GENERALIZED MOSER LEMMA

MATHIEU STIÉNON

ABSTRACT. We show how the classical Moser lemma from symplectic geometry extends to generalized complex structures (GCS) on arbitrary Courant algebroids. For this, we extend the notion of a Lie derivative to sections of the tensor bundle $(\otimes^i E) \otimes (\otimes^j E^*)$ with respect to sections of the Courant algebroid E using the Dorfman bracket. We then give a cohomological interpretation of the existence of one-parameter families of GCS on E and of flows of automorphisms of E identifying all GCS of such a family. In the particular case of symplectic manifolds, we recover the results of Moser. Finally, we give a criterion to detect the local triviality of arbitrary GCS which generalizes the Darboux-Weinstein theorem.

1. INTRODUCTION

The classical Moser lemma for symplectic manifolds [24] describes a cohomological condition for two symplectic structures to be equivalent. It can be stated as follows. For two symplectic forms Ω_0 and Ω_1 on a compact symplectic manifold M , if there exists a smooth one-parameter family ω_t of symplectic forms on M , all with the same periods and such that $\Omega_0 = \omega_0$ and $\Omega_1 = \omega_1$, then there is a global diffeomorphism φ of M , diffeotopic to the identity and such that $\Omega_1 = \varphi^* \Omega_0$.

Recently there has been increasing interest in generalized complex structures [2, 4, 5, 7, 8, 9, 10, 13, 14, 15, 16], which comprise both symplectic and complex structures as special cases. It is natural to ask if Moser's lemma extends to generalized complex geometry. The aim of the present paper is to give an affirmative answer to this question. While up to now, most of the research on generalized complex structures focused on exact Courant algebroids $TM \oplus T^*M$ [28], in this paper, we work on generic Courant algebroids. A generalized complex structure on a Courant algebroid E is a bundle map $J : E \rightarrow E$ satisfying $J^2 = -1$, which is orthogonal with respect to the symmetric pairing and whose Nijenhuis torsion vanishes [3].

As is well known, the proof of the classical Moser lemma involves the Lie derivative of symplectic forms with respect to a vector field. Here, as a first step, we introduce a concept of a Lie derivative of generalized complex structures with respect to sections of a Courant algebroid. Indeed, we introduce the Lie derivative

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of sections of the tensor bundle $(\otimes^i E) \otimes (\otimes^j E^*)$ with respect to sections of the Courant algebroid E . We hope this construction will be of independent interest in the future. Such a Lie derivative can be defined exactly as in the classical case. Namely, via the Dorfman bracket, one can think of $\Gamma(E)$ as a subset of $\mathbf{aut}(E)$, the infinitesimal automorphisms of the Courant algebroid E . Thus any section of E generates a local flow of automorphisms of the Courant algebroid E . We define the Lie derivative of any section of $(\otimes^i E) \otimes (\otimes^j E^*)$ as the time-derivative at $t = 0$ of the pull-back of this section by the flow. For any $X, Y \in \Gamma(E)$, the Lie derivative $\mathcal{L}_X Y$ is simply the Dorfman bracket $X \circ Y$. The Lie derivative of a bundle map $J : E \rightarrow E$, seen as a section of $E^* \otimes E$, is given by $(\mathcal{L}_X J)(Y) = \mathcal{L}_X J(Y) - J(\mathcal{L}_X Y)$, $\forall X, Y \in \Gamma(E)$. This formula reduces exactly to the usual Lie derivative of the symplectic form when J corresponds to a symplectic structure and X is a vector field.

With the help of the Lie derivative, we are able to translate the infinitesimal isomorphism condition for J_t ,

$$\dot{J}_t + \mathcal{L}_{x_t} J_t = 0, \quad \forall t \text{ with } x_t \in \Gamma(E),$$

as the exactness, at every time t , of a family $t \rightarrow \dot{\omega}_t$ of Lie algebroid 2-cocycles relative to the one-parameter family of Lie algebroids $t \mapsto L_t$, where L_t is the $+i$ -eigenbundle of J_t . Hence we recover exactly the same situation as in the classical Moser lemma context [24].

As first examples, we consider symplectic and complex structures on a Lie algebroid A . The Courant algebroid E involved is the double $A \oplus A^*$ of A . When A is the tangent bundle Lie algebroid TM , we recover the usual Moser lemma for symplectic manifolds and the classical result of Kodaira [19], respectively. Other examples of generalized complex structures are given by Hamiltonian operators in the sense of Liu-Weinstein-Xu [22]. We describe conditions when such operators induce isomorphic generalized complex structures. Holomorphic Poisson structures are a special case.

As another application, we give a Darboux-Weinstein style theorem for generalized complex structures. More precisely, we describe local cohomological conditions which guarantee the local triviality of a generalized complex structure on the standard Courant algebroid. This cohomological condition is always satisfied for symplectic manifolds. On the other hand, for an integrable complex structure, this condition should be related to Kohn's proof [20] of the Newlander-Nirenberg theorem [26].

2. LIE DERIVATIVES

The symmetry groups of generalized Courant algebroids were first studied by Roytenberg from the point of view of supergeometry [27]. In this section, we recall some basic ingredients which are needed later on. Here we choose the more conventional description of symmetries in terms of linear vector fields on Courant algebroids, which better fits our needs. See also [17, 18, 5] for discussions on a similar topic.

Definition 2.1 ([22]). A Courant algebroid is a triple consisting of a vector bundle $E \rightarrow M$ equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, a skew-symmetric bracket $[\![\cdot, \cdot]\!]$ on $\Gamma(E)$, and a smooth bundle map $E \xrightarrow{\rho} TM$ called the anchor. These induce a natural differential operator $\mathcal{D} : C^\infty(M) \rightarrow \Gamma(E)$ defined

by

$$\langle \mathcal{D}f, a \rangle = \frac{1}{2} \rho(a) f$$

for all $f \in C^\infty(M)$ and $a \in \Gamma(E)$.

These structures must be compatible in the following sense: $\forall a, b, c \in \Gamma(E)$ and $\forall f, g \in C^\infty(M)$,

$$\begin{aligned} \rho(\llbracket a, b \rrbracket) &= [\rho(a), \rho(b)], \\ \llbracket \llbracket a, b \rrbracket, c \rrbracket + \llbracket \llbracket b, c \rrbracket, a \rrbracket + \llbracket \llbracket c, a \rrbracket, b \rrbracket &= \frac{1}{3} \mathcal{D}(\langle \llbracket a, b \rrbracket, c \rangle + \langle \llbracket b, c \rrbracket, a \rangle + \langle \llbracket c, a \rrbracket, b \rangle), \\ \llbracket a, fb \rrbracket &= f \llbracket a, b \rrbracket + (\rho(a)f)b - \langle a, b \rangle \mathcal{D}f, \\ \rho \circ \mathcal{D} &= 0, \text{ i.e. } \langle \mathcal{D}f, \mathcal{D}g \rangle = 0, \\ (2.1) \quad \rho(a) \langle b, c \rangle &= \langle \llbracket a, b \rrbracket + \mathcal{D} \langle a, b \rangle, c \rangle + \langle b, \llbracket a, c \rrbracket + \mathcal{D} \langle a, c \rangle \rangle. \end{aligned}$$

The main example is due to T. Courant [11]. Given a smooth manifold M , the bundle $TM \oplus T^*M \rightarrow M$ carries a natural Courant algebroid structure, where the anchor is the projection onto the tangent component and the pairing and bracket are given, respectively, by

$$\begin{aligned} \langle X + \xi, Y + \eta \rangle &= \frac{1}{2} (\xi(Y) + \eta(X)), \\ \llbracket X + \xi, Y + \eta \rrbracket &= [X, Y] + L_X \eta - L_Y \xi + \frac{1}{2} d(\xi(Y) - \eta(X)), \end{aligned}$$

$\forall X, Y \in \mathfrak{X}(M)$, $\forall \xi, \eta \in \Omega^1(M)$.

