BANDO-FUTAKI INVARIANTS ON HYPERSURFACES

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ABSTRACT. In this paper, the Bando-Futaki invariants on hypersurfaces are derived in terms of the degree of the defining polynomials, the dimension of the underlying projective space, and the given holomorphic vector field. In addition, the holomorphic invariant introduced by Tian and Chen (2002) is proven to be the Futaki invariant on compact Kähler manifolds with positive first Chern class.

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1. INTRODUCTION

In 1983, Futaki introduced the well-known Futaki invariant [5], which is an obstacle to the existence of Kähler-Einstein metrics on a compact complex manifold with positive first Chern class. Other generalizations of the Futaki invariant were introduced later, all of which are obstructions to certain geometric structures. The Calabi-Futaki invariant [2] is an obstruction to the existence of Kähler metrics of constant scalar curvature on a compact Kähler manifold. The Bando-Futaki invariants, raised by Bando [1] in 1983, are obstructions to the harmonicity of the higher order Chern forms. The Bando-Futaki invariants vanish if and only if the short-time solutions of the almost Kähler-Einstein metrics exist (cf. Leung [11]). Tian and Zhu found a holomorphic invariant [16], which is an obstruction to the existence of a Kähler-Ricci soliton. Recently, Futaki [7] generalized the Bando-Futaki invariants and the Futaki-Morita invariants [8]. The new invariants give

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obstructions to asymptotic Chow semi-stability when the invariant polynomials are Todd polynomials.

Efficient methods for computing the Futaki invariant and the generalized Futaki invariants are essential to characterizing the existence of certain geometric structures. Lu [12] constructed a formula to evaluate the Futaki invariant on complete intersections. The formula depends on the dimension of the projective space, the degree of the defining polynomials, and the given tangent holomorphic vector field. Concurrently, Yotov [17] derived the same result with a different approach. On complete intersections, Phong and Sturm [14] formularized the Futaki invariant and the Mabuchi energy functional using the Deligne pairing. Their methods may lead to a complete solution to the problem of computing the Futaki invariant.

The main part of this paper is the computation of the Bando-Futaki invariants on hypersurfaces in $\mathbb{CP}^n$. The value, stated in Theorem 1.1, is in terms of the dimension n , the degree of the defining polynomial of the hypersurface, and the tangent vector field. In Theorem 1.2, we prove that Chen and Tian's holomorphic invariants introduced in [3, section 5] are the Futaki invariants.

Let M be an m -dimensional compact complex manifold with positive first Chern class $c_1(M) > 0$. Let ω_M be a Kähler form on M such that $[\omega_M] \in c_1(M)$. Let Θ be the curvature tensor with respect to the Kähler metric ω_M on the holomorphic tangent bundle $T^{1,0}(M)$. Let $c_q(\Theta) = P^q(\Theta)$ denote the q -th Chern form of Θ , where $P^q(\Theta)$ are the elementary polynomials defined by

$$\det(\Theta + t \cdot I) = \sum_{q=0}^m P^q(\Theta) \cdot t^{m-q}.$$

Define the Chern classes $c_q(M)$ by

$$c_q(M) = [P^q(\Theta)] \in H_{DR}^{2q}(M),$$

where we set $c_0(M) = [1] \in H_{DR}^0(M)$. Let $Hc_q(\Theta)$ be the harmonic part of $c_q(\Theta)$ in the Hodge decomposition. Since M is Kähler, there exists a real $(q-1, q-1)$ form f_q such that

$$(1.1) \quad c_q(\Theta) - Hc_q(\Theta) = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f_q,$$

where f_q is unique up to a $\partial \bar{\partial}$ -closed form.

Definition 1.1. Let $\mathcal{F}_q : H^0(M, T^{1,0}(M)) \rightarrow \mathbb{C}$. The q -th Bando-Futaki invariant is defined as

$$(1.2) \quad \mathcal{F}_q(X) = \int_M L_X f_q \wedge \omega_M^{m+1-q}$$

for $q = 1, \dots, m$.

Each $\mathcal{F}_q$ is well-defined on the Lie algebra of holomorphic vector fields and is independent of the choice of the Kähler form in the Kähler class. This property was proved by Bando [1] and can also be found in Futaki's book [6]. In particular, $\mathcal{F}_1$ is known as the Futaki invariant.

Theorem 1.1. *Let M be a hypersurface in $\mathbb{CP}^n$ defined by a homogeneous polynomial F of degree d with $d \leq n$. Let X be a holomorphic vector field on $\mathbb{CP}^n$ such that*

$$XF = \kappa F$$

for a constant κ . Then the q -th Bando-Futaki invariant is

$$\mathcal{F}_q(X) = -(n+1-d)^{n-q} \frac{(d-1)(n+1)}{n} \sum_{j=0}^{q-1} (-d)^j (j+1) \binom{n}{q-j-1} \kappa.$$

A hypersurface M defined by the homogeneous polynomial $F = 0$ of degree d has positive first Chern class if $d \leq n$. In fact, M has nonzero holomorphic vector fields.

A summary of the proof is as follows. The first step is to find the potential forms f_q defined in (1.1) for $1 \leq q \leq n-1$. In order to do this, we compute the elementary symmetric polynomials by using the curvature tensors of the hypersurface in terms of local coordinates. Then we find the extra holomorphic forms so that the potentials f_q can be expressed globally. The second step is to evaluate the Bando-Futaki invariants. We take the contraction of (1.1) with vector field X . Then we may write the resulting function as a $\bar{\partial}$ -equation of $(q-1, q-1)$ -forms. That is,

$$\bar{\partial}[-q\tilde{P}^q(\nabla X, \Theta, \dots, \Theta) + q\alpha_{qq}\theta\omega^{q-1} - i(X)\partial f_q] = 0,$$

where θ is the Hamiltonian function of ω , $\tilde{P}^q$ is the polarization of the q -th elementary symmetric polynomial, and α_{qq} is a constant. We denote that $\omega_M = (n+1-d)\omega$ and $\omega = \omega_{FS}|_M$, where ω_{FS} is the Fubini-Study metric on $\mathbb{CP}^n$. By Hodge decomposition, we have

$$-q\tilde{P}^q(\nabla X, \Theta, \dots, \Theta) + q\alpha_{qq}\theta\omega^{q-1} - i(X)\partial f_q = \psi_q + \bar{\partial}\varphi_q,$$

where ψ_q is the harmonic part and $\bar{\partial}\varphi_q$ is the exact part. In Lemma 2.5, we show that $\psi_q = C(q)\omega^{q-1}$ for some constant $C(q)$. In Lemma 2.6, we prove that

$$(1.3) \quad q \int_M \tilde{P}^q(\nabla X, \Theta, \dots, \Theta) \wedge \omega_M^{n-q} = 0.$$

Then we reduce (1.2) to

$$\begin{aligned} \mathcal{F}_q(X) &= \int_M L_X(f_q) \wedge \omega_M^{n-q} \\ &= q\alpha_{qq}(n+1-d)^{n-q} \int_M \theta\omega^{n-1} - C(q)(n+1-q)^{n-q} \int_M \omega^{n-1}. \end{aligned}$$

By [12, Theorem 5.1], we have

$$\int_M \theta\omega^{n-1} = \frac{\kappa}{n} \quad \text{and} \quad \int_M \omega^{n-1} = d.$$

Thus the Bando-Futaki invariant is

$$\int_M L_X f_q \wedge \omega^{n-q} = (n+1-d)^{n-q} \left(\frac{q\alpha_{qq}\kappa}{n} - C(q)d \right).$$

This can be computed explicitly. We overcome two obstacles in our proof. The first obstacle is to show that the equality holds in (1.3). Note that it is not true that

for any invariant polynomial φ ,

$$\int_M \varphi(\nabla X, \Theta, \dots, \Theta) \wedge \omega^{n-q} = 0^1$$

For $q = 1$, it is trivial to see that

$$\int_M \tilde{P}^1(\nabla X) \wedge \omega_M^{n-1} = \int_M \operatorname{div}(X) \omega_M^{n-1} = \int_M L_X(\omega_M^{n-1}) = 0.$$

However, (1.3) is an unexpected result for the polarization $\tilde{P}^q(\nabla X, \Theta, \dots, \Theta)$ of the elementary polynomial for $2 \leq q \leq n-1$. Without this result, it would be very difficult to find the formula. The second obstacle is to compute $C(q)$ and to write φ_q as a globally defined form.

Corollary 1.1 (Lu, [12]). *Given the conditions of Theorem 1.1 and $q = 1$, the first Bando-Futaki invariant is the same as the Futaki invariant given as*

$$\mathcal{F}_1(X) = -(n+1-d)^{n-1} \frac{(n+1)(d-1)}{n} \kappa.$$

Our method can be applied, in principle, to the case of complete intersections and toric varieties. However, the computation is complicated using the current notation. Therefore our goal is to create an abstract setting from our current ideas before applying the invariant formula to more general cases.

In [6], the Futaki invariants (and Bando-Futaki invariants) are formularized as equivalent Chern numbers, which can be computed using the Atiyah-Bott localization formula. The method used in this paper is completely different from Atiyah and Bott's method and is an alternative way to compute the equivalent Chern numbers.

Remark 1.1. All Bando-Futaki invariants on a hypersurface vanish if the hypersurface is K-semistable.

Here we restate the definition [13]:

Definition 1.2. We say M is K-semistable if every holomorphic tangent vector field X on M satisfies

$$(1.4) \quad \lim_{t \rightarrow 0} t \frac{d}{dt} M(\omega_0, \omega_t) \geq 0,$$

where $M(\omega_0, \omega_t)$ is the K-energy with respect to ω_0 and ω_t , and $\sigma(t)^*\omega_0 = \omega_t$, where the one-parameter family of automorphisms $\sigma(t)$ is generated by the holomorphic vector field X .

In section 3, we study the holomorphic invariants that were introduced by Chen and Tian [3].

Definition 1.3. Let M be an n -dimensional simply connected Kähler manifold with a Kähler form ω and X be a holomorphic vector field over M . Since M is simply connected, there exists a smooth function θ_X such that $i(X)\omega = \frac{\sqrt{-1}}{2\pi} \bar{\partial}\theta_X$.²

¹If $\varphi(\nabla X, \Theta, \dots, \Theta) \wedge \omega^{n-q} = \operatorname{div} X \cdot (c_1(\Theta))^{n-1}$, then $\int_M \varphi(\nabla X, \Theta, \dots, \Theta) \wedge \omega^{n-q} \neq 0$.

²In order to keep the original definition, we use the equation $\theta_X = -\theta$, which is the opposite sign from the Hamiltonian function θ defined as above.

For any positive integer k , define a new holomorphic invariant to be

$$(1.5) \quad \begin{aligned} F_k(X, \omega) &= (n-k) \int_M \theta_X \omega^n + (k+1) \int_M \Delta \theta_X Ric(\omega)^k \wedge \omega^{n-k} \\ &\quad - (n-k) \int_M \theta_X Ric(\omega)^{k+1} \wedge \omega^{n-k-1}. \end{aligned}$$

Theorem 1.2. *Let M be a compact complex Kähler manifold with a Kähler form $[\omega] \in c_1(M) > 0$. Then the Chen-Tian holomorphic invariants are the Futaki invariants:*

$$F_k(X, \omega) = (k+1) \int_M X(f_\omega) \omega^n,$$

where f_ω is a potential function such that $Ric(\omega) - \omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f_\omega$.

2. BANDO-FUTAKI INVARIANTS

2.1. Curvature tensor on hypersurfaces. The following setting is adopted from [15] and [13]. Let $Z = [Z_0, \dots, Z_n]$ be the homogeneous coordinate of $\mathbb{CP}^n$. Let $U_0 = \{Z \in \mathbb{CP}^n | Z_0 \neq 0\}$ and $z = (z_1, \dots, z_n)$, where $z_i = Z_i/Z_0$ for $i = 1, \dots, n$. Without loss of generality, assume that we work on the coordinate chart (U_0, z) . Under this coordinate system, the Fubini-Study metric is given by

$$\omega_{FS} = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^n g_{i\bar{j}} dz_i \wedge d\bar{z}_j = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^n \left(\frac{\delta_{ij}}{1+|z|^2} - \frac{\bar{z}_i z_j}{(1+|z|^2)^2} \right) dz_i \wedge d\bar{z}_j,$$

where $|z|^2 = \sum_{i=1}^n |z_i|^2$. Let f be the defining polynomial of M on U_0 , where

$$f(z) = F \left[1, \frac{Z_1}{Z_0}, \dots, \frac{Z_n}{Z_0} \right] = \frac{1}{Z_0^n} F[Z_0, \dots, Z_n].$$

By the implicit function theorem, there exists an open set $V \subseteq \{(z_2, \dots, z_n) \in \mathbb{C}^{n-1}\}$ of U_0 such that z' solves the equation $f(z_1, \dots, z_n) = 0$ whenever $\frac{\partial f}{\partial z_1}(z) \neq 0$. Let $z_1 = z'(z_2, \dots, z_n)$. Therefore, z_1 is a holomorphic function on V . Under the coordinate system $(V, (z_2, \dots, z_n))$, let the restricted Kähler form be

$$\omega = \omega_{FS}|_M = \sum_{i,j=2}^n \frac{\sqrt{-1}}{2\pi} \tilde{g}_{i\bar{j}} dz_i \wedge d\bar{z}_j,$$

where

$$(2.1) \quad \tilde{g}_{i\bar{j}} = \frac{\delta_{ij} + a_i \bar{a}_j}{1+|z|^2} - \frac{(\bar{z}_i + \bar{z}_1 a_i)(z_j + z_1 \bar{a}_j)}{(1+|z|^2)^2} \quad \text{for } i, j = 2, \dots, n,$$

where $a_i = \frac{\partial z_1}{\partial z_i}$ for $i = 2, \dots, n$. In order to compute the curvature tensor with respect to the metric $\tilde{g}_{i\bar{j}}$, it is critical to find the inverse matrix of $\tilde{g}_{i\bar{j}}$.

Lemma 2.1. *Using the same notation as above, the inverse of $\tilde{g}_{i\bar{j}}$ is*

$$\begin{aligned} \tilde{g}^{i\bar{j}} &= \frac{1}{\varrho} (\varrho(1+|z|^2)\delta_{ji} - a_j \bar{a}_i + \bar{z}_j z_i(1+|a|^2)) \\ &\quad - \frac{1}{\varrho} \left(a_j z_i \left(\sum_{k=2}^n \bar{a}_k \bar{z}_k - \bar{z}_1 \right) + \bar{z}_j \bar{a}_i \left(\sum_{k=2}^n a_k z_k - z_1 \right) \right), \end{aligned}$$

where $\varrho = \frac{\sum_{k=0}^n |F_k|^2}{(1+|z|^2)|F_1|^2}$, $|a|^2 = \sum_{i=2}^n |a_i|^2$, and $F_k = \frac{\partial F}{\partial Z_k}$ for $k = 0, 1, \dots, n$.

Proof. Consider $\tilde{g}_{i\bar{j}}$ as a matrix A_{ij} for $2 \leq i, j \leq n$. Since $\tilde{g}_{i\bar{j}}$ is a matrix of a linear combination of matrices δ_{ij} , $a_i \bar{a}_j$, $\bar{z}_i z_j$, $a_i z_j$, and $\bar{z}_i \bar{a}_j$ pointwisely, its adjugate and inverse can be written as linear combinations of δ_{ij} , $a_i \bar{a}_j$, $\bar{z}_i z_j$, $a_i z_j$, and $\bar{z}_i \bar{a}_j$ pointwisely. More precisely, let $A_{ij} = (\gamma_1 \delta_{ij} + \gamma_2 a_i \bar{a}_j + \gamma_3 \bar{z}_i z_j + \gamma_4 a_i z_j + \gamma_5 \bar{z}_i \bar{a}_j)$, where

$$(2.2) \quad (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) \\ = \left(\frac{1}{1+|z|^2}, \frac{1+|z|^2-|z_1|^2}{(1+|z|^2)^2}, -\frac{1}{(1+|z|^2)^2}, -\frac{\bar{z}_1}{(1+|z|^2)^2}, -\frac{z_1}{(1+|z|^2)^2} \right).$$

By definition, the adjugate matrix is given by

$$(\text{adj} A)_{ji} = (-1)^{i+j} \sum_{\sigma \in S_{n-2}} \text{sgn}(\sigma) A_{\sigma(i_3)j_3} \cdots A_{\sigma(i_n)j_n},$$

where S_{n-2} are all permutations of $\{i_3, \dots, i_n\} = \{2, \dots, n\} - \{i\}$ and $\{j_3, j_4, \dots, j_n\} = \{2, \dots, n\} - \{j\}$. There exist functions $\eta_1, \dots, \eta_5$ such that

$$(\text{adj} A)_{ji} = (\eta_1 \delta_{ji} + \eta_2 a_j \bar{a}_i + \eta_3 \bar{z}_j z_i + \eta_4 a_j z_i + \eta_5 \bar{z}_j \bar{a}_i).$$

Apply the following formulas given in [13]:

$$(2.3) \quad \sum_{k=2}^n a_k z_k - z_1 = \frac{F_0}{F_1}, \\ 1 + |a|^2 + \left| \sum_{k=2}^n a_k z_k - z_1 \right|^2 = \frac{\sum_{k=0}^n |F_k|^2}{|F_1|^2}, \\ \det \tilde{g}_{i\bar{j}} = \frac{1}{(1+|z|^2)^n} \frac{\sum_{k=0}^n |F_k|^2}{|F_1|^2}.$$

Then we can compute the coefficients $\eta_1, \dots, \eta_5$ by solving the following linear equation system:

$$(2.4) \quad \begin{aligned} \tilde{g}_{i\bar{j}} \tilde{g}^{k\bar{j}} &= \tilde{g}_{i\bar{j}} \frac{1}{\det \tilde{g}_{i\bar{j}}} (\text{adj} A)_{jk} \\ &= \frac{1}{\det \tilde{g}_{i\bar{j}}} (\gamma_1 \delta_{ij} + \gamma_2 a_i \bar{a}_j + \gamma_3 \bar{z}_i z_j + \gamma_4 a_i z_j + \gamma_5 \bar{z}_i \bar{a}_j) \\ &\quad \times (\eta_1 \delta_{jk} + \eta_2 a_j \bar{a}_k + \eta_3 \bar{z}_j z_k + \eta_4 a_j z_k + \eta_5 \bar{z}_j \bar{a}_k) \\ &= \delta_{ik}. \end{aligned}$$

Furthermore, we may rewrite (2.4) as

$$(2.5) \quad \begin{aligned} (\gamma_1 + \gamma_2 |a|^2 + \gamma_4 \sum_{j=2}^n a_j z_j) \eta_2 + (\gamma_2 \sum_{j=2}^n \bar{a}_j \bar{z}_j + \gamma_4 \sum_{j=2}^n |z_j|^2) \eta_5 &= -\gamma_2 \eta_1, \\ (\gamma_1 + \gamma_3 \sum_{j=2}^n |z_j|^2 + \gamma_5 \sum_{j=2}^n \bar{a}_j \bar{z}_j) \eta_3 + (\gamma_5 |a|^2 + \gamma_3 \sum_{j=2}^n a_j z_j) \eta_4 &= -\gamma_3 \eta_1, \\ (\gamma_2 \sum_{j=2}^n \bar{a}_j \bar{z}_j + \gamma_4 \sum_{j=2}^n |z_j|^2) \eta_3 + (\gamma_1 + \gamma_2 |a|^2 + \gamma_4 \sum_{j=2}^n a_j z_j) \eta_4 &= -\gamma_4 \eta_1, \\ (\gamma_3 \sum_{j=2}^n a_j z_j + \gamma_5 |a|^2) \eta_2 + (\gamma_1 + \gamma_3 \sum_{j=2}^n |z_j|^2 + \gamma_5 \sum_{j=2}^n \bar{a}_j \bar{z}_j) \eta_5 &= -\gamma_5 \eta_1, \end{aligned}$$

where $\eta_1 = \det \tilde{g}_{i\bar{j}} \frac{1}{\gamma_1} = (1 + |z|^2) \det \tilde{g}_{i\bar{j}}$. Hence by using (2.2) and (2.4), the solutions for (2.5) are

$$(\eta_2, \eta_3, \eta_4, \eta_5) = \frac{1}{(1 + |z|^2)^{n-1}} \left(-1, 1 + |a|^2, -\left(\sum_{k=2}^n \bar{a}_k \bar{z}_k - \bar{z}_1\right), -\left(\sum_{k=2}^n a_k z_k - z_1\right) \right).$$

Then we may obtain

$$\tilde{g}^{i\bar{j}} = \frac{1}{\det \tilde{g}_{i\bar{j}}} (\eta_1 \delta_{ji} + \eta_2 a_j \bar{a}_i + \eta_3 \bar{z}_j z_i + \eta_4 a_j z_i + \eta_5 \bar{z}_j \bar{a}_i).$$

□

The following lemma is important for computing higher order Chern forms of the hypersurface M and for evaluating the Bando-Futaki invariants.

