CHARACTERIZATION AND SEMIADDITIVITY OF THE $\mathcal{C}^1$ -HARMONIC CAPACITY

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ABSTRACT. The $\mathcal{C}^1$ -harmonic capacity κ^c plays a central role in problems of approximation by harmonic functions in the $\mathcal{C}^1$ -norm in $\mathbb{R}^{n+1}$. In this paper we prove the comparability between the capacity κ^c and its positive version κ_+^c . As a corollary, we deduce the semiadditivity of κ^c . This capacity can be considered as a generalization in $\mathbb{R}^{n+1}$ of the continuous analytic capacity α in $\mathbb{C}$. Moreover, we also show that the so-called inner boundary conjecture fails for dimensions $n > 1$, unlike in the case $n = 1$.

1. INTRODUCTION

In 1992, Paramonov [P] introduced the notions of Lip_1 -harmonic capacity κ and $\mathcal{C}^1$ -harmonic capacity κ^c . He showed that these capacities are useful in connection with problems of approximation in the $\mathcal{C}^1$ -norm by harmonic functions. In [MP], some of their geometric properties were studied and the capacities κ_+ and κ_+^c were introduced. A. Volberg proved in [V] the comparability between κ and κ_+ , i.e. that there exists a constant $C > 0$ such that for any compact set $E \subset \mathbb{R}^{n+1}$,

$$C^{-1}\kappa_+(E) \leq \kappa(E) \leq C\kappa_+(E).$$

As a corollary one deduces that κ is semiadditive, that is to say, there exists an absolute constant $C > 0$ such that for all compact sets $E, F \subset \mathbb{R}^{n+1}$,

$$\kappa(E \cup F) \leq C(\kappa(E) + \kappa(F)).$$

However, as shown in [P], for the problems of approximation mentioned above the capacity κ^c is more useful than the capacity κ .

In this paper we will show that results analogous to the ones of [V] for the capacity κ also hold for the capacity κ^c . To state them in detail we introduce some definitions and notation. We denote by $Lip_{loc}(\mathbb{R}^{n+1})$ the space of locally Lipschitz functions.

Definition 1. Given a compact set $E \subset \mathbb{R}^{n+1}$ we define

$$\begin{aligned}\kappa(E) &:= \sup\{|\langle \Delta f, 1 \rangle| : f \in Lip_{loc}(\mathbb{R}^{n+1}), \text{supp}(\Delta f) \subset E, \|\nabla f\|_\infty \leq 1\}, \\ \kappa^c(E) &:= \sup\{|\langle \Delta f, 1 \rangle| : f \in \mathcal{C}^1(\mathbb{R}^{n+1}), \text{supp}(\Delta f) \subset E, \|\nabla f\|_\infty \leq 1\}.\end{aligned}$$

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We also set

$$\begin{aligned}\kappa_+(E) &:= \sup\{|\langle \Delta f, 1 \rangle| : f \in Lip_{loc}(\mathbb{R}^{n+1}), \text{supp}(\Delta f) \subset E, \|\nabla f\|_\infty \leq 1, \\ &\quad \Delta f \text{ is a positive measure}\}, \\ \kappa_+^c(E) &:= \sup\{|\langle \Delta f, 1 \rangle| : f \in C^1(\mathbb{R}^{n+1}), \text{supp}(\Delta f) \subset E, \|\nabla f\|_\infty \leq 1, \\ &\quad \Delta f \text{ is a positive measure}\}.\end{aligned}$$

In the preceding definition, Δf must be understood in the sense of distributions and $\langle \cdot, \cdot \rangle$ stands for the usual duality between distributions and test functions. An immediate consequence of the definition is that $\kappa_+(E) \leq \kappa(E)$ and $\kappa_+^c(E) \leq \kappa^c(E)$.

Our main result is the following.

Theorem 1. *There exists an absolute constant $C > 0$ such that for any compact set $E \subset \mathbb{R}^{n+1}$,*

$$\kappa^c(E) \leq C\kappa_+^c(E).$$

Thus, κ^c and κ_+^c are comparable.

The Lip_1 -harmonic and C^1 -harmonic capacities can be considered as generalizations in $\mathbb{R}^{n+1}$ of the analytic and continuous analytic capacities in $\mathbb{C}$ (denoted by γ and α , respectively). The second author proved in [T2], [T3] the comparability of both analytic capacities with their respective positive versions: there exists a constant $C > 0$ such that for every compact set $E \subset \mathbb{C}$, $\gamma(E) \leq C\gamma_+(E)$ and $\alpha(E) \leq C\alpha_+(E)$. From those results it turns out that there exists an absolute constant $C > 0$ such that for every compact set $E \subset \mathbb{R}^2$, $C^{-1}\kappa(E) \leq \gamma(E) \leq C\kappa(E)$ and $C^{-1}\kappa^c(E) \leq \alpha(E) \leq C\kappa^c(E)$. Also, let us remark that in the work [MPV], the capacity κ_+^c was also studied, and it was shown to be useful in connection with problems on C^1 -approximation by subharmonic functions.

Now we are going to introduce one of the main tools used to study the capacities κ and κ^c . Given a finite measure μ and a function $f \in L^2(\mu)$, the Riesz transform of f with respect to μ is

$$R_\mu^j f(x) = \int \frac{x^j - y^j}{|x - y|^{n+1}} f(y) d\mu(y),$$

for $j = 1, \dots, n+1$ and $x \notin \text{supp}(f d\mu)$. Also, for $x \in \mathbb{R}^{n+1}$ and $\varepsilon > 0$, we set

$$R_{\mu, \varepsilon}^j f(x) = \int_{|x-y|>\varepsilon} \frac{x^j - y^j}{|x - y|^{n+1}} f(y) d\mu(y).$$

Finally we denote $R_\mu f(x) = \{R_\mu^j f(x)\}_j$ and $R^j \mu(x) = R_\mu^j 1(x)$. Notice that the integral defining $R^j \mu(x)$ is absolutely convergent for $x \notin \text{supp}(\mu)$, but not if $x \in \text{supp}(\mu)$ in general. However, using Fubini it is easy to check that it is absolutely convergent for almost every $x \in \mathbb{R}^{n+1}$ with respect to the Lebesgue measure. We say that R_μ is bounded in $L^2(\mu)$ if all the operators $R_{\mu, \varepsilon}$ are bounded in $L^2(\mu)$ with norm bounded above by some constant independent of ε .

Consider the fundamental solution for the Laplace equation $\Delta f = 0$ in $\mathbb{R}^{n+1}$,

$$(1) \quad \phi(x) = \begin{cases} -\frac{1}{2\pi} \log \frac{1}{|x|}, & n = 1, \\ -\frac{1}{(n-1)\sigma_{n+1}|x|^{n-1}}, & n \geq 2, \end{cases}$$

where σ_{n+1} stands for the n -dimensional surface area of the unit sphere in $\mathbb{R}^{n+1}$. Given a measure μ supported on E , take $f(x) = \phi * \mu(x)$ and notice that $\langle \Delta f, 1 \rangle = \mu(E)$ and $\nabla f = R\mu$. This leads us to give an equivalent of κ_+ and κ_+^c .

Definition 2. Given a compact set $E \subset \mathbb{R}^{n+1}$, we set

$$\begin{aligned}\kappa_+(E) &:= \sup\{\mu(E) : \text{supp}(\mu) \subset E, \|R\mu\|_{L^\infty(\mathbb{R}^{n+1})} \leq 1\}, \\ \kappa_+^c(E) &:= \sup\{\mu(E) : \text{supp}(\mu) \subset E, \|R\mu\|_{L^\infty(\mathbb{R}^{n+1})} \leq 1, R\mu \text{ is continuous}\}.\end{aligned}$$

To prove Theorem 1 first one needs to obtain a new characterization of κ_+^c . To this end, we introduce a new capacity κ_0^c . Given a Radon measure μ , we denote

$$\theta_\mu(x) = \lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^n}$$

whenever the limit exists. We set

$$\kappa_0^c(E) := \sup \mu(E),$$

where the supremum is taken over those positive measures supported in E such that $\|R_\mu\|_{L^2(\mu), L^2(\mu)} \leq 1$ and for all $x \in \mathbb{R}^{n+1}$ and $r > 0$, $\mu(B(x, r)) \leq r^n$ and $\theta_\mu(x) = 0$.

Theorem 2. *There exists a constant $C > 0$ such that for any compact set $E \subset \mathbb{R}^{n+1}$,*

$$C^{-1}\kappa_+^c(E) \leq \kappa_0^c(E) \leq C\kappa_+^c(E).$$

It is easy to check that κ_0^c is countably semiadditive, and κ_+^c is too because of the previous theorem. From this result and Theorem 1 we deduce the countable semiadditivity of κ^c :

Corollary 1. *Let $E_i, i \geq 1$, be Borel subsets of $\mathbb{R}^{n+1}$ such that $\bigcup_i E_i$ is a bounded set. Then*

$$\kappa^c\left(\bigcup_i E_i\right) \leq C \sum_i \kappa^c(E_i),$$

where $C > 0$ is an absolute constant.

In this paper we also show that the so-called inner boundary conjecture fails for $n > 1$, unlike in the case $n = 1$. To state this result in detail, we need to introduce additional terminology. Given a compact set $E \subset \mathbb{R}^{n+1}$, denote by $\mathcal{C}^1(E)$ the Banach space of functions which have a $\mathcal{C}^1$ -extension¹ to some neighborhood of E , and let $h^1(E)$ be the closure in the $\mathcal{C}^1$ -norm (restricted to E) of the subspace of functions which are harmonic in some neighborhood of E . Also, let $\mathcal{C}^1 h(E)$ be the subspace of functions in $\mathcal{C}^1(E)$ which are harmonic in E^0 . Clearly we have $h^1(E) \subset \mathcal{C}^1 h(E)$. In 1992, P.V. Paramonov [P] characterized the compact sets E satisfying the equality $h^1(E) = \mathcal{C}^1 h(E)$ in terms of the capacity κ^c (see Theorem 10 below).

Denote by $\partial_o E$ the set of points in the boundary of E that also belong to the boundary of a connected component of $\mathbb{R}^{n+1} \setminus E$, and by $\partial_i E = \partial E \setminus \partial_o E$, the *inner boundary* of E . In the case $n = 1$, the inner boundary conjecture asserts that, given $E \subset \mathbb{C}$, if $\alpha(\partial_i E) = 0$, then every function f analytic in E^0 and continuous in E can be approximated uniformly in E by functions analytic in a neighborhood of E . This conjecture was proved to be true in [T2], as a corollary of the semiadditivity of the capacity α .

For $n \geq 2$, we have the following result.

Theorem 3. *For every $n \geq 2$ there exists a compact set $K \subset \mathbb{R}^{n+1}$ satisfying $\kappa^c(\partial_i K) = 0$ and $h^1(K) \neq \mathcal{C}^1 h(K)$.*

¹Recall Whitney's extension theorem.

2. PRELIMINARIES

For any set $A \subset \mathbb{R}^{n+1}$ we denote by $|A|$ its Lebesgue measure and by $\mathcal{H}^s(A)$ its s -dimensional Hausdorff measure. We say that a measure μ in $\mathbb{R}^{n+1}$ has polynomial growth of order n and constant C_0 (or simply, polynomial growth) if for all $x \in \mathbb{R}^{n+1}$ and $r > 0$, $\mu(B(x, r)) \leq C_0 r^n$. We will denote by $\mathcal{M}(E)$ by the set of Radon measures supported on E . We will also denote by $\Sigma^n(E)$ the set of all positive Radon measures μ supported on E with polynomial growth of order n and constant $C_0 = 1$.

We will say that a ball is μ -doubling with constant C if $\mu(B(x, 2r)) \leq C\mu(B(x, r))$. The measure μ is doubling if all the balls centered in $\text{supp}(\mu)$ are doubling with a uniform constant.

We consider the maximal operator $M\nu(x) = \sup_{r>0} \frac{|\nu|(B(x, r))}{r^n}$, for any measure ν .

A kernel $k(\cdot, \cdot) : L^1_{loc}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \setminus \{(x, y) : x = y\}) \rightarrow \mathbb{R}$ is called a Calderón-Zygmund kernel if for all $x \neq y$,

- $|k(x, y)| \leq \frac{C}{|x-y|^n}$;
- there exists $0 < \delta \leq 1$ such that if $|x - x'| \leq |x - y|/2$, then

$$|k(x, y) - k(x', y)| + |k(y, x) - k(y, x')| \leq C \frac{|x - x'|^\delta}{|x - y|^{n+\delta}}.$$

The Calderón-Zygmund operator (CZO) associated to the kernel $k(\cdot, \cdot)$ and a measure μ is defined as

$$Tf(x) = \int k(x, y)f(y)\mu(y).$$

The above integral may not be convergent for some functions f , so one introduces for $\varepsilon > 0$,

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} k(x, y)f(y)\mu(y),$$

and one says that T is bounded in $L^2(\mu)$ if all the operators T_ε are bounded in $L^2(\mu)$ with constant independent of ε . We also set

$$T_*f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|.$$

The Riesz transform is a particular case of CZO.

In some cases we will need a regularized version of the Riesz transform. Take $\varphi \in C^\infty(\mathbb{R}^{n+1})$, $0 \leq \varphi \leq 1$, radial, satisfying $\varphi = 0$ in $B(0, 1/2)$ and $\varphi = 1$ in $\mathbb{R}^{n+1} \setminus B(0, 1)$. Given $\varepsilon > 0$, define

$$\overline{R}_{\varepsilon, \mu} f(x) = \int \varphi\left(\frac{x-y}{\varepsilon}\right) \frac{x-y}{|x-y|^{n+1}} f(y) d\mu(y), \text{ for } f \in L^1(d\mu),$$

and

$$\overline{R}_\varepsilon \mu(x) = \overline{R}_{\varepsilon, \mu} 1(x).$$

Since the new kernel $\varphi\left(\frac{x-y}{\varepsilon}\right) \frac{x-y}{|x-y|^{n+1}}$ is continuous with respect to x , we can deduce that $\overline{R}_{\varepsilon, \mu}(x)$ is a continuous function. If the measure μ has polynomial growth it is easy to check that for all $x \in \mathbb{R}^{n+1}$ and $f \in L^1(\mu)$,

$$(2) \quad |\overline{R}_{\mu, \varepsilon} f(x) - R_{\mu, \varepsilon} f(x)| \leq CM(f d\mu)(x).$$

Given $a, b > 0$ we use the notation $a \lesssim b$ if there exists an absolute constant $C > 0$ satisfying $a \leq Cb$. If also $b \lesssim a$, then we write $a \approx b$.

3. THE CAPACITY κ_0^c

In this section we will prove Theorem 2.

3.1. The first inequality in Theorem 2.

Proposition 1. *For any compact set $E \subset \mathbb{R}^{n+1}$,*

$$\kappa_+^c(E) \lesssim \kappa_0^c(E).$$

Proof. Let μ be a measure supported on E such that $R\mu$ is continuous, $\|R\mu\|_{L^\infty(\mathbb{R}^{n+1})} \leq 1$ and $\mu(E) \geq \kappa_+^c(E)/2$. Let $x \in \mathbb{R}^{n+1}$ and $r > 0$ be fixed. We call $B := B(x, r)$ and we denote by ν the normal vector on ∂B and by ∂_ν the normal derivative. Take $\varphi \in \mathcal{C}^\infty$ radial, satisfying $0 \leq \varphi$, $\int \varphi(x)dx = 1$, and denote $\varphi_t(x) := \frac{1}{t^{n+1}}\varphi(\frac{x}{t})$.

- (1) In order to prove that $\mu \in \Sigma^n(E)$, we want to apply Green's formula to $v = \phi * \mu$. It can be shown that $\Delta v = \mu$ and $\nabla v = R\mu$ in a distributional sense. Since $\varphi_t * v \in C^\infty$ by Green's formula:

$$\begin{aligned} \left| \int_B \varphi_t * \mu(y)dy \right| &= \left| \int_B \Delta(\varphi_t * v)(y)dy \right| \\ &= \left| \int_{\partial B} \varphi_t * \partial_\nu v(s) d\mathcal{H}^n(s) \right| \leq \int_{\partial B} \|\varphi\|_{L^1} |R\mu|(s) d\mathcal{H}^n(s) \leq Cr^n. \end{aligned}$$

Since $\lim_{t \rightarrow \infty} \int_B \varphi_t * \mu(y)dy = \mu(B)$, we infer that $\mu(B) \leq Cr^n$.

- (2) Let us see that $\theta_\mu(x) = 0$. Take $\varepsilon > 0$ and fix $x \in \mathbb{R}^{n+1}$. We know that for r small enough, if $|x-y| < r$, $|R\mu(x) - R\mu(y)| < \varepsilon$. Define $G_x(y) = \langle R\mu(x), y \rangle$ and observe that $\nabla_y G_x = R\mu(x)$. Then, applying Green's formula,

$$\int_{\partial B} \varphi_t * \partial_\nu G_x(s) d\mathcal{H}^n(s) = \int_B \varphi_t * \Delta G_x dy = 0$$

and

$$\begin{aligned} \left| \int_B \varphi_t * \mu(y)dy \right| &= \left| \int_{\partial B} \varphi_t * (\partial_\nu v - \partial_\nu G_x) d\mathcal{H}^n(s) \right| \\ &\leq \int_{\partial B} |\varphi_t| * |\nu(s)(R\mu(s) - R\mu(x))| d\mathcal{H}^n(s) \\ &\leq \varepsilon \int_{\partial B} \|\varphi_t\|_{L^1} \|\nu(s)\|_2 d\mathcal{H}^n(s) \leq \varepsilon Cr^n, \end{aligned}$$

and letting $t \rightarrow \infty$ we obtain the result.

