A NOTE ON THE VERLINDE BUNDLES ON ELLIPTIC CURVES

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ABSTRACT. We study the splitting properties of the Verlinde bundles over elliptic curves. Our methods rely on the explicit description of the moduli space of semistable vector bundles on elliptic curves, and on the analysis of the symmetric powers of the Schrödinger representation of the Theta group.

1. INTRODUCTION

Recently, Popa defined and studied a class of vector bundles on the Jacobians of curves, which he termed the Verlinde bundles [15]. The fibers of these vector bundles are the spaces of nonabelian Theta functions on the moduli spaces of bundles with fixed determinant over the curve, as the determinant varies in the Jacobian. Popa investigated the splitting properties of these bundles under certain étale pullbacks. He further used these results to prove the Strange Duality conjecture at level 1 and to study the basepointfreeness of the pluri-Theta series.

In this note, we study the Verlinde vector bundles in genus 1, correcting some of the statements of [15]. The techniques and results of our work are also useful to the understanding of the higher genus case, as we will demonstrate in [13].

To set the stage, consider a smooth complex projective curve X of genus $g \geq 1$ and write $U_X(r, r(g-1))$ for the moduli space of rank r , degree $r(g-1)$ semistable bundles on X . This moduli space comes equipped with a canonical Theta divisor supported on the locus

$$(1.1) \quad \Theta_r = \{V \in U_X(r, r(g-1)), \text{ such that } h^0(V) = h^1(V) \neq 0\}.$$

Following Popa [15], we define the level k Verlinde bundles on the Jacobian as the pushforwards

$$(1.2) \quad E_{r,k} = \det_* (\Theta_r^k)$$

under the determinant morphism

$$\det : U_X(r, r(g-1)) \rightarrow \text{Jac}^{r(g-1)}(X) \cong \text{Jac}(X).$$

Among the results Popa proved, we mention:

- (i) the pullback of $E_{r,k}$ under the multiplication morphism

$$r : \text{Jac}(X) \rightarrow \text{Jac}(X)$$

splits as a sum of line bundles;

Received by the editors September 11, 2008.

2000 *Mathematics Subject Classification*. Primary 14H60; Secondary 14H40.

This work was supported by NSF grant DMS-0701114.

- (ii) $E_{r,k}$ is globally generated if $k \geq r+1$ and is normally generated if $k \geq 2r+1$;
- (iii) $E_{r,k}$ is ample, polystable with respect to any polarization on the Jacobian, and satisfies IT_0 .

In addition, it is known that the Verlinde bundles enjoy the following level-rank symmetry:

- (iv) there is an isomorphism

$$SD : E_{r,k}^\vee \cong \widehat{E_{k,r}}.$$

The hat decorating the bundle on the right hand side denotes the Fourier-Mukai transform with kernel the normalized Poincaré bundle on the Jacobian.

The morphism (iv), sometimes termed “*Strange Duality*,” was constructed in this form by Popa. Proofs that SD is an isomorphism can be found in [10], [3]. The case of elliptic curves, which will be relevant for us, is simpler; a discussion is contained in [7].

To explain the results of this note, assume from now on that X is a smooth complex projective curve of genus 1. For reasons which will become clear only later, let us temporarily write h for the rank of the bundles making up the moduli space. We will first show:

Theorem 1.1. *Let k , h and q be positive integers. The Verlinde bundle $E_{h,k}$ splits as a sum of line bundles iff the level k is divisible by the rank h . When $k = h(q-1)$, we have*

$$(1.3) \quad E_{h,h(q-1)} \cong \Theta^{q-1} \otimes \left(\bigoplus L_\xi^{\oplus m_\xi} \right).$$

Here, Θ is the canonical Theta bundle on the Jacobian, and the L_ξ 's are the h -torsion line bundles. Each line bundle L_ξ of order ω occurs with multiplicity

$$(1.4) \quad m_\xi = \sum_{\delta|h} \frac{1}{q\delta^2} \binom{q\delta}{\delta} \left\{ \frac{h/\omega}{h/\delta} \right\},$$

provided that either h or q is odd. If both h and q are even, then

$$(1.5) \quad m_\xi = \sum_{\delta|h} \frac{(-1)^\delta}{q\delta^2} \binom{q\delta}{\delta} \left\{ \frac{h/\omega}{h/\delta} \right\}.$$

The symbol $\{\}$ appearing in the above statement is defined as follows. For any integer $h \geq 2$, we decompose

$$h = p_1^{a_1} \cdots p_n^{a_n}$$

into powers of primes. We set

$$(1.6) \quad \left\{ \frac{\lambda}{h} \right\} = \begin{cases} 0 & \text{if } p_1^{a_1-1} \cdots p_n^{a_n-1} \text{ does not divide } \lambda, \\ \prod_{i=1}^n \left(\epsilon_i - \frac{1}{p_i^i} \right) & \text{otherwise.} \end{cases}$$

Here,

$$\epsilon_i = \begin{cases} 1 & \text{if } p_i^{a_i} | \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

If $h = 1$, the symbol is always defined to be 1.

Note that it was expected that the splitting of the Verlinde bundles should involve only h -torsion line bundles. In fact, Popa proved the isomorphism

$$h^*E_{h,h(q-1)} \cong h^*\Theta^N,$$

where $N = \frac{1}{q} \binom{hq}{h}$. However, the multiplicities m_ξ of the nontrivial bundles L_ξ were incorrectly claimed to be 0 in Proposition 2.7 of [15]. This led to an erroneous statement in Proposition 5.3. Our note corrects this oversight.

As an example, when h is an odd prime, all nontrivial h -torsion line bundles appear in the decomposition (1.3) with the same (nonzero) multiplicity. This follows for instance by the arguments of [2], upon analyzing the action of a symplectic group on the h -torsion points. This is consistent with the theorem above, which specializes to

$$E_{h,h(q-1)} = \Theta^{q-1} \otimes \left(\bigoplus_{\xi \neq 0} L_\xi^{\oplus n} \oplus \mathcal{O}^{\oplus m} \right).$$

Here,

$$n = \frac{1}{h^2} \left(\frac{1}{q} \binom{qh}{h} - 1 \right) \text{ and } m = \frac{1}{h^2} \left(\frac{1}{q} \binom{qh}{h} - 1 \right) + 1.$$

Our proof will show that

$$m = \dim \left(\text{Sym}^{h(q-1)} S_h \right)^{H_h},$$

with S_h being the Schrödinger representation of the Heisenberg group H_h . If h is not prime, the ensuing formulas for multiplicities are more complicated, and their integrality is not immediately clear.

Theorem 1.1 is stated for the moduli spaces of bundles of degree zero. The case of arbitrary rank and degree, and of arbitrary Theta divisors will be the subject of Theorem 3.1 in Section 3.1.