The nonsymmetric law

$$a \circ b := \llbracket a, b \rrbracket + \mathcal{D} \langle a, b \rangle$$

is called the *Dorfman bracket* [27]. The following result is due to Roytenberg [27].

Proposition 2.2. *For all $a, b, c \in \Gamma(E)$ and $f \in C^\infty(M)$, one has:*

$$\begin{aligned} \mathcal{D}f \circ a &= 0, & a \circ \llbracket b, c \rrbracket &= \llbracket a \circ b, c \rrbracket + \llbracket b, a \circ c \rrbracket, \\ a \circ (fb) &= f(a \circ b) + (\rho(a)f)b, & \rho(a) \langle b, c \rangle &= \langle a \circ b, c \rangle + \langle b, a \circ c \rangle. \end{aligned}$$

Any section of the dual bundle $E^* \rightarrow M$ can be seen as a fiberwise linear function on E and vice versa. In other words, $\Gamma(E^*)$ is naturally a subspace of $C^\infty(E)$. Recall that an infinitesimal automorphism of a vector bundle $E \xrightarrow{\pi} M$ corresponds exactly to a vector field on E (i.e., a derivation of $C^\infty(E)$) under which the subspaces $\pi^* C^\infty(M)$ and $\Gamma(E^*)$ are stable [23, Proposition 2.2]. The latter is equivalent to a covariant differential operator on E^* [23], i.e. a pair of differential operators $\delta^0 : C^\infty(M) \rightarrow C^\infty(M)$ and $\delta^1 : \Gamma(E^*) \rightarrow \Gamma(E^*)$ satisfying

$$\begin{aligned} \delta^0(fg) &= f\delta^0(g) + \delta^0(f)g, & \forall f, g \in C^\infty(M), \\ \delta^1(f\alpha) &= f\delta^1(\alpha) + \delta^0(f)\alpha, & \forall \alpha \in \Gamma(E^*), \forall f \in C^\infty(M). \end{aligned}$$

From now on, we assume that E is a Courant algebroid. By $\mathfrak{aut}(E)$, we denote the Lie algebra of infinitesimal automorphisms of the Courant algebroid E . The symmetric pairing $\langle \cdot, \cdot \rangle$ identifies $\Gamma(E)$ with $\Gamma(E^*)$ by $x \mapsto \langle x, \cdot \rangle$. Therefore, one easily recovers the following theorem of Roytenberg [27], which he proved using super-geometry.

Proposition 2.3. *The Lie algebra $\mathbf{aut}(E)$ consists of those covariant differential operators $\delta = (\delta^0, \delta^1)$ on E , where $\delta^0 : C^\infty(M) \rightarrow C^\infty(M)$ and $\delta^1 : \Gamma(E) \rightarrow \Gamma(E)$, satisfying the additional properties:*

$$\delta^0 \langle x, y \rangle = \langle \delta^1 x, y \rangle + \langle x, \delta^1 y \rangle$$

and

$$\delta^1 \llbracket x, y \rrbracket = \llbracket \delta^1 x, y \rrbracket + \llbracket x, \delta^1 y \rrbracket,$$

for all $x, y \in \Gamma(E)$.

For any $z \in \Gamma(E)$, define

$$\delta_z^0 : C^\infty(M) \rightarrow C^\infty(M) \quad \text{and} \quad \delta_z^1 : \Gamma(E) \rightarrow \Gamma(E)$$

by

$$\delta_z^0(f) = \rho(z)(f) \quad \text{and} \quad \delta_z^1(x) = z \circ x, \quad \forall f \in C^\infty(M), \forall x \in \Gamma(E).$$

It follows from Propositions 2.2 and 2.3 that $\delta_z = (\delta_z^0, \delta_z^1)$ is an infinitesimal automorphism of the Courant algebroid E , i.e. $\delta_z \in \mathbf{aut}(E)$.

By ϕ_t we denote the (local) flow

$$\begin{array}{ccc} E & \xrightarrow{\phi_t} & E \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{\varphi_t} & M \end{array}$$

generated by the vector field on E corresponding to δ_z . The one-parameter group $t \mapsto \phi_t$ acts on the space of sections of $E \rightarrow M$ by

$$\phi_t^* : \Gamma(E) \rightarrow \Gamma(E) : \sigma \mapsto \phi_t^{-1} \circ \sigma \circ \varphi_t.$$

By abuse of notation, we use the same symbol ϕ_t (resp. ϕ_t^*) to denote the induced flow on the tensor bundles $E_j^i = (\otimes^i E) \otimes (\otimes^j E^*)$ ($i, j \in \{0, 1, 2, \dots\}$) (resp. the induced action on the spaces of sections of the E_j^i 's).

For any section $\sigma \in \Gamma(E_j^i)$, define $\mathcal{L}_z \sigma \in \Gamma(E_j^i)$ by

$$\mathcal{L}_z \sigma = \left. \frac{d}{d\tau} \phi_\tau^* \sigma \right|_{\tau=0}$$

(see [1, Theorem 2.2.20]). Thus we have the usual identity:

$$\left. \frac{d}{d\tau} \phi_\tau^* \sigma \right|_{\tau=t} = \phi_t^* (\mathcal{L}_z \sigma).$$

In the following proposition, we give a list of important properties of this Lie derivative, which will be useful in future discussions.

Proposition 2.4. *For all $f, g \in C^\infty(M)$ and $x, y, z \in \Gamma(E)$, we have:*

$$\begin{aligned} \mathcal{L}_z f &= \rho(z)f, & \mathcal{L}_z \langle x, y \rangle &= \langle \mathcal{L}_z x, y \rangle + \langle x, \mathcal{L}_z y \rangle, \\ \mathcal{L}_z x &= z \circ x, & \mathcal{L}_z \llbracket x, y \rrbracket &= \llbracket \mathcal{L}_z x, y \rrbracket + \llbracket x, \mathcal{L}_z y \rrbracket, \\ \mathcal{L}_{\mathcal{D}f} &= 0, & \mathcal{L}_f x &= f \mathcal{L}_x y - (\mathcal{L}_y f) x + 2\langle x, y \rangle \mathcal{D}f, \\ \mathcal{L}_{\llbracket x, y \rrbracket} &= [\mathcal{L}_x, \mathcal{L}_y]. \end{aligned}$$

Moreover,

$$\mathcal{L}_z(\sigma \otimes \tau) = \mathcal{L}_z \sigma \otimes \tau + \sigma \otimes \mathcal{L}_z \tau \quad \text{for all } \sigma, \tau \in \bigoplus_{i,j} E_j^i,$$

$$\text{and } [\delta, \mathcal{L}_z] = \mathcal{L}_{\delta^1 z} \text{ for all } \delta \in \mathbf{aut}(E).$$

In particular, if $J : E \rightarrow E$ is a bundle map over the identity $M \xrightarrow{\text{id}} M$, i.e. $J \in \Gamma(E^* \otimes E)$, then

$$(\mathcal{L}_x J)(y) = \mathcal{L}_x(Jy) - J(\mathcal{L}_x y), \quad \forall x, y \in \Gamma(E).$$

Example 2.5. For the standard Courant algebroid structure on $TM \oplus T^*M$, one has

$$\mathcal{L}_{X+\xi}(Y + \eta) = L_X(Y + \eta) - i_Y d\xi, \quad \forall X, Y \in \mathfrak{X}(M), \forall \xi, \eta \in \Omega^1(M),$$

where L denotes the usual Lie derivative.