- (3) We want to see that $\|R_\varepsilon \mu\|_{L^\infty(\mu)} \leq C$ with C independent of ε in order to apply a nondoubling version of the $T(1)$ theorem (for instance, [NTV2] or [T1]). Let $x \in \mathbb{R}^{n+1}$, $B := B(x, \varepsilon)$ and $y \in B_2 := B(x, \varepsilon/2)$. Since μ has polynomial growth, it can be shown that if $|x-y| < \varepsilon/2$, then

$$|R_{\mu, \varepsilon}(x) - R_{\mu, \varepsilon}(y)| \leq C.$$

So if for a function f and a set A we denote $m_A(f) = \frac{1}{|A|} \int_A f(y)dy$, then we have:

$$|R_\varepsilon \mu(x)| \leq |R_\varepsilon \mu(x) - m_{B_2}(R_\varepsilon \mu)| + |m_{B_2}(R_\varepsilon \mu)| \leq C + |m_{B_2}(R_\varepsilon \mu)|.$$

To estimate the last term, we set

$$(3) \quad R_\varepsilon \mu(y) = R\mu(y) - R_\mu(\chi_B)(y) + R_\mu(\chi_B)(y) - R_\mu(\chi_{B(y,\varepsilon)})(y)$$

and also

$$\begin{aligned} |R_\mu(\chi_B)(y) - R_\mu(\chi_{B(y,\varepsilon)})(y)| &\leq \int_{|x-z|<\varepsilon, |y-z|>\varepsilon} |k(x, z)| d\mu(z) \\ &\quad + \int_{|y-z|<\varepsilon, |x-z|>\varepsilon} |k(y, z)| d\mu(z). \end{aligned}$$

We deal with both integrals in the same way. Let us proceed with the first one. Necessarily $|x - z| > \varepsilon/2$ and so

$$\int_{|x-z|<\varepsilon, |x-z|>\varepsilon/2} |k(x, z)| d\mu(z) \leq \frac{1}{\varepsilon^n} \mu(B(x, \varepsilon)) \leq C.$$

Integrating (3) in the variable y , using the last estimate and the fact that $\|R\mu\|_{L^\infty(\mathbb{R}^{n+1})} \leq 1$, we obtain

$$|m_{B_2}(R_\varepsilon \mu)| \leq C + |m_{B_2}(R_\mu(\chi_B))|.$$

Finally,

$$\begin{aligned} |m_{B_2}(R_\mu(\chi_B))| &= \left| \frac{1}{|B_2|} \int_{B_2} \int_B k(y, z) d\mu(z) dy \right| \\ &\leq C \frac{1}{|B_2|} \int_B \int_{B_2} \frac{1}{|y - z|^n} dy d\mu(z) \leq C \frac{\varepsilon \mu(B)}{|B_2|} \leq C. \end{aligned}$$

Thus $\|R_\varepsilon \mu\|_{L^\infty(\mu)} \leq C$ uniformly on ε and by the $T(1)$ theorem we deduce that R_μ is bounded in $L^2(\mu)$. $\square$

3.2. Technical lemmas. The second inequality of Theorem 2 is a little bit harder to prove. In this section we will provide the necessary technical lemmas. The first one is a vectorial version of a well-known result. The proof can be found with minor modifications in [MP].

Lemma 1. *Let μ be a positive Radon measure on a locally compact Hausdorff space X and let $T_i : \mathcal{M}(X) \rightarrow \mathcal{C}(X)$, $i = 1, \dots, n$ be linear bounded operators. Suppose that each $T_i^* : \mathcal{M}(X) \rightarrow \mathcal{C}_0(X)$ is $(1, 1)$ weak type with respect to the μ measure, that is to say, that there exists a constant A such that*

$$(4) \quad \mu\{x : |T_i^* \nu(x)| > \alpha\} \leq A\alpha^{-1} \|\nu\|$$

for $i = 1, \dots, n$, $\alpha > 0$ and $\nu \in \mathcal{M}$. Then, for every $\tau > 0$ and every Borel set $E \subset X$ with $0 < \mu(E) < \infty$, there exists $0 \leq h \leq 1$ satisfying $h(x) = 0$ for $x \in X \setminus E$,

$$\int_E h d\mu \geq \mu(E)/(1 + \tau)$$

and

$$\|T_i(hd\mu)\|_\infty \leq (n + 1)A/\tau, \quad i = 1, \dots, n.$$

Given a measure μ with polynomial growth with constant $C_0 > 0$ and a CZO T_μ , consider the following set:

$$X_0 = \{x \in \mathbb{R}^{n+1}; T_* \mu(x) > \alpha\},$$

and define

$$\varepsilon_0(x) = \sup\{\varepsilon : \varepsilon > 0, |T_\varepsilon \mu(x)| > \alpha\}.$$

If $x \notin X_0$, we define $\varepsilon_0(x) = 0$. Consider the exceptional set $X = \bigcup_{x \in X_0} B(x, \varepsilon_0(x))$.

Lemma 2. *If $y \in X$, then $T_* \mu(y) > \alpha - DC_0$, where D is a constant not depending on either α or y .*

A proof can be found with minor modifications in [T3], Lemma 10.1. We will talk now about the *suppressed kernels*. They were introduced by F. Nazarov, S. Treil and A. Volberg. Given a Calderón-Zygmund kernel $k(x, y)$ and a Lipschitz function, $\rho : \mathbb{R}^{n+1} \rightarrow (0, +\infty)$, we consider the suppressed kernel:

$$k_\rho(x, y) := k(x, y) \frac{1}{1 + k^2(x, y) \rho(x)^n \rho(y)^n}.$$

Then $k_\rho(x, y)$ is a Calderón-Zygmund kernel with constants independent of ρ (see Lemma 8.2 from [V]). We denote by T_μ^ρ the associated CZO.

Lemma 3. *Let μ be a measure with polynomial growth with constant C_0 , ρ a nonnegative Lipschitz function, and T_μ^ρ the operator associated to the suppressed kernel. Suppose that $|T_\varepsilon \mu(x)| \leq \alpha$ for some $x \in \mathbb{R}^{n+1}$ and all $\varepsilon \geq r_0$ with $r_0 \geq 0$. If $\rho(x) \geq \tau r_0$ for a $\tau > 0$, then*

$$|T_\varepsilon^\rho \mu(x)| \leq C_0 C_\tau + \alpha$$

for all $\varepsilon > 0$, with C_τ depending only on τ and the CZ constants of the kernel.

A proof can be found in [V] or [T3]. The following result is one of the key steps for the proof of Theorem 2.

Lemma 4. *Set $E = \text{supp}(\mu)$ and suppose that $\theta_\mu(x) = 0$ for all $x \in E$. Let $T_\mu : L^2(\mu) \rightarrow L^2(\mu)$ be a bounded Calderón-Zygmund operator with antisymmetric kernel ($k(x, y) = -k(y, x)$). Then, given $\delta > 0$ we can find $F \subset E$ with $\mu(E \setminus F) < \delta$ such that*

- a) $\lim_{r \rightarrow 0} \frac{\mu(B(x, r) \cap F)}{r^n} = 0$ uniformly on $x \in \mathbb{R}^{n+1}$,
- b) $\lim_{r \rightarrow 0} \|T_\mu\|_{L^2(\mu|_{B(x, r) \cap F}) \rightarrow L^2(\mu|_{B(x, r) \cap F})} = 0$, uniformly on $x \in \mathbb{R}^{n+1}$.

Proof. Let $F_1 \subset E$ such that $\lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^n} = 0$ uniformly on $x \in F_1$ and $\mu(E \setminus F_1) < \delta$ and denote $\sigma = \mu|_{F_1}$. There exists some sequence $\{\varepsilon_m\}_m$, $\varepsilon_m \rightarrow 0$, such that the operators $T_{\sigma, \varepsilon_m}$ converge to some operator T_σ^w weakly in $\mathcal{B}(L^2(\sigma))^*$. Then for all $x \notin \text{supp}(f)$, $\lim_{\varepsilon_m \rightarrow 0} T_{\sigma, \varepsilon_m}(f)(x)$ exists and coincides σ -a.e. with $T_\sigma^w(f)(x)$. Let $F \subset F_1$ with $\sigma(F_1 \setminus F) < \delta$ satisfying

$$\lim_{r \rightarrow 0} \frac{1}{\sigma(B(x, r))} \int_{B(x, r)} T^w \sigma \, d\sigma = T^w \sigma(x)$$

uniformly on $x \in F$.

Claim 1. For all η with $0 < \eta \leq 2$, there exists $R > 0$ small enough such that for all $x_0 \in F_1$, setting $B := B(x_0, R)$, for σ -a.e. $x \in F \cap B$ we have

$$(5) \quad |T_{\sigma, \varepsilon}(\chi_{2B})(x) - T_\sigma^w(\chi_{2B})(x)| \leq C\eta, \text{ if } \varepsilon \leq \eta R.$$

We will prove the claim below. Fix $0 < \eta < 2$ and suppose that R is small enough such that $\frac{\sigma(B(x, r))}{r^n} \leq \eta$ for all $r \leq 2R$ and all $x \in F$. Let $x_0 \in F$ and $B := B(x_0, R)$ be fixed. Then, for $x \in F \cap \eta B$ we have that

$$(6) \quad |T_{\sigma, \varepsilon}(\chi_{2\eta B})(x)| \leq C_1 \eta, \quad \forall \varepsilon > 0,$$

since:

- a) if $\varepsilon \geq \eta R$, then $|T_{\sigma, \varepsilon}(\chi_{2\eta B})(x)| \leq \frac{\sigma(2\eta B)}{(\eta R)^n} \leq C\eta$;
- b) if $\varepsilon < \eta R$, applying the preceding claim,

$$\begin{aligned} |T_{\sigma, \varepsilon}(\chi_{2\eta B})(x)| &= |T_{\sigma}(\chi_{2\eta B \setminus B(x, \varepsilon)})(x)| \\ &= |T_{\sigma}(\chi_{2\eta B \setminus B(x, \eta R)})(x) + T_{\sigma}(\chi_{B(x, \eta R) \setminus B(x, \varepsilon)})(x)| \\ &\leq |T_{\sigma}(\chi_{2\eta B \setminus B(x, \eta R)})(x)| + |T_{\sigma, \varepsilon}(\chi_{2B})(x) - T_{\sigma, \eta R}(\chi_{2B})(x)| \\ &\leq |T_{\sigma}(\chi_{2\eta B \setminus B(x, \eta R)})(x)| + 2C\eta \leq \frac{\sigma(2\eta B)}{(\eta R)^n} + 2C\eta \leq C\eta. \end{aligned}$$

We wish to apply the $T(1)$ theorem. To this end we would like the last estimate to hold for all $x \in \mathbb{R}^{n+1}$. We will solve this question by means of the technique of suppressed kernels of [NTV2] and [V]. Let us denote $\nu = \mu|_{F_1 \cap 2\eta B} = \sigma|_{2\eta B}$ and

$$X_0 = \{x \in \mathbb{R}^{n+1} : T_*\nu(x) > C_1\eta + D\eta\},$$

where C_1 appears in (6) and D in Lemma 2. For $x \in X_0$ fixed, let

$$\varepsilon_0(x) = \sup\{\varepsilon : \varepsilon > 0, |T_{\varepsilon}\nu(x)| > C_1\eta + D\eta\}.$$

If $x \notin X_0$, we set $\varepsilon_0(x) = 0$. Consider the exceptional set $X = \bigcup_{x \in X_0} B(x, \varepsilon_0(x))$ and define

$$\rho(x) = d(x, \mathbb{R}^{n+1} \setminus X).$$

This function is Lipschitz and satisfies $\rho(x) \geq \varepsilon_0(x)$ for all x . Indeed, if $x \notin X_0$, $\varepsilon_0(x) = 0$; and if $x \in X_0$, $B(x, \varepsilon_0(x)) \subset X$. Finally define

$$k_{\rho}(x, y) := \frac{k(x, y)}{1 + k^2(x, y)\rho(x)^n\rho(y)^n},$$

and denote by T_{ν}^{ρ} the associated operator.

The kernels $k(\cdot, \cdot)$ and $k_{\rho}(\cdot, \cdot)$ coincide in $\eta B \cap F$, since $\rho(x) = 0$ for all $x \in \eta B \cap F$. Indeed, Lemma 2 tells us that for all $y \in X$, $T_*\nu(y) > C_1\eta$. But if $x \in \eta B \cap F$, by (6) we have $T_*\nu(x) \leq C_1\eta$. So, $\eta B \cap F \subset \mathbb{R}^{n+1} \setminus X$ and $\rho(x) = 0$.

By Lemma 3, for all $x \in \mathbb{R}^{n+1}$, $T_*\nu(x) \leq C\eta$. Then, by the nonhomogeneous $T(1)$ theorem (for example [NTV2] or [T1]), T_{ν}^{ρ} is bounded on $L^2(\nu)$ with norm $\leq C\eta$. Thus T_{σ} is bounded in $L^2(\mu|_{\eta B \cap F})$ with norm $\leq C\eta$ because T_{ν}^{ρ} coincides with T_{σ} in $\eta B \cap F$. This proves the lemma. $\square$

To prove Claim 1 we need to introduce the notion of thin boundaries. Given some positive constant C_{30} and some Borel measure μ , we say that the ball $B(x, r)$ has *thin boundaries* if

$$(7) \quad \mu\{x \in B(x, 1.5r) : d(x, \partial B(x, r)) \leq \lambda\} \leq C_{30}\lambda\mu(B(x, 1.5r)).$$

Of course not all the balls satisfy the preceding condition. However, as the following lemma shows, there are many balls with thin boundaries. See [NTV1, Lemma 2.8] for the proof.

Lemma 5. *Suppose that C_{30} is big enough. Given any Radon measure μ in $\mathbb{R}^{n+1}$ and any ball $B(x, r)$, one can find $B(x, r')$ with thin boundaries with $r' \in [r, 1.1r]$.*

In the rest of the paper, when talking about the thin boundaries condition, we assume that C_{30} is big enough so that the conclusion of the preceding lemma holds.

The proof of the next lemma can be found in [NTV1, Lemma 2.9].

Lemma 6. *Let $B(x, \varepsilon)$ be a ball in $\mathbb{R}^{n+1}$ with thin boundaries and μ a positive measure with polynomial growth with constant C_0 . Then*

$$\int_{B(x, 1.5\varepsilon) \setminus B(x, \varepsilon)} \int_{B(x, \varepsilon)} \frac{1}{|y - z|^n} d\mu(z) d\mu(y) \leq C_0 C\mu(B(x, 1.5\varepsilon)).$$

We will also need the following well-known technical result.

Lemma 7. *If a measure μ has polynomial growth with constant C_0 , for $r_0 > 0$ we have*

$$\int_{|x-z| \geq r_0} \frac{1}{|x-z|^{n+1}} d\mu(z) \leq CC_0/r_0.$$

Proof of Claim 1. The arguments are quite similar to the ones for the proof of Cotlar's inequality. However, instead of the L^2 boundedness of T_σ we will use its antisymmetry.

Fix $\eta > 0$, and take $R > 0$ small enough such that for all $0 < r \leq 2R$ and $x \in F$, $\sigma(B(x, r)) \leq \eta r^n$, and

$$(8) \quad |m_{\sigma, B(x, r)}(T^w \sigma) - T^w \sigma(x)| \leq \eta,$$

where $m_{\sigma, B(x, r)}(f) = \frac{1}{\sigma(B(x, r))} \int_{B(x, r)} f d\sigma$.

Recall that for any positive Radon measure with polynomial growth, fixed $x \in \mathbb{R}^{n+1}$ and a constant $C' > 2^{n+1}$, we can find a sequence of doubling balls with constant C' centered in x and radii tending to zero. For $x \in F$ and $\varepsilon \leq \eta R$ fixed, let $\varepsilon_2 \leq \varepsilon$ be the maximal $\tilde{\varepsilon} \leq \varepsilon$ such that $B(x, \tilde{\varepsilon})$ is σ -doubling with constant 2^{n+2} (it is easy to check that indeed this maximum exists). Let $\varepsilon_3 \in [\varepsilon_2, 1.1\varepsilon_2]$ such that $B(x, \varepsilon_3)$ has thin boundaries for σ .

We now proceed as follows:

$$\begin{aligned} |T_{\sigma, \varepsilon}(\chi_{2B})(x) - T_\sigma^w(\chi_{2B})(x)| &\leq |T_{\sigma, \varepsilon}(\chi_{2B})(x) - T_{\sigma, \varepsilon_2}(\chi_{2B})(x)| \\ &\quad + |T_{\sigma, \varepsilon_2}(\chi_{2B})(x) - T_{\sigma, 1.5\varepsilon_3}(\chi_{2B})(x)| \\ &\quad + |T_{\sigma, 1.5\varepsilon_3}(\chi_{2B})(x) - m_{\sigma, B(x, \varepsilon_3)} T_\sigma^w(\chi_{2B})| \\ &\quad + |m_{\sigma, B(x, \varepsilon_3)} T_\sigma^w(\chi_{2B}) - T_\sigma^w(\chi_{2B})(x)| \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

We will see that $I_i \leq C\eta$ for $i = 1, \dots, 4$.

First we deal with I_1 . Denote $\varepsilon'_2 = 2\varepsilon_2$, and consider $N \geq 0$ such that $2^{N-1}\varepsilon'_2 \leq \varepsilon < 2^N\varepsilon'_2$. Then,

$$I_1 \leq \int_{\varepsilon_2 < |x-z| \leq \varepsilon'_2} \frac{1}{|x-z|^n} d\sigma(z) + \int_{\varepsilon'_2 < |x-z| \leq \varepsilon} \frac{1}{|x-z|^n} d\sigma(z).$$

We estimate the first integral:

$$\int_{\varepsilon_2 < |x-z| \leq \varepsilon'_2} \frac{1}{|x-z|^n} d\sigma(z) \leq C \frac{\sigma(B(x, 2\varepsilon_2))}{(2\varepsilon_2)^n} \leq C\eta.$$

If $\varepsilon'_2 \neq \varepsilon$, then $B(x, 2^k \varepsilon'_2)$ is nondoubling for $0 \leq k \leq N-1$, and so $\sigma(B(x, 2^k \varepsilon'_2)) \leq \sigma(B(x, 2^N \varepsilon'_2))/2^{(n+2)(N-k)}$ for these k 's. Thus,

$$\begin{aligned} \int_{\varepsilon'_2 < |x-z| \leq \varepsilon} \frac{1}{|x-z|^n} d\sigma(z) &\leq \sum_{k=1}^N \int_{\varepsilon_2^{k-1} < |x-z| \leq \varepsilon_2^k} \frac{1}{|x-z|^n} d\sigma(z) \\ &\leq \sum_{k=1}^N \frac{\sigma(B(x, 2^k \varepsilon'_2))}{(2^{k-1} \varepsilon'_2)^n} \leq \sum_{k=1}^N \frac{2^{(n+2)(k-N)} \sigma(B(x, 2^N \varepsilon'_2))}{2^{(k-1-N)n} (2^N \varepsilon'_2)^n} \\ &= 2^n \frac{\sigma(B(x, 2^N \varepsilon'_2))}{(2^N \varepsilon'_2)^n} \sum_{k=1}^N \left(\frac{1}{2^2}\right)^{N-k} \leq C\eta \sum_{k=1}^{\infty} \left(\frac{1}{2^2}\right)^k = C\eta. \end{aligned}$$

The estimate of I_2 is straightforward:

$$I_2 = \left| \int_{\varepsilon_2 \leq |x-z| \leq 1.5\varepsilon_3} k(x, z) d\sigma(z) \right| \leq \frac{\sigma(B(x, 1.5\varepsilon_3))}{\varepsilon_2^n} \leq C \frac{\sigma(B(x, 2\varepsilon_2))}{(2\varepsilon_2)^n} \leq C\eta.$$