Lemma 2.2. *The curvature of the hypersurface is*

$$\sum_{i,j=2}^n R_{ki\bar{j}}^\ell dz_i \wedge d\bar{z}_j = \sum_{i,j=2}^n \left(\delta_{k\ell} \tilde{g}_{i\bar{j}} + \delta_{i\ell} \tilde{g}_{k\bar{j}} - \frac{1}{\rho} \frac{\partial a_k}{\partial z_i} \frac{\partial \bar{a}_s}{\partial \bar{z}_j} \tilde{g}^{\ell\bar{s}} \right) dz_i \wedge d\bar{z}_j$$

for $2 \leq k, \ell \leq n$, where $\rho = (1 + |z|^2) \frac{\sum_{k=0}^n |F_k|^2}{|F_1|^2}$.

Proof. Recall the curvature formula

$$(2.6) \quad R_{ki\bar{j}}^\ell = -\frac{\partial^2 \tilde{g}_{k\bar{s}}}{\partial z_i \partial \bar{z}_j} \tilde{g}^{\ell\bar{s}} + \frac{\partial \tilde{g}_{k\bar{q}}}{\partial z_i} \frac{\partial \tilde{g}_{p\bar{s}}}{\partial \bar{z}_j} \tilde{g}^{p\bar{q}} \tilde{g}^{\ell\bar{s}}$$

on Kähler manifolds. By using the metric $\tilde{g}_{i\bar{j}}$ defined in (2.1), we obtain

$$(2.7) \quad \frac{\partial \tilde{g}_{k\bar{q}}}{\partial z_i} = -\frac{\bar{z}_i + \bar{z}_1 a_i}{1 + |z|^2} \tilde{g}_{k\bar{q}} - \frac{\bar{z}_k + \bar{z}_1 a_k}{1 + |z|^2} \tilde{g}_{i\bar{q}} + \frac{\partial a_k}{\partial z_i} \frac{(1 + |z|^2 - |z_1|^2) \bar{a}_q - \bar{z}_1 z_q}{(1 + |z|^2)^2},$$

$$(2.8) \quad \frac{\partial \tilde{g}_{p\bar{s}}}{\partial \bar{z}_j} = -\frac{z_j + z_1 \bar{a}_j}{1 + |z|^2} \tilde{g}_{p\bar{s}} - \frac{z_s + z_1 \bar{a}_s}{1 + |z|^2} \tilde{g}_{p\bar{j}} + \frac{\partial \bar{a}_s}{\partial \bar{z}_j} \frac{(1 + |z|^2 - |z_1|^2) a_p - z_1 \bar{z}_p}{(1 + |z|^2)^2}.$$

Use (2.8) to compute the second derivative

$$(2.9) \quad \begin{aligned} \frac{\partial^2 \tilde{g}_{k\bar{s}}}{\partial z_i \partial \bar{z}_j} &= -\tilde{g}_{i\bar{j}} \tilde{g}_{k\bar{s}} - \tilde{g}_{i\bar{s}} \tilde{g}_{k\bar{j}} \\ &\quad + \frac{(z_j + z_1 \bar{a}_j)(\bar{z}_i + \bar{z}_1 a_i)}{(1 + |z|^2)^2} \tilde{g}_{k\bar{s}} + \frac{(z_j + z_1 \bar{a}_j)(\bar{z}_k + \bar{z}_1 a_k)}{(1 + |z|^2)^2} \tilde{g}_{i\bar{s}} \\ &\quad + \frac{(z_s + z_1 \bar{a}_s)(\bar{z}_i + \bar{z}_1 a_i)}{(1 + |z|^2)^2} \tilde{g}_{k\bar{j}} + \frac{(z_s + z_1 \bar{a}_s)(\bar{z}_k + \bar{z}_1 a_k)}{(1 + |z|^2)^2} \tilde{g}_{i\bar{j}} \\ &\quad - \frac{(z_j + z_1 \bar{a}_j)}{1 + |z|^2} \frac{\partial a_k}{\partial z_i} \frac{(1 + |z|^2 - |z_1|^2) \bar{a}_s - \bar{z}_1 z_s}{(1 + |z|^2)^2} \\ &\quad - \frac{(z_s + z_1 \bar{a}_s)}{1 + |z|^2} \frac{\partial a_k}{\partial z_i} \frac{(1 + |z|^2 - |z_1|^2) \bar{a}_j - \bar{z}_1 z_j}{(1 + |z|^2)^2} \\ &\quad + \frac{\partial a_k}{\partial z_i} \frac{\partial \bar{a}_s}{\partial \bar{z}_j} \frac{1 + |z|^2 - |z_1|^2}{(1 + |z|^2)^2} + \frac{\partial \bar{a}_s}{\partial \bar{z}_j} \frac{a_k \bar{z}_i - a_i \bar{z}_k}{(1 + |z|^2)^2} \\ &\quad - 2 \frac{\partial \bar{a}_s}{\partial \bar{z}_j} (\bar{z}_i + \bar{z}_1 a_i) \frac{(1 + |z|^2 - |z_1|^2) a_k - z_1 \bar{z}_k}{(1 + |z|^2)^3}. \end{aligned}$$

By using (2.7) and (2.8) in (2.9), we obtain

$$(2.10) \quad \frac{\partial^2 \tilde{g}_{k\bar{s}}}{\partial z_i \partial \bar{z}_j} = -\tilde{g}_{i\bar{j}} \tilde{g}_{k\bar{s}} - \tilde{g}_{i\bar{s}} \tilde{g}_{k\bar{j}} + \frac{\partial \tilde{g}_{k\bar{q}}}{\partial z_i} \frac{\partial \tilde{g}_{p\bar{s}}}{\partial \bar{z}_j} \tilde{g}^{p\bar{q}} + \frac{\partial a_k}{\partial z_i} \frac{\partial \bar{a}_s}{\partial \bar{z}_j} \frac{1 + |z|^2 - |z_1|^2}{(1 + |z|^2)^2} \\ - \frac{\partial a_k}{\partial z_i} \frac{\partial \bar{a}_s}{\partial \bar{z}_j} \left(\frac{(1 + |z|^2 - |z_1|^2) a_p - z_1 \bar{z}_p}{(1 + |z|^2)^2} \right) \left(\frac{(1 + |z|^2 - |z_1|^2) \bar{a}_q - \bar{z}_1 z_q}{(1 + |z|^2)^2} \right) \tilde{g}^{p\bar{q}}.$$

In fact, we have

$$(2.11) \quad \left(\frac{(1 + |z|^2 - |z_1|^2) a_p - z_1 \bar{z}_p}{(1 + |z|^2)^2} \right) \left(\frac{(1 + |z|^2 - |z_1|^2) \bar{a}_q - \bar{z}_1 z_q}{(1 + |z|^2)^2} \right) \tilde{g}^{p\bar{q}} - \frac{1 + |z|^2 - |z_1|^2}{(1 + |z|^2)^2} \\ = - \frac{|F_1|^2}{(1 + |z|^2) \sum_{k=0}^n |F_k|^2} = - \frac{1}{\rho}.$$

By using (2.11) in (2.10), we get

$$- \frac{\partial^2 \tilde{g}_{k\bar{s}}}{\partial z_i \partial \bar{z}_j} \tilde{g}^{\ell\bar{s}} + \frac{\partial \tilde{g}_{k\bar{q}}}{\partial z_i} \frac{\partial \tilde{g}_{p\bar{s}}}{\partial \bar{z}_j} \tilde{g}^{p\bar{q}} \tilde{g}^{\ell\bar{s}} = \delta_{k\ell} \tilde{g}_{i\bar{j}} + \delta_{i\ell} \tilde{g}_{k\bar{j}} - \frac{1}{\rho} \frac{\partial a_k}{\partial z_i} \frac{\partial \bar{a}_s}{\partial \bar{z}_j} \tilde{g}^{\ell\bar{s}}.$$

□

The Ricci curvature of a hypersurface has been shown in [13, 15]. It also directly follows from Lemma 2.2.

Remark 2.1. Given the conditions as in Theorem 1.1, the Ricci curvature on the hypersurface is

$$Ric(\omega) = (n + 1 - d)\omega - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi,$$

where

$$\xi = \log \left(\frac{\sum_{k=0}^n |F_k|^2}{(\sum_{k=0}^n |Z_k|^2)^{d-1}} \right).$$

Proof. The trace of the curvature form is the Ricci curvature. By using Lemma 2.2, we compute

$$\sum_{k=2}^n \sum_{i,j=2}^n R_{k\bar{i}j}^k dz_i \wedge d\bar{z}_j = \sum_{k=2}^n \sum_{i,j=2}^n \left(\delta_{kk} \tilde{g}_{i\bar{j}} + \delta_{ik} \tilde{g}_{j\bar{k}} - \frac{1}{\rho} \frac{\partial a_k}{\partial z_i} \frac{\partial \bar{a}_s}{\partial \bar{z}_j} \tilde{g}^{k\bar{s}} \right) dz_i \wedge d\bar{z}_j \\ = \sum_{i,j=2}^n \left(n \tilde{g}_{i\bar{j}} - \frac{1}{\rho} \sum_{k,s=2}^n \frac{\partial a_k}{\partial z_i} \frac{\partial \bar{a}_s}{\partial \bar{z}_j} \tilde{g}^{k\bar{s}} \right) dz_i \wedge d\bar{z}_j,$$

where

$$\frac{1}{\rho} \frac{\sqrt{-1}}{2\pi} \sum_{k,s=2}^n \frac{\partial a_k}{\partial z_i} \frac{\partial \bar{a}_s}{\partial \bar{z}_j} \tilde{g}^{k\bar{s}} = \partial_i \bar{\partial}_j \log \left(\frac{\sum_{k=0}^n |F_k|^2}{|F_1|^2} \right).$$

We can find an extra globally defined function such that

$$(2.12) \quad \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left(\frac{\sum_{k=0}^n |F_k|^2}{|F_1|^2} \right) = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left(\frac{\sum_{k=0}^n |F_k|^2}{(\sum_{k=0}^n |Z_k|^2)^{d-1}} \frac{(\sum_{k=0}^n |Z_k|^2)^{d-1}}{|F_1|^2} \right) \\ = (d - 1)\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi.$$

□

2.2. Chern forms.

Lemma 2.3. *Given the conditions in Theorem 1.1, the q -th Chern form on a hypersurface is*

$$c_q(\Theta) = \sum_{k=0}^q \alpha_{qk} \omega^k \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{q-k},$$

where

$$\begin{aligned} \alpha_{00} &= 1, \\ \alpha_{qq} &= \binom{n+1}{q} - d\alpha_{(q-1)(q-1)}, \\ \alpha_{q(q-k)} &= -[d\alpha_{(q-1)(q-k-1)} + \alpha_{(q-1)(q-k)}] \quad \text{for } k = 1, \dots, q-1, \\ \alpha_{q0} &= (-1)^q, \end{aligned}$$

for $q = 1, \dots, n-1$.

Before we prove this lemma, we need several steps. Let $\Theta = \sum_{k,\ell=2}^n \Theta_k^\ell \frac{\partial}{\partial z_\ell} \otimes dz_k$ be the curvature tensor of M such that

$$\Theta_k^\ell = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=2}^n R_{kij}^\ell dz_i \wedge d\bar{z}_j$$

represents a $(1,1)$ -form-valued matrix for $2 \leq k, \ell \leq n$. In order to save some space, we denote the (k,k) -form-valued matrix $\sum_{i_2, \dots, i_k=2}^n \Theta_{i_1}^{i_2} \wedge \dots \wedge \Theta_{i_k}^{i_{k+1}}$ by $\sum_{i_2, \dots, i_k=2}^n \Theta_{i_1}^{i_2} \dots \Theta_{i_k}^{i_{k+1}}$. The q -th Chern polynomial defined in [9, pp. 402, 417] is

$$(2.13) \quad P^q(\Theta) = \frac{1}{q!} \sum_{\sigma \in S_q} \sum_{i_1, \dots, i_q=2}^n \text{sgn}(\sigma) \Theta_{i_1}^{i_{\sigma(1)}} \Theta_{i_2}^{i_{\sigma(2)}} \dots \Theta_{i_q}^{i_{\sigma(q)}}.$$

In Sublemma 2.1, we derive a formula to compute higher order Chern polynomials in terms of the lower order Chern polynomials. The idea is as follows. Let

$$\phi_j = \sum_{i_1, \dots, i_j=2}^n \Theta_{i_1}^{i_2} \Theta_{i_2}^{i_3} \dots \Theta_{i_j}^{i_1}$$

be a (j,j) -form corresponding to a cycle of order j . Every nontrivial permutation $\sigma \in S_q$ is a product of cyclic permutations which operate on disjoint indices: $\sigma = \sigma_1 \sigma_2 \dots \sigma_k$. If $\sigma_1, \dots, \sigma_k$ are cycles of length $j_1, \dots, j_k$, respectively, then $\sum_{i_1, \dots, i_q=2}^n \Theta_{i_1}^{i_{\sigma(1)}} \dots \Theta_{i_q}^{i_{\sigma(q)}} = \phi_{j_1} \dots \phi_{j_k}$. For each $1 \leq \ell \leq k$, let $\sigma = \sigma_\ell \tau_\ell$, where $\tau_\ell = \sigma_1 \dots \sigma_{\ell-1} \sigma_{\ell+1} \dots \sigma_k \in S_{q-j_\ell}$. Since τ_ℓ operates on $q-j_\ell$ many indices which are different from what σ_ℓ operates on, without loss of generality, assume that τ_ℓ permutes $\{1, \dots, q-j_\ell\}$ and $\sigma_\ell = (q-j_\ell+1 \dots q)$. Then $\text{sgn}(\tau_\ell) \sum_{i_1, \dots, i_{q-j_\ell}=2}^n \Theta_{i_1}^{i_{\tau_\ell(1)}} \dots \Theta_{i_{q-j_\ell}}^{i_{\tau_\ell(q-j_\ell)}}$ is one term in the expansion of $P^{q-j_\ell}(\Theta)$ as in (2.13). Therefore, we have

$$\sum_{i_1, \dots, i_q=2}^n \text{sgn}(\sigma) \Theta_{i_1}^{i_{\sigma(1)}} \dots \Theta_{i_q}^{i_{\sigma(q)}} = \text{sgn}(\sigma_\ell) \text{sgn}(\tau_\ell) \phi_\ell \sum_{i_1, \dots, i_{q-j_\ell}=2}^n \Theta_{i_1}^{i_{\tau_\ell(1)}} \dots \Theta_{i_{q-j_\ell}}^{i_{\tau_\ell(q-j_\ell)}},$$

which belongs to one term in the expansion of $\text{sgn}(\sigma_\ell) \phi_{j_\ell} P^{q-j_\ell}(\Theta)$. For every $\sigma \in S_q$, $\sum_{i_1, \dots, i_q=2}^n \text{sgn}(\sigma) \Theta_{i_1}^{i_{\sigma(1)}} \dots \Theta_{i_q}^{i_{\sigma(q)}}$ can be written as one term in the expansion of

$\text{sgn}(\sigma_1)\phi_{j_1}P^{q-j_1}(\Theta), \dots$, or $\text{sgn}(\sigma_k)\phi_{j_k}P^{q-j_k}(\Theta)$. Then we conclude that $P^q(\Theta) = \sum_{j=1}^q (-1)^{j-1} a_j \phi_j P^{q-j}(\Theta)$ for some rational number a_j for $j = 1, \dots, q$.

Sublemma 2.1. *The q -th Chern polynomial can be written as*

$$(2.14) \quad P^q(\Theta) = \sum_{j=1}^q (-1)^{j-1} a_j \phi_j P^{q-j}(\Theta) = \frac{1}{q} \sum_{j=1}^q (-1)^{j-1} \phi_j P^{q-j}(\Theta)$$

for $q = 1, \dots, n-1$. Let $P^0(\Theta) = 1$ for convention.

