ON THE CHARACTERIZATION OF ALGEBRAICALLY INTEGRABLE PLANE FOLIATIONS

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ABSTRACT. We give a characterization theorem for non-degenerate plane foliations of degree different from 1 having a rational first integral. Moreover, we prove that the degree r of a non-degenerate foliation as above provides the minimum number, $r + 1$, of points in the projective plane through which pass infinitely many algebraic leaves of the foliation.

1. INTRODUCTION

At the end of the 19th century, Darboux [9], Poincaré [20, 21, 22], Painlevé [19] and Autonne [1] studied the problem that, nowadays, can be stated as follows: to characterize those algebraic foliations on the projective plane over the field of complex numbers (plane foliations, in the sequel) which have a rational first integral. Most recent contributions related to that problem are focused on solving the so-called Poincaré problem, which consists of bounding the degrees of the irreducible algebraic leaves of a foliation in terms of numerical data related to it [7, 5, 17, 11]. From this bound, one can try to get a rational first integral through an algebraic computation.

In this note, we only consider non-degenerate plane foliations $\mathcal{F}$. When $\mathcal{F}$ has a rational first integral R , the singular points of $\mathcal{F}$ are, on the one hand, the indeterminacies of the rational map $f_R : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ which R defines ($Ind(R)$) and, on the other hand, the singular points of the fibers of this rational map ($Sing(R)$). In the case considered in this paper, non-degenerate foliations, $Ind(R)$ ($Sing(R)$, respectively) coincides with the set of *non-reduced* (*reduced*, respectively) singularities of $\mathcal{F}$, denoted by $NRed(\mathcal{F})$ ($Red(\mathcal{F})$, respectively).

In Proposition 3.6, for a plane foliation $\mathcal{F}$ of degree different from 1, we consider the set of rational functions R such that $Ind(R) = NRed(\mathcal{F})$ and $Sing(R) = Red(\mathcal{F})$ and characterize when an element of that set is a rational first integral of $\mathcal{F}$ in terms of local conditions on the singularities of the closures of the fibers of f_R . These conditions only depend on the analytic type of the mentioned singularities. An important ingredient of the proof is the fact, proved by Campillo and Olivares

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in [6], that foliations of degree different from one are determined by the scheme associated with the indeterminacies of the polarity map of the foliation.

If we know Seidenberg's resolution of $\mathcal{F}$ [24], the ideas used to prove the above result can be strengthened to get our main result, Theorem 3.7, which is a characterization theorem for foliations of degree different from 1 having a rational first integral. The characterization conditions are two numerical ones (related to the eigenvalues of the non-reduced singularities) and other conditions that have to do with a certain sheaf on the surface obtained after Seidenberg's resolution process. Note that when $\mathcal{F}$ is of degree one and all its singularities are non-degenerate, with rational quotients of eigenvalues, then it always has a rational first integral.

In the previous results, an important object is the set $NRed(\mathcal{F})$. For this reason, we end this paper by proving in Theorem 3.13 that, under the assumption that $\mathcal{F}$ is algebraically integrable, the cardinality of $NRed(\mathcal{F})$ is greater than the degree of $\mathcal{F}$. Notice that this means that the degree of a non-degenerate foliation, r , provides the minimum number, $r + 1$, of points in $\mathbb{P}^2$ through which pass infinitely many leaves of the foliation.

2. PLANE FOLIATIONS AND ALGEBRAIC INTEGRABILITY

Along this note, $\mathbb{P}^2$ will denote the complex projective plane, $\mathcal{O}_{\mathbb{P}^2}$ ($\mathcal{O}_{\mathbb{P}^2}^{an}$, respectively) its structural sheaf as an algebraic variety over $\mathbb{C}$ (its sheaf of holomorphic functions, respectively). A *plane foliation* or a *foliation* on $\mathbb{P}^2$, $\mathcal{F}$, will be an (algebraic) foliation (with singularities and singular set of codimension two) of degree r ($r \geq 0$) on $\mathbb{P}^2$. That is, $\mathcal{F}$ is a non-trivial map of vector bundles $\mathcal{F}: H^{\otimes(-r+1)} \rightarrow T\mathbb{P}^2$ whose cokernel is torsion-free, where $T\mathbb{P}^2$ and H are the corresponding bundles to the tangent and hyperplane sheaves on $\mathbb{P}^2$ and $H^{\otimes n}$ means the n -fold tensor product of H if $n > 0$, the $(-n)$ -fold tensor product of $H^\vee$ if $n < 0$ and the trivial line bundle if $n = 0$.

A plane foliation $\mathcal{F}$ provides a tangent direction (or, equivalently, a projective line) to all points in $\mathbb{P}^2$ but finitely many called *singularities of $\mathcal{F}$* , a set denoted by $Sing(\mathcal{F})$. This allows us to define the so-called *polarity map* $\phi: \mathbb{P}^2 \cdots \rightarrow (\mathbb{P}^2)^\vee$, that is, the rational map which sends each point $p \in \mathbb{P}^2 \setminus Sing(\mathcal{F})$ to that point in $(\mathbb{P}^2)^\vee$ corresponding to the projective line that $\mathcal{F}$ associates with p . The scheme-theoretic fibers ϕ^*E of the lines E in $(\mathbb{P}^2)^\vee$ are degree $-(r+1)$ curves on $\mathbb{P}^2$ and so, since lines in $(\mathbb{P}^2)^\vee$ can be identified with points in $\mathbb{P}^2$, we can assign to each point $p \in \mathbb{P}^2$ a curve of degree $r+1$ called the *polar of p with respect to $\mathcal{F}$* . The indeterminacy ideal $\mathcal{J}$ of the polarity map gives a subscheme of $\mathbb{P}^2$, named the *singular subscheme of $\mathcal{F}$* , which determines the foliation [6].

