A SOLVABLE VERSION OF THE BAER–SUZUKI THEOREM

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ABSTRACT. Suppose that G is a finite group and $x \in G$ has prime order $p \geq 5$. Then x is contained in the solvable radical of G , $O_\infty(G)$, if (and only if) $\langle x, x^g \rangle$ is solvable for all $g \in G$. If G is an almost simple group and $x \in G$ has prime order $p \geq 5$, then this implies that there exists $g \in G$ such that $\langle x, x^g \rangle$ is not solvable. In fact, this is also true when $p = 3$ with very few exceptions, which are described explicitly.

1. INTRODUCTION

The Baer–Suzuki theorem provides a useful characterization of the Fitting subgroup of a finite (or linear) group. It can be stated as follows:

Theorem 1 (Baer–Suzuki). *Let G be a finite (or linear) group. Suppose that for some $x \in G$, $\langle x, x^g \rangle$ is nilpotent for all $g \in G$. Then $\langle x^G \rangle$ is nilpotent. That is, x is contained in the Fitting subgroup of G .*

It is natural to ask if there is an analogous result if the nilpotency condition is replaced with solvability. However, it is easy to find counterexamples. For example, any two involutions generate a dihedral group. So if G is a non-abelian simple group and x is an involution in G , then $\langle x^G \rangle = G$ is not solvable yet $\langle x, x^g \rangle$ is solvable for all $g \in G$.

There are also counterexamples when x has order 3. Suppose that $x \in SL(n, 3)$ ($n \geq 3$) has order 3 and acts trivially on some hyperplane; that is, x is a transvection. Then x and any conjugate x^g generate a group that acts trivially on a subspace of codimension at most 2. Thus $\langle x, x^g \rangle$ is solvable since it has a normal abelian subgroup N such that $\langle x, x^g \rangle/N$ is isomorphic to a subgroup of $GL(2, 3)$. However, since x is not central, it is not contained in the solvable radical of $SL(n, 3)$ and $\langle x^G \rangle$ is not solvable. The aim is to prove the following:

Theorem A. *Let G be a finite group. Suppose that $x \in G$ has prime order $p \geq 5$. If $\langle x, x^g \rangle$ is solvable for all $g \in G$, then $\langle x^G \rangle$ is solvable. Equivalently, if $x \notin O_\infty(G)$, then there exists $g \in G$ such that $\langle x, x^g \rangle$ is not solvable.*

Received by the editors January 25, 2008 and, in revised form, September 14, 2008.

2000 *Mathematics Subject Classification.* Primary 20F14, 20D10.

The author was partially supported by the NSF grant DMS 0653873.

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It is worth noting that Theorem A implies the following result:

Corollary 1. *Let G be a finite (or linear) group. Then G is solvable if and only if any two conjugates generate a solvable group.*

Proof. Let G be a minimal counterexample to the version of the theorem for finite groups. Thus G is a finite simple group by minimality, and therefore G contains an element x of prime order $p \geq 5$. So Theorem A implies that there exists $g \in G$ such that $\langle x, x^g \rangle$ is not solvable, and thus G is not a minimal counterexample. The version of the theorem for linear groups follows from the finite group version using a standard argument (see [FGG, Corollary 1.2] for example). $\square$

Also note that a minimal counterexample in Theorem 1 must be one of the minimal simple groups described by Thompson in the N -group paper [Tho68]. Thus one could prove Theorem 1 without relying on the full Classification theorem by ruling out all of the minimal simple groups.

Theorem A is also used in [FGG] to prove:

Theorem 2. *Let G be a finite or linear group. Then $x \in G$ is contained in the solvable radical of G if and only if $\langle x, x^{g_1}, x^{g_2}, x^{g_3} \rangle$ is solvable for all $g_1, g_2, g_3 \in G$.*

The proof in [FGG] relies on the Classification of Finite Simple Groups; however, a weaker version of the theorem for finite groups is also given in [FGG] that does not rely on the Classification theorem:

Theorem 3. *Let G be a finite group. Then $x \in G$ is contained in the solvable radical of G if and only if every 7 conjugates of x generate a solvable group.*

Theorems A and 2 were announced in [GPS07] (see Theorems 7.3 and 7.4). They have been obtained independently in [GGKP08b] and [GGKP08a], also using the Classification theorem.

2. REDUCTION

Lemma 1 below simplifies matters considerably. It reduces the proof to a situation where G is an almost simple group.