Proof. Prove (2.14) by induction. For $q = 1$, $P^1(\Theta) = \sum_{i=1}^n \Theta_i^i = \phi_1$. Assume that $P^k(\Theta) = \frac{1}{k} \sum_{j=1}^k (-1)^{j-1} \phi_j P^{k-j}(\Theta)$ holds for $k = 2, \dots, q-1$. Actually, there are $\binom{q}{j}(j-1)!$ many cycles of order j in S_q . For any positive integer numbers $j_1, \dots, j_k$ such that $j_1 + \dots + j_k = q$, the coefficient for $\phi_{j_1} \dots \phi_{j_k}$ on the left-hand side of (2.14) is

$$\begin{cases} \frac{1}{q!} \binom{q}{j_1} (j_1-1)! \binom{q-j_1}{j_2} (j_2-1)! \dots \binom{j_k}{j_k} (j_k-1)!, & \text{if } j_t \neq j_s, 1 \leq s, t \leq k; \\ \frac{1}{\ell! q!} \binom{q}{j_1} (j_1-1)! \binom{q-j_1}{j_2} (j_2-1)! \dots \binom{j_k}{j_k} (j_k-1)!, & \text{if } \exists \ell \leq k, j_1 = \dots = j_\ell; \end{cases}$$

and so on. Therefore, the coefficient for ϕ_q is $a_q = \frac{1}{q!} \binom{q}{q} (q-1)! = \frac{1}{q}$. For $q > j > q/2$, the coefficient for $\phi_j \phi_{q-j}$ on the left-hand side of (2.14) is $\frac{1}{q!} \binom{q}{j} (j-1)! (q-j-1)! = \frac{1}{j(q-j)}$. On the other hand, since the term $\phi_j \phi_{q-j}$ only appears in the expansion of $\phi_j P^{q-j}(\Theta)$ and $\phi_{q-j} P^j(\Theta)$, by using the induction hypothesis, compare the coefficient of $\phi_j \phi_{q-j}$ on both sides of (2.14). That is,

$$(2.15) \quad \frac{1}{j(q-j)} = \frac{a_j}{q-j} + \frac{a_{q-j}}{j}.$$

If $q = 2k$ for some integer k , the coefficient of $\phi_k \phi_k$ is

$$\frac{1}{2! q!} \binom{q}{k} (k-1)! (k-1)! = \frac{1}{k} a_k$$

since $\phi_k \phi_k$ only appears in $\phi_k P^k(\Theta)$. Then $a_k = \frac{1}{2k} = \frac{1}{q}$. For $j = q-2$, since $\phi_{q-2} \phi_1 \phi_1$ only appears once in $\phi_{q-2} P^2(\Theta)$ and twice in $\phi_1 P^{q-2}(\Theta)$, the coefficient is

$$(2.16) \quad \frac{1}{2(q-2)} = \frac{a_{q-2}}{2} + \frac{a_1}{q-1} + \frac{a_1}{(q-1)(q-2)}.$$

Furthermore, for $q-2 > j > q-j-1 > 1$, the term $\phi_j \phi_{q-j-1} \phi_1$ appears twice in $\phi_j P^{q-j}(\Theta)$, $\phi_{q-j-1} P^{j+1}(\Theta)$ and $\phi_1 P^{q-1}(\Theta)$. The coefficient for $\phi_j \phi_{q-j-1} \phi_1$ is

$$\begin{aligned} (2.17) \quad & \frac{1}{j(q-j-1)} \\ &= \frac{a_j}{q-j} + \frac{a_j}{(q-j)(q-j-1)} + \frac{a_{q-j-1}}{j+1} + \frac{a_{q-j-1}}{(j+1)j} + \frac{a_1}{(q-1)(q-j-1)} + \frac{a_1}{(q-1)j} \\ &= \frac{a_j}{q-j-1} + \frac{a_{q-j-1}}{j} + \frac{a_1}{(q-j-1)j}. \end{aligned}$$

By using (2.15), (2.16), and (2.17), we obtain the relation of a_j

$$(2.18) \quad a_{q-1} = t,$$

$$(2.19) \quad a_{q-j} = \frac{-(j-1) + j(q-1)t}{q-j} \quad \text{for } j > q/2,$$

$$(2.20) \quad a_j = 1 - (q-1)t \quad \text{for } j < q/2.$$

The term $\underbrace{\phi_1 \cdots \phi_1}_q$ only appears in $\phi_1 P^{q-1}(\Theta)$; the coefficient of it is

$$\frac{1}{q!q!} \binom{q}{1} \binom{q-1}{1} \cdots \binom{1}{1} = \frac{1}{q!} = \frac{a_1}{(q-1)(q-2) \cdots 1}.$$

We get $a_1 = \frac{1}{q}$. By using (2.18), (2.19) and (2.20), we obtain $a_j = \frac{1}{q}$ for $j = 2, \dots, q$. $\square$

Therefore, we only need to compute ϕ_j for $j = 2, \dots, q$ and use (2.14) to formulate $P^q(\Theta)$.

Sublemma 2.2. *For $3 \leq j \leq q$, we have*

$$\begin{aligned} \sum_{i_2, \dots, i_{j-1}=2}^n \Theta_{i_1}^{i_2} \cdots \Theta_{i_{j-1}}^{i_j} &= \omega \sum_{i_3, \dots, i_{j-1}=2}^n \Theta_{i_1}^{i_3} \Theta_{i_3}^{i_4} \cdots \Theta_{i_{j-1}}^{i_j} \\ &\quad - \frac{\sqrt{-1}}{2\pi} \frac{1}{\rho} \sum_{\alpha, \beta=2}^n \left(\frac{\partial a_{i_1}}{\partial z_\alpha} \frac{\partial \bar{a}_s}{\partial \bar{z}_\beta} \tilde{g}^{i_j \bar{s}} dz_\alpha \wedge d\bar{z}_\beta \right) \wedge (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{j-2}, \end{aligned}$$

where $\rho = (1 + |z|^2) \frac{\sum_{k=0}^n |F_k|^2}{|F_1|^2}$, $\xi = \log \left(\frac{\sum_{k=0}^n |F_k|^2}{(\sum_{k=0}^n |Z_k|^2)^{d-1}} \right)$, and

$$\Theta_i^j = \frac{\sqrt{-1}}{2\pi} \sum_{\alpha, \beta=2}^n R_{i\alpha\bar{\beta}}^j dz_\alpha \wedge d\bar{z}_\beta.$$

Proof. Prove by induction. For $j = 3$, use Lemma 2.2 and (2.12) to compute

$$\begin{aligned}
 (2.21) \quad & \sum_{i_2=2}^n \Theta_{i_1}^{i_2} \Theta_{i_2}^{i_3} = \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \sum_{i_2, \alpha, \beta, \lambda, \eta, s, t=2}^n \left(\delta_{i_1 i_2} \tilde{g}_{\alpha \bar{\beta}} + \delta_{\alpha i_2} \tilde{g}_{i_1 \bar{\beta}} - \frac{1}{\rho} \frac{\partial a_{i_1}}{\partial z_\alpha} \frac{\partial \bar{a}_s}{\partial \bar{z}_\beta} \tilde{g}^{i_2 \bar{s}} \right) \\
 & \cdot \left(\delta_{i_2 i_3} \tilde{g}_{\lambda \bar{\eta}} + \delta_{\lambda i_3} \tilde{g}_{i_2 \bar{\eta}} - \frac{1}{\rho} \frac{\partial a_{i_2}}{\partial z_\lambda} \frac{\partial \bar{a}_t}{\partial \bar{z}_\eta} \tilde{g}^{i_3 \bar{t}} \right) dz_\alpha \wedge d\bar{z}_\beta \wedge dz_\lambda \wedge d\bar{z}_\eta \\
 & = \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \sum_{i_2, \alpha, \beta, \lambda, \eta=2}^n \left[\delta_{i_1 i_3} \tilde{g}_{\alpha \bar{\beta}} \tilde{g}_{\lambda \bar{\eta}} + \delta_{i_3 \lambda} \tilde{g}_{\alpha \bar{\beta}} \tilde{g}_{i_1 \bar{\eta}} \right. \\
 & \quad \left. - \frac{1}{\rho} \sum_{t=2}^n \frac{\partial a_{i_1}}{\partial z_\lambda} \frac{\partial \bar{a}_t}{\partial \bar{z}_\eta} \tilde{g}^{i_3 \bar{t}} \tilde{g}_{\alpha \bar{\beta}} - \frac{1}{\rho} \sum_{s=2}^n \frac{\partial a_{i_1}}{\partial z_\alpha} \frac{\partial \bar{a}_s}{\partial \bar{z}_\beta} \tilde{g}^{i_3 \bar{s}} \tilde{g}_{\lambda \bar{\eta}} \right. \\
 & \quad \left. + \frac{1}{\rho^2} \sum_{s, t=2}^n \left(\frac{\partial a_{i_1}}{\partial z_\alpha} \frac{\partial \bar{a}_t}{\partial \bar{z}_\eta} \tilde{g}^{i_3 \bar{t}} \right) \left(\frac{\partial a_{i_2}}{\partial z_\lambda} \frac{\partial \bar{a}_s}{\partial \bar{z}_\beta} \tilde{g}^{i_2 \bar{s}} \right) \right] dz_\alpha \wedge d\bar{z}_\beta \wedge dz_\lambda \wedge d\bar{z}_\eta \\
 & = \omega \wedge \Theta_{i_1}^{i_3} - \frac{1}{\rho} \sum_{\alpha, \beta, s=2}^n \frac{\partial a_{i_1}}{\partial z_\alpha} \frac{\partial \bar{a}_s}{\partial \bar{z}_\beta} \tilde{g}^{i_3 \bar{s}} dz_\alpha \wedge d\bar{z}_\beta \wedge \omega \\
 & \quad - \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \sum_{\alpha, \beta, t, \lambda, \eta=2}^n \frac{1}{\rho} \frac{\partial a_{i_1}}{\partial z_\alpha} \frac{\partial \bar{a}_t}{\partial \bar{z}_\eta} \tilde{g}^{i_3 \bar{t}} \left(\partial_\lambda \bar{\partial}_\beta \log \frac{\sum_{k=0}^n |F_k|^2}{|F_1|^2} \right) dz_\alpha \wedge d\bar{z}_\eta \wedge dz_\lambda \wedge d\bar{z}_\beta \\
 & = \omega \wedge \Theta_{i_1}^{i_3} - \frac{\sqrt{-1}}{2\pi} \frac{1}{\rho} \sum_{\alpha, \eta, t=2}^n \left(\frac{\partial a_{i_1}}{\partial z_\alpha} \frac{\partial \bar{a}_t}{\partial \bar{z}_\eta} \tilde{g}^{i_3 \bar{t}} dz_\alpha \wedge d\bar{z}_\eta \right) \wedge (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi).
 \end{aligned}$$

Suppose that the statement is true for $j - 1$. By the induction hypothesis, we get

$$\begin{aligned}
 & \sum_{i_2, \dots, i_{j-1}=2}^n \Theta_{i_1}^{i_2} \cdots \Theta_{i_{j-1}}^{i_j} = \left(\omega \wedge \sum_{i_3, \dots, i_{j-1}=2}^n \Theta_{i_1}^{i_3} \Theta_{i_3}^{i_4} \cdots \Theta_{i_{j-2}}^{i_{j-1}} \right) \Theta_{i_{j-1}}^{i_j} \\
 & \quad - \frac{\sqrt{-1}}{2\pi} \frac{1}{\rho} \sum_{i_{j-1}=2}^n \sum_{s, \alpha, \beta=2}^n \left(\frac{\partial a_{i_1}}{\partial z_\alpha} \frac{\partial \bar{a}_s}{\partial \bar{z}_\beta} \tilde{g}^{i_{j-1} \bar{s}} dz_\alpha \wedge d\bar{z}_\beta \right) \wedge (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{j-3} \wedge \Theta_{i_{j-1}}^{i_j} \\
 & = \omega \wedge \sum_{i_3, \dots, i_{j-1}=2}^n \Theta_{i_1}^{i_3} \Theta_{i_3}^{i_4} \cdots \Theta_{i_{j-1}}^{i_j} \\
 & \quad - \frac{\sqrt{-1}}{2\pi} \frac{1}{\rho} \sum_{\alpha, \beta=2}^n \left(\frac{\partial a_{i_1}}{\partial z_\alpha} \frac{\partial \bar{a}_t}{\partial \bar{z}_\beta} \tilde{g}^{i_j \bar{t}} dz_\alpha \wedge d\bar{z}_\beta \right) \wedge (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{j-2},
 \end{aligned}$$

where

$$\begin{aligned}
 (2.22) \quad & \sum_{i_{j-1}, \alpha, \beta, s=2}^n \left(\frac{\partial a_{i_1}}{\partial z_\alpha} \frac{\partial \bar{a}_s}{\partial \bar{z}_\beta} \tilde{g}^{i_{j-1} \bar{s}} dz_\alpha \wedge d\bar{z}_\beta \right) \wedge \Theta_{i_{j-1}}^{i_j} \\
 & = \sum_{\alpha, \beta, s=2}^n \left(\frac{\partial a_{i_1}}{\partial z_\alpha} \frac{\partial \bar{a}_s}{\partial \bar{z}_\beta} \tilde{g}^{i_j \bar{s}} dz_\alpha \wedge d\bar{z}_\beta \right) \wedge (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)
 \end{aligned}$$

is part of (2.21). We omit the details of the proof. $\square$

Theorem 2.1. *With the curvature given in Lemma 2.2, the trace of the wedge product of j copies of curvature tensors on the hypersurface M is*

$$\sum_{i_1, \dots, i_j=2}^n \Theta_{i_1}^{i_2} \cdots \Theta_{i_{j-1}}^{i_j} \Theta_{i_j}^{i_1} = (n+1)\omega^j - (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^j$$

for $2 \leq j \leq q$.

Proof. Prove by induction. For $j = 2$, compute it directly from (2.21). That is,

$$\begin{aligned} \sum_{i_1, i_2=2}^n \Theta_{i_1}^{i_2} \Theta_{i_2}^{i_1} &= \omega \wedge ((n+1-d)\omega - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi) \\ &\quad - ((d-1)\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi) (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi) \\ &= (n+1)\omega^2 - (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^2. \end{aligned}$$

Suppose the statement is true for $j-1$. By Sublemma 2.2, the induction hypothesis and (2.22), we get

$$\begin{aligned} &\sum_{i_1, i_j=2}^n \left(\sum_{i_2, \dots, i_{j-1}=2}^n \Theta_{i_1}^{i_2} \Theta_{i_2}^{i_3} \cdots \Theta_{i_{j-1}}^{i_j} \right) \Theta_{i_j}^{i_1} \\ &= \omega \wedge \sum_{i_1, i_3, \dots, i_j=2}^n \Theta_{i_1}^{i_3} \Theta_{i_3}^{i_4} \cdots \Theta_{i_{j-1}}^{i_j} \Theta_{i_j}^{i_1} \\ &\quad - \frac{\sqrt{-1}}{2\pi} \sum_{i_1, i_j, \alpha, \beta, s=2}^n \frac{\partial a_{i_1}}{\partial z_\alpha} \frac{\partial \bar{a}_s}{\partial \bar{z}_\beta} \tilde{g}^{i_j \bar{s}} dz_\alpha \wedge d\bar{z}_\beta \wedge (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{j-2} \wedge \Theta_{i_j}^{i_1} \\ &= \omega \wedge [(n+1)\omega^{j-1} - (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{j-1}] \\ &\quad - [(d-1)\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi] \wedge (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{j-1} \\ &= (n+1)\omega^j - (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^j. \end{aligned}$$

□

Proof of Lemma 2.3. Prove by induction. For $q = 1$,

$$P^1(\Theta) = (n+1-d)\omega - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi = \alpha_{10} \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi + \alpha_{11} \omega,$$

where $\alpha_{10} = (-1)$, $\alpha_{11} = \binom{n+1}{1} - d\alpha_{00}$ and $\alpha_{00} = 1$. Suppose that

$$P^i(\Theta) = \sum_{k=0}^i \alpha_{ik} \omega^k \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{i-k}$$

holds for $2 \leq i \leq q-1$. By Sublemma 2.1, Theorem 2.1 and the induction hypothesis, we have

$$(2.23) \quad P^q(\Theta) = \frac{1}{q} \sum_{j=1}^q (-1)^{j-1} \phi_j P^{q-j}(\Theta) \\ = \frac{1}{q} \sum_{j=1}^q (-1)^{j-1} [(n+1)\omega^j - (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^j] \sum_{k=0}^{q-j} \alpha_{(q-j)k} \omega^k \wedge (\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{q-j-k},$$

where

$$(2.24) \quad \alpha_{(q-j)k} = \begin{cases} -(d\alpha_{(q-j-1)(k-1)} + \alpha_{(q-j-1)k}), & \text{if } k < q-j; \\ \binom{n+1}{q-j} - d\alpha_{(q-j-1)(q-j-1)}, & \text{if } k = q-j. \end{cases} \\ = (-1)^{q-j} \sum_{\ell=0}^k \binom{q-j-\ell}{k-\ell} (-1)^\ell d^{k-\ell} \binom{n+1}{\ell}$$

for $j = 1, \dots, q-1$. We rewrite (2.23) as $P^q(\Theta) = \sum_{k=0}^q \alpha_{qk} \omega^k (\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{q-k}$, where

$$(2.25) \quad \alpha_{qk} = \frac{1}{q} \sum_{j=1}^k (-1)^{j-1} (n+1 - d^j) \alpha_{(q-j)(k-j)} \\ + \frac{1}{q} \sum_{i=1}^{q-k} \sum_{j=i}^{k+i} (-1)^j d^{j-i} \binom{j}{j-i} \alpha_{(q-j)(k-j+i)}.$$

By using (2.24), we compute (2.25)

$$(2.26) \quad \alpha_{qk} = \frac{n+1}{q} (-1)^{q-1} \sum_{j=1}^k \sum_{\ell=0}^{k-j} \binom{q-j-\ell}{k-j-\ell} (-1)^\ell d^{k-j-\ell} \binom{n+1}{\ell}$$

$$(2.27) \quad + \frac{(-1)^q}{q} \sum_{i=0}^{q-k} \sum_{j=i}^{k+i} \sum_{\ell=0}^{k-j+i} (-1)^\ell d^{k-\ell} \binom{j}{j-i} \binom{q-j-\ell}{k-j+i-\ell} \binom{n+1}{\ell}$$

$$(2.28) \quad - \frac{(-1)^q}{q} \sum_{\ell=0}^k \binom{q-\ell}{k-\ell} \binom{n+1}{\ell} d^{k-\ell} (-1)^\ell.$$

In fact, we have

$$(2.29) \quad \sum_{\ell=0}^{\lambda-1} (-1)^\ell \binom{n+1}{\ell} = (-1)^{\lambda-1} \binom{n}{\lambda-1}.$$