$\mathcal{F}$, in analytic terms, can be defined, up to multiplication by a non-zero complex number, by a reduced homogeneous vector field (up to a suitable multiple of the radial vector field) or, equivalently, by a reduced homogeneous 1-form, $\Omega \equiv A dX + B dY + C dZ$, ($X : Y : Z$) being projective coordinates on $\mathbb{P}^2$, such that A, B, C are homogeneous polynomials of degree $(r+1)$ in X, Y, Z without common factors and satisfying the Euler condition $XA + YB + ZC = 0$. This last condition is what allows us to recover Ω from local data, which will be considered over the corresponding affine charts of $\mathbb{P}^2$. For example, Ω over the chart $Z \neq 0$ will be expressed by $\omega = a(x, y)dx + b(x, y)dy$, where $x = X/Z$, $y = Y/Z$, $a(x, y) = A(X, Y, 1)$, $b(x, y) = B(X, Y, 1)$. As a consequence, a point p in the above chart will be a singularity of $\mathcal{F}$ if, and only if, $a(p) = b(p) = 0$. Recall that the colength of the stalk $\mathcal{J}_p$ of the

above-defined indeterminacy ideal (which is the ideal $(a, b)\mathcal{O}_{\mathbb{P}^2, p}$ in the coordinates $\{x, y\}$) in the local ring $\mathcal{O}_{\mathbb{P}^2, p}$, $\mu_p(\mathcal{F})$, is called the *Milnor number* of $\mathcal{F}$ at the point p .

For fixed homogeneous coordinates on $\mathbb{P}^2$, a rational function R of $\mathbb{P}^2$ is defined by the quotient of two homogeneous polynomials in these coordinates without common components and with the same degree, F and G . Associated with R , we consider the linear pencil $\mathcal{P}(R)$ consisting of the projective curves with equations $\alpha F + \beta G = 0$, where (α, β) runs over $\mathbb{C}^2 \setminus \{0\}$. The curves in $\mathcal{P}(R)$ are the topological closures of the fibers of the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ that R defines. The base locus of $\mathcal{P}(R)$ is the set of indeterminacies of this map. As we have said, it will be denoted by $Ind(R)$. Also, we shall denote by $Sing(R)$ the set of singular points of the curves of the pencil $\mathcal{P}(R)$ which are not in $Ind(R)$.

By definition, a foliation $\mathcal{F}$ has a *rational (or meromorphic) first integral* if there exists a rational function S of $\mathbb{P}^2$ such that $dS \wedge \Omega = 0$. In this case, there exists a rational function $R = F/G$ such that all the rational first integrals of $\mathcal{F}$ have the form $\frac{P(F, G)}{Q(F, G)}$, where P, Q are homogeneous polynomials of the same degree; such a function R will be called a *primitive rational first integral* and it satisfies that its associated pencil of plane curves $\mathcal{P}(R)$ is irreducible (which means that it has irreducible general elements). Moreover, the leaves of $\mathcal{F}$ are the algebraic curves whose irreducible components are components of some curve of the mentioned pencil.

We only consider foliations $\mathcal{F}$, which we shall call *non-degenerate* ones, such that the Milnor numbers of all its singularities equal one. For each singularity p of $\mathcal{F}$, the pair (δ, ρ) of eigenvalues of the linear part of any local vector field defining $\mathcal{F}$ at p is named the (pair of) *eigenvalues associated with p* . Notice that $\delta \neq 0 \neq \rho$. We are interested in algebraic integrability; so we assume that δ and ρ are integers and that $\gcd(\delta, \rho) = 1$, after dividing by its greatest common divisor if it is necessary. When δ/ρ is not positive, then we shall say that $\mathcal{F}$ has a *reduced* singularity at p . We shall denote by $Red(\mathcal{F})$ ($NRed(\mathcal{F})$, respectively) the set of reduced (non-reduced, respectively) singularities.

A well-known fact, which we shall use systematically throughout the paper, is that if $\mathcal{F}$ is a non-degenerate foliation with rational first integral R , then $Ind(R) = NRed(\mathcal{F})$ and $Sing(R) = Red(\mathcal{F})$.

3. CHARACTERIZATION OF FOLIATIONS WITH A RATIONAL FIRST INTEGRAL

In this section, we study the problem of characterizing when a non-degenerate plane foliation has a rational first integral. First of all, we recall some definitions that we shall use in the sequel.

Definition 3.1. A polynomial $P \in \mathbb{C}[x, y]$ is *weighted-homogeneous* with weights (w_1, w_2) , where w_1, w_2 are fixed rational numbers, if it can be expressed as a linear combination of monomials $x^\alpha y^\beta$ for which $\frac{\alpha}{w_1} + \frac{\beta}{w_2} = 1$.

Definition 3.2. A plane curve singularity is *quasi-homogeneous* if it is analytically equivalent to a singularity whose local equation is defined by a weighted-homogeneous polynomial.

Let $\mathbb{Z}_+$ stand for the set of positive integers. Given $a, b, k \in \mathbb{Z}_+$ such that $\gcd(a, b) = 1$, we shall denote by $S(a, b, k)$ the topological singularity type of a plane curve singularity with equation $x^{ka} + y^{kb} = 0$, where $\{x, y\}$ are local coordinates.

We shall also consider the following set of topological singularity types:

$$\mathcal{S} = \{S(a, b, k) \mid a, b, k \in \mathbb{Z}_+ \text{ and } \gcd(a, b) = 1\}.$$

The topology of a quasi-homogeneous singularity determines the weights of all quasi-homogeneous polynomials defining the singularity [25]. Therefore, the quasi-homogeneous singularities with singularity type $S(a, b, k)$ correspond to the weights (ka, kb) .