Lemma 1. *Suppose that G is a finite group such that the Fitting subgroup $F(G)$ is trivial. Let L be a component of G .*

(a) *If x is an element of G such that $x \notin N_G(L)$ and $x^2 \notin C_G(L)$, then there exists an element g in G such that $\langle x, x^g \rangle$ is not solvable.*

(b) *If x is an element of G such that $x \notin N_G(L)$ and $x^2 \in C_G(L)$, then there exist elements g_1 and g_2 in G such that $\langle x, x^{g_1}, x^{g_2} \rangle$ is not solvable.*

Proof. Write $E(G)$ for the subgroup of G generated by its components. Then the generalized Fitting subgroup is $F^*(G) = E(G)F(G)$. Since $F(G) = 1$, it follows that $Z(F^*(G)) = Z(E(G))$ is a normal abelian subgroup of G and is therefore trivial. Also, $Z(E(G))$ is generated by the centers of each component of G and so all of the components of G are simple. Moreover, $E(G)$ must be a direct product of the components of G . So G is embedded in $\text{Aut}(F^*(G)) = \text{Aut}(E(G))$. It suffices to assume that $G = \langle L, x \rangle$. Thus if $t := |\{L^{x^i} : \text{for } i = 1, 2, \dots\}|$, then $E(G) = L \times \dots \times L^{x^{t-1}}$ and $\text{Aut}(E(G)) \cong \text{Aut}(L) \wr S_t$. Since x does not normalize

L , it follows that $t \geq 2$. Moreover, it suffices to assume that $x = (\sigma_1, \dots, \sigma_t)\tau$, where $\sigma_i \in \text{Aut}(L^{x^{i-1}})$ and τ is the t -cycle $(12 \cdots t)$. Now observe that

$$\begin{aligned} x^{(u_1, \dots, u_t)} &= (u_1, \dots, u_t)(\sigma_1, \dots, \sigma_t)\tau(u_1, \dots, u_t)^{-1}\tau^{-1}\tau \\ &= (u_1, \dots, u_t)(\sigma_1, \dots, \sigma_t)(u_t^{-1}, u_1^{-1}, \dots, u_{t-1}^{-1})\tau. \end{aligned}$$

So if

$$\begin{aligned} u_t &= 1, u_{t-1} = \sigma_t, u_{t-2} = (\sigma_t \sigma_{t-1}), u_{t-3} = (\sigma_t \sigma_{t-1} \sigma_{t-2}), \dots, \\ u_1 &= (\sigma_t \sigma_{t-1} \cdots \sigma_1), \end{aligned}$$

then $x^{(u_1, \dots, u_t)} = (y, 1, 1, \dots, 1)\tau$ for some $y \in \text{Aut}(L)$. Thus, it suffices to assume that x is of this form.

Now let $g := (w_1, \dots, w_t) \in \text{Aut}(L) \times \cdots \times \text{Aut}(L^{x^{t-1}})$ so that

$$x^{-1}(w_1, \dots, w_t)x(w_1^{-1}, \dots, w_t^{-1}) = (w_2 w_1^{-1}, \dots)$$

and

$$(w_1, \dots, w_t)x(w_1, \dots, w_t)^{-1}x^{-1} = (w_1 y w_t^{-1} y^{-1}, \dots).$$

First, suppose that $t \geq 3$. By [AG84, Theorem B], there exist l_1 and l_2 in L such that $L = \langle l_1, l_2 \rangle$. So define $w_1 = 1$, $w_2 = l_1$, and $w_t = y^{-1} l_2 y$. Thus $\langle x, x^g \rangle$ contains $\langle l_1, \dots \rangle$ and $\langle l_2, \dots \rangle$ and is not solvable. If $t = 2$ and $x^2 \notin C_G(L)$, then $x = (y, 1)\tau$ and since $x^2 = (y, y)$, it follows that $y \neq 1$. Now $\langle y, L \rangle$ is almost simple, so by [GK00] there exists $z \in \langle y, L \rangle$ such that $\langle y, z \rangle$ contains L . Observe that there exists $l \in L$ such that $z = y^k l$. So define $w_1 := 1$ and $w_2 := l$ and then

$$\begin{aligned} x^{2k-1}x^{(w_1, w_2)} &= (y^k, y^{k-1})\tau(w_1, w_2)(y, 1)\tau(w_1^{-1}, w_2^{-1}) \\ &= (y^k w_2 w_1^{-1}, \cdot) = (z, \cdot) \end{aligned}$$

and so $\langle x, x^{(w_1, w_2)} \rangle$ cannot be solvable. This proves part (a).

To prove (b), suppose that x does not normalize L and $x^2 \in C_G(L)$. So it suffices to assume that $t = 2$ and $x = \tau$. If $g_1 := (1, l_1)$ and $g_2 := (1, l_2)$, then

$$x^{-1}x^{g_1} = (l_1, \cdot); \quad x^{-1}x^{g_2} = (l_2, \cdot)$$

and thus $\langle x, x^{g_1}, x^{g_2} \rangle$ is not solvable. This proves part (b) of Lemma 1. $\square$

Lemma 2. *Suppose that (x, G) is a minimal counterexample. Then G is almost simple.*

