DEPTH-ZERO BASE CHANGE FOR RAMIFIED $U(2, 1)$

JEFFREY D. ADLER AND JOSHUA M. LANSKY

ABSTRACT. We give an explicit description of L -packets and quadratic base change for depth-zero representations of ramified unitary groups in two and three variables. We show that this base change lifting is compatible with a certain lifting of families of representations of finite groups. We conjecture that such a compatibility is valid in much greater generality.

1. INTRODUCTION

Given a finite Galois extension E/F of local fields and a reductive algebraic F -group $\underline{G}$, “base change” is, roughly, a (sometimes only conjectural) mapping from (packets of) representations of $G = \underline{G}(F)$ to those of $\underline{G}(E)$. This mapping is essentially the same as the Shintani lifting (as introduced in [43] and extended in [19, 21, 24–27] for finite groups) in those cases in which both are known to exist (see [3, 16, 22, 31, 40]).

Correspondences like base change that are associated to the Langlands program can be difficult to describe explicitly, even in cases where they are known to exist. However, the Bernstein decomposition of the category of smooth representations of G gives rise to a partition of the set of irreducible admissible representations, and one hopes that base change is compatible with this partition. More specifically, given a Bushnell-Kutzko *type* [11] that is conjecturally associated to a piece of this partition, there should be a corresponding type for a piece of the corresponding partition of the representations of $\underline{G}(E)$. The work of Bushnell and Henniart [10, 12–14] provides evidence for such a compatibility when $\underline{G} = \mathrm{GL}(n)$, and Blasco [6] addresses the case of positive-depth representations of $U(3)$.

Depth-zero representations are particularly interesting, since they are associated to cuspidal representations of finite groups. One can then ask whether base change for depth-zero representations is compatible with base change for finite groups. In the situation we consider in [1] and the present article, there is indeed such a compatibility.

To state this relationship, we briefly recall some notation. Let k_F and k_E denote the residue fields of F and E . Given a point x in the building of $\underline{G}(F)$, one has an associated parahoric subgroup $G_x \subseteq \underline{G}(F)$. This is a compact open subgroup whose reductive quotient is the group of rational points of a connected reductive k_F -group $\underline{G}_x$. Thus, given a representation σ of $\underline{G}_x(k_F)$, one can inflate to obtain a representation $\mathrm{infl}(\sigma)$ of G_x . If $\tilde{\underline{G}}$ is the restriction of scalars $R_{E/F}\underline{G}$ of $\underline{G}$, then one

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similarly has a parahoric subgroup $\tilde{G}_x \subseteq \tilde{G} = \underline{G}(E)$ whose reductive quotient is $\tilde{\underline{G}}_x(k_F)$. If E/F is unramified, then $\tilde{\underline{G}}_x = R_{k_E/k_F} \underline{G}_x$, so this last reductive quotient is equal to $\underline{G}_x(k_E)$.

If K is a compact open subgroup of $\underline{G}(F)$ and ρ is a smooth representation of K , then we will say that a representation π of $\underline{G}(F)$ contains the pair (K, ρ) if the restriction $\pi|_K$ contains ρ . We will say that π has *depth zero* if π contains a pair of the form $(G_x, \text{infl}(\sigma))$, where σ is an irreducible, cuspidal representation of $\underline{G}_x(k_F)$. It makes sense to refer to *depth-zero L -packets*, since one expects that either all or none of the representations in a given L -packet will have depth zero.

Conjecture 1.1. *Suppose E/F is an unramified extension of local fields, and there is a base change lifting $\text{BC}_{E/F}$ that takes L -packets for $\underline{G}(F)$ to L -packets for $\underline{G}(E)$. Let Π denote a depth-zero L -packet for $\underline{G}(F)$, and let $\pi \in \Pi$. Suppose that π contains $(G_x, \text{infl}(\sigma))$, where σ is an irreducible, cuspidal representation of $\underline{G}_x(k_F)$. Then some representation $\tilde{\pi} \in \text{BC}_{E/F}(\Pi)$ contains $(\tilde{G}_x, \text{infl}(\text{BC}_{k_E/k_F}(\sigma)))$, where BC_{k_E/k_F} is the base change lifting associated to the extension k_E/k_F and the k_F -group $\underline{G}_x$.*

If E/F is quadratic and $\underline{G}$ is a unitary group in two or three variables defined with respect to E/F , then Rogawski [40] has shown that a base change lifting exists, and he has derived some of its properties. The main theorem of [1] can be briefly stated as follows:

Theorem 1.2. *Conjecture 1.1 is true when E/F is unramified quadratic and when $\underline{G}$ is a unitary group in 2 or 3 variables defined with respect to E/F .*

In order to prove this theorem, we had to make several assumptions.

We assumed $|k_F| > 59$, but only to avoid some technical complications. A weaker hypothesis is probably sufficient. In order to be able to make use of Rogawski's construction of the quadratic base change lifting for $U(2, 1)$, [40], we assumed that F has characteristic zero. We also needed to assume that base change for unitary groups in two and three variables, as well as various endoscopic transfers, preserve depth, at least for depth-zero L -packets.

In the present paper, we consider the case where E/F is ramified. Now the problem is completely different, since we no longer have $\tilde{\underline{G}}_x = R_{k_E/k_F} \underline{G}_x$.

Conjecture 1.3. *Suppose E/F is a cyclic, tamely ramified extension of local fields, and there is a base change lifting $\text{BC}_{E/F}$ that takes L -packets for $\underline{G}(F)$ to L -packets for $\underline{G}(E)$. Let Π denote a depth-zero L -packet for $\underline{G}(F)$, and let $\pi \in \Pi$. Suppose that π contains $(G_x, \text{infl}(\sigma))$, where σ is an irreducible, cuspidal representation of $\underline{G}_x(k_F)$. Then some representation $\tilde{\pi} \in \text{BC}_{E/F}(\Pi)$ contains $(\tilde{G}_x, \text{infl}(\tilde{\sigma}))$ for some irreducible representation $\tilde{\sigma}$ of $\tilde{G}_x(k_F)$ contained in $\ell(\sigma)$.*

Here, $\ell(\sigma)$ is a collection of representations of $\tilde{G}_x(k_F)$ that arises via a certain lifting (which we will later call the ε -lifting) of representations of finite reductive groups. When E/F is unramified, ℓ is compatible with the base change lifting BC_{k_E/k_F} . The case where E/F is ramified quadratic and $\underline{G}$ is a unitary group in 2 or 3 variables gives rise to liftings for several finite groups. In the 3-variable case, we describe these liftings explicitly in §8. We prove basic properties of ℓ in general elsewhere (see [2]).

Our main theorem is the following:

Theorem 1.4. *Suppose Hypothesis 1.5. Conjecture 1.3 is true when F has characteristic zero, the order of k_F is greater than 3 and not equal to 7, E/F is a totally and tamely ramified quadratic extension, and $\underline{G}$ is the quasi-split unitary group $U(2, 1)_{E/F}$ associated to E/F .*

However, we go somewhat further, describing base change explicitly for each depth-zero L -packet for $U(2, 1)_{E/F}$ (see §7).

As in [1], we assume that F has characteristic zero only so that we can apply results of Rogawski [40]. Our calculations apply equally well if F is a function field of characteristic more than 3.

In order to prove the theorem, we need to assume the following.

Hypothesis 1.5. *All representations within an L -packet have the same depth. Considering unitary groups with respect to a totally, tamely ramified quadratic extension E/F , the following endoscopic transfers, defined with respect to a depth-zero character (see §4.8 of [40]), take depth-zero (resp. positive-depth) L -packets to depth-zero (resp. positive-depth) L -packets:*

- from $(U(1, 1) \times U(1))(F)$ to $U(2, 1)(F)$;
- from $(U(1) \times U(1) \times U(1))(F)$ to $(U(1, 1) \times U(1))(F)$.

Note that we do not include quadratic base change for $U(1, 1)(F)$ or $U(2, 1)(F)$ (as defined in [40]) among the transfers listed above.

We can leave out the quadratic base change lifting from $U(1, 1)(F)$ to $GL(2, E)$ because we have just received a work of Blasco [7] that describes this base change. In particular, Théorème 4.5 *loc. cit.* implies that depth is preserved in the above sense.

The quadratic base change lifting from $U(2, 1)(F)$ to $GL(3, E)$ is not included in the hypothesis, because all depth zero representations of $U(2, 1)(F)$ are obtained via parabolic induction from a maximal torus or via transfer from $(U(1, 1) \times U(1))(F)$. As a result, the base change lifts of depth-zero L -packets of $U(2, 1)(F)$ can be expressed in terms of the base change lifts of associated packets of these simpler groups (see Propositions 7.1, 7.3, and 7.4) for which depth-zero preservation is known. (For unramified $U(2, 1)$ this is not the case, and therefore in [1] the above assumption had to be made for the base change lifting for $U(2, 1)(F)$ as well.)

More generally, one expects that base change will multiply depth by the ramification degree of E/F (see [32]), but we do not need such a strong statement here.

We expect that other correspondences of representations can be made more explicit by examinations of compact open data. For example, Silberger and Zink [44] have made the Abstract Matching Theorem [5, 18, 39] explicit for depth-zero discrete series representations, and Pan [37] has shown that the theta correspondence for depth-zero representations is compatible with that for finite groups.

The structure of this article is as follows. In §2, we present our notation, as well as some basic facts about Bruhat-Tits buildings and restriction of scalars. We fix a choice of nonarchimedean local field F . In §3, we review the general notion of Shintani lifting for reductive groups over local and finite fields. Given a quasi-semisimple automorphism ε of a finite reductive group, we also describe

the corresponding ε -lifting of (families of) representations. In §4, we fix a ramified quadratic extension E/F , an associated ramified unitary group $\underline{G} = U(2, 1)$, and a particular F -automorphism ε of $\tilde{\underline{G}} = R_{E/F}\underline{G}$. We discuss the structure of these groups. We also describe the reductive quotients of the stabilizer groups of points in the Bruhat-Tits building of $\underline{G}$ and classify their representations. We then use this to describe the representations of $\underline{G}(F)$ of depth zero.

In §5, we give an explicit description of the depth-zero L -packets for ramified unitary groups in two variables, as well as their base change lifts to $\mathrm{GL}(2, F)$. There are two reasons to do so. First, in order to describe explicit base change for all representations of $U(2, 1)$ (not just of depth zero), one would need to understand depth-zero base change for these smaller groups, since they arise in the constructions by J.-K. Yu [48] and S. Stevens [47] of positive-depth supercuspidal representations. Second, some of our understanding of base change for depth-zero representations comes via endoscopy, and $U(1, 1) \times U(1)$ is an endoscopic group for $U(2, 1)$. In order to handle the case of the anisotropic group $U(2)$, we must assume the existence of a Jacquet-Langlands-like correspondence between depth-zero, discrete-series representations of $U(1, 1)$ and those of $U(2)$. Conjecturally, such a correspondence is determined by the theta correspondence between $U(2)$ and $U(1, 1)$. See §5.4.

In §6, we explicitly determine the depth-zero L -packets and A -packets for $\underline{G}(F)$. Certain A -packets and L -packets of G have size 2 and contain both a supercuspidal and a nonsupercuspidal representation. Such packets are parametrized by certain (Weyl equivalence classes of) characters λ of the diagonal torus of $U(2, 1)(F)$. Proposition 6.5 completely determines the relationship between the two representations in such packets, and this is sufficient for the verification of this case of Theorem 1.4 in §7. In particular, an understanding of the explicit dependence of the elements of such a packet on the character λ is not required. For completeness, we nonetheless provide such a description in Proposition 6.6. The proof, which we do not include here, involves computations of the values of certain irreducible characters of $\mathrm{SL}(2, k_F)$ on unipotent elements.

In §7, we determine the base change lift of each of the depth-zero packets of $\underline{G}(F)$. In §8 we verify that the types of the representations in a given packet are related to those of the base change lift of the packet according to Theorem 1.4.

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2. GENERAL NOTATION AND FACTS

Given a field K , we will denote its algebraic closure by $\overline{K}$. We will generally use underlined letters to denote algebraic groups and corresponding ordinary letters to denote groups of rational points, where the field of rationality will be clear from the context. For any algebraic K -group $\underline{H}$, we will let $\underline{H}^\circ$ denote the connected component.

Let ε be a K -automorphism of $\underline{H}$ of finite order. We will say that an element h of $\underline{H}$ is ε -semisimple if $h\varepsilon$ is a semisimple element of the semidirect product $\underline{H} \rtimes \langle \varepsilon \rangle$. Let h be such an element, and let $\underline{C}$ be the fixed point group of $\mathrm{Int}(h) \circ \varepsilon$ in $\underline{H}$.

Then we will call $h \in \underline{H}$ ε -regular (resp. *strongly* ε -regular) if $\underline{C}^\circ$ (resp. $\underline{C}$) is a torus. If ε is trivial we will drop the ε in the above terminology. Note that h is ε -regular if and only if $h\varepsilon$ is a regular element of $\underline{H} \rtimes \langle \varepsilon \rangle$. Let $\underline{H}^{\varepsilon\text{-reg}}$ denote the set of ε -regular elements of $\underline{H}$. If ε is trivial, this is just the set $\underline{H}^{\text{reg}}$ of regular elements of $\underline{H}$.

We will say that two elements $g, h \in H$ are ε -conjugate (resp. *stably* ε -conjugate) if $h = xg\varepsilon(x)^{-1}$ for some x in H (resp. $\underline{H}(\overline{K})$).

Let $\underline{G}$ be an algebraic K -group, and let L/K be a finite extension. Let $\tilde{\underline{G}} = R_{L/K}(\underline{G})$, the algebraic K -group obtained from $\underline{G}$ via restriction of scalars from L to K . Then $\tilde{\underline{G}}(K)$ can be identified with $\underline{G}(L)$. Moreover, $\tilde{\underline{G}}$ is L -isomorphic to the direct product $\underline{G} \times \cdots \times \underline{G}$ of $[L : K]$ copies of $\underline{G}$. We will be interested in the case where L/K is cyclic. In this case, let ι be a generator of $\text{Gal}(L/K)$. Then $\tilde{\underline{G}}$ is equipped with a K -automorphism ε whose action on $\tilde{\underline{G}}(K)$ corresponds to that of ι on $\underline{G}(L)$. Moreover, the fixed-point group $\tilde{\underline{G}}^\varepsilon$ can be identified with $\underline{G}$.

If L/K is a quadratic extension, we let $U(1, L/K)$ denote the kernel of the norm map $R_{L/K}(\text{GL}(1)) \rightarrow \text{GL}(1)$ of K -groups given by $x \mapsto x\varepsilon(x)$, where ε is the automorphism of $R_{L/K}(\text{GL}(1))$ corresponding to the nontrivial element of $\text{Gal}(L/K)$. We will sometimes denote $U(1, L/K)(K)$ by L^1 when K is understood.

For any reductive algebraic group $\underline{G}$ defined over either a finite or nonarchimedean local field K , we have the following notation and terminology.

- $\mathcal{E}(G)$ will denote the set of irreducible representations of G .
- $\mathbf{1}_G$ will denote the trivial representation of G .
- St_G will denote the Steinberg representation of G .
- For any representation σ of a subgroup H of G , $\text{ind}_H^G \sigma$ will denote the representation of G obtained from σ via normalized compact induction. If σ is irreducible, then ω_σ will denote the central character of σ .
- For any admissible, finite-length representation π of G , let θ_π denote the character of π , considered either as a function on the set of elements or conjugacy classes of G (of G^{reg} in the local-field case), or as a distribution on an appropriate function space on G .
- Suppose ε is an automorphism of G . Then ε acts in a natural way on the set of equivalence classes of irreducible, admissible representations of G . Suppose π is such a representation and $\pi \cong {}^\varepsilon \pi$. Let $\pi(\varepsilon)$ denote an intertwining operator from π to ${}^\varepsilon \pi$. If ε has order ℓ , then we can and will normalize $\pi(\varepsilon)$ by requiring that the scalar $\pi(\varepsilon)^\ell$ equal 1. Then $\pi(\varepsilon)$ is well determined up to a scalar ℓ th root of unity. The ε -twisted character of π is the distribution $\theta_{\pi, \varepsilon}$ defined by $\theta_{\pi, \varepsilon}(f) = \text{Tr}(\pi(f)\pi(\varepsilon))$ for $f \in C_c^\infty(G)$, the space of compactly supported, locally constant functions on G . As with the character, the twisted character can be represented by a function (again denoted $\theta_{\pi, \varepsilon}$) on G ($G^{\varepsilon\text{-reg}}$ in the local-field case), or on its set of ε -conjugacy classes.
- A class function $f : G \rightarrow \mathbb{C}$ will be called *stable* if it is constant on every stable conjugacy class of G .
- A locally constant function f on G will be called an ε -stable class function, or simply ε -stable, if $f(x) = f(y)$ for all $x, y \in G$ such that $y = gx\varepsilon(g)^{-1}$ for some $g \in \underline{G}(\overline{F})$.