Definition 3.3. We shall say that a plane curve singularity is *nodal* if its local equations have the form $g_1^n g_2^m = 0$, where m, n are positive integers and g_1, g_2 define analytically irreducible regular germs which are transversal.

Definition 3.4. Let $\mathcal{F}$ be a non-degenerate foliation on $\mathbb{P}^2$ and p a non-reduced singular point of $\mathcal{F}$ with associated eigenvalues (δ, ρ) . $\mathcal{F}$ is said to be *linearizable* at p if there exist analytical coordinates $\{u, v\}$ at p such that $\mathcal{F}_p$ is defined by the differential form $\rho v du - \delta u dv$.

Although the results of the next lemma are well known for the specialists, we include its proof for the reader's convenience.

Lemma 3.5. *Let $\mathcal{F}$ be a non-degenerate foliation on $\mathbb{P}^2$ admitting a rational first integral R . Then,*

- (a) $\mathcal{F}$ is linearizable at every non-reduced singularity.
- (b) A point $p \in \mathbb{P}^2$ is a reduced singularity of $\mathcal{F}$ if, and only if, it is not a base point of $\mathcal{P}(R)$ and the unique curve H of $\mathcal{P}(R)$ passing through p has a nodal singularity at this point.

Proof. For non-resonant singularities, (a) follows from Poincaré's linearization theorem (see, for instance, [4]) and otherwise, bearing in mind that the singularity is non-reduced, from the Poincaré-Dulac normal form theorem (see [4] and [3, pages 16 and 17]).

To prove (b), assume that p is a reduced singularity. Set $h := u \prod_{i=1}^s h_i^{e_i} = 0$, a local equation of the germ at p of H , where u is a unit and $h_i = 0$ an analytically irreducible germ for $i = 1, 2, \dots, s$. Then, $d(u^{-1}h) = w \prod_{i=1}^s h_i^{e_i-1}$, where $w = adx + bdy$ with $a, b \in \mathcal{O}_{\mathbb{P}^2, p}^{an}$ and $\gcd(a, b) = 1$, $\{x, y\}$ being local coordinates (see [13, Section 2.4]). Therefore, the multiplicity of $\mathcal{F}$ at p (i.e., the minimum of the m_p -adic orders of a and b , m_p being the maximal ideal of $\mathcal{O}_{\mathbb{P}^2, p}^{an}$) is $\nu = -1 + \sum_{i=1}^s \nu(h_i)$, where $\nu(h_i)$ denotes the multiplicity of the germ h_i . Since $\nu = 1$, one has that $\sum_{i=1}^s \nu(h_i) = 2$ and, in particular, either $s = 1$ or $s = 2$.

If $s = 1$, then h is $u(x^2 + f)^{e_1}$ for some local coordinates $\{x, y\}$, $f \in (x, y)^3$ and u a unit. This shows a local expression of $\mathcal{F}$ at p , which, clearly, cannot hold since $\mathcal{F}$ is non-degenerate. If $s = 2$, either $h = 0$ defines a nodal singularity, or $h = u(x + f_1)^{e_1}(x + f_2)^{e_2}$ with $f_1, f_2 \in (x, y)^2$, which again contradicts the fact that $\mathcal{F}$ is non-degenerate.

The converse is clear since, in certain coordinates $\{x, y\}$, a local equation for H is $h = 0$, with $h = (x + f)^{e_1}(y + g)^{e_2}$ and $f, g \in (x, y)^2 \subseteq \mathcal{O}_{\mathbb{P}^2, p}^{an}$, and hence, p is a reduced singularity of $\mathcal{F}$. $\square$

Recall that if C is a curve, p a point in C , $\eta : \overline{C} \rightarrow C$ the normalization map and γ the class map, that is, the composition on the canonical sheaf $\gamma : \Omega_C^1 \rightarrow \eta_* \Omega_{\overline{C}}^1 \rightarrow \omega_C$, the *Milnor (Tjurina, respectively) number* of C at p is $\mu(C, p) :=$

$l(\text{Cok}(\gamma \circ d)_p) (\tau(C, p) := l(\text{Ext}_{\mathcal{O}_{C,p}}^1(\Omega_{C,p}^1, \mathcal{O}_{C,p}))$, respectively), where $d : \mathcal{O}_C \rightarrow \Omega_C^1$ is the universal derivation and l the length.

Proposition 3.6. *Let $\mathcal{F}$ be a non-degenerate plane algebraic foliation of degree different from 1 and let R be a rational function such that $\text{Ind}(R) = N\text{Red}(\mathcal{F})$ and $\text{Sing}(R) = \text{Red}(\mathcal{F})$. Then, R is a rational first integral of $\mathcal{F}$ if and only if the following conditions are satisfied:*

1. *The singularity at every point in $\text{Sing}(R)$ of the curve of $\mathcal{P}(R)$ passing through it is nodal.*
2. *There exist two curves of the pencil $\mathcal{P}(R)$ (say, with equations $H_1 = 0$ and $H_2 = 0$) such that, for each $p \in \text{Ind}(R)$, the germs at p of the curves with equations $H_1 = 0$ and $H_2 = 0$ are equisingular and the one of $H_1 H_2 = 0$ satisfies the following properties:*
 - (a) *it is a reduced germ whose topological singularity type belongs to $\mathcal{S}$,*
 - (b) *its associated Milnor and Tjurina numbers coincide.*

Proof. Assume that R is a rational first integral for $\mathcal{F}$. Condition 1 is satisfied by Part (b) of Lemma 3.5.