The following result appeared in [17].

Proposition 2.6. *Let E denote the standard Courant algebroid structure on $TM \oplus T^*M$. Then, for all $X \in \mathfrak{X}(M)$ and $\xi \in \Omega^1(M)$, the one-parameter groups integrating the infinitesimal automorphisms $\delta_X, \delta_\xi, \delta_{X+\xi} \in \mathfrak{aut}(E)$ are given respectively by*

$$\begin{aligned} e^{t\delta_X}(Y + \eta) &= (\varphi_t)_* Y + (\varphi_t^{-1})^* \eta, \\ e^{t\delta_\xi}(Y + \eta) &= Y + \eta + t i_Y d\xi, \\ e^{t\delta_{X+\xi}}(Y + \eta) &= (\varphi_t)_* Y + (\varphi_t^{-1})^* \eta + \int_0^t (\varphi_{t-\tau}^{-1})^* (i_{\varphi_{\tau*} Y} d\xi) d\tau, \end{aligned}$$

where $t \mapsto \varphi_t$ denotes the flow of the vector field X , i.e. satisfies

$$X_{\varphi_t(m)} = \left. \frac{d}{dt} \varphi_\tau(m) \right|_{\tau=t}.$$

In case the section $X_t + \xi_t \in \Gamma(E)$ is time-dependent, its flow

$$\begin{array}{ccc} TM \oplus T^*M & \xrightarrow{\phi_{a,b}} & TM \oplus T^*M \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi_{a,b}} & M \end{array}$$

from time a to b is given by

$$(2.2) \quad \phi_{a,b}(Y + \eta) = (\varphi_{a,b})_* Y + (\varphi_{b,a})^* \eta + \int_a^b (\varphi_{b,\tau})^* (i_{(\varphi_{a,\tau})_* Y} d\xi_\tau) d\tau,$$

where $\varphi_{a,b}$ denotes the flow of the time-dependent vector field X_t from time a to b .

3. MAIN THEOREM

Let E be a Courant algebroid on a smooth manifold M and let $J : E \rightarrow E$ be a vector bundle map over the identity such that $J^2 = -\text{id}$. Then the complexification $E_{\mathbb{C}} := E \otimes \mathbb{C}$ with the extended $\mathbb{C}$ -linear Courant algebroid structure decomposes as the direct sum $L \oplus \bar{L}$ of the eigenbundles of J . Here L is associated to the eigenvalue $+i$ and its complex conjugate $\bar{L}$ to $-i$. The bundle map J is called a *generalized complex structure* if J is orthogonal with respect to $\langle \cdot, \cdot \rangle$ (this forces L and $\bar{L}$ to be isotropic) and the spaces of sections $\Gamma(L)$ and $\Gamma(\bar{L})$ are closed under the Courant bracket, or equivalently, J is “integrable”:

$$[Jx, Jy] - [x, y] - J([Jx, y] + [x, Jy]) = 0, \quad \forall x, y \in \Gamma(E).$$

We refer the reader to [2, 4, 5, 7, 8, 9, 10, 13, 14, 15, 16] for more details.

The group of automorphisms of the Courant algebroid acts on its set of generalized complex structures in a natural way.

Given a smooth family $t \mapsto J_t$ of generalized complex structures on E , we define a 2-form $\omega_t \in \Gamma(\wedge^2 E^*)$ by

$$\omega_t(x, y) = \langle x, J_t y \rangle, \quad \forall x, y \in \Gamma(E).$$

It is clear that ω_t is skew-symmetric. By L_t we denote the $+i$ eigenbundle of $J_t : E_{\mathbb{C}} \rightarrow E_{\mathbb{C}}$. Thus L_t is a Dirac structure on $E_{\mathbb{C}}$. Indeed $(L_t, \overline{L_t})$ is a (complex) Lie bialgebroid [23].

The restriction of the time derivative of $\omega_t \in \Gamma(\wedge^2 E^*)$ to the subbundle L_t defines a 2-form $\dot{\omega}_t \in \Gamma(\wedge^2 L_t^*)$:

$$\dot{\omega}_t(v, w) = \langle v, \dot{J}_t w \rangle, \quad \forall v, w \in \Gamma(L_t).$$

Remark 3.1. Differentiating $J_t J_t = -\text{id}$, we obtain $\dot{J}_t J_t + J_t \dot{J}_t = 0$. Therefore, the map $\dot{J}_t$ swaps the eigenbundles L_t and $\overline{L_t}$.

The cohomology of the canonical differential complex

$$\dots \xrightarrow{d_{L_t}} \Gamma(\wedge^{\bullet-1} L_t^*) \xrightarrow{d_{L_t}} \Gamma(\wedge^{\bullet} L_t^*) \xrightarrow{d_{L_t}} \Gamma(\wedge^{\bullet+1} L_t^*) \xrightarrow{d_{L_t}} \dots$$

associated to the Lie algebroid L_t [6] is the *Lie algebroid cohomology* of L_t .

Proposition 3.2. $d_{L_t} \dot{\omega}_t = 0, \forall t$.

Proof. A straightforward computation using Definition 2.1 and the orthogonality of J_t gives

$$(3.1) \quad (d_{L_t} \dot{\omega}_t)(v, w, z) = \langle v, [\dot{J}_t z, w] + [z, \dot{J}_t w] - \dot{J}_t [z, w] \rangle \quad \forall v, w, z \in \Gamma(L_t).$$

Since J_t is integrable $\forall t$, we have

$$[J_t v, J_t w] - [v, w] - J_t([J_t v, w] + [v, J_t w]) = 0.$$

Differentiating w.r.t. t , we get

$$[\dot{J}_t v, J_t w] + [J_t v, \dot{J}_t w] - \dot{J}_t([J_t v, w] + [v, J_t w]) - J_t([\dot{J}_t v, w] + [v, \dot{J}_t w]) = 0.$$

Now, taking $v, w \in \Gamma(L_t)$, the above relation becomes

$$i[\dot{J}_t v, w] + i[v, \dot{J}_t w] - 2i\dot{J}_t[v, w] - J_t([\dot{J}_t v, w] + [v, \dot{J}_t w]) = 0$$

and, multiplying by $(-i)$, we obtain

$$\frac{1+iJ_t}{2}([\dot{J}_t v, w] + [v, \dot{J}_t w]) = \dot{J}_t[v, w].$$

Hence, since L_t is isotropic,

$$(3.2) \quad \langle z, [\dot{J}_t v, w] + [v, \dot{J}_t w] - \dot{J}_t[v, w] \rangle = 0, \quad \forall v, w, z \in \Gamma(L_t).$$

The result follows from (3.1) and (3.2). $\square$

Lemma 3.3. *Let $t \mapsto J_t$ ($t \in [0, 1]$) be a smooth family of generalized complex structures on a Courant algebroid E . There exists a smooth path $t \mapsto \beta_t$ in $\Gamma(L_t^*)$ such that*

$$(3.3) \quad \dot{\omega}_t = d_{L_t} \beta_t, \quad \forall t$$

if, and only if,

$$\dot{J}_t + \mathcal{L}_{x_t} J_t = 0, \quad \forall t,$$

where $x_t = \frac{1}{2i}(z_t - \overline{z_t}) \in \Gamma(E)$ is the imaginary part of the unique section $z_t \in \Gamma(L_t)$ such that $\beta_t = \langle \overline{z_t}, \cdot \rangle$.