The case when the level is not divisible by the rank is slightly more involved, and requires additional ideas. We will consider this most general situation separately, in Section 3.2. To explain the final result, let us first change the notation, writing hr for the rank of the bundles making up the moduli space, and letting hk be the level. If $\gcd(r, k) = 1$, then, for any h -torsion line bundle ξ , there is a unique stable bundle $W_{r,k,\xi}$ on the Jacobian, having rank r and determinant $\Theta^k \otimes \xi$. We will show:

Theorem 1.2. *Assume that $\gcd(r, k) = 1$. The Verlinde bundle of level hk splits as*

$$(1.7) \quad E_{hr,hk} \cong \bigoplus_{\xi} W_{r,k,\xi}^{\oplus m_\xi}.$$

For each h -torsion line bundle ξ on the Jacobian, having order ω , the multiplicity of the bundle $W_{r,k,\xi}$ in the above decomposition equals

$$(1.8) \quad m_\xi = \sum_{\delta|h} \frac{(-1)^{(h+1)kr\delta}}{(r+k)\delta^2} \binom{(r+k)\delta}{r\delta} \left\{ \frac{h/\omega}{h/\delta} \right\}.$$

Finally, the methods of this work make use of the characteristic zero hypothesis. In positive characteristic, it is likely that the answer is different and that it depends on the Hasse invariant of the curve.

After this paper was written, Fabrizio Catanese informed us that computations of similar flavor appear in [5], in connection with certain moduli spaces of complex surfaces; we would like to thank him for pointing out the reference. In this note, however, we cover most general numerics, making use of stronger representation-theoretic results for Heisenberg groups. Furthermore, we handle the combinatorics involved in the calculation differently, expressing the final answer in closed form.

2. THE PROOF OF THEOREM 1.1

The title of this section is self-explanatory. The proof of Theorem 1.1 to be given below relies on two essential ingredients:

- (i) first, the *geometric* input is provided by the explicit description of the moduli space of bundles over elliptic curves, as found in [1], [16];
- (ii) second, it will be crucial to understand the symmetric powers of the Schrödinger representation of the Heisenberg group. An *algebraic* computation will determine their characters, which are related to the decomposition (1.3).

We will discuss these two items at some length in the next sections, attempting to keep the exposition reasonably self-contained. Our arguments are quite elementary, so it is plausible that some of the results below may already exist in the literature; we tried to provide references whenever possible.

2.1. Geometry. Fix an elliptic curve (X, o) . Throughout the paper we identify $X \cong \text{Jac}^0(X)$ in the usual way,

$$p \rightarrow \mathcal{O}_X(p - o).$$

In [1], [16], Atiyah and Tu showed that the moduli space $U_X(h, 0)$ of rank h , degree 0 semistable vector bundles on X is isomorphic to the symmetric product

$$U_X(h, 0) \cong \text{Sym}^h X.$$

Up to S -equivalence, the isomorphism can be realized explicitly as

$$(2.1) \quad \text{Sym}^h X \ni (p_1, \dots, p_h) \rightarrow \mathcal{O}_X(p_1 - o) \oplus \dots \oplus \mathcal{O}_X(p_h - o) \in U_X(h, 0).$$

Under these identifications, the morphism taking bundles to their determinants

$$\det : U_X(h, 0) \rightarrow \text{Jac}(X)$$

is the Abel-Jacobi map, which in this case becomes the addition

$$a : \text{Sym}^h X \rightarrow X, (p_1, \dots, p_h) \mapsto p_1 + \dots + p_h.$$

Note that the fiber of the morphism a over the point $p \in X$ is the linear series

$$(2.2) \quad |[p] + (h-1)[o]| = |[p] - [o] + h[o]|.$$

In fact, as an Abel-Jacobi map, the morphism a has the structure of a projective bundle

$$\mathbb{P}(\mathbf{V}_h) \rightarrow X,$$

where $\mathbf{V}_h$ is a rank h vector bundle on X . To describe $\mathbf{V}_h$, we let $\mathcal{P}$ be the Poincaré bundle over $X \times X$, normalized in the usual way,

$$\mathcal{P} = \mathcal{O}_{X \times X}(\Delta - \{o\} \times X - X \times \{o\}),$$

with $\Delta \hookrightarrow X \times X$ being the diagonal. The Fourier-Mukai transform with kernel $\mathcal{P}$, denoted

$$\mathbf{RS} : \mathbf{D}(X) \rightarrow \mathbf{D}(X),$$

is given by

$$\mathbf{RS}(\alpha) = \mathbf{R}p_{2!}(p_1^* \alpha \otimes \mathcal{P}),$$

for $\alpha \in \mathbf{D}(X)$. Here

$$p_1, p_2 : X \times X \rightarrow X$$

are the projections on the two factors. Using (2.2), we conclude that

$$(2.3) \quad \mathbf{V}_h = \mathbf{RS}(\mathcal{O}_X(h[o])).$$

A standard computation shows that $\mathbf{V}_h$ has rank h and determinant $-[o]$; the reference [9] spells out the details. Furthermore, we note that $\mathbf{V}_h$ is simple since it is the Fourier-Mukai transform of a simple bundle [11]. In fact, by Atiyah's classic study [1], there is a unique bundle on X with all the above properties, defined inductively as the (unique) nontrivial extension

$$(2.4) \quad 0 \rightarrow \mathbf{V}_{h-1} \rightarrow \mathbf{V}_h \rightarrow \mathcal{O}_X \rightarrow 0,$$

starting with $\mathbf{V}_1 = \mathcal{O}_X(-[o])$. Alternatively, this exact sequence is obtained as the Fourier-Mukai transform of

$$0 \rightarrow \mathcal{O}_X((h-1)[o]) \rightarrow \mathcal{O}_X(h[o]) \rightarrow \mathcal{O}_{\{o\}} \rightarrow 0.$$

Note that the line bundle (2.1) has a section precisely when $p_i = o$ for some $1 \leq i \leq h$. It follows from (1.1) that the canonical Theta divisor Θ_h on $U_X(h, 0)$ is the image of the symmetric sum

$$(2.5) \quad [o] + \mathrm{Sym}^{h-1} X \hookrightarrow \mathrm{Sym}^h X.$$

Thus, the Theta line bundle Θ_h agrees, at least fiberwise, with $\mathcal{O}_{\mathbb{P}(\mathbf{V}_h)}(1)$. In fact, one can show the isomorphism

$$\Theta_h \cong \mathcal{O}_{\mathbb{P}(\mathbf{V}_h)}(1).$$

Moreover, the canonical section vanishing along the Theta divisor (2.5) is the composition

$$\mathcal{O}_{\mathbb{P}(\mathbf{V}_h)} \rightarrow \mathcal{O}_{\mathbb{P}(\mathbf{V}_h)}(1) \otimes a^* \mathbf{V}_h \rightarrow \mathcal{O}_{\mathbb{P}(\mathbf{V}_h)}(1),$$

with the second arrow given by (2.4). These observations allow us to compute the level k Verlinde bundle

$$(2.6) \quad \mathbf{E}_{h,k} = a_* (\Theta_h^k) = a_* (\mathcal{O}_{\mathbb{P}(\mathbf{V}_h)}(k)) = \mathrm{Sym}^k \mathbf{V}_h^\vee.$$

For convenience, we will write $\mathbf{W}_h = \mathbf{V}_h^\vee$ for the unique stable bundle on X of rank h and determinant $\mathcal{O}_X([o])$. More generally, if $\gcd(h, d) = 1$, we let $\mathbf{W}_{h,d}$ be the unique stable bundle of rank h and determinant $\mathcal{O}_X(d[o])$. The bundles $\mathbf{W}_{h,d}$ can be constructed inductively as successive extensions; cf. [14], page 180. Indeed, consider two consecutive terms $0 \leq \frac{d_1}{h_1} < \frac{d_2}{h_2} < 1$ in the Farey sequence; *i.e.*, assume that

$$h_1 d_2 - h_2 d_1 = 1.$$

Set $h = h_1 + h_2, d = d_1 + d_2$. Then $\mathbf{W}_{h,d}$ is the unique nontrivial extension

$$0 \rightarrow \mathbf{W}_{h_1, d_1} \rightarrow \mathbf{W}_{h, d} \rightarrow \mathbf{W}_{h_2, d_2} \rightarrow 0.$$

With these preliminaries out of the way, we proceed to investigate the splitting behavior of the Verlinde bundles $E_{h,k}$. Our analysis relies on the multiplicative structure of the Atiyah bundles [1], which may not be immediately obvious.