To deal with I_3 we will argue as in the proof of Cotlar's inequality: for all σ a.e. $y \in B(x, \varepsilon_3)$ we have

$$\begin{aligned} &|T_{\sigma, 1.5\varepsilon_3}(\chi_{2B})(x) - T_{\sigma}^w(\chi_{2B \setminus B(x, 1.5\varepsilon_3)})(y)| \\ &= \left| \int_{\substack{|z-x_0| \leq 2R \\ |z-x| \geq 1.5\varepsilon_3}} k(x, z) d\sigma(z) - \int_{\substack{|z-x_0| \leq 2R \\ |z-x| \geq 1.5\varepsilon_3}} k(y, z) d\sigma(z) \right| \\ &\leq C \int_{1.5\varepsilon_3 \leq |z-x|} \frac{\varepsilon_3}{|x-z|^{n+1}} d\sigma|_B(z) \leq C\eta, \end{aligned}$$

by Lemma 7. Then,

$$(9) \quad |T_{\sigma, 1.5\varepsilon_3}(\chi_{2B})(x) - m_{\sigma, B(x, \varepsilon_3)}(T_{\sigma}^w(\chi_{2B \setminus B(x, 1.5\varepsilon_3)}))| \leq C\eta.$$

From the antisymmetry of T_{σ}^w we have $\int_{B(x, \varepsilon_3)} T_{\sigma}^w(\chi_{B(x, \varepsilon_3)}) d\mu = 0$, and since $\sigma(B(x, \delta)) \leq \eta\delta^n$, for $\delta \leq 2R$,

$$\begin{aligned} |m_{\sigma, B(x, \varepsilon_3)}(T_{\sigma}^w(\chi_{B(x, 1.5\varepsilon_3)}))| &= |m_{\sigma, B(x, \varepsilon_3)}(T_{\sigma}^w(\chi_{B(x, 1.5\varepsilon_3) \setminus B(x, \varepsilon_3)}))| \\ &= \frac{1}{\sigma(B(x, \varepsilon_3))} \left| \int_{B(x, 1.5\varepsilon_3) \setminus B(x, \varepsilon_3)} T_{\sigma}^w(\chi_{B(x, \varepsilon_3)}) d\sigma(y) \right|. \end{aligned}$$

Since $B(x, \varepsilon_3)$ has thin boundaries and $B(x, \varepsilon_2)$ is doubling, from Lemma 6 we deduce

$$\begin{aligned} &\frac{1}{\sigma(B(x, \varepsilon_3))} \left| \int_{B(x, 1.5\varepsilon_3) \setminus B(x, \varepsilon_3)} T_{\sigma}^w(\chi_{B(x, \varepsilon_3)}) d\sigma(y) \right| \\ &\leq \frac{1}{\sigma(B(x, \varepsilon_3))} \int_{B(x, 1.5\varepsilon_3) \setminus B(x, \varepsilon_3)} \int_{B(x, \varepsilon_3)} \frac{1}{|y-z|^n} d\sigma(z) d\sigma(y) \\ &\leq C\eta \frac{\sigma(B(x, 1.5\varepsilon_3))}{\sigma(B(x, \varepsilon_3))} \leq C\eta \frac{\sigma(B(x, 2\varepsilon_2))}{\sigma(B(x, \varepsilon_2))} \leq C\eta. \end{aligned}$$

Therefore, $I_3 \leq C\eta$.

It only remains to estimate I_4 . For $y \in B(x, \varepsilon_3)$, by Lemma 7, we have

$$\begin{aligned} |T_\sigma^w(\chi_{F_1 \setminus 2B})(x) - T_\sigma^w(\chi_{F_1 \setminus 2B})(y)| &= \left| \int_{|z-x_0| > 2R} (k(x, z) - k(y, z)) d\sigma(z) \right| \\ &\leq \varepsilon_3 \int_{|z-x| > R} \frac{1}{|z-x|^{n+1}} d\sigma(z) \leq C\eta\varepsilon_3/R \leq C\eta. \end{aligned}$$

Finally, taking the mean on $y \in B(x, \varepsilon_3)$ with respect to σ , and using (8), we obtain

$$\begin{aligned} |m_{\sigma, B(x, \varepsilon_3)} T_\sigma^w(\chi_{2B}) - T_\sigma^w(\chi_{2B})(x)| &\leq |m_{\sigma, B(x, \varepsilon_3)} T^w \sigma - T^w \sigma(x)| \\ &+ |m_{\sigma, B(x, \varepsilon_3)} T_\sigma^w(\chi_{F_1 \setminus 2B}) - T_\sigma^w(\chi_{F_1 \setminus 2B})(x)| \leq C\eta. \quad \square \end{aligned}$$

We now show the last lemma of this section. This is one of the key steps for the proof of Theorem 2.

Lemma 8. *Let $F = \text{supp}(\mu)$ and $T = (T^1, \dots, T^{n+1})$ be a vectorial Calderón-Zygmund operator. Suppose that*

- a) $\theta_\mu(x) = \lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^n} = 0$ uniformly on $x \in F$,
- b) $\lim_{r \rightarrow 0} \|T\mu\|_{L^2(\mu|_{B(x, r)}), L^2(\mu|_{B(x, r)})} = 0$ uniformly on $x \in F$.

Let f be a bounded function supported on F such that $\|T_\varepsilon^j f\|_{L^\infty(\mathbb{R}^{n+1})} < C$ for all j, ε . Then, given $0 < \tau \leq 1$, there exists $\delta > 0$ and a function g supported on F satisfying $\int g d\mu = \int f d\mu$, $0 \leq g \leq \|f\|_{L^\infty(\mu)} + \tau$, $\|T_^j g\|_{L^\infty(\mathbb{R}^{n+1})} \leq \|T_*^j f\|_{L^\infty(\mathbb{R}^{n+1})} + \tau$, and also*

$$(10) \quad |T_\varepsilon^j g(x) - T_\varepsilon^j g(y)| \leq \tau, \quad \text{if } |x - y| \leq \delta \text{ and } \varepsilon > 0,$$

and

$$(11) \quad |T_\varepsilon^j g(x) - T_\varepsilon^j g(y)| \leq \sup_{\varepsilon' > 0} |T_{\varepsilon'}^j f(x) - T_{\varepsilon'}^j f(y)| + \tau, \quad \forall x, y \in \mathbb{R}^{n+1}, \varepsilon > 0,$$

for each $j = 1, \dots, n+1$.

Proof. Let N be a big integer and $\varepsilon_1 \ll \tau/\|f\|_{L^\infty(\mu)}$ a small constant, both chosen below. Consider the standard dyadic lattice made up of cubes $\{Q_i\}_{i \in I}$ of side length l . We choose l so that for $x \in F$, $0 < r \leq Nl$ we have $\frac{\mu(B(x, r))}{r^n} \leq \varepsilon_1$ and also $\|T\mu\|_{L^2(\mu|_{Q_i}) \rightarrow L^2(\mu|_{Q_i})} \leq \varepsilon_2$ with $\varepsilon_2 \rightarrow 0$ as $\varepsilon_1 \rightarrow 0$. Now we construct the function g . Fix i and Q_i with $\mu(Q_i) \neq 0$. Since T is bounded in Q_i , we can apply Lemma 1 and find a function φ_i supported on Q_i , with $0 \leq \varphi_i \leq 1$ and satisfying $\int \varphi_i d\mu \geq (1 + \tau)^{-1} \mu(Q_i)$ and $\|T_\varepsilon^j \varphi_i\|_{L^\infty(\mathbb{R}^{n+1})} \leq \varepsilon_3$, where ε_3 will be chosen below and $\varepsilon_3 \rightarrow 0$ as $\|T\|_{L^2(\mu|_{Q_i}), L^2(\mu|_{Q_i})} \rightarrow 0$. If we set

$$g_i = \frac{\int_{Q_i} f d\mu}{\int \varphi_i d\mu} \varphi_i,$$

we get

$$\|T_\varepsilon g_i\|_{L^\infty(\mathbb{R}^{n+1})} \leq \frac{\int_{Q_i} f d\mu}{\int \varphi_i d\mu} \varepsilon_3 \leq \varepsilon_3(1 + \tau)\|f\|_{L^\infty(\mu)} \leq 2\varepsilon_3\|f\|_{L^\infty(\mu)}.$$

We define

$$g = \sum_{i \in I} g_i.$$

By construction $g \geq 0$, $\int g d\mu = \int f d\mu$ and also

$$\|g\|_{L^\infty(\mu)} = \sup_{i \in I} \|g_i\|_{L^\infty(\mu)} \leq (1 + \tau) \|f\|_{L^\infty(\mu)}.$$

Let us prove inequality (10) for $\delta = l/2$. We call $g_a = g\chi_{3Q_i}$ and $g_b = g - g_a = g\chi_{\mathbb{R}^{n+1} \setminus 3Q_i}$. Given $x \in Q_i$ and y such that $|x - y| \leq \delta$, we have

$$|T_\varepsilon g(x) - T_\varepsilon g(y)| \leq |T_\varepsilon g_a(x) - T_\varepsilon g_a(y)| + |T_\varepsilon g_b(x) - T_\varepsilon g_b(y)|.$$

First we estimate the term involving g_a :

$$\begin{aligned} |T_\varepsilon g_a(x) - T_\varepsilon g_a(y)| &\leq |T_\varepsilon g_a(x)| + |T_\varepsilon g_a(y)| \\ &\leq \sum_{k: Q_k \subset 3Q_i} (|T_\varepsilon g_a(x)| + |T_\varepsilon g_a(y)|) \leq 3^{n+1} \varepsilon_3 \|f\|_{L^\infty(\mu)}. \end{aligned}$$

Now we deal with the other term:

$$\begin{aligned} |T_\varepsilon g_b(x) - T_\varepsilon g_b(y)| &\leq \|g\|_{L^\infty(\mu)} \int_{\mathbb{R}^{n+1} \setminus 3Q_i} |k_\varepsilon(x - z) - k_\varepsilon(y - z)| d\mu(z) \\ &\leq Cl \|f\|_{L^\infty(\mu)} \int_{\mathbb{R}^{n+1} \setminus 3Q_i} \frac{1}{|x - z|^{n+1}} d\mu(z). \end{aligned}$$

We split the last integral into two parts:

$$\int_{\mathbb{R}^{n+1} \setminus 3Q_i} \frac{1}{|x - z|^{n+1}} d\mu(z) \leq \left(\int_{l \leq |x - z| \leq Nl} + \int_{|x - z| > Nl} \right) \frac{1}{|x - z|^{n+1}} d\mu(z).$$

We estimate the first integral in the following way. If we take $N = 2^M$,

$$\begin{aligned} \int_{l \leq |x - z| \leq Nl} \frac{1}{|z - x|^{n+1}} d\mu(z) &\leq \sum_{k=0}^M \int_{2^k l \leq |x - z| \leq 2^{k+1} l} \frac{1}{|z - x|^{n+1}} d\mu(z) \\ &\leq \sum_{k=0}^M \frac{\varepsilon_1 (2^{k+1} l)^n}{(2^k l)^{n+1}} \leq \frac{2^n \varepsilon_1}{l} \sum_{k=0}^{\infty} \frac{1}{2^k}. \end{aligned}$$

By Lemma 7 we get

$$\int_{\mathbb{R}^{n+1} \setminus 3Q_i} \frac{1}{|x - z|^{n+1}} d\mu(z) \leq C \left(\frac{\varepsilon_1}{l} + \frac{1}{Nl} \right).$$

Choosing ε_1 small enough and N big enough,

$$|T_\varepsilon g_b(x) - T_\varepsilon g_b(y)| \leq C \|f\|_{L^\infty(\mu)} \left(\varepsilon_1 + \frac{1}{N} \right) \leq \tau/2.$$

Thus, if ε_3 is small enough,

$$|T_\varepsilon g(x) - T_\varepsilon g(y)| \leq \tau.$$

Now we are going to prove (11). Given $x \in Q_i$ and $y \in Q_j$,

$$\begin{aligned} |T_\varepsilon g(x) - T_\varepsilon g(y)| &\leq |T_\varepsilon (g\chi_{\mathbb{R}^{n+1} \setminus 3Q_i})(x) - T_\varepsilon (g\chi_{\mathbb{R}^{n+1} \setminus 3Q_j})(y)| \\ &\quad + |T_\varepsilon (g\chi_{3Q_i})(x)| + |T_\varepsilon (g\chi_{3Q_j})(y)|. \end{aligned}$$

For the second term we have

$$(12) \quad |T_\varepsilon(g\chi_{3Q_i})(x)| \leq \sum_{k=1}^{3^{n+1}} |T_\varepsilon(g_{i_k})(x)| \leq 3^{n+1} 2\varepsilon_3 \|f\|_{L^\infty(\mu)}.$$

The third term is estimated in the same way. We deal now with the first one:

$$\begin{aligned} & |T_\varepsilon(g\chi_{\mathbb{R}^{n+1} \setminus 3Q_i})(x) - T_\varepsilon(g\chi_{\mathbb{R}^{n+1} \setminus 3Q_j})(y)| \\ & \leq |T_\varepsilon(f\chi_{\mathbb{R}^{n+1} \setminus 3Q_i})(x) - T_\varepsilon(f\chi_{\mathbb{R}^{n+1} \setminus 3Q_j})(y)| \\ & \quad + |T_\varepsilon(g\chi_{\mathbb{R}^{n+1} \setminus 3Q_i})(x) - T_\varepsilon(f\chi_{\mathbb{R}^{n+1} \setminus 3Q_i})(x)| \\ & \quad + |T_\varepsilon(g\chi_{\mathbb{R}^{n+1} \setminus 3Q_j})(y) - T_\varepsilon(f\chi_{\mathbb{R}^{n+1} \setminus 3Q_j})(y)| \\ & \leq I + II + III. \end{aligned}$$

First we consider I . We set $\varepsilon' = \max(\varepsilon, l)$. We have

$$|T_\varepsilon(f\chi_{\mathbb{R}^{n+1} \setminus 3Q_i})(x) - T_{\varepsilon'}(f)(x)| \leq \|f\|_{L^\infty(\mu)} \int |k_{\varepsilon'}(x, z) - k_\varepsilon(x, z)\chi_{\mathbb{R}^{n+1} \setminus 3Q_i}(z)| d\mu(z).$$

If $\varepsilon \geq l$, then $\varepsilon' = \varepsilon$, and we may assume that $|x - z| \geq \varepsilon \geq l$ in the preceding integral. In this case we have

$$\int_{3Q_i} |k_\varepsilon(x, z)| d\mu(z) \leq \frac{\mu(3Q_i)}{l^n} \leq C\varepsilon_1.$$

If $\varepsilon \leq l$, $k_{\varepsilon'}(x, z) = k_\varepsilon(x, z)$ if $z \notin 3Q_i$, $|x - z| \geq l$ and we get

$$\int_{3Q_i} |k_{\varepsilon'}(x, z)| d\mu(z) \leq \frac{\mu(3Q_i)}{l^n} \leq C\varepsilon_1.$$

In any case we deduce

$$(13) \quad |T_\varepsilon(f\chi_{\mathbb{R}^{n+1} \setminus 3Q_i})(x) - T_{\varepsilon'}(f)(x)| \leq C\varepsilon_1 \|f\|_{L^\infty(\mu)},$$

and so, proceeding in the same way with y , and taking ε_1 small enough,

$$|T_\varepsilon(f\chi_{\mathbb{R}^{n+1} \setminus 3Q_i})(x) - T_\varepsilon(f\chi_{\mathbb{R}^{n+1} \setminus 3Q_j})(y)| \leq \sup_{\varepsilon' > 0} |T_{\varepsilon'} f(x) - T_{\varepsilon'} f(y)| + \tau/4.$$

Both II and III are estimated in the same way:

$$\begin{aligned} II &= \left| \int_{\mathbb{R}^{n+1} \setminus 3Q_i} k_\varepsilon(x, z)(g(z) - f(z)) d\mu(z) \right| \\ &\leq \sum_{k=1}^{\infty} \left| \int_{3^{k+1}Q_i \setminus 3^kQ_i} k_\varepsilon(x, z)(g(z) - f(z)) d\mu(z) \right| =: \sum_{k=1}^{\infty} A_k. \end{aligned}$$

Calling z_h the center of Q_h , since $\int_{Q_h} g(z) d\mu(z) = \int_{Q_h} f(z) d\mu(z)$, then $k(x, z_h) \int_{Q_h} (g - f)(z) d\mu(z) = 0$, and so:

$$\begin{aligned} A_k &\leq \sum_{h: Q_h \subset 3^{k+1}Q_i \setminus 3^kQ_i} \left| \int_{Q_h} k_\varepsilon(x, z)(g(z) - f(z)) d\mu(z) \right| \\ &\leq \sum_{h: Q_h \subset 3^{k+1}Q_i \setminus 3^kQ_i} \int_{Q_h} |k_\varepsilon(x, z) - k(x, z_h)| |g(z) - f(z)| d\mu(z) \\ &\leq C \sum_{h: Q_h \subset 3^{k+1}Q_i \setminus 3^kQ_i} \frac{l}{(3^k l)^{n+1}} \mu(Q_h) \|f\|_{L^\infty(\mu)} \leq C \frac{\mu(3^{k+1}Q_i)}{3^k (3^{k+1}l)^n} \|f\|_{L^\infty(\mu)}. \end{aligned}$$

Taking $N \approx 3^{M'}$ with M' big enough, we obtain

$$II \leq C \left(\sum_{k=1}^{M'} 3^{-k} \varepsilon_1 + \sum_{k=M'+1}^{\infty} 3^{-k} \right) \|f\|_{L^\infty(\mu)} \leq C(\varepsilon_1 + 1/N) \|f\|_{L^\infty(\mu)}.$$

So we deduce that $II \leq \tau/4$ if we choose ε_1 small enough and N big enough.

Finally, we consider the last inequality: $\|T_*g\|_{L^\infty(\mathbb{R}^{n+1})} \leq \|T_*f\|_{L^\infty(\mathbb{R}^{n+1})} + \tau$. Given $\varepsilon > 0$, we set

$$\begin{aligned} |T_\varepsilon g(x)| &\leq |T_\varepsilon(g\chi_{3Q_i})(x)| + |T_\varepsilon(g\chi_{\mathbb{R}^{n+1} \setminus 3Q_i})(x) - T_\varepsilon(f\chi_{\mathbb{R}^{n+1} \setminus 3Q_i})(x)| \\ &\quad + |T_\varepsilon(f\chi_{\mathbb{R}^{n+1} \setminus 3Q_i})(x) - T_{\varepsilon'}(f)(x)| + T_*f(x). \end{aligned}$$

From (12), the estimate for II above, and (13), we get $|T_\varepsilon g(x)| \leq \|T_*f\|_{L^\infty(\mathbb{R}^{n+1})} + \tau$. $\square$

3.3. The second inequality in Theorem 2.

Proposition 2. *For any compact set $E \subset \mathbb{R}^{n+1}$,*

$$\kappa_0^c(E) \lesssim \kappa_+^c(E).$$

Proof. Let μ be a positive Radon measure satisfying

$$\mu(E) \geq \kappa_0^c(E)/2, \quad \mu \in \Sigma_+^n(E), \quad \theta_\mu(x) = 0 \quad \forall x \in \mathbb{R}^{n+1}, \quad \|R_\mu\|_{L^2(\mu), L^2(\mu)} \leq 1.$$

Then, all the operators R_μ^j are bounded in $L^2(\mu)$ with a constant independent of j .