Proof. For $v, w \in \Gamma(L_t)$, one has $\dot{\omega}_t(v, w) = \langle v, \dot{J}_t w \rangle$ and

$$\begin{aligned} (d_{L_t} \beta_t)(v, w) &= \rho(v) \beta_t(w) - \rho(w) \beta_t(v) - \beta_t(\llbracket v, w \rrbracket) \\ &= \rho(v) \langle \bar{z}_t, w \rangle - \rho(w) \langle \bar{z}_t, v \rangle - \langle \bar{z}_t, \llbracket v, w \rrbracket \rangle \\ &= 2 \langle \mathcal{D} \langle \bar{z}_t, w \rangle, v \rangle - (\langle \mathcal{L}_w \bar{z}_t, v \rangle + \langle \bar{z}_t, \mathcal{L}_w v \rangle) - \langle \bar{z}_t, \llbracket v, w \rrbracket \rangle \\ &= 2 \langle \mathcal{D} \langle \bar{z}_t, w \rangle, v \rangle - \langle \llbracket w, \bar{z}_t \rrbracket + \mathcal{D} \langle w, \bar{z}_t \rangle, v \rangle - \langle \bar{z}_t, \llbracket w, v \rrbracket \rangle - \langle \bar{z}_t, \llbracket v, w \rrbracket \rangle \\ &= \langle v, \llbracket \bar{z}_t, w \rrbracket + \mathcal{D} \langle \bar{z}_t, w \rangle \rangle = \langle v, \mathcal{L}_{\bar{z}_t} w \rangle. \end{aligned}$$

Hence

$$\begin{aligned} \dot{\omega}_t &= d_{L_t} \beta_t \\ \Leftrightarrow \quad \langle v, \dot{J}_t w - \mathcal{L}_{\bar{z}_t} w \rangle &= 0, \quad \forall v \in \Gamma(L_t) \\ \Leftrightarrow \quad \dot{J}_t w - \mathcal{L}_{\bar{z}_t} w &\in \Gamma(L_t) \\ \Leftrightarrow \quad (\dot{J}_t w)_{\overline{L_t}} &= (\mathcal{L}_{\bar{z}_t} w)_{\overline{L_t}}, \end{aligned}$$

for all $w \in \Gamma(L_t)$. But $\dot{J}_t w \in \Gamma(\overline{L_t})$ as $w \in \Gamma(L_t)$. Therefore,

$$(3.4) \quad \dot{J}_t w = (\mathcal{L}_{\bar{z}_t} w)_{\overline{L_t}}, \quad \forall w \in \Gamma(L_t).$$

Since $\Gamma(L_t)$ is $\llbracket \cdot, \cdot \rrbracket$ -closed and L_t is $\langle \cdot, \cdot \rangle$ -isotropic,

$$\mathcal{L}_{z_t} w = \llbracket z_t, w \rrbracket + \mathcal{D} \langle z_t, w \rangle = \llbracket z_t, w \rrbracket \in \Gamma(L_t)$$

and

$$(3.5) \quad (\mathcal{L}_{z_t} w)_{\overline{L_t}} = 0,$$

for all $w \in \Gamma(L_t)$. Subtracting (3.5) from (3.4), we get

$$\begin{aligned} \dot{J}_t w &= (\mathcal{L}_{(\bar{z}_t - z_t)} w)_{\overline{L_t}} = -2i (\mathcal{L}_{x_t} w)_{\overline{L_t}} = -2i \frac{1+iJ_t}{2} (\mathcal{L}_{x_t} w) \\ &= -\mathcal{L}_{x_t} (i w) + J_t (\mathcal{L}_{x_t} w) = -\mathcal{L}_{x_t} (J_t w) + J_t (\mathcal{L}_{x_t} w) = -(\mathcal{L}_{x_t} J_t)(w), \end{aligned}$$

for all $w \in \Gamma(L_t)$. Thus,

$$\dot{J}_t v + (\mathcal{L}_{x_t} J_t)(v) = 0, \quad \forall v \in \Gamma(L_t).$$

Since x_t is a real section of E and the endomorphisms J_t and $\dot{J}_t$ are real, we also have

$$\dot{J}_t \bar{v} + (\mathcal{L}_{x_t} J_t)(\bar{v}) = 0, \quad \forall \bar{v} \in \Gamma(\overline{L_t}).$$

Therefore, $\dot{J}_t + \mathcal{L}_{x_t} J_t = 0$, for all t . $\square$

Theorem 3.4. Assume that M is a compact manifold and that E is a Courant algebroid over M . Assume that J_0 and J_1 are two generalized complex structures on E , which are connected by a smooth family of generalized complex structures $t \mapsto J_t$ ($t \in [0, 1]$) on E . Assume there exists a smooth path $t \mapsto \beta_t$ in $\Gamma(L_t^*)$ such that $\dot{\omega}_t = d_{L_t} \beta_t$, $\forall t$. Then J_0 and J_1 are isomorphic.

Proof. The idea of the proof is to construct a smooth one-parameter family of automorphisms $t \mapsto \phi_t$ in $\text{Aut}(E)$ such that $\phi_t \circ J_0 = J_t \circ \phi_t$, $\forall t$. Since $L_t^* \simeq \overline{L_t}$, there exists a unique time-dependent section $z_t \in \Gamma(L_t)$ such that $\beta_t = \langle \bar{z}_t, \cdot \rangle$. Set $x_t = \frac{1}{2i}(z_t - \bar{z}_t) \in \Gamma(E)$. It follows from Lemma 3.3 that

$$\dot{J}_t + \mathcal{L}_{x_t} J_t = 0, \quad \forall t.$$

Therefore

$$\frac{d}{d\tau} \phi_\tau^{-1} \circ J_\tau \circ \phi_\tau \Big|_{\tau=t} = \phi_t^{-1} \circ (\dot{J}_t + \mathcal{L}_{x_t} J_t) \circ \phi_t = 0,$$

where $\phi_t \in \text{Aut}(E)$ is the flow of $\delta_{x_t} \in \mathfrak{aut}(E)$ (as in Section 2), which always exists since M is compact. $\square$

Remark 3.5. When M is not compact, the above theorem is still valid if x_t is complete, i.e. if its flow exists for arbitrary time.

Remark 3.6. In the standard Courant algebroid $TM \oplus T^*M$ case, a remarkable deformation/obstruction theory for generalized complex manifolds was developed by Gualtieri in his thesis [13]. It would be interesting to explore the connection between the theorem above and Gualtieri's deep results. Moreover, a special situation of Theorem 3.4, one where the path J_t is constant, was considered by Hu in [18, Remark 3.6] using a different method.

4. EXAMPLES

Example 1. A symplectic form on a Lie algebroid $A \rightarrow M$ is a smooth section Ω of $\wedge^2 A^* \rightarrow M$, which is d_A -closed and nondegenerate: $a = 0$ if $\Omega(a, b) = 0$ for all $b \in \Gamma(A)$ [25]. Here $d_A : \Gamma(\wedge^k A^*) \rightarrow \Gamma(\wedge^{k+1} A^*)$ is the Lie algebroid cohomology differential.