Proof. Since (x, G) is a minimal counterexample, the solvable radical of G is trivial. Let N be a minimal normal subgroup. So $N \cong L \times \cdots \times L$ for some non-abelian simple group L . If $x \in N$, then $G = N$ since otherwise $\langle x^N \rangle$ would be a solvable normal subgroup of N , and N does not have any such subgroups. Thus, if $x \in N$, then G is simple since G has no non-trivial normal subgroups. Now assume that $x \notin N$ and let $H := \langle x, N \rangle$. If $G \neq H$, then $\langle x^H \rangle \cap N$ is a solvable normal subgroup of N and is thus trivial. Thus $[x, N] = 1$, which is not possible, because it would follow that $[\langle x^G \rangle, N] = 1$. Since N is a minimal normal subgroup, $\langle x^G \rangle \cap N$ would be trivial and thus $\langle (xN)^{G/N} \rangle \cong \langle x^G \rangle N/N \cong \langle x^G \rangle$. This is not possible since $\langle (xN)^{G/N} \rangle$ is solvable by minimality. So $G = H = \langle x, N \rangle$. Note that the Fitting subgroup of G is trivial since the solvable radical is trivial and thus x normalizes

TABLE 1. List of exceptions to Theorem A*

G_0	x
$PSL(n, 3), n > 2$	transvection
$PSp(2n, 3), n > 1$	transvection
$PSU(n, 3), n > 2$	transvection
$PSU(n, 2), n > 3$	reflection of order 3
$P\Omega^\epsilon(n, 3), n > 6$	x a long root element
$E_l(3), F_4(3), {}^2E_6(3), {}^3D_4(3)$	x a long root element
$G_2(3)$	x a long or short root element
$G_2(2)' \cong PSU(3, 3)$	transvection

every component by Lemma 1. So L is normal in G , $N = L$ and $G = \langle x, L \rangle$. Now G is almost simple since L is the unique minimal normal subgroup of G . $\square$

The Classification of Finite Simple Groups can be used to determine the possibilities for the socle G_0 of G , and thus eliminate each possibility case by case. In fact, the following theorem is slightly stronger and implies Theorem A.

Theorem A*. *Let G be a finite almost simple group with socle G_0 . Suppose that x is an element of odd prime order in G . Then one of the following holds.*

- (i) *There exists $g \in G$ such that $\langle x, x^g \rangle$ is not solvable.*
- (ii) *$p = 3$ and (x, G_0) belongs to a short list of exceptions given in Table 1. Moreover, there exist $g_1, g_2 \in G$ such that $\langle x, x^{g_1}, x^{g_2} \rangle$ is not solvable, unless $G_0 \cong PSU(n, 2), PSp(2n, 3)$. In any case, there exist $g_1, g_2, g_3 \in G$ such that $\langle x, x^{g_1}, x^{g_2}, x^{g_3} \rangle$ is not solvable.*

Corollary 2. *Let G be an almost simple group, and suppose that $x \in G$ has prime order $p \geq 5$. Suppose that x is contained in the solvable radical of all proper subgroups M containing x . Then there exists $g \in G$ such that $\langle x, x^g \rangle = G$.*

Proof. By Theorem A*, there exists $g \in G$ such that $\langle x, x^g \rangle$ is not solvable. If $\langle x, x^g \rangle \neq G$, then it is contained in some maximal subgroup M . However, the hypothesis implies that $x \in O_\infty(M)$, which would mean that $\langle x, x^g \rangle$ would be solvable. Thus $\langle x, x^g \rangle = G$. $\square$

Clearly, we only need to check that the hypothesis in the corollary is true for all maximal subgroups. Indeed, if $x \in M$ and $M < M' < G$, then $\langle x^{M'} \rangle$ is solvable; therefore $\langle x^M \rangle$ is solvable and thus $x \in O_\infty(M)$.

3. PRELIMINARIES

Let $\overline{G}$ be a simple classical algebraic group of adjoint type over the algebraic closure of $\mathbb{F}_q$. Let σ be a Frobenius morphism of $\overline{G}$ such that $\overline{G}_\sigma := \{g \in \overline{G} : g^\sigma = g\}$ is a finite almost simple classical group over $\mathbb{F}_q$. Write G_0 for the socle of $\overline{G}_\sigma$ and note that $\overline{G}_\sigma$ is the group $\text{Inndiag}(G_0)$ of inner diagonal automorphisms of G_0 . A collection of lemmas, definitions, and theorems is listed below, which will be very useful in the sequel.

Lemma 3. Let $x \in \overline{G}_\sigma$ have odd prime order r . Define $(G, \hat{G})$ as follows:

G_0	$PSL_n^\epsilon(q)$	$PSp_n(q)$	$P\Omega_n^\epsilon(q)$
$(G, \hat{G})$	$(\overline{G}_\sigma, GL_n^\epsilon(q))$	$(G_0, Sp_n(q))$	$(G_0, \Omega_n^\epsilon(q))$

(a) Then one of the following holds: (i) x lifts to an element $\hat{x} \in \hat{G}$ of order r such that $|x^G| = |\hat{x}^{\hat{G}}|$.

