (z_1, z_2) be holomorphic coordinates on $\mathbb{C}^2$, and $(k, l) \in \mathbb{Z}_n \times \mathbb{Z}_m$. Define

$$\begin{aligned} (k, l) : (z_1, z_2) &\longrightarrow \left(\exp\left(\frac{2k\pi i}{n}\right) z_1, \exp\left(\frac{2l\pi i}{m}\right) z_2 \right) \\ (\bar{z}_1, \bar{z}_2) &\longrightarrow \left(\exp\left(-\frac{2k\pi i}{n}\right) \bar{z}_1, \exp\left(-\frac{2l\pi i}{m}\right) \bar{z}_2 \right). \end{aligned}$$

The fixed point subspace of $(k, l) \in \mathbb{Z}_n \times \mathbb{Z}_m$ can be described explicitly:

- (1) if $k \neq 0, l \neq 0$, the fixed point set of (k, l) consists of only one point, the origin;
- (2) if $k = 0, l \neq 0$, $(0, l)$'s fixed point set is $\mathbb{C} \times \{0\} \subset \mathbb{C} \times \mathbb{C}$;
- (3) if $k \neq 0, l = 0$, $(k, 0)$'s fixed point set is $\{0\} \times \mathbb{C} \subset \mathbb{C} \times \mathbb{C}$;
- (4) if $k = l = 0$, $(0, 0)$'s fixed point space is $\mathbb{C} \times \mathbb{C}$.

To look for noncommutative Poisson structures on $\text{Poly}(\mathbb{R}^4) \rtimes (\mathbb{Z}_n \times \mathbb{Z}_m)$, we only need to consider the fixed point space of the identity, which is $\mathbb{C}^2$, and the codimension 2 fixed point subspace of the form $\{0\} \times \mathbb{C}$ of $(k, 0)$, and the fixed point subspace $\mathbb{C} \times \{0\}$ of $(0, l)$. We consider the following collection of bivector fields, where $\alpha, \beta, \lambda_k, \mu_l$ are real constants.

- (1) On the fixed point subspace of $(0, 0)$, we consider $\Pi_{0,0}^\alpha = i\alpha |z_2|^2 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial \bar{z}_1}$. We notice that $\Pi_{0,0}$ is $\mathbb{Z}_n \times \mathbb{Z}_m$ invariant, and satisfies $[\Pi_{0,0}, \Pi_{0,0}] = 0$.
- (2) On the fixed point subspace of $(k, 0)$, we consider $\Pi_{k,0} = i\lambda_k |z_2|^2 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial \bar{z}_1}$, which is a smooth section of the determinant bundle of the normal bundle over $\{0\} \times \mathbb{C} \subset \mathbb{C} \times \mathbb{C}$. Again we notice that $\Pi_{k,0}$ is $\mathbb{Z}_n \times \mathbb{Z}_m$ invariant, and $[\Pi_{0,0}, \Pi_{k,0}] = 0$.

- (3) On the fixed point subspace of $(0, l)$, we consider $\Pi_{0,l} = i\mu_l |z_1|^2 \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial \bar{z}_2}$, which is a smooth section of the determinant bundle of the normal bundle over $\mathbb{C} \times \{0\} \subset \mathbb{C} \times \mathbb{C}$. We notice that $\Pi_{0,l}$ is $\mathbb{Z}_n \times \mathbb{Z}_m$ invariant, but

$$\begin{aligned} [\Pi_{0,0}, \Pi_{0,l}] = & -\alpha\mu_l \left[|z_2|^2 \left(z_1 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} \right) \wedge \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial \bar{z}_2} \right. \\ & \left. + |z_1|^2 \left(z_2 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right) \wedge \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial \bar{z}_1} \right] \neq 0. \end{aligned}$$

However, if we look at the restriction of $[\Pi_{0,0}, \Pi_{0,l}]$ to the fixed point subspace of $(0, l)$, which is $\mathbb{C} \times \{0\}$, it does vanish. We have that

$$pr^{(0,l)}([\Pi_{0,0}, \Pi_{0,l}]|_{\mathbb{C} \times \{0\}}) = 0.$$

By Theorem 5.1, we conclude that the collection of $(\Pi_{0,0}^\alpha, \Pi_{k,0}, \Pi_{0,l})$ defines a family of noncommutative Poisson structures on $Poly(\mathbb{R}^4) \rtimes (\mathbb{Z}_n \times \mathbb{Z}_m)$. One can also easily check that if we replace $\Pi_{0,0}^\alpha$ by $\Pi_{0,0}^\beta = i\beta |z_1|^2 \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial \bar{z}_2}$, $(\Pi_{0,0}^\beta, \Pi_{k,0}, \Pi_{0,l})$ defines another family of noncommutative Poisson structures on $Poly(\mathbb{R}^4) \rtimes (\mathbb{Z}_n \times \mathbb{Z}_m)$.

There are various ways to generalize the above families of examples to higher dimensions. For example, one can consider actions of $\mathbb{Z}_n \times \mathbb{Z}_m$ which act on the first two components of $\mathbb{C}^k$ as above, but act trivially on the left $\mathbb{C}^{k-2}$ component. Then the above two families of noncommutative Poisson structures naturally extend to $Poly(\mathbb{R}^{2k}) \rtimes (\mathbb{Z}_n \times \mathbb{Z}_m)$. We will leave the more nontrivial generalizations to the future [11].

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