Let Ω_0, Ω_1 be a pair of symplectic forms on A , which can be connected by a smooth family $t \mapsto \Omega_t$ of symplectic 2-forms. Having endowed A^* with the trivial Lie algebroid structure, the pair (A, A^*) becomes a Lie bialgebroid. On its “double”, the Courant algebroid $E = A \oplus A^*$, we get the one-parameter family of generalized complex structures

$$J_t = \begin{pmatrix} 0 & -\pi_t^\# \\ \Omega_t^\flat & 0 \end{pmatrix}.$$

Here $\pi_t^\# : A^* \rightarrow A$ denotes the inverse of the bundle map $\Omega_t^\flat : A \rightarrow A^*$ associated to the symplectic form Ω_t .

It is well known that

$$\lambda_t : A_{\mathbb{C}} \rightarrow L_t : X \mapsto X - i\Omega_t^\flat X$$

is a Lie algebroid isomorphism. Therefore, the diagram

$$\begin{array}{ccc} \Gamma(\wedge^k L_t^*) & \xrightarrow{\lambda_t^*} & \Gamma(\wedge^k A^*) \\ d_{L_t} \downarrow & & \downarrow d_A \\ \Gamma(\wedge^{k+1} L_t^*) & \xrightarrow{\lambda_t^*} & \Gamma(\wedge^{k+1} A^*) \end{array}$$

is commutative.

Observe that $\lambda_t^* \dot{\omega}_t = -\dot{\Omega}_t$. Indeed,

$$\lambda_t^* \dot{\omega}_t(X, Y) = \langle \lambda_t X, j_t \lambda_t Y \rangle = \left\langle \begin{pmatrix} X \\ -i\Omega_t^\flat X \end{pmatrix}, \begin{pmatrix} 0 & -\dot{\pi}_t^\# \\ \dot{\Omega}_t^\flat & 0 \end{pmatrix} \begin{pmatrix} Y \\ -i\Omega_t^\flat Y \end{pmatrix} \right\rangle = -\dot{\Omega}_t(X, Y).$$

Here we have used the identity $\dot{\Omega}_t^\flat = -\Omega_t^\flat \circ \dot{\pi}_t^\# \circ \Omega_t^\flat$, which is easily obtained from $\pi_t^\# \circ \Omega_t^\flat = \text{id}$. Hence

$$d_A \dot{\Omega}_t = -d_A \lambda_t^* \dot{\omega}_t = -\lambda_t^* d_{L_t} \dot{\omega}_t = 0.$$

Now assume there exists a family of smooth sections $\xi_t \in \Gamma(A^*)$ such that $\dot{\Omega}_t = -d_A \xi_t$. It is easy to check that $\xi_t = \lambda_t^* \beta_t$, where $\beta_t = \langle i\lambda_t \pi_t^\# \xi_t, \cdot \rangle$. Hence

$$\lambda_t^* \dot{\omega}_t = -\dot{\Omega}_t = d_A \xi_t = d_A \lambda_t^* \beta_t = \lambda_t^* d_{L_t} \beta_t;$$

i.e., $\dot{\omega}_t = d_{L_t} \beta_t$.

By Theorem 3.4, we have $\phi_{t*} J_t = J_0$, where ϕ_t is the Courant algebroid automorphism generated by $\delta_{\pi_t^\# \xi_t}$. Since $\pi_t^\# \xi_t \in \Gamma(A)$, ϕ_t is actually induced by a Lie algebroid automorphism of A , which will also be denoted by ϕ_t by abuse of notation.

We have proved the following Lie algebroid version of Moser's lemma [24].

Corollary 4.1. *Assume that M is a compact manifold and that A is a Lie algebroid over M . Assume that Ω_0 and $\Omega_1 \in \Gamma(\wedge^2 A^*)$ are two symplectic structures on the Lie algebroid A which can be connected by a smooth family of symplectic structures $t \mapsto \Omega_t$ ($t \in [0, 1]$) on A . Assume there exists a smooth path $t \mapsto \xi_t$ in $\Gamma(A^*)$ such that $\dot{\Omega}_t = -d_A \xi_t$, $\forall t$. Then Ω_0 and Ω_1 are isomorphic. That is, there is a Lie algebroid automorphism ϕ , isotopic to the identity, such that $\phi^* \Omega_1 = \Omega_0$.*

Example 2. Let $A \rightarrow M$ be a Lie algebroid endowed with an integrable almost complex structure, i.e. a bundle map $j : A \rightarrow A$, over the identity, whose Nijenhuis tensor vanishes and such that $j^2 = -\text{id}$. Its complexification $A_{\mathbb{C}} = A \otimes \mathbb{C}$ decomposes as the direct sum $A^{1,0} \oplus A^{0,1}$ of the eigenbundles of j with eigenvalues $+i$ and $-i$, respectively. Both $A^{1,0}$ and $A^{0,1}$ are Lie subalgebroids of $A_{\mathbb{C}}$. The vector bundle $A^{1,0}$ is naturally an $A^{0,1}$ -module: the representation of $A^{0,1}$ on $A^{1,0}$ is the map

$$\Gamma(A^{0,1}) \times \Gamma(A^{1,0}) \rightarrow \Gamma(A^{1,0}) : (X, Y) \mapsto X \triangleright Y := \text{pr}^{1,0}[X, Y],$$

where $\text{pr}^{1,0}$ stands for the canonical projection $A_{\mathbb{C}} = A^{0,1} \oplus A^{1,0} \rightarrow A^{1,0}$.

Now consider the dual vector bundle $A^* \rightarrow M$ as a trivial Lie algebroid; i.e., the anchor and the bracket are both zero. The pair (A, A^*) is a Lie bialgebroid and the endomorphism

$$J = \begin{pmatrix} -j & 0 \\ 0 & j^* \end{pmatrix}$$

is a generalized complex structure on its double $E := A \oplus A^*$. The $+i$ -eigenbundle L of J is naturally isomorphic, as a Lie algebroid, to the semi-direct product Lie algebroid $A^{0,1} \ltimes (A^{1,0})^*$, where $(A^{1,0})^*$ is the $A^{0,1}$ -module dual to $A^{1,0}$.

The following lemma can be easily verified.

Lemma 4.2. *Let A be a Lie algebroid, let M be an A -module and let $A \ltimes M^*$ be the resulting semi-direct product Lie algebroid. Then the diagram*

$$\begin{array}{ccc} C^{k-1}(A, M) & \xrightarrow{\alpha} & C^k(A \ltimes M^*, \mathbb{C}) \\ D \downarrow & & \downarrow D \\ C^k(A, M) & \xrightarrow{\alpha} & C^{k+1}(A \ltimes M^*, \mathbb{C}) \end{array}$$

commutes for all $k \geq 0$. Here $C^k(A, M) := \Gamma(\wedge^k A^* \otimes M)$, the D 's are the coboundary operators for the Lie algebroid cohomologies of A (with values in the module M) and $A \ltimes M^*$, while α is the skew-symmetrization map

$$(\wedge^k A^*) \otimes M \rightarrow (\wedge^k A^*) \wedge M \hookrightarrow \wedge^{k+1}(A \ltimes M^*)^*.$$

Let $t \mapsto j_t$, $t \in [0, 1]$ be a smooth family of integrable almost complex structures on A . Let

$$J_t = \begin{pmatrix} -j_t & 0 \\ 0 & j_t^* \end{pmatrix}$$

be the associated family of generalized complex structures on the Courant algebroid $E = A \oplus A^*$.