Lemma 2.1. *The Verlinde bundle $E_{h,k}$ splits as a sum of line bundles if and only if h divides k .*

Proof. This result will be reproved later in the paper. A more direct argument is given below. First, observe that $E_{h,k}$ is a direct summand of $W_h^{\otimes k}$. It suffices to show that these tensor powers split as sums of line bundles iff h divides k . In fact, something more general is true:

Claim 2.2. Assuming $\gcd(h, d) = 1$, the tensor powers $W_{h,d}^{\otimes k}$ split as sums of rank h' bundles of the form $W_{h',dk'} \otimes M$, where the M 's are various degree 0 line bundles. Here, we set

$$h' = \frac{h}{\gcd(h, k)}, \quad k' = \frac{k}{\gcd(h, k)}.$$

To prove the *Claim*, we first decompose $h = h_1 \dots h_s$ into powers of primes, and pick integers $d_1, \dots, d_s$ such that

$$\frac{d_1}{h_1} + \dots + \frac{d_s}{h_s} = \frac{d}{h}.$$

Then,

$$W_{h,d} = W_{h_1,d_1} \otimes \dots \otimes W_{h_s,d_s}.$$

This could be argued as follows: both sides have the same (coprime) rank and determinant, and are moreover semistable, in fact stable. Therefore, they should coincide by Atiyah's classification. With this understood, one checks that it is enough to take h to be a power of a prime p .

For the latter case, we will need the following rephrasing of Theorems 13 and 14 in [1]. Assume e_1, e_2, e are integers not divisible by p and that

$$\frac{e_1}{p^{a_1}} + \frac{e_2}{p^{a_2}} = \frac{e}{p^a}.$$

Then, Atiyah showed that for certain degree 0 line bundles M , we have

$$(2.7) \quad W_{p^{a_1}, e_1} \otimes W_{p^{a_2}, e_2} = \bigoplus_M W_{p^a, e} \otimes M.$$

Thus, when h is a power of a prime, the *Claim* follows from (2.7), by a straightforward induction on k . $\square$

Remark 2.3. Using a sharper version of Atiyah's results, one can prove that when h is odd, the M 's appearing in the *Claim* above are representatives for the cosets of h -torsion line bundles on X modulo the twisting action of the group of h' -torsion line bundles. The same statement should hold true for h even, but Atiyah's results only show that the orders of the M 's divide $2h$. In particular, for h odd and $\gcd(h, k) = 1$, we immediately conclude that

$$(2.8) \quad E_{h,k} \cong \bigoplus_{i=1}^m W_{h,k},$$

with $m = \frac{1}{h+k} \binom{h+k}{h}$. Equation (2.8) is a particular case of Theorem 1.2.

We will identify the splitting of $E_{h,k} = \text{Sym}^k W_h$ when the level k is divisible by the rank h . We set

$$q = 1 + \frac{k}{h}.$$

Let X_h be the group of h -torsion points on the elliptic curve. Let G_h be the Theta group of the line bundle $\mathcal{O}_X(h[o])$, which is a central extension

$$1 \rightarrow \mathbb{C}^* \rightarrow G_h \rightarrow X_h \rightarrow 1.$$

The assignment

$$\eta \rightarrow \eta^{2h}$$

defines an endomorphism of G_h , whose image lies in the center of G_h . This is easily checked for instance using (2.9) below. Let H_h be the kernel of this endomorphism. It corresponds to an extension

$$1 \rightarrow \mu_{2h} \rightarrow H_h \rightarrow X_h \rightarrow 1,$$

where $\mu_{2h} \hookrightarrow \mathbb{C}^*$ is the group of $2h$ -roots of 1. Finally, let S_h denote the h -dimensional Schrödinger representation of G_h , *i.e.* the unique representation such that the center of G_h acts by its natural character. We refer the reader to Mumford's paper [12] or to Chapter 6 of [4] for more details regarding Heisenberg groups and their Schrödinger representations.

Picking Theta structures, we identify G_h with the Heisenberg group

$$(2.9) \quad G_h \cong \mathbb{C}^* \times \mathbb{Z}/h\mathbb{Z} \times \mathbb{Z}/h\mathbb{Z}.$$

The multiplication on the right hand side is defined as

$$(\alpha, x, y)(\alpha', x', y') = (\alpha\alpha'\zeta^{y'x}, x+x', y+y').$$

Here, we set

$$\zeta = \exp\left(\frac{2\pi i}{h}\right).$$

The Schrödinger representation S_h is realized on the space of functions

$$f : \mathbb{Z}/h\mathbb{Z} \rightarrow \mathbb{C};$$

cf. [4], page 164. The action of the element $(\alpha, x, y) \in G_h$ on a function f is given by the new function

$$F : \mathbb{Z}/h\mathbb{Z} \rightarrow \mathbb{C}, \quad F(a) = \alpha\zeta^{ya} \cdot f(x+a).$$

We will first compute the pullbacks of the Verlinde bundles under the morphism $h : X \rightarrow X$ which multiplies by h on the elliptic curve. Using the description of V_h as a Fourier-Mukai transform provided by (2.3), and Theorem 3.11 in [11], we obtain

$$h^*V_h \cong \mathcal{O}_X(-h[o])^{\oplus h}.$$

In fact, we claim that G_h -equivariantly, we have

$$(2.10) \quad h^*V_h \cong S_h \otimes \mathcal{O}_X(-h[o]);$$

see also [14], page 162. Indeed, consider the trivial bundle

$$h^*V_h \otimes \mathcal{O}_X(h[o]) \cong V \otimes \mathcal{O}_X,$$

where V is an h -dimensional vector space. Both factors of the tensor product on the left carry a G_h -action covering the translation X_h -action on the base X . Therefore, endowing the structure sheaf appearing on the right with the trivial G_h -action, we obtain a G_h -representation on V . Moreover, note that the center of G_h acts

on V by homotheties. Therefore, $V \cong S_h$, by the uniqueness of the Schrödinger representation. This establishes (2.10).