In order to prove 2, pick two distinct general elements (with equations $H_i = 0$, $i = 1, 2$) of the pencil $\mathcal{P}(R)$ such that the local equation at each point $p \in \text{Ind}(R)$ of the curve C defined by $H_1 H_2 = 0$ is reduced. Consider a point $p \in \text{Ind}(R)$. By Part (a) of Lemma 3.5, there exist local coordinates $u, v \in \mathcal{O}_{\mathbb{P}^2, p}^{\text{an}}$ such that the foliation $\mathcal{F}$ is defined locally at p by the differential 1-form $\rho v du - \delta u dv$, where (δ, ρ) is the pair of eigenvalues associated with p . If h_1 (h_2 , respectively) in $\mathcal{O}_{\mathbb{P}^2, p}^{\text{an}}$ defines the germ provided by H_1 (H_2 , respectively), since $H_1 = 0$ and $H_2 = 0$ are general elements of $\mathcal{P}(R)$ one has that h_1 and h_2 give equisingular reduced germs (and, obviously, without common factors). Moreover, since $\frac{u\rho}{v\delta}$ is a primitive local rational first integral of the germ of foliation $\mathcal{F}_p$, there exist a unit z and homogeneous polynomials in two variables P and Q of the same degree and without common factors such that $h_1 = zP(u^\rho, v^\delta)$ and $h_2 = zQ(u^\rho, v^\delta)$ (see, for instance, [13, Section 2.9]). $z^{-2}h_1 h_2$ factorizes into a product of polynomials of the type $L(u^\rho, v^\delta)$, L being a homogeneous polynomial of degree 1. Hence, it is clear that Condition 2(a) is satisfied. Finally, notice that $h_1 h_2$ defines a quasi-homogeneous singularity and, then, Condition 2(b) follows from the second “Satz” in [23, page 123].

Conversely, consider the foliation $\mathcal{H}$ provided by the derivation of the rational function R . Taking into account the hypotheses of the statement, the equality $\text{Sing}(\mathcal{H}) = \text{Sing}(\mathcal{F})$ clearly holds. Next, we shall prove the equality $\mathcal{F} = \mathcal{H}$ and, hence, the result. By applying [6, Th. 3.5], it is enough to show that $\mathcal{H}$ is a non-degenerate foliation.

Let H_1 and H_2 be as in Condition 2 and take an arbitrary point $p \in \text{Ind}(R)$. Consider the curve C of equation $H_1 H_2 = 0$. By Condition 2(b) and the second “Satz” in [23, page 123], C has a quasi-homogeneous singularity at p . Therefore, there exist local analytic coordinates $\{u, v\}$ and a quasi-homogeneous polynomial P such that $P(u, v) = 0$ defines this singularity. Moreover, applying Condition 2(a), its singularity type belongs to $\mathcal{S}$ and there exist three positive integers ρ, δ and k such that $\text{gcd}(\rho, \delta) = 1$ and $(k\delta, k\rho)$ is the pair of weights corresponding to $P(u, v)$. It is straightforward to see that $P(u, v) = Q(u^\rho, v^\delta)$, where Q is a homogeneous polynomial in two variables of degree k . Therefore, $P(u, v) = \prod_{i=1}^k L_i(u^\rho, v^\delta)$, L_i

being distinct homogeneous polynomials of degree 1. Now, k is even and $P(u, v) = Q_1(u^\rho, v^\delta)Q_2(u^\rho, v^\delta)$, where Q_1 and Q_2 are homogeneous polynomials of degree $k/2$ and $Q_1(u^\rho, v^\delta) = 0$ ($Q_2(u^\rho, v^\delta) = 0$, respectively) is an equation of the germ at p of the curve $H_1 = 0$ ($H_2 = 0$, respectively) in the local coordinates u and v (notice that these facts are true because the mentioned germs are equisingular). Taking into account that the derivation of $Q_1(u^\rho, v^\delta)/Q_2(u^\rho, v^\delta)$ defines the same local foliation as the germ $\mathcal{H}_p$, we deduce that u^ρ/v^δ is a primitive local first integral of $\mathcal{H}_p$ and, therefore, $\mu_p(\mathcal{H}) = 1$. Finally, Condition 1 ensures that $\mu_p(\mathcal{H}) = 1$ for all $p \in \text{Sing}(R)$, which concludes the proof. $\square$

Let $\mathcal{F}$ be a non-degenerate plane foliation. By Seidenberg's resolution process [24], there exists a sequence of point blow-ups whose composition, $\pi : Y \rightarrow \mathbb{P}^2$, satisfies that the foliation on Y $\pi^*(\mathcal{F})$ has only reduced singularities (see [3, pages 12 and 13]). Such a composition morphism is called a *minimal resolution* of $\mathcal{F}$ if it is minimal with respect to the number of involved blow-ups. We shall denote by $\pi_{\mathcal{F}} : X_{\mathcal{F}} \rightarrow \mathbb{P}^2$ a minimal resolution of singularities of $\mathcal{F}$ and by $\mathcal{C}_{\mathcal{F}}$ the set of centers of the blow-ups that are involved in it (notice that this set is, essentially, unique). $\mathcal{C}_{\mathcal{F}}$ is a disjoint union $\bigcup \mathcal{C}_{\mathcal{F},p}$, p running over $N\text{Red}(\mathcal{F})$, where $\mathcal{C}_{\mathcal{F},p} := \{q \in \mathcal{C}_{\mathcal{F}} \mid q \geq p\}$ and $\geq$ is the partial ordering on $\mathcal{C}_{\mathcal{F}}$ given by $q \geq r$ if and only if q is infinitely near to r [8, 3.3].