Taking $A = A_t^{0,1}$ and $M = A_t^{1,0}$ in Lemma 4.2, we get that

$$\begin{array}{ccc} C^0(A_t^{0,1}, A_t^{1,0}) & \xrightarrow{\alpha} & C^1(L_t, \mathbb{C}) \\ \bar{\partial}_t \downarrow & & \downarrow d_{L_t} \\ C^1(A_t^{0,1}, A_t^{1,0}) & \xrightarrow{\alpha} & C^2(L_t, \mathbb{C}) \end{array}$$

is commutative for all $t \in [0, 1]$. Here $\bar{\partial}$ denotes the differential operator of the Lie algebroid cohomology of $A_t^{0,1}$ with values in the module $A_t^{1,0}$.

The time derivative $\dot{j}_t$ may be seen as an element of $\Gamma((A_t^{0,1})^* \otimes A_t^{1,0}) = C^1(A_t^{0,1}, A_t^{1,0})$ because $\dot{j}_t$ swaps $A_t^{0,1}$ and $A_t^{1,0}$. Moreover, $\dot{\omega}_t = -\alpha(\dot{j}_t)$. Indeed, for all $X, Y \in \Gamma(A_t^{0,1})$ and all $\xi, \eta \in \Gamma((A_t^{1,0})^*)$, one has

$$\begin{aligned} \dot{\omega}_t(X + \xi, Y + \eta) &= \langle X + \xi, \dot{J}_t(Y + \eta) \rangle = \langle X + \xi, \dot{j}_t Y - \dot{j}_t^* \eta \rangle \\ &= \frac{1}{2}(\xi(\dot{j}_t Y) - \eta(\dot{j}_t X)) = -\alpha(\dot{j}_t)(X + \xi, Y + \eta). \end{aligned}$$

Assume there exists a family $t \in [0, 1] \mapsto z_t \in \Gamma(A_t^{0,1})$ such that

$$\bar{\partial}_t(\frac{1}{2}\overline{z_t}) = -\dot{j}_t.$$

(Remark that $\frac{1}{2}\overline{z_t} \in \Gamma(A_t^{1,0}) = C^0(A_t^{0,1}, A_t^{1,0})$.) Then

$$\dot{\omega}_t = -\alpha(\dot{j}_t) = \alpha \circ \bar{\partial}_t(\frac{1}{2}\overline{z_t}) = d_{L_t} \circ \alpha(\frac{1}{2}\overline{z_t}) = d_{L_t}(\langle \overline{z_t}, \cdot \rangle),$$

since $\langle \overline{z_t}, X + \xi \rangle = \frac{1}{2}\xi(\overline{z_t}) = \frac{1}{2}\alpha(\overline{z_t})(\xi)$ for all $X + \xi \in L_t = A_t^{0,1} \ltimes (A_t^{1,0})^*$.

Set $x_t = \frac{1}{2i}(z_t - \overline{z_t}) \in \Gamma(A)$. According to Theorem 3.4, we have $\phi_{t*}J_t = J_0$, where ϕ_t is the flow of automorphisms of the Courant algebroid $E = A \oplus A^*$ generated by δ_{x_t} . Here, since the section x_t of E actually lies in the Lie algebroid A , ϕ_t is induced by a family of Lie algebroid isomorphisms.

We have proved

Corollary 4.3. *Let A be a Lie algebroid over a compact manifold M . Let j_0 and j_1 be two integrable almost complex structures on A , which are connected by a smooth family j_t of integrable almost complex structures. Assume there exists a smooth family $z_t \in \Gamma(A_t^{0,1})$ such that $\bar{\partial}_t \overline{z_t} = -2\dot{j}_t$. Then j_0 and j_1 are isomorphic; i.e., there exists a Lie algebroid automorphism ϕ such that $j_1 = \phi_* j_0$ ($:= \phi^{-1} \circ j_0 \circ \phi$).*

When A is the tangent Lie algebroid TM , this is a classical result [19].

5. HAMILTONIAN OPERATORS

Let J be a generalized complex structure on a Courant algebroid E . Then $E = L \oplus \overline{L}$ and $L^* \simeq \overline{L}$, where L is the $+i$ -eigenbundle of J and (L, L^*) is a complex Lie bialgebroid.

Modulo the isomorphism $E \rightarrow E^* : e \mapsto \langle e, \cdot \rangle$, which identifies $\overline{L}$ with L^* (resp. L with $\overline{L}^*$), any section $H \in \Gamma(\wedge^2 L^*)$ (resp. $\overline{H} \in \Gamma(\wedge^2 \overline{L}^*)$) can be seen as a map

$H' : L \rightarrow \overline{L}$ (resp. $\overline{H}' : \overline{L} \rightarrow L$). The graph $L_H = \{(v, H'v)\}_{v \in L} \subset L \oplus \overline{L}$ of $H' : L \rightarrow \overline{L}$ is a Dirac structure of E if, and only if, the Maurer-Cartan equation

$$d_L H + \frac{1}{2}[H, H]_{L^*} = 0$$

holds [22]. In this case, H is called a *Hamiltonian operator*. Here $d_L : \Gamma(\wedge^\bullet L^*) \rightarrow \Gamma(\wedge^{\bullet+1} L^*)$ is the Lie algebroid differential of L and $[\cdot, \cdot]_{L^*} : \Gamma(\wedge^k L^*) \otimes \Gamma(\wedge^l L^*) \rightarrow \Gamma(\wedge^{k+l-1} L^*)$ is the Schouten bracket of the Lie algebroid L^* .

When H is a Hamiltonian operator, L_H is a Lie algebroid. The vector bundle isomorphism

$$L \rightarrow L_H : v \mapsto (v, H'v)$$

induces the cochain complex isomorphism

$$(5.1) \quad \begin{array}{ccccccc} \dots & \xrightarrow{d_{L_H}} & \Gamma(\wedge^\bullet L_H^*) & \xrightarrow{d_{L_H}} & \Gamma(\wedge^{\bullet+1} L_H^*) & \xrightarrow{d_{L_H}} & \dots \\ & & \downarrow \simeq & & \downarrow \simeq & & \\ \dots & \xrightarrow{d_H} & \Gamma(\wedge^\bullet L^*) & \xrightarrow{d_H} & \Gamma(\wedge^{\bullet+1} L^*) & \xrightarrow{d_H} & \dots, \end{array}$$

where $d_H = d_L + [H, \cdot]_{L^*}$ and d_{L_H} is the Lie algebroid coboundary operator of L_H .

Lemma 5.1. *Assume $H \in \Gamma(\wedge^2 L^*)$ is a Hamiltonian operator. Then L_H is the $+i$ -eigenbundle of a generalized complex structure on E if, and only if, the bundle map $\overline{H}' \circ H' - 1 : L \rightarrow L$ is invertible.*

Proof. First, note that $\overline{(L_H)} = (\overline{L})_{\overline{H}}$. Hence $(v, H'v) \in L_H \cap \overline{L_H}$ iff $\exists w \in \overline{L}$ such that $(v, H'v) = (\overline{H}'w, w)$, which holds if, and only if, $(\overline{H}' \circ H' - 1)v = 0$. It thus follows that $L_H \cap \overline{L_H} = \{0\}$ iff $\overline{H}' \circ H' - 1$ is invertible. $\square$

Example 5.2. Consider the generalized complex structure $J = \begin{pmatrix} -j & 0 \\ 0 & j^* \end{pmatrix}$ on the standard Courant algebroid $E = TM \oplus T^*M$ corresponding to a complex structure $j : TM \rightarrow TM$. Then we have $L = T^{0,1} \ltimes (T^{1,0})^*$ and $\overline{L} = T^{1,0} \ltimes (T^{0,1})^*$, where the Lie algebroid structure on L is the semi-direct product, similarly for $\overline{L}$. Let $\pi \in \Gamma(\wedge^2 T^{1,0})$. It is simple to see that $\overline{\pi}' \circ \pi' = 0$. Hence $\overline{\pi}' \circ \pi' - 1 = -1$ is invertible. On the other hand, π satisfies the Maurer-Cartan equation if and only if $\bar{\partial}\pi = 0$ and $[\pi, \pi] = 0$. That is, π is a holomorphic Poisson structure. Thus we recover the well-known fact that a holomorphic Poisson structure is a generalized complex structure [13, 12] on the standard Courant algebroid $TM \oplus T^*M$.