- For any maximal K -torus $\underline{T}$ of a K -group $\underline{G}$, let $W(\underline{T}, \underline{G})$ denote the quotient of $\underline{T}$ in its normalizer in $\underline{G}$, and let $W_K(\underline{T}, \underline{G})$ denote the group of K -points of the absolute Weyl group $N_{\underline{G}}(\underline{T})/\underline{T}$.

Now fix a nonarchimedean local field F of residual characteristic p . For any algebraic extension E/F , let $\mathcal{O}_E$ denote the ring of integers of E , $\mathfrak{p}_E$ the prime ideal in $\mathcal{O}_E$, and $k_E = \mathcal{O}_E/\mathfrak{p}_E$ the residue field. Let q_E denote the order of k_E . Let $|\cdot|_E$ denote the absolute value on E , normalized so that primes in $\mathcal{O}_E$ have absolute value q_E^{-1} . In the case $E = F$, we will drop the subscripts E above. For any abelian extension E/F , let $\omega_{E/F}$ denote the character of $F^\times$ associated to E/F by local class field theory.

Let $\widehat{\underline{G}}$ denote the complex dual group of $\underline{G}$. The L -group of $\underline{G}$ is the group ${}^L\underline{G} = \widehat{\underline{G}} \rtimes W_F$, where W_F is the Weil group of F . We note that if $\widetilde{\underline{G}} = R_{E/F}\widehat{\underline{G}}$, then the dual group of $\widetilde{\underline{G}}$ is the direct product of $[E:F]$ copies of $\widehat{\underline{G}}$, and we have a natural diagonal embedding of ${}^L\underline{G}$ in ${}^L\widetilde{\underline{G}}$.

For every reductive algebraic F -group $\underline{G}$ and every algebraic extension E/F of finite ramification degree, one has an associated *extended affine building* $\mathcal{B}(\underline{G}, E)$, as defined by Bruhat and Tits [8, 9]. As a $\underline{G}(E)$ -set, $\mathcal{B}(\underline{G}, E)$ is a direct product of an affine space (on which $\underline{G}^\circ(E)$ acts via translation) and the *reduced building* $\mathcal{B}^{\text{red}}(\underline{G}, E)$, which depends only on $\underline{G}/\underline{Z}$, where $\underline{Z}$ is the center of $\underline{G}$. Note that $\underline{Z}(E)$ fixes $\mathcal{B}^{\text{red}}(\underline{G}, E)$. The building $\mathcal{B}(\underline{G}, F)$ always has a natural embedding into $\mathcal{B}(\underline{G}, E)$. To every point $x \in \mathcal{B}(\underline{G}, E)$ there is an associated parahoric subgroup $\underline{G}(E)_x$ of $\underline{G}(E)$. The stabilizer $\text{stab}_{\underline{G}(E)}(x)$ of x in $\underline{G}(E)$ contains $\underline{G}(E)_x$ with finite index.

The pro- p -radical of G_x is denoted G_{x+} , and the quotient $\text{stab}_G(x)/G_{x+}$ is the group $\mathbf{G}_x$ of rational points of a reductive k_F -group $\underline{\mathbf{G}}_x$. The quotient G_x/G_{x+} is the group $\mathbf{G}_x^\circ$ of rational points of $(\underline{\mathbf{G}}_x)^\circ$. (Note that it is more common to use $\underline{\mathbf{G}}_x$ to denote what we are calling $\underline{\mathbf{G}}_x^\circ$.)

More generally, one could deal with parahoric subgroups of $\underline{G}(E)$ and could form the quotient $\text{stab}_{\underline{G}(E)}(x)/\underline{G}(E)_{x+}$, which is isomorphic to the group of rational points of a reductive k_E -group that we will for the moment denote $\underline{\mathbf{G}}_x^E$ (we will not need this notation later). Suppose E/F is a Galois extension, and let $\Gamma = \text{Gal}(E/F)$. If E/F is tamely ramified, then by Corollary 2.3 of [48], the group of fixed points $(\underline{\mathbf{G}}_x^E)^\Gamma$ for the action of Γ on $\underline{\mathbf{G}}_x^E$ is equal to $\mathbf{G}_x$. Note that if E/F is unramified, then $\underline{\mathbf{G}}_x^E = R_{k_E/k_F}\underline{\mathbf{G}}_x$, so that $\underline{\mathbf{G}}_x^E = \underline{\mathbf{G}}_x(k_E)$. Moreover, we may identify Γ with $\text{Gal}(k_E/k_F)$, and the resulting actions of Γ on $\underline{\mathbf{G}}_x^E$ and on $\underline{\mathbf{G}}_x(k_E)$ are the same. But when E/F is ramified, the situation is more complicated. See §4.2 for examples.

These objects do not depend specifically on x . Rather, they depend on the facet $\mathcal{F}$ in $\mathcal{B}(\underline{G}, E)$ containing x . Thus, we will feel free to replace x by $\mathcal{F}$ in the subscripts above. In case $\underline{G}^\circ$ is a torus, the building contains only one facet, and we may write G_0 , G_{0+} , and $\underline{\mathbf{G}}$ instead of G_x , G_{x+} , and $\underline{\mathbf{G}}_x$, since the latter are independent of the choice of x .

In general, given an E -group and a point or facet in the building, we will use sans-serif lettering to denote the corresponding k_E -group.

Let $\underline{T}$ be an F -torus. If a character λ of $\underline{T}$ is trivial on T_{0+} , we let $\bar{\lambda}$ denote the corresponding character of $\mathbf{T} = T_0/T_{0+}$.

Following the notation in [40], if $\underline{T}$ is an F -torus in $\underline{G}$, let $\mathcal{D}_{\underline{G}}(\underline{T}/F)$ denote the kernel of the natural map $H^1(F, \underline{T}) \rightarrow H^1(F, \underline{G})$.

If $\underline{G}$ is a connected reductive k_F -group, $\underline{T}$ is a maximal k -torus in $\underline{G}$, and θ is a (complex) character of $\underline{T}$, then let $R_{\underline{T}}^{\underline{G}}\theta$ denote the corresponding Deligne-Lusztig virtual character of $\underline{G}$ [17].

Given a surjection $A \rightarrow B$ between abstract groups A and B , and a representation ρ of B , we can pull back (or “inflate”) ρ to form a representation of A , denoted $\text{infl}(\rho)$. Whenever this notation is used, the surjection will be clear from the context.

3. LIFTINGS

3.1. Norm mappings and Shintani lifting. Let $\underline{H}$ be a connected reductive group defined over a field K , and let ε be a K -automorphism of $\underline{H}$ of finite order ℓ . One can define a norm mapping $N_{\varepsilon}^{\underline{H}}$ from $\underline{H}$ to $\underline{H}$ by

$$h \mapsto h \cdot \varepsilon(h) \cdots \varepsilon^{\ell-1}(h).$$

We note that if elements h and h' in $\underline{H}(\overline{K})$ are ε -conjugate, then $N_{\varepsilon}^{\underline{H}}(h)$ and $N_{\varepsilon}^{\underline{H}}(h')$ are conjugate in $\underline{H}(\overline{K})$. Thus the norm gives a map from the set of ε -conjugacy classes in $\underline{H}$ to the set of conjugacy classes in $\underline{H}$.

Suppose $\underline{H}$ is abelian, and let $\underline{C} = (\underline{H}^{\varepsilon})^{\circ}$ be the identity component of the group of ε -fixed points of $\underline{H}$. Then $N_{\varepsilon}^{\underline{H}}$ gives an F -homomorphism from $\underline{H}$ to $\underline{C}$. For any character λ of $\underline{C}$, define the lifted character $\tilde{\lambda}$ of $\underline{H}$ by $\tilde{\lambda} = \lambda \circ N_{\varepsilon}^{\underline{H}}$. Note the abuse of notation in this definition—the group $\underline{C}$ does not, of course, determine $\underline{H}$ or ε . However, whenever we use this notation, $\underline{H}$ and ε will be clear from the context.

Now suppose L/K is a finite cyclic extension of degree ℓ of local or finite fields. Let ι be a generator of $\text{Gal}(L/K)$. As in §2, let $\underline{G}$ be a connected reductive K -group and $\tilde{\underline{G}} = R_{L/K}\underline{G}$. Let ε be the K -automorphism of $\tilde{\underline{G}}$ corresponding to ι . Under various conditions (for example, if $\underline{G}$ is quasi-split and has a simply connected derived group), the map $N_{\varepsilon}^{\tilde{\underline{G}}}$ determines an injective map $\mathcal{N} = \mathcal{N}_{L/K}^{\underline{G}}$ from the set of stable, strongly ε -regular ε -conjugacy classes of $\tilde{\underline{G}}$ to the set of stable, strongly regular conjugacy classes in $\underline{G}$.

If Π and $\tilde{\Pi}$ are finite sets of irreducible (smooth if K is local) representations of $\underline{G}$ and $\tilde{\underline{G}}$, respectively, we say that $\tilde{\Pi}$ is a *Shintani lift* of Π if

$$\Theta_{\tilde{\Pi}, \varepsilon}(g) = \Theta_{\Pi}(\mathcal{N}(g))$$

for all $g \in \tilde{\underline{G}}^{\varepsilon\text{-reg}}$ (all $g \in \tilde{\underline{G}}$ in the finite-field case), where Θ_{Π} and $\Theta_{\tilde{\Pi}, \varepsilon}$ are nontrivial stable (resp. ε -stable) linear combinations of the characters (resp. ε -twisted characters) of the elements of Π (resp. $\tilde{\Pi}$). Note that, at least in the p -adic case, it is easily seen that the Shintani lifting is independent of the choice of generator ι .

In most cases, one expects the Shintani and base change liftings to coincide, although exceptions to this are known (see [29] and [40, §13.2]). Rogawski has shown that this is indeed the case for all irreducible smooth tempered representations of quasi-split p -adic unitary groups in two and three variables [40, §11, 13].

3.2. A related lifting for representations of finite groups of Lie type. For a connected reductive k -group $\underline{H}$, let $\underline{H}^*$ denote the dual k -group. The dual group is not canonical, but no statement we make will depend on its explicit realization.

There is a canonical one-to-one correspondence between the rational conjugacy classes of maximal k -tori of $\underline{H}$ and those of $\underline{H}^*$ [15, Prop. 4.3.4]. Let $\underline{T}$ be a maximal k -torus of $\underline{H}$ and let $\underline{T}^*$ be a maximal k -torus of $\underline{H}^*$ whose rational conjugacy class corresponds to that of $\underline{T}$. We will say that $\underline{T}$ and $\underline{T}^*$ are k -dual to each other. Once a fixed embedding of $k^\times$ in $\mathbb{C}^\times$ is chosen, we may identify $\underline{T}^*$ with the group of characters of $\underline{T}$.

By work of Lusztig [33] and Digne-Lehrer-Michel [20], the set $\mathcal{E}(\underline{H})$ of irreducible representations of $\underline{H}$ can be partitioned into a collection of families $\mathcal{E}_{(s)}(\underline{H})$, called *rational series*, parametrized by semisimple conjugacy classes (s) in $\underline{H}^*$. The set $\mathcal{E}_{(s)}(\underline{H})$ consists of those elements of $\mathcal{E}(\underline{H})$ whose characters occur in a Deligne-Lusztig virtual character $R_{\underline{T}}^{\underline{H}}\chi$, where $\underline{T}$ is a maximal k -torus of $\underline{H}$ that is k -dual to a maximal k -torus $\underline{T}^*$ of $\underline{H}^*$ containing s , and χ is the character of $\underline{T}$ that corresponds to $s \in \underline{T}^*$.

Now, as in §3.1, let $\tilde{\underline{G}}$ denote a connected reductive k -group, and ε a k -automorphism of $\tilde{\underline{G}}$. We will assume that ε is *quasi-semisimple*, meaning that ε fixes some pair $(\tilde{\underline{B}}_0, \tilde{\underline{T}}_0)$ consisting of a Borel subgroup $\tilde{\underline{B}}_0$ of $\tilde{\underline{G}}$ and a maximal torus $\tilde{\underline{T}}_0$ of $\tilde{\underline{B}}_0$. Let $\underline{\tilde{G}} = (\tilde{\underline{G}}^\varepsilon)^\circ$ denote the connected part of the group of ε -fixed points in $\tilde{\underline{G}}$. From [46, Corollary 9.4] or [30, Theorem 1.1.A(1)], $\underline{\tilde{G}}$ is reductive. Let $\underline{T} \subseteq \underline{\tilde{G}}$ denote a maximal k -torus, and let $\tilde{\underline{T}}$ denote the centralizer of $\underline{T}$ in $\tilde{\underline{G}}$. Then from [30, Thm. 1.1.A], $\tilde{\underline{T}}$ is a maximal k -torus in $\underline{\tilde{G}}$.

The maximal k -tori $\underline{T}$ and $\tilde{\underline{T}}$ determine, up to k -conjugacy, maximal k -tori $\underline{T}^*$ and $\tilde{\underline{T}}^*$ in the dual groups $\underline{\tilde{G}}^*$ and $\tilde{\underline{G}}^*$, respectively. The map $N_{\varepsilon}^{\tilde{\underline{T}}}: \tilde{\underline{T}} \rightarrow \underline{T}$ determines a map $N_{\underline{T}^*, \tilde{\underline{T}}^*}^*: \underline{T}^* \rightarrow \tilde{\underline{T}}^*$.

Theorem 3.1. *There is a map*

$$\begin{aligned} \mathcal{N}^*: \{ \text{semisimple conjugacy classes in } \underline{\mathbf{G}}^* \} \\ \longrightarrow \{ \text{semisimple conjugacy classes in } \tilde{\underline{\mathbf{G}}}^* \} \end{aligned}$$

that is compatible with the maps $N_{\underline{T}^*, \tilde{\underline{T}}^*}^*$ in the sense that if $s \in \underline{\mathbf{T}}^*$, then $N_{\underline{T}^*, \tilde{\underline{T}}^*}^*(s) \in \mathcal{N}^*((s))$.

Thus, we have a lifting of families of irreducible representations from $\underline{\mathbf{G}}$ to $\tilde{\underline{\mathbf{G}}}$:

$$\mathcal{E}_{(s)}(\underline{\mathbf{G}}) \mapsto \mathcal{E}_{\mathcal{N}^*((s))}(\tilde{\underline{\mathbf{G}}}).$$

We will refer to this as the ε -lifting.

We will prove Theorem 3.1 elsewhere [2]. We will also refine the ε -lifting by showing in many cases that one can lift subsets of $\mathcal{E}_{(s)}(\underline{\mathbf{G}})$ to subsets of $\mathcal{E}_{\mathcal{N}^*((s))}(\underline{\mathbf{G}})$. For now, note that if the character of an irreducible representation π of $\underline{\mathbf{G}}$ appears in $R_{\underline{T}}^{\underline{\mathbf{G}}}\chi$, then the character of the ε -lifting $\tilde{\pi}$ of π is contained in $R_{\tilde{\underline{T}}}^{\tilde{\underline{\mathbf{G}}}}\tilde{\chi}$, where $\tilde{\chi} = \chi \circ N_{\varepsilon}^{\tilde{\underline{T}}}$.

4. UNITARY GROUPS IN THREE VARIABLES

4.1. Specific notation. We will assume from now on that the characteristic of the nonarchimedean local field F is zero and the residual characteristic p is odd. Let $\bar{F}$ denote a fixed algebraic closure of F . Let ϖ denote a uniformizer of F . Let E and E' denote the two ramified quadratic extensions of F in $\bar{F}$. Then $k_E = k_{E'} = k$. Fix

uniformizers ϖ_E and $\varpi_{E'}$ for E and E' (respectively) such that $\varpi_E^2 = \varpi_{E'}^2 = \varpi$. Also, fix a character Ω of $E^\times$ extending $\omega_{E/F}$ that is trivial on $1 + \mathfrak{p}_E$. Note that there are two such characters. Let ι denote the generator of $\Gamma := \text{Gal}(E/F)$. For any extension L/F , we have an action of ι on $E \otimes L$ given by $x \otimes y \mapsto \iota(x) \otimes y$.

Let ε' be the F -automorphism of $R_{E/F}\text{GL}(3)$ corresponding to ι . Let

$$\Phi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \text{GL}(3, F) \subset (R_{E/F}\text{GL}(3))(F).$$

From now on, let $\underline{G}$ denote the group of fixed points in $R_{E/F}\text{GL}(3)$ of the automorphism $\varepsilon : x \mapsto \text{Int}(\Phi)({}^t\varepsilon'(x)^{-1})$, where ty denotes the transpose of the matrix y . Then $\underline{G}$ is the unique (up to isomorphism) unitary group $U(2, 1)$ in three variables with respect to E/F , and $\tilde{\underline{G}}$ is isomorphic to $R_{E/F}\text{GL}(3)$. For any extension L/F , we can and will explicitly realize $\underline{G}(L)$ as the group

$$\{g \in \text{GL}(3, E \otimes L) \mid g \Phi {}^t(\iota(g)) = \Phi\},$$

where the action of ι on $\text{GL}(3, E \otimes L)$ is coordinatewise. Then the action of ε on $\text{GL}(3, E \otimes L)$ is given by $g \mapsto \Phi {}^t\iota(g)^{-1} \Phi^{-1}$.