Taking determinants in (2.10), we obtain

$$(2.11) \quad h^* \mathcal{O}_X(-[o]) \cong \Lambda^h S_h \otimes \mathcal{O}_X(-h[o])^h.$$

This identification is a priori only G_h -equivariant, but, since the center of G_h acts trivially, the isomorphism is in fact X_h -equivariant. Similarly, dualizing and taking symmetric powers in (2.10), we obtain an X_h -equivariant identification

$$(2.12) \quad h^* \mathrm{Sym}^k W_h \cong \mathrm{Sym}^k S_h^\vee \otimes \mathcal{O}_X(h[o])^k \cong \mathrm{Sym}^k S_h^\vee \otimes (\Lambda^h S_h)^{q-1} \otimes h^* \mathcal{O}_X([o])^{q-1}.$$

Observe that the action of the central elements α of G_h on the Heisenberg module

$$M_k = \mathrm{Sym}^k S_h^\vee \otimes (\Lambda^h S_h)^{q-1}$$

is trivial, since

$$(2.13) \quad \alpha^{-k} \cdot (\alpha^h)^{q-1} = 1.$$

Therefore M_k is an X_h -module. The X_h -action splits into eigenspaces indexed by the characters ξ of X_h , each appearing with multiplicity m_ξ :

$$(2.14) \quad M_k \cong \bigoplus_{\xi} \xi^{\oplus m_\xi}.$$

Let us write $\widehat{X}_h$ for the group of characters of X_h . For each $\xi \in \widehat{X}_h$, let L_ξ denote the corresponding h -torsion line bundle on X . The pullback $h^* L_\xi$ is the trivial bundle endowed with the X_h -character ξ . Using (2.12) and (2.14), we obtain an X_h -equivariant identification

$$h^* \mathrm{Sym}^k W_h \cong h^* \left(\bigoplus_{\xi} L_\xi^{\oplus m_\xi} \right) \otimes h^* \mathcal{O}_X([o])^{q-1}.$$

This equivariant isomorphism determines the Verlinde bundle on the left, by general considerations about the Picard group of finite quotients. We can also give a direct argument as follows. Pushing forward the previous equation by h , we obtain the X_h -isomorphism

$$(2.15) \quad \mathrm{Sym}^k W_h \otimes h_* \mathcal{O}_X \cong \bigoplus_{\xi} L_\xi^{\oplus m_\xi} \otimes \mathcal{O}_X([o])^{q-1} \otimes h_* \mathcal{O}_X.$$

Note that X_h -equivariantly,

$$(2.16) \quad h_* \mathcal{O}_X \cong \bigoplus_{\xi \in \widehat{X}_h} L_\xi.$$

Comparing (2.15) and (2.16), and singling out the X_h -invariant part, we conclude that

$$(2.17) \quad E_{h,k} \cong \mathrm{Sym}^k W_h \cong \left(\bigoplus_{\xi \in \widehat{X}_h} L_\xi^{\oplus m_\xi} \right) \otimes \mathcal{O}_X([o])^{q-1}.$$

2.2. Algebra. It remains to determine the multiplicities m_ξ appearing in (2.14). Regarding M_k as a representation of the finite group H_h and using the standard representation theory for finite groups as in [8, page 17], we obtain

$$(2.18) \quad m_\xi = \frac{1}{|H_h|} \sum_{\eta \in H_h} \xi(\eta^{-1}) \text{Tr}_{M_k}(\eta).$$

We will compute this sum explicitly with the aid of the following.

Lemma 2.4. *Let $\eta \in H_h$ be an element whose image under the map $H_h \rightarrow X_h$ has order exactly h/δ in X_h . The trace of η on M_k equals*

$$\text{Tr}_{M_k}(\eta) = \frac{1}{q} \binom{q\delta}{\delta},$$

provided that either h or q is odd.

Proof. We pick Theta structures, so that G_h and S_h are given by (2.9). Consider the basis $f_1, \dots, f_h$ of S_h given by

$$f_i(j) = \delta_{i,j}.$$

By definition, the action of

$$\eta = (\alpha, x, y) \in \mu_{2h} \times \mathbb{Z}/h\mathbb{Z} \times \mathbb{Z}/h\mathbb{Z}$$

is given as

$$(2.19) \quad \eta \cdot f_i = \alpha \zeta^{y(i-x)} \cdot f_{i-x}.$$

To compute the trace of η on M_k , we may assume that $\alpha = 1$, since the scaling action of the center of H_h is trivial, as remarked in (2.13).

We begin by computing the trace $\text{Tr Sym}^k \eta$ of the action of η on $\text{Sym}^k S_h^\vee$. For simplicity, we will first treat the case $x = 0$. The eigenvalues of the action of η on S_h are $1, \zeta_y, \dots, \zeta_y^{h-1}$. Here, we set

$$\zeta_y = \zeta^y.$$

Therefore,

$$(2.20) \quad \text{Tr Sym}^k \eta = \sum_{1 \leq i_1 \leq \dots \leq i_k \leq h} \zeta_y^{-(i_1 + \dots + i_k)} = \sum_{j_1 + \dots + j_h = k} \zeta_y^{-(j_1 + 2j_2 + \dots + hj_h)}.$$

In the above, j_r denotes the number of i 's which equal r . Now, we compute the generating series

$$\sum_k \text{Tr Sym}^k \eta \cdot t^k = \frac{1}{1 - \zeta_y^{-1}t} \cdot \frac{1}{1 - \zeta_y^{-2}t} \cdots \frac{1}{1 - \zeta_y^{-h}t}.$$

Write

$$h = lm,$$

where

$$m = \gcd(h, y) \text{ and } \gcd(l, y) = 1.$$

Then $\epsilon = \zeta_y$ is a primitive root of 1 of order l . Therefore, the product in the denominator above becomes

$$\begin{aligned} (1 - \zeta_y^{-1}t) \cdots (1 - \zeta_y^{-h}t) &= \zeta_y^{-h(h-1)/2} (-1)^h ((t-1)(t-\epsilon) \cdots (t-\epsilon^{l-1}))^m \\ &= (-1)^{h+y(h-1)} (t^l - 1)^m. \end{aligned}$$

We can extract the coefficient of t^k :

$$(2.21) \quad \text{Tr Sym}^k \eta = (-1)^{h+y(h-1)+m+\frac{k}{l}} \binom{-m}{\frac{k}{l}} = (-1)^{\frac{k}{l}} \binom{-m}{\frac{k}{l}} = \frac{1}{q} \binom{qm}{m}.$$

In particular, this computation implies that the sum (2.20) is 1 when $m = \gcd(h, y) = 1$. Moreover, the argument shows that the sum (2.20) vanishes if k is not divisible by $l = \frac{h}{\gcd(h, y)}$.

We will now consider the η 's in H_h for which $x \neq 0$. For these, the computation is notationally more involved. To begin, we write

$$x = x's \text{ and } h = h's,$$

where $s = \gcd(h, x)$. Let u be any constant with

$$u^{h'} = (-1)^{yx'(h'+1)}.$$

Note in particular that $u^k = 1$ for h odd. For h even, we have

$$(2.22) \quad u^k = (-1)^{xy(q-1)}.$$

Now, it is easy to see that the eigenvalues of η on S_h are

$$(2.23) \quad \lambda_{i,j} = u \zeta_y^i \sigma^j, \quad 1 \leq i \leq s, 1 \leq j \leq h',$$

where

$$\sigma = \exp\left(\frac{2\pi i}{h'}\right).$$

In fact, we can exhibit an eigenvector for $\lambda = \lambda_{i,j}$, namely

$$v_\lambda = \sum_{k=0}^{h'-1} \lambda^{-k} \zeta_y^{ki - \frac{k(k+1)}{2}x} \cdot f_{i-kx}.$$