Now, let $t \in [0, 1] \mapsto H_t \in \Gamma(\wedge^2 L^*)$ be a family of Hamiltonian operators such that $\overline{H}_t' \circ H_t' - 1$ is invertible for all $t \in [0, 1]$. By Lemma 5.1, we obtain a family of generalized complex structures $t \mapsto J_t$. In this case, its corresponding $\dot{\omega}_t$ has a very simple description.

Lemma 5.3. $\Theta_t^* \dot{\omega}_t = -i\dot{H}_t \in \Gamma(\wedge^2 L^*)$.

Here Θ_t is the vector bundle map

$$L \rightarrow E = L \oplus \overline{L} : v \mapsto (v, H_t'v)$$

identifying L with L_{H_t} .

Proof. By definition, we have

$$J_t(v, H'_t v) = i(v, H'_t v).$$

Differentiating w.r.t. t , we get

$$\dot{J}_t(v, H'_t v) + J_t(0, \dot{H}'_t v) = i(0, \dot{H}'_t v).$$

It thus follows that

$$\begin{aligned} \Theta_t^* \dot{\omega}_t(v, w) &= \langle (v, H'_t v), \dot{J}_t(w, H'_t w) \rangle = \langle (v, H'_t v), (i - J_t)(w, H'_t w) \rangle \\ &= \langle (i + J_t)(v, H'_t v), (w, H'_t w) \rangle = 2i \langle (v, H'_t v), (w, H'_t w) \rangle = -i \dot{H}_t(v, w). \end{aligned}$$

□

From (5.1) and Lemma 5.3, it follows that (3.3) is equivalent to

$$\frac{dH_t}{dt} = d_L Y_t + [H_t, Y_t]_{L^*},$$

where $Y_t = i\Theta_t^* \beta_t \in \Gamma(L^*)$. We have proved

Theorem 5.4. *Let E be a Courant algebroid over a compact manifold M whose complexification $E_{\mathbb{C}}$ is the direct sum $L \oplus \bar{L}$ of a pair of complex conjugated complex Lie algebroids. Let $H_t \in \Gamma(\wedge^2 L^*)$ be a smooth family of Hamiltonian operators such that $\overline{H'_t} \circ H'_t - 1$ is invertible. Assume there exists a smooth family $Y_t \in \Gamma(L^*)$ solving the equation*

$$(5.2) \quad \frac{dH_t}{dt} = d_L Y_t + [H_t, Y_t]_{L^*}.$$

Then the generalized complex structures associated to H_0 and H_1 are isomorphic.

Example 5.5. Take $L = T^{0,1} \ltimes (T^{1,0})^*$ as in Section 4. Let $\pi_t \in \Gamma(\wedge^2 T^{1,0})$ be a family of holomorphic Poisson bivector fields. It is simple to check that $Y_t = X_t^{1,0} + \xi_t^{0,1} \in \bar{L} = T^{1,0} \ltimes (T^{0,1})^*$ satisfies (5.2) iff

$$\bar{\partial} X_t^{1,0} = 0, \quad \bar{\partial} \xi_t^{0,1} = 0 \quad \text{and} \quad \dot{\pi}_t = [\pi_t, X_t^{1,0}].$$

In particular, one may take $\xi_t^{0,1} = 0$. Thus we obtain the following standard result: if there exists a family of complete holomorphic vector fields $X_t^{1,0}$ satisfying the equation

$$\dot{\pi}_t = [\pi_t, X_t^{1,0}],$$

then the holomorphic Poisson structures π_0 and π_1 are isomorphic. Note that, in this case, $x_t = 2 \operatorname{Re} X_t^{1,0} \in \mathfrak{X}(M)$ and x_t is complete iff $X_t^{1,0}$ is complete.

6. DARBOUX-WEINSTEIN THEOREM

As an application of Theorem 3.4, we present a Darboux-Weinstein style theorem for generalized complex structures.

Let J be a generalized complex structure on the standard Courant algebroid $TM \oplus T^*M \rightarrow M$. Then $\Gamma(TM \oplus T^*M)$ is a module over the ring $C^\infty(M)$. Given a point $o \in M$, choose local coordinates $(x_1, \dots, x_n)$ identifying an open neighbourhood U of o in M with a unit ball in $\mathbb{R}^n$ centered at the origin (and

mapping o to 0). Then $\{e_1 = \partial_{x^1}, \dots, e_n = \partial_{x^n}, e_{n+1} = dx^1, \dots, e_{2n} = dx^n\}$ is a local frame of $TM \oplus T^*M$ over U , which enjoys the following remarkable properties:

$$(6.1) \quad \llbracket e_i, e_j \rrbracket = 0, \quad \forall i, j \in \{1, \dots, 2n\};$$

$$(6.2) \quad \langle e_i, e_j \rangle \text{ is constant on } U, \quad \forall i, j \in \{1, \dots, 2n\}.$$

Now consider the matrix representation of J in this local frame:

$$J e_j|_x = \sum_i a_j^i(x) e_i|_x, \quad x \in U.$$

The a_j^i 's are smooth functions on $U \subset \mathbb{R}^n$. Set

$$(6.3) \quad J_t e_j|_x = \sum_i a_j^i(tx) e_i|_x,$$

for all $t \in [0, 1]$ and $x \in U$. Clearly, $J_t^2 = -\text{id}$, $J_1 = J$ and J_0 is represented by the constant matrix $a_j^i(0)$. Moreover, (6.1) implies that J_0 is integrable.

Lemma 6.1. *All J_t 's are generalized complex structures on the restriction of $TM \oplus T^*M$ to U .*

Proof. Let $\{e^k\}_{k=1, \dots, 2n}$ be the dual frame to $\{e_i\}_{i=1, \dots, 2n}$. From a tedious computation using the relations (6.1), (6.2) and

$$\llbracket fA, gB \rrbracket = fg\llbracket A, B \rrbracket + f(\rho(A)g)B - g(\rho(B)f)A + \langle A, B \rangle (g\mathcal{D}f - f\mathcal{D}g)$$

(where $f, g \in C^\infty(M)$ and $A, B \in \Gamma(TM \oplus T^*M)$), it follows that

$$\begin{aligned} (e^k, \llbracket J_t e_i, J_t e_j \rrbracket)(x) &= t(e^k, \llbracket J e_i, J e_j \rrbracket)(tx), \\ (e^k, \llbracket J_t e_i, e_j \rrbracket)(x) &= t(e^k, \llbracket J e_i, e_j \rrbracket)(tx), \end{aligned}$$

and

$$(e^k, J_t \llbracket J_t e_i, e_j \rrbracket)(x) = t(e^k, J \llbracket J e_i, e_j \rrbracket)(tx).$$