(ii) $G_0 = PSL_n^\epsilon(q)$, $r \mid \gcd(q - \epsilon, n)$ and x is $\overline{G}$ -conjugate to $[I_{\frac{n}{r}}, \omega I_{\frac{n}{r}}, \dots, \omega^{r-1} I_{\frac{n}{r}}]$, where ω is a primitive r th root of unity.

(b) If $r \nmid q$, then $x^{G_0} = x^{\overline{G}_\sigma}$.

Proof. See [Bur04, 3.11] and [GLS98, 4.2.2(j)]. $\square$

Definition 1. Let $\mathcal{A}$ be the set of pairs (x, H) such that:

- (i) x is an element of odd prime order contained in a group H ;
- (ii) $H/O_\infty(H)$ is almost simple;
- (iii) x is not contained in $O_\infty(H)$;
- (iv) $(x, H/O_\infty(H))$ is not one of the examples in Table 1.

Lemma 4. (a) If $x \in G$ is an inner-diagonal automorphism of G_0 and $|x^{G_0}| = |x^{\overline{G}_\sigma}|$, then it suffices to take $G = \overline{G}_\sigma$.

(b) If y is some $\text{Aut}(G_0)$ -conjugate of x and there exists $l \in G_0$ such that $\langle y, y^l \rangle$ is not solvable, then there exists $l' \in G_0$ such that $\langle x, x^{l'} \rangle$ is not solvable.

(c) If x is contained in H , a proper subgroup of G , and $(x, H) \in \mathcal{A}$, then G cannot be a minimal counterexample to Theorem A*.

Proof. (a) Suppose that the theorem is true for $\overline{G}_\sigma$. If x is contained in G , then $x \in \overline{G}_\sigma$ and so there exists $g \in \overline{G}_\sigma$ such that $\langle x^g, x \rangle$ is not solvable. But then there exists $g_1 \in G_0$ such that $x^{g_1} = x^g$ by the condition.

(b) Suppose that $y = x^g$ for some $g \in \text{Aut}(G_0)$. Then $\langle y, y^{l'} \rangle^{g^{-1}} = \langle x, x^{l'^g} \rangle = \langle x, x^{l'} \rangle$ since $lg^{-1} = g^{-1}glg^{-1} = g^{-1}l'$.

(c) Trivial. $\square$

Lemma 5. Let $X_1, \dots, X_k$ be representatives for the conjugacy classes of maximal subgroups containing x . Let n_i be the number of conjugates of X_i that contain x . If

$$|x^G|^2 > \sum_i n_i |x^G \cap X_i| = \sum |x^G \cap X_i|^2 [G : X_i],$$

then there exists $g \in G$ such that $\langle x, x^g \rangle = G$.

Proof. Let $X_{i1}, \dots, X_{in_i}$ be the conjugates of X_i that contain x . The aim is to show that x^G cannot be contained in $\bigcup_{i,j} X_{ij}$, since this proves the lemma. It is not hard to show that $n_i/[G : X_i] = |x^G \cap X_i|/|x^G|$. It then follows that

$$\begin{aligned} |x^G \cap \bigcup_{i,j} X_{ij}| &\leq \sum_i n_i |x^G \cap X_i| \\ &= \sum |x^G \cap X_i|^2 [G : X_i] / |x^G| \end{aligned}$$

and so if x^G were contained in $\bigcup_{i,j} X_{ij}$, then

$$\begin{aligned} |x^G| &= |x^G \cap \bigcup_{i,j} X_{ij}| \leq \sum_i n_i |x^G \cap X_i| \\ &= \sum |x^G \cap X_i|^2 [G : X_i] / |x^G|. \end{aligned}$$

However, this implies that

$$|x^G|^2 \leq \sum_i n_i |x^G \cap X_i| = \sum_i |x^G \cap X_i|^2 [G : X_i],$$

which contradicts the hypothesis. $\square$

Remark. If

$$|G|/|C_G(x)|^2 > \sum_i |x^G \cap X_i|,$$

then the conclusion of the theorem holds.

Lemma 6. *Let G_0 be a simple group of Lie type and suppose that G satisfies $G_0 \trianglelefteq G \leq \text{Inndiag}(G_0)$.*

(i) *Suppose that $x \in G$ is unipotent and P_1 and P_2 are distinct maximal parabolic subgroups containing a common Borel subgroup, with unipotent radicals U_1 and U_2 , respectively. Then there exists $i \in \{1, 2\}$ such that x is G -conjugate to an element of $P_i \setminus U_i$.*

(ii) *Suppose that $x \in G$ is semisimple and is contained in a parabolic subgroup of G . Suppose further that the Lie rank of G_0 is at least 2. Then there exists a maximal parabolic subgroup P with a Levi complement J such that x is conjugate to an element of J not centralized by any component of J .*

Proof. See [GS03, Lemma 2.2]. $\square$

Theorem 4. *Let G be an almost simple group and let $x \in G$ with $x \neq 1$. If $x^G \subseteq M_1 \cup M_2$ for subgroups M_1 and M_2 of G , then G_0 is contained in M_i for $i = 1$ or 2 .*

Proof. See [Gur98, Theorem 2.1]. $\square$

To begin the proof of Theorem A*, let (x, G) be a minimal counterexample. Then G is almost simple with socle G_0 . If $p \geq 5$, then Theorem A holds for any group containing fewer elements than G .