We order the indices (i, j) lexicographically. The trace $\text{Tr Sym}^k \eta$ is obtained by summing all products

$$\left(\lambda_{1,j_1^1}^{-1} \cdots \lambda_{1,j_1^1}^{-1}\right) \left(\lambda_{2,j_2^1}^{-1} \cdots \lambda_{2,j_2^1}^{-1}\right) \cdots \left(\lambda_{s,j_s^1}^{-1} \cdots \lambda_{s,j_s^1}^{-1}\right),$$

where

$$1 \leq j_1^i \leq j_2^i \leq \cdots \leq j_s^i \leq h'.$$

Let a_1 be the number of terms in the product whose first index is 1; $a_2, \dots, a_s$ have a similar meaning. We require $a_1 + \cdots + a_s = k$. After substituting (2.23) in the product above, we sum over the j 's, keeping the a 's fixed. We have seen already in the derivation of (2.20) that the sum

$$\sum_{1 \leq j_1^i \leq \cdots \leq j_{a_i}^i \leq h'} \sigma^{-(j_1^i + \cdots + j_{a_i}^i)}$$

is 0 if h' does not divide a_i , and it equals 1 otherwise. Therefore, writing $a_i = h'a'_i$, we need to evaluate

$$\sum_{a'_1 + \cdots + a'_s = \frac{k}{h'}} \zeta_y^{-h'a'_1} \zeta_y^{-2h'a'_2} \cdots \zeta_y^{-sh'a'_s} = \sum_{a'_1 + \cdots + a'_s = \frac{k}{h'}} \gamma_y^{-(a'_1 + \cdots + sa'_s)}.$$

Here, we set $\gamma = \exp\left(\frac{2\pi i}{s}\right)$, so that $\zeta_y^{h'} = \gamma_y$. We have already computed sums of this type in (2.20). We obtained the answer

$$(2.24) \quad \frac{1}{q} \binom{q\delta}{\delta}$$

for $\delta = \gcd(s, y) = \gcd(h, x, y)$. This expression gives the trace $\text{Tr Sym}^k \eta$ when h is odd. The formula includes the previously considered case $x = 0$, for which $\delta = m$. The sign change (2.22) is required when h is even.

Finally, the trace of η on $\Lambda^h S_h$ is computed using (2.19):

$$(2.25) \quad \eta \cdot f_1 \wedge \dots \wedge f_h = (-1)^{x(h+1)} \prod_{i=1}^h \zeta_y^{i-x} \cdot f_1 \wedge \dots \wedge f_h = (-1)^{(h+1)(x+y)} f_1 \wedge \dots \wedge f_h.$$

This completes the proof when h is odd. When h is even, we take into account the sign corrections of the previous paragraph and (2.22). We append formula (2.24) by the overall sign

$$(-1)^{xy(q-1)} \cdot (-1)^{(x+y)(h+1)(q-1)} = (-1)^{(xy+x+y)(q-1)}.$$

This does not change (2.24) when q is odd, proving the lemma. When h and q are both even, we note, for further use, that the overall sign of (2.24) can be rewritten as

$$(2.26) \quad (-1)^{\gcd(h, x, y)} = (-1)^\delta.$$

□

We proceed to calculate the sum (2.18). We *claim* that the multiplicity m_ξ depends only on the order of the character $\xi \in \widehat{X}_h$. To this end, consider the group $\text{Aut}(H_h, \mu_{2h})$ of automorphisms of H_h which restrict to the identity on the center μ_{2h} . As essentially remarked in [2], the characters appearing in the X_h -representation M_k are exchanged by the action of $\text{Aut}(H_h, \mu_{2h})$. Beauville's argument is based on the observation that for each $F \in \text{Aut}(H_h, \mu_{2h})$, the standard H_h -module structure of S_h , $\rho : H_h \rightarrow \text{GL}(S_h)$, is isomorphic to the twisted module structure $F \circ \rho : H_h \rightarrow \text{GL}(S_h)$. This follows by examining the character of the center of H_h , and by making use of the uniqueness of the Schrödinger representation. The same observation applies to the associated H_h -module M_k . With this understood, our *claim* is a consequence of the lemma below. This result is possibly known, yet for completeness we decided to include the argument. Note that the lemma is not indispensable for the proofs to follow, yet it allows for some simplification of the formulas.

Lemma 2.5. *Under the action of $\text{Aut}(H_h, \mu_{2h})$, two characters of X_h belong to the same orbit if and only if they have the same order in $\widehat{X}_h$.*

Proof. Fix two characters χ_1, χ_2 of X_h :

$$\chi_i : X_h \rightarrow \mathbb{C}^\star, (x, y) \rightarrow \zeta^{a_i x + b_i y}, 1 \leq i \leq 2.$$

The condition on the orders of χ_1 and χ_2 translates into

$$\gcd(h, a_1, b_1) = \gcd(h, a_2, b_2) := \tau.$$

This implies that we can solve the equations below, with the Greek letters as the unknowns:

$$(2.27) \quad a_1 \lambda + b_1 \mu = a_2 \pmod{h}, \quad a_1 \nu + b_1 \gamma = b_2 \pmod{h}.$$

We claim that we may further achieve

$$(2.28) \quad \lambda\gamma - \mu\nu = 1 \pmod{h}.$$

This can be seen for instance as follows. By the Chinese Remainder Theorem, we may take h to be a power of a prime. In this case, assume first that $\tau = 1$. Starting with any solution of (2.27), define a new quadruple

$$\lambda' = \lambda + b_1x, \mu' = \mu - a_1x, \nu' = \nu + b_1y, \gamma' = \gamma - a_1y.$$

The assumption $\tau = 1$ implies that we can find a pair (x, y) such that (2.28) holds:

$$\lambda'\gamma' - \mu'\nu' = (\lambda\gamma - \mu\nu) + b_2x - a_2y \equiv 1 \pmod{h}.$$

For arbitrary τ , after dividing by τ , and using the case we already proved, we may assume that (2.27) is satisfied mod h , and that (2.28) holds true mod h/τ . We lift the solution using Hensel's lemma, ensuring that (2.28) is also satisfied mod h .

Finally, define $F : \mathbf{H}_h \rightarrow \mathbf{H}_h$ by

$$F(\alpha, x, y) = (\alpha\zeta^{\frac{1}{2}(\lambda\mu x^2 + \nu\gamma y^2 + 2\mu\nu xy)}, \lambda x + \nu y, \mu x + \gamma y).$$

Equation (2.28) is used to prove that F is an automorphism of $\mathbf{H}_h$, while equation (2.27) shows that F sends χ_1 to χ_2 . $\square$

Henceforth, for the computation of (2.18), we will take ξ to be the character

$$\xi = \xi_\lambda : \mathbb{Z}/h\mathbb{Z} \times \mathbb{Z}/h\mathbb{Z} \ni (x, y) \mapsto \zeta_\lambda^{x+y} = \zeta^{\lambda(x+y)} \in \mathbb{C}^*.$$

Here, we assume that λ divides h , so that the character ξ has order

$$\omega = \frac{h}{\lambda}.$$

Assume that either h or q is odd. Using Lemma 2.4, we rewrite (2.18) as

$$(2.29) \quad m_\xi = \frac{1}{h^2} \sum_{\delta|h} \frac{1}{q} \binom{q\delta}{\delta} \left(\sum_{\gcd(h, x, y) = \delta} \xi_\lambda(x, y) \right).$$

If both h and q are even, each term in (2.29) is multiplied by the sign $(-1)^\delta$, as it follows from (2.26). In this case,

$$(2.30) \quad m_\xi = \frac{1}{h^2} \sum_{\delta|h} \frac{(-1)^\delta}{q} \binom{q\delta}{\delta} \left(\sum_{\gcd(h, x, y) = \delta} \xi_\lambda(x, y) \right).$$

We will evaluate formulas (2.29) and (2.30) in terms of the character (1.6) defined in the introduction.