Let $\underline{Z}$ denote the center of $\underline{G}$. So, following our notational conventions, $\tilde{\underline{Z}}$ can be identified with the center of $\tilde{\underline{G}}$, and Z is the reductive quotient of $\tilde{Z}$. Note that, since Z is isomorphic to E^1 , Z has only two elements.

Let $\mathcal{B} = \mathcal{B}(\underline{G}, F)$ and $\tilde{\mathcal{B}} = \mathcal{B}(\tilde{\underline{G}}, F)$. Then $\tilde{\mathcal{B}}$ can be identified with $\mathcal{B}(\underline{G}, E)$, and the action of ε on $\tilde{\mathcal{B}}$ corresponds to the action of ι on $\mathcal{B}(\underline{G}, E)$. Thus from [41] we may and will identify the set of fixed points $\tilde{\mathcal{B}}^\varepsilon$ with $\mathcal{B}$.

For any F -group $\underline{Q}$ for which the norm map is defined, let $\mathcal{N}^{\underline{Q}} = \mathcal{N}_{E/F}^{\underline{Q}}$. When $\underline{Q} = \underline{G}$, we simply write $\mathcal{N}$.

There is a unique homomorphism $\tilde{\underline{G}} \rightarrow \text{GL}(1)$ that agrees with the usual determinant map on $\tilde{\underline{G}}$. Given any algebraic subgroup $\underline{S}$ of $\tilde{\underline{G}}$, we will denote by $\det_{\underline{S}}$ the restriction of this homomorphism to $\underline{S}$. We will omit the subscript when it is clear from the context.

Since $\underline{G}$ is F -quasi-split, it contains F -Borel subgroups. In particular, $\tilde{\mathcal{B}}$ must contain some ε -invariant apartment $\tilde{\mathcal{A}}$ with more than one ε -fixed point. Choose an ε -fixed point y in an ε -invariant minimal facet in $\tilde{\mathcal{A}}$ and an ε -invariant alcove $\tilde{\mathcal{F}}$ in $\tilde{\mathcal{A}}$ such that the closure of $\tilde{\mathcal{F}}$ contains y . Let $\mathcal{F}$ denote the set of ε -fixed points of $\tilde{\mathcal{F}}$. Then these choices determine an F -Borel subgroup $\underline{B}$ together with a Levi factor $\underline{M}$ of $\underline{B}$. Note that $\underline{M}$ is isomorphic to $E^\times \times E^1$. We may assume that our choices of y and $\tilde{\mathcal{F}}$ are such that $\underline{B}$ is the group of upper-triangular matrices in $\underline{G}$, $\underline{M}$ is the group of diagonal matrices, and $\tilde{G}_y$ is $\text{GL}(3, \mathcal{O}_E)$.

The boundary of $\mathcal{F}$ contains two points: the previously chosen point y , and another point that we will denote z . Note that $\tilde{\mathcal{F}}$ is the direct product of a one-dimensional affine space and an ε -invariant equilateral triangle Δ in the reduced building of $\tilde{\underline{G}}$ (which we will identify with a subset of $\tilde{\mathcal{B}}$), y is the ε -fixed vertex of Δ , and z is the midpoint of the wall of Δ that is opposite y . In the building $\mathcal{B}$, y and z are both special vertices, but neither is hyperspecial.

Let $\underline{\mathcal{B}}_y$ (resp. $\underline{\mathcal{B}}_z$) denote the Borel subgroup of $\underline{\mathcal{B}}_y$ (resp. $\underline{\mathcal{B}}_z$) determined by $\mathcal{F}$.

Let $\tilde{\underline{H}}$ be the subgroup of $\tilde{\underline{G}}$ whose L -rational points (for any extension L/F) consist of those matrices of the form

$$\begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix} \subset \text{GL}(3, E \otimes L).$$

Then $\tilde{H}$ is a Levi factor of a parabolic F -subgroup $\tilde{P}$. Let $\underline{H}$ denote the group $\tilde{H} \cap \underline{G}$. Note that $\tilde{H}$ is isomorphic to $R_{E/F}\underline{H}$. We have that $\underline{H}$ is an E -Levi, but not F -Levi, subgroup of $\underline{G}$. It is an endoscopic group for $\underline{G}$, isomorphic to a direct product of $\underline{Z} \cong U(1, E/F)$ and the F -subgroup $\underline{H}^0$ of $\underline{H}$ consisting of matrices of the form

$$\begin{pmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{pmatrix}.$$

We note that $\underline{H}^0$ is a quasi-split ramified unitary group in two variables.

4.2. Reductive quotients of stabilizers and their representations. For us, $\tilde{\mathbf{G}}_x$ will always mean the reductive quotient of $\tilde{G}$ corresponding to the point $x \in \mathcal{B}(\underline{G}, F)$, rather than, say, the restriction of scalars $R_{k_E/k_F}\tilde{\mathbf{G}}_x$, though these are the same when E/F is unramified. For x equal to either of the points y or z , or the facet $\mathcal{F}$ in between them, we will compute $\mathbf{G}_x$ and describe its irreducible representations.

4.2.1. The group $\mathbf{G}_y$. Recall that $\tilde{G}_y$ is the group $\mathrm{GL}(3, \mathcal{O}_E)$. Explicitly, $\tilde{G}_y$ is the stabilizer in $\mathrm{GL}(3, E)$ of the lattice $\mathcal{L} = \mathcal{O}_E \oplus \mathcal{O}_E \oplus \mathcal{O}_E$ inside $V = E^3$. Then $\tilde{G}_{y+}$ is the subgroup of elements whose image in $\mathrm{GL}(\mathcal{L}/\varpi_E\mathcal{L})$ is trivial. Thus, with our identifications, $\tilde{\mathbf{G}}_y = \mathrm{GL}(\mathcal{L}/\varpi_E\mathcal{L}) \cong \mathrm{GL}(3, k)$. The action of ε is given by $g \mapsto \Phi({}^t g^{-1})\Phi^{-1}$, so $\mathbf{G}_y = \mathrm{O}(2, 1)(k)$, the orthogonal group determined by the symmetric form whose matrix with respect to the standard basis is Φ , and $\mathbf{G}_y^\circ = \mathrm{SO}(2, 1)(k)$. In particular, $\mathbf{G}_y = \mathbf{G}_y^\circ \times \mathbf{Z}$.

We now briefly classify the irreducible representations of $\mathbf{G}_y$. Since any such representation is of the form $\tau \otimes \nu$ for some representation τ of $\mathbf{G}_y^\circ$ and one of the two characters ν of $\mathbf{Z}$, we turn our attention to the irreducible representations of $\mathbf{G}_y^\circ$. Since $\mathrm{SO}(2, 1)(k)$ is isomorphic to $\mathrm{PGL}(2, k)$, representations of $\mathbf{G}_y^\circ$ correspond to representations of $\mathrm{GL}(2, k)$ having trivial central character. The following classification is easily derived from [45, Chapter VIII].

The group $\underline{\mathbf{M}}^\circ$ embeds in the Borel subgroup $\underline{\mathbf{B}}_y^\circ$ of $\underline{\mathbf{G}}_y^\circ$. If χ is a character of $\mathbf{M}^\circ$, then the induced representation $\mathrm{ind}_{\underline{\mathbf{B}}_y^\circ}^{\mathbf{G}_y^\circ} \chi$ is irreducible except when χ extends to a character χ_0 of $\mathbf{G}_y^\circ$ (i.e., when χ has order dividing 2). In this case, the induced representation is the sum of χ_0 and $\mathrm{St}_{\mathbf{G}_y^\circ} \cdot \chi_0$.

The remaining representations are cuspidal of the following form. If l/k is a quadratic extension, then $\underline{\mathbf{G}}_y^\circ$ contains an elliptic torus $\underline{\mathbf{I}} \cong U(1, l/k)$. For any character χ of $\mathbf{T}$ of order greater than 2 (i.e., χ with trivial stabilizer in $W_k(\underline{\mathbf{I}}, \underline{\mathbf{G}}_y^\circ)$), we have a cuspidal Deligne-Lusztig representation of $\mathbf{G}_y^\circ$ of degree $q - 1$ whose character is $-R_{\underline{\mathbf{I}}}^{\underline{\mathbf{G}}_y^\circ} \chi$. We note that $R_{\underline{\mathbf{I}}}^{\underline{\mathbf{G}}_y^\circ} \chi = R_{\underline{\mathbf{I}}}^{\underline{\mathbf{G}}_y^\circ} \chi'$ if and only if $\chi' = \chi^{\pm 1}$.

4.2.2. The group $\mathbf{G}_z$. The group $\tilde{G}_z$ is the intersection of the stabilizers in $\mathrm{GL}(3, E)$ of two lattices: $\mathcal{L}' := \mathcal{O}_E \oplus \mathcal{O}_E \oplus \varpi_E \mathcal{O}_E$, and $\mathcal{L}'' := \mathcal{O}_E \oplus \varpi_E \mathcal{O}_E \oplus \varpi_E \mathcal{O}_E$. Explicitly, we have

$$\tilde{G}_z = \mathrm{GL}(3, E) \cap \begin{bmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathfrak{p}_E^{-1} \\ \mathfrak{p}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathfrak{p}_E & \mathfrak{p}_E & \mathcal{O}_E \end{bmatrix} \text{ and } \tilde{G}_{z+} = 1 + \begin{bmatrix} \mathfrak{p}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathfrak{p}_E & \mathfrak{p}_E & \mathcal{O}_E \\ \mathfrak{p}_E^2 & \mathfrak{p}_E & \mathfrak{p}_E \end{bmatrix}.$$

In particular, $\tilde{\mathbf{G}}_z \cong \mathrm{GL}(2, k) \times \mathrm{GL}(1, k)$. Note that ε acts on the first factor via $g \mapsto g/\det(g)$ and on the second factor via $g \mapsto g^{-1}$. Thus, $\mathbf{G}_z \cong \mathrm{SL}(2, k) \times \{\pm 1\}$ and $\mathbf{G}_z^\circ \cong \mathrm{SL}(2, k)$. As in the case of $\mathbf{G}_y$, we have $\mathbf{G}_z = \mathbf{G}_z^\circ \times \mathbf{Z}$. Note that here the

factor $Z \cong \{\pm 1\}$ embeds as the group of scalar matrices in G_z , not as the group $\begin{pmatrix} 1 & \\ & \pm 1 \\ & & 1 \end{pmatrix}$.

As in the case of the vertex y , an irreducible representation of G_z can be expressed in the form $\tau \otimes \nu$, where τ is an irreducible representation of G_z° and ν is a character of Z . The representations of G_z° are as follows.

If χ is a character of M , then the representation $\text{ind}_{B_z}^{G_z^\circ} \chi$ is irreducible unless χ has order dividing 2. In this case, the representation is a sum of two irreducible components. If χ is trivial, then these components are $\mathbf{1}$ and $\text{St}_{G_z^\circ}$, while if χ has order 2, then the components ϑ and ϑ' each have degree $(q+1)/2$.

Again, G_z° contains an elliptic torus $\mathbb{T} \cong U(1, l/k)$. For any character χ of $\mathbb{T}$ of order greater than 2, we have a cuspidal Deligne-Lusztig representation of G_z° of degree $q-1$ whose character is $-R_{\mathbb{T}}^{G_z^\circ} \chi$. As with G_y , $R_{\mathbb{T}}^{G_z^\circ} \chi = R_{\mathbb{T}}^{G_z^\circ} \chi'$ if and only if $\chi' = \chi^{\pm 1}$. The remaining (cuspidal) representations of G_z° are the two degree- $(q-1)/2$ components of $-R_{\mathbb{T}}^{G_z^\circ} \chi$ when χ has order 2.

4.2.3. The group $G_{\mathcal{F}}$. The group $\tilde{G}_{\mathcal{F}}$ is the standard, upper-triangular Iwahori subgroup of $\text{GL}(3, E)$, so $\tilde{G}_{\mathcal{F}}$ is the reductive quotient of the upper-triangular Borel subgroup of $\tilde{G}_y$. Therefore,

$$G_{\mathcal{F}} = \left\{ \begin{pmatrix} a & \pm 1 & \\ & a^{-1} & \\ & & 1 \end{pmatrix} \mid a \in k^\times \right\} \quad \text{and} \quad G_{\mathcal{F}}^\circ = \left\{ \begin{pmatrix} a & 1 & \\ & a^{-1} & \\ & & 1 \end{pmatrix} \mid a \in k^\times \right\}.$$

We may thus write $G_{\mathcal{F}} = G_{\mathcal{F}}^\circ \times Z$. The representation theory of $G_{\mathcal{F}}$ is elementary.

4.3. Maximal tori. Up to stable conjugacy, there are four families of maximal tori of $\underline{G}$. According to the classification in [40], they are isomorphic to:

(4.3-0) $R_{E/F}(\text{GL}(1)) \times U(1, E/F)$,

(4.3-1) $U(1, E/F) \times U(1, E/F) \times U(1, E/F)$,

(4.3-2) $R_{K/F}(U(1, EK/K)) \times U(1, E/F)$ for $K \neq E$ a quadratic extension of F ,

(4.3-3) $R_{K/F}(U(1, EK/K))$ for K a cubic extension of F .

Note that all maximal tori of $\underline{G}$ are elliptic except for those of type (4.3-0).

4.3.1. A torus of type (4.3-1). Up to conjugacy, $\underline{H}$ contains two F -tori in the family (4.3-1) (see [40, §3.6]). Let $\underline{C}$ be the maximal F -torus of $\underline{H}$ whose group of F -points C consists of those matrices γ of the form

$$\gamma = \begin{pmatrix} \frac{\gamma_1 + \gamma_3}{2} & 0 & \frac{\gamma_1 - \gamma_3}{2} \\ 0 & \gamma_2 & 0 \\ \frac{\gamma_1 - \gamma_3}{2} & 0 & \frac{\gamma_1 + \gamma_3}{2} \end{pmatrix},$$

where $\gamma_i \in E^1$. The group $\underline{C}$ is of type (4.3-1), and we identify C with $E^1 \times E^1 \times E^1$ via the map $\gamma \mapsto (\gamma_1, \gamma_2, \gamma_3)$. The Weyl group $W(C, G)$ (resp. $W(C, H)$) acts on C by permuting the γ_i (resp. γ_1 and γ_3) transitively. We note that $\underline{C}$ is an endoscopic group of $\underline{H}$.

4.3.2. Ramified quadratic tori. Up to conjugacy, $\underline{H}$ contains one F -torus of type (4.3-2) with $K = E'$ (see [40, §3.6]). Let $\underline{S}$ be the maximal F -torus of $\underline{H}$ such that the group of F -points S consists of those matrices of the form

$$\begin{pmatrix} \frac{1}{2}(\alpha + \iota'(\alpha)) & 0 & \frac{1}{2\varpi_{E'}}(\alpha - \iota'(\alpha)) \\ 0 & \gamma & 0 \\ \frac{\varpi_{E'}}{2}(\alpha - \iota'(\alpha)) & 0 & \frac{1}{2}(\alpha + \iota'(\alpha)) \end{pmatrix},$$

where $\alpha \in \ker(N_{EE'/E'})$, $\gamma \in \ker(N_{E/F})$ and ι' is the generator of the group $\text{Gal}(EE'/E)$. Then $\underline{S}$ is a torus of the above type and $S \subset \text{stab}_G(z)$. Moreover, the group $S = S^\circ \times \{\pm 1\} \subset G_z$ is isomorphic to $U(1, l/k) \times \{\pm 1\}$, where l is a quadratic extension of k .

Proposition 4.1. *The group $\underline{G}$ contains exactly two F -conjugacy classes of F -tori of type (4.3-2) with $K = E'$. There exists a torus $\underline{S}' \subset \underline{G}$ of this type such that S' fixes the vertex y . Moreover, $\underline{S}'$ is stably conjugate but not F -conjugate to $\underline{S}$.*

The first statement follows from [40, §3.6]. To establish the existence of $\underline{S}'$, one can use the construction of Morris [34, §1]. We omit the details.

We will need the following lemma concerning the values of characters of S° at regular elements. We leave the elementary proof to the reader.