4. ALTERNATING GROUPS

Suppose that $G_0 = A_n$. Then x is contained in A_n since it has odd order. Firstly, consider the case where $p \geq 5$. The cycle structure of x will consist only of p -cycles. So it suffices to assume that $x = (12 \dots p)\sigma$ for some $\sigma \in \text{Alt}\{p+1, \dots, n\}$. Observe that if $g := (123)$, then $xgx^{-1}g^{-1} = (2p3)$. Thus $\langle x, x^g \rangle$ contains $\text{Alt}\{1, 2, \dots, p\}$ since a primitive permutation group of degree $p \geq 5$ containing a 3-cycle contains A_p (see [Wie64, Theorem 13.9] for example). So (x, G) cannot be a counterexample in this case.

Now suppose that $p = 3$. Then the cycle structure of x consists of only 3-cycles. If x is the product of more than one 3-cycle, then it suffices to assume that $G = A_6$ and x is the product of two 3-cycles. But then x is conjugate to a 3-cycle in $\text{Aut}(A_6)$. Thus we may assume that x is a 3-cycle in A_5 and without loss of generality, that $x = (123)$. If $g := (14253)$, then $x^g = (451)$. Thus, $xx^g = (12345)$ and $\langle x, x^g \rangle \cong A_5$.

5. $PSL(n, q)$

If $G_0 \cong PSL(n, q)$, then it is convenient to treat the cases where $n = 2$ and $n \geq 3$ separately.

5.1. $G_0 \cong PSL(2, q)$. Suppose that x is in $\text{Inndiag}(PSL(2, q)) \cong PGL(2, q)$. Since x has odd order, it must lie in $PSL(2, q)$.

5.1.1. $x \in \text{Inndiag}(PSL(2, q))$ and $p \mid q$. If $p \mid q$ and $p \geq 5$, then it suffices to assume that x is contained in $PSL(2, p)$. Consider the possibilities for the maximal subgroups of $PSL(2, p)$ containing $\langle x, x^g \rangle$, which are described in [GLS98, Theorem 6.5.1]. By the order of x and since (x, G) is a minimal counterexample, the only type of maximal subgroup possible is a Borel subgroup, B . Now since $p \mid q$, x and x^g must lie inside the kernel K of B which is (elementary) abelian. So any p -elements lying in a common Borel subgroup must commute. Thus, since there must exist a conjugate of x that does not commute with x (otherwise $[x^G, x^G] = 1$ and G_0 would be abelian) there exists $g \in G$ such that $\langle x, x^g \rangle = PSL(2, p)$, which is not solvable for $p \geq 5$.

If $p = 3$, then $q = 3^a$, where $a > 1$ and since $x \in PSL(2, q)$, it suffices to assume that $G = PSL(2, q)$. Now $A_6 \cong PSL(2, 9)$, so let us assume that $q > 9$. If $q = 3^a$ and a is not prime, then there exists a conjugate x^g of x that is contained in a subfield subgroup H with (x^g, H) in $\mathcal{A}$. So a must be an odd prime. Now we may assume that

$$x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

There are two classes of transvections in G , and since -1 is not a square in $\mathbb{F}_q$, x and x^{-1} are not conjugate. Thus if we let

$$y := \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix},$$

then x or x^{-1} is conjugate to y . So there exists $g \in G$ such that $\langle x, x^g \rangle$ contains

$$xy = \begin{pmatrix} 1+s & 1 \\ s & 1 \end{pmatrix},$$

which is semisimple and has trace $s + 2$. In particular we can choose s so that xy has order $\frac{q+1}{2}$ and an inspection of the maximal subgroups of G shows that $\langle x, x^g \rangle = G$.

5.1.2. $x \in \text{Inndiag}(PSL(2, q))$ and $p \nmid q$. Suppose now that $p \nmid q$. Then either $p \mid q - 1$ or $p \mid q + 1$. If $p \mid q - 1$, then x is contained in a split torus. Examining the character table of $PSL(2, q)$ shows that for an element z of order $(q + 1)/(2, q - 1)$,

$$\sum_{\chi \in \text{Irr}(G_0)} \frac{\overline{\chi(z)} |\chi(x)|^2}{\chi(1)} \neq 0,$$

so there exists $g \in G$ such that $[x, g]$ has order $(q + 1)/(2, q - 1)$. That is, $[x, g]$ generates a non-split torus. It follows that $\langle x, x^g \rangle$ generates $PSL(2, q)$. Indeed, $\langle x, x^g \rangle$ contains an irreducible torus, and it also contains x , which does not normalize this torus. An inspection of the maximal subgroups of $PSL(2, q)$ yields that $\langle x, x^g \rangle$ must generate the whole group for $q \geq 11$, and $q = 8$. It suffices to assume that $q \neq 4, 5$ or 9 since in those cases G is isomorphic to an alternating group. When $q = 7$, the normalizer of a non-split torus is not maximal, but is contained in subgroups isomorphic to S_4 . However, since $p \mid q - 1$, $\langle x, x^g \rangle$ cannot be contained in S_4 since S_4 does not contain two elements of order 3 whose product has order 4.