Lemma 2.6. *We have*

$$\sum_{\gcd(h, x, y) = \delta} \xi_\lambda(x, y) = \frac{h^2}{\delta^2} \left\{ \frac{h/\omega}{h/\delta} \right\}.$$

Proof. Replacing h , x and y by h/δ , x/δ and y/δ , respectively, we may assume $\delta = 1$. To solve this case, let us set

$$(2.31) \quad N_\lambda(h) = \sum_{\gcd(h, x, y) = 1} \xi_\lambda(x, y) = \sum_{\gcd(h, x, y) = 1} \zeta_\lambda^{x+y}.$$

It suffices to show that

$$(2.32) \quad \mathbf{N}_\lambda(h) = h^2 \left\{ \frac{\lambda}{h} \right\}.$$

This is immediate when $h = p^a$ is a power of a prime. In this case, if $p^a | \lambda$, the left hand side of (2.32) counts the pairs $1 \leq x, y \leq p^a$ such that $\gcd(p^a, x, y) = 1$. Their number is $p^{2a-2}(p^2 - 1)$, which equals the right hand side. Otherwise, since the distinct roots of unity add up to 0, we have

$$\sum_{(x,y,p^a)=1} \zeta_\lambda^{x+y} = - \sum_{p|(x,y)} \zeta_\lambda^{x+y}.$$

If $p^{a-1} | \lambda$, then all terms in the last sum are equal to 1, hence giving the answer $-p^{2a-2}$. Finally, if p^{a-1} does not divide λ , then replacing ζ_λ by $\zeta_{p\lambda}$, we sum all distinct roots of unity of order $p^{a-1}/\gcd(p^{a-1}, \lambda)$, each appearing with equal multiplicity. This gives the answer 0.

The general case follows by induction on the number of prime factors of h , once we establish the multiplicativity in h of the function $\mathbf{N}_\lambda(h)$. Let $h = h_1 h_2$ with $\gcd(h_1, h_2) = 1$. Choose integers u, v such that

$$h_1 u + h_2 v = 1.$$

By the Chinese Remainder Theorem, the pairs $(x, y) \bmod h$ are in one-to-one correspondence with the pairs $(x_1, y_1) \bmod h_1$, $(x_2, y_2) \bmod h_2$ such that

$$\begin{aligned} x &\equiv x_1 \bmod h_1, \quad x \equiv x_2 \bmod h_2, \\ y &\equiv y_1 \bmod h_1, \quad y \equiv y_2 \bmod h_2. \end{aligned}$$

Explicitly, we have

$$x = h_1 u x_2 + h_2 v x_1 \bmod h, \quad y = h_1 u y_2 + h_2 v y_1 \bmod h.$$

The condition $\gcd(h, x, y) = 1$ is equivalent to

$$\gcd(h_1, x_1, y_1) = 1, \quad \gcd(h_2, x_2, y_2) = 1.$$

We compute

$$\begin{aligned} \mathbf{N}_\lambda(h) &= \sum_{\gcd(h, x, y)=1} \zeta_\lambda^{x+y} = \sum_{\gcd(h_1, x_1, y_1)=1, \gcd(h_2, x_2, y_2)=1} \zeta_\lambda^{h_2 v(x_1+y_1)} \cdot \zeta_\lambda^{h_1 u(x_2+y_2)} \\ &= \mathbf{N}_{\lambda v}(h_1) \mathbf{N}_{\lambda u}(h_2) = h_1^2 \left\{ \frac{\lambda v}{h_1} \right\} \cdot h_2^2 \left\{ \frac{\lambda u}{h_2} \right\} = h^2 \left\{ \frac{\lambda}{h_1} \right\} \left\{ \frac{\lambda}{h_2} \right\} = h^2 \left\{ \frac{\lambda}{h} \right\}. \end{aligned}$$

In the last line, we used the fact that the factors u and v do not change the symbol $\{\}$ since these numbers are prime to h_2 and h_1 , respectively. $\square$

Putting together (2.17), (2.29), (2.30) and Lemma 2.6, we complete the proof of Theorem 1.1.

3. ARBITRARY NUMERICS

3.1. Arbitrary rank and degree. We will now discuss a variant of Theorem 1.1, which covers the case of arbitrary rank and degree. Let r, d be two integers with

$$h = \gcd(r, d).$$

Write

$$r = hr', d = hd', \text{ where } \gcd(r', d') = 1.$$

We will consider Theta divisors on the moduli space $U_X(r, d)$. Their definition requires the choice of a twisting vector bundle N of complementary slope

$$\mu(N) = -\frac{d}{r}.$$

We set

$$(3.1) \quad \Theta_{r,N} = \{V \in U_X(r, d), \text{ such that } h^0(V \otimes N) = h^1(V \otimes N) \neq 0\}.$$

To avoid confusion, even though it may be notationally cumbersome, we decorate the Theta's by the twisting bundles N , and by the rank of the bundles in the moduli space.

It is convenient to assume that N has the minimal possible rank r' . The level k Verlinde bundle

$$E_{r,k}^N = \det_* (\Theta_{r,N}^k)$$

is obtained by pushing forward the pluri-Theta bundle Θ_N^k on $U_X(r, d)$ via the morphism

$$\det : U_X(r, d) \rightarrow \text{Jac}^d(X).$$

As before, we have an isomorphism

$$(3.2) \quad U_X(r, d) \cong \text{Sym}^h X.$$

Set-theoretically, this isomorphism is essentially defined by twisting (2.1) by the unique indecomposable vector bundle $W_{r',d'}$ of rank r' and determinant $d'[o]$ on X . More precisely, if $(p_1, \dots, p_h)$ are h points of X , pick $(q_1, \dots, q_h)$ such that

$$r' \cdot q_i = p_i, \quad 1 \leq i \leq h.$$

Then, the isomorphism (3.2) is given by

$$(3.3) \quad \text{Sym}^h X \ni (p_1, \dots, p_h) \mapsto W_{r',d'} \otimes \mathcal{O}_X(q_1 - o) \oplus \dots \oplus W_{r',d'} \otimes \mathcal{O}_X(q_h - o) \in U_X(r, d).$$

Note that the answer on the right hand side of (3.3) is independent of the choice of q_i . Indeed, any two q_i 's must differ by an r' -torsion point χ . However, by Atiyah's classification,

$$(3.4) \quad W_{r',d'} \otimes L_\chi \cong W_{r',d'},$$

as both bundles are simple, of the same rank and determinant. It was observed in [16], and it is clear from (3.3), that the determinant

$$\det : U_X(r, d) \rightarrow \text{Jac}^d(X)$$

becomes the addition morphism

$$a : \text{Sym}^h X \rightarrow X, (p_1, \dots, p_h) \rightarrow p_1 + \dots + p_h.$$

Here, we used the identification

$$X \cong \text{Jac}(X) \cong \text{Jac}^d(X),$$

with the second arrow given by twisting degree zero line bundles by $\mathcal{O}_X(d[o])$. Via this identification, the divisor $\Theta_{1,\mathcal{O}(-d[o])}$ on $\text{Jac}^d(X)$ corresponds to the canonical Theta on $\text{Jac}(X)$.