Lemma 4.2. *Let χ and ψ be nontrivial characters of S° , a cyclic group of order $q+1$, and suppose that the order of χ is greater than 2. Let X denote the set of all $s \in S^\circ$ with $s^2 \neq 1$. Let m be any map from $X \rightarrow \{\pm 1\}$.*

- (i) *The function $\chi + \chi^{-1}$ cannot vanish at every element of X if $q > 3$.*
- (ii) *The functions $\chi + \chi^{-1}$ and $-\psi - \psi^{-1}$ cannot agree on all of X if $q \neq 7$.*
- (iii) *If $\chi + \chi^{-1} = \psi + \psi^{-1}$ on X , then $\chi = \psi^{\pm 1}$.*
- (iv) *If $\chi + \chi^{-1} = m \cdot (\psi + \psi^{-1})$ on X , then $\chi = \psi^{\pm 1} \mu$, where μ has order dividing 2. If, in addition, the range of m is $\{\pm 1\}$ and $q > 7$, then μ is nontrivial.*
- (v) *The functions $\chi + \chi^{-1}$ and $m\psi$ cannot agree on all of X .*

4.3.3. Ramified cubic tori. Let K be the extension of F obtained by adjoining a cube root ϖ_K of ϖ_F in $\overline{F}$, and let $\underline{T}$ be an F -torus of type (4.3-3) associated to K . Then by [40, §3.6], $\underline{T}$ is the unique such torus up to conjugacy. Let $K' = EK = E(\varpi_{K'})$, where $\varpi_{K'}$ is the uniformizer ϖ_E/ϖ_K of K' . Then $T \cong \ker N_{K'/K}$ and $\tilde{T} \cong (K')^\times$.

We will realize the latter isomorphism (and hence the torus $\tilde{T}$) explicitly by specifying that it take $a + b\varpi_{K'} + c\varpi_{K'}^2$ ($a, b, c \in E$ not all 0) to

$$\begin{pmatrix} a & b & c \\ c\varpi_E & a & b \\ b\varpi_E & c\varpi_E & a \end{pmatrix} \in \text{GL}(3, E).$$

We note that under this isomorphism, the action of ε on $\tilde{T}$ corresponds to the action $\alpha \mapsto \iota(\alpha)^{-1}$ on $(K')^\times$, where we view ι as the generator of $\text{Gal}(K'/K)$. It is easily checked that $\tilde{T} \subset \text{stab}_{\tilde{G}}(\tilde{\mathcal{F}})$. Hence $T \subset \text{stab}_G(\mathcal{F})$, so T_0 is a subgroup of both G_y and G_z .

Lemma 4.3. *For $x = y$ or z , the image of T_0 in G_x is the unipotent radical of B_x .*

The proof is a straightforward calculation.

4.4. Depth-zero representations of G .

Principal series of G . For $\lambda \in \text{Hom}(M, \mathbb{C}^\times)$, there exist unique characters $\lambda_1 \in \text{Hom}(E^\times, \mathbb{C}^\times)$ and $\lambda_2 \in \text{Hom}(E^1, \mathbb{C}^\times)$ such that

$$(4.4.1) \quad \lambda \left(\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \iota(\alpha)^{-1} \end{pmatrix} \right) = \lambda_1(\alpha) \lambda_2(\alpha \iota(\alpha)^{-1} \beta),$$

where $\alpha \in E^\times$ and $\beta \in E^1$. By [28], $\text{ind}_B^G \lambda$ is irreducible except for the following cases:

- (4.4PS-1) $\lambda_1 = |\cdot|_E^{\pm 1}$,
- (4.4PS-2) $\lambda_1|_{F^\times} = \omega_{E/F} |\cdot|_F^{\pm 1}$,
- (4.4PS-3) λ_1 is nontrivial and $\lambda_1|_{F^\times}$ is trivial.

We note that in cases (4.4PS-1) and (4.4PS-2), changing the sign of the exponent has the effect of changing λ into one of its conjugates under the action of the Weyl group. Thus, the choice of exponent can affect the representation $\text{ind}_B^G \lambda$, but not its set of irreducible constituents.

In case (4.4PS-1), $\text{ind}_B^G \lambda$ has two constituents: the one-dimensional representation $\psi = \lambda_2 \circ \det$ and the square-integrable representation $\text{St}_G \cdot \psi$.

In case (4.4PS-2), $\text{ind}_B^G \lambda$ also has two constituents: a square-integrable representation $\pi^2(\lambda)$ and a nontempered unitary representation $\pi^n(\lambda)$.

In case (4.4PS-3), $\text{ind}_B^G \lambda$ decomposes into a direct sum $\pi_1(\lambda) \oplus \pi_2(\lambda)$.

By [36], $\text{ind}_B^G \lambda$ has depth zero if and only if λ has depth zero. Hence, in each of these cases, λ_2 is a character of $E^1/(E^1 \cap (1 + \mathfrak{p}_E)) \cong \{\pm 1\}$, so there are only two possibilities for λ_2 . Thus in case (4.4PS-1), there are only two Weyl group orbits of characters λ , hence four distinct constituents of principal series: two characters of G and two special representations.

In case (4.4PS-2), there are two possibilities for $\lambda_1|_{F^\times}$. Since λ_1 is trivial on $1 + \mathfrak{p}_E$ and $|E^\times/F^\times(1 + \mathfrak{p}_E)| = 2$, there are four possibilities for λ_1 on $E^\times$, hence four Weyl group orbits of characters λ . Therefore, there are four distinct representations $\pi^2(\lambda)$ and four distinct $\pi^n(\lambda)$.

In case (4.4PS-3), λ_1 must equal $\omega_{K/E}$, where K/E is unramified quadratic. Thus there are two possibilities for λ , and these characters are not in the same Weyl group orbit. Therefore, there are two distinct representations $\pi_1(\lambda)$ and two distinct $\pi_2(\lambda)$.

Supercuspidal representations of G . Since G has no nonminimal proper parabolic subgroups, the remaining depth-zero irreducible representations are supercuspidal. Any such representation has a unique expression of the form $\text{ind}_{\text{stab}_G(x)}^G \sigma$, where x is either y or z and where σ is the inflation to $\text{stab}_G(x)$ of a cuspidal representation of $\mathbf{G}_x$. We can write the latter representation as $\bar{\sigma} \otimes \nu$ for some character ν of $\mathbf{Z}$ and some cuspidal representation $\bar{\sigma}$ of $\mathbf{G}_x^\circ$ (see §4.2.1 and §4.2.2). Based on the classification of such cuspidal representations (see §4.2), we have the following kinds of supercuspidal representations of depth zero:

- (4.4SC-1) $\text{ind}_{\text{stab}_G(y)}^G \sigma$, where $\bar{\sigma}$ is a cuspidal representation of $\mathbf{G}_y$.
- (4.4SC-2) $\text{ind}_{\text{stab}_G(z)}^G \sigma$, where $\bar{\sigma}$ is a cuspidal representation of $\mathbf{G}_z$ of degree $q - 1$.
- (4.4SC-3) $\text{ind}_{\text{stab}_G(z)}^G \sigma$, where $\bar{\sigma}$ is a cuspidal representation of $\mathbf{G}_z$ of degree $(q - 1)/2$.

As discussed in §4.2, the representation $\bar{\sigma}$ of $\mathbf{G}_x^\circ$ ($x = y$ or z) is a component of a cuspidal representation arising via Deligne-Lusztig induction from a regular character χ of an elliptic torus in $\mathbf{G}_x^\circ$. We note that this torus can be taken to be $\mathbf{S}^\circ$ if $x = z$ and $(\mathbf{S}')^\circ$ if $x = y$. In case (4.4SC-3), χ is the character ϕ of $\mathbf{S}^\circ$ of order 2.

4.5. Base change and endoscopic transfers. In [40], Rogawski establishes the base change transfers of L -packets from H to $\tilde{H}$ and from G to $\tilde{G}$ associated to the

natural embeddings ${}^L\underline{H} \rightarrow {}^L\widetilde{\underline{H}}$ and ${}^L\underline{G} \rightarrow {}^L\widetilde{\underline{G}}$, respectively. In addition, a certain L -homomorphism ${}^L\underline{H} \rightarrow {}^L\underline{G}$ (resp. ${}^L\underline{C} \rightarrow {}^L\underline{H}$) is associated to any character ς of $E^\times$ extending $\omega_{E/F}$ (see [40, §4.8.1]), and a corresponding transfer of L -packets from H to G (resp. C to H) is established. Let Ξ_G (resp. Ξ_H) denote this transfer of L -packets in the particular case $\varsigma = \Omega$.

Due to the particular form of these L -homomorphisms, the Langlands philosophy suggests Hypothesis 1.5, which we are assuming throughout.

5. DESCRIPTION OF DEPTH-ZERO L -PACKETS AND EXPLICIT BASE CHANGE FOR UNITARY GROUPS IN TWO VARIABLES

In this section we describe the L -packets for the quasi-split group $H^0 = U(1, 1)(F)$ (defined with respect to the Hermitian form whose matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$) as well as their base change lifts to $\mathrm{GL}(2, E)$.

We do the same for the group $\underline{H}^1(F)$, where $\underline{H}^1$ denotes the F -anisotropic inner form of $\underline{H}^0$. One can realize $\underline{H}^1(F)$ explicitly as the compact unitary group $U(2)$, defined with respect to the Hermitian form whose matrix is $\begin{pmatrix} 1 & 0 \\ 0 & -\epsilon \end{pmatrix}$, where ϵ is a nonsquare unit in $F^\times$. Note that $\underline{H}^0(F) \cong \widetilde{\underline{H}}^1(F) \cong \mathrm{GL}(2, E)$.

We note that this explicit description of base change implies that the analogue of Theorem 1.4 holds for unitary groups in two variables, although we omit the verifications as they are entirely analogous to (but less complicated than) those for $U(2, 1)$.

Recall that $\underline{H} = \underline{H}^0 \times \underline{Z}$. For every subgroup $\underline{L}$ of $\underline{G}$, let $\underline{L}^0$ denote the subgroup of $\underline{H}^0$ obtained by projecting $\underline{L} \cap \underline{H}$ onto the $\underline{H}^0$ component of $\underline{H}$.

5.1. Depth-zero L -packets of $U(1, 1)$. The L -packets of H^0 are the orbits of $\mathrm{PGL}(2, F)$ on the set of equivalence classes of irreducible admissible representations of H^0 [40, §11.1]. We first describe the principal series L -packets.

Let $\lambda \in \mathrm{Hom}(M^0, \mathbb{C}^\times) = \mathrm{Hom}(E^\times, \mathbb{C}^\times)$. According to [40, §11.1], the principal series $\mathrm{ind}_{B^0}^{H^0} \lambda$ is irreducible except in the cases

- (1) $\lambda|_{F^\times} = |\cdot|_F^{\pm 1}$,
- (2) $\lambda|_{F^\times} = \omega_{E/F}$.

In the first case, $\mathrm{ind}_{B^0}^{H^0} \lambda$ has two constituents: the one-dimensional representation $\psi = \mu \circ \det_{\underline{H}^0}$, where μ is the character of E^1 satisfying $\mu \circ \mathcal{N}^{U(1)}|_{E^\times} = \lambda|_E^{\mp 1/2}$; and the representation $\mathrm{St}_{H^0} \cdot \psi$. In the second case, $\mathrm{ind}_{B^0}^{H^0} \lambda$ decomposes into a direct sum $\pi_1(\lambda) \oplus \pi_2(\lambda)$ of irreducible representations. By [36], $\mathrm{ind}_{B^0}^{H^0} \lambda$ has depth zero if and only if λ has depth zero.

The principal series L -packets of G are as follows [40, §11.1]. (Here λ and ψ denote one-dimensional representations of M^0 and H^0 , respectively.)

- (1) $\{\mathrm{ind}_{B^0}^{H^0} \lambda\}$, where $\mathrm{ind}_{B^0}^{H^0} \lambda$ is irreducible;
- (2) $\{\psi\}$;
- (3) $\{\mathrm{St}_{H^0} \cdot \psi\}$;
- (4) $\{\pi_1(\lambda), \pi_2(\lambda)\}$, where $\mathrm{ind}_{B^0}^{H^0} \lambda$ is reducible of the second type described above.

The remaining irreducible representations and L -packets of H^0 are supercuspidal. The depth-zero supercuspidals of H^0 have a unique expression of the form

$\text{ind}_{H_z^0}^{H^0} \sigma$, where σ is the inflation to H_z^0 of an irreducible cuspidal representation $\bar{\sigma}$ of $H_z^0 \cong \text{SL}(2, k)$. Note that since $H_z^0 \cong G_z^\circ$, the results of §4.2 on G_z° apply to H_z^0 as well. Therefore, we have the following kinds of supercuspidal representations of H^0 :

- (5.1SC-1) $\text{ind}_{G_z}^G \sigma$, where $\bar{\sigma}$ is a cuspidal representation of H_z^0 of dimension $q - 1$.
- (5.1SC-2) $\text{ind}_{G_z}^G \sigma$, where $\bar{\sigma}$ is a cuspidal representation of H_z^0 of dimension $(q-1)/2$.

It is easily checked that $\text{PGL}(2, F)$ acts trivially on the equivalence class of any supercuspidal of the form (5.1SC-1); hence these representations are elements of L -packets of size 1. On the other hand, if ϵ is a nonsquare unit of $F^\times$, then $\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$ exchanges the two classes consisting of representations of the form (5.1SC-2). Thus H^0 has one depth-zero supercuspidal L -packet of size 2.

5.2. Base change lifts for $U(1, 1)$. By [40, §11.4], the base change lifts of principal series L -packets of H^0 are as follows. Let $\lambda \in \text{Hom}(M^0, \mathbb{C}^\times)$, and let $\tilde{\lambda}$ denote the character of $\tilde{M}^0$ lifted from λ (see §3.1).

- (i) If $\text{ind}_{B^0}^{H^0} \lambda$ is irreducible and $\text{ind}_{\tilde{B}^0}^{\tilde{H}^0} \tilde{\lambda}$ is irreducible, then the base change lift of the L -packet $\{\text{ind}_{B^0}^{H^0} \lambda\}$ is $\text{ind}_{\tilde{B}^0}^{\tilde{H}^0} \tilde{\lambda}$.
- (ii) If $\text{ind}_{B^0}^{H^0} \lambda$ is irreducible but $\text{ind}_{\tilde{B}^0}^{\tilde{H}^0} \tilde{\lambda}$ is reducible, it must be the case that $\lambda|_{F^\times} = |\cdot|_F^{\pm 1} \omega_{E/F}$, and the base change lift of the L -packet $\{\text{ind}_{B^0}^{H^0} \lambda\}$ is $\left(\lambda|_{E^\times} |\cdot|_E^{\mp 1/2}\right) \circ \det_{\tilde{H}^0}$.
- (iii) If $\lambda|_{F^\times} = |\cdot|_F^{\pm 1}$, let ψ be the one-dimensional representation $\mu \circ \det_{\tilde{H}^0}$, where $\mu \circ \mathcal{N}^{U(1)}|_{E^\times} = \lambda|_{E^\times} |\cdot|_E^{\mp 1/2}$. Then the lift of the L -packet consisting of the constituent ψ (resp., the constituent $\text{St}_{H^0} \cdot \psi$) of $\text{ind}_{B^0}^{H^0} \lambda$ is the one-dimensional constituent $\tilde{\psi} = (\lambda|_{E^\times} |\cdot|_E^{\mp 1/2}) \circ \det_{\tilde{H}^0}$ (resp., the constituent $\text{St}_{\tilde{H}^0} \cdot \tilde{\psi}$) of $\text{ind}_{\tilde{B}^0}^{\tilde{H}^0} \tilde{\lambda}$.
- (iv) If $\lambda|_{F^\times} = \omega_{E/F}$, then the lift of the L -packet $\{\pi_1(\lambda), \pi_2(\lambda)\}$ is $\text{ind}_{\tilde{B}^0}^{\tilde{H}^0} \tilde{\lambda}$.

Now let Π be the supercuspidal L -packet comprising the two representations of the form (5.1SC-2).

Using Hypothesis 1.5 and a process of elimination, it can be checked that Π is the transfer via Ξ_H from the endoscopic group $\underline{C}^0$ of the character $\varphi = \mathbf{1} \otimes \phi$ of $C^0 \cong E^1 \times E^1$, where ϕ is the nontrivial depth-zero character of E^1 of order 2 (see [40, §11.1]). The torus $\tilde{M}$ is $\tilde{H}^0$ -conjugate to the torus $\tilde{C}$, say $\tilde{C} = {}^h \tilde{M}$, where $h \in \tilde{H}^0$. It follows from [40, Prop. 11.4.1(a)] that the base change lift of Π is the representation

$$(\text{ind}_{\tilde{B}^0}^{\tilde{H}^0} \tilde{\varphi}) \cdot (\Omega^{-1} \circ \det_{\tilde{H}^0}),$$

where $\tilde{\varphi}$ is the character of $\tilde{C}$ lifted from φ . Note that there are two possible choices for Ω , but up to equivalence, this lift does not depend on the choice.