If $p \mid q + 1$, then the character table implies that there exists $g \in G$ such that $[x, g]$ has order $(q - 1)/(2, q - 1)$. Thus $\langle x, x^g \rangle$ contains a split torus, and since $p \mid q + 1$, it acts irreducibly. Therefore, an inspection of the maximal subgroups shows that for $q \geq 13$ and $q = 8$, $\langle x, x^g \rangle = PSL(2, q)$. Again, the cases when $q = 4, 5$, and 9 do not concern us. Also note that $q \neq 7$ since $p \mid q + 1$. If $q = 11$, then $\langle x, x^g \rangle$ contains a maximal split torus and acts irreducibly. The list of maximal subgroups then implies that either $\langle x, x^g \rangle = PSL(2, q)$ or A_5 . There are no other possibilities for $x \in \text{Inndiag}(PSL(2, q))$.

5.1.3. x an outer automorphism of $PSL(2, q)$. Suppose that x is not contained in $\text{Inndiag}(G_0)$. Then by [GL83, 7.2], and since x has odd order, there exists an element $g \in PGL(2, q)$ such that x^g is a standard field automorphism. So it suffices to assume that x is a standard field automorphism by Lemma 4, and moreover, that $G = \langle G_0, x \rangle$. Write $q = q_1^p$ and consider the set $\Gamma = \{y \in x^{G_0} \mid \langle x, y \rangle \neq G\}$. The aim is to bound the cardinality of Γ and show that this is smaller than $|x^{G_0}|$. Now if $y \in \Gamma$, then consider the possibilities for subgroups H of G_0 containing $\langle x, y \rangle \cap G_0$. Observe that $\langle x, y \rangle \cap G_0$ cannot be dihedral. Indeed, since a dihedral group has a characteristic cyclic subgroup of index 2, K say, K would be normal in $\langle x, y \rangle$. Now $\langle x, y \rangle / K$ has a normal subgroup of order 2 and a subgroup of order p , which is normal since it has index 2. So, $\langle x, y \rangle / K$ is abelian of order $2p$, but this is impossible since it is generated by two elements of order p . Thus, it suffices to assume that H is a Borel subgroup, a cyclic group of order $(q + 1)/(2, q - 1)$, or a subfield subgroup. Since p is odd, any A_5 or S_4 will be contained in a subfield subgroup and any cyclic group of order $(q - 1)/(2, q - 1)$ will be contained in a Borel subgroup. Now let H be a Borel, non-split torus or subfield subgroup of the form $L(2, q^{1/r})$, where r is a prime distinct from p . Observe that we may assume that there are no subfield subgroups of the form $L(2, q^{1/r})$ ($r \neq p$) since some conjugate of x will be a non-trivial field automorphism of the simple subgroup $L(2, q^{1/r})$ contradicting the minimality of (x, G) . Now the conjugates of H fixed by x form one $C_{G_0}(x)$ orbit. This follows from the fact that any two conjugates of x in $H\langle x \rangle$ are in fact conjugate by an element of H , which is a consequence of Lang's Theorem (see [GL83, 7.2]). So if k is the number of conjugates of H fixed by x , then

$$\begin{aligned} |\{y \in x^{G_0} : \langle x, y \rangle \cap G_0 \text{ is contained in a conjugate of } H\}| &\leq |x^H| \cdot k \\ &= \frac{|H||C_{G_0}(x)|}{|C_H(x)|^2}. \end{aligned}$$

Moreover x does not fix any non-trivial conjugate of $C_{G_0}(x) = PSL(2, q^{1/p})$, so

$$|\{y \in x^{G_0} : \langle x, y \rangle \cap G_0 \text{ is contained in some conjugate of } C_{G_0}(x)\}| \leq |C_{G_0}(x)|.$$

Therefore if the representatives for the conjugacy classes of the subgroups above are denoted by $H_1, \dots, H_m, H_{m+1} := C_{G_0}(x)$, then

$$\begin{aligned} |\Gamma| &= |\{y \in x^{G_0} : G_0 \cap \langle x, y \rangle \text{ is contained in some conjugate of some } H_i\}| \\ &\leq |C_{G_0}(x)| + \sum_{i=1}^m |H_i||C_{G_0}(x)|/|C_{H_i}(x)|^2. \end{aligned}$$