Finally, we can easily identify the Theta divisors on $U_X(r, d)$. There is a natural choice for the twisting bundle N , namely the Atiyah bundle $N_0 = W_{r', -d'}$. It was shown in [1], and it follows from equation (3.4), that the tensor product

$$(3.5) \quad W_{r', -d'} \otimes W_{r', d'} = \bigoplus_{\chi} L_{\chi}$$

splits as the direct sum of all r' -torsion line bundles L_{χ} . As a consequence of (3.1), (3.3), (3.5), we see that for the bundles V in the Theta divisor, we have $q_i = \chi$ for some r' -torsion point χ , and some $1 \leq i \leq h$. Thus, Θ_{r, N_0} is the image of the symmetric sum

$$[o] + \text{Sym}^{h-1} X \hookrightarrow \text{Sym}^h X.$$

We have therefore recovered (2.5), and thus reduced the computation to the case we already studied.

Theorem 3.1. *Fix r and d to be two integers with $h = \gcd(r, d)$, and let N be a vector bundle of slope*

$$\mu(N) = -\frac{d}{r}$$

and of minimal rank. Then, $E_{r, k}^N$ splits as a sum of line bundles iff h divides k . If $k = h(q-1)$, then

$$E_{r, k}^N \cong (\Theta_{1, (\det N)^h})^{q-1} \otimes \left(\bigoplus_{\xi \in \tilde{X}_h} L_{\xi}^{\oplus m_{\xi}} \right).$$

Here the m_{ξ} are given by the same formulas (1.4) and (1.5) as in Theorem 1.1.

Proof. When $N_0 = W_{r', -d'}$, the statement is a consequence of the above discussion and the proof of Theorem 1.1. The general case follows from here, since both the Verlinde bundle and the right hand side only change by translations. To see this, set

$$L = \det N \otimes (\det N_0)^{-1}.$$

On the one hand, formulas of Drezet-Narasimhan [6] imply that

$$E_{r, k}^N = \det_{\star} (\Theta_{r, N}^k) = \det_{\star} (\Theta_{r, N_0} \otimes \det^{\star} L)^k = E_{r, k}^{N_0} \otimes L^k.$$

In the above, we view the degree 0 line bundle L on X as a line bundle on the Jacobian in the standard way. On the other hand, we have

$$\Theta_{1, (\det N)^h} = \Theta_{1, (\det N_0)^h} \otimes L^h.$$

The theorem follows by putting these observations together. $\square$

3.2. Arbitrary level and rank. In this subsection we will prove Theorem 1.2. We will keep the same notation as in the introduction, writing hr for the rank, and letting hk be the level, with $\gcd(r, k) = 1$. We will determine the splitting type of the Verlinde bundle

$$E_{hr, hk} = \det_{\star} (\Theta_{hr}^{hk}) = \text{Sym}^{hk} W_{hr},$$

obtained by pushing forward tensor powers of the canonical Theta bundle Θ_{hr} via

$$\det : U_X(hr, 0) \rightarrow \text{Jac}(X) \cong X.$$

The case of nonzero degree and arbitrary Theta's is entirely similar, and we will leave the details to the interested reader.

Proof of Theorem 1.2. We first consider the case when r is odd. The arguments used to prove Theorem 1.1 go through with only minor changes. It suffices to check that the decomposition (1.7),

$$E_{hr,hk} \cong \bigoplus_{\xi} W_{r,k,\xi}^{m_{\xi}},$$

holds X_{hr} -equivariantly, after pullback by the morphism hr . The pullback of the left hand side is evaluated G_{hr} -equivariantly via (2.12):

$$(3.6) \quad (hr)^* E_{hr,hk} \cong (hr)^* \mathrm{Sym}^{hk} W_{hr} \cong \mathcal{O}_X(hr[o])^{hk} \otimes \mathrm{Sym}^{hk} S_{hk}^{\vee}.$$

For the right hand side, recall first that $W_{r,k,\xi}$ has rank r and determinant $\mathcal{O}_X(k[o]) \otimes \xi$. By comparing ranks and degrees, we see that

$$(3.7) \quad W_{r,k,\xi} \cong W_{r,k} \otimes L_{\chi}.$$

Here L_{χ} is any hr -torsion line bundle with $L_{\chi}^r = L_{\xi}$. Note that χ is uniquely defined only up to r -torsion line bundles. The ambiguity inherent in the choice of χ will be shown to be inessential later. Observe that the pullback $(hr)^* L_{\chi}$ is the trivial bundle, endowed with the X_{hr} -character χ .

We will determine the pullback of $W_{r,k}$ by the morphism hr . As a first step, we show that nonequivariantly

$$(3.8) \quad r^* W_{r,k} \cong \mathcal{O}_X(kr[o])^{\oplus r}.$$

The ingredients needed for the proof of (3.8) are found in Lemma 22 of Atiyah's paper [1]. There, it is explained that all indecomposable factors of $r^* W_{r,k}$ have the same rank r' and degree k' . Therefore,

$$r^* W_{r,k} \cong \bigoplus_{i=1}^{r/r'} W_{r',k'} \otimes M_i$$

for some line bundles M_i . In fact, examining Atiyah's arguments, one can prove a little bit more. Using (3.5), we observe that

$$r^* W_{r,k} \otimes r^* W_{r,k}^{\vee} \cong \bigoplus_1^{r^2} \mathcal{O}_X.$$

The above tensor product contains $W_{r',k'} \otimes W_{r',k'}^{\vee} \otimes M_i \otimes M_j^{-1}$ as a direct summand, for any i and j . Now, applying equation (3.5) again, we see that

$$W_{r',k'} \otimes W_{r',k'}^{\vee} \cong \bigoplus_{\rho} L_{\rho},$$

the sum being taken over the r' -torsion points ρ . This clearly gives a contradiction, unless $r' = 1$ and the bundles M_i and M_j coincide. In conclusion, we proved that

$$(3.9) \quad r^* W_{r,k} \cong \bigoplus_{i=1}^r M,$$

for a suitable line bundle M . Taking determinants we obtain that

$$M \cong \mathcal{O}_X(kr[o]) \otimes P,$$

for some r -torsion line bundle P . We claim that P is symmetric, *i.e.* $(-1)^*P \cong P$. When r is odd, these two facts together imply that P must be trivial, proving (3.8). The symmetry of P is a consequence of (3.9) and of the symmetry of $W_{r,k}$. Indeed,

$$(-1)^*W_{r,k} \cong W_{r,k},$$

as both bundles are simple, and have the same rank and determinant.

Having established (3.8), we compute

$$(3.10) \quad (hr)^*W_{r,k} \cong \mathcal{O}_X(hr[o])^{hk} \otimes R,$$

where R is an r -dimensional vector space. In fact, R carries a representation of the Theta group G_{hr} , such that the center acts with weight $-hk$. However, this does not determine the representation R uniquely, not even as a representation of H_{hr} . In fact, one can show that there are precisely h^2 representations $R_{i,j}$ of H_{hr} with central weight $-hk$ [17]; they will be indexed by integers $i, j \in \mathbb{Z}/h\mathbb{Z} \times \mathbb{Z}/h\mathbb{Z}$.