By Lemma 4 there are subsets $F_j \subset E$ such that $\lim_{r \rightarrow 0} \frac{\mu(B(x,r) \cap F_j)}{r^n} = 0$ uniformly on $x \in \mathbb{R}^{n+1}$ and

$$(14) \quad \lim_{r \rightarrow 0} \|R_{\mu, \varepsilon}^j\|_{L^2(\mu|_{B(x,r) \cap F_j}), L^2(\mu|_{B(x,r) \cap F_j})} = 0,$$

uniformly on $x \in \mathbb{R}^{n+1}$ for all j , so that the set $F := \bigcap F_j$ satisfies $\mu(E \setminus F) < \varepsilon$.

Now we wish to apply Lemma 1. For technical reasons we need to work with the regularized Riesz transform $\overline{R}_\varepsilon$. By equation (2), $\overline{R}_{\mu, \varepsilon}$ is also bounded in $L^2(\mu)$, and this implies (see, for example Theorem 1.1 in [T4]) that $\overline{R}_\varepsilon$ is bounded from $\mathcal{M}(\mathbb{R}^{n+1})$ into $L^{1,\infty}(\mu)$. So we have that for any finite measure ν ,

$$\mu\{x : |\overline{R}_\varepsilon \nu(x)| > \lambda\} \leq A \frac{\|\nu\|}{\lambda}$$

with A independent of ε and tending to zero if $\|\overline{R}_\mu\|_{L^2(\mu) \rightarrow L^2(\mu)}$ tends to zero and $\theta_\mu(x) = 0$ for all x . We now apply Lemma 1, so for each $\varepsilon > 0$ we can find $0 \leq f_1^\varepsilon \leq 1$, $\|\overline{R}_{\mu, \varepsilon} f_1^\varepsilon\|_{L^\infty(\mathbb{R}^{n+1})} \leq 1$, $\int f_1^\varepsilon d\mu \geq C^{-1} \mu(F)$. Applying again equation (2), since $M_\mu f_1^\varepsilon(x) \leq 1$, we get $\|R_{\mu, \varepsilon} f_1^\varepsilon\|_{L^\infty(\mathbb{R}^{n+1})} \leq C$. Let f_1 be the limit in the weak* topology of $L^\infty(\mu)$ of a suitable subsequence of $\{f_1^\varepsilon\}_\varepsilon$. It is easy to check that for all $\varepsilon > 0$, we have $0 \leq f_1 \leq 1$, $\|R_{\mu, \varepsilon} f_1\|_{L^\infty(\mathbb{R}^{n+1})} \leq C$ and $\int f_1 d\mu \geq C^{-1} \mu(F)$.

Define $\tau_m = 2^{-m}$. Then applying Lemma 8, we can construct a sequence $\{f_m\}_m$ in a way such that $\int f_{m+1} d\mu = \int f_m d\mu$, $0 \leq f_{m+1} \leq \|f_m\|_{L^\infty(\mu)} + \tau_m$, $\|T_* f_{m+1}\|_{L^\infty(\mathbb{R}^{n+1})} \leq \|T_* f_m\|_{L^\infty(\mathbb{R}^{n+1})} + \tau_m$, and also

$$|R_{\mu, \varepsilon} f_m(x) - R_{\mu, \varepsilon} f_m(y)| \leq \tau_m, \quad \text{if } |x - y| \leq \delta_m, \quad \varepsilon > 0,$$

and

$$|R_{\mu, \varepsilon} f_m(x) - R_{\mu, \varepsilon} f_m(y)| \leq \sup_{\varepsilon' > 0} |R_{\mu, \varepsilon'} f_{m-1}(x) - R_{\mu, \varepsilon'} f_{m-1}(y)| + \tau_m,$$

$$\forall x, y \in \mathbb{R}^{n+1}, \quad \varepsilon > 0.$$

Moreover, we assume $\delta_m \leq \delta_{m-1}$. Since for all m , $\|f_m\|_\infty \leq 2$, we know that there exists a subsequence, which for simplicity we also call $\{f_m\}_m$, converging to f in the weak* topology of $L^\infty(\mu)$. Then we have

$$R_{\mu,\varepsilon} f_m(x) \rightarrow R_{\mu,\varepsilon} f(x)$$

when $m \rightarrow \infty$. The function f also satisfies $\int f = \int f_1 \geq C^{-1} \mu(E) \geq C^{-1} \kappa_0^c(E)/2$, and $\|R_{\mu,\varepsilon} f\|_{L^\infty(\mathbb{R}^{n+1})} \leq C$, since

$$\|R_{\mu,\varepsilon} f_m\|_{L^\infty(\mathbb{R}^{n+1})} \leq \|R_{\mu,*} f_1\| + \sum_{l=1}^{m-1} 2^{-l} \leq C.$$

On the other hand, if $|x - y| \leq \delta_m$, we have that $|R_{\mu,\varepsilon} f_m(x) - R_{\mu,\varepsilon} f_m(y)| \leq 2^{-m}$ for all ε , and for $k \geq m$,

$$|R_{\mu,\varepsilon} f_k(x) - R_{\mu,\varepsilon} f_k(y)| \leq \sup_{\varepsilon' > 0} |R_{\mu,\varepsilon'} f_m(x) - R_{\mu,\varepsilon'} f_m(y)| + \sum_{l=m}^{k-1} 2^{-l} \leq 2^{-m+2}.$$

So,

$$|R_{\mu,\varepsilon} f(x) - R_{\mu,\varepsilon} f(y)| \leq 2^{-m+2}, \text{ if } |x - y| \leq \delta_m.$$

By now we have found a measure $d\nu = f d\mu$ with $\nu(E) \geq \mu(E)/C$, $\|R_\varepsilon \nu\|_{L^\infty(\mathbb{R}^{n+1})} \leq C$ for all ε , and the family of functions $R_\varepsilon \nu(x)$ is equicontinuous in x . Since $R\nu(x)$ is well defined almost everywhere with respect to the Lebesgue measure, it coincides with $\lim_{\varepsilon \rightarrow 0} R_\varepsilon \nu(x)$ for almost every x , so $\|R\nu\|_{L^\infty(\mathbb{R}^{n+1})} \leq C$. We also have that $R\nu(x)$ is continuous in $x \notin \text{supp}(\nu)$. It remains to see that $R\nu(x)$ coincides almost everywhere with a continuous function on E .

Consider the family of functions $\{R_{\mu,\varepsilon} f\}_\varepsilon$ on $\overline{B}(0, N)$, where N satisfies $E \subset B(0, N)$. This family is uniformly bounded and equicontinuous on $\overline{B}(0, N)$. Applying the Ascoli-Arzelà theorem, we can find a subsequence $\{\varepsilon_k\}_k$, with $\varepsilon_k \rightarrow 0$ in a way such that $\{R_{\mu,\varepsilon_k} f\}_k$ converges uniformly to a function g continuous on $\overline{B}(0, N)$. Since $R\nu(x)$ coincides almost everywhere with respect to the Lebesgue measure on $x \in B(0, N)$ with $\lim_{\varepsilon \rightarrow 0} R_\varepsilon \nu(x) = g(x)$, we obtain the result. $\square$

3.4. Some consequences.

Corollary 2. *Let $E_i, i \geq 1$, be Borel subsets of $\mathbb{R}^{n+1}$ such that $\bigcup_i E_i$ is a bounded set. Then*

$$\kappa_+^c \left(\bigcup_i E_i \right) \lesssim \sum_i \kappa_+^c(E_i).$$

Proof. Let μ be a measure supported on $\bigcup_i E_i$ with $\theta_\mu(x) = 0$ for all x , $\|R_\mu\|_{L^2(\mu) \rightarrow L^2(\mu)} \leq 1$ and $\mu(\bigcup_i E_i) \geq \kappa_+^0(\bigcup_i E_i)/2$. Then the measures $\mu_i := \mu|_{E_i}$ are supported on E_i , $\theta_{\mu_i}(x) = 0$ and $\|R_{\mu_i}\|_{L^2(\mu_i) \rightarrow L^2(\mu_i)} \leq 1$, and so:

$$\kappa_+^0 \left(\bigcup_i E_i \right) \leq 2\mu \left(\bigcup_i E_i \right) \leq 2 \sum_i \mu(E_i) \leq 2 \sum_i \kappa_0^c(E_i).$$

Applying Theorem 2 we obtain the result. $\square$

Corollary 3. *Let $E \subset \mathbb{R}^{n+1}$. Suppose that for any compact subset $F \subset E$ with $\mathcal{H}^n(F) < \infty$ we have that $\kappa(F) = 0$. Then*

$$\kappa(E) \approx \kappa^c(E).$$

Proof. We only have to prove that $\kappa(E) \lesssim \kappa^c(E)$. By Volberg's results in [V], there exists a positive measure $\mu \in \Sigma^n(E)$ satisfying $\mu(E) \geq \kappa(E)/2$ and $\|R_\mu\|_{L^2(\mu), L^2(\mu)} \leq 1$. It is enough to see that $\theta_\mu(x) = 0$ μ -almost everywhere. For each $k \geq 1$, consider the sets

$$E_k = \{x \in E : \theta_\mu^*(x) > 1/k\}.$$

These sets satisfy $\mathcal{H}^n(E_k) < \infty$, so we have that $\kappa(E_k) = 0$. Using the fact that $R_{\mu|_{E_k}}$ is bounded on $L^2(\mu|_{E_k})$ with norm equal to or less than 1, we deduce that $\mu(E_k) = 0$ and the corollary follows. $\square$

The next results are about a generalization of the 1/4 planar Cantor set in $\mathbb{R}^2$ (see [M]). We construct this set as follows: consider a sequence $\lambda = (\lambda_k)_k$, $0 \leq \lambda_k \leq 1/2$, and in the first generation take the 2^{n+1} cubes contained in $E_0 = [0, 1]^{n+1}$ with sides parallel to the axes, with side length λ_1 , in a way such that each cube contains a vertex of E_0 . For each cube repeat the process using λ_2 . In the k -th generation we will have $2^{(n+1)k}$ cubes of side length $\sigma_k = \lambda_1 \lambda_2 \dots \lambda_k$. Denote by $E_N = E_N(\lambda_1, \lambda_2, \dots, \lambda_k)$ the union of these cubes and set $\mathcal{C}(\lambda) = \bigcap_k E_k$. Denote $p_N := \mathcal{H}_{|E_N}^{n+1} / \mathcal{H}^{n+1}(E_N)$ and let p be the weak limit of the sequence $\{p_N\}_N$. We need to recall two theorems before stating our result.

Theorem 4 ([MT]). *Let $E(\lambda)$ with $2^{-(n+1)/n} \leq \lambda_k \leq \lambda_0 \leq 1/2$ be a Cantor set as above. There exists a constant C , only depending on λ_0 and n , such that for all $N = 1, 2, \dots$ we have*

$$C^{-1} \left(\sum_{k=1}^N \frac{1}{(2^{k(n+1)} \sigma_k^n)^2} \right)^{1/2} \leq \|R_{p_N}\|_{L^2(p_N) \rightarrow L^2(p_N)} \leq C \left(\sum_{k=1}^N \frac{1}{(2^{k(n+1)} \sigma_k^n)^2} \right)^{1/2}.$$

Theorem 5 ([MT]). *If for all k , $2^{-(n+1)/n} \leq \lambda_k \leq \lambda_0 \leq 1/2$, then for all $N = 1, 2, \dots$,*

$$\begin{aligned} C^{-1} \left(\sum_{k=1}^N \frac{1}{(2^{k(n+1)} \sigma_k^n)^2} \right)^{-1/2} &\leq \kappa_+(E_N(\lambda)) \\ &\leq \kappa(E_N(\lambda)) \leq C \left(\sum_{k=1}^N \frac{1}{(2^{k(n+1)} \sigma_k^n)^2} \right)^{-1/2}. \end{aligned}$$

From Theorems 4 and 5 and Theorem 2 we deduce:

Theorem 6. *Let λ be such that $2^{-(n+1)/n} \leq \lambda_k \leq \lambda_0 \leq 1/2$. Then:*

$$\kappa(E(\lambda)) \approx \kappa^c(E(\lambda)) \approx \left(\sum_{k=1}^{\infty} \frac{1}{(2^{k(n+1)} \sigma_k^n)^2} \right)^{-1/2}.$$

Proof. In [P] it is shown that if $\{A_k\}_k$ is a decreasing sequence of sets with $A = \lim_k A_k$, then $\kappa(A) = \lim_k \kappa(A_k)$ and so from Theorems 4 and 5 it follows that, if

$$\tau := \left(\sum_{k=1}^{\infty} \frac{1}{(2^{k(n+1)} \sigma_k^n)^2} \right)^{1/2},$$

then

$C^{-1}\tau \leq \|R_p\|_{L^2(p) \rightarrow L^2(p)} \leq C\tau$ and $C^{-1}\tau^{-1} \leq \kappa_+(E(\lambda)) \leq \kappa(E(\lambda)) \leq C\tau^{-1}$,
with $\tau^{-1} = 0$ if $\tau = \infty$.

The inequality $\kappa^c(E(\lambda)) \leq \kappa(E(\lambda))$ follows by definition. To prove the converse estimate we may assume that $\tau < \infty$. Otherwise, $\kappa^c(E(\lambda)) \leq \kappa(E(\lambda)) = 0$ implies $\kappa^c(E(\lambda)) = 0$. Consider $\mu := \tau^{-1}p$. Then it is easy to check that $\theta_\mu(x) = 0$ for all $x \in \mathbb{R}^{n+1}$ and by Theorem 4, $\|R_\mu\|_{L^2(\mu) \rightarrow L^2(\mu)} \leq C$. We also have that $\mu(E(\lambda)) = 1/\tau$ and so, applying Theorem 5,

$$\kappa(E(\lambda)) \approx \mu(E(\lambda)),$$

and by Theorem 2,

$$\mu(E(\lambda)) \lesssim \kappa_0^c(E(\lambda)) \lesssim \kappa_+^c(E(\lambda)) \leq \kappa^c(E(\lambda)). \quad \square$$

4. THE CAPACITIES κ^h

In order to prove Theorem 1 we need to introduce some new capacities. Let $h : (0, +\infty) \rightarrow (0, +\infty)$ be a function satisfying the following properties (we will call them h -properties): for all $r > 0$, $h(r)/r^n$ is nondecreasing, $h(r) \leq r^n$, $h(2r) \leq 2^{n+1}h(r)$, $\lim_{r \rightarrow 0+} h(r)/r^n = 0$.

Definition 3. Given a compact set $E \subset \mathbb{R}^{n+1}$, we set

$$\kappa^h(E) = \sup_{f \in A^h(E)} |\langle \Delta f, 1 \rangle|,$$

where $A^h(E)$ is the class of Lipschitz functions satisfying $\text{supp}(\Delta f) \subset E$, $\|\nabla f\|_{L^\infty} \leq 1$, and for any ball $B(x_0, r)$ and any $\varphi \in \mathcal{C}_c^\infty(B(x_0, r))$,

$$\left| \int f(x) \Delta \varphi(x) dx \right| \leq h(r)r^2 \|\Delta \varphi\|_\infty.$$

If, in addition, $f = \phi * \mu$ where μ is a positive measure supported on E , we will say that $f \in A_+^h(E)$. We also define

$$\kappa_+^h(E) = \sup_{f \in A_+^h(E)} |\langle \Delta f, 1 \rangle|.$$

Given a positive Radon measure μ and a function h satisfying the h -properties, we say that $\mu \in \Sigma^h$ if $M^h \mu(x) := \sup_{r>0} \frac{\mu(B(x,r))}{h(r)} \leq 1$ for all x (notice that $\Sigma^h \subset \Sigma^n$). We write $\mu \in \Sigma^h(E)$ if moreover $\text{supp}(\mu) \subset E$.

The following result relates the new capacities κ^h with κ^c .

Lemma 9. For any compact set $E \subset \mathbb{R}^{n+1}$,

$$\kappa^c(E) \lesssim \sup_h \kappa^h(E),$$

where the supremum is taken over all functions h satisfying the h -properties.

Proof. We take $f \in \mathcal{C}^1$, $\|\nabla f\|_\infty \leq 1$, $\text{supp}(\Delta f) \subset E$, $x_0 \in \mathbb{R}^{n+1}$, $r > 0$ and $\varphi \in \mathcal{C}_c^\infty(B(x_0, r))$. For $s > 0$, consider

$$w_{\nabla f}(s) := \sup_{|x-y| \leq s} |\nabla f(x) - \nabla f(y)|,$$

the modulus of continuity of ∇f . The function $h(s) = \frac{s^n w_{\nabla f}(s)}{2}$ satisfies the h -properties. Then, for all $\varphi \in \mathcal{C}_c^\infty(B(x_0, r))$, applying Green's formula we see that

$\int_{B(x_0, r)} \Delta \varphi(x) dx = \int_{B(x_0, r)} [\nabla f(x_0) \cdot (x - x_0)] \Delta \varphi(x) dx = 0$, so we get for a suitable $\xi_x \in [x, x_0]$,

$$\begin{aligned} \left| \int_{B(x_0, r)} f(x) \Delta \varphi(x) dx \right| &= \left| \int_{B(x_0, r)} (f(x) - f(x_0) - [\nabla f(x_0) \cdot (x - x_0)]) \Delta \varphi(x) dx \right| \\ &= \left| \int_{B(x_0, r)} [(\nabla f(\xi_x) - \nabla f(x_0)) \cdot (x - x_0)] \Delta \varphi(x) dx \right| \\ &\lesssim w_{\nabla f}(r) \|\Delta \varphi\|_{\infty} r^{n+2} \lesssim h(r) r^2 \|\Delta \varphi\|_{\infty}. \end{aligned}$$

Therefore $f \in A^h(E)$ and the lemma follows. $\square$

Definition 4. For any compact set $E \subset \mathbb{R}^{n+1}$, we set

$$\kappa_0^h(E) := \sup\{\mu(E) : \mu \text{ positive measure, } \|R_{\mu}\|_{L^2(\mu), L^2(\mu)} \leq 1, \mu \in \Sigma^h(E)\}.$$

Lemma 10. For any compact set $E \subset \mathbb{R}^{n+1}$,

$$\kappa_0^h(E) \approx \kappa_+^h(E).$$

Proof. First we will show that

$$\kappa_0^h(E) \lesssim \kappa_+^h(E).$$

Let μ be a measure supported on E such that $\|R_{\mu}\|_{L^2(\mu), L^2(\mu)} \leq 1$, and $M\mu^h(x) \leq 1$, for all $x \in \mathbb{R}^{n+1}$. Since $\|R_{\mu}\|_{L^2(\mu), L^2(\mu)} \leq 1$, applying Lemma 1 we find a function f supported on E with $0 \leq f \leq 1$ satisfying $\mu(E)/2 \leq \int f d\mu$ and $\|R_{\mu}f\|_{\infty} \leq C$. Consider $g = \phi * (fd\mu)$. Then $\|\nabla g\|_{\infty} = \|R_{\mu}f\|_{\infty} \leq C$ and $\Delta g = fd\mu$. For each $\varphi \in \mathcal{C}_c^{\infty}(B(x_0, r))$, in a distributional sense,

$$\left| \int g(x) \Delta \varphi(x) dx \right| = \left| \int \Delta g(x) \varphi(x) dx \right| \leq \mu(B(x_0, r)) \|\varphi\|_{\infty} \leq Ch(r) r^2 \|\Delta \varphi\|_{\infty},$$

where we used the estimate $\|\varphi\|_{\infty} \leq Cr^2 \|\Delta \varphi\|_{\infty}$. Thus $Cg \in A^h(E)$ and so $\kappa_+^h(E) \geq C\mu(E)$.