We now discuss the base change lifts of the singleton depth-zero supercuspidal L -packets. Let ρ be a depth-zero supercuspidal representation of H^0 arising from a cuspidal representation $\bar{\sigma}$ of H_z^0 of degree $q - 1$. Then (as in the case of G_z°), $\theta_{\bar{\sigma}} = -R_{\tilde{S}^0}^{\tilde{H}^0} \chi$ for some regular character χ of S^0 . Since the character $\tilde{\chi}$ of $\tilde{S}^0$ lifted from χ via $N_{\tilde{E}}^{\tilde{S}^0}$ is in general position, there is a cuspidal representation $\tilde{\sigma}$ of $\tilde{H}_z^0$

such that $\theta_{\tilde{\sigma}} = -R_{\tilde{S}^0}^{\tilde{H}_z^0} \tilde{\chi}$. We note that $\tilde{\sigma}$ depends only on $\bar{\sigma}$ and not on the choice of χ . Let $\tilde{\sigma}$ be the inflation to $\tilde{H}_z^0$ of $\tilde{\sigma}$. We can extend $\tilde{\sigma}$ uniquely to a representation (which we will also denote by $\tilde{\sigma}$) of $\tilde{H}_z^0 \tilde{Z}^0$ in such a way that $\omega_{\tilde{\sigma}}(\varpi_E) = \chi(-1)$. (Here we are identifying $\tilde{Z}^0$ with $E^\times$ and Z^0 with E^1 .)

Proposition 5.1. *The base change lift of $\{\rho\}$ is $\text{ind}_{\tilde{H}_z^0 \tilde{Z}^0}^{\tilde{H}^0} \tilde{\sigma}$.*

Proof. Let $\tilde{\rho}_*$ be the base change lift of $\{\rho\}$ and let $\tilde{\rho} = \text{ind}_{\tilde{H}_z^0 \tilde{Z}^0}^{\tilde{H}^0} \tilde{\sigma}$. According to [40, §11.4], $\tilde{\rho}_*$ is the Shintani lift of ρ , so we can choose $\tilde{\rho}_*(\varepsilon)$ so that $\theta_{\tilde{\rho}_*, \varepsilon} = \theta_\rho \circ \mathcal{N}^{\tilde{H}^0}$. Choose γ in S^0 such that the image $\bar{\gamma}$ of γ in H_z^0 is regular. It follows from §4.3 that $S^0 \cong \ker N_{EE'/E'}$ and $\tilde{S}^0 \cong (EE')^\times$. Furthermore, these isomorphisms can be chosen so that the map $\mathcal{N}^{\tilde{S}^0} : \tilde{S}^0 \rightarrow S^0$ corresponds to the map $(EE')^\times \rightarrow \ker N_{EE'/E'}$ given by $x \mapsto x/\iota(x)$. Since the latter is surjective even when restricted to $\mathcal{O}_{EE'}^\times$, $\mathcal{N}^{\tilde{S}^0}$ must map $\tilde{S}_0^0$ onto S^0 . Thus, there exists $\delta \in \tilde{S}_0^0 \cap (\tilde{H}^0)^{\varepsilon\text{-reg}}$ with $\mathcal{N}(\delta) = \gamma$. By [1, Lemma 2.1 and Prop. 4.1], γ is contained in a unique parahoric subgroup of H^0 , namely H_z^0 . By [1, Prop. 7.1], it follows that $\theta_\rho(\gamma) = \theta_\sigma(\gamma)$. Therefore,

$$(5.2.1) \quad \theta_{\tilde{\rho}_*, \varepsilon}(\delta) = \theta_\rho(\gamma) = \theta_\sigma(\gamma) = \theta_{\tilde{\sigma}}(\bar{\gamma}) = -\chi(\bar{\gamma}) - \chi^{-1}(\bar{\gamma}),$$

where the last equality is the Deligne-Lusztig character formula (see [45, Thm. 6.8]).

From [40, Prop. 11.4.1(c)], $\tilde{\rho}_*$ is square integrable and is the base change lift of a unique square integrable representation. Hence by (iii) above, $\tilde{\rho}_*$ cannot be a twist of the Steinberg representation, and so it must be supercuspidal. From Hypothesis 1.5, $\tilde{\rho}_*$ has depth zero, so it must therefore be of the form $\text{ind}_{\tilde{H}_z^0 \tilde{Z}^0}^{\tilde{H}^0} \tilde{\sigma}_*$, where $\tilde{\sigma}_*$ is an extension to $\tilde{H}_z^0 \tilde{Z}^0$ of the representation of $\tilde{H}_z^0$ inflated from a cuspidal representation $\tilde{\sigma}_*$ of $\tilde{H}_z^0$.

Since $\tilde{\rho}_*$ is ε -invariant, it follows from [35, Thm. 5.2] that ${}^\varepsilon \tilde{\sigma}_*$ and $\tilde{\sigma}_*$ are associate, and this easily implies that $\tilde{\sigma}_*$ is ε -invariant. Since $\tilde{\sigma}_*$ is an irreducible cuspidal representation of $\tilde{H}_z^0 \cong \text{GL}(2, k)$, it must be the case that $\tilde{\sigma}_*$ arises via Deligne-Lusztig induction from a character φ of $\tilde{S}^0$ in general position with respect to the action of $W(\tilde{S}^0, \tilde{H}_z^0)$. Since $\tilde{\sigma}_*$ is ε -invariant, we must have that ${}^\varepsilon \varphi$ is in the $W(\tilde{S}^0, \tilde{H}_z^0)$ -orbit of φ . If ${}^\varepsilon \varphi \neq \varphi$, it is easy to see that φ has order dividing 2, hence is not in general position. Thus ${}^\varepsilon \varphi = \varphi$. It follows that $\varphi = \tilde{\psi}$ for some character ψ of S^0 in general position with respect to the action of $W(S^0, H_z^0)$. Let $\bar{\sigma}_*$ be the cuspidal representation of H_z^0 obtained via Deligne-Lusztig induction from ψ .

Extend $\tilde{\rho}_*$ to a representation (also denoted $\tilde{\rho}_*$) of $\tilde{H}^0 \langle \varepsilon \rangle$ such that for ε -regular $h \in \tilde{H}^0$, we have

$$\theta_{\tilde{\rho}_*}(h\varepsilon) = \theta_{\tilde{\rho}_*, \varepsilon}(h).$$

As a representation of $\tilde{H}^0 \langle \varepsilon \rangle$,

$$\tilde{\rho}_* = \text{ind}_{\tilde{H}_z^0 \tilde{Z}^0 \langle \varepsilon \rangle}^{\tilde{H}^0 \langle \varepsilon \rangle} \tilde{\sigma}_*,$$

where $\tilde{\sigma}_*$ is extended compatibly from $\tilde{H}_z^0 \tilde{Z}^0$ to $\tilde{H}_z^0 \tilde{Z}^0 \langle \varepsilon \rangle$. This extension, in turn, determines an extension of $\tilde{\sigma}_*$ to $\tilde{H}_z^0 \langle \varepsilon \rangle$.

Note that $\delta \in \tilde{S}_0^0 \subset \tilde{H}_z^0$. Furthermore, an argument in the proof of [1, Prop. 5.5] shows that $\delta\varepsilon$ lies in a unique conjugate of $\tilde{H}_z^0 \tilde{Z}^0 \langle \varepsilon \rangle$ (namely, $\tilde{H}_z^0 \tilde{Z}^0 \langle \varepsilon \rangle$ itself).

Therefore, [1, Prop. 7.1] implies that

$$(5.2.2) \quad \theta_{\tilde{\rho}_*, \varepsilon}(\delta) = \theta_{\tilde{\rho}_*}(\delta\varepsilon) = \theta_{\tilde{\sigma}_*}(\delta\varepsilon) = \theta_{\tilde{\sigma}_*}(\bar{\delta}\varepsilon),$$

where $\bar{\delta}$ is the image of δ in $\tilde{H}_z^0$.

To compute $\theta_{\tilde{\sigma}_*}(\bar{\delta}\varepsilon)$, consider the representation ${}^\varepsilon\tilde{\sigma}_*$. Since $\tilde{\psi}$ factors through $\mathcal{N}$, $\tilde{\psi}$ is trivial on $\tilde{Z}^0$, and so $\tilde{\sigma}_*$ has trivial central character. As in the case of $\tilde{G}_z$ (see §4.2.2), ε acts on $\tilde{H}_z^0$ via $x \rightarrow x/\det(x)$. Therefore, for any $h \in \tilde{H}_z^0$ we have that ${}^\varepsilon h h^{-1} = 1/\det h \in \tilde{Z}^0$. Thus $\tilde{\sigma}_*({}^\varepsilon h h^{-1})$ is the identity so that ${}^\varepsilon\tilde{\sigma}_* = \tilde{\sigma}_*$. (Note that before we only had that ${}^\varepsilon\tilde{\sigma}_*$ and $\tilde{\sigma}_*$ were equivalent.) In other words, $\tilde{\sigma}_*(\varepsilon) = \pm \text{id}$. This together with (5.2.2) and the Deligne-Lusztig character formula imply that $\theta_{\tilde{\rho}_*, \varepsilon}(\delta) = \theta_{\tilde{\sigma}_*}(\bar{\delta}\varepsilon)$ is equal to

$$(5.2.3) \quad \pm\theta_{\tilde{\sigma}_*}(\bar{\delta}) = \pm(\tilde{\psi}(\bar{\delta}) + {}^w\tilde{\psi}(\bar{\delta})) = \pm(\psi(\bar{\gamma}) + \psi^{-1}(\bar{\gamma})),$$

where w is the nontrivial element of $W(\tilde{S}^0, \tilde{H}_z^0)$. It follows that from (5.2.1) and (5.2.3) we obtain $\chi(\bar{\gamma}) + \chi^{-1}(\bar{\gamma}) = \pm(\psi(\bar{\gamma}) + \psi^{-1}(\bar{\gamma}))$. Letting γ vary over all elements of S^0 with regular image in H_z^0 , parts (ii) and (iii) of Lemma 4.2 then imply that $\psi = \chi^{\pm 1}$ if $q \neq 7$ (and that the sign in (5.2.3) must be $+$). Thus $\tilde{\sigma}_*$ is equivalent to $\tilde{\sigma}$.

Moreover, since $\omega_{\tilde{\rho}_*} = \omega_\rho \circ \mathcal{N}^{\mathbb{Z}^0}$,

$$\omega_{\tilde{\sigma}_*}(\varpi_E) = \omega_{\tilde{\rho}_*}(\varpi_E) = \omega_\rho(-1) = \omega_\sigma(-1) = \chi(-1) = \omega_{\tilde{\sigma}}(\varpi_E).$$

It follows that the extensions of $\tilde{\sigma}$ and $\tilde{\sigma}_*$ to $\tilde{H}_z^0 \tilde{Z}^0$ are equivalent. Therefore, $\tilde{\rho}_*$ and $\tilde{\rho}$ are equivalent. $\square$

5.3. Depth-zero L -packets for $U(2)$. The group $H^1 = U(2)(F)$ is compact, so its building has only one point: y . The corresponding parahoric subgroup has index two in H^1 . The quotient H^1/H_y^{1+} is isomorphic to the orthogonal group $O(2)(k)$ defined with respect to the form $\begin{pmatrix} 1 & 0 \\ 0 & -\bar{\epsilon} \end{pmatrix}$. Here $\bar{\epsilon}$ is the image in k of the nonsquare unit $\epsilon \in F^\times$. We may identify $SO(2)(k)$ with the group of norm-one elements of the quadratic extension of k . Fix an element h of $O(2)(k)$ that does not belong to $SO(2)(k)$. Then conjugation by h acts on $SO(2)(k)$ by inversion. This allows us to classify the representations of $O(2)(k)$, and thus the depth-zero representations of H^1 , and to gather these latter representations into L -packets. Our L -packets are determined by characters of $SO(2)(k)$. Let χ denote such a character.

- (1) Suppose that $\chi \neq \chi^{-1}$. Then $\text{infl}_{SO(2)(k)}^{O(2)(k)} \chi$ is irreducible and forms a singleton L -packet Π_χ^1 .
- (2) Suppose that χ is trivial. Then χ extends to $O(2)(k)$ in two ways, yielding a trivial character χ_+ and a quadratic character χ_- . Inflating these extensions, we obtain two singleton L -packets for H^1 , which we will denote by Π_+^1 and Π_-^1 , respectively.
- (3) Suppose that χ is the nontrivial quadratic character of $SO(2)(k)$. Then again χ extends to $O(2)(k)$ in two ways. Inflating each character to H^1 , we obtain an L -packet Π_χ^1 of size two.

It is easy to see that for each L -packet above, the sum of the characters in an L -packet is stable. Moreover, our L -packets are minimal with respect to this property.

5.4. Base change lifts for $U(2)$ via a Jacquet-Langlands-like correspondence. Since $\underline{H}^1$ is an inner form of $\underline{H}^0$, we can obtain a base change lift if we can associate each depth-zero L -packet for H^1 to one for H^0 . This association will be similar to the Jacquet-Langlands correspondence (see [4, 18, 39]). That is, given a depth-zero L -packet Π^1 for H^1 , we want to find a depth-zero L -packet Π^0 for H^0 such that

$$(5.4.1) \quad \sum_{\pi \in \Pi^1} \theta_{\pi}(g_1) = - \sum_{\pi \in \Pi^0} \theta_{\pi}(g_0)$$

for all regular $g_1 \in H^1$ and $g_0 \in H^0$ whose stable conjugacy classes are associated in a natural way.

Define a map JL from the depth-zero L -packets of H^1 to those of H^0 as follows. We let $\text{JL}(\Pi_+^1)$ be the Steinberg representation of H^0 , and we let $\text{JL}(\Pi_-^1)$ be the twist of this representation by the depth-zero quadratic character of H^0 . If $\chi \neq \chi^{-1}$, then $\text{JL}(\Pi_{\chi}^1)$ is the supercuspidal representation of the form (5.1SC-1) coming from the character χ . If χ is the nontrivial quadratic character, then $\text{JL}(\Pi_{\chi}^1)$ is the L -packet consisting of the two representations of the form (5.1SC-2).

Using Lemma 4.2, it is not difficult to see that if $q > 3$ and $q \neq 7$, then JL is the only correspondence that satisfies (5.4.1) for all $g_1 \in H^1$ whose image in $\text{SO}(2)(k)$ is regular. Thus, if we assume that there is a Jacquet-Langlands-like correspondence from the depth-zero L -packets of H^0 to those of H^1 , then it must be JL.

According to Conjecture 1 of [38], JL is related to the theta correspondence between H^0 and H^1 in the following way. Given an L -packet Π for H^1 , some representation $\pi \in \Pi$ shows up in the theta correspondence, and $\Theta(\pi) \in \text{JL}(\Pi)$.

6. DESCRIPTION OF DEPTH-ZERO L -PACKETS FOR RAMIFIED UNITARY GROUPS IN THREE VARIABLES

6.1. L -packets consisting of principal series constituents. The following proposition is due to Rogawski [40, §12.2].

Proposition 6.1. *The L -packets of G that consist entirely of principal series constituents all have one of the following forms (where λ and ψ denote one-dimensional representations of M and G , respectively):*

(6.1-1) $\{\text{ind}_P^G \lambda\}$, where $\text{ind}_P^G \lambda$ is irreducible;

(6.1-2) $\{\psi\}$;

(6.1-3) $\{\text{St}_G \cdot \psi\}$;

(6.1-4) $\{\pi_1(\lambda), \pi_2(\lambda)\}$, where $\text{ind}_P^G \lambda$ is reducible of type (4.4PS-3).

(6.1-5) $\{\pi^n(\lambda)\}$, where $\text{ind}_P^G \lambda$ is reducible of type (4.4PS-2). The representation $\pi^n(\lambda)$ is contained in the A -packet $\Pi(\lambda) = \{\pi^n(\lambda), \pi^s(\lambda)\}$, where $\pi^s(\lambda)$ is a supercuspidal representation that sits inside an L -packet with the square-integrable principal series constituent $\pi^2(\lambda)$.

In the depth-zero setting, the representation $\pi^s(\lambda)$ will be explicitly described in §6.3.

6.2. Supercuspidal L -packets. In order to compute the depth-zero supercuspidal L -packets of G , we need two preliminary results. The first gives the values of the characters of depth-zero supercuspidal representations on “very regular” elements of certain elliptic tori. The second determines the sizes of such L -packets and asserts that they are all transfers via Ξ_G of L -packets of H .