Now we shall define a family of sheaves on $X_{\mathcal{F}}$ associated with this minimal resolution. For each $p \in N\text{Red}(\mathcal{F})$, we distinguish two cases:

- $\mathcal{F}$ is linearizable at p . Then there exist local coordinates $u, v \in \mathcal{O}_{\mathbb{P}^2, p}^{an}$ such that $\mathcal{F}_p$ is defined by the differential 1-form $\rho v du - \delta u dv$, (δ, ρ) being the pair of eigenvalues associated with p . Let J_p be the ideal of $\mathcal{O}_{\mathbb{P}^2, p}^{an}$ generated by u^ρ and v^δ . Elementary computations show that the infinitely near points needed to eliminate the base points of the ideal J_p (see [8, 7.2]) coincide with those in $\mathcal{C}_{\mathcal{F},p}$. As a consequence, $\mathcal{C}_{\mathcal{F},p}$ is totally ordered by the relation $\geq$.
- $\mathcal{F}$ is not linearizable at p . Then there exist local analytic coordinates u, v providing the Poincaré-Dulac normal form of $\mathcal{F}_p$, which is $(nu + v^n)du - vdv$ for a certain positive integer n . Using this fact, it is straightforward to see that, as in the previous case, the set $\mathcal{C}_{\mathcal{F},p}$ is totally ordered by the relation $\geq$. Observe that this case does not occur when $\mathcal{F}$ admits a rational first integral.

Each set $\mathcal{C}_{\mathcal{F},p}$ defines, then, a valuation of the fraction field of $\mathcal{O}_{\mathbb{P}^2, p}$ and a simple complete primary ideal I_p of that local ring [26, pages 389 to 391]; in fact, when $\mathcal{F}$ is linearizable at p , $I_p \mathcal{O}_{\mathbb{P}^2, p}^{an}$ coincides with the integral closure of J_p in $\mathcal{O}_{\mathbb{P}^2, p}^{an}$. Set $D_p(\mathcal{F})$ to be the unique exceptionally supported divisor on $X_{\mathcal{F}}$ such that $I_p \mathcal{O}_{X_{\mathcal{F}}} = \mathcal{O}_{X_{\mathcal{F}}}(-D_p(\mathcal{F}))$ and, for each positive integer d and each map $\mathbf{k} : N\text{Red}(\mathcal{F}) \rightarrow \mathbb{Z}_+$, denote by $\mathcal{L}(\mathcal{F}, d, \mathbf{k})$ the sheaf $\pi_{\mathcal{F}}^* \mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{O}_{X_{\mathcal{F}}}(-\sum \mathbf{k}(p) D_p(\mathcal{F}))$, where the sum is taken over the set of non-reduced singularities of $\mathcal{F}$.

We shall use the above notation in the statement and proof of the next result, which characterizes those non-degenerate plane foliations of degree $r \neq 1$ admitting a rational first integral.

Theorem 3.7. *Let $\mathcal{F}$ be a non-degenerate foliation on $\mathbb{P}^2$ of degree $r \neq 1$ and let $\{(\delta_p, \rho_p)\}_{p \in N\text{Red}(\mathcal{F})}$ be the set of pairs of eigenvalues associated with the non-reduced singularities. Then, $\mathcal{F}$ has a rational first integral if and only if there exist a positive integer d and a map $\mathbf{k} : N\text{Red}(\mathcal{F}) \rightarrow \mathbb{Z}_+$ such that:*

- (a) $d^2 = \sum_{p \in N\text{Red}(\mathcal{F})} \mathbf{k}(p)^2 \rho_p \delta_p$.
- (b) $d(r+2) = \sum_{p \in N\text{Red}(\mathcal{F})} \mathbf{k}(p)(\rho_p + \delta_p)$.
- (c) $h^0(X_{\mathcal{F}}, \mathcal{L}(\mathcal{F}, d, \mathbf{k})) = 2$.
- (d) *There exist two curves of $\pi_{\mathcal{F}*}|\mathcal{L}(\mathcal{F}, d, \mathbf{k})|$ (say, with equations $H_1 = 0$ and $H_2 = 0$) such that, for each $p \in N\text{Red}(\mathcal{F})$, the germs at p of the curves with equations $H_1 = 0$ and $H_2 = 0$ are equisingular and the one of $H_1 H_2 = 0$ satisfies the following conditions: it is reduced, its associated Milnor and Tjurina numbers coincide and its topological singularity type is $S(\rho_p, \delta_p, 2\mathbf{k}(p))$.*
- (e) *Each point in $\text{Red}(\mathcal{F})$ is a singular point of some curve in $\pi_{\mathcal{F}*}|\mathcal{L}(\mathcal{F}, d, \mathbf{k})|$.*

Moreover, in this case, a primitive rational first integral of $\mathcal{F}$ is F/G , where F and G are two homogeneous polynomials defining any two different curves in $\pi_{\mathcal{F}*}|\mathcal{L}(\mathcal{F}, d, \mathbf{k})|$.

Proof. Assume first that $\mathcal{F}$ has a rational first integral and let $R = F/G$ be a primitive one. Let d denote the degree of F and G . The morphism $\pi_{\mathcal{F}}$ defined above is the one eliminating the base points of the pencil $\mathcal{P}(R)$ or, equivalently, eliminating the indeterminacies of the rational map $\mathbb{P}^2 \cdots \rightarrow \mathbb{P}^1$ provided by R [2, II.7]. Moreover, for each $p \in N\text{Red}(\mathcal{F})$, if k_p denotes the number of branches through p of a general curve C of the pencil $\mathcal{P}(R)$, the multiplicities m_q of the strict transforms of C at the points $q \in \mathcal{C}_{\mathcal{F},p}$ coincide with the ones of the strict transforms of any general element of the ideal $I_p^{k_p}$ (since $I_p \mathcal{O}_{\mathbb{P}^2,p}^{an}$ is the integral closure of J_p). Thus, taking the function $\mathbf{k} : N\text{Red}(\mathcal{F}) \rightarrow \mathbb{Z}_+$ such that $\mathbf{k}(p) = k_p$ for all $p \in N\text{Red}(\mathcal{F})$, the following equality of sheaves holds: $\mathcal{L}(\mathcal{F}, d, \mathbf{k}) = \pi_{\mathcal{F}}^* \mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{O}_{X_{\mathcal{F}}}(\sum m_q E_q^*)$, where the sum is taken over the points $q \in \mathcal{C}_{\mathcal{F}}$ and E_q^* denotes the pull-back on $X_{\mathcal{F}}$ of the exceptional divisor appearing in the blow-up centered at q . Since $\mathcal{P}(R)$ coincides with the pencil $\mathcal{P}_{\mathcal{F}}$ given in [12, Lem. 1], we can apply this result to deduce that $\mathcal{P}(R) = \pi_{\mathcal{F}*}|\mathcal{L}(\mathcal{F}, d, \mathbf{k})|$; hence (c) is satisfied. Clauses (d) and (e) follow from Proposition 3.6, (a) is a consequence of applying Bézout's Theorem to two general curves in $\mathcal{P}(R)$ and (b) follows from a result in [21].