Let us now consider the inequality $\kappa_+^h(E) \lesssim \kappa_0^h(E)$. Let $f = \phi * \mu$, where μ is a positive measure supported on E such that $\|\nabla f\|_{\infty} = \|R_{\mu}\|_{L^{\infty}} \leq 1$, and for any ball $B(x_0, r)$ and all $\varphi \in \mathcal{C}_c^{\infty}(B(x_0, r))$,

$$\left| \int f(x) \Delta \varphi(x) dx \right| \leq h(r) r^2 \|\Delta \varphi\|_{\infty}.$$

Arguing as in Proposition 1 we deduce that for all $\varepsilon > 0$, $\|R_{\mu, \varepsilon} 1\|_{L^{\infty}} \leq C$ and μ has polynomial growth. Applying the $T(1)$ theorem for nondoubling measures, we infer that $\|R_{\mu}\|_{L^2(\mu), L^2(\mu)} \leq C$. Take $\varphi \in \mathcal{C}_c^{\infty}(B(x_0, 2r))$ such that $\chi_{B(x_0, r)} \leq \varphi \leq 1$, $\|\Delta \varphi\|_{\infty} \leq C/r^2$. Then,

$$\mu(B(x_0, r)) \leq \int \varphi d\mu = |\langle \Delta f, \varphi \rangle| = \left| \int f \Delta \varphi dx \right| \leq h(2r) (2r)^2 \|\Delta \varphi\|_{\infty} \leq Ch(r).$$

$\square$

Lemma 11. For any compact set $E \subset \mathbb{R}^{n+1}$,

$$\kappa_+^c(E) \approx \sup_h \kappa_+^h(E).$$

Proof. • $\kappa_+^c(E) \leq C \sup_h \kappa_+^h$ is proved as in Lemma 9.
 • Given h we want to see that $\kappa_+^h(E) \leq C \kappa_+^c(E)$. From the fact that $M^h \mu(x) \leq 1$ we deduce $M\mu(x) \leq 1$ and $\theta_\mu(x) = 0$ for all x . Therefore

$$\kappa_+^h(E) \approx \kappa_0^h(E) \leq \kappa_0^c(E) \approx \kappa_+^c(E). \quad \square$$

5. PROPERTIES AND FIRST ESTIMATES FOR THE CAPACITIES κ^h

First of all we want to notice that since $\frac{h(r)}{r^n}$ is nondecreasing, for all $s \geq r$, $h(r) \leq h(s)$. In this section, we will also use the fact that given $\lambda > 1$, for all $t > 0$,

$$(15) \quad h(\lambda t) \leq (2\lambda)^{n+1} h(t),$$

which has a direct proof.

Given $\varphi \in C^\infty$, we consider the following localization operator:

$$V_\varphi f(x) = \phi * (\varphi \Delta f)(x).$$

Proposition 3. *There exists $C > 0$ such that if $E \subset \mathbb{R}^{n+1}$ is compact, $f \in A^h(E)$ and $\varphi \in C^\infty(B(x_0, r))$, satisfying $\|\Delta \varphi\|_\infty \leq C/r^2$, then*

$$C^{-1} V_\varphi f \in A^h(E \cap B(x_0, r)).$$

Proof. This is a consequence of the following:

- $\text{supp}(\Delta V_\varphi f) \subset E \cap B(x_0, r)$, since $\Delta V_\varphi f = \varphi \Delta f$.
- $\|\nabla V_\varphi f\|_\infty \leq A w_{\nabla f}(r) \|\Delta \varphi\|_\infty r^2 \leq C$. A proof can be found in [P].
- Let $\psi \in C^\infty(B(y_0, s))$ and $d = \min(r, s)$. Then, since $\text{supp}(\varphi \psi) \subset B(x_0, r) \cap B(y_0, s)$, in a distributional sense,

$$\begin{aligned} \left| \int V_\varphi f \Delta \psi \, dx \right| &= \left| \int \Delta f \varphi \psi \, dx \right| = \left| \int f \Delta(\varphi \psi) \, dx \right| \leq h(d) d^2 \|\Delta(\varphi \psi)\|_\infty \\ &\leq h(d) d^2 (\|\Delta \psi\|_\infty \|\varphi\|_\infty + 2 \|\nabla \psi \nabla \varphi\|_\infty + \|\psi\|_\infty \|\Delta \varphi\|_\infty) \\ &\lesssim h(d) \|\Delta \psi\|_\infty (d^2 + d^2 \frac{s}{r} + d^2 \frac{s^2}{r^2}) \lesssim h(s) s^2 \|\Delta \psi\|_\infty. \quad \square \end{aligned}$$

Of course, in the preceding proposition, balls can be replaced by cubes.

Lemma 12. *Let $E \subset \mathbb{R}^{n+1}$, $x_0 \in \mathbb{R}^{n+1}$, $r > 0$, $B = B(x_0, r)$.*

- a) $\kappa^h(E) \leq C_2 h(\text{diam}(E))$.
- b) $\kappa^h(B(x_0, r)) \approx \kappa_+^h(B(x_0, r)) \approx h(r)$.
- c) *Let R be an $(n+1)$ -dimensional rectangle $((n+1)$ -rectangle) with side lengths $l_1 \leq \dots \leq l_{n+1}$. Then*

$$\kappa^h(R) \approx \kappa_+^h(R) \approx \frac{l_2 \dots l_{n+1}}{l_1^n} h(l_1) = \frac{\mathcal{H}^{n+1}(R)}{l_1^{n+1}} h(l_1).$$

- d) $\kappa^h(E \cup B) \lesssim \kappa^h(E) + \kappa^h(B)$.
- e) *Suppose $R_1, \dots, R_m$ are closed $(n+1)$ -rectangles and $m \geq 0$. Then $\kappa^h(R_1 \cup \dots \cup R_m) \leq C_m (\kappa^h(R_1) + \dots + \kappa^h(R_m))$.*
- f) *Let $\{E_k\}_k$ be a sequence of compact sets such that $E_{k+1} \subset E_k$ for all k . If $E = \bigcap_k E_k$, then $\kappa_+^h(E) = \lim_k \kappa_+^h(E_k)$.*

Proof. a) Suppose that $E \subset B$. Let us take $\varphi \in \mathcal{C}^\infty(B(x_0, 2r))$, $\chi_{B(x_0, r)} \leq \varphi \leq \chi_{B(x_0, 2r)}$, $\|\Delta\varphi\|_\infty \leq C/r^2$. If $f \in A^h(E)$, then

$$|\langle \Delta f, 1 \rangle| = \left| \int_{B(x_0, 2r)} \varphi(x) \Delta f(x) dx \right| \leq h(2r)(2r)^2 \|\Delta\varphi\|_\infty \leq Ch(r).$$

b) Since $\kappa_+^h(B) \leq \kappa^h(B) \lesssim h(r)$, we only need to prove $h(r) \leq \kappa_+^h(B)$. Take $d\mu = \frac{h(r)}{r^{n+1}} d\mathcal{H}_{|B}^{n+1}$. We want to see that $M^h\mu(x) \leq C$ for all x and $\|R_\mu\|_{L^2(\mu), L^2(\mu)} \leq C$. If $s \leq r$, since $h(r) = h(\frac{r}{s}s) \lesssim (\frac{r}{s})^{n+1}h(s)$, we get

$$\mu(B(x, s)) \lesssim h(r)(s/r)^{n+1} \lesssim h(s).$$

Otherwise, if $s > r$, then

$$\mu(B(x, s)) \leq \mu(B(x_0, r)) \leq Ch(r) \leq Ch(s).$$

For all $x \in B(x_0, 2r)$ and $j = 1, \dots, n+1$,

$$|R\mu(x)| \leq C \frac{h(r)}{r^{n+1}} \int_B \frac{1}{|x-y|^n} dy \leq C \frac{h(r)}{r^n} \leq C.$$

Applying the $T(1)$ theorem, for example, we deduce that $\|R_\mu\|_{L^2(\mu), L^2(\mu)} \leq C$.

c) Let us see that $\kappa^h(R) \lesssim \frac{\mathcal{H}^{n+1}(R)}{l_1^{n+1}} h(l_1)$.

Case 1. Suppose that $l_1 = \dots = l_{n+1} = l$. Then we can find $x_0 \in \mathbb{R}^{n+1}$ such that $B(x_0, l/2) \subset R \subset B(x_0, \frac{\sqrt{n}}{2}l)$. Then

$$h(l) \approx h(l/2) \approx \kappa_+^h(B(x_0, l/2)) \leq \kappa_+^h(R) \leq \kappa^h(R) \leq \kappa^h(B(x_0, \frac{\sqrt{n}}{2}l)) \approx h(l).$$

Case 2. Suppose that for all $i = 2, \dots, n+1$, $l_i = m_i l_1$ with $m_i \in \mathbb{N}$. Then we can decompose R in $N = m_2 \dots m_{n+1}$ cubes with disjoint interiors $R = \bigcup_{i=1}^N Q_i$. Let $\psi_i \in \mathcal{C}^\infty(2Q_i)$ be nonnegative such that $\sum_i \psi_i = 1$ in R and $\|\Delta\psi_i\| \leq C/l_1^2$. If $f \in A^h(R)$, then $C^{-1}V_{\psi_i}f \in A^h(2Q_i)$. Therefore,

$$|\langle \Delta V_{\psi_i} f, 1 \rangle| = |\langle \psi_i \Delta f, 1 \rangle| \lesssim \kappa^h(2Q_i) \approx h(l_1)$$

and

$$|\langle \Delta f, 1 \rangle| \leq \sum_i |\langle \psi_i \Delta f, 1 \rangle| \leq Nh(l_1) = \frac{l_2 \dots l_{n+1}}{l_1^n} h(l_1).$$

Case 3. Let m_i be integers such that $l_i/l_1 \leq m_i \leq 2l_i/l_1$ for $i = 2, \dots, n+1$. Consider an $(n+1)$ -rectangle $R_0 \supset R$ with sides $l_1, m_2 l_1, \dots, m_{n+1} l_1$. Then

$$\kappa^h(R) \leq \kappa^h(R_0) \leq m_2 \dots m_{n+1} h(l_1) \leq 2^n \frac{l_2}{l_1} \dots \frac{l_{n+1}}{l_1} h(l_1).$$

It remains to see that $\frac{\mathcal{H}^{n+1}(R)}{l_1^{n+1}} h(l_1) \lesssim \kappa^h(R)$. Suppose R is centered at the origin. Consider $\Pi = \{x_1 = 0\} \cap \frac{1}{2}R$ and the measure $d\mu = d\mathcal{H}_{|\Pi}^n$. Then, it is easy to check that R_μ is bounded in $L^2(\mu)$ and applying Lemma 1 we can find a function b such that $0 \leq b \leq 1$, $\int b d\mu \geq \mu(R)/2$ and $\|R(bd\mu)\|_{L^\infty} \leq 1$. Take $\varphi_0 \in \mathcal{C}^\infty(B(0, 1))$ with $0 \leq \varphi_0 \leq 2$ and $\int \varphi_0 = 1$. Denote $\varphi(x) = \frac{10^{n+1}}{l_1^{n+1}} \varphi_0(\frac{10x}{l_1})$. Then $\int \varphi dx = 1$, $\text{supp}(\varphi) \subset B(0, l_1/10)$ and

$\|\varphi\|_\infty \leq C/l_1^{n+1}$. Define $g(x) = \frac{h(l_1)}{l_1^n}(\varphi * bd\mu)(x)$ and $G = \phi * g$; observe that $\text{supp } g \subset R$. Since we can express for a measure ν , $R\nu(x) = \frac{z}{|z|^{n+1}} * \nu(x)$ (in a vectorial sense), we have that

$$\|\nabla G\|_{L^\infty(\mathbb{R}^{n+1})} = C\|R_{\mathcal{H}^{n+1}}(g)\|_{L^\infty(\mathbb{R}^{n+1})} = \frac{h(l_1)}{l_1^n}\|\varphi * R(bd\mu)\|_{L^\infty(\mathbb{R}^{n+1})} \leq C,$$

and we also have

$$\langle \Delta G, 1 \rangle = \int g(x)dx \approx \frac{h(l_1)}{l_1^n} l_2 \dots l_{n+1}.$$

It remains to see that for any $x_0 \in \mathbb{R}^{n+1}$, $r > 0$ and $\psi \in \mathcal{C}^\infty(B(x_0, r))$,

$$\int G(x)\Delta\psi(x)dx \leq Ch(r)r^2\|\Delta\psi\|_\infty,$$

and we will have finished.

(i) Suppose that $l_1 \leq r$. Then

$$\begin{aligned} \left| \int G(x)\Delta\psi(x)dx \right| &= \left| \int \Delta G(x)\psi(x)dx \right| \leq \|\psi\|_\infty \frac{h(l_1)}{l_1^n} \int_{B(x_0, r)} (\varphi * bd\mu)(x)dx \\ &= \|\psi\|_\infty \frac{h(l_1)}{l_1^n} \int_{|x-x_0| < r} \int_{|x-y| < l_1/10} \varphi(x-y)b(y)d\mu(y)dx \\ &\leq Cr^2\|\Delta\psi\|_\infty \frac{h(l_1)}{l_1^n} \int_{|x_0-y| < 2r} b(y)d\mu(y) \\ &\leq Cr^2\|\Delta\psi\|_\infty \frac{h(l_1)}{l_1^n} r^n \leq Ch(r)r^2\|\Delta\psi\|_\infty, \end{aligned}$$

where we used the inequality $\frac{h(l_1)}{l_1^n} \leq \frac{h(r)}{r^n}$.

(ii) Suppose that $r \leq l_1$. Then

$$\begin{aligned} \left| \int G(x)\Delta\psi(x)dx \right| &\leq \|\psi\|_\infty \frac{h(l_1)}{l_1^n} \int_{|x-x_0| < r} \int_{|x-y| < l_1/10} \varphi(x-y)b(y)d\mu(y)dx \\ &\leq r^2\|\Delta\psi\|_\infty \frac{h(l_1)}{l_1^n} \int_{|x_0-x| < r} \|\varphi\|_\infty \mu(B(x, l_1/10))dx \\ &\leq Cr^2\|\Delta\psi\|_\infty \frac{h(l_1)}{l_1^n} \frac{r^{n+1}}{l_1} \leq Ch(r)r^2\|\Delta\psi\|_\infty, \end{aligned}$$

using that μ has polynomial growth of order n , $\|\varphi\|_\infty \leq C/l_1^{n+1}$ and, since $r \leq l_1$, $h(l_1) \leq 2^{n+1}(\frac{l_1}{r})^{n+1}h(r)$.

d) Let $f \in A^h(E \cup B)$ and $\psi \in \mathcal{C}^\infty$ such that $\chi_B \leq \psi \leq \chi_{2B}$, $\|\Delta\psi\|_\infty \leq C/r^2$. We define $f_1 = V_\psi f$ and $f_2 = f - f_1$. So we have $\text{supp}(\Delta f_2) \subset E$, $\|\nabla f_2\|_\infty \leq \|\nabla f\|_\infty + \|\nabla f_1\|_\infty$. Since for all $\varphi \in \mathcal{C}^\infty$, $|\int f_2 \Delta\varphi| \leq |\int f \Delta\varphi| + |\int f_1 \Delta\varphi|$, we conclude that $C^{-1}f_2 \in A^h(E)$. Using the fact that $C^{-1}f_1 \in A^h(2B)$,

$$|\langle \Delta f, 1 \rangle| \leq |\langle \Delta f_1, 1 \rangle| + |\langle \Delta f_2, 1 \rangle| \lesssim \kappa^h(2B) + \kappa^h(E) \approx \kappa^h(B) + \kappa^h(E).$$

e) We will prove the case with only two $(n+1)$ -rectangles R, T . The case with $n+1$ $(n+1)$ -rectangles follows from analogous arguments. Let $l_1^R \leq \dots \leq l_{n+1}^R$ be the sides of R and $l_1^T \leq \dots \leq l_{n+1}^T$ the ones from T . First we suppose that $l_i^\alpha = m_i^\alpha l_1^\alpha$, $\alpha \in \{R, T\}$, $m_i^\alpha \in \mathbb{N}$ and $i = 2, \dots, n+1$. We

can decompose $R = \bigcup_{j=1}^M R_j$ and $T = \bigcup_{j=1}^N T_j$, where $M = m_2^R \dots m_{n+1}^R$ and $N = m_2^T \dots m_{n+1}^T$ and R_j are cubes with side lengths l_1^R and T_j cubes with sides l_1^T . We also suppose that $l_1^T \leq l_1^R$. Take functions $\phi_j \in C^\infty$ satisfying $\chi_{\frac{1}{2}R_j} \leq \phi_j \leq \chi_{2R_j}$, $\|\Delta\phi_j\|_\infty \leq C/(l_1^R)^2$ and $\sum_j \phi_j = 1$ in R . Take functions $\tilde{\psi}_j$ satisfying analogous properties for T , and define $\psi_j = \tilde{\psi}_j (1 - \sum_k \phi_k)$. Then $\sum_k \phi_k + \sum_j \psi_j = 1$ in $R \cup T$. If $f \in A^h(R \cup T)$, then

$$f = \sum_k^M V_{\phi_k} f + \sum_j^N V_{\psi_j} f.$$

We know that $C^{-1}V_{\phi_k} f \in A^h(2R_k)$ and as $l_1^T \leq l_1^R$ and each ϕ_k has an overlap with at most C_n different ϕ_i ,

$$\|\Delta\psi_j\|_\infty \lesssim \|\Delta\tilde{\psi}_j\|_\infty + \|\nabla\tilde{\psi}_j\|_\infty \left(\sum_k \|\nabla\phi_k\|_\infty \right) + \|\sum_k \Delta\phi_k\|_\infty \leq C/(l_1^T)^2,$$

so $C^{-1}V_{\psi_k} f \in A^h(2T_j)$. Then

$$\begin{aligned} |\langle \Delta f, 1 \rangle| &\leq \sum_k^M |\langle \phi_k \Delta f, 1 \rangle| + \sum_j^N |\langle \psi_j \Delta f, 1 \rangle| \\ &\lesssim Mh(l_1^R) + Nh(l_1^T) \approx \kappa^h(R) + \kappa^h(T). \end{aligned}$$

In the general case we can find $(n+1)$ -rectangles $R_0 \supset R$ and $T_0 \supset T$ with side lengths $m_i^\alpha l_1^\alpha$, $\alpha \in \{R_0, T_0\}$, $m_i^\alpha \in \mathbb{N}$, satisfying $m_i^\alpha l_1^\alpha \geq l_i^\alpha \geq m_i^\alpha l_1^\alpha / 2$, and so

$$\kappa^h(R \cup T) \leq \kappa^h(R_0 \cup T_0) \lesssim \kappa^h(R_0) + \kappa^h(T_0) \lesssim \kappa^h(R) + \kappa^h(T).$$

f) This proof follows from standard arguments. $\square$

Now we are going to prove some estimates for $f \in A^h(E)$. Given $\alpha = (\alpha_1, \dots, \alpha_{n+1}) \in \mathbb{Z}_+^{n+1}$ and $x \in \mathbb{R}^{n+1}$, we set $x^\alpha = x_1^{\alpha_1} \dots x_{n+1}^{\alpha_{n+1}}$, $\alpha! = \alpha_1! \dots \alpha_{n+1}!$, $|\alpha| = \sum_i \alpha_i$, and $\partial^\alpha = \partial^{\alpha_1} / \partial x_1^{\alpha_1} \dots \partial^{\alpha_{n+1}} / \partial x_{n+1}^{\alpha_{n+1}}$. We will also denote for $1 \leq k \leq n+1$, $\vec{k} = (0, \dots, 1, \dots, 0)$ with the 1 in the k -th coordinate.