Let $\underline{S}$ and $\underline{S}'$ be the ramified quadratic tori defined in §4.3.2. Let $\gamma \in S$ be a regular element of G whose image $\bar{\gamma}$ in $\mathbf{S} \cong \mathbf{S}^\circ \times \mathbf{Z}$ lies in $\mathbf{S}^\circ$ and is regular in $\mathbf{G}_z^\circ$. Then [1, Lemma 2.1] implies that γ lies in a unique parahoric subgroup of G , namely G_z . By Proposition 4.1, there is an element $g \in \underline{G}(E)$ such that $gSg^{-1} = S'$. Set $\gamma' = g\gamma g^{-1}$. Then similarly, we have that γ' lies in a unique maximal compact subgroup of G , namely G_y .

The next lemma concerns character values of supercuspidal representations of depth zero. Recall that a regular character χ of $\mathbf{S}^\circ$ or $(\mathbf{S}')^\circ$, together with a character ν of $\mathbf{Z}$, gives rise to a supercuspidal representation of G as explained in §4.4.

Lemma 6.2. *Let π be a supercuspidal representation of G of depth zero. Then, in the notation of §4.4,*

$$\theta_\pi(\gamma) = \begin{cases} 0 & \text{if } \pi \text{ is of type (4.4SC-1),} \\ -[\chi(\bar{\gamma}) + \chi^{-1}(\bar{\gamma})] & \text{if } \pi \text{ is of type (4.4SC-2),} \\ -\phi(\bar{\gamma}) & \text{if } \pi \text{ is of type (4.4SC-3),} \end{cases}$$

$$\theta_\pi(\gamma') = \begin{cases} -[\chi(\bar{\gamma}) + \chi^{-1}(\bar{\gamma})] & \text{if } \pi \text{ is of type (4.4SC-1),} \\ 0 & \text{if } \pi \text{ is of type (4.4SC-2),} \\ 0 & \text{if } \pi \text{ is of type (4.4SC-3),} \end{cases}$$

where ϕ is the character of $\mathbf{S}^\circ$ of order 2.

Proof. By the discussion in §4.4, π is compactly induced from a representation σ of G_v , where $v = y$ or z . Let γ^* be either γ or γ' and, correspondingly, let x be either z or y . Since γ^* lies in G_x and no other parahoric subgroup, [1, Prop. 7.1] implies that $\theta_\pi(\gamma^*) = \theta_\sigma(\gamma^*)$ if $v = x$ and $\theta_\pi(\gamma^*) = 0$ otherwise. In the former case $\theta_\sigma(\gamma^*) = \theta_{\bar{\sigma}}(\bar{\gamma}^*)$, where $\bar{\gamma}^*$ is the image of γ^* in $\mathbf{G}_x^\circ$. In cases (4.4SC-1) and (4.4SC-2), the character of $\bar{\sigma}$ is (up to a sign) the Deligne-Lusztig character of $\mathbf{G}_x^\circ$ corresponding to χ , and the result follows from the character formula for such representations [45, Thm. 6.8]. In case (4.4SC-3), $\bar{\sigma}$ is one of the two irreducible components of the Deligne-Lusztig representation of $\mathbf{G}_z^\circ$ corresponding to ϕ , and the result follows from this character formula together with the fact that these two components take the same value on semisimple elements of $\mathbf{G}_z^\circ$. $\square$

Lemma 6.3. *Let ρ be a depth-zero supercuspidal representation of H such that $\{\rho\}$ is an L -packet. Then the transfer Π of $\{\rho\}$ via Ξ_G is a depth-zero supercuspidal L -packet of G of size two. Moreover, every depth-zero supercuspidal L -packet Π arises in this way.*

Proof. The first statement follows from [40, §12.2], given our assumption on Ξ_G in Hypothesis 1.5.

Now suppose that Π is a depth-zero supercuspidal L -packet of G . An immediate consequence of Lemma 6.2 is that none of the depth-zero supercuspidal characters are stable. Since the sum of the characters of the elements of a tempered L -packet is stable (see [40, §12–13]), it follows that G has no depth-zero supercuspidal L -packets of size 1. According to [40, §12.2, Lemmas 13.1.1 and 13.1.3] and Hypothesis 1.5, a nonsingleton depth-zero supercuspidal L -packet Π must be the transfer via Ξ_G of a depth-zero supercuspidal L -packet R of the endoscopic group H . From [40, §12.1], we have that the L -packet R has size 1 or 2. But if $|R|$ were 2, then R would be the transfer via Ξ_H of a (depth-zero) $W(C, H)$ -regular character ψ of C . Moreover,

since Π is supercuspidal, ψ would have to be $W(C, G)$ -regular. But since $W(C, G)$ acts on $C = E^1 \times E^1 \times E^1$ by permuting coordinates transitively, and since $C \cong \{\pm 1\}^3$, C has no $W(C, G)$ -regular characters of depth zero. Hence $|R|$ must be 1 and $|\Pi| = 2|R| = 2$ by [40, Lemma 13.1.2]. $\square$

Using the results of §5.1 and [40, §12.1], we see that the singleton supercuspidal depth-zero L -packets of H are those of the form $\{\rho\}$, where $\rho = \rho^0 \otimes \psi$, ρ^0 is a supercuspidal of H^0 of type (5.1SC-1), and ψ is a depth-zero character of Z .

We now determine the depth-zero supercuspidal L -packets of G . In the following, we will view the restriction of Ω to E^1 as a character of Z ; hence $\bar{\Omega}$ is a character of Z . Also, for any character ν of Z , $\hat{\nu}$ will denote the inflation of ν to Z . Now fix a character ν of Z , and let χ be a character of S^0 of order greater than 2. Note that $S^\circ = S^0$, so we may view χ as a character of S° . By identifying S with S' via conjugation by the element $g \in \underline{G}(E)$ from the beginning of this section, we may also view χ as a character of $(S')^\circ$. (This character of $(S')^\circ$ depends on the choice of g , but its Weyl group orbit is uniquely determined by χ . The following representations constructed from χ depend only on this orbit.) We fix an identification of S^0 with l^1 , where l/k is a quadratic extension. This determines identifications of S° and $(S')^\circ$ with l^1 .

If $x = y$ (resp. if $x = z$), let $\pi_x = \pi_x(\chi, \nu)$ be the supercuspidal representation of G of the form (4.4SC-1) (resp. (4.4SC-2)) associated to χ and ν . That is, $\pi_x = \text{ind}_{\text{stab}_G(x)}^G (\bar{\sigma}_x \otimes \nu)$, where $\bar{\sigma}_x$ is the cuspidal Deligne-Lusztig representation of G_x° obtained from χ . Let $\rho = \rho(\chi, \nu) = \text{ind}_{\text{stab}_H(z)}^H \tau$, where τ is defined as follows. Let $\bar{\tau}$ be the Deligne-Lusztig cuspidal representation of H_z° associated to $\chi\phi$ (ϕ the character of S^0 of order 2), and define τ to be the inflation to $\text{stab}_H(z)$ of the representation $\bar{\tau} \otimes \nu\bar{\Omega}$ of $H_z = H_z^\circ \times Z$.

Proposition 6.4. *The singleton $\{\rho(\chi, \nu)\}$ is a depth-zero supercuspidal L -packet of H . The image of $\{\rho(\chi, \nu)\}$ under Ξ_G is the depth-zero supercuspidal L -packet $\{\pi_y(\chi, \nu), \pi_z(\chi, \nu)\}$ of G . Moreover, every depth-zero supercuspidal L -packet of G is of this form.*

Proof. By Lemma 6.3 and the subsequent discussion, $\{\rho\}$ is an L -packet of H whose transfer via Ξ_G is a supercuspidal L -packet $\Pi = \{\pi_1, \pi_2\}$ of size 2, and all such L -packets arise in this way. Therefore, to prove the proposition, it suffices to show that $\Pi = \{\pi_y, \pi_z\}$.

From Hypothesis 1.5, Π has depth zero. As discussed in §4.4, π_i ($i = 1, 2$) is associated to a cuspidal representation $\bar{\sigma}_i$ of $G_{x_i}^\circ$ ($x_i = y$ or z) and a character ν_i of Z . To determine ν_i , note that the central character ω_{π_i} of π_i is equal to $\hat{\nu}_i$, while $\omega_\rho = \hat{\nu}\Omega$. By [40, Prop. 4.9.1], ω_{π_i} must satisfy $\omega_{\pi_i} = \Omega\omega_\rho = \Omega\hat{\nu}\Omega = \hat{\nu}$. Thus $\nu_i = \nu$.

We now determine the $\bar{\sigma}_i$. By [40, Lemma 12.7.2], up to re-ordering,

$$\theta_\rho^G = \theta_{\pi_1} - \theta_{\pi_2},$$

where θ_ρ^G is the distribution on G coming from θ_ρ via the transfer Ξ_G . Let γ be an element of S^0 whose image $\bar{\gamma}$ in G_z is a regular element of S^0 . Let $\gamma' = g\gamma g^{-1}$ be as in Lemma 6.2, and let δ be the element of $\mathcal{D}_{\underline{G}}(\underline{S}/F)$ represented by the cocycle $s \mapsto s(g)g^{-1}$, $s \in \text{Gal}(E/F)$. Let $\chi_0 = \chi\phi$. A result analogous to Lemma 6.2 shows that $\theta_\rho(\gamma) = -\chi_0(\bar{\gamma}) - \chi_0^{-1}(\bar{\gamma})$. Moreover, due to the particular form of γ and the

fact (see [40, Prop. 3.7.1]) that $W_F(\underline{S}, \underline{H}) = W_F(\underline{S}, \underline{G})$, it follows from [40, §4.9, Lemma 12.5.1] that

$$\theta_\rho^G(\gamma) = n(\gamma)\theta_\rho(\gamma), \quad \theta_\rho^G(\gamma') = n(\gamma)\kappa(\delta)\theta_\rho(\gamma),$$

where κ is the element of the dual of $\mathcal{D}_{\underline{G}}(\underline{S}/F)$ corresponding to the endoscopic group $\underline{H}$, and

$$n(\gamma) = \Omega(-N_{EE'/E}(\gamma^{-1} - 1)) = \bar{\Omega}(-N_{l/k}(\bar{\gamma}^{-1} - 1)) = \bar{\Omega}(-N_{l/k}(\bar{\gamma} - 1)).$$

(In this formula, we are identifying $\gamma \in S^0$ (resp. $\bar{\gamma} \in S^0$) with the element of $\ker(N_{EE'/E})$ (resp. l^1) to which it corresponds.)

Note that $n(\gamma) \in \{\pm 1\}$. Moreover, as γ ranges over all elements of S^0 with regular image in $\mathbf{G}_z$, $\bar{\gamma}$ ranges over all of $l^1 \setminus \{\pm 1\}$, and an elementary argument shows that $n(\gamma) = \bar{\Omega}(-N_{l/k}(\bar{\gamma} - 1))$ must assume both the values 1 and -1 if $q > 5$. We also note that δ is not in the image of $\mathcal{D}_{\underline{H}}(\underline{S}, F)$ and, consequently, we have $\kappa(\delta) = -1$ by [40, §4.10, Lemma 3.13]. Thus we have

$$(6.2.1) \quad \begin{aligned} \theta_{\pi_1}(\gamma) - \theta_{\pi_2}(\gamma) &= \theta_\rho^G(\gamma) = n(\gamma)\theta_\rho(\gamma) = -n(\gamma)(\chi_0(\bar{\gamma}) + \chi_0^{-1}(\bar{\gamma})), \\ \theta_{\pi_1}(\gamma') - \theta_{\pi_2}(\gamma') &= \theta_\rho^G(\gamma') = -n(\gamma)\theta_\rho(\gamma) = n(\gamma)(\chi_0(\bar{\gamma}) + \chi_0^{-1}(\bar{\gamma})). \end{aligned}$$

Assume that both π_i are of the form (4.4SC-1). Then it follows from Lemma 6.2 that $\theta_{\pi_i}(\gamma) = 0$. By Lemma 4.2(i), this contradicts (6.2.1) if $q > 3$. A similar argument with γ' replacing γ shows that at least one of the representations π_i must be of the form (4.4SC-1). Hence precisely one of the π_i , say π_j , is of this form, and the other, say $\pi_{j'}$, is of the form (4.4SC-2) or (4.4SC-3).

We then have $x_j = y$, and as above, $\bar{\sigma}_j$ is associated to a character χ_j of $S^0 \cong (S')^\circ$ of order greater than 2. By Lemma 6.2 $\theta_{\pi_{j'}}$ vanishes at γ' , and by (6.2.1) we have

$$\pm(\chi_j(\bar{\gamma}) - \chi_j^{-1}(\bar{\gamma})) = \theta_{\pi_1}(\gamma') - \theta_{\pi_2}(\gamma') = n(\gamma)(\chi_0(\bar{\gamma}) + \chi_0^{-1}(\bar{\gamma})).$$

By Lemma 4.2(iv), this implies that $\chi_j = \chi_0^{\pm 1}\phi = \chi^{\pm 1}$ if $q > 7$. Thus $\bar{\sigma}_j \cong \bar{\sigma}_y$. Since $\nu_j = \nu$, we have $\pi_j \cong \pi_y$.

Since $x_j = y$, it must be the case that $x_{j'} = z$. An argument similar to that in the preceding paragraph using both parts (iv) and (v) of Lemma 4.2 shows that $\pi_{j'}$ must be of the form (4.4SC-2) and that if $\bar{\sigma}_{j'}$ is associated to the character $\chi_{j'}$ of $S^0 = S^\circ$, then $\chi_{j'} = \chi^{\pm 1}$. It follows that $\bar{\sigma}_{j'} \cong \bar{\sigma}_z$ and, hence, that $\pi_{j'} \cong \pi_z$. $\square$

6.3. Nonsupercuspidal L -packets containing supercuspidals. It remains to determine the depth-zero L -packets of G of the form $\{\pi^s(\lambda), \pi^2(\lambda)\}$, i.e., the nonsupercuspidal depth-zero L -packets containing a supercuspidal representation. In this case, λ is a depth-zero character of M such that the restriction of the character λ_1 of $E^\times$ to $F^\times$ is $\omega_{E/F} \cdot |\cdot|_F^{\pm 1}$ (see (4.4PS-2) and (6.1-5)). In other words, $\lambda_1 = \eta \cdot |\cdot|_E^{\pm 1/2}$ for some character η of $E^\times$ satisfying $\eta|_{F^\times} = \omega_{E/F}$. Recall from Proposition 6.1 that the discrete series representation $\pi^2(\lambda)$ is a constituent of the principal series representation $\pi = \text{ind}_B^G \lambda$, the only other constituent being a non-tempered representation, denoted $\pi^n(\lambda)$, which is paired with $\pi^s(\lambda)$ in an A -packet.

The representation $\pi^s(\lambda)$ must be of the form (4.4SC-3), as those of the form (4.4SC-1) and (4.4SC-2) are precisely the representations that lie in depth-zero supercuspidal L -packets by Proposition 6.4. There are four equivalence classes of representations of the form (4.4SC-3), since there are two equivalence classes

of cuspidal representations of G_z° of degree $(q-1)/2$, and two characters of $Z \cong \{\pm 1\}$. This agrees with our determination in §4.4 that there are four different representations of the form $\pi^2(\lambda)$ (resp. $\pi^n(\lambda)$).

In the following proposition, we determine $\pi^s(\lambda)$ in terms of the representation $\pi^n(\lambda)$.

Proposition 6.5. *Let λ be a depth-zero character of M such that $\lambda_1|_{F^\times} = \omega_{E/F}|\cdot|_F^{\pm 1}$.*

- (i) *As a representation of G_z° , $\pi^n(\lambda)^{G_{z+}}$ is equivalent to an irreducible component ϑ of the reducible principal series representation $\text{ind}_{B_z^\circ}^{G_z^\circ} \chi$, where χ is the character of M° of order 2.*
- (ii) *The representation $\pi^s(\lambda)$ is supercuspidal of the form (4.4SC–3). In particular, $\pi^s(\lambda) \cong \text{ind}_{\text{stab}_G(z)}^G \text{infl}(\bar{\sigma} \otimes \nu)$, where ν is the character $\bar{\lambda}_1|_{\{\pm 1\}} \bar{\lambda}_2$ of Z and $\bar{\sigma}$ is the unique irreducible cuspidal representation of G_z° whose character agrees with $-\theta_\vartheta$ on the set of nontrivial unipotent elements in G_z° .*

Note that, as one would expect, $\pi^s(\lambda)$ is independent of the power of $|\cdot|_E$ occurring in λ_1 .

Proof. As explained in the preceding discussion, $\pi^s = \pi^s(\lambda)$ must be supercuspidal of the form (4.4SC–3). That is, $\pi^s(\lambda) = \text{ind}_{\text{stab}_G(z)}^G \text{infl}(\bar{\sigma} \otimes \nu)$ for some cuspidal representation $\bar{\sigma}$ of G_z° of degree $(q-1)/2$ and some character ν of Z . Our task is to determine $\bar{\sigma}$ and ν in terms of $\pi^n = \pi^n(\lambda)$.