Conversely, assume the existence of a positive integer d , a map $\mathbf{k} : N\text{Red}(\mathcal{F}) \rightarrow \mathbb{Z}_+$ and curves $H_1 = 0$ and $H_2 = 0$ as in the statement. Set $R = H_1/H_2$ and let $\mathcal{H}$ be the foliation provided by the derivation of the rational function R . Using Condition (d) and similar arguments to those given in the last part of the proof of Proposition 3.6, one can deduce that, for each $p \in N\text{Red}(\mathcal{F})$, the Milnor number $\mu_p(\mathcal{H})$ equals 1, the germs at p of $H_1 = 0$ and $H_2 = 0$ have singularity type $S(\rho_p, \delta_p, \mathbf{k}(p))$ and, moreover, they have the same minimal resolution of singularities. From the last assertion one has also that $i_p(H_1 = 0, H_2 = 0) = \mathbf{k}(p)^2 \rho_p \delta_p$, where i_p stands for the intersection multiplicity at p . This fact, Bézout's Theorem and Condition (a) show that the base points on $\mathbb{P}^2$ of the pencil $\mathcal{P}(R) = \pi_{\mathcal{F}*}|\mathcal{L}(\mathcal{F}, d, \mathbf{k})|$ (the equality holds from (c)) are exactly those in $N\text{Red}(\mathcal{F})$. Hence, $N\text{Red}(\mathcal{H}) = N\text{Red}(\mathcal{F})$.

It follows from the above paragraph that the pair of eigenvalues attached to each non-reduced singularity p of $\mathcal{H}$ is (δ_p, ρ_p) and the number of branches through p

of a general curve of the pencil $\mathcal{P}(R)$ is $\mathbf{k}(p)$. Then, if r' denotes the degree of the foliation $\mathcal{H}$, the equality $d(r' + 2) = \sum_{p \in N\text{Red}(\mathcal{F})} \mathbf{k}(p)(\rho_p + \delta_p)$ holds [21]. By Condition (b), we deduce that $r = r'$. Therefore,

$$(3.1) \quad \sum_{p \in \text{Sing}(\mathcal{H})} \mu_p(\mathcal{H}) = r^2 + r + 1$$

and so, $\#\text{Red}(\mathcal{H}) \leq \#\text{Red}(\mathcal{F})$, where $\#$ stands for the cardinality. But, by Condition (e) and Part (b) of Lemma 3.5, each point in $\text{Red}(\mathcal{F})$ is also a singular point of $\mathcal{H}$. Hence one gets that $\text{Ind}(R) = N\text{Red}(\mathcal{H}) = N\text{Red}(\mathcal{F})$, $\text{Sing}(R) = \text{Red}(\mathcal{H}) = \text{Red}(\mathcal{F})$ and, taking into account (3.1), $\mathcal{H}$ is a non-degenerate foliation. Thus, by Proposition 3.6, R is a rational first integral of $\mathcal{F}$.

Finally, since $h^0(X, \mathcal{L}(\mathcal{F}, d, \mathbf{k})) = 2$ we deduce that R is a primitive rational first integral of $\mathcal{F}$ by the proof of [12, Th. 2 (a)]. $\square$

Remark 3.8. Note that, if Condition (d) in the statement of the above result is satisfied for two curves of $\pi_{\mathcal{F}*}|\mathcal{L}(\mathcal{F}, d, \mathbf{k})|$, then it holds for any pair of general elements of this linear system.

Example 3.9. The pair of Diophantine equations of a fixed non-degenerate foliation $\mathcal{F}$ given by clauses (a) and (b) of Theorem 3.7, where the unknowns are d and $\{\mathbf{k}(p)\}_{p \in N\text{Red}(\mathcal{F})}$, can have infinitely many solutions. Indeed, foliations in the set $\{\mathcal{F}_\alpha^r\}_{\alpha \in E}$, $2 \leq r \leq 4$, given in [17] (E is a countable and dense set of parameters in the set of complex numbers), have singularities of fixed analytic type [17, Def. 1], rational first integral and their degrees can be chosen arbitrarily large. Therefore we get the Diophantine equations

$$\begin{aligned} d^2 = 2 \sum_{i=1}^2 k_i^2 + 6 \sum_{i=3}^5 k_i^2 & \parallel d^2 = \sum_{i=1}^3 k_i^2 + 2 \sum_{i=4}^8 k_i^2 & \parallel d^2 = \sum_{i=1}^{12} k_i^2 \\ 4d = 3 \sum_{i=1}^2 k_i + 5 \sum_{i=3}^5 k_i & \parallel 5d = 2 \sum_{i=1}^3 k_i + 3 \sum_{i=4}^8 k_i & \parallel 3d = \sum_{i=1}^{12} k_i \end{aligned}$$

associated, respectively, to the cases $r = 2, 3, 4$, where for simplicity we have set $\mathbf{k}(p_i) = k_i$, which have infinitely many solutions (attached to the corresponding first integrals).