Hence the integrability of J at the point tx implies the integrability of J_t at the point x . Since J is integrable on U , U is the unit ball around 0 in $\mathbb{R}^n$ and $t \in [0, 1]$, the conclusion follows. $\square$

As an immediate consequence of Theorem 3.4, we have

Corollary 6.2. *Assume there exists a smooth path $t \in [0, 1] \mapsto \beta_t \in \Gamma(L_t^*|_U)$ such that $\dot{\omega}_t = d_{L_t}\beta_t$. Then the restriction of J to U is isomorphic to the constant generalized complex structure J_0 .*

Theorem 6.3. *Let J be a generalized complex structure (with $+i$ -eigenbundle L) on the standard Courant algebroid $TM \oplus T^*M \rightarrow M$. Let $(x_1, \dots, x_n)$ be local coordinates on M identifying an open domain U with the unit ball centered at the origin in $\mathbb{R}^n$ and define J_t as in (6.3). If there exists $\beta \in \Gamma(L^*|_U)$ such that*

$$d_L \beta = \dot{\omega}_1, \quad \beta|_0 = 0 \quad \text{and} \quad d\beta|_0 = 0,$$

then $J|_U$ is isomorphic to a constant generalized complex structure.

Remark 6.4. Since, at a singular point (for the underlying Poisson structure [2, 4, 5, 7, 8, 9, 10, 13, 14, 15, 16]), a generalized complex structure cannot be isomorphic to a constant one, the condition of Theorem 6.3 must fail in any neighbourhood of any singular point.

The proof is split into several lemmas.

Lemma 6.5. *Let E_1 and E_2 be two Courant algebroids over the same base manifold M , with anchor maps ρ_1 and ρ_2 , respectively. Let*

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi} & M \end{array}$$

be an isomorphism of vector bundles such that

$$\rho_2 \circ \phi = \varphi_* \circ \rho_1$$

and

$$[\phi^{-1} \circ \alpha \circ \varphi, \phi^{-1} \circ \beta \circ \varphi] = \phi^{-1} \circ [\alpha, \beta] \circ \varphi, \quad \forall \alpha, \beta \in \Gamma(E_2).$$

Then ϕ establishes a 1-1 correspondence between the subbundles of E_1 and E_2 which are Lie algebroids.

Lemma 6.6. *The vector bundle map*

$$\begin{array}{ccc} TM \oplus T^*M|_U & \xrightarrow{\phi_t} & TM \oplus T^*M|_U \\ \downarrow & & \downarrow \\ U & \xrightarrow{\varphi_t} & U, \end{array}$$

where

$$\varphi_t(x_1, \dots, x_n) = (tx_1, \dots, tx_n)$$

and

$$\phi_t(e_i|_x) = te_i|_{tx},$$

satisfies the assumptions of Lemma 6.5. Moreover, we have

$$J \circ \phi_t = \phi_t \circ J_t.$$

Corollary 6.7. *The “multiplication by t ” map*

$$\begin{array}{ccc} L_t & \xrightarrow{\phi_t} & L \\ \downarrow & & \downarrow \\ U & \xrightarrow{\varphi_t} & U \end{array}$$

is a Lie algebroid isomorphism.

Therefore we have

$$\phi_t^* \circ d_L = d_{L_t} \circ \phi_t^*.$$

Lemma 6.8.

$$t^3 \dot{\omega}_t = \phi_t^* \dot{\omega}_1.$$

Proof. Differentiating (6.3) w.r.t. t , we get

$$\dot{J}_t e_j|_x = \sum_{k,l} x^l \left. \frac{\partial a_j^k}{\partial x^l} \right|_{tx} e_k|_x.$$

Therefore

$$\dot{\omega}_t(e_i|_x, e_j|_x) = \langle e_i|_x, \dot{J}_t e_j|_x \rangle = b_{ij} x^l \left. \frac{\partial a_j^k}{\partial x^l} \right|_{tx}.$$

Substituting 1 for t and tx for x in the last relation, one gets

$$\dot{\omega}_1(e_i|_{tx}, e_j|_{tx}) = b_{ij} tx^l \left. \frac{\partial a_j^k}{\partial x^l} \right|_{tx}.$$

Now, multiplying by t^2 , we obtain

$$\dot{\omega}_1(te_i|_{tx}, te_j|_{tx}) = b_{ij} t^3 x^l \left. \frac{\partial a_j^k}{\partial x^l} \right|_{tx}$$

or, equivalently,

$$\dot{\omega}_1(\phi_t e_i|_x, \phi_t e_j|_x) = t^3 \dot{\omega}_t(e_i|_x, e_j|_x).$$

□

Proof of Theorem 6.3. According to Lemma 6.8,

$$t^3 \dot{\omega}_t = \phi_t^* \dot{\omega}_1 = \phi_t^* d_L \beta = d_{L_t} \phi_t^* \beta.$$

Now set $\beta_t = t^{-3} \phi_t^* \beta$. We have $\beta_t(e_i|_x) = t^{-3} \beta(\phi_t e_i|_x) = t^{-2} \beta(e_i|_{tx})$, which shows that β_t is well defined for $t = 0$ since $\beta|_0 = 0$ and $d\beta|_0 = 0$. The conclusion follows from Theorem 3.4 since $\dot{\omega}_t = d_{L_t} \beta_t$. □

Remark 6.9. When J is the generalized complex structure corresponding to a symplectic structure, it is straightforward to check that the Poincaré lemma for de Rham cohomology ensures that the conditions of Theorem 6.3 hold at any point $m \in M$. As a consequence, we recover the Darboux-Weinstein theorem. However the one-parameter family of generalized complex structures (or family of symplectic structures) considered here is different from the one used in [29].

On the other hand, when J is the generalized complex structure corresponding to an integrable complex structure $j : TM \rightarrow TM$, Kohn gave a proof of the Newlander-Nirenberg theorem using exactly the same j_t as ours [20]. Kohn's essential idea and major achievement was to prove the Poincaré lemma for Dolbeault cohomology using the description of this cohomology in terms of the integrable almost complex structure rather than holomorphic charts. As a consequence, he was able to deform the local chart used to define the family of local almost complex structures $t \mapsto j_t$ joining the given j to the constant j_0 into a local chart in which j is constant. It would be interesting to compare our condition with the one in [20] in that case.

Remark 6.10. Note that the degree 2 Lie algebroid cohomology of L may not always be locally trivial for generic generalized complex structures even though the Poincaré lemma holds for both the symplectic and complex cases. For instance, when J corresponds to a holomorphic Poisson structure π , the Lie algebroid cohomology of L is simply the Poisson cohomology of π [21], which may not always be locally trivial.

On the other hand, the condition of Theorem 6.3 is stronger than just requiring $H^2(L|_U, \mathbb{C})$ to vanish locally on a neighbourhood U . For instance, when J is the generalized complex structure corresponding to the holomorphic Lie Poisson structure of a complex simple Lie algebra, $H^2(L|_U, \mathbb{C})$ vanishes on a neighbourhood U of 0. However J cannot be locally trivial on neighbourhoods of 0.

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DEPARTEMENT MATHEMATIK, E.T.H. ZÜRICH, 8092 ZÜRICH, SWITZERLAND

E-mail address: `stienon@math.ethz.ch`

Current address: Institut de Mathématiques de Jussieu, Université Paris-Diderot, 75013 Paris, France

E-mail address: `stienon@math.jussieu.fr`