We change the index in (2.26) and use (2.29) to get

$$\begin{aligned}
 & \frac{n+1}{q} (-1)^{q-1} \sum_{j=1}^k \sum_{\ell=0}^{k-j} \binom{q-j-\ell}{k-j-\ell} (-1)^\ell d^{k-j-\ell} \binom{n+1}{\ell} \\
 &= \frac{n+1}{q} (-1)^{q-1} \sum_{\lambda=1}^k \binom{q-\lambda}{k-\lambda} d^{k-\lambda} \sum_{\ell=0}^{\lambda-1} (-1)^\ell \binom{n+1}{\ell} \\
 &= \frac{n+1}{q} (-1)^{q-1} \sum_{\lambda=1}^k \binom{q-\lambda}{k-\lambda} d^{k-\lambda} (-1)^{\lambda-1} \binom{n}{\lambda-1} \\
 (2.30) \quad &= \frac{1}{q} (-1)^q \sum_{\lambda=1}^k \binom{q-\lambda}{k-\lambda} d^{k-\lambda} (-1)^\lambda \binom{n+1}{\lambda}.
 \end{aligned}$$

Change the summing order in (2.27) and get

$$\begin{aligned}
 (2.31) \quad & \frac{(-1)^q}{q} \sum_{i=0}^{q-k} \sum_{j=i}^{k+i} \sum_{\ell=0}^{k-j+i} (-1)^\ell d^{k-\ell} \binom{j}{j-i} \binom{q-j-\ell}{k-j+i-\ell} \binom{n+1}{\ell} \\
 &= \frac{(-1)^q}{q} \sum_{j=0}^k (-1)^\ell d^{k-\ell} \sum_{i=0}^{q-k} \sum_{\ell=0}^j \binom{k+i-j}{k-j} \binom{q-k-i+j-\ell}{j-\ell} \binom{n+1}{\ell} \\
 &= \frac{(-1)^q}{q} \sum_{\ell=0}^k (-1)^\ell d^{k-\ell} \sum_{j=\ell}^k \sum_{i=0}^{q-k} \binom{k+i-j}{k-j} \binom{q-k-i+j-\ell}{j-\ell} \binom{n+1}{\ell}.
 \end{aligned}$$

In fact, we have

$$(2.32) \quad \sum_{i=0}^d \binom{s+i}{s} = \binom{s+d+1}{d+1}.$$

Reassemble the following term:

$$\begin{aligned}
 (2.33) \quad & \sum_{i=0}^{q-k} \binom{k+i-j}{k-j} \binom{q-k-i+j-\ell}{j-\ell} = \sum_{i=0}^{q-k} \binom{k+i-j}{k-j} \\
 & + \sum_{s=1}^{q-k} \sum_{i=0}^{q-k-s} \binom{k+i-j}{k-j} \left[\binom{s+j-\ell}{j-\ell} - \binom{s-1+j-\ell}{j-\ell} \right].
 \end{aligned}$$

Then we may apply (2.32) to each assorted item in (2.33) to get

$$\begin{aligned}
 (2.34) \quad & \sum_{i=0}^{q-k} \binom{k+i-j}{k-j} \binom{q-k-i+j-\ell}{j-\ell} \\
 &= \sum_{s=0}^{q-k} \binom{k+s-(j-1)}{k-(j-1)} \binom{q-k-s+(j-1)-\ell}{(j-1)-\ell}.
 \end{aligned}$$

If we repeat the procedure in (2.34) for r times for $r = 1, \dots, j - \ell$, we get

$$\begin{aligned} \sum_{i=0}^{q-k} \binom{k+i-j}{k-j} \binom{q-k-i+j-\ell}{j-\ell} \\ = \sum_{i=0}^{q-k} \binom{k+i-(j-r)}{k-(j-r)} \binom{q-k-i+(j-r)-\ell}{(j-r)-\ell}. \end{aligned}$$

For each $j = r + \ell$, we have

$$\sum_{i=0}^{q-k} \binom{k+i-j}{k-j} \binom{q-k-i+j-\ell}{j-\ell} = \sum_{i=0}^{q-k} \binom{k+i-\ell}{k-\ell} \binom{q-k-i}{0} = \binom{q-\ell+1}{k-\ell+1}.$$

Then we obtain

$$(2.35) \quad \sum_{j=\ell}^k \sum_{i=0}^{q-k} \binom{k+i-j}{k-j} \binom{q-k-i+j-\ell}{j-\ell} = (k-\ell+1) \binom{q-\ell+1}{k-\ell+1}.$$

Use (2.35) in (2.31) to get

$$\begin{aligned} (2.36) \quad & \frac{(-1)^q}{q} \sum_{i=0}^{q-k} \sum_{j=i}^{k+i} \sum_{\ell=0}^{k-j+i} (-1)^\ell d^{k-\ell} \binom{j}{j-i} \binom{q-j-\ell}{k-j+i-\ell} \binom{n+1}{\ell} \\ & = \frac{(-1)^q}{q} \sum_{\ell=0}^k (-1)^\ell d^{k-\ell} (k-\ell+1) \binom{q-\ell+1}{k-\ell+1} \binom{n+1}{\ell} \\ & = \frac{(-1)^q}{q} \sum_{\ell=0}^k (-1)^\ell d^{k-\ell} (q-\ell+1) \binom{q-\ell}{k-\ell} \binom{n+1}{\ell}. \end{aligned}$$

By adding (2.30), (2.36) and (2.28), we can get the coefficient

$$\begin{aligned} (2.37) \quad & \alpha_{qk} = \frac{(-1)^q}{q} \sum_{\ell=1}^k \binom{q-\ell}{k-\ell} d^{k-\ell} (-1)^\ell \ell \binom{n+1}{\ell} \\ & + \frac{(-1)^q}{q} d^k (q+1) \binom{q}{k} + \frac{(-1)^q}{q} \sum_{\ell=1}^k (-1)^\ell d^{k-\ell} (q-\ell+1) \binom{q-\ell}{k-\ell} \binom{n+1}{\ell} \\ & - \frac{(-1)^q}{q} \sum_{\ell=0}^k (-1)^\ell d^{k-\ell} \binom{q-\ell}{k-\ell} \binom{n+1}{\ell} \\ & = (-1)^q \sum_{\ell=0}^k (-1)^\ell d^{k-\ell} \binom{q-\ell}{k-\ell} \binom{n+1}{\ell}. \end{aligned}$$

Comparing (2.37) and (2.24), we get

$$\alpha_{qk} = \begin{cases} \binom{n+1}{q} - d\alpha_{(q-1)(q-1)}, & \text{if } k = q; \\ -(d\alpha_{(q-1)(k-1)} + \alpha_{(q-1)k}), & \text{if } 0 \leq k < q. \end{cases}$$

□

2.3. Computation of the Bando-Futaki invariants. In Lemma 2.3, we obtain the q -th Chern polynomial

$$\begin{aligned} c_q(\Theta) &= \sum_{k=0}^q \alpha_{qk} \omega^k \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{q-k} \\ &= \alpha_{qq} \omega^q + d\psi_q, \end{aligned}$$

where $\psi_q = -\frac{\sqrt{-1}}{2\pi} \partial \xi \wedge \sum_{k=0}^{q-1} \alpha_{qk} \omega^k \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{q-k-1}$. Then $c_q(\Theta)$ and $\alpha_{qq} \omega^q$ are in the same cohomology class in $H_{DR}^{2q}(M)$. On any compact Kähler manifold, ω^q is harmonic for each $1 \leq q \leq n-1$. The Hodge theorem says that the harmonic representation of $c_q(\Theta)$ is unique on any compact Kähler manifold. That is, $Hc_q(\Theta) = \alpha_{qq} \omega^q$. Actually, we have the Hodge decomposition

$$(2.38) \quad c_q(\Theta) = \alpha_{qq} \omega^q + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f_q,$$

where $f_q = \sum_{k=0}^{q-1} \alpha_{qk} \xi \omega^k \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{q-k-1}$ is globally defined. However, we will not directly compute the q -th Bando-Futaki invariant

$$\mathcal{F}_q(X) = \int_M \mathcal{L}_X f_q \wedge \omega_M^{n-q-1},$$

where $\omega_M = (n+1-d)\omega \in c_1(M)$ and $\omega = \omega_{FS}|_M$.

First, take the contraction map on (2.38) with X :

$$(2.39) \quad i(X)c_q(\Theta) - i(X)\alpha_{qq}\omega_q = \frac{\sqrt{-1}}{2\pi} \bar{\partial} i(X) \partial f_q.$$

Let us compute each term in (2.39) separately. We take the contraction of ω^q with X :

$$i(X)\omega^q = qi(X)(\omega)\omega^{q-1} = q\left(-\frac{\sqrt{-1}}{2\pi} \bar{\partial} \theta\right)\omega^{q-1},$$

where $i(X)\omega = -\frac{\sqrt{-1}}{2\pi} \bar{\partial} \theta$. More precisely, we can express a holomorphic vector field

$$(2.40) \quad \tilde{X} = \sum_{i=0}^n \lambda_i Z_i \frac{\partial}{\partial Z_i}$$

over $\mathbb{CP}^n$ with $\sum_{k=0}^n \lambda_k = 0$. The restriction of $\tilde{X}$ on $M \cap V$ is given by

$$\tilde{X}|_V = X = \sum_{i=2}^n (\lambda_i - \lambda_0) z_i \left(a_i \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_i} \right).$$

Hence the Hamilton function can be expressed explicitly as

$$\begin{aligned} (2.41) \quad \theta &= -\tilde{X} \log \left(\sum_{k=0}^n |Z_k|^2 \right) = -\frac{\sum_{k=0}^n \lambda_k |Z_k|^2}{\sum_{k=0}^n |Z_k|^2} \\ &= -\frac{\sum_{k=1}^n \lambda_k |z_k|^2 + \lambda_0}{1 + |z|^2} = -\frac{\sum_{k=1}^n (\lambda_k - \lambda_0) |z_k|^2}{1 + |z|^2} - \lambda_0. \end{aligned}$$

Let $\tilde{P}(\underbrace{\Theta, \dots, \Theta}_q)$ be the polarization of its elementary invariant polynomial $P^q(\Theta)$.

Take the contraction of the curvature with X :

$$i(X)\Theta_k^\ell = i(X)\frac{\sqrt{-1}}{2\pi}\sum_{i,j=2}^n R_{k\bar{i}\bar{j}}^\ell dz_i \wedge d\bar{z}_j = \frac{\sqrt{-1}}{2\pi}\sum_{i,j=2}^n X^i R_{k\bar{i}\bar{j}}^\ell d\bar{z}_j = -\frac{\sqrt{-1}}{2\pi}\bar{\partial}X_k^\ell,$$

where

$$(2.42) \quad X_k^\ell = \frac{\partial X^\ell}{\partial z_k} + \sum_i X^i \Gamma_{ik}^\ell = -\sum_{j=2}^n \bar{g}^{\ell\bar{j}} \partial_k \bar{\partial}_j \theta$$

for $2 \leq k, \ell \leq n$. Let

$$\nabla X = \sum_{k,\ell} X_k^\ell dz_k \otimes \frac{\partial}{\partial z_\ell} = \sum_{k,\ell} \left(\frac{\partial X^\ell}{\partial z_k} + \sum_i X^i \Gamma_{ki}^\ell \right) dz_k \otimes \frac{\partial}{\partial z_\ell}.$$

Then we can take the contraction map on the q -th Chern form

$$i(X)c_q(\Theta) = q\tilde{P}^q(i(X)\Theta, \Theta, \dots, \Theta) = -q\frac{\sqrt{-1}}{2\pi}\bar{\partial}\tilde{P}^q(\nabla X, \Theta, \dots, \Theta).$$

We may rewrite (2.39) as

$$(2.43) \quad \bar{\partial} \left(-q\tilde{P}^q(\nabla X, \Theta, \dots, \Theta) + q\alpha_{qq}\theta\omega^{q-1} - i(X)\partial f_{q,\Theta} \right) = 0.$$

In fact, the space of harmonic forms $\mathcal{H}^{p,q}(M)$ and the Dolbeault cohomology $H_{\bar{\partial}}^{p,q}(M)$ groups are isomorphic. By the Hodge decomposition, the $\bar{\partial}$ -closed form in (2.43) can be written as

$$(2.44) \quad -q\tilde{P}^q(\nabla X, \Theta, \dots, \Theta) + q\alpha_{qq}\theta\omega^{q-1} - i(X)\partial f_{q,\Theta} = \psi_q + \bar{\partial}\varphi_q,$$

where ψ_q is the harmonic part of the left-hand side of (2.44) and φ_q is a $(q-1, q-2)$ form. In Lemma 2.5, we will show that $\psi_q = C(q)\omega^{q-1}$ for some constant C_q . Instead of computing the Bando-Futaki invariants directly, we will compute the following:

$$(2.45) \quad \int_M L_X f_q \wedge \omega_M^{n-q} = \int_M (di(X)f_q + i(X)\partial f_q) \wedge \omega_M^{n-q} \\ = \int_M \left(-q\tilde{P}^q(\nabla X, \Theta, \dots, \Theta) + q\alpha_{qq}\theta\omega^{q-1} - C(q)\omega^{q-1} - \bar{\partial}\varphi_q \right) \wedge \omega_M^{n-q}.$$

Finding $C(q)$ and showing $q \int_M \tilde{P}^q(\nabla X, \Theta, \dots, \Theta) \wedge \omega_M^{n-q} = 0$ are the next two steps to compute (2.45). In order to evaluate $C(q)$, it is necessary to express $\tilde{P}^q(\nabla X, \Theta, \dots, \Theta)$ explicitly.

Lemma 2.4. *The covariant derivative of the polarization of the elementary polynomial P^q is given by*

$$(2.46) \quad q\tilde{P}^q(\nabla X, \Theta, \dots, \Theta) = -\operatorname{div}(X)\gamma_{q1} + \theta\gamma_{q2} \\ - \sum_{j=2}^q (-1)^{j-1} \eta_{q-j} \wedge \left((\omega^{j-2} + n\zeta_{j-2}) \wedge (\partial\bar{\partial}\theta) - \zeta_{j-2} \wedge (\partial\bar{\partial}\Delta\theta) \right),$$

where

$$\begin{aligned}
\gamma_{q1} &= \sum_{k=0}^{q-1} \alpha_{qk} (q-k) \omega^k \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{q-1-k}, \\
\gamma_{q2} &= \sum_{k=0}^{q-1} \left((q-k)(n+1-d) \alpha_{qk} + (k+1) \alpha_{q(k+1)} \right) \omega^k \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{q-1-k}, \\
\eta_{q-j} &= \sum_{k=0}^{q-j} \alpha_{(q-j)k} \frac{\sqrt{-1}}{2\pi} \omega^k \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{q-j-k}, \\
\zeta_{j-2} &= \sum_{k=0}^{j-2} (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{j-2-k} \wedge \omega^k, \\
\Delta \theta &= - \sum_{\alpha, \beta=2}^n \tilde{g}^{\alpha\beta} \partial_\alpha \bar{\partial}_\beta \theta.
\end{aligned}$$

Proof. According to Sublemma 2.1, we have

$$\begin{aligned}
(2.47) \quad q \tilde{P}^q(\nabla X, \Theta, \dots, \Theta) &= \frac{q}{q!} \sum_{\sigma \in S_q} \sum_{i_1, \dots, i_q=2}^n \text{sgn}(\sigma) X_{i_1}^{i_{\sigma(1)}} \Theta_{i_2}^{i_{\sigma(2)}} \dots \Theta_{i_q}^{i_{\sigma(q)}} \\
&= \sum_{i_1=2}^n X_{i_1}^{i_1} P^{q-1}(\Theta) - \sum_{i_1, i_2}^n X_{i_1}^{i_2} \Theta_{i_2}^{i_1} P^{q-2}(\Theta) + \sum_{j=3}^q (-1)^{j-1} E_j P^{q-j}(\Theta),
\end{aligned}$$

where

$$E_j = \sum_{i_1, \dots, i_j=2}^n X_{i_1}^{i_2} \Theta_{i_2}^{i_3} \dots \Theta_{i_j}^{i_1}$$

for $j = 3, \dots, q$. Let $E_1 = \sum_{i=2}^n X_i^i$ and $E_2 = \sum_{i,j=2}^n X_i^j \Theta_i^j$. In order to express E_j as the differential forms that we have known, we need the following:

Sublemma 2.3. *For $j = 3, \dots, q$, we have a regressive relation for E_j :*

$$(2.48) \quad E_j = \omega \wedge E_{j-1} + \Phi \wedge \left(d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{j-2}$$

$$(2.49) \quad = \omega^{j-2} \wedge E_2 + \Phi \wedge \sum_{k=0}^{j-3} \left(d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{j-2-k} \omega^k,$$

where

$$(2.50) \quad \Phi = -\frac{\sqrt{-1}}{2\pi} \frac{1}{\rho} \sum_{k, \ell, p, q=2}^n X_k^\ell \frac{\partial a_\ell}{\partial z_p} \frac{\partial \bar{a}_s}{\partial \bar{z}_q} \tilde{g}^{k\bar{s}} dz_p \wedge d\bar{z}_q.$$

Proof. Equation (2.48) will be proved by induction. For $j = 3$, use Sublemma 2.2 to compute

$$\begin{aligned}
E_3 &= \sum_{i,j,k=2}^3 X_i^j \Theta_j^k \Theta_k^i = \omega \wedge \sum_{i,j=2}^n (X_i^j \Theta_j^i) \\
&\quad - \frac{\sqrt{-1}}{2\pi} \frac{1}{\rho} \sum_{\alpha, \beta=2}^n X_i^j \left(\frac{\partial a_j}{\partial z_\alpha} \frac{\partial \bar{a}_s}{\partial \bar{z}_\beta} \tilde{g}^{i\bar{s}} dz_\alpha \wedge d\bar{z}_\beta \right) \wedge \left(d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right) \\
&= \omega \wedge E_2 + \Phi \wedge \left(d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right).
\end{aligned}$$

Assume that (2.48) holds for $j - 1$. By Sublemma 2.2, the induction hypothesis, and (2.22), compute that

$$\begin{aligned}
 E_j &= \sum_{i_1, \dots, i_j=2}^n X_{i_1}^{i_2} \left(\Theta_{i_2}^{i_3} \cdots \Theta_{i_{j-1}}^{i_j} \right) \Theta_{i_j}^{i_1} \\
 &= \omega \sum_{i_1, i_2, i_4, \dots, i_j=2}^n X_{i_1}^{i_2} \Theta_{i_2}^{i_4} \Theta_{i_4}^{i_5} \cdots \Theta_{i_{j-1}}^{i_j} \Theta_{i_j}^{i_1} \\
 &\quad - \frac{\sqrt{-1}}{2\pi} \frac{1}{\rho} \sum_{i_1, i_2, i_j, \alpha, \beta=2}^n X_{i_1}^{i_2} \left(\frac{\partial a_{i_2}}{\partial z_\alpha} \frac{\partial \bar{a}_s}{\partial \bar{z}_\beta} \tilde{g}^{i_j \bar{s}} dz_\alpha \wedge d\bar{z}_\beta \right) \wedge (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{j-3} \wedge \Theta_{i_j}^{i_1} \\
 &= \omega \wedge E_{j-1} - \frac{\sqrt{-1}}{2\pi} \frac{1}{\rho} \sum_{i_1, i_2, i_j, \alpha, \beta=2}^n X_{i_1}^{i_2} \left(\frac{\partial a_{i_2}}{\partial z_\alpha} \frac{\partial \bar{a}_s}{\partial \bar{z}_\beta} \tilde{g}^{i_1 \bar{s}} dz_\alpha \wedge d\bar{z}_\beta \right) \wedge (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{j-2} \\
 &= \omega \wedge E_{j-1} + \Phi \wedge (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{j-2}.
 \end{aligned}$$

Equation (2.49) follows directly from (2.48). $\square$

Before we start to compute E_j , we need to find Φ .