To end this example, we examine clauses in Theorem 3.7 for the foliation $\mathcal{F}_0^4$ that is given by the 1-form $AdX + BdY + CdZ$, where $A = (Y^3 - Z^3)YZ$, $B = (Z^3 - X^3)XZ$ and $C = (X^3 - Y^3)XY$. It has 12 points in $N\text{Red}(\mathcal{F})$ and 9 points in $\text{Red}(\mathcal{F})$. As $\mathcal{F}_0^4$ has rational first integral, clauses (a) to (e) in Theorem 3.7 must hold. In fact, $d = 6$, $k_1 = k_2 = k_3 = 3$ and $k_i = 1$, $4 \leq i \leq 12$ are solutions for the above last Diophantine equations. Forms $H_1 = 3X^3Y^3 - X^3Z^3 - 2Y^3Z^3$ and $H_2 = 2X^3Y^3 - X^3Z^3 - Y^3Z^3$ span the vector space $H^0(X_{\mathcal{F}}, \mathcal{L}(\mathcal{F}, 6, \mathbf{k}))$ for $\mathbf{k}$ defined as above and a suitable ordering of the points in $N\text{Red}(\mathcal{F})$. The equation of the germ of the curve defined by H_1 (H_2 , respectively) at the point $p = (0 : 0 : 1) \in N\text{Red}(\mathcal{F})$ (whose image by $\mathbf{k}$ equals 3) in suitable local coordinates is given by $h_1 = x^3 + 2y^3 - 3x^3y^3$ ($h_2 = x^3 + y^3 - 2x^3y^3$, respectively). The germ defined by h_1h_2 factorizes into a product of 6 smooth transversal analytically irreducible germs passing through p (having, thus, singularity type $S(1, 1, 6)$) and its Milnor and Tjurina numbers are both equal to 25 (we have used SINGULAR [14] to do the computation); obviously the germs defined by h_1 and h_2 are equisingular. The same situation happens for the two remaining points of $N\text{Red}(\mathcal{F})$ whose image by $\mathbf{k}$ is 3. The germs of H_1 and H_2 at any point $q \in N\text{Red}(\mathcal{F})$ such that $\mathbf{k}(q) = 1$ are analytically irreducible, smooth and transversal. This shows that the conditions given in (d) are satisfied for H_1 and H_2 . Finally (e) also holds because the points

$(j^r : 0 : 1)$ $((1 : j^r : 0),$ respectively) $((0 : j^r : 1),$ respectively), where $r \in \{0, 1, 2\}$ and $j := e^{2\pi i/3}$, are singular points of $Y^3(X^3 - Z^3)$ ($Z^3(Y^3 - X^3)$, respectively) ($X^3(Y^3 - Z^3)$, respectively).

Remark 3.10. The unique non-degenerate foliations of degree one are defined, up to projective isomorphism, by the differential 1-forms $aYZ dX + bXZ dY - (a + b)XY dZ$, where a and b are positive integers, $(X : Y : Z)$ being projective coordinates on the complex projective plane (see [6, Sect. 4]). All of them are algebraically integrable, since $X^a Y^b / Z^{a+b}$ is a rational first integral.

Remark 3.11. Theorem 3.7 allows us to decide whether a non-degenerate foliation $\mathcal{F}$ of degree $r \neq 1$, defined by a projective differential 1-form Ω , does or does not have a rational first integral $\frac{F}{G}$ such that the degree of F and G is less than a fixed value t (and to compute it, if it exists). The procedure should be as follows: firstly, perform a minimal resolution of $\mathcal{F}$ and compute the divisors $D_p(\mathcal{F})$ for all $p \in NRed(\mathcal{F})$. Secondly, consider the integers $d < t$ and the finite set of maps $\mathbf{k} : NRed(\mathcal{F}) \rightarrow \mathbb{Z}_+$ satisfying the conditions (a), (b) and (c) of Theorem 3.7. Then, $\mathcal{F}$ has a rational first integral with the above condition if and only if $d(\frac{F}{G}) \wedge \Omega = 0$ for some basis $\{F, G\}$ of a linear system of the type $\pi_{\mathcal{F}*}|\mathcal{L}(\mathcal{F}, d, \mathbf{k})|$.

Example 3.12. Take projective coordinates $(X : Y : Z)$ on the complex projective plane and consider the non-degenerate foliation $\mathcal{F}$ defined by the projective differential 1-form $\Omega = AdX + BdY + CdZ$, where

$$A = Z(2X^3 + 2Y^3 - YZ^2), \quad B = Z(XZ^2 - 6XY^2) \quad \text{and} \quad C = 4XY^3 - 2X^4.$$

It has 6 non-reduced singularities $p_1, p_2, \dots, p_6$ and the morphism $\pi_{\mathcal{F}} : X_{\mathcal{F}} \rightarrow \mathbb{P}^2$ is the composition of the blow-ups with centers $p_1, p'_1, p_2, p'_2, p_3, p'_3, p_4, p_5$ and p_6 , where p'_i is infinitely near to p_i for $1 \leq i \leq 3$. The pair of eigenvalues associated with p_i , $1 \leq i \leq 3$ ($4 \leq i \leq 6$, respectively), is $(1, 2)$ $((1, 1),$ respectively). Using results of [18] we see that the divisor $D_{p_i}(\mathcal{F})$ is $E_{p_i}^* + E_{p'_i}^*$ ($E_{p_i}^*$, respectively) for $1 \leq i \leq 3$ ($4 \leq i \leq 6$, respectively), where E_q^* denotes the total transform on $X_{\mathcal{F}}$ of the exceptional divisor appearing in the blow-up centered at q . The integer $d = 3$ and the map $\mathbf{k} \equiv 1$ satisfy the conditions (a), (b) and (c) of Theorem 3.7 and the linear system $\pi_{\mathcal{F}*}|\mathcal{L}(\mathcal{F}, 3, \mathbf{k})|$ is spanned by the curves with equations $F := X^3 - 2Y^3 + YZ^2 = 0$ and $G := XZ^2 = 0$. After checking it, we assert that F/G is a rational first integral of $\mathcal{F}$.