If $q_0 := q^{1/p}$, then

$$\begin{aligned} |H||C_{G_0}(x)|/|C_H(x)|^2 &= (q_0^p + 1)q_0(q_0^2 - 1)/(q_0 + 1)^2 \\ &= (q_0^p + 1)q_0(q_0 - 1)/(q_0 + 1) \end{aligned}$$

when H is a non-split torus. Similarly if H is a Borel subgroup, then

$$|H||C_{G_0}(x)|/|C_H(x)|^2 = q_0^p(q_0^p - 1)(q_0 + 1)/q_0(q_0 - 1).$$

So,

$$|\Gamma| \leq q_0(q_0^2 - 1) + \frac{q_0^p(q_0^p - 1)(q_0 + 1)}{q_0(q_0 - 1)} + \frac{(q_0^p + 1)q_0(q_0 - 1)}{(q_0 + 1)}.$$

However, $|x^{G_0}| = |G_0|/|C_{G_0}(x)| = \frac{q_0^p(q_0^{2p} - 1)}{q_0(q_0^p - 1)}$ and $q \geq 8$, so it follows that $|x^{G_0}| > |\Gamma|$, as required. Thus, if x is an outer automorphism of $PSL(2, q)$, then (x, G) cannot be a minimal counterexample.

6. OUTER AUTOMORPHISMS

If (x, G) is a minimal counterexample and x is an outer automorphism of G_0 , then the work for $G_0 = PSL(2, q)$ allows a considerable narrowing of the possibilities for G_0 . This is demonstrated in Lemma 7 below.

Lemma 7. *If x is an outer automorphism of G_0 that is not inner-diagonal and (x, G) is a minimal counterexample, then G_0 is a Suzuki–Ree group.*

Proof. Since $x \notin \text{Inndiag}(G_0)$ and x has odd prime order, either x is a field automorphism or, $G_0 \cong D_4(q)$ or ${}^3D_4(q)$ and x is a graph or graph-field automorphism. Since the case where $G_0 \cong PSL(2, q)$ has already been eliminated, the Lie rank is at least 2. If x is a field automorphism, then by [GL83, 7.2] and Lemma 4 it suffices to assume that x is a standard field automorphism. So if G_0 is not a Suzuki–Ree group, then x will act non-trivially as a field automorphism on a fundamental SL_2 -subgroup, by [GLS98, Theorem 3.2.8]. So (x, G) cannot be a minimal counterexample.

If $G_0 \cong {}^3D_4(q)$ and x is a graph automorphism of order 3, then [GL83, 9.1] describes the conjugacy classes of such elements. Let γ be the standard triality automorphism and $g = h_{\beta_0}(\omega)$, where ω is a primitive cube root of unity and β_0 is the γ -invariant fundamental root. Thus, if $3 \nmid q$, then it suffices to assume that x is either γ or $g\gamma$. Also, if $3 \mid q$, then it suffices to assume that x is either γ or $x_\beta(1)\gamma$, where β is the highest root. In all cases, x normalizes the maximal parabolic corresponding to β_0 . Moreover x acts non-trivially on the Levi complement in all these cases and so (x, G) cannot be a minimal counterexample. The only case left is where $G_0 \cong D_4(q)$ and x is a graph or field-graph automorphism of order 3, in which case, using [GL83], it suffices to assume that x is either the standard triality (and $C_{G_0}(x) = G_2(q)$) or it normalizes but does not centralize a subgroup isomorphic to $G_2(q)$. In the latter case x induces a non-trivial automorphism on $G_2(q)$, so (x, G) cannot be a minimal counterexample. In the former case, since $G_2(q)$ does not contain a Sylow 3-subgroup, x normalizes more than one conjugate of $G_2(q)$. Since it only centralizes one $G_2(q)$ subgroup, it follows that x induces a non-trivial automorphism on some subgroup isomorphic to $G_2(q)$, and so (x, G) cannot be a minimal counterexample. $\square$

7. $PSL(n, q)$, $n \geq 3$

7.1. $x \in PGL(n, q)$, $p \nmid q$, $n \geq 3$. Now suppose that (x, G) is a minimal counterexample with $G_0 = PSL(n, q)$ and $n \geq 3$.

Lemma 8. *For $n \geq 3$, if one can lift x to an element of order p in $GL(n, q)$ and x does not act irreducibly, then (x, G) cannot be a minimal counterexample.*

Proof. Suppose that one can lift x to an element of $GL(n, q)$ of order p . Now the minimal polynomial $m_x(t)$ of x divides $(t^p - 1)$, so suppose that $(t^p - 1)/(t - 1)$ factors into irreducibles $g_1(t) \dots g_k(t)$. Then each non-linear $g_i(t)$ is the minimal polynomial of some primitive p th root of unity ζ_p . Thus