From [40, Thm. 13.1.1, Prop. 13.1.2], the distribution characters of π^s and π^n satisfy

$$(6.3.1) \quad \theta_{\pi^s} = \theta_\xi^G - \theta_{\pi^n},$$

where ξ is a certain character of H associated to λ and θ_ξ^G is the transfer of ξ via Ξ_G . The same equation holds for the functions on G^{reg} that represent these distributions. Since θ_ξ^G transforms via a character under Z [40, Lemma 12.5.1], it follows that π^s and π^n have the same central character. But $\omega_{\pi^n} = \lambda_1|_{E^1} \lambda_2$, while ω_{π^s} is the inflation of ν to Z . Hence $\nu = \bar{\lambda}_1|_{\{\pm 1\}} \bar{\lambda}_2$.

Let $\gamma \in G_z$ be a regular elliptic element of G whose image in G_z° is regular. By [1, Lemma 2.1], the only conjugate of $\text{stab}_G(z)$ that contains γ is $\text{stab}_G(z)$ itself. It follows from [1, Prop. 7.1] that $\theta_{\pi^s}(\gamma) = \theta_{\bar{\sigma}}(\bar{\gamma})$, where $\bar{\gamma}$ is the image of γ in G_z° . We now determine $\bar{\sigma}$ by evaluating the right-hand side of (6.3.1) at appropriate regular elliptic elements of G .

Let $\underline{T}$ be a torus of type (4.3–3) associated to the ramified cubic extension $F(\sqrt[3]{\varpi_F})/F$ for some cube root $\sqrt[3]{\varpi_F}$ of ϖ_F in $\bar{F}$. As indicated in §4.3.3, we may choose $\underline{T}$ so that $T_0 \subset G_{\mathcal{F}}$, where $\mathcal{F}$ is the fundamental alcove in $\mathcal{B}(\underline{G}, F)$. By Lemma 4.3, the image of T_0 in G_x° ($x = y$ or z) is the unipotent radical of B_x , whence T_0 has trivial image in $G_{\mathcal{F}}$. Choose $\gamma \in T_0$ with nontrivial image in G_z .

First note that $\theta_\xi^G(\gamma)$ vanishes since the stable conjugacy class of γ does not meet H . Indeed, $\underline{H}$ has no maximal F -tori of type (4.3–3). We now compute the value of $\theta_{\pi^n}(\gamma)$, thus determining the value of the right-hand side of (6.3.1) at γ .

According to [42, Lemma III.4.10, Thm. 4.16], we have that

$$(6.3.2) \quad \theta_{\pi^n}(\gamma) = -\text{Tr}(\gamma | (\pi^n)^{G_{\mathcal{F}+}}) + \text{Tr}(\gamma | (\pi^n)^{G_{y+}}) + \text{Tr}(\gamma | (\pi^n)^{G_{z+}}).$$

We now compute each term of this alternating sum.

Since π^2 and π^n are the constituents of a principal series representation, [35] implies that, as representations of $G_{\mathcal{F}}^{\circ} \cong M^{\circ}$, their spaces of $G_{\mathcal{F}+}$ -fixed vectors are nontrivial sums of one-dimensional characters. As in the proof of [1, Prop. 4.4], we can use Mackey theory and Frobenius reciprocity to see that

$$\mathrm{Hom}_{G_{\mathcal{F}+}}(\mathbf{1}, \mathrm{ind}_B^G \lambda) = \bigoplus_{B \backslash G / G_{\mathcal{F}+}} \mathbb{C},$$

so that $|B \backslash G / G_{\mathcal{F}+}|$ is the dimension of the space of $G_{\mathcal{F}+}$ -fixed vectors in $\mathrm{ind}_B^G \lambda$. Since y is special, we have that $G = G_y B$. Therefore,

$$|B \backslash G / G_{\mathcal{F}+}| = |(B \cap G_y) \backslash G_y / G_{\mathcal{F}+}| = |B_y \backslash G_y / B_y| = 2,$$

by the Bruhat decomposition. Thus the space of $G_{\mathcal{F}+}$ -fixed vectors in $\mathrm{ind}_B^G \lambda$ is two dimensional, so $(\pi^2)^{G_{\mathcal{F}+}}$ and $(\pi^n)^{G_{\mathcal{F}+}}$ are both one dimensional. In particular, since the image of γ in $G_{\mathcal{F}}^{\circ}$ is trivial, we have

$$(6.3.3) \quad \mathrm{trace}(\gamma | (\pi^n)^{G_{\mathcal{F}+}}) = 1.$$

An argument contained in the proof of [1, Prop. 4.4] shows that the representation $\pi^{G_{y+}}$ of $G_y / G_{y+} = G_y^{\circ}$ is equivalent to $\mathrm{ind}_{B_y^{\circ}}^{G_y^{\circ}} \bar{\lambda}_1$. Since $\bar{\lambda}_1 = \bar{\eta}$ is the character of M of order 2, it follows from §4.2 that the latter representation decomposes as $\phi \oplus \mathrm{St}_{G_y^{\circ}} \cdot \phi$, where ϕ is the character of G_y° of order 2. As in the proof of [1, Prop. 4.4], it follows that $(\pi^n)^{G_{y+}} \cong \phi$, while $(\pi^2)^{G_{y+}} \cong \mathrm{St}_{G_y^{\circ}} \cdot \phi$. In particular, since the image $\bar{\gamma}$ of γ in G_y is unipotent,

$$(6.3.4) \quad \mathrm{Tr}(\gamma | (\pi^n)^{G_{y+}}) = \phi(\bar{\gamma}) = 1.$$

As in the case of G_y , the representation $\pi^{G_{z+}}$ of G_z° is equivalent to $\mathrm{ind}_{B_z^{\circ}}^{G_z^{\circ}} \bar{\lambda}_1$. Since $\bar{\lambda}_1$ has order 2, it follows from §4.2 that the latter representation decomposes as $\vartheta \oplus \vartheta'$, where ϑ and ϑ' are the two inequivalent irreducible representations of G_z° of degree $(q+1)/2$. Again, $(\pi^n)^{G_{z+}}$ must be equivalent to one of these representations, say ϑ . This proves (i). In particular, we have

$$(6.3.5) \quad \mathrm{Tr}(\gamma | (\pi^n)^{G_{z+}}) = \theta_{\vartheta}(\bar{\gamma}),$$

where $\bar{\gamma}$ is the image of γ in G_z° .

Therefore, by (6.3.2), (6.3.3), (6.3.4), and (6.3.5), we have

$$\theta_{\pi^n}(\gamma) = -1 + 1 + \theta_{\vartheta}(\bar{\gamma}) = \theta_{\vartheta}(\bar{\gamma}).$$

Hence (6.3.1) implies that

$$\theta_{\bar{\sigma}}(\bar{\gamma}) = \theta_{\pi^s}(\gamma) = -\theta_{\vartheta}(\bar{\gamma}).$$

This completes the proof of (ii). $\square$

Since Proposition 6.5 determines the relationship between π^s and π^n , hence between π^s and π^2 , it is sufficient for the verification of this case of Theorem 1.4 in §7. In particular, we do not need to know precisely how these representations depend on the character λ . For completeness, however, we provide such a description in the following result, which determines ϑ and hence $\bar{\sigma}$ explicitly in terms of the inducing character λ . It is enough to compute the values of ϑ and $\bar{\sigma}$ on any nontrivial unipotent element of G_z° , as these values uniquely determine these representations (see [23, §5.2]). The proof involves a careful analysis of intertwining operators on the representations $\pi = \mathrm{ind}_B^G \lambda$ of G and $\mathrm{ind}_{B_z^{\circ}}^{G_z^{\circ}} \bar{\eta}$ of G_z° .

It follows from the discussion in §4.2 that the group G_z° may be explicitly realized as the group of matrices of the form

$$\begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} \in \tilde{G}_z^\circ = \mathrm{GL}(2, k) \times \mathrm{GL}(1, k)$$

such that $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$. Moreover, B_z is identified with the group of lower-triangular matrices in G_z° in this realization. Choose an element γ of the cubic torus T from the proof of the preceding proposition such that $\bar{\gamma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \in B_z \subset G_z^\circ$.

Proposition 6.6. *Let λ be a depth-zero character of M such that $\lambda_1 = \eta| \cdot |_E^{\pm 1/2}$, where $\eta|_{F^\times} = \omega_{E/F}$. Then the representation $\vartheta = \pi^n(\lambda)^{G_{z^+}}$ of G_z° satisfies $\theta_\vartheta(\bar{\gamma}) = (1 - \eta(2\varpi_E)q^{1/2})/2$. Thus the cuspidal representation $\bar{\sigma}$ of G_z° from Proposition 6.5(ii) satisfies $\theta_{\bar{\sigma}}(\bar{\gamma}) = (-1 + \eta(2\varpi_E)q^{1/2})/2$.*

7. EXPLICIT BASE CHANGE FOR RAMIFIED UNITARY GROUPS IN THREE VARIABLES

7.1. L -packets consisting of principal series constituents. The following proposition gives the base change lifts of the principal series L -packets of G . Recall that $\tilde{H}$ is an F -Levi subgroup of $\tilde{P}$. Note that there is a natural isomorphism $\tilde{H} \rightarrow \tilde{H}^0 \times \tilde{Z}$. In particular, $\tilde{H} \cong \mathrm{GL}(2, E) \times \mathrm{GL}(1, E)$, and in the statement of the proposition, we identify these groups.

Proposition 7.1. *Let λ be a character of M and let $\tilde{\lambda}$ denote the character of $\tilde{M}$ lifted from λ . Let λ_1 and λ_2 be the characters of $E^\times$ and E^1 , respectively, associated to λ . Let $\tilde{\lambda}_2$ be the character $\lambda_2 \circ \mathcal{N}^{U(1)}|_{E^\times}$ of $E^\times$.*

- (i) *If $\mathrm{ind}_B^G \lambda$ is irreducible and $\mathrm{ind}_{\tilde{B}}^{\tilde{G}} \tilde{\lambda}$ is irreducible, then the base change lift of the L -packet $\{\mathrm{ind}_B^G \lambda\}$ is $\mathrm{ind}_{\tilde{B}}^{\tilde{G}} \tilde{\lambda}$.*
- (ii) *If $\mathrm{ind}_B^G \lambda$ is irreducible but $\mathrm{ind}_{\tilde{B}}^{\tilde{G}} \tilde{\lambda}$ is reducible, then $\lambda_1|_{F^\times} = | \cdot |_E^{\pm 1}$, and the base change lift of $\{\mathrm{ind}_B^G \lambda\}$ is the constituent*

$$\mathrm{ind}_{\tilde{P}}^{\tilde{G}} \left((\lambda_1 \tilde{\lambda}_2 | \cdot |_E^{\mp 1/2} \circ \det_{\tilde{H}^0}) \otimes \tilde{\lambda}_2 \right)$$

of $\mathrm{ind}_{\tilde{B}}^{\tilde{G}} \tilde{\lambda}$.

- (iii) *If $\lambda_1 = | \cdot |_E^{\pm 1}$, then the lift of the L -packet consisting of the one-dimensional constituent $\psi = \lambda_2 \circ \det_{\underline{G}}$ (respectively, the constituent $\mathrm{St}_G \cdot \psi$) of $\mathrm{ind}_B^G \lambda$ is the one-dimensional constituent $\tilde{\psi} = \tilde{\lambda}_2 \circ \det_{\underline{G}}$ (respectively, the constituent $\mathrm{St}_{\tilde{G}} \cdot \tilde{\psi}$) of $\mathrm{ind}_{\tilde{B}}^{\tilde{G}} \tilde{\lambda}$.*
- (iv) *If $\lambda_1|_{F^\times}$ is trivial and λ_1 is nontrivial, then the lift of the L -packet $\{\pi_1(\lambda), \pi_2(\lambda)\}$ is $\mathrm{ind}_{\tilde{B}}^{\tilde{G}} \tilde{\lambda}$.*

Proof. The statement and proof of this proposition are identical to those of [1, Prop. 4.1]. $\square$

7.2. Supercuspidal L -packets. Let ρ be an irreducible admissible representation of $H = H^0 \times Z$. Suppose ρ factors as $\rho^0 \otimes \psi$ for some irreducible admissible representations ρ^0 of H^0 and ψ of Z . The group $\underline{H} = \underline{H}^0 \times \underline{Z}$ may also be factored as $\underline{H} = \underline{H}^0 \times \underline{J}$, where $\underline{J} \cong \underline{Z} \cong U(1, E/F)$ is the F -subgroup of $\underline{H}$ consisting of matrices of the form $\begin{pmatrix} 1 & 0 & 1 \\ 0 & * & 0 \\ 1 & 0 & 1 \end{pmatrix}$. The factorization of ρ with respect to the latter decomposition of H (denoted by the symbol $\otimes'$) is

$$\rho^0 \otimes' \psi \omega_{\rho^0}^{-1},$$

where we view ψ and ω_{ρ^0} as representations of J via the obvious isomorphism of J with Z . An analogous result holds for $\tilde{H}$ when one converts between the decompositions $\tilde{H}^0 \times \tilde{Z}$ and $\tilde{H}^0 \times \tilde{J}$ of $\tilde{H}$, where $\tilde{J}$ is the F -subgroup of $\tilde{H}$ consisting of matrices of the form $\begin{pmatrix} 1 & 0 & 1 \\ 0 & * & 0 \\ 1 & 0 & 1 \end{pmatrix}$. We will again use the notation $\otimes'$ to denote factorization with respect to the second decomposition.

We now recall the labile base change lifting for the groups H^0 and H (see [40, §11.4, 12.1]). If ρ^0 is an element of a singleton L -packet, let $\tilde{\rho}^0$ denote the (stable) base change lift of ρ^0 to $\tilde{H}^0$. The labile base change lift $\tilde{\rho}^0$ is defined to be $\tilde{\rho}^0 \cdot (\Omega \circ \det_{\tilde{H}^0})$. Similarly, the labile base change lift of the irreducible representation $\rho \otimes' \psi$ of $H = H^0 \times J$ is defined to be the representation $\tilde{\rho}^0 \cdot (\Omega \circ \det_{\tilde{H}^0}) \otimes' \tilde{\psi}$ of $\tilde{H}^0 \times \tilde{J}$. Here, as usual, $\tilde{\psi}$ denotes the character of the torus $\tilde{J}$ lifted from ψ .

We will need the following lemma to determine the base change lift of a supercuspidal L -packet of G . For the remainder of this section, $\det$ will be understood to mean $\det_{\tilde{H}^0}$.

Lemma 7.2. *Let $\rho = \rho^0 \otimes \psi$ be an irreducible admissible representation of $H = H^0 \times Z$. If $\{\rho^0\}$ is an L -packet of H^0 , then the labile base change lift $\tilde{\rho}'$ of ρ is*

$$\tilde{\rho}^0 \cdot (\Omega \circ \det) \otimes \tilde{\psi} \Omega^2.$$

Proof. By the preceding discussion, we may factor ρ as $\rho^0 \otimes' \psi \omega_{\rho^0}^{-1}$ with respect to the decomposition $H = H^0 \times J$. Then $\tilde{\rho}'$ is equal to

$$\tilde{\rho}^0 \cdot (\Omega \circ \det) \otimes' \tilde{\psi} \tilde{\omega}_{\rho^0}^{-1}.$$

Thus, with respect to the decomposition $\tilde{H} = \tilde{H}^0 \times \tilde{Z}$, $\tilde{\rho}'$ factors as

$$\tilde{\rho}^0 \cdot (\Omega \circ \det) \otimes \tilde{\psi} \tilde{\omega}_{\rho^0}^{-1} \omega_{\tilde{\rho}^0 \cdot (\Omega \circ \det)}.$$

But

$$\omega_{\tilde{\rho}^0 \cdot (\Omega \circ \det)} = \tilde{\omega}_{\rho^0} \Omega^2,$$

where, on the right side, we view Ω as a character of $\tilde{Z} = E^\times$. The result follows. $\square$

We now consider a supercuspidal L -packet of G . Fix a character ν of Z and a character χ of S° of order greater than 2. We recall the notation of §6.2. As in §4.4, χ and ν determine a cuspidal representation $\bar{\sigma}_x$ of G_x , where x is either y or z . Similarly, as in the discussion before Proposition 6.4, $\chi\phi$ and $\nu\bar{\Omega}$ determine a cuspidal representation $\bar{\tau}$ of H_z , where ϕ is the character of S^0 of order 2. Let

$$\pi_x = \text{ind}_{\text{stab}_G(x)}^G \sigma_x, \quad \rho = \text{ind}_{\text{stab}_H(z)}^H \tau,$$

where σ_x and τ are the inflations of $\bar{\sigma}_x$ and $\bar{\tau}$, respectively. Then by Proposition 6.4, the L -packet $\Pi = \{\pi_y, \pi_z\}$ is the transfer via Ξ_G of $\{\rho\}$ from H . Note that we can express ρ as the representation $\rho^0 \otimes \widehat{\nu}\Omega$ (relative to the decomposition $H = H^0 \times Z$), where ρ^0 is the representation of H^0 of the form (5.1SC-1) coming from the cuspidal Deligne-Lusztig representation $\bar{\sigma}$ of H^0 associated to $\chi\phi$.