If $\text{supp}(\Delta f) \subset B(a, \delta)$, then, for all x satisfying $|x - a| > 3\sqrt{n+1}\delta$, we can write

$$f(x) = \sum_{|\alpha| \geq 0} c_\alpha \partial^\alpha \phi(x - a),$$

where

$$c_\alpha = c_\alpha(f, a) = \frac{(-1)^{|\alpha|}}{\alpha!} \langle \Delta f(y), (y - a)^\alpha \rangle$$

in a distributional sense. This result can be found in [P]. The following results also follow by arguments similar to the ones in [P].

Lemma 13. Suppose that $E \subset B(a, \delta)$ and $f \in A^h(E)$. Then

$$|c_\alpha| \lesssim \frac{|\alpha|^2 + 1}{\alpha!} (2\delta)^\alpha \kappa^h(E).$$

Proof. Suppose that $a = 0$. By definition $|c_\alpha(f)| = \frac{1}{\alpha!} |\langle \Delta f, y^\alpha \rangle|$. Consider $g = \phi * (y^\alpha \varphi \Delta f) = V_{y^\alpha \varphi} f$, where $\varphi \in C^\infty$ satisfies $\chi_{B(a, \delta)} \leq \varphi \leq \chi_{B(a, 2\delta)}$, $\|\Delta \varphi\|_\infty \lesssim \frac{1}{\delta^2}$. Consequently

$$\|\Delta(y^\alpha \varphi)\|_\infty = \|\Delta(y^\alpha) \varphi + 2\nabla \varphi \nabla y^\alpha + y^\alpha \Delta \varphi\|_\infty \leq C(|\alpha|^2 + 1)(2\delta)^{|\alpha|-2},$$

and $C^{-1} \frac{g}{(|\alpha|^2+1)(2\delta)^{|\alpha|}} \in A^h(E)$. Since $|\frac{c_0(g)}{\alpha!}| = |c_\alpha(f)|$, we conclude that

$$\frac{1}{\alpha!} \left[\frac{c_0(g)}{(|\alpha|^2+1)(2\delta)^{|\alpha|}} \right] \leq C \frac{\kappa^h(E)}{\alpha!}. \quad \square$$

Lemma 14. *If $f \in A^h(E)$, $d(x, E) \geq 7\sqrt{n+1}\text{diam}(E)$ and $c_0 = \langle \Delta f, 1 \rangle = 0$, then*

$$|\nabla f(x)| \leq \frac{\text{diam}(E)\kappa^h(E)}{\text{dist}(x, E)^{n+1}}.$$

Proof. Considering $a \in E$, $\delta = \text{diam}(E)$ and $|x - a| > 3\delta\sqrt{n+1}$, we can write

$$f(x) = \sum_{|\alpha| \geq 1} c_\alpha \partial^\alpha \phi(x - a).$$

Let us observe that $|\partial^\alpha \phi(x - a)| \leq C\alpha! \frac{(3\sqrt{n+1})^{|\alpha|}}{|x-a|^{n-1+|\alpha|}}$. So if $1 \leq k \leq n+1$,

$$\begin{aligned} |\partial_k f(x)| &\leq \sum_{|\alpha| \geq 2} |c_{\alpha-k} \partial^\alpha \phi(x - a)| \\ &\leq C \sum_{|\alpha| \geq 2} \frac{(|\alpha| - 1)^2 + 1}{(\alpha - k)!} (2\delta)^{|\alpha|-1} \kappa^h(E) \alpha! \frac{(3\sqrt{n+1})^{|\alpha|}}{|x-a|^{n-1+|\alpha|}} \\ &\leq C \frac{\kappa^h(E)\delta}{|x-a|^{n+1}} \sum_{|\alpha| \geq 2} (|\alpha|^3) \left(\frac{6\delta\sqrt{n+1}}{|x-a|} \right)^{|\alpha|} \left(\frac{|x-a|}{\delta} \right)^2 \\ &\leq C \frac{\kappa^h(E)\delta}{|x-a|^{n+1}} \sum_{|\alpha| \geq 0} (|\alpha|^3) \left(\frac{6\delta\sqrt{n+1}}{|x-a|} \right)^{|\alpha|} \leq C \frac{\kappa^h(E)\delta}{|x-a|^{n+1}}. \quad \square \end{aligned}$$

6. POTENTIALS

The rest of the paper, with the exception of section 9, is devoted to proving the comparability

$$\kappa^h \approx \kappa_+^h,$$

with constants independent of h . Theorem 1 follows from this result and Lemmas 9 and 11.

This section uses some ideas from [V], in particular from chapters 5.1 and 5.2. Our objective is to describe κ_+^h in terms of the potentials $\mathcal{U}_\sigma^h$ which will be defined below. The results in this section can be easily proved with minor modifications from [V] (Theorems 5.3, 5.4, 5.5, 5.7 and Lemma 5.6), noticing that $\Sigma^h(E) \subset \Sigma(E)$, so we skip the proof.

Given a measure σ and a CZO operator T_σ , we denote:

$$\kappa_{T_1}^h(E) = \sup\{\|\sigma\| : \sigma \in \Sigma^h(E), \int_E |T_\sigma 1| d\sigma \leq \|\sigma\|\}.$$

Theorem 7. *There exist constants $C_4, C_5 > 0$ such that for any compact set $E \subset \mathbb{R}^{n+1}$,*

$$C_4^{-1} \kappa_{R_1}^h(E) \leq \liminf_{\varepsilon \rightarrow 0} \kappa_{R_\varepsilon 1}^h(E) \leq \limsup_{\varepsilon \rightarrow 0} \kappa_{R_\varepsilon 1}^h(E) \leq C_4 \kappa_{R_1}^h(E)$$

and

$$C_5^{-1} \kappa_0^h \leq \kappa_{R_1}^h(E) \leq C_5 \kappa_0^h(E).$$

Once we know the theorem, given any compact set E , we can choose $\varepsilon_0 = \varepsilon_0(E)$ small enough such that for $S := \overline{R}_{\varepsilon_0}$, $\kappa_{S_1}^h(E) \leq C_6 \kappa_+^h(E)$, with C_6 depending on C_4, C_5 , etc. Given a measure σ , define the functional

$$F(\sigma) = \frac{\|\sigma\|^2}{\|\sigma\| + \int_E |S_\sigma 1|^2 d\sigma}.$$

Theorem 8. *Given a compact set $E \subset \mathbb{R}^{n+1}$, $\varepsilon_0(E)$ and S as above,*

$$\kappa_0^h(E) \approx \sup_{\sigma \in \Sigma^h(E)} F(\sigma),$$

and the maximum is attained.

From now on we will call σ_0 a maximizer of the functional F . From the preceding theorem we deduce

$$\int_E |S_{\sigma_0} 1|^2 d\sigma_0 \leq \|\sigma_0\|,$$

and consequently $F(\sigma_0) \approx \|\sigma_0\|$.

Let us introduce some notation:

$$S_{\sigma_0} S_{\sigma_0} 1(x) = (S_{\sigma_0}^j(S_{\sigma_0}^j)(x))_j,$$

which is a vector-valued operator, and also consider

$$\begin{aligned} S_{\sigma_0,*} f &= S_{\sigma_0,*}^1 + \dots + S_{\sigma_0,*}^{n+1}, \\ S_{\sigma_0,*} S_{\sigma_0} f &= S_{\sigma_0,*}^1(S_{\sigma_0}^1 f) + \dots + S_{\sigma_0,*}^{n+1}(S_{\sigma_0}^{n+1} f). \end{aligned}$$

Finally define the potentials

$$\begin{aligned} \mathcal{U}_\sigma^h(x) &= M^h \sigma(x) + S_\sigma^* 1(x) + S_\sigma^* S_\sigma 1(x), \\ \overline{\mathcal{U}}_\sigma^h(x) &= \mathcal{U}_\sigma^h(x) + M^h(|S_\sigma 1| d\sigma)(x). \end{aligned}$$

We will use the following lemma.

Lemma 15. *Suppose that H is a positive measure supported on E such that for all $\lambda \in [0, \lambda_0]$, $\lambda_0 > 0$, $\sigma_\lambda = \sigma_0 + \lambda H \in \Sigma^h(E)$, then*

$$\frac{\|\sigma_0\| + 2 \int_E |S_{\sigma_0} 1|^2 d\sigma_0}{\|\sigma_0\|} \leq \int [|S_{\sigma_0} 1|^2 - 2 S_{\sigma_0} S_{\sigma_0} 1] \frac{dH}{H(E)}.$$

The next lemma is proved as inequality (5.39) of [V].

Lemma 16. *Let σ be a positive measure satisfying $\int_E |S_\sigma 1|^2 d\sigma \leq \|\sigma\|$. (Recall that this inequality holds for σ_0 .) Then:*

$$\kappa_+^h(\{x \in \mathbb{R}^{n+1} : \overline{\mathcal{U}}_\sigma^h(x) > \lambda\}) \leq C \frac{\|\sigma\|}{\lambda}.$$

Remark 1. Recall that $S_\sigma := \overline{R}_{\sigma, \varepsilon_0}$ depends on $\varepsilon_0(E)$, and so does $\overline{\mathcal{U}}_\sigma^h$. It is easy to check that Lemma 16 also holds if we replace S_σ by $\overline{R}_{\sigma, \varepsilon}$, for any $\varepsilon > 0$, and we modify $\overline{\mathcal{U}}_\sigma^h$ accordingly.

Lemma 17. *Suppose that E is a finite union of cubes. Then there exists a universal constant $\alpha > 0$ such that*

$$\mathcal{U}_{\sigma_0}^h(x) \geq \alpha \text{ for all } x \in E.$$

Recall that σ_0 is a maximizer of the functional F .

7. FIRST PART OF THE PROOF OF $\kappa^h \approx \kappa_+^h$

Roughly speaking, the arguments in this section correspond to the ones of the First Main Lemma in [T3] for the capacities γ^h .

Consider the measure σ_0 given in the previous section and, for a small $\lambda > 0$ which will be fixed below, the open set

$$\Omega_\lambda = \{x \in \mathbb{R}^{n+1} : \overline{\mathcal{U}}_{\sigma_0}^h(x) > \lambda\}.$$

By Lemma 17, if λ is small enough, $E \subset \Omega_\lambda$. Consider the Whitney decomposition $\Omega_\lambda = \bigcup_{j \in J} Q_j$ into cubes with disjoint interiors satisfying $20Q_j \subset \Omega_\lambda$, $1000\sqrt{n}Q_j \cap (\mathbb{R}^{n+1} \setminus \Omega_\lambda) \neq \emptyset$ and $\sum_j \chi_{100\sqrt{n}Q_j} \leq C$. Among these cubes choose a finite subfamily $I \subset J$ such that $2Q_i \cap E \neq \emptyset$ for all $i \in I$ and define

$$F := \bigcup_{i \in I} Q_i.$$

By construction, $E \subset F$ and, applying Lemma 16, $\kappa_+^h(F) \leq \kappa_+^h(\Omega_\lambda) \leq \frac{C}{\lambda} \|\sigma_0\| \approx \frac{C}{\lambda} \kappa_+^h(E)$.

Lemma 18. *Suppose that $\kappa_+^h(E) \leq C_3 h(\text{diam}(E))$ with C_3 small enough. Then the compact set $F = \bigcup_{i \in I} Q_i$ defined above satisfies:*

- (a) $E \subset F$ and $\kappa_+^h(F) \leq C \kappa_+^h(E)$,
- (b) $\sum_i \kappa^h(E \cap 2Q_i) \leq C \kappa_+^h(E)$,
- (c) $\text{diam}(Q_i) \leq \text{diam}(E)/10$ for all $i \in I$.

Proof. As explained above, the first statement is proved choosing λ small enough. Let us show (c). Observe that for $x \notin E$, $\overline{\mathcal{U}}_{\sigma_0}^h \leq C \sigma_0(E)/h(\text{dist}(x, E))$. In fact,

$$M^h \sigma_0(x) = \sup_{r > \text{dist}(x, E)} \frac{\sigma_0(B(x, r))}{h(r)} \leq \frac{\sigma_0(E)}{h(\text{dist}(x, E))}$$

and for $\varepsilon > \varepsilon_0(E)$,

$$\overline{\mathcal{R}}_{\varepsilon, \sigma_0} 1(x) \leq \int |\varphi\left(\frac{x-y}{\varepsilon}\right) k(x, y)| d\sigma(y) \leq C \frac{\sigma(E)}{\text{dist}(x, E)^n} \leq C \frac{\sigma(E)}{h(\text{dist}(x, E))}.$$

We can estimate the terms that are left using the inequality $\int_E |S_{\sigma_0} 1|(x) d\sigma_0(x) \leq \sigma_0(E)$.

For $x \in \Omega_\lambda \setminus E$, we have

$$\lambda < \overline{\mathcal{U}}_{\sigma_0}^h(x) \leq \frac{C \sigma_0(E)}{h(\text{dist}(x, E))}.$$

Therefore, if C_3 is small enough, applying (15),

$$h(\text{dist}(x, E)) \leq \frac{C \sigma_0(E)}{\lambda} \leq \frac{C \kappa_+^h(E)}{\lambda} \leq \frac{h(\text{dist}(x, E))}{(20)^{n+1}} \leq h\left(\frac{\text{diam}(E)}{10}\right).$$

So $\text{dist}(x, E) \leq \text{diam}(E)/10$ and $\text{diam}(\Omega_\lambda) \leq 6\text{diam}(E)/5$. Since $20Q_i \subset \Omega_\lambda$ for all i ,

$$20\text{diam}(Q_i) \leq \text{diam}(\Omega_\lambda) \leq 6\text{diam}(E)/5,$$

which proves (c).

In order to prove (b), we need two more results related with some localized potentials. Denote $\sigma_j = \sigma_0|_{4Q_j}$ and

$$\mathcal{W}_{\sigma_j}^h(x) = M^h \sigma_j(x) + S_{\sigma_j}^* 1(x) + S_{\sigma_j}^* S_{\sigma_0} 1(x) + M^h(|S_{\sigma} 1| d\sigma_j)(x).$$

Notice the difference between $\mathcal{W}_{\sigma_j}^h$ and $\overline{\mathcal{U}}_{\sigma_j}^h$. Arguing as in [V], pp. 39–41 (using also Remark 1), with minor changes we get

Lemma 19. *Let α be the constant in Lemma 17. Then there exists a constant α_1 depending only on n such that*

$$\mathcal{W}_{\sigma_j}^h(x) \geq \alpha_1 \alpha \quad \text{for all } x \in 3Q_j \cap E.$$

Using the same arguments as in Lemma 16 (see also (5.50) of [V]) we can prove that the following estimate holds:

$$\kappa_+^h(\{x \in \mathbb{R}^{n+1} : \mathcal{W}_{\sigma_j}^h(x) > \lambda\}) \leq C \frac{(\|\sigma_j\| + \| |S_{\sigma_0} 1| d\sigma_j \|)}{\lambda}.$$

Applying Lemma 19 we have that $2Q_j \cap E \subset \{x \in \mathbb{R}^{n+1} : \mathcal{W}_{\sigma_j}^h(x) > \alpha_1 \alpha\}$, so:

$$\kappa_+^h(2Q_j \cap E) \leq \frac{C}{\alpha_1 \alpha} (\|\sigma_j\| + \| |S_{\sigma_0} 1| d\sigma_j \|).$$

Thus

$$\begin{aligned} \sum_j \kappa_+^h(2Q_j \cap E) &\leq \frac{C}{\alpha_1 \alpha} \sum_j (\|\sigma_j\| + \| |S_{\sigma_0} 1| d\sigma_j \|) \\ &\leq \frac{C}{\alpha_1 \alpha} (\|\sigma_0\| + \int |S_{\sigma_0} 1| d\sigma_0) \leq \frac{C}{\alpha_1 \alpha} \|\sigma_0\|. \end{aligned}$$

Thus (b) in Lemma 18 follows. $\square$

Lemma 20. *Let A be a fixed constant. Suppose that $\kappa^h(E \cap 2Q_i) \leq A\kappa_+^h(E \cap 2Q_i)$ for all $i \in I$. If $\kappa^h(E) \geq A\kappa_+^h(E)$, then there exist measures μ and ν , both supported on F satisfying:*

- (d) $\mu(F) \approx \kappa^h(E)$,
- (e) $d\nu = b d\mu$ and $\|b\|_{L^\infty(\mu)} \leq C$,
- (f) $|\nu(F)| \geq \kappa^h(E)/2$.
- (g) For any function $\varphi \in \mathcal{C}^\infty(B(x_0, r))$, $|\int \varphi d\nu| \leq Ch(r)r^2 \|\Delta\varphi\|_\infty$.