Sublemma 2.4. *We compute Φ explicitly as follows:*

$$\begin{aligned}
 \Phi &= \operatorname{div}(X) \left((d-1)w + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right) - n \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \theta + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \Delta \theta \\
 &\quad - (n+1)\theta \left((d-1)w + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right).
 \end{aligned}$$

Proof. Using (2.42) in (2.50), Φ can be computed alternately by

$$\Phi = \frac{\sqrt{-1}}{2\pi} \frac{1}{\rho} \sum_{k, \ell, j, p, q=2}^n \tilde{g}^{\ell \bar{j}} \partial_k \bar{\partial}_j \theta \frac{\partial a_\ell}{\partial z_p} \frac{\partial \bar{a}_s}{\partial \bar{z}_q} \tilde{g}^{k \bar{s}} dz_p \wedge d\bar{z}_q.$$

By using the definition of $\tilde{g}^{k \bar{s}}$ given in Lemma 2.1, we get

$$\begin{aligned}
 (2.51) \quad \varrho \frac{\partial a_\ell}{\partial z_p} \frac{\partial \bar{a}_s}{\partial \bar{z}_q} \tilde{g}^{k \bar{s}} &= \frac{\sum_{i=0}^n |F_i|^2}{|F_1|^2} \frac{\partial a_\ell}{\partial z_p} \frac{\partial \bar{a}_k}{\partial \bar{z}_q} - \frac{\partial a_\ell}{\partial z_p} \bar{a}_k \sum_{s=2}^n \frac{\partial \bar{a}_s}{\partial \bar{z}_q} a_s + (1+|a|^2) \frac{\partial a_\ell}{\partial z_p} z_k \sum_{s=2}^n \frac{\partial \bar{a}_s}{\partial \bar{z}_q} \bar{z}_s \\
 &\quad - \frac{\bar{F}_0}{\bar{F}_1} \frac{\partial a_\ell}{\partial z_p} z_k \sum_{s=2}^n \frac{\partial \bar{a}_s}{\partial \bar{z}_q} a_s - \frac{F_0}{F_1} \frac{\partial a_\ell}{\partial z_p} \bar{a}_k \sum_{s=2}^n \frac{\partial \bar{a}_s}{\partial \bar{z}_q} \bar{z}_s.
 \end{aligned}$$

By using the definition of θ in (2.41), we get

$$\bar{\partial}_j \theta = -\frac{(\lambda_j - \lambda_0)z_j}{1+|z|^2} - \frac{(\lambda_1 - \lambda_0)z_1 \bar{a}_j}{1+|z|^2} - (\theta + \lambda_0) \frac{z_j + z_1 \bar{a}_j}{1+|z|^2}.$$

Then we compute

$$\begin{aligned}
 \partial_k \bar{\partial}_j \theta &= -\frac{\delta_{kj}(\lambda_j - \lambda_0)}{1 + |z|^2} + \frac{(\lambda_j - \lambda_0)z_j(\bar{z}_k + \bar{z}_1 a_k)}{(1 + |z|^2)^2} \\
 &\quad - \frac{(\lambda_1 - \lambda_0)a_k \bar{a}_j}{1 + |z|^2} + \frac{(\lambda_1 - \lambda_0)z_1 \bar{a}_j(\bar{z}_k + \bar{z}_1 a_k)}{(1 + |z|^2)^2} \\
 &\quad + \frac{(\lambda_k - \lambda_0)\bar{z}_k}{1 + |z|^2} \frac{z_j + z_1 \bar{a}_j}{1 + |z|^2} + \frac{(\lambda_1 - \lambda_0)\bar{z}_1 z_k}{1 + |z|^2} \frac{z_j + z_1 \bar{a}_j}{1 + |z|^2} \\
 &\quad + 2(\theta + \lambda_0) \frac{(\bar{z}_k + \bar{z}_1 a_k)(z_j + z_1 \bar{a}_j)}{(1 + |z|^2)^2} - (\theta + \lambda_0) \frac{\delta_{kj} + a_k \bar{a}_j}{1 + |z|^2}.
 \end{aligned}
 \tag{2.52}$$

A straightforward computation gives that

$$\begin{aligned}
 \tilde{g}^{\ell\bar{j}} \partial_k \bar{\partial}_j \theta &= -\delta_{\ell k}(\lambda_\ell - \lambda_0) + \frac{(\lambda_\ell - \lambda_0)z_\ell(\bar{z}_k + \bar{z}_1 a_k)}{1 + |z|^2} - \delta_{\ell k}(\theta + \lambda_0) \\
 &\quad + \bar{a}_\ell \frac{|F_1|^2}{\sum_{i=0}^n |F_i|^2} a_k (\lambda_k - \lambda_0) - (\lambda_1 - \lambda_0) a_k \bar{a}_\ell \frac{|F_1|^2}{\sum_{i=0}^n |F_i|^2} \\
 &\quad - z_\ell \frac{|F_1|^2}{\sum_{i=0}^n |F_i|^2} \frac{\bar{F}_0}{F_1} a_k ((\lambda_1 - \lambda_0) - (\lambda_k - \lambda_0)).
 \end{aligned}
 \tag{2.53}$$

Hence the Laplacian of θ is

$$\begin{aligned}
 \Delta \theta &= -\tilde{g}^{k\bar{j}} \partial_k \bar{\partial}_j \theta = -\left(\lambda_0 + \frac{|F_1|^2}{\sum_{i=0}^n |F_i|^2} \sum_{k=1}^n (\lambda_k - \lambda_0) |a_k|^2 - n\theta \right) \\
 &= -\left(\frac{\sum_{k=0}^n \lambda_k |F_k|^2}{\sum_{i=0}^n |F_i|^2} - n\theta \right).
 \end{aligned}
 \tag{2.54}$$

Denote $\operatorname{div}(X) = \sum_{i=2}^n X_i^i = \Delta \theta$.

Observe that

$$\begin{aligned}
 \partial_p \bar{\partial}_q \Delta \theta &= n \partial_p \bar{\partial}_q \theta - \frac{|F_1|^2}{\sum_{i=0}^n |F_i|^2} \sum_{k=2}^n (\lambda_k - \lambda_0) \frac{\partial a_k}{\partial z_p} \frac{\partial \bar{a}_k}{\partial \bar{z}_q} \\
 &\quad + \left(\frac{|F_1|^2}{\sum_{i=0}^n |F_i|^2} \right)^2 \sum_{k=2}^n (\lambda_k - \lambda_0) \frac{\partial a_k}{\partial z_p} \bar{a}_k \left(\sum_{s=2}^n \frac{\partial \bar{a}_s}{\partial \bar{z}_q} a_s + \frac{F_0}{F_1} \sum_{s=2}^n \frac{\partial \bar{a}_s}{\partial \bar{z}_q} \bar{z}_s \right) \\
 &\quad + \left(\frac{|F_1|^2}{\sum_{i=0}^n |F_i|^2} \right)^2 \sum_{k=2}^n (\lambda_k - \lambda_0) \frac{\partial \bar{a}_k}{\partial \bar{z}_q} a_k \left(\sum_{s=2}^n \frac{\partial a_s}{\partial z_p} \bar{a}_s + \frac{\bar{F}_0}{F_1} \sum_{s=2}^n \frac{\partial a_s}{\partial z_p} z_s \right) \\
 &\quad - \sum_{k=1}^n (\lambda_k - \lambda_0) |a_k|^2 \left(\frac{|F_1|^2}{\sum_{i=0}^n |F_i|^2} \right)^3 \\
 &\quad \cdot \left(\sum_{s=2}^n \frac{\partial a_s}{\partial z_p} \bar{a}_s + \frac{\bar{F}_0}{F_1} \sum_{s=2}^n \frac{\partial a_s}{\partial z_p} z_s \right) \left(\sum_{s=2}^n \frac{\partial \bar{a}_s}{\partial \bar{z}_q} a_s + \frac{F_0}{F_1} \sum_{s=2}^n \frac{\partial \bar{a}_s}{\partial \bar{z}_q} \bar{z}_s \right) \\
 &\quad + \sum_{k=1}^n (\lambda_k - \lambda_0) |a_k|^2 \frac{|F_1|^2}{\sum_{i=0}^n |F_i|^2} \partial_p \bar{\partial}_q \log \left| \frac{\nabla F}{F_1} \right|^2.
 \end{aligned}$$

Then multiply (2.51) and (2.53) together to get

$$\begin{aligned} \frac{1}{\rho} \bar{g}^{\ell\bar{j}} \partial_k \bar{\partial}_j \theta \frac{\partial a_\ell}{\partial z_p} \frac{\partial \bar{a}_s}{\partial \bar{z}_q} \bar{g}^{k\bar{s}} &= \partial_p \bar{\partial}_q \Delta \theta - n \partial_p \bar{\partial}_q \theta - (\theta + \lambda_0) \partial_p \bar{\partial}_q \log \frac{\sum_{i=0}^n |F_i|^2}{|F_1|^2} \\ &\quad - \sum_{i=1}^n (\lambda_i - \lambda_0) |a_i|^2 \frac{|F_1|^2}{\sum_{i=0}^n |F_i|^2} \partial_p \bar{\partial}_q \log \frac{\sum_{i=0}^n |F_i|^2}{|F_1|^2}. \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} \Phi &= \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \Delta \theta - \frac{\sqrt{-1}}{2\pi} n \partial \bar{\partial} \theta - (\theta + \lambda_0) \partial \bar{\partial} \log \frac{\sum_{i=0}^n |F_i|^2}{|F_1|^2} \\ &\quad - \sum_{i=1}^n (\lambda_i - \lambda_0) |a_i|^2 \frac{|F_1|^2}{\sum_{i=0}^n |F_i|^2} \partial \bar{\partial} \log \frac{\sum_{i=0}^n |F_i|^2}{|F_1|^2} \\ &= \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \Delta \theta - \frac{\sqrt{-1}}{2\pi} n \partial \bar{\partial} \theta - (n+1) \theta \left((d-1) \omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right) \\ &\quad + \left(n \theta - \sum_{i=1}^n (\lambda_i - \lambda_0) |a_i|^2 \frac{|F_1|^2}{\sum_{i=0}^n |F_i|^2} - \lambda_0 \right) \left((d-1) \omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right). \end{aligned}$$

Then by using (2.54) and $\Delta \theta = \operatorname{div}(X)$, we get Φ . $\square$

Sublemma 2.5. For $j = 3, \dots, q$, write E_j explicitly as follows:

$$\begin{aligned} E_j &= \operatorname{div}(X) \psi_{j-1} - (n+1) \theta (\psi_{j-1} - \omega^{j-1}) \\ (2.55) \quad &\quad - \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \theta \right) \wedge (n \zeta_{j-2} + \omega^{j-2}) + \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \Delta \theta \right) \wedge \zeta_{j-2}, \end{aligned}$$

where $\zeta_i = \sum_{k=0}^i (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{i-k} \omega^k$ and $\psi_i = (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^i$.

Proof. E_2 can be obtained directly from Φ :

$$E_2 = \sum_{i,j=2}^n X_i^j \Theta_j^i = (\operatorname{div}(X) \omega - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \theta + \Phi).$$

Then by using (2.49) and Sublemma 2.4, we get

$$\begin{aligned} E_j &= E_2 \wedge \omega^{j-2} + \Phi \wedge \sum_{k=0}^{j-3} (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{j-2-k} \omega^k \\ &= \Phi \wedge \sum_{k=0}^{j-2} (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{j-2-k} \omega^k + \operatorname{div}(X) \omega^{j-1} - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \theta \wedge \omega^{j-2} \\ &= \operatorname{div}(X) (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{j-1} - (n+1) \theta \left((d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{j-1} - \omega^{j-1} \right) \\ &\quad + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \theta \wedge \left(n \sum_{k=0}^{j-2} (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{j-2-k} \omega^k - \omega^{j-2} \right) \\ &\quad + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \Delta \theta \wedge \sum_{k=0}^{j-2} (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{j-2-k} \omega^k. \end{aligned}$$

$\square$

We return to the proof of Lemma 2.4. Denote $E_1 = \sum_{i=2}^n X_i^i = \operatorname{div} X = \Delta\theta$. In fact, we can write down E_2 explicitly:

$$\begin{aligned} E_2 &= \operatorname{div} X \left(d\omega + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\xi \right) - (n+1) \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\theta \\ &\quad + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\Delta\theta - (n+1)\theta \left((d-1)\omega + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\xi \right). \end{aligned}$$

Then we can use Sublemma 2.5 to compute $\sum_{j=1}^q (-1)^{j-1} E_j P^{q-j}(\Theta)$. The coefficient for $\operatorname{div}(X)$ in (2.47) is given by

$$\begin{aligned} (2.56) \quad & \sum_{j=1}^q (-1)^{j-1} \psi_{j-1} \sum_{k=0}^{q-j} \alpha_{(q-j)k} \omega^k \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\xi \right)^{q-j-k} \\ &= \sum_{j=1}^q (-1)^{j-1} \left(d\omega + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\xi \right)^{j-1} \sum_{k=0}^{q-j} \alpha_{(q-j)k} \omega^k \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\xi \right)^{q-j-k} \\ &= \sum_{t=1}^q \sum_{\ell=0}^{t-1} \sum_{j=t-\ell}^{q-\ell} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t} \omega^{q-t} \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\xi \right)^{t-1}. \end{aligned}$$

We derive several formulas before we perform further computations. For each fixed index $j \geq t$, use (2.24) to get

$$\begin{aligned} (2.57) \quad & \sum_{\ell=0}^{t-1} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t} \\ &= \sum_{\ell=1}^{t-1} (-1) \left(d\alpha_{(q-j-1)(q-j-\ell-1)} + \alpha_{(q-j-1)(q-j-\ell)} \right) (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t} \\ &\quad + (-1)^{j-1} d^{j-t} \left(\binom{n+1}{q-j} - d\alpha_{(q-j-1)(q-j-1)} \right) \binom{j-1}{j-t}. \end{aligned}$$

Rewrite (2.57) to get

$$\begin{aligned} (2.58) \quad & \sum_{\ell=0}^{t-1} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t} \\ &= \sum_{\ell=1}^{t-2} \alpha_{(q-j-1)(q-j-\ell-1)} (-1)^j d^{j+1+\ell-t} \left[\binom{j}{j+1+\ell-t} - \binom{j-1}{j+\ell+1-t} \right] \\ &\quad + \alpha_{(q-j-1)(q-j-t)} (-d)^j + \alpha_{(q-j-1)(q-j-1)} (-1)^j d^{j+1-t} \binom{j-1}{j+1-t} \\ &\quad + \sum_{\ell=1}^{t-2} \alpha_{(q-j-1)(q-j-\ell-1)} (-1)^j d^{j+\ell+1-t} \binom{j-1}{j+\ell+1-t} \\ &\quad + (-1)^{j-1} d^{j-t} \left(\binom{n+1}{q-j} - d\alpha_{(q-j-1)(q-j-1)} \right) \binom{j-1}{j-t}. \end{aligned}$$

Cancelling out the same terms in (2.58), we obtain

$$\begin{aligned}
 (2.59) \quad & \sum_{\ell=0}^{t-1} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t} \\
 &= \sum_{\ell=0}^{t-1} \alpha_{(q-j-1)(q-j-\ell-1)} (-1)^j d^{j+1+\ell-t} \binom{j}{j+1+\ell-t} \\
 &\quad + (-1)^{j-1} d^{j-t} \binom{n+1}{q-j} \binom{j-1}{j-t} \quad \text{for } j \geq t.
 \end{aligned}$$

Similarly to (2.59), we have

$$\begin{aligned}
 (2.60) \quad & \sum_{\ell=t-j}^{t-1} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t} \\
 &= \sum_{\ell=t-j-1}^{t-1} \alpha_{(q-j-1)(q-j-\ell-1)} (-1)^j d^{j+1+\ell-t} \binom{j}{j+1+\ell-t} \quad \text{for } j < t.
 \end{aligned}$$

For each $1 \leq s \leq q - 2t + 1$, we divide the following summation into two parts:

$$\begin{aligned}
 (2.61) \quad & \sum_{\ell=0}^{t-1} \sum_{j=t-\ell}^{q-\ell} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t} \\
 &= \sum_{\ell=0}^{t-1} \sum_{j=t+s}^{q-\ell} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t} \\
 &\quad + \sum_{j=1}^{t+s-1} \sum_{\ell=\max\{0, t-j\}}^{t-1} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t}.
 \end{aligned}$$

For $s = 1$ in the second part of (2.61), by (2.60) for each fixed j , each summation in ℓ is equal to each other. Then we use $j = t$ as the representation.