To determine R , we will use the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & G_{hr} & \longrightarrow & X_{hr} \longrightarrow 0 \\ & & \uparrow i & & \uparrow & & \uparrow \wr \\ 0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & G_h & \longrightarrow & X_h \longrightarrow 0. \end{array}$$

Here, the morphism i is the r -fold cover $\alpha \mapsto \alpha^r$, and the middle arrow is the natural morphism of Theta groups $G_h \rightarrow G_{hr}$. Via this diagram, we may consider the action of the group G_h on both sides of (3.10). Recall from equation (2.11) that G_h -equivariantly, we have

$$\mathcal{O}_X(hr[o])^{hk} \cong \mathcal{O}_X(h[o])^{hkr} \cong h^*\mathcal{O}_X([o])^{kr} \otimes (\Lambda^h S_h)^{kr}.$$

Therefore, using (3.10), we see that G_h -equivariantly,

$$R \otimes (\Lambda^h S_h)^{kr} = h^*(r^*W_{r,k} \otimes \mathcal{O}_X(-kr[o])).$$

Note that the left hand side is an X_h -module, since the center of G_h acts trivially; to this end, recall that the morphism i is an r -fold covering of the centers. By equation (3.8), the right hand side is the pullback of a trivial vector bundle, carrying a trivial X_h -action. Consequently, the X_h -representation $R \otimes (\Lambda^h S_h)^{kr}$ is trivial.

This latter observation pins down the H_{hr} -representation R . Let us again pick Theta structures, identifying the Theta group H_{hr} with the Heisenberg group. The characters of the h^2 representations $R_{i,j}$ were computed in Theorem 3 in [17]. There it was proved that the trace of $\eta = (\alpha, x, y) \in H_{hr} \cong \mu_{2hr} \times \mathbb{Z}/hr\mathbb{Z} \times \mathbb{Z}/hr\mathbb{Z}$ equals (3.11)

$$\text{Trace}_{R_{i,j}}(\eta) = \begin{cases} r\alpha^{-hk}\zeta^{ix+jy} & \text{if } (x, y) \in X_h, \text{ i.e. if } (x, y) \in r\mathbb{Z}/hr\mathbb{Z} \times r\mathbb{Z}/hr\mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\zeta = \exp(\frac{2\pi i}{hr})$. The character of the H_h -representation $(\Lambda^h S_h)^{kr}$ was calculated in (2.25):

$$\text{Trace}(\eta) = \alpha^{hkr}(-1)^{(h+1)(x+y)kr}.$$

Since $R \otimes (\Lambda^h S_h)^{kr}$ is a trivial X_h -module, we must have $i = j = \frac{hrk(h+1)}{2}$. Then, the trace of R becomes

$$(3.12) \quad \text{Trace}_R(\eta) = \begin{cases} r\alpha^{-hk}(-1)^{(h+1)kr(x+y)} & \text{if } (x, y) \in X_h, \\ 0 & \text{otherwise.} \end{cases}$$

Making use of (3.6) and (3.8), we can now check that both sides of (1.7) agree equivariantly after pullback by hr . It remains to prove that H_{hr} -equivariantly:

$$(3.13) \quad \text{Sym}^{hk} S_{hr}^\vee \cong R \otimes \bigoplus_{\chi} \chi^{\oplus m_\chi}.$$

In this sum, the χ 's are h^2 representatives of the characters of X_{hr} , modulo those characters of X_{hr} which restrict trivially to the subgroup $X_h \hookrightarrow X_{hr}$. Taking representatives is necessary to avoid repetitions. Indeed, by comparing characters, we see that

$$R \otimes \chi \cong R$$

iff χ restricts trivially to the subgroup X_h . This equation also takes care of the ambiguity seemingly present in the pullback of (3.7) by hr . Note moreover that each representative χ appearing in the sum (3.13) restricts to a well-defined character ξ of X_h , hence giving rise to an h -torsion line bundle L_ξ on X . We will write ω for the order of this line bundle.

In (3.13), the multiplicities m_χ are claimed to have the expressions given in equation (1.8) of the theorem. Checking (3.13) amounts to a character calculation. For the left hand side, the character was essentially computed in Lemma 2.4. Going through the proof of the lemma, we see that the trace of $\eta = (\alpha, x, y) \in H_{hr}$ on $\text{Sym}^{hk} S_{hr}^\vee$ is zero, unless (x, y) is an h -torsion point, say of order h/δ in X_h . In the latter case,

$$(3.14) \quad \text{Trace}(\eta) = (-1)^{xyk(h+1)} \alpha^{-hk} \cdot \frac{r}{r+k} \binom{(r+k)\delta}{r\delta}.$$

It suffices to check that the formula

$$m_\chi = \frac{1}{2h^{3r}} \sum_{\eta=(\alpha, x, y) \in \mu_{2hr} \times X_h} \text{Trace}_{\text{Sym}^{hk} S_{hr}}(\eta) \cdot \text{Trace}_R(\eta)^{-1} \cdot \chi(\eta)^{-1}$$

yields the same answer as (1.8). Substituting (3.12) and (3.14), and recalling that ξ denotes the restriction of χ to X_h , we obtain

$$m_\chi = \frac{1}{h^2} \sum_{\delta|h} \frac{(-1)^{(h+1)kr\delta}}{r+k} \binom{(r+k)\delta}{r\delta} \sum_{(x,y) \text{ has order } h/\delta} \xi(x, y).$$

By Lemma 2.6, this expression can be rewritten as

$$m_\chi = \sum_{\delta|h} \frac{(-1)^{(h+1)kr\delta}}{(r+k)\delta^2} \binom{(r+k)\delta}{r\delta} \left\{ \frac{h/\omega}{h/\delta} \right\}.$$

This completes the proof when r is odd.

When r is even, k must be odd, since $\gcd(r, k) = 1$. Therefore, the theorem holds true for the Verlinde bundle $E_{hk, hr}$. We will now use the level-rank symmetry of the Verlinde bundles under the Fourier-Mukai transform

$$E_{hr, hk}^\vee \cong \widehat{E_{hk, hr}},$$

which was explained in item (iv) of the introduction. We claim that the Atiyah bundles enjoy the analogous symmetry under Fourier-Mukai:

$$W_{r,k,\xi}^\vee \cong \widehat{W_{k,r,\xi}}.$$

Indeed, the case of trivial ξ is the following well-known isomorphism generalizing (2.3):

$$W_{r,k}^\vee \cong \widehat{W_{r,k}}.$$

This is a consequence of the fact that both bundles are simple, of the same rank, and same determinant; alternatively, one may argue using the construction of the Atiyah bundles as successive extensions, explained in Section 2.1. The case of general ξ is an immediate corollary, since the bundles involved differ only by translations. To see this, pick any line bundle M with $M^k = \xi$, and let τ_M denote the translation induced by M on the elliptic curve. We compute

$$\widehat{W_{k,r,\xi}} \cong \widehat{W_{k,r}} \otimes M \cong \tau_M^* \widehat{W_{k,r}} \cong \tau_M^* W_{r,k}^\vee \cong W_{r,k,\xi}^\vee.$$

The first and last isomorphism follow as usual by Atiyah's classification, while the second is a general fact about the Fourier-Mukai transform [11].

We conclude the proof of the theorem by collecting the above observations. $\square$

ACKNOWLEDGEMENTS

We would like to thank Alina Marian for conversations related to this topic and for many useful suggestions, and Mihnea Popa for e-mail correspondence.

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