Proposition 7.3. *The base change lift of the L -packet $\Pi = \{\pi_y, \pi_z\}$ is the representation $\text{ind}_{\tilde{P}}^{\tilde{G}}(\tilde{\rho}^0 \cdot (\Omega \circ \det) \otimes \tilde{\nu})$, where $\tilde{\rho}^0$ is the base change lift of ρ^0 to $\tilde{H}^0$.*

It is readily checked that the equivalence class of $\tilde{\rho}^0 \cdot (\Omega \circ \det)$, and hence that of $\text{ind}_{\tilde{P}}^{\tilde{G}}(\tilde{\rho}^0 \cdot (\Omega \circ \det) \otimes \tilde{\nu})$, is independent of the choice of Ω .

Proof. Since Π is the endoscopic transfer of ρ , [40, Prop. 13.2.2(c)] implies that the base change lift of Π is $\text{ind}_{\tilde{P}}^{\tilde{G}}(\tilde{\rho}')$, where $\tilde{\rho}'$ is the labile base change lift of ρ . It follows from Lemma 7.2 that the labile base change lift of $\rho = \rho^0 \otimes \widehat{\nu}\Omega$ is

$$\tilde{\rho}^0 \cdot (\Omega \circ \det) \otimes \tilde{\nu}\tilde{\Omega}\Omega^2.$$

Now as a character of $E^\times$, $\tilde{\Omega}$ is the map $a \mapsto \Omega(a/\iota(a))$. Since $\Omega|_{F^\times} = \omega_{E/F}$, it follows that for any $a \in E^\times$, $\tilde{\Omega}\Omega^2(a) = \tilde{\Omega}^{-1}\Omega^2(a) = \Omega(N_{E/F}(a)) = 1$. This proves the proposition. $\square$

7.3. Nonsupercuspidal L -packets containing supercuspidals.

Proposition 7.4. *Let λ be a character of M of depth zero such that $\lambda_1|_{F^\times} = \omega_{E/F}| \cdot |\cdot|_F^{\pm 1}$.*

(i) *The base change lift of the L -packet $\{\pi^2(\lambda), \pi^s(\lambda)\}$ is*

$$\text{ind}_{\tilde{P}}^{\tilde{G}}\left(\text{St}_{\tilde{H}} \cdot \left((\lambda_1 \tilde{\lambda}_2) \cdot |_E^{\mp 1/2} \circ \det_{\tilde{H}^0} \otimes \tilde{\lambda}_2\right)\right).$$

(ii) *The base change lift of the A -packet $\{\pi^n(\lambda), \pi^s(\lambda)\}$ is*

$$\text{ind}_{\tilde{P}}^{\tilde{G}}\left((\lambda_1 \tilde{\lambda}_2) \cdot |_E^{\mp 1/2} \circ \det_{\tilde{H}^0} \otimes \tilde{\lambda}_2\right).$$

Moreover, the above two base change lifts are precisely the irreducible constituents of the principal series representation $\text{ind}_{\tilde{B}}^{\tilde{G}} \tilde{\lambda}$.

Note that the proposition has the same content if we restrict the choice of exponent in the hypothesis to be +1 (or to be -1).

Proof. Identical to the proof of [1, Prop. 4.2]. $\square$

8. COMPATIBILITY OF BASE CHANGE AND MINIMAL K -TYPES

8.1. Principal series L -packets. As in §7.1, suppose Π consists entirely of constituents of the depth-zero principal series representation $\text{ind}_B^G \lambda$. Since each element of Π has depth zero, $\text{ind}_B^G \lambda$ and hence λ have depth zero by [36, Theorem 5.2]. It follows from [36] that $(G_{\mathcal{F}}, \text{infl } \bar{\lambda})$ is a minimal K -type contained in each element of Π , where we have identified M° with $G_{\mathcal{F}}/G_{\mathcal{F}+}$. Similarly, $(\tilde{G}_{\mathcal{F}}, \text{infl } \tilde{\lambda})$ is a minimal K -type of each constituent of $\tilde{\pi} = \text{ind}_{\tilde{P}}^{\tilde{G}} \tilde{\lambda}$ (see Proposition 7.1), where $\tilde{G}_{\mathcal{F}}/\tilde{G}_{\mathcal{F}+}$

is identified with $\tilde{M}$. But the image of $\bar{\lambda}$ under the ε -lifting from $\underline{M}^\circ$ to $\tilde{M}$ of is $\tilde{\lambda}$. Thus the main theorem holds in this case.

8.2. Supercuspidal L -packets. Recalling the notation of §7.2, suppose that Π is the supercuspidal L -packet $\{\pi_y(\chi, \nu), \pi_z(\chi, \nu)\}$. For $x = y$ or z , π_x contains the minimal K -type (G_x, σ_x) , where $\bar{\sigma}_x$ is the cuspidal Deligne-Lusztig representation of $\mathbf{G}_x$ associated to the character χ of $\mathbf{S}^0 = \mathbf{S}^\circ \cong (\mathbf{S}')^\circ$.

By Proposition 7.3, the base change lift $\tilde{\pi}$ of Π is $\text{ind}_{\tilde{P}}^{\tilde{G}} \left(\tilde{\rho}^0 \cdot (\Omega \circ \det) \otimes \tilde{\nu} \right)$, where ρ^0 is the supercuspidal representation of H^0 coming from the cuspidal Deligne-Lusztig representation $\bar{\sigma}$ of $\mathbf{H}^0$ associated to $\chi\phi$. Therefore, by Proposition 5.1, $\tilde{\rho}^0 \cdot (\Omega \circ \det)$ is the representation

$$\begin{aligned} & \left(\text{ind}_{\tilde{H}_z^0 \tilde{Z}^0}^{\tilde{H}^0} \text{ext infl} \left(R_{\tilde{\mathbf{S}}^0}^{\tilde{\mathbf{H}}_z^0} \tilde{\chi\phi} \right) \right) \cdot (\Omega \circ \det) \\ &= \text{ind}_{\tilde{H}_z^0 \tilde{Z}^0}^{\tilde{H}^0} \left(\text{ext infl} \left(R_{\tilde{\mathbf{S}}^0}^{\tilde{\mathbf{H}}_z^0} (\tilde{\chi\phi}) \right) \cdot (\Omega \circ \det) \right). \end{aligned}$$

Here the notation “infl” denotes the inflation from $\tilde{\mathbf{H}}_z^0$ to $\tilde{H}_z^0$, and “ext” denotes the unique extension of the given representation of $\tilde{H}_z^0$ to $\tilde{H}_z^0 \tilde{Z}$ such that the element of $\tilde{Z}$ corresponding to ϖ_E acts via $(\chi\phi)(-1)$. Now

$$\text{ext infl} \left(R_{\tilde{\mathbf{S}}^0}^{\tilde{\mathbf{H}}_z^0} (\tilde{\chi\phi}) \right) \cdot (\Omega \circ \det) = \text{ext}' \text{infl} \left(R_{\tilde{\mathbf{S}}^0}^{\tilde{\mathbf{H}}_z^0} (\tilde{\chi\phi}) \cdot (\bar{\Omega} \circ \det) \right),$$

where “ext'” denotes the extension such that ϖ_E acts via $(\chi\phi)(-1)\Omega(\varpi_E^2) = (\chi\phi)(-1)\Omega(-1) = -\chi(-1)$. Since the restriction of $\bar{\Omega} \circ \det$ to $\tilde{\mathbf{S}}^0$ is quadratic, the latter representation is equivalent to $\text{ext}' \text{infl} \left(R_{\tilde{\mathbf{S}}^0}^{\tilde{\mathbf{H}}_z^0} (\tilde{\chi}) \right)$. Thus $\tilde{\rho}^0 \cdot (\Omega \circ \det) \otimes \tilde{\nu}$ is equivalent to

$$\text{ind}_{\tilde{H}_z^0 \tilde{Z}}^{\tilde{H}} \left(\text{ext}' \text{infl} \left(R_{\tilde{\mathbf{S}}^0}^{\tilde{\mathbf{H}}_z^0} \tilde{\chi} \right) \otimes \tilde{\nu} \right).$$

Since $\tilde{\nu}$ is trivial on the maximal compact subgroup $\tilde{Z}_0$ of $\tilde{Z}$, the representation

$$\text{ext}' \text{infl} \left(R_{\tilde{\mathbf{S}}^0}^{\tilde{\mathbf{H}}_z^0} \tilde{\chi} \right) \otimes \tilde{\nu}$$

of $\tilde{H}_z^0 \tilde{Z}$ can be expressed as an extension to $\tilde{H}_z^0 \tilde{Z}$ of the representation

$$\text{infl} \left(R_{\tilde{\mathbf{S}}^0}^{\tilde{\mathbf{H}}_z^0} \tilde{\chi} \right) \otimes \mathbf{1}_{\tilde{Z}_0} = \text{infl} R_{\tilde{\mathbf{S}}}^{\tilde{\mathbf{H}}_z} (\tilde{\chi} \otimes \mathbf{1}_{\tilde{Z}}).$$

Since $\tilde{\pi}$ is parabolically induced from the representation $\tilde{\rho}^0 \cdot (\Omega \circ \det) \otimes \tilde{\nu}$, which contains

$$\left(\tilde{H}_z, \text{infl} \left(R_{\tilde{\mathbf{S}}}^{\tilde{\mathbf{H}}_z} (\tilde{\chi} \otimes \mathbf{1}_{\tilde{Z}}) \right) \right),$$

and since $\tilde{\mathbf{G}}_z \cong \tilde{\mathbf{H}}_z$, we have from [36, Thm. 6.11] that $\tilde{\pi}$ contains the K -type

$$(8.2.1) \quad \left(\tilde{G}_z, \text{infl} \left(R_{\tilde{\mathbf{S}}}^{\tilde{\mathbf{G}}_z} (\tilde{\chi} \otimes \mathbf{1}_{\tilde{Z}}) \right) \right).$$

But $R_{\tilde{\mathbf{S}}}^{\tilde{\mathbf{G}}_z} (\tilde{\chi} \otimes \mathbf{1}_{\tilde{Z}})$ is the ε -lift of $\bar{\sigma}_z$ since the character $\tilde{\chi} \otimes \mathbf{1}_{\tilde{Z}}$ of $\tilde{\mathbf{S}}$ is the ε -lift of the character χ of $\mathbf{S}$.

We now consider the vertex y . Note that

$$R_{\tilde{S}'}^{\tilde{G}_y}(\tilde{\chi} \otimes \mathbf{1}_{\tilde{Z}}) \cong \text{ind}_{\tilde{P}_{z'}}^{\tilde{G}_y}(R_{\tilde{S}'}^{\tilde{G}_{z'}}(\tilde{\chi} \otimes \mathbf{1}_{\tilde{Z}})),$$

where $S' = gSg^{-1}$ is the stable conjugate of S fixing y (see 4.3.2), $z' = g \cdot z$, and $\tilde{P}_{z'}$ is the image of $\tilde{G}_{z'}$ in $\tilde{G}_y$. Now $\tilde{\pi}$ must contain the minimal K -type

$$\left(\tilde{G}_{z'}, \text{infl}(R_{\tilde{S}'}^{\tilde{G}_{z'}}(\tilde{\chi} \otimes \mathbf{1}_{\tilde{Z}}))\right),$$

since it is an associate of (8.2.1). An application of Frobenius reciprocity then shows that $\tilde{\pi}$ must also contain

$$\left(\tilde{G}_y, \text{infl}(R_{\tilde{S}'}^{\tilde{G}_y}(\tilde{\chi} \otimes \mathbf{1}_{\tilde{Z}}))\right),$$

and $R_{\tilde{S}'}^{\tilde{G}_y}(\tilde{\chi} \otimes \mathbf{1})$ is the ε -lift of $\bar{\sigma}_y$.

8.3. Nonsupercuspidal L -packets containing supercuspidals. Now suppose that Π is an L -packet of the form $\{\pi^2(\lambda), \pi^s(\lambda)\}$ or an A -packet of the form $\{\pi^n(\lambda), \pi^s(\lambda)\}$ for some $\lambda \in \text{Hom}(M, \mathbb{C}^\times)$ of depth zero (see case (4.4PS-2) and §6.1). In either case, let $\tilde{\pi}$ denote the base change lift of Π . Both $\pi^2(\lambda)$ and $\pi^n(\lambda)$ are constituents of the principal series representation $\text{ind}_B^G \lambda$. It follows from [36] that $\text{ind}_B^G \lambda$ has depth zero and that, for any $x \in \mathcal{F}$, $(G_x, \text{infl } \bar{\lambda})$ is a minimal K -type for both $\pi^2(\lambda)$ and $\pi^n(\lambda)$. By Proposition 7.4, $\tilde{\pi}$ is a constituent of the principal series $\text{ind}_{\tilde{B}}^{\tilde{G}} \tilde{\lambda}$. Therefore, as in §8.1, $(\tilde{G}_x, \text{infl } \tilde{\lambda})$ is a minimal K -type for $\tilde{\pi}$. The main theorem now follows for $\pi^2(\lambda)$ and $\pi^n(\lambda)$ exactly as it did in §8.1.

It remains to consider $\pi^s(\lambda)$ (both as an element of $\{\pi^s(\lambda), \pi^2(\lambda)\}$ and as one of $\{\pi^s(\lambda), \pi^n(\lambda)\}$). As shown in Proposition 6.5, $\pi^s(\lambda)$ is a supercuspidal representation of the form (4.4SC-3). Therefore, $\pi^s(\lambda)$ contains a minimal K -type of the form $(G_z, \text{infl}(\bar{\sigma} \otimes \nu))$, where $\bar{\sigma}$ is a cuspidal representation of $\mathbf{G}_z^\circ$ of degree $(q-1)/2$ and ν is a character of $\mathbf{Z}$. Therefore, $(G_z, \text{infl } \bar{\sigma})$ is a minimal K -type of $\pi^s(\lambda)$.

Now $\bar{\sigma}$ is a subrepresentation of $R_{\tilde{S}^\circ}^{\mathbf{G}_z^\circ} \phi$, where ϕ is the nontrivial quadratic character of $\mathbf{S}^\circ$. Hence $\bar{\sigma}$ is a member of the rational series of representations of $\mathbf{G}_z^\circ$ attached to ϕ (see §3.2). The image of this rational series under the ε -lifting is the rational series of representations of $\tilde{\mathbf{G}}_z$ attached to the character $\tilde{\phi}$ of $\tilde{\mathbf{S}}$. Since $\tilde{\phi}$ is the nontrivial quadratic character $\tilde{\eta}$ of $\tilde{\mathbf{S}}$, it extends to $\tilde{\mathbf{G}}_z$, and it is easily seen that the latter series consists of the representations $\tilde{\eta}$ and $\text{St}_{\tilde{\mathbf{G}}_z} \cdot \tilde{\eta}$.

Since the base change lift of the L - or A -packet Π is an irreducible constituent of $\text{ind}_B^G \lambda$, we have (as in the proof of Proposition 6.5)

$$\left(\text{ind}_B^G \lambda\right)^{\tilde{G}_z^+} \cong \text{ind}_{\tilde{\mathbf{B}}_z}^{\tilde{\mathbf{G}}_z} \tilde{\lambda},$$

so $\tilde{\pi}^{\tilde{G}_z^+}$ is a subrepresentation of $\text{ind}_{\tilde{\mathbf{B}}_z}^{\tilde{\mathbf{G}}_z} \tilde{\lambda}$.

Since $\lambda_1|_{F^\times} = \omega_{E/F} \cdot |\cdot|_F^{\pm 1}$, one readily computes that the character $\tilde{\lambda}$ of $\tilde{\mathbf{M}}$ is equal to $(\tilde{\eta} \circ \det) \otimes \mathbf{1}$, where the factorization is with respect to the decomposition

$\tilde{M} = \tilde{M}^0 \times \tilde{Z}$. Thus

$$\mathrm{ind}_{\tilde{B}_z}^{\tilde{G}_z} \tilde{\lambda} \cong \tilde{\eta} \oplus \left(\mathrm{St}_{\tilde{G}_z} \cdot \tilde{\eta} \right).$$

Hence $\tilde{\pi}$ contains a representation in the ε -lift of the rational series containing $\bar{\sigma}$.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, AMERICAN UNIVERSITY, WASHINGTON, DC
20016-8050

E-mail address: jadler@american.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, AMERICAN UNIVERSITY, WASHINGTON, DC
20016-8050

E-mail address: lansky@american.edu