That is,

$$\begin{aligned}
 (2.62) \quad & \sum_{j=1}^t \sum_{\ell=t-j}^{t-1} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t} \\
 &= t \sum_{\ell=0}^{t-1} \alpha_{(q-t)(q-t-\ell)} (-1)^{t-1} d^{\ell} \binom{t-1}{\ell}.
 \end{aligned}$$

By (2.62), we have

$$\begin{aligned}
 (2.63) \quad & \sum_{j=1}^{t+s-1} \sum_{\ell=\max\{0, t-j\}}^{t-1} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t} \\
 &= t \sum_{\ell=0}^{t-1} \alpha_{(q-t)(q-t-\ell)} (-1)^{t-1} d^{\ell} \binom{t-1}{\ell} \\
 &\quad + \sum_{j=t+1}^{t+s-1} \sum_{\ell=0}^{t-1} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t}.
 \end{aligned}$$

By using (2.59) in (2.63), we obtain

$$\begin{aligned}
 (2.64) \quad & \sum_{j=1}^{t+s-1} \sum_{\ell=\max\{0, t-j\}}^{t-1} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t} \\
 &= t \sum_{\ell=0}^{t-1} \alpha_{(q-t-1)(q-t-1-\ell)} (-1)^t d^{\ell+1} \binom{t}{\ell+1} + t(-1)^{t-1} \binom{n+1}{q-t} \\
 &\quad + \sum_{j=t+1}^{t+s-1} \sum_{\ell=0}^{t-1} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t} \\
 &= (t+1) \sum_{\ell=0}^{t-1} \alpha_{(q-(t+1))(q-(t+1)-\ell)} (-1)^t d^{\ell+1} \binom{t}{\ell+1} + t(-1)^{t-1} \binom{n+1}{q-t} \\
 &\quad + \sum_{j=t+2}^{t+s-1} \sum_{\ell=0}^{t-1} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t}.
 \end{aligned}$$

Repeat the procedure in (2.64) and put it back into (2.61) to get

$$\begin{aligned}
 (2.65) \quad & \sum_{\ell=0}^{t-1} \sum_{j=t-\ell}^{q-\ell} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t} \\
 &= \sum_{\ell=0}^{t-1} \sum_{j=t+s}^{q-\ell} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t} \\
 &\quad + (t+s-1) \sum_{\ell=0}^{t-1} \alpha_{(q-(t+s-1))(q-(t+s-1)-\ell)} (-1)^{t-s-2} d^{s+\ell-1} \binom{t-s-2}{s+\ell-1} \\
 &\quad + \sum_{j=t}^{t+s-2} j (-1)^{j-1} d^{j-t} \binom{n+1}{q-j} \binom{j-1}{j-t} \quad \text{for } 1 \leq s \leq q-2t+1.
 \end{aligned}$$

For $s = q-2t+1$ in (2.65), we change the indexes in the first term on the right-hand side of it. That is,

$$\begin{aligned}
 (2.66) \quad & \sum_{\ell=0}^{t-1} \sum_{j=t+s}^{q-\ell} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t} \\
 &= \sum_{\ell=0}^{t-1} \sum_{j=q-t+1}^{q-\ell} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t} \\
 &= \sum_{j=0}^{t-1} \sum_{\ell=0}^{t-1-j} \alpha_{(t-1-j)(t-1-j-\ell)} (-1)^{q-t+j} d^{q-2t+j+\ell+1} \binom{q-t+j}{q-2t+j+\ell+1}.
 \end{aligned}$$

With the same procedure as we do in (2.59), we have

$$\begin{aligned}
 (2.67) \quad & \sum_{\ell=0}^{t-1-\lambda} \alpha_{(t-1-\lambda)(t-1-\lambda-\ell)} (-1)^{q-t+\lambda} d^{q-2t+\lambda+\ell+1} \binom{q-t+\lambda}{q-2t+\lambda+\ell+1} \\
 &= \sum_{\ell=0}^{t-2-\lambda} \alpha_{(t-2-\lambda)(t-2-\lambda-\ell)} (-1)^{q-t+\lambda+1} d^{q-2t+\lambda+\ell+2} \binom{q-t+\lambda+1}{q-2t+\lambda+\ell+2} \\
 &\quad + (-1)^{q-t+j} d^{q-2t+\lambda+\ell+1} \binom{q-t+\lambda}{q-2t+\lambda+\ell+1} \binom{n+1}{t-1-\lambda}
 \end{aligned}$$

for $j = 0, 1, \dots, t-1$. Put (2.66) back into (2.65) and use (2.67) to reiterate the procedure as (2.65). That is,

$$\begin{aligned}
 (2.68) \quad & \sum_{\ell=0}^{t-1} \sum_{j=t-\ell}^{q-\ell} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t} \\
 &= \sum_{j=0}^{t-1} \sum_{\ell=0}^{t-1-j} \alpha_{(t-1-j)(t-1-j-\ell)} (-1)^{q-t+j} d^{q-2t+j+\ell+1} \binom{q-t+j}{q-2t+j+\ell+1} \\
 &\quad + (q-t) \sum_{\ell=0}^{t-1} \alpha_{(t-1)(t-1-\ell)} (-1)^{q-t} d^{q-2t+\ell+1} \binom{q-t}{q-2t+\ell+1} \\
 &\quad + \sum_{j=t}^{q-t} j (-1)^{j-1} d^{j-t} \binom{n+1}{q-j} \binom{j-1}{j-t} \\
 &= \sum_{j=\lambda}^{t-1} \sum_{\ell=0}^{t-1-j} \alpha_{(t-1-j)(t-1-j-\ell)} (-1)^{q-t+j} d^{q-2t+j+\ell+1} \binom{q-t+j}{q-2t+j+\ell+1} \\
 &\quad + (q-t+\lambda) \sum_{\ell=0}^{t-1-\lambda} \alpha_{(t-1-\lambda)(t-1-\lambda-\ell)} (-1)^{q-t+\lambda} d^{q-2t+\lambda+\ell+1} \binom{q-t+\lambda}{q-2t+\lambda+\ell+1} \\
 &\quad + \sum_{j=t}^{q-t+\lambda} j (-1)^{j-1} d^{j-t} \binom{n+1}{q-j} \binom{j-1}{j-t}
 \end{aligned}$$

for $\lambda = 1, \dots, t-1$.

By substituting $\lambda = t-1$ in (2.68), we get

$$\begin{aligned}
 (2.69) \quad & \sum_{\ell=0}^{t-1} \sum_{j=t-\ell}^{q-\ell} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t} \\
 &= (-1)^{q-1} d^{q-t} \binom{q-1}{q-t} + (q-1) (-1)^{q-1} d^{q-t} \binom{q-1}{q-t} \\
 &\quad + \sum_{j=t}^{q-1} j (-1)^{j-1} d^{j-t} \binom{n+1}{q-j} \binom{j-1}{j-t} \\
 &= \sum_{j=t}^q j (-1)^{j-1} d^{j-t} \binom{n+1}{q-j} \binom{j-1}{j-t} = -t \alpha_{q(q-t)}.
 \end{aligned}$$

By substituting (2.69) into (2.56), we get the coefficient for $\text{div}(X)$:

$$\begin{aligned}
 (2.70) \quad & \sum_{j=1}^q (-1)^{j-1} \psi_{j-1} \sum_{k=0}^{q-j} \alpha_{(q-j)k} \omega^k \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{q-j-k} \\
 &= \sum_{t=1}^q \sum_{\ell=0}^{t-1} \sum_{j=t-\ell}^{q-\ell} \alpha_{(q-j)(q-j-\ell)} (-1)^{j-1} d^{j+\ell-t} \binom{j-1}{j+\ell-t} \omega^{q-t} \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{t-1} \\
 &= - \sum_{t=1}^q t \alpha_{q(q-t)} \omega^{q-t} \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{t-1} \\
 &= - \sum_{t=0}^{q-1} (q-t) \alpha_{qt} \omega^t \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{q-t-1}.
 \end{aligned}$$

The coefficient for θ in (2.47) is

$$\begin{aligned}
 (2.71) \quad & - (n+1) \sum_{j=2}^q (-1)^{j-1} (\psi_{j-1} - \omega^{j-1}) \sum_{k=0}^{q-j} \alpha_{(q-j)k} \omega^k \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{q-j-k} \\
 &= - (n+1) \sum_{j=2}^q (-1)^{j-1} \psi_{j-1} \sum_{k=0}^{q-j} \alpha_{(q-j)k} \omega^k \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{q-j-k} \\
 &\quad + (n+1) \sum_{j=2}^q (-1)^{j-1} \omega^{j-1} \sum_{k=0}^{q-j} \alpha_{(q-j)k} \omega^k \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{q-j-k}.
 \end{aligned}$$

Using (2.24) in the second part of (2.71), we get

$$\begin{aligned}
 (2.72) \quad & (n+1) \sum_{j=1}^q (-1)^j \sum_{k=0}^{q-j} \alpha_{(q-j)k} \omega^{k+j-1} \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{q-j-k} \\
 &= - (n+1) \sum_{t=0}^{q-1} \sum_{k=0}^t \alpha_{(q-t-1+k)k} (-1)^{t-k} \omega^t \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{q-t-1} \\
 &= (-1)^q (n+1) \sum_{t=0}^{q-1} \sum_{k=0}^t \binom{q-1-k}{t-k} \binom{n}{k} (-1)^k d^{t-k} \omega^t \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{q-t-1}.
 \end{aligned}$$

Use (2.24) to compute

$$\begin{aligned}
 (2.73) \quad & d(q-t)\alpha_{qt} - (t+1)\alpha_{q(t+1)} \\
 &= (-1)^q d(q-t) \sum_{k=0}^t \binom{q-k}{t-k} \binom{n+1}{k} d^{t-k} (-1)^k \\
 &\quad - (-1)^q (t+1) \sum_{k=0}^{t+1} \binom{q-k}{t+1-k} \binom{n+1}{k} d^{t+1-k} (-1)^k \\
 &= (-1)^q \sum_{k=0}^{t-1} (-1)^{k+1} \left((q-t) \binom{q-1-k}{t-1-k} - (t+1) \binom{q-1-k}{t-k} \right) \binom{n+1}{k+1} \\
 &\quad + (-1)^{q+t} (t+1) \binom{q-1-t}{0} \binom{n+1}{t+1} \\
 &= (-1)^q \sum_{k=0}^t (-1)^k (n+1) \binom{q-1-k}{t-k} \binom{n}{k}.
 \end{aligned}$$

Comparing the coefficients in (2.73) and (2.72), we get

$$\begin{aligned}
 (2.74) \quad & (n+1) \sum_{j=1}^q (-1)^j \sum_{k=0}^{q-j} \alpha_{(q-j)k} \omega^{k+j-1} \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{q-j-k} \\
 &= - \sum_{t=0}^{q-1} (d(q-t)\alpha_{qt} - (t+1)\alpha_{q(t+1)}) \omega^t \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{q-t-1}.
 \end{aligned}$$

Substituting (2.70) and (2.74) in (2.72), the coefficient for θ in (2.47) is

$$(2.75) \quad \sum_{t=0}^{q-1} ((n+1-d)(q-t)\alpha_{qt} + (t+1)\alpha_{q(t+1)}) \omega^t \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{q-t-1}.$$

By using Sublemma 2.4 and (2.49), the coefficient for $-\partial \bar{\partial} \theta$ in (2.47) is

$$\begin{aligned}
 (2.76) \quad & \frac{\sqrt{-1}}{2\pi} \sum_{j=2}^q (-1)^{j-1} \left(n \sum_{\ell=0}^{j-2} (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{j-2-\ell} \omega^\ell + \omega^{j-2} \right) \\
 & \cdot \sum_{k=0}^{q-j} \alpha_{(q-j)k} \omega^k \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{q-j-k} = \sum_{j=2}^q (-1)^{j-1} (n\zeta_{j-2} + \omega^{j-2}) \wedge \eta_{q-j},
 \end{aligned}$$

where

$$\begin{aligned}
 \eta_{q-j} &= \sum_{k=0}^{q-j} \alpha_{(q-j)k} \omega^k \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi \right)^{q-j-k}, \\
 \zeta_{j-2} &= \frac{\sqrt{-1}}{2\pi} \sum_{k=0}^{j-2} (d\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \xi)^{j-2-k} \omega^k.
 \end{aligned}$$

By using Sublemma 2.4 and (2.49), the coefficient for $\partial\bar{\partial}\Delta\theta$ in (2.47) is

$$(2.77) \quad \begin{aligned} & \frac{\sqrt{-1}}{2\pi} \sum_{j=2}^q (-1)^{j-1} \sum_{\ell=0}^{j-2} \left(d\omega + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\xi \right)^{j-2-\ell} \omega^\ell \sum_{k=0}^{q-j} \alpha_{(q-j)k} \omega^k \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\xi \right)^{q-j-k} \\ & = \sum_{j=2}^q (-1)^{j-1} \zeta_{j-2} \wedge \eta_{q-j}. \end{aligned}$$

By adding $-\text{div}(X) \cdot (2.70) + \theta \cdot (2.75) + \partial\bar{\partial}\theta \wedge (2.76) + \partial\bar{\partial}\Delta\theta \wedge (2.77)$, we obtain $q\tilde{P}^q(\nabla X, \Theta, \dots, \Theta)$. $\square$

Lemma 2.5. *The Hodge decomposition of equation (2.44) can be computed as follows:*

$$(2.78) \quad -q\tilde{P}^q(\nabla X, \Theta, \dots, \Theta) + q\alpha_{qq}\theta\omega^{q-1} - i(X)\partial f_q = -\kappa\alpha_{q(q-1)}\omega^{q-1} + \bar{\partial}\varphi_q,$$

where

$$(2.79) \quad \begin{aligned} \varphi_q &= \kappa \frac{\sqrt{-1}}{2\pi} \sum_{k=0}^{q-2} (q-k)\omega^k \wedge (\partial\xi) \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\xi \right)^{q-k-2} \\ &+ \frac{\sqrt{-1}}{2\pi} \sum_{k=1}^{q-1} k\alpha_{qk}\theta(\partial\xi) \wedge \omega^{k-1} \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\xi \right)^{q-k-1} \\ &- \frac{\sqrt{-1}}{2\pi} \sum_{k=0}^{q-1} \alpha_{qk}(q-k-1)X(\xi)\partial\xi\omega^k \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\xi \right)^{q-k-2} \\ &- \sum_{j=1}^q (-1)^{j-1} [(\partial\theta) \wedge (\omega^{j-1} + n\zeta_{j-2}) - (\partial\Delta\theta) \wedge \zeta_{j-2}] \wedge \eta_{q-j} \end{aligned}$$

is a globally defined form.

Proof. In (2.38), the potential form is obtained:

$$f_q = \sum_{k=0}^{q-1} \alpha_{qk}\xi\omega^k \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\xi \right)^{q-k-1}$$

for $q = 1, \dots, n-1$. By direction computation, we get

$$(2.80) \quad \begin{aligned} i(X)\partial f_q &= \sum_{k=0}^{q-1} \alpha_{qk}X(\xi)\omega^k \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\xi \right)^{q-k-1} \\ &+ \sum_{k=1}^{q-1} k\alpha_{qk} \frac{\sqrt{-1}}{2\pi} (\partial\xi) \wedge (\bar{\partial}\theta) \wedge \omega^{k-1} \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\xi \right)^{q-k-1} \\ &- \sum_{k=0}^{q-2} (q-k-1)\alpha_{qk} \frac{\sqrt{-1}}{2\pi} (\partial\xi) \wedge (\bar{\partial}X(\xi)) \wedge \omega^k \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\xi \right)^{q-k-2}. \end{aligned}$$

Using Lemma 2.4 and (2.80), we obtain

$$\begin{aligned}
& -q\tilde{P}^q(\nabla X, \Theta, \dots, \Theta) + q\alpha_{qq}\theta\omega^{q-1} - i(X)\partial f_q \\
& = \operatorname{div}(X) \sum_{k=0}^{q-1} (q-k)\alpha_{qk}\omega^k \wedge \left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\xi\right)^{q-1-k} \\
& \quad - \theta \sum_{k=0}^{q-1} [(q-k)(n+1-d)\alpha_{qk} + (k+1)\alpha_{q(k+1)}] \omega^k \wedge \left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\xi\right)^{q-1-k} \\
& \quad + \sum_{j=1}^q (-1)^{j-1} [(\partial\bar{\partial}\theta) \wedge (\omega^{j-1} + n\zeta_{j-2}) - (\partial\bar{\partial}\Delta\theta) \wedge \zeta_{j-2}] \wedge \eta_{q-j} \\
& \quad + q\alpha_{qq}\theta\omega^{q-1} - X(\xi) \sum_{k=0}^{q-1} [(q-k) - (q-k-1)]\alpha_{qk}\omega^k \wedge \left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\xi\right)^{q-1-k} \\
& \quad - \frac{\sqrt{-1}}{2\pi} \sum_{k=1}^{q-1} k\alpha_{qk}(\partial\xi) \wedge (\bar{\partial}\theta) \wedge \omega^{k-1} \left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\xi\right)^{q-k-1} \\
(2.81) \quad & + \frac{\sqrt{-1}}{2\pi} \sum_{k=0}^{q-2} (q-k-1)\alpha_{qk}(\partial\xi) \wedge (\bar{\partial}X(\xi)) \wedge \omega^k \wedge \left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\xi\right)^{q-k-2}.
\end{aligned}$$

In fact, we have the formula [12, Theorem 4.1]

$$(2.82) \quad \operatorname{div}(X) - X(\xi) - (n-d+1)\theta = -\kappa.$$

By using (2.82) in (2.81), we have

$$\begin{aligned}
& -q\tilde{P}^q(\nabla X, \Theta, \dots, \Theta) + q\alpha_{qq}\theta\omega^{q-1} - i(X)\partial f_q \\
& = -\kappa \sum_{k=0}^{q-1} (q-k)\alpha_{qk}\omega^k \wedge \left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\xi\right)^{q-1-k} \\
& \quad - \theta \sum_{k=1}^{q-1} k\alpha_{qk}\omega^{k-1} \wedge \left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\xi\right)^{q-1-k} \\
& \quad - \frac{\sqrt{-1}}{2\pi} \sum_{k=1}^{q-1} k\alpha_{qk}(\partial\xi) \wedge (\bar{\partial}\theta) \wedge \omega^{k-1} \left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\xi\right)^{q-k-1} \\
& \quad + \sum_{j=1}^q (-1)^{j-1} [(\partial\bar{\partial}\theta) \wedge (\omega^{j-1} + n\zeta_{j-2}) - (\partial\bar{\partial}\Delta\theta) \wedge \zeta_{j-2}] \wedge \eta_{q-j} \\
& \quad + X(\xi) \sum_{k=0}^{q-2} (q-k-1)\alpha_{qk}\omega^k \wedge \left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\xi\right)^{q-1-k} \\
(2.83) \quad & + \frac{\sqrt{-1}}{2\pi} \sum_{k=0}^{q-2} (q-k-1)\alpha_{qk}(\partial\xi) \wedge (\bar{\partial}X(\xi)) \wedge \omega^k \wedge \left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\xi\right)^{q-k-2}.
\end{aligned}$$

We can decompose (2.83) as $-\kappa\alpha_{q(q-1)}\omega^{q-1} + \bar{\partial}\varphi_q$, where φ_q is given in (2.79). Since ω^q is harmonic, the harmonic part of (2.83) is $-\kappa\alpha_{q(q-1)}\omega^q$. $\square$

Lemma 2.6. *We define $q\tilde{P}^q(\nabla X, \Theta, \dots, \Theta)$ in (2.47). For $q = 1, \dots, n-1$, we have*

$$(2.84) \quad \int_M q\tilde{P}^q(\nabla X, \Theta, \dots, \Theta) \wedge \omega_M^{n-q} = 0,$$

where $\omega_M = (n+1-d)\omega$.