Proof. For each $i \in I$ choose a function $g_i \in \mathcal{C}^\infty(2Q_i)$, $0 \leq g_i \leq 1$, $\|\Delta g_i\|_\infty \leq C/l(Q_i)^2$ and $\sum_i g_i = 1$ on Ω_λ . Also take $f \in A^h(E)$ with $|\langle \Delta f, 1 \rangle| \geq \kappa^h(E)/2$ and Δ_i a disc concentric with Q_i and radius $r_i \leq l(Q_i)/2$ satisfying $h(r_i) = \kappa^h(E \cap 2Q_i)/C$. We define

$$d\mu = \sum_i \frac{h(r_i)}{\mathcal{H}^{n+1}(\Delta_i)} d\mathcal{H}_{|\Delta_i}^{n+1}$$

and

$$\nu = \sum_i \frac{\langle \Delta f, g_i \rangle}{\mathcal{H}^{n+1}(\Delta_i)} d\mathcal{H}_{|\Delta_i}^{n+1}.$$

Observe that $\kappa_+^h(\Delta_i) \approx h(r_i) = \mu(\Delta_i)$ and $\text{supp}(\nu) \subset \text{supp}(\mu) \subset F$. Let us prove (f):

$$|\nu(F)| = \left| \sum_i \langle \Delta f, g_i \rangle \right| = |\langle \Delta f, 1 \rangle| \geq \kappa^h(E)/2.$$

Defining $b = \frac{\langle \Delta f, g_i \rangle}{h(r_i)}$ on Δ_i and noticing that $|\langle \Delta f, g_i \rangle| \leq C\kappa^h(E \cap 2Q_i) = Ch(r_i)$, we obtain (e). The first statement (d) follows from:

$$\begin{aligned} \kappa^h(E) &\leq 2|\langle \Delta f, 1 \rangle| = 2 \left| \sum_i \langle \Delta f, g_i \rangle \right| \leq 2 \sum_i |\langle \Delta f, g_i \rangle| \leq C \sum_i \kappa^h(E \cap 2Q_i) \\ &= \mu(F) \leq CA \sum_i \kappa_+^h(E \cap 2Q_i) \leq CA \kappa_+^h(E) \leq C\kappa^h(E). \end{aligned}$$

Now we are going to prove (g). Take $\varphi \in \mathcal{C}^\infty(B(x_0, r))$. We distinguish two different cases.

Suppose that there exists a cube Q_{i_0} with side length $l(Q_{i_0}) \geq r$, $Q_{i_0} \cap B(x_0, r) \neq \emptyset$. Then $B(x_0, r) \subset 5Q_{i_0}$ and by the finite overlapping of the $10Q_i$, $\#\{i \in I : Q_i \cap \text{supp}(\varphi) \neq \emptyset\} \leq C$. So, if we show that

$$\left| \int_{Q_i} \varphi d\nu \right| \leq Ch(r)r^2 \|\Delta\varphi\|_\infty,$$

we will have finished. Let us prove this inequality:

$$\begin{aligned} \left| \int_{Q_i} \varphi d\nu \right| &\leq C\mu(Q_i \cap B(x_0, r)) \|\varphi\|_\infty \leq C \frac{\min(r, r_i)^{n+1} h(r_i)}{r_i^{n+1}} r^2 \|\Delta\varphi\|_\infty \\ &\leq Ch(r)r^2 \|\Delta\varphi\|_\infty, \end{aligned}$$

using (15) in the last estimate.

If otherwise $l(Q_i) < r$ for all i such that $Q_i \cap B(x_0, r) \neq \emptyset$, we will prove that for all $\{z_i\}_i$ with $z_i \in Q_i$,

$$\left| \int f \Delta\varphi dx - \sum_i \varphi(z_i) \nu(Q_i) \right| \leq Ch(r)r^2 \|\Delta\varphi\|_\infty.$$

This fact together with $f \in A^h(E)$ implies $|\sum_i \varphi(z_i) \nu(Q_i)| \leq Ch(r)r^2 \|\Delta\varphi\|_\infty$. Observing that by the definition of ν and by continuity for each $i \in I$ there exists $z_i \in Q_i$ satisfying $\varphi(z_i) \nu(Q_i) = \int_{Q_i} \varphi d\nu$, the statement follows. We also have

$$\begin{aligned} \left| \int f \Delta\varphi dx - \sum_i \varphi(z_i) \nu(Q_i) \right| &\leq \sum_i \left| \int V_{g_i} f \Delta\varphi dx - \varphi(z_i) \nu(Q_i) \right| \\ &= \sum_i \left| \int f \Delta([\varphi - \varphi(z_i)]g_i) dx \right|. \end{aligned}$$

We call $\psi_i = [\varphi - \varphi(z_i)]g_i$, which is supported on $2Q_i$. Since $f \in A^h(E)$, just checking $\|\Delta\psi\|_\infty \leq C\|\Delta\varphi\|_\infty$, the proof is finished; in fact,

$$\begin{aligned} \left| \int f \Delta\varphi dx - \sum_i \varphi(z_i) \nu(Q_i) \right| &\leq C \sum_i h(l_i) l_i^2 \|\Delta\psi\|_\infty \leq C \|\Delta\varphi\|_\infty \sum_i \frac{h(r)}{r^n} l_i^{n+2} \\ &\leq Cr \frac{h(r)}{r^n} \mathcal{H}^{n+1}(B(x_0, Cr)) \|\Delta\varphi\|_\infty \\ &\leq Ch(r)r^2 \|\Delta\varphi\|_\infty, \end{aligned}$$

but

$$\|\Delta\psi\|_\infty \leq \|g_i \Delta\varphi\|_\infty + 2\|\nabla\varphi \nabla g_i\|_\infty + \|(\varphi - \varphi(z_i)) \Delta g_i\|_\infty \leq C\|\Delta\varphi\|_\infty. \quad \square$$

Now we need to introduce a modified version of the Riesz transform and prove a technical lemma. Take $\psi \in C^\infty(B(0, 1))$ radial, $0 \leq \psi \leq 2$, $\int \psi dx = 1$, $\|\Delta \psi\|_\infty \leq C$. Define $\psi_\varepsilon(x) = \frac{1}{\varepsilon^{n+1}} \psi(\frac{x}{\varepsilon})$ and

$$K_\varepsilon \nu(x) = \psi_\varepsilon * R\nu(x) = \psi_\varepsilon * \frac{y}{|y|^{n+1}} * \nu(x).$$

We define the maximal operator in the usual way:

$$K_* \nu(x) = \sup_{\varepsilon > 0} |K_\varepsilon \nu(x)|.$$

Lemma 21. *Let $i \in I$, $z \in \mathbb{R}^{n+1} \setminus (14\sqrt{n+1}Q_i)$, and $\nu_i = \nu|_{Q_i}$. Then*

$$K_*(\nu_i - g_i \Delta f)(z) \leq C \frac{l(Q_i) \mu(Q_i)}{d(z, 2Q_i)^{n+1}}.$$

Proof. If we set $\alpha_i = \nu_i - g_i \Delta f$, then $\text{supp}(\alpha_i) \subset (E \cap 2Q_i) \cup \Delta_i$ and $\langle \alpha_i, 1 \rangle = 0$. To estimate $K_\varepsilon(\alpha_i)(z)$ we will distinguish two different cases:

(a) $\varepsilon < \text{dist}(z, 2Q_i)/4$.

Remember from Proposition 3 that $\|R(g_i \Delta f)\|_\infty \leq C$. We can also easily see that $\|R(\nu_i)\|_\infty \leq C$ and so $|R(\alpha_i)(w)| \leq C$ for all $w \in \text{supp}(\alpha_i)$. By Lemma 14,

$$|R(\alpha_i)(w)| \leq C \frac{\text{diam}(\text{supp}(\alpha_i)) \kappa^h(\text{supp}(\alpha_i))}{\text{dist}(w, \text{supp}(\alpha_i))^{n+1}},$$

for all w such that $\text{dist}(w, \text{supp}(\alpha_i)) \geq 7\sqrt{n+1} \text{diam}(\text{supp}(\alpha_i))$, in particular for $w \in B(z, \varepsilon)$. But by statement (d) of Lemma 12, $\kappa^h(\text{supp}(\alpha_i)) \lesssim \kappa^h(\Delta_i) + \kappa^h(E \cap 2Q_i) \lesssim \mu(Q_i)$, since $\kappa^h(\Delta_i) \approx h(r_i) \leq C \kappa^h(E \cap 2Q_i)$. If $w \in B(z, \varepsilon)$, then $\text{dist}(w, 2Q_i) \approx \text{dist}(z, 2Q_i)$ and consequently

$$|R(\alpha_i)(w)| \leq C \frac{l(Q_i) \mu(Q_i)}{d(z, 2Q_i)^{n+1}}.$$

Making the convolution with ψ_ε we obtain the required estimate.

(b) $\varepsilon \geq \text{dist}(z, 2Q_i)/4$.

If we define $h_i = \psi_\varepsilon * \alpha_i$, then $K_\varepsilon \alpha_i = R(h_i d\mathcal{H}^{n+1})$. Observe that since $l_i = l(Q_i) \leq \varepsilon$ (in this case),

$$\begin{aligned} \text{supp}(h_i) &\subset \text{supp}(\psi_\varepsilon) + \text{supp}(\alpha_i) \subset B(0, \varepsilon) + B(z_i, 2\sqrt{n+1}l_i) \\ &= B(z_i, \varepsilon + 2\sqrt{n+1}l_i) \subset B(z_i, C\varepsilon) \end{aligned}$$

and

$$|K_\varepsilon \alpha_i(z)| \leq \int_{B(z_i, C\varepsilon)} \frac{|h_i(\xi)|}{|\xi - z|^n} d\xi \leq C \|h_i\|_\infty \varepsilon.$$

Now we need to estimate $\|h_i\|_\infty$. Let $\eta_i \in C^\infty$ satisfying $\chi_{2Q_i} \leq \eta_i \leq \chi_{3Q_i}$ and $\|\Delta \eta_i\|_\infty \leq C/l_i^2$. Then, using $\langle \alpha_i, 1 \rangle = 0$,

$$\begin{aligned} h_i(w) &= \langle \alpha_i, \psi_\varepsilon(\cdot - w) \rangle = \langle \alpha_i, \psi_\varepsilon(\cdot - w) - \psi_\varepsilon(z_i - w) \rangle \\ &= \frac{l_i}{\varepsilon^{n+2}} \langle \alpha_i, \frac{\varepsilon^{n+2}}{l_i} (\psi_\varepsilon(\cdot - w) - \psi_\varepsilon(z_i - w)) \eta_i \rangle =: \frac{l_i}{\varepsilon^{n+2}} \langle \alpha_i, \varphi_w \eta_i \rangle. \end{aligned}$$

If we denote $\varrho = \phi * (\alpha_i \varphi_w \eta_i)$ we only need to check that $C^{-1} \varrho \in A^h(\text{supp}(\alpha_i))$. If this is the case,

$$|h_i(w)| = \frac{l_i}{\varepsilon^{n+2}} |\langle \alpha_i \varphi_w \eta_i, 1 \rangle| \leq C \frac{l_i}{\varepsilon^{n+2}} \kappa^h(\text{supp}(\alpha_i)) \leq C \frac{l_i \mu(Q_i)}{\varepsilon^{n+2}}.$$

Therefore,

$$|K_\varepsilon \alpha_i(z)| \leq C \frac{l_i \mu(Q_i)}{\varepsilon^{n+2}} \varepsilon \leq C \frac{l(Q_i) \mu(Q_i)}{d(z, 2Q_i)^{n+1}}.$$

It remains to show that $C^{-1}\varrho \in A^h(\text{supp}(\alpha_i))$. One easily checks that $C^{-1}\phi * \nu_i \in A^h(Q_i)$. Recall that $C^{-1}V_{g_i}f = C^{-1}\phi * (g_i\Delta f) \in A^h(2Q_i)$, so $C^{-1}(\phi * \alpha_i) \in A^h(\text{supp}(\alpha_i))$. Since $\text{supp}(\varphi_w \eta_i) \subset (3Q_i)$, we only need to check that $\|\Delta(\varphi_w \eta_i)\|_\infty \leq C/l(Q_i)^2$ and then apply Proposition 3: if $z \in 3Q_i$,

$$|\varphi_w(z)| = \left| \frac{\varepsilon^{n+2}}{l_i} (\psi_\varepsilon(z-w) - \psi_\varepsilon(z_i-w)) \right| \leq \frac{\varepsilon^{n+2}}{l_i} \frac{1}{\varepsilon^{n+2}} \|\nabla \psi\|_\infty |z - z_i| \leq C,$$

$$|\nabla \varphi_w(z)| \leq \frac{\varepsilon^{n+2}}{l_i} \frac{1}{\varepsilon^{n+2}} \|\nabla \psi\|_\infty \leq \frac{C}{l_i},$$

and

$$|\Delta \varphi_w(z)| = \frac{\varepsilon^{n+2}}{l_i} |\Delta \psi_\varepsilon(z-w)| \leq \frac{1}{l_i \varepsilon} \|\Delta \psi\|_\infty \leq \frac{C}{l_i^2}.$$

Finally, using $\|\Delta \eta_i\|_\infty \leq C/l_i^2$ (and so $\|\nabla \eta_i\|_\infty \leq C/l_i$), we get

$$\|\Delta(\varphi_w \eta_i)\|_\infty = \|\eta_i \Delta \varphi_w + 2\nabla \varphi_w \nabla \eta_i + \varphi_w \Delta \eta_i\|_\infty \leq C/l_i^2. \quad \square$$

In the next lemma we construct a set H following some ideas from [NTV1]. Our set H will be closely related with the growth of the measure μ constructed in the last theorem.

Lemma 22. *There exists a subset $H \subset F$ satisfying :*

- (h) *If $\mu(B(x, r)) > C_h h(r)$ (for a big fixed constant C_h), then $B(x, r) \cap F \subset H$.*
- (i) *H is of the form $H = \bigcup_{k \in I_H} B(x_k, r_k)$, with $\sum_{k \in I_H} h(r_k) \leq \varepsilon \mu(F)$, for $0 < \varepsilon$ arbitrarily small, if C_h is chosen big enough.*
- (j) $\int_{F \setminus H} R_* \nu d\mu \leq C \mu(F)$.

Proof. Let C_h be a big constant to be fixed below. Given $x \in F$ and $r > 0$ consider

$$\mathcal{R}(x) = \sup\{r > 0 : \mu(B(x, r)) > C_h h(r)\},$$

and set $\mathcal{R}(x) = 0$ if $\mu(B(x, r)) \leq C_h h(r)$ for all $r > 0$. $\mathcal{R}(x)$ will be called the *Ahlfors radius* of x . For $r \geq \text{diam}(F)/100$,

$$\mu(B(x, r)) \leq \mu(F) \leq C \kappa^h(E) \leq C \kappa^h(F) \leq C h(\text{diam}(F)) \leq C_8 h(r),$$

so choosing $C_h \geq C_8$ we get $\mathcal{R}(x) \leq \text{diam}(F)/100$. Denote

$$H_0 = \bigcup_{x \in F, \mathcal{R}(x) > 0} B(x, \mathcal{R}(x)).$$

Applying Vitali's covering theorem, we can cover $H_0 \subset \bigcup_k B(x_k, 5\mathcal{R}(x_k)) =: H$, where $\{B(x_k, \mathcal{R}(x_k))\}_k$ is a disjoint family of balls. We remark that all the non-Ahlfors disks (those for which $\mu(B(x, r)) > C_h h(r)$) are contained in H since $H_0 \subset H$. Now we can prove (i):

$$\sum_k h(\mathcal{R}(x_k)) \leq \frac{1}{C_h} \sum_k \mu(B(x_k, \mathcal{R}(x_k))) \leq \frac{1}{C_h} \mu(F),$$

and choosing C_h big enough we are done.

Now we are going to prove (j). To this end we will show that

$$\int_{F \setminus H} K_* \nu d\mu \leq C\mu(F).$$

Then, using $|R_*\nu(z) - K_*\nu(z)| \leq M\nu(z)$ and $M\nu(z) \leq C_h C$ for all $z \in F \setminus H$, by construction of H and (e) of Lemma 20, the proof is finished.

First of all, observe that $\|K_\varepsilon(\Delta f)\|_\infty \leq 1$. This follows from the fact that $\|\nabla f\|_\infty = \|R(\Delta f)\|_\infty$ and $K_\varepsilon(\Delta f) = \psi_\varepsilon * R(\Delta f)$. Therefore,

$$\begin{aligned} \int_{F \setminus H} K_* \nu d\mu &\leq \int_{F \setminus H} K_*(\Delta f) d\mu + \int_{F \setminus H} K_*(\nu - \Delta f) d\mu \\ &\leq \mu(F \setminus H) + \sum_{i \in I} \int_{F \setminus H} K_*(\nu_i - g_i \Delta f) d\mu. \end{aligned}$$

Fix $i \in I$ and define $c_n = 14\sqrt{n+1}$. Since $\|K_*(\nu_i - g_i \Delta f)\|_\infty \leq \|K_*(\nu_i)\|_\infty + \|K_*(g_i \Delta f)\|_\infty \leq C$ (see the first paragraph of the proof of (a) in Lemma 21) and applying Lemma 21,

$$\int_{F \setminus H} K_*(\nu_i - g_i \Delta f) d\mu \leq C\mu(c_n Q_i) + C \int_{F \setminus (c_n Q_i \cup H)} \frac{l(Q_i)\mu(Q_i)}{\text{dist}(z, 2Q_i)^{n+1}} d\mu.$$

Let $N \geq 1$ be the least integer such that $(c_n^{N+1} Q_i \setminus c_n^N Q_i) \setminus H \neq \emptyset$ and take $z_0 \in (c_n^{N+1} Q_i \setminus c_n^N Q_i) \setminus H$. Then

$$\begin{aligned} \int_{F \setminus (c_n Q_i \cup H)} \frac{1}{\text{dist}(z, 2Q_i)^{n+1}} d\mu(z) &\leq \sum_{k \geq N} \int_{(c_n^{k+1} Q_i \setminus c_n^k Q_i) \setminus H} \frac{1}{\text{dist}(z, 2Q_i)^{n+1}} d\mu(z) \\ &\leq C \sum_{k \geq N} \frac{\mu(c_n^{k+1} Q_i)}{l(c_n^{k+1} Q_i)^{n+1}} \\ &\leq C \sum_{k \geq N} \frac{\mu(B(z_0, Cl(c_n^{k+1} Q_i)))}{l(c_n^{k+1} Q_i)^{n+1}} \\ &\leq C_h C \sum_{k \geq 0} \frac{l(c_n^{k+1} Q_i)^n}{l(c_n^{k+1} Q_i)^{n+1}} \leq C_h C \frac{1}{l(Q_i)}. \end{aligned}$$

Finally, using $\sum_i \chi_{c_n Q_i} \leq C$ we obtain:

$$\int_{F \setminus H} K_* \nu d\mu \leq \sum_i \mu(c_n Q_i) \leq \mu(F). \quad \square$$

8. SECOND PART OF THE PROOF OF $\kappa^h \approx \kappa_+^h$

The second part of the proof of $\kappa^h \approx \kappa_+^h$ requires a $T(b)$ -type theorem of Nazarov, Treil and Volberg. In order to apply it, we need to introduce some additional notions. Consider a dyadic lattice $\mathcal{D}$ in $\mathbb{R}^{n+1}$. We define two exceptional sets: $H_{\mathcal{D}}, T_{\mathcal{D}}$.