When a non-degenerate plane foliation $\mathcal{F}$ has a rational first integral, the cardinality of the set $NRed(\mathcal{F})$ of non-reduced singularities is not arbitrary, as the next result shows. Notice that this gives an easy to check criterion to decide that a foliation has no rational first integral.

Theorem 3.13. *Let $\mathcal{F}$ be a degree r non-degenerate foliation on $\mathbb{P}^2$ which has a rational first integral. Then*

$$r + 1 \leq n,$$

where n is the number of non-reduced singularities of $\mathcal{F}$.

Proof. Let $\{(\delta_p, \rho_p)\}_{p \in NRed(\mathcal{F})}$ be the set of pairs of eigenvalues associated with the non-reduced singularities of the foliation $\mathcal{F}$. Consider a primitive rational first integral of $\mathcal{F}$, R , and let C be a general curve of the pencil $\mathcal{P}(R)$. For each $p \in NRed(\mathcal{F})$, let k_p be the number of branches of the singularity of C at p . Assume that C has degree d . Bertini's Theorem proves that the unique singularities of C are

the non-reduced singularities of $\mathcal{F}$ and similar arguments to some used in the proof of Proposition 3.6 show that each non-reduced singularity p is quasi-homogeneous with associated weights $(k_p\rho_p, k_p\delta_p)$. So, in suitable coordinates $\{x, y\}$ in $\mathcal{O}_{\mathbb{P}^2, p}^{an}$, the germ of C at p is given by $H(x^{\rho_p}, y^{\delta_p})$, H being a homogeneous polynomial in two indeterminates of degree k_p . So the Milnor and Tjurina numbers coincide at each singularity of C .

Let μ denote the sum of Milnor numbers of the singularities of C . Then,

$$(3.2) \quad (d-1)(d-r-1) \leq \mu$$

by a result of du Plessis and Wall [10]. C is a general curve of the pencil $\mathcal{P}(R)$ and, so, the Milnor-Jung formula states that

$$\mu_p = 2\bar{\delta}_p - k_p + 1$$

for all $p \in NRed(\mathcal{F})$, where $\mu_p = \mu(C, p)$ ($\bar{\delta}_p$, respectively) is the Milnor number (genus diminution or δ -invariant, respectively) of C at the singularity p . Hence,

$$\mu = \sum_{p \in NRed(\mathcal{F})} (2\bar{\delta}_p - k_p) + n.$$

On the one hand, by the genus formulae, one gets:

$$2g - 2 = C \cdot C + C \cdot K_{\mathbb{P}^2} - \sum_{p \in NRed(\mathcal{F})} k_p^2 \rho_p \delta_p + \sum_{p \in NRed(\mathcal{F})} k_p(\rho_p + \delta_p - 1),$$

where g is the geometrical genus of C and $K_{\mathbb{P}^2}$ a canonical divisor of the projective plane (see [15, pages 279 and 280]). On the other hand, it follows that

$$2g = (d-1)(d-2) - 2 \sum_{p \in NRed(\mathcal{F})} \bar{\delta}_p$$

(see, for instance, [16, Sect. IV, Ex. 1.8]). Then, we obtain the following equality:

$$\mu = \sum_{p \in NRed(\mathcal{F})} [k_p^2 \rho_p \delta_p - k_p(\rho_p + \delta_p)] + n.$$

Finally,

$$\mu = d^2 - (r+2)d + n,$$

since $d^2 = \sum_{p \in NRed(\mathcal{F})} k_p^2 \rho_p \delta_p$ and $\sum_{p \in NRed(\mathcal{F})} k_p(\rho_p + \delta_p) = (r+2)d$ [21], which concludes the proof after replacing in (3.2) the value of μ . $\square$

Example 3.14. For each $a \in \mathbb{C} \setminus \{0, 1, -1\}$, consider the non-degenerate plane foliation $\mathcal{F}_a$ defined by the projective 1-form $\Omega = AdX + BdY + CdZ$, $(X : Y : Z)$ being projective coordinates in the complex projective plane, where:

$$\begin{aligned} A &= Z(aXZ - Y^2 + Z^2), & B &= Z(X^2 - Z^2) & \text{and} \\ C &= XY^2 - aX^2Z - XZ^2 - X^2Y + YZ^2. \end{aligned}$$

The cardinality of $NRed(\mathcal{F}_a)$ is 1 or 2, depending on the value of the parameter a (see [12, Example 5]). Since the degree of the foliation is 2, Theorem 3.13 allows us to discard the existence of a rational first integral for any foliation in the family $\{\mathcal{F}_a\}_{a \in \mathbb{C} \setminus \{0, 1, -1\}}$.

Finally, we give an example which shows that the lower bound of Theorem 3.13 turns out to be sharp.

Example 3.15. Consider any foliation $\mathcal{F}$ of the family given in Remark 3.10. $\mathcal{F}$ has degree 1 and it has three singular points: $(0 : 0 : 1)$, $(0 : 1 : 0)$ and $(1 : 0 : 0)$. A rational first integral of $\mathcal{F}$ is $\frac{X^a Y^b}{Z^{a+b}}$ (for suitable a and b) and, so, the non-reduced singularities are $(0 : 1 : 0)$ and $(1 : 0 : 0)$.

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