Proof. Consider

$$\begin{aligned} 0 &= \int_M L_X(P^{q-1}(\Theta) \wedge \omega^{n-q}) \\ &= \int_M (L_X P^{q-1}(\Theta)) \wedge \omega^{n-q} + \int_M P^{q-1}(\Theta) \wedge L_X(\omega^{n-q}) \\ &= -(q-1) \frac{\sqrt{-1}}{2\pi} \int_M \partial \bar{\partial} \tilde{P}^{q-1}(\nabla X, \Theta, \dots, \Theta) \wedge \omega^{n-q} \\ (2.85) \quad &\quad - (n-q) \frac{\sqrt{-1}}{2\pi} \int_M P^{q-1}(\Theta) \wedge \partial \bar{\partial} \theta \wedge \omega^{n-q-1}. \end{aligned}$$

By Stoke's Theorem, we have $\int_M \partial \bar{\partial} \tilde{P}^{q-1}(\nabla X, \Theta, \dots, \Theta) \wedge \omega^{n-q} = 0$. By using (2.13) and (2.42), we compute that

$$\begin{aligned} (2.86) \quad &-(n-q) \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \theta \wedge P^{q-1}(\Theta) \wedge \omega^{n-q-1} \\ &= -\frac{(n-q)(n-1-q)!}{(q-1)!(n-1)!} \sum_{\sigma \in S_{q-1}, \tau \in S_q} \sum_{i_1, j_1, k_1, \dots, k_{q-1}, i_q, j_q=2}^n \text{sgn}(\sigma) \text{sgn}(\tau) \\ &\quad \cdot \partial_{i_1} \bar{\partial}_{j_1} \theta R_{k_1 i_2 \bar{j}_2}^{k_{\sigma(1)}} \dots R_{k_{q-1} i_q \bar{j}_q}^{k_{\sigma(q-1)}} \tilde{g}^{i_{\tau(1)} \bar{j}_1} \dots \tilde{g}^{i_{\tau(q)} \bar{j}_q} \omega^{n-1} \\ &= \frac{(n-q)!}{(q-1)!(n-1)!} \sum_{\sigma \in S_{q-1}, \tau \in S_q} \sum_{i_1, k_1, s_1, \dots, s_{q-1}, i_q, k_{q-1}=2}^n \text{sgn}(\sigma) \text{sgn}(\tau) \\ &\quad \cdot X_{i_1}^{i_{\tau(1)}} R_{i_2 k_1 \bar{s}_1}^{i_{\tau(2)}} \dots R_{i_q k_{q-1} \bar{s}_{q-1}}^{i_{\tau(q)}} \tilde{g}^{k_{\sigma(1)} \bar{s}_1} \dots \tilde{g}^{k_{\sigma(q-1)} \bar{s}_{q-1}} \omega^{n-1}. \end{aligned}$$

By definition, we have

$$\begin{aligned} (2.87) \quad \tilde{P}^q(\nabla X, \Theta, \dots, \Theta) &= \frac{1}{q!} \left(\frac{\sqrt{-1}}{2\pi} \right)^{q-1} \sum_{\tau \in S_q} \sum_{i_1, k_1, s_1, \dots, k_{q-1}, s_{q-1}, i_q=2}^n \text{sgn}(\tau) \\ &\quad \cdot X_{i_1}^{i_{\tau(1)}} R_{i_2 k_1 \bar{s}_1}^{i_{\tau(2)}} \dots R_{i_q k_{q-1} \bar{s}_{q-1}}^{i_{\tau(q)}} dz_{k_1} \wedge d\bar{z}_{s_1} \wedge \dots \wedge dz_{k_{q-1}} \wedge d\bar{z}_{s_{q-1}}. \end{aligned}$$

By comparing (2.86) and (2.87), we obtain that

$$-(n-q) \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \theta \wedge P^{q-1}(\Theta) \wedge \omega^{n-q-1} = q\tilde{P}^q(\nabla X, \Theta, \dots, \Theta) \wedge \omega^{n-q}.$$

Therefore, we have

$$\int_M q\tilde{P}^q(\nabla X, \Theta, \dots, \Theta) \wedge \omega_M^{n-q} = \int_M L_X(P^{q-1}(\Theta) \wedge \omega_M^{n-q}) = 0.$$

□

Lemma 2.6 can be considered as a result of a generalized divergence theorem. Please see the Appendix for a further study.

Proof of Theorem 1.1. From (2.45), we can compute the q -th Bando-Futaki invariant by

$$(2.88) \quad \int_M L_X f_q \wedge \omega_M^{n-q} = (n+1-d)^{n-q} \left(-C(q) \int_M \omega^{n-1} + q\alpha_{qq} \int_M \theta \omega^{n-1} \right) \\ - q \int_M \tilde{P}^q(\nabla X, \Theta, \dots, \Theta) \wedge \omega_M^{n-q} - \int_M \bar{\partial} \varphi_q \wedge \omega_M^{n-q}.$$

From Lemma 2.5, we have $C(q) = -\kappa\alpha_{q(q-1)}$. By [12, Theorem 5.1], we have $\int_M \theta \omega^{n-1} = \frac{\kappa}{n}$. From Lemma 2.6, we have $q \int_M \tilde{P}^q(\nabla X, \Theta, \dots, \Theta) \wedge \omega_M^{n-q} = 0$. In addition, the divergence theorem implies that $\int_M \bar{\partial} \varphi_q \wedge \omega_M^{n-q} = 0$. With this information, we get

$$\int_M L_X f_q \wedge \omega_M^{n-q} = \kappa(n+1-d)^{n-q} (\alpha_{qq} \frac{q}{n} + d\alpha_{q(q-1)}).$$

Using (2.24), the q -th Bando-Futaki invariant can be computed directly by

$$\begin{aligned} \mathcal{F}_q(X) &= -(n+1-d)^{n-q} \frac{(d-1)}{n} \sum_{j=0}^{q-1} (-d)^j (j+1)(q-j) \binom{n+1}{q-j} \kappa \\ &= -(n+1-d)^{n-q} \frac{(d-1)(n+1)}{n} \sum_{j=0}^{q-1} (-d)^j (j+1) \binom{n}{q-1-j} \kappa. \end{aligned}$$

□

Corollary 2.1. *With the notation as in Theorem 1.1, if $d = 1$, all the Bando-Futaki invariants are zero.*

Corollary 2.2. *With the notation as in Theorem 1.1, if M is a cubic surface in $\mathbb{CP}^3$, then the Bando-Futaki invariants are*

$$\mathcal{F}_1(X) = -\frac{8}{3}\kappa \quad \text{and} \quad \mathcal{F}_2(X) = 16\kappa.$$

The following example is due to [4] and [12].

Example 2.1. Let

$$M = \{Z \in \mathbb{CP}^3 \mid F(Z) = Z_0 Z_1^2 + Z_2^2 Z_3 - Z_2 Z_3^2 = 0\}$$

be a hypersurface in $\mathbb{CP}^3$, where $Z = [Z_0, Z_1, Z_2, Z_3]$ are the homogeneous coordinates of $\mathbb{CP}^3$. M is a Kähler orbifold with $c_1(M) > 0$. Let

$$X = -7Z_0 \frac{\partial}{\partial Z_0} + 5Z_1 \frac{\partial}{\partial Z_1} + Z_2 \frac{\partial}{\partial Z_2} + Z_3 \frac{\partial}{\partial Z_3}$$

be a holomorphic vector field in $\mathbb{CP}^3$, which satisfies $XF = 3F$. The Bando-Futaki invariants are

$$\mathcal{F}_1(X) = -8 \quad \text{and} \quad \mathcal{F}_2(X) = 48.$$

Example 2.2. Let

$$M = \{Z \in \mathbb{CP}^4 \mid F(Z) = Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 = 0\}$$

be a hypersurface in $\mathbb{CP}^4$. Let

$$X = Z_0 \frac{\partial}{\partial Z_0} + Z_1 \frac{\partial}{\partial Z_1} + Z_2 \frac{\partial}{\partial Z_2} + Z_3 \frac{\partial}{\partial Z_3} + Z_4 \frac{\partial}{\partial Z_4}$$

be a holomorphic vector field on $\mathbb{CP}^4$, which satisfies

$$X(F) = 2F.$$

The Bando-Futaki invariants are

$$\mathcal{F}_1(X) = -270, \quad \mathcal{F}_2(X) = 135, \quad \text{and} \quad \mathcal{F}_3(X) = -\frac{285}{2}.$$

3. CHEN AND TIAN'S HOLOMORPHIC INVARIANTS

The following holomorphic invariants were introduced by Chen and Tian [3]. We prove that they are the Futaki invariants.

Definition 3.1. Let M be an n -dimensional simply connected Kähler manifold with positive first Chern class $c_1(M)$. Let ω be a Kähler form such that $[\omega] \in c_1(M)$. There exists a smooth function θ_X such that $i(X)\omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \theta_X$.³ Define

$$\begin{aligned} \mathcal{F}_k(X, \omega) &= (n-k) \int_M \theta_X \omega^n + (k+1) \int_M \Delta \theta_X Ric(\omega)^k \wedge \omega^{n-k} \\ &\quad - (n-k) \int_M \theta_X Ric(\omega)^{k+1} \wedge \omega^{n-k-1}. \end{aligned} \quad (3.1)$$

These new holomorphic invariants are independent of the choices of the Kähler metrics in the Kähler class $[\omega]$, which were shown in [3].

Proof of Theorem 1.2. There exists a smooth real-valued function f over M , such that

$$Ric(\omega) - \omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f. \quad (3.2)$$

Taking the inner derivative on both sides, we have

$$\operatorname{div} X + \theta_X + X(f) = \beta, \quad (3.3)$$

where β is a constant if M is compact. We need the following two formulas:

$$\begin{aligned} 0 &= \int_M (i(X)[\partial f(\partial \bar{\partial} f)^{j-1} \omega^{n-j+1}]) \\ &= j \int_M X(f)(\partial \bar{\partial} f)^{j-1} \omega^{n-j+1} + (n-j+1) \int_M \bar{\partial} \theta_X \partial f(\partial \bar{\partial} f)^{j-1} \omega^{n-j} \\ &= j \int_M X(f)(\partial \bar{\partial} f)^{j-1} \omega^{n-j+1} + (n-j+1) \int_M \theta_X (\partial \bar{\partial} f)^j \omega^{n-j} \end{aligned} \quad (3.4)$$

for $1 \leq j \leq k+1$, and

$$\operatorname{div} X = \Delta \theta_X + c, \quad (3.5)$$

where c is a constant and $\Delta \theta_X = g^{i\bar{j}} \partial_i \bar{\partial}_j \theta_X$ if $\omega = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^n g_{i\bar{j}} dz_i \wedge d\bar{z}_j$. By using (3.2), the new holomorphic invariants can be written as

$$\begin{aligned} \mathcal{F}_k(X, \omega) &= (k+1) \int_M \Delta \theta_X \sum_{i=1}^k \binom{k}{i} \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f \right)^i \omega^{n-i} \\ &\quad - (n-k) \int_M \theta_X \sum_{i=1}^{k+1} \binom{k+1}{i} \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f \right)^i \omega^{n-i}. \end{aligned}$$

³In order to maintain the definition as in the original paper, we have opposite signs of the notation for θ_X and Δ from what we used in the previous section.

For each $1 \leq i \leq k+1$, using $(n-i+1) = (n-k) + (k-i-1)$ and (3.4), we get

$$(3.6) \quad \begin{aligned} & -(n-k) \int_M \theta_X \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f \right)^i \omega^{n-i} \\ &= (k-i-1) \int_M \theta_X \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f \right)^i \omega^{n-i} + i \int_M X(f) (\partial \bar{\partial} f)^{i-1} \omega^{n-i+1}. \end{aligned}$$

By using (3.5) and (3.6), we have

$$(3.7) \quad \begin{aligned} \mathcal{F}_k(X, \omega) &= (k+1) \int_M \operatorname{div} X \sum_{i=1}^k \binom{k}{i} \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f \right)^i \omega^{n-i} \\ &\quad + \int_M \sum_{i=1}^{k+1} i \binom{k+1}{i} X(f) \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f \right)^{i-1} \omega^{n-i+1} \\ &\quad + \int_M \sum_{i=1}^{k+1} (k-i+1) \binom{k+1}{i} \theta_X \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f \right)^i \omega^{n-i}. \end{aligned}$$

Since $\sum_{i=1}^{k+1} i \binom{k+1}{i} = \sum_{i=0}^k (i+1) \binom{k+1}{i+1} = (k+1) + \sum_{i=1}^k (k+1) \binom{k}{i}$, we divide the term with $X(f)$ in (3.7) into two parts and combine one part with $\operatorname{div}(X)$. That is,

$$(3.8) \quad \begin{aligned} \mathcal{F}_k(X, \omega) &= (k+1) \int_M (\operatorname{div} X + X(f)) \sum_{i=1}^k \binom{k}{i} \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f \right)^i \omega^{n-i} \\ &\quad + (k+1) \int_M X(f) \omega^n \\ &\quad + \int_M \sum_{i=1}^{k+1} (k+1) \binom{k}{i} \theta_X \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f \right)^i \omega^{n-i}. \end{aligned}$$

By using (3.3) in (3.8), we get

$$\begin{aligned} \mathcal{F}_k(X, \omega) &= (k+1) \int_M X(f) \omega^n - (k+1) \int_M \theta_X \sum_{i=1}^k \binom{k}{i} \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f \right)^i \omega^{n-i} \\ &\quad + \int_M \sum_{i=1}^k (k+1) \binom{k}{i} \theta_X \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f \right)^i \omega^{n-i} \\ &= (k+1) \int_M X(f) \omega^n. \end{aligned}$$

These holomorphic invariants are simply the Futaki invariants on compact Kähler manifolds with $c_1(M) > 0$. Indeed, Kobayashi [10] proved that a compact Kähler manifold with positive Ricci curvature is simply connected. $\square$

4. APPENDIX

Lemma 4.1. *For $q = 2, \dots, n-1$, let*

$$Y = \sum_{i_1=2}^n Y^{i_1} \frac{\partial}{\partial z_{i_1}} \in T^{1,0}(M) \otimes \wedge^{q-1}(T^{1,0}(M)^* \otimes \overline{T^{1,0}(M)}^*)$$

be a holomorphic vector field with $(q-1, q-1)$ -valued form, where

$$Y^{i_1} = X^{i_1} P^{q-1}(\Theta) + \sum_{j=2}^q \sum_{i_2, \dots, i_j=2}^n (-1)^{j-1} X^{i_2} \Theta_{i_2}^{i_3} \dots \Theta_{i_j}^{i_1} P^{q-j}(\Theta).$$

Then we have $q\tilde{P}^q(\nabla X, \Theta, \dots, \Theta) = \operatorname{div}(Y)$. For $q = 1$, we have $Y = X$ and $\tilde{P}^1(\nabla X) = \operatorname{div}(X)$.

Proof. We have

$$\begin{aligned} \operatorname{div}(Y) &= \sum_{i_1=2}^n \nabla_{i_1} Y^{i_1} \\ &= \sum_{i_1=2}^n X^{i_1} P^{q-1} + \sum_{i_1=2}^n X^{i_1} \nabla_{i_1} P^{q-1} \\ &\quad + \sum_{j=2}^q \sum_{i_1, \dots, i_j=2}^n (-1)^{j-1} X^{i_2} \Theta_{i_2}^{i_3} \dots \Theta_{i_j}^{i_1} P^{q-j}(\Theta) \\ &\quad + \sum_{j=2}^q \sum_{i_1, \dots, i_j=2}^n (-1)^{j-1} X^{i_2} \nabla_{i_1} (\Theta_{i_2}^{i_3} \dots \Theta_{i_j}^{i_1}) P^{q-j}(\Theta) \\ (4.1) \quad &\quad + \sum_{j=2}^q \sum_{i_1, \dots, i_j=2}^n (-1)^{j-1} X^{i_2} \Theta_{i_2}^{i_3} \dots \Theta_{i_j}^{i_1} \nabla_{i_1} (P^{q-j}(\Theta)). \end{aligned}$$

By using the definition of $q\tilde{P}^q(\nabla X, \Theta, \dots, \Theta)$ in (2.47), we get

$$\begin{aligned} \operatorname{div}(Y) &= q\tilde{P}^q(\nabla X, \Theta, \dots, \Theta) + \sum_{i_1=2}^n X^{i_1} \nabla_{i_1} P^{q-1} \\ &\quad - \sum_{j=2}^q \sum_{i_1, \dots, i_j=2}^n (-1)^j X^{i_2} \nabla_{i_1} (\Theta_{i_2}^{i_3} \dots \Theta_{i_j}^{i_1}) P^{q-j}(\Theta) \\ (4.2) \quad &\quad - \sum_{j=2}^q \sum_{i_1, \dots, i_j=2}^n (-1)^j X^{i_2} \Theta_{i_2}^{i_3} \dots \Theta_{i_j}^{i_1} \nabla_{i_1} P^{q-j}(\Theta). \end{aligned}$$

This is equivalent to showing the following sublemma.

Sublemma 4.1. For $q = 2, \dots, n-1$, we have

$$\begin{aligned} \sum_{i_1=2}^n X^{i_1} \nabla_{i_1} P^{q-1} &= \sum_{j=2}^q \sum_{i_1, \dots, i_j=2}^n (-1)^j X^{i_2} \nabla_{i_1} (\Theta_{i_2}^{i_3} \dots \Theta_{i_j}^{i_1}) P^{q-j}(\Theta) \\ (4.3) \quad &\quad + \sum_{j=2}^q \sum_{i_1, \dots, i_j=2}^n (-1)^j X^{i_2} \Theta_{i_2}^{i_3} \dots \Theta_{i_j}^{i_1} \nabla_{i_1} P^{q-j}(\Theta). \end{aligned}$$

For $q = 1$, (4.3) is zero.