($H_{\mathcal{D}}$) Consider the family of dyadic cubes $R \in \mathcal{D}$ for which there exists a ball $B(x_k, \mathcal{R}(x_k))$ satisfying $B(x_k, \mathcal{R}(x_k)) \cap R \neq \emptyset$ and $10\mathcal{R}(x_k) < l(R) \leq 20\mathcal{R}(x_k)$. Among these cubes choose a maximal disjoint family $\{R_j\}_{j \in I_H(\mathcal{D})}$. Then we define

$$H_{\mathcal{D}} := \bigcup_{j \in I_H(\mathcal{D})} R_j.$$

By construction $H \subset H_{\mathcal{D}}$, and since for each ball $B(x_k, \mathcal{R}(x_k))$ there originate at most 4^{n+1} cubes R_j with $j \in I_H(\mathcal{D})$, we obtain:

$$\sum_{j \in I_H(\mathcal{D})} h(l(R_j)) \leq 4^{n+1} \sum_k h(20\mathcal{R}(x_k)) \leq C \sum_k h(\mathcal{R}(x_k)) \leq \frac{C}{C_h} \mu(F) \leq \varepsilon \mu(F),$$

choosing C_h big enough, for any given $\varepsilon > 0$.

($T_{\mathcal{D}}$) Let us now deal with the accretivity condition. We say that a cube $R \subset \mathbb{R}^{n+1}$ is accretive if

$$\mu(R) \leq C_d |\nu(R)|,$$

where C_d is a big constant to be chosen later. Among those nonaccretive cubes $R \in \mathcal{D}$ choose a maximal disjoint family and define

$$T_{\mathcal{D}} = \bigcup_{m \in I_T(\mathcal{D})} R_m.$$

Now we introduce randomness on the dyadic lattice. Consider the usual dyadic lattice $\mathcal{D}_{n+1}$ with cubes of the form $[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}) \times \dots \times [\frac{k_{n+1}}{2^j}, \frac{k_{n+1}+1}{2^j})$, $k_1, \dots, k_{n+1}, j \in \mathbb{Z}$. Suppose that $F \subset B(0, 2^{N-3})$, $N \in \mathbb{N}$, and set $Q^0(w) = w + [-2^N, 2^N]^{n+1}$ with $w \in [-2^{N-1}, 2^{N-1}]^{n+1} =: \Omega$. We consider a new dyadic lattice $\mathcal{D}(w)$ that has Q^0 as starting cube and cubes of the form $w + Q \subset Q^0$, $Q \in \mathcal{D}_{n+1}$. Let P be the probability measure given by the normalized Lebesgue measure on Ω .

Lemma 23. *If C_h and C_d are chosen big enough we have the following estimates for the expectation of the size of $H_{\mathcal{D}} \cup T_{\mathcal{D}}$:*

$$\int_{\Omega} |\nu(H_{\mathcal{D}} \cup T_{\mathcal{D}})| dP(w) \leq \frac{1}{2} |\nu(F)|$$

and

$$\int_{\Omega} \mu(H_{\mathcal{D}} \cup T_{\mathcal{D}}) dP(w) \leq \delta_0 \mu(F),$$

where $\delta_0 < 1$ is a fixed constant.

The proof can be found with minor modifications in [T2]. We will also need the following $T(b)$ type theorem.

Theorem 9. *Let μ be a positive measure supported on $F \subset \mathbb{R}^{n+1}$. Suppose that there exists a measure ν and, for each $w \in \Omega$, two exceptional sets $H_{\mathcal{D}(w)}, T_{\mathcal{D}(w)}$ made up of dyadic cubes from $\mathcal{D}(w)$ such that:*

- (a) *Every ball B_r of radius r such that $\mu(B_r) > C_h r$ is contained in $H_{\mathcal{D}(w)}$ for all w .*
- (b) *$d\nu = b d\mu$ with $\|b\|_{L^\infty(\mu)} \leq C \mu(F)$.*
- (c) *$\int_{\mathbb{R}^{n+1} \setminus H_{\mathcal{D}(w)}} R_* \nu d\mu \leq C \mu(F)$, for all $w \in \Omega$.*
- (d) *If $Q \in \mathcal{D}(w)$ is such that $Q \not\subset T_{\mathcal{D}(w)}$, then $\mu(Q) \leq C_d |\nu(F)|$.*
- (e) *$\int_{\Omega} \mu(H_{\mathcal{D}(w)} \cup T_{\mathcal{D}(w)}) dP(w) \leq \delta_0 \mu(F)$ with $\delta_0 < 1$.*

Then there exists a subset $G \subset F \setminus \bigcap_{w \in \Omega} (H_{\mathcal{D}(w)} \cup T_{\mathcal{D}(w)})$ with $\mu(G) \geq C_{10}^{-1} \mu(F)$, such that the Riesz transform is bounded in $L^2(\mu|_G)$ with its norm and C_{10} depending only on the constants above.

The proof of this theorem can be deduced from [NTV2], where instead of (e) one asks for the stronger condition (e'): $\mu(H_{\mathcal{D}(w)} \cup T_{\mathcal{D}(w)}) \leq \delta_0 \mu(F)$. In [T3] we can find the arguments needed to prove the theorem by changing the condition (e) by (e'). Now we can state the main lemma of the second part of the proof.

Lemma 24. Assume $\kappa_+^h(E) \leq Ch(\text{diam}(E))$, $\kappa^h(E) \geq A\kappa_+^h(E)$, and $\kappa^h(E \cap 2Q_i) \leq A\kappa_+^h(E \cap 2Q_i)$ for all i and a fixed $A > 1$. Then there exists a subset $G \subset F$ with $\mu(F) \leq C_{10}\mu(G)$ such that $\mu(G \cap B(x, r)) \leq C_h h(r)$ for all $x \in G$, $r > 0$ and the Riesz transform is bounded in $L^2(\mu|_G)$, with $\|R\|_{L^2(\mu|_G), L^2(\mu|_G)} \leq C_{11}$, and C_{10} and C_{11} not depending on A .

Proof. The theorem follows from Theorem 9. We only need to remark that the set G obtained in the last theorem satisfies

$$G \subset \mathbb{R}^{n+1} \setminus \left(\bigcap_{w \in \Omega} H_{\mathcal{D}(w)} \right) \subset \mathbb{R}^{n+1} \setminus H,$$

and so $\mu(B(x, r)) \leq C_h h(r)$ for all $x \in G$ and $r > 0$. $\square$

Lemma 25. There exists an absolute constant B such that if $A \geq 1$ is any fixed constant and

- a) $\kappa_+^h(E) \leq C_3 h(\text{diam}(E))$,
- b) $\kappa^h(E \cap Q) \leq A\kappa_+^h(E \cap Q)$ for all cubes Q with $\text{diam}(Q) \leq \text{diam}(E)/10$,
- c) $\kappa^h(E) \geq A\kappa_+^h(E)$,

then $\kappa^h(E) \leq B\kappa_+^h(E)$.

Proof. We can find a measure μ and sets F, G such as the ones constructed in section 7, and then we apply Lemma 24 so that

- (a) $E \subset F$ and $\kappa_+^h(E) \approx \kappa_+^h(F)$,
- (b) $\mu(F) \approx \kappa^h(E)$,
- (c) $G \subset F$ and $\mu(G) \geq \mu(F)$,
- (d) $\mu(G \cap B(x, r)) \leq C_h h(r)$ for all $x \in G$, $r > 0$ and $\|R\|_{L^2(\mu|_G), L^2(\mu|_G)} \leq C$.

Finally we get

$$\kappa^h(E) \leq C\mu(F) \leq C\mu(G) \leq C\kappa_+^h(F) \leq B\kappa_+^h(E). \quad \square$$

Lemma 26. There exists an absolute constant A_0 such that if $\kappa^h(E \cap Q) \leq A_0\kappa_+^h(E \cap Q)$ for all cubes Q with $\text{diam}(Q) \leq \text{diam}(E)/10$, then $\kappa^h(E) \leq A_0\kappa_+^h(E)$.

Proof. If $\kappa_+^h(E) > C_3 h(\text{diam}(E))$, since $\kappa^h(E) \leq C_2 h(\text{diam}(E))$ we are done. Denote $A_0 = \max[C_2/C_3, B]$. If $\kappa_+^h(E) \leq C_3 h(\text{diam}(E))$ and $\kappa^h(E) > A_0\kappa_+^h(E)$ we get a contradiction, since by applying the lemma above we deduce $\kappa^h(E) \leq B\kappa_+^h(E) \leq A_0\kappa_+^h(E)$. $\square$

Proof of $\kappa^h \approx \kappa_+^h$, and of Theorem 1. As mentioned above, Theorem 1 is a direct consequence of the comparability $\kappa^h \approx \kappa_+^h$ and Lemmas 9 and 11. We only have to prove the inequality $\kappa^h \lesssim \kappa_+^h$ since the converse is trivial.

For $d > 0$ consider $U_d(E) = \{x : d(x, E) \leq d\}$. By the approximation property in Lemma 12, if d is small enough for a given ε , $\kappa_+^h(U_d(E)) \leq \kappa_+^h(E) + \varepsilon$. Let E_0 be a compact set made up of all closed dyadic cubes from some dyadic lattice of side length $\sqrt{n+1}d/2$ that intersect E . If we prove $\kappa^h(E_0) \leq C\kappa_+^h(E_0)$, we are done, since

$$\kappa^h(E) \leq \kappa^h(E_0) \leq C\kappa_+^h(E_0) \leq C(\kappa_+^h(E) + \varepsilon).$$

To prove $\kappa^h(E_0) \leq C\kappa_+^h(E_0)$ we will show by an induction argument that if R is a closed $(n+1)$ -rectangle with sides parallel to the axes, then

$$(16) \quad \kappa^h(R \cap E_0) \leq A_0\kappa_+^h(R \cap E_0).$$

If $\text{diam}(R) \leq d/2^{n+1}$, then $R \cap E_0$ is a union of at most 2^{n+1} (this constant may not be sharp) closed $(n+1)$ -rectangles $R_1, \dots, R_{2^{n+1}}$. Then applying the properties from Lemma 12,

$$\begin{aligned} \kappa^h(R_1 \cup \dots \cup R_{2^{n+1}}) &\leq C(\kappa^h(R_1) + \dots + \kappa^h(R_{2^{n+1}})) \leq C(\kappa_+^h(R_1) + \dots + \kappa_+^h(R_{2^{n+1}})) \\ &\leq C\kappa_+^h(R_1 \cup \dots \cup R_{2^{n+1}}). \end{aligned}$$

Now we argue by induction on m . Suppose that all $(n+1)$ -rectangles T with $\text{diam}(T) \leq 2^{m-1}d$ satisfy (16). Let R be an $(n+1)$ -rectangle with $\text{diam}(R) \leq 2^m d$. We can apply Lemma 26 to $R \cap E_0$: in fact if Q is a cube with $\text{diam}(Q) \leq \text{diam}(R \cap E_0)/10$, by the induction hypothesis,

$$\kappa^h(Q \cap R \cap E_0) \leq A_0 \kappa_+^h(Q \cap R \cap E_0),$$

since $Q \cap R$ is an $(n+1)$ -rectangle with $\text{diam}(Q \cap R) \leq \text{diam}(R \cap E_0)/10 \leq 2^m d/10 \leq 2^{m-1}d$, and so

$$\kappa^h(R \cap E_0) \leq A_0 \kappa_+^h(R \cap E_0). \quad \square$$

9. THE INNER BOUNDARY CONJECTURE IN $\mathbb{R}^{n+1}$

In this section we will prove Theorem 3, which asserts that, for $n \geq 2$, there exists a compact set $K \subset \mathbb{R}^{n+1}$ satisfying $\kappa^c(\partial_i K) = 0$ and $h^1(K) \neq \mathcal{C}^1 h(K)$. First we need to recall the following result, proved by P.V. Paramonov [P] in 1992.

Theorem 10. *Let $E \subset \mathbb{R}^{n+1}$. Then the following are equivalent:*

- (a) $\mathcal{C}^1 h(E) = h^1(E)$.
- (b) *For each bounded open set D in $\mathbb{R}^{n+1}$,*

$$\kappa^c(D \setminus E^0) = \kappa^c(D \setminus E).$$

- (c) *There exist constants $A > 0$ and $r \geq 1$ such that for each ball $B(a, \delta)$,*

$$\kappa^c(B(a, \delta) \setminus E^0) \leq A \kappa^c(B(a, r\delta) \setminus E).$$

- (d) *For each point $x \in \partial E$, there exists $r \geq 1$ such that*

$$\liminf_{\delta \rightarrow 0} \frac{\kappa^c(B(a, r\delta) \setminus E)}{\kappa^c(B(a, \delta) \setminus E^0)} > 0.$$

Proof of Theorem 3. We will prove the theorem in $\mathbb{R}^3$. Analogous arguments can be used for $\mathbb{R}^{n+1}$ with $n \geq 3$. Consider $T = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ and the usual dyadic lattice $\mathcal{D}_T$ in $\mathbb{R}^2$ generated by T (i.e. for each $k \geq 0$, $T = \bigcup_{Q \in \mathcal{D}_{T,k}} Q$, where $\#\mathcal{D}_{T,k} = 4^k$, every $Q \in \mathcal{D}_{T,k}$ is a square with sides parallel to the axes that has side length 2^{-k} , and $\mathcal{D}_T = \bigcup_{k \geq 0} \mathcal{D}_{T,k}$, etc.). Take $0 < \varepsilon < 1$ and $k > 1$, and for each square $Q \in \mathcal{D}_{T,k}$, consider a concentric open square R_Q with side length $\varepsilon/4^{2k}$. We define a subfamily I of the squares $Q \in \mathcal{D}_T$ as follows. We denote $I_0 = \mathcal{D}_{T,0}$, and for $k \geq 1$,

$$I_k = \{Q \in \mathcal{D}_{T,k} : R_Q \text{ does not intersect another } R_{Q'} \text{ for } Q' \in \mathcal{D}_{T,j}, j < k\}.$$

We set $I = \bigcup_{k \geq 0} I_k$. Notice that our construction ensures that the family of squares $\{R_Q\}_{Q \in I}$ is pairwise disjoint. We consider the set

$$K = (T \setminus \bigcup_{Q \in I} R_Q) \times [0, 1] \subset \mathbb{R}^3.$$

Observe that K is connected, has empty interior, and since $\mathbb{R}^3 \setminus K$ is connected and $\partial K = \partial(\mathbb{R}^3 \setminus K)$, we have $\partial_i K = \emptyset$, so $\kappa^c(\partial_i K) = 0$.

Now consider $D = (0, 1)^3 \subset \mathbb{R}^3$. We will see that $\kappa^c(D \setminus K^0) \neq \kappa^c(D \setminus K)$ and then, by Theorem 10, we will have $h^1(K) \neq C^1 h(K)$. Remember that if B is any ball of radius r , or R is any cube in $\mathbb{R}^3$ with side length r , then $\kappa^c(R) \approx \kappa^c(B(x, r)) \approx r^2$. Remember also that for any compact set E , $\kappa(E) \leq C \mathcal{H}_\infty^2(E)$, where $\mathcal{H}_\infty^2$ denotes the two-dimensional Hausdorff content. Since $K^0 = \emptyset$,

$$1 \leq C \kappa^c(D) = C \kappa^c(D \setminus K^0).$$

Now observe that if $Q \subset \mathbb{R}^2$ is a square with side length $l(Q) \leq 1$, decomposing $Q \times (0, 1) \subset \mathbb{R}^3$ into at most $2/l(Q)$ cubes, then

$$\mathcal{H}_\infty^2(Q \times (0, 1)) \leq \frac{2}{l(Q)} \mathcal{H}_\infty^2(Q \times (0, l(Q))) \leq Cl(Q).$$

So,

$$\begin{aligned} \kappa^c(D \setminus K) &\leq C \mathcal{H}_\infty^2\left(\bigcup_{Q \in I} R_Q \times (0, 1)\right) \leq C \sum_{k \geq 1} \sum_{Q \in \mathcal{D}_{T,k}} \mathcal{H}_\infty^2(R_Q \times (0, 1)) \\ &\leq C \sum_{k \geq 1} 4^k \frac{\varepsilon}{4^{2k}} \leq C\varepsilon. \end{aligned}$$

Taking ε small enough we are done. $\square$

REFERENCES

- [M] P. Mattila, *On the analytic capacity and curvature of some Cantor sets with non- σ -finite length*, Publ. Mat., Vol. 40 (1996), 195–204. MR1397014 (97d:30052)
- [MP] P. Mattila, P.V. Paramonov, *On geometric properties of harmonic Lip_1 -capacity*, Pacific J. Math. 171, No. 2, (1995), 469–491. MR1372240 (97b:31005)
- [MT] J. Mateu, X. Tolsa, *Riesz transforms and harmonic Lip_1 -capacity in Cantor sets*, Proc. London Math. Soc. 89 (3) (2004), 676–696. MR2107011 (2005g:42038)
- [MPV] Dzh. Verdera, M.S. Mel'nikov, P.V. Paramonov, *C^1 -approximation and the extension of subharmonic functions*. (Russian) Mat. Sb. 192 (2001), no. 4, 37–58; translation in Sb. Math. 192 (2001), no. 3-4, 515–535. MR1834090 (2002g:41030)
- [NTV1] F. Nazarov, S. Treil, A. Volberg, *The Tb-theorem on non-homogeneous spaces*, Acta Math. 190 (2), (2003), 151–239. MR1998349 (2005d:30053)
- [NTV2] F. Nazarov, S. Treil, A. Volberg, *The Tb-theorem on non-homogeneous spaces that proves a conjecture of Vitushkin*, preprint n. 519, Centre de Recerca Matemàtica, Barcelona 2002.
- [P] P.V. Paramonov, *Harmonic approximation in the C^1 -norm*, Math. USSR-Sb., 71 (1992), 183–207. MR1085885 (92j:31006)
- [T1] X. Tolsa, *Littlewood-Paley theory and the $T(1)$ theorem with non-doubling measures*, Adv. Math. 164, 57–116, (2001). MR1870513 (2003e:42029)
- [T2] X. Tolsa, *The semiadditivity of continuous analytic capacity and the inner boundary conjecture*, Amer. J. Math., (2002), 523–567. MR2058383 (2005f:30052)
- [T3] X. Tolsa, *Painlevé's problem and the semiadditivity of analytic capacity*, Acta Math. 190 (1), (2003), 105–149. MR1982794 (2005c:30020)
- [T4] X. Tolsa, *A proof of the weak $(1,1)$ inequality for singular integrals with non doubling measures based on a Calderón-Zygmund decomposition*, Publ. Mat. 45 (2001), 163–174. MR1829582 (2002d:42019)

- [V] A. Volberg, *Calderón-Zygmund capacities and operators in nonhomogeneous spaces*, Regional Conference Series in Mathematics, No. 100, Amer. Math. Soc., 2003. MR2019058 (2005c:42015)

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