We need two formulas before proving Sublemma 4.1. In fact, equation (4.4) gives

$$\begin{aligned}\nabla\Theta &= \sum_{i,j,k=2}^n \nabla_i \Theta_j^k \frac{\partial}{\partial z_k} \otimes dz_i \wedge dz_j \\ &= - \sum_{i,j,k=2}^n \nabla_i \Theta_j^k \frac{\partial}{\partial z_k} \otimes dz_j \wedge dz_i \\ &= - \sum_{i,j,k=2}^n \nabla_j \Theta_i^k \frac{\partial}{\partial z_k} \otimes dz_j \wedge dz_i = -\nabla\Theta.\end{aligned}$$

This is the Bianchi identity $\nabla\Theta = 0$.

Sublemma 4.2. *For $2 \leq i \leq n$ and $q = 1, \dots, n-1$, we compute the covariant derivative on the curvature form and the Chern forms*

$$(4.4) \quad \nabla_i \Theta_j^k = \nabla_j \Theta_i^k,$$

$$(4.5) \quad \nabla_i P^q(\Theta) = \sum_{\ell=1}^q \sum_{i_1, \dots, i_\ell=2}^n (-1)^{\ell-1} \frac{1}{\ell} \nabla_i (\Theta_{i_1}^{i_2} \Theta_{i_2}^{i_3} \dots \Theta_{i_\ell}^{i_1}) P^{q-\ell}(\Theta),$$

where $\Theta_j^k = \frac{\sqrt{-1}}{2\pi} \sum_{s,t=2}^n R_{jst}^k dz_s \wedge d\bar{z}_t = -\frac{\sqrt{-1}}{2\pi} \sum_{s,t=2}^n \bar{\partial}_t \Gamma_{sj}^k dz_s \wedge d\bar{z}_t$. Since M is Kähler, $\Gamma_{sj}^k = \Gamma_{js}^k = \frac{\partial \bar{g}_{s\bar{j}}}{\partial z_j} \bar{g}^{k\bar{s}}$.

Proof. First, we compute

$$\begin{aligned}\nabla_i R_{jst}^k &= -\partial_i \bar{\partial}_t \Gamma_{js}^k + \Gamma_{i\lambda}^k R_{jst}^\lambda - \Gamma_{ij}^\lambda R_{\lambda st}^k - \Gamma_{is}^\lambda R_{j\lambda t}^k \\ &= -\bar{\partial}_t \left(\frac{\partial^2 \bar{g}_{s\bar{j}}}{\partial z_i \partial z_j} \bar{g}^{k\bar{s}} \right) + \bar{\partial}_t \left(\frac{\partial \bar{g}_{s\bar{j}}}{\partial z_j} \frac{\partial \bar{g}_{\alpha\bar{i}}}{\partial z_i} \bar{g}^{k\bar{\alpha}} \bar{g}^{\alpha\bar{s}} \right) \\ &\quad + \Gamma_{i\lambda}^k R_{jst}^\lambda - \Gamma_{ij}^\lambda R_{\lambda st}^k - \Gamma_{is}^\lambda R_{j\lambda t}^k \\ &= -\bar{\partial}_t \left(\frac{\partial^2 \bar{g}_{s\bar{j}}}{\partial z_i \partial z_j} \bar{g}^{k\bar{s}} \right) - \Gamma_{i\alpha}^k R_{jst}^\alpha - \Gamma_{js}^\alpha R_{i\alpha t}^k \\ &\quad + \Gamma_{i\lambda}^k R_{jst}^\lambda - \Gamma_{ij}^\lambda R_{\lambda st}^k - \Gamma_{is}^\lambda R_{j\lambda t}^k.\end{aligned}\tag{4.6}$$

Similarly, we get

$$\begin{aligned}\nabla_j R_{ist}^k &= -\bar{\partial}_t \left(\frac{\partial^2 \bar{g}_{s\bar{j}}}{\partial z_j \partial z_i} \bar{g}^{k\bar{s}} \right) - \Gamma_{j\alpha}^k R_{ist}^\alpha - \Gamma_{is}^\alpha R_{j\alpha t}^k \\ &\quad + \Gamma_{j\lambda}^k R_{ist}^\lambda - \Gamma_{ji}^\lambda R_{\lambda st}^k - \Gamma_{js}^\lambda R_{i\lambda t}^k.\end{aligned}\tag{4.7}$$

Since M is Kähler, $\Gamma_{ij}^\lambda = \Gamma_{ji}^\lambda$. Therefore, (4.6) and (4.7) are equal. Then we get (4.4). Next, we prove (4.5) by induction. For $q = 1$, $\nabla_i P^1(\Theta) = \nabla_i \sum_{i_1=2}^n \Theta_{i_1}^{i_1}$. Assume that (4.5) is true for $2 \leq k \leq q-1$. By using (2.14), we compute the covariant derivative on the q -Chern form:

$$\begin{aligned}\nabla_i P^q(\Theta) &= \frac{1}{q} \sum_{j=1}^q (-1)^{j-1} \nabla_i (\Theta_{i_1}^{i_2} \Theta_{i_2}^{i_3} \dots \Theta_{i_j}^{i_1}) P^{q-j}(\Theta) \\ &\quad + \frac{1}{q} \sum_{j=1}^q (-1)^{j-1} \Theta_{i_1}^{i_2} \Theta_{i_2}^{i_3} \dots \Theta_{i_j}^{i_1} \nabla_i P^{q-j}(\Theta).\end{aligned}\tag{4.8}$$

By using the induction hypothesis on the second term of (4.8), we get

$$\begin{aligned}
 & \frac{1}{q} \sum_{j=1}^q (-1)^{j-1} \Theta_{i_1}^{i_2} \Theta_{i_2}^{i_3} \cdots \Theta_{i_j}^{i_1} \nabla_i P^{q-j}(\Theta) \\
 &= \frac{1}{q} \sum_{j=1}^q (-1)^{j-1} \Theta_{i_1}^{i_2} \cdots \Theta_{i_j}^{i_1} \sum_{\ell=1}^{q-j} (-1)^{\ell-1} \frac{1}{\ell} \nabla_i (\Theta_{i_{j+1}}^{i_{j+2}} \cdots \Theta_{i_{j+\ell}}^{i_{j+1}}) P^{q-j-\ell}(\Theta) \\
 &= \frac{1}{q} \sum_{\ell=1}^q (-1)^{\ell-1} \frac{1}{\ell} \nabla_i (\Theta_{i_{j+1}}^{i_{j+2}} \cdots \Theta_{i_{j+\ell}}^{i_{j+1}}) \sum_{j=1}^{q-\ell} (-1)^{j-1} \Theta_{i_1}^{i_2} \cdots \Theta_{i_j}^{i_1} P^{q-j-\ell}(\Theta) \\
 (4.9) \quad &= \frac{1}{q} \sum_{\ell=1}^q (-1)^{\ell-1} \frac{q-\ell}{\ell} \nabla_i (\Theta_{i_1}^{i_2} \cdots \Theta_{i_\ell}^{i_1}) P^{q-\ell}(\Theta).
 \end{aligned}$$

By substituting (4.9) back into (4.8), we obtain (4.5). $\square$

Proof of Sublemma 4.1. The proof is by induction. For $q = 2$, by using (4.3), we get

$$X^i \nabla_i \Theta_j^j = X^i \nabla_j \Theta_i^j = X^j \nabla_i \Theta_j^i.$$

Assume that (4.3) holds for $k = 2, \dots, q-1$. By using (4.5), we compute

$$\begin{aligned}
 (4.10) \quad & \sum_{i_1=2}^n X^{i_1} \nabla_{i_1} P^{q-1}(\Theta) - \sum_{j=2}^q \sum_{i_1, \dots, i_j=2}^n (-1)^j X^{i_2} \nabla_{i_1} (\Theta_{i_2}^{i_3} \cdots \Theta_{i_j}^{i_1}) P^{q-j}(\Theta) \\
 & - \sum_{j=2}^q \sum_{i_1, \dots, i_j=2}^n (-1)^j X^{i_2} \Theta_{i_2}^{i_3} \cdots \Theta_{i_j}^{i_1} \nabla_{i_1} P^{q-j}(\Theta) \\
 &= \sum_{j=2}^q \sum_{i_1, \dots, i_j=2}^n (-1)^j \frac{1}{j-1} X^{i_1} \nabla_{i_1} (\Theta_{i_2}^{i_3} \cdots \Theta_{i_j}^{i_2}) P^{q-j}(\Theta) \\
 & - \sum_{j=2}^q \sum_{i_1, \dots, i_j=2}^n (-1)^j X^{i_2} \nabla_{i_1} (\Theta_{i_2}^{i_3} \cdots \Theta_{i_j}^{i_1}) P^{q-j}(\Theta) \\
 & - \sum_{j=2}^q \sum_{i_1, \dots, i_j=2}^n (-1)^j X^{i_2} \Theta_{i_2}^{i_3} \cdots \Theta_{i_j}^{i_1} \sum_{\ell=1}^{q-j} (-1)^{\ell-1} \frac{1}{\ell} \nabla_{i_1} (\Theta_{i_{j+1}}^{i_{j+2}} \cdots \Theta_{i_{j+\ell}}^{i_{j+1}}) P^{q-j-\ell}(\Theta).
 \end{aligned}$$

In fact, we have

$$(4.11) \quad \nabla_{i_1} \left(\sum_{i_2, \dots, i_j=2}^n \Theta_{i_2}^{i_3} \cdots \Theta_{i_j}^{i_2} \right) = (j-1) \sum_{i_2, \dots, i_j=2}^n (\nabla_{i_1} \Theta_{i_2}^{i_3}) \cdots \Theta_{i_j}^{i_2}$$

and

$$\begin{aligned}
 (4.12) \quad \nabla_{i_1} (\Theta_{i_2}^{i_3} \cdots \Theta_{i_j}^{i_1}) &= (\nabla_{i_1} \Theta_{i_2}^{i_3}) \Theta_{i_3}^{i_4} \cdots \Theta_{i_j}^{i_1} + \sum_{k=3}^{j-1} \Theta_{i_2}^{i_3} \cdots \Theta_{i_{k-1}}^{i_k} (\nabla_{i_1} \Theta_{i_k}^{i_{k+1}}) \cdots \Theta_{i_j}^{i_1} \\
 &\quad + \Theta_{i_2}^{i_3} \cdots \Theta_{i_{j-1}}^{i_j} (\nabla_{i_1} \Theta_{i_j}^{i_1}).
 \end{aligned}$$

Using (4.11) in the first term on the right-hand side of (4.10) and using (4.12) in the second term of (4.10), we get

$$\begin{aligned}
 (4.13) \quad & \sum_{i_1, i_2=2}^n X^{i_1} \nabla_{i_1} \Theta_{i_2}^{i_2} P^{q-2}(\Theta) + \sum_{j=3}^q \sum_{i_1, \dots, i_j=2}^n (-1)^j X^{i_1} (\nabla_{i_1} \Theta_{i_2}^{i_3}) \dots \Theta_{i_j}^{i_2} P^{q-j}(\Theta) \\
 & - \sum_{j=2}^q \sum_{i_1, \dots, i_j=2}^n (-1)^j X^{i_2} (\nabla_{i_1} \Theta_{i_2}^{i_3}) \dots \Theta_{i_j}^{i_1} P^{q-j}(\Theta) \\
 & - \sum_{j=2}^q \sum_{i_1, \dots, i_j=2}^n (-1)^j \sum_{k=3}^{j-1} X^{i_2} \Theta_{i_2}^{i_3} \dots \Theta_{i_{k-1}}^{i_k} (\nabla_{i_1} \Theta_{i_k}^{i_{k+1}}) \dots \Theta_{i_j}^{i_1} P^{q-j}(\Theta) \\
 & - \sum_{j=2}^q \sum_{i_1, \dots, i_j=2}^n (-1)^j X^{i_2} \Theta_{i_2}^{i_3} \dots (\nabla_{i_1} \Theta_{i_j}^{i_1}) P^{q-j}(\Theta) \\
 & - \sum_{j=2}^q \sum_{i_1, \dots, i_{j+\ell}=2}^n (-1)^j X^{i_2} \Theta_{i_2}^{i_3} \dots \Theta_{i_j}^{i_1} \sum_{\ell=1}^{q-j} (-1)^{\ell+1} (\nabla_{i_1} \Theta_{i_{j+1}}^{i_{j+2}}) \dots \Theta_{i_{j+\ell}}^{i_{j+1}} P^{q-j}(\Theta).
 \end{aligned}$$

By using (4.4) on the third term of (4.13), the first three terms of (4.13) are cancelled out. Changing the indices in the last term of (4.13),

$$\begin{aligned}
 & - \sum_{j=2}^q \sum_{i_1, \dots, i_{j+\ell}=2}^n (-1)^j X^{i_2} \Theta_{i_2}^{i_3} \dots \Theta_{i_j}^{i_1} \sum_{\ell=1}^{q-j} (-1)^{\ell+1} (\nabla_{i_1} \Theta_{i_{j+1}}^{i_{j+2}}) \dots \Theta_{i_{j+\ell}}^{i_{j+1}} P^{q-j}(\Theta) \\
 & = - \sum_{j=2}^q \sum_{i_1, \dots, i_j=2}^n \sum_{k=3}^{j-1} (-1)^{j-1} X^{i_2} \Theta_{i_2}^{i_3} \dots \Theta_{i_{k-1}}^{i_k} (\nabla_{i_1} \Theta_{i_k}^{i_{k+1}}) \dots \Theta_{i_j}^{i_1} P^{q-j}(\Theta) \\
 (4.14) \quad & - \sum_{j=2}^q \sum_{i_1, \dots, i_j=2}^n (-1)^{j-1} X^{i_2} \Theta_{i_2}^{i_3} \dots \Theta_{i_{j-1}}^{i_1} (\nabla_{i_1} \Theta_{i_j}^{i_j}) P^{q-j}(\Theta).
 \end{aligned}$$

By using (4.4) in (4.14), we obtain

$$\begin{aligned}
 & - \sum_{j=2}^q \sum_{i_1, \dots, i_{j+\ell}=2}^n (-1)^j X^{i_2} \Theta_{i_2}^{i_3} \dots \Theta_{i_j}^{i_1} \sum_{\ell=1}^{q-j} (-1)^{\ell+1} (\nabla_{i_1} \Theta_{i_{j+1}}^{i_{j+2}}) \dots \Theta_{i_{j+\ell}}^{i_{j+1}} P^{q-j}(\Theta) \\
 & = \sum_{j=2}^q \sum_{i_1, \dots, i_j=2}^n \sum_{k=3}^j (-1)^{j-1} X^{i_2} \Theta_{i_2}^{i_3} \dots \Theta_{i_{k-1}}^{i_k} (\nabla_{i_k} \Theta_{i_1}^{i_{k+1}}) \dots \Theta_{i_j}^{i_k} P^{q-j}(\Theta) \\
 (4.15) \quad & + \sum_{j=2}^q \sum_{i_1, \dots, i_j=2}^n (-1)^{j-1} X^{i_2} \Theta_{i_2}^{i_3} \dots \Theta_{i_j}^{i_1} (\nabla_{i_j} \Theta_{i_1}^{i_j}) P^{q-j}(\Theta).
 \end{aligned}$$

By substituting (4.15) back into (4.13), the last three terms of (4.13) are cancelled out. Therefore, (4.13) is equal to 0. $\square$

$\square$

By Lemma 4.1, we have

$$\int_M q \tilde{P}^q(\nabla X, \Theta, \dots, \Theta) \wedge \omega_M^{n-q} = \int_M \operatorname{div}(Y) \wedge \omega_M^{n-q}.$$

Theorem 4.1 (General divergence theorem). *Given a holomorphic vector field with $(q-1, q-1)$ -valued form Y defined in Lemma 4.1, we have*

$$\int_M \operatorname{div}(Y) \wedge \omega_M^{n-q} = \int_M L_Y(\omega^{n-q}) = 0.$$

Proof. Let $\tilde{Y} = \sum_{i_1=2}^n \sum_{j=2}^q \sum_{i_2, \dots, i_j=2}^n (-1)^{j-1} X^{i_2} \Theta_{i_2}^{i_3} \dots \Theta_{i_j}^{i_1} P^{q-j}(\Theta) \otimes \frac{\partial}{\partial z_{i_1}}$. We have $Y = \sum_{i_1=2}^n X^{i_1} P^{q-1}(\Theta) \otimes \frac{\partial}{\partial z_{i_1}} + \tilde{Y}$. Compute

$$\begin{aligned} \int_M L_{\tilde{Y}}(\omega^{n-q}) &= (n-q) \int_M \sum_{i_2, \dots, i_j=2}^n (-1)^{j-1} X^{i_2} \Theta_{i_2}^{i_3} \dots \Theta_{i_j}^{i_1} \\ &\quad \wedge (\tilde{g}_{i_1 \bar{\lambda}} d\bar{z}_\lambda) \wedge P^{q-j}(\Theta) \wedge \omega^{n-1-q}. \end{aligned}$$

On Kähler manifolds, we have

$$\begin{aligned} \sum_{i_1, \lambda=2}^n \Theta_{i_j}^{i_1} \wedge (\tilde{g}_{i_1 \bar{\lambda}} d\bar{z}_\lambda) &= \frac{\sqrt{-1}}{2\pi} \sum_{i_1, \alpha, \beta, \lambda=2}^n R_{i_j \bar{\lambda} \alpha \bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta \wedge d\bar{z}_\lambda \\ &= \frac{\sqrt{-1}}{2\pi} \sum_{i_1, \alpha, \beta, \lambda=2}^n R_{i_j \bar{\beta} \alpha \bar{\lambda}} dz_\alpha \wedge d\bar{z}_\beta \wedge d\bar{z}_\lambda = 0. \end{aligned}$$

Therefore, we obtain $\int_M L_{\tilde{Y}}(\omega^{n-q}) = 0$. By the linearity of vector addition and (2.85), we have

$$\begin{aligned} \int_M L_Y(\omega^{n-q}) &= \int_M L_{P^{q-1}(\Theta) \otimes X}(\omega^{n-q}) \\ &= \int_M P^{q-1}(\Theta) \wedge L_X(\omega^{n-q}) \\ (4.16) \quad &= \int_M L_X(P^{q-1}(\Theta) \wedge \omega^{n-q}). \end{aligned}$$

By (4.16), Lemma 4.1 and Lemma 2.6, we obtain

$$\begin{aligned} \int_M L_Y(\omega^{n-q}) &= \int_M L_X(P^{q-1}(\Theta) \wedge \omega^{n-q}) \\ &= q \int_M \tilde{P}^q(\nabla X, \Theta, \dots, \Theta) \wedge \omega^{n-q} \\ &= \int_M \operatorname{div}(Y) \wedge \omega^{n-q} \\ &= 0. \end{aligned}$$

□

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