$$\deg g_i(t) = [\mathbb{F}_q(\zeta_p) : \mathbb{F}_q].$$

But $\mathbb{F}_q(\zeta_p)$ is just the finite field of q^e elements, where e is the smallest positive integer such that $p \mid q^e - 1$. So all of the $g_i(t)$'s have degree e . Now $m_x(t)$ is a product of some $g_i(t)$'s and possibly $t - 1$. By considering the rational canonical form of x , it is clear that there is an e -dimensional subspace U of V on which x acts invariantly, non-trivially and irreducibly. If $2 \leq e < n$, then consider the induced transformation of U , x_U so that $x_U \in GL(e, q)$. Now observe that if $(e, q) \neq (2, 2)$ or $(2, 3)$, then $(x_U, GL(e, q)) \in \mathcal{A}$. If $(e, q) = (2, 3)$, then p would be 2. So the only case of concern is $(e, q) = (2, 2)$ and then for $n \geq 4$ one can just reduce to the case where $G_0 = PSL(4, 2)$. However $PSL(4, 2) \cong A_8$ and $PSL(3, 2) \cong PSL(2, 7)$, which have already been eliminated. If $e = 1$, then since $p \mid q - 1$, $q \geq 4$. Now x will act non-trivially on a 2-dimensional subspace U' ; thus $x_{U'} \in GL(2, q)$ and $(x_{U'}, GL(2, q)) \in \mathcal{A}$. So unless $e = n \geq 3$, (x, G) cannot be a minimal counterexample. $\square$

Now observe that the proof above shows that if $(x, PGL(n, q))$ is a minimal counterexample and x lifts to an element of order p in $GL(n, q)$, then p is a primitive prime divisor of $q^n - 1$ and x acts irreducibly. Also, if x acts irreducibly and n is not prime, then some conjugate of x is contained in a field extension subgroup $PGL(\frac{n}{r}, q^r)$. Thus, if $(x, PGL(n, q))$ is a minimal counterexample, then n is prime.

The results in [GPPS99] state that any subgroup of $GL(n, q)$ that has order divisible by a primitive prime divisor of $q^e - 1$ must be one of nine types (2.1–2.9). The results of [GPPS99] will be used frequently and are summarized in Table 2. The notation of [GPPS99] will be used, namely, that the element of $GL(d, q)$ that is a primitive prime divisor of $q^e - 1$ will be referred to as a $\text{ppd}(d, q, e)$ -element. The only elements that are of interest are $\text{ppd}(n, q, n)$ -elements, where n is (an odd) prime. So what are the possibilities for a maximal subgroup M of $GL(n, q)$ containing x ?

Lemma 9. *Suppose that x is a $\text{ppd}(n, q, n)$ -element contained in a subgroup M of G , where G is a classical group of dimension $n \geq 3$ and (x, G) is a minimal counterexample. Then $p \geq 5$ and M cannot be of type 2.2, 2.3, 2.4(a), 2.6, 2.7, 2.8, or 2.9.*

Proof. Firstly, if $p = 3$, then since $p \nmid q$, Fermat's Little Theorem implies that $p \mid q^2 - 1$; thus p cannot be a primitive prime divisor of $q^n - 1$ for $n \geq 3$. If G is a classical group, then $M \leq G \leq GL(n, q)$ for some q , and so M must be one of the examples in [GPPS99]. All of the subgroups M of type 2.6–2.9 are almost simple modulo scalars, so it suffices to check that $(x, M/(M \cap Z)) \in \mathcal{A}$.

TABLE 2. Summary of descriptions in [GPPS99] of subgroup types containing $\text{ppd}(d, q, e)$ -elements

Type	Rough description	Conditions on d, q, e
Classical (2.1(a))	$SL(d, q_0) \trianglelefteq M$	p a $\text{ppd}(q_0, d, e)$ -element
Classical (2.1(b))	$Sp(d, q_0) \trianglelefteq M$	d, e both even; p a $\text{ppd}(q_0, d, e)$ -element
Classical (2.1(c))	$SU(d, q_0) \trianglelefteq M$	q_0 a square; e odd; p a $\text{ppd}(q_0, d, e)$ -element
Classical (2.1(d))	$\Omega^\epsilon(d, q_0) \trianglelefteq M$	$\epsilon = \pm$ when d even; $\epsilon = 0$ when dq is odd; e even; p a $\text{ppd}(q_0, d, e)$ -element
Reducible (2.2)	M reducible	
Imprimitive (2.3)	$M \leq GL(1, q) \text{ wr } S_d$	$p = e + 1 \leq d$
Extension Field (2.4(a))	$M \leq GL(1, q^d).d$	$p = d = e + 1$
Extension Field (2.4(b))	$M \leq GL(d/b, q^b).b$	$b \mid \gcd(d, e)$
Symplectic type (2.5)		$d = 2^a$; q odd not a square; $p = d + 1 = e + 1$ or $p = d - 1 = e + 1$
Nearly simple (2.6–2.9)	$M/(M \cap Z)$ simple	Possibilities listed in tables in [GPPS99]

If M is of type 2.6 or 2.7, then $F^*(M/(M \cap Z)) \cong A_d$ for some d , or a sporadic group and so $(x, M/(M \cap Z)) \in \mathcal{A}$. The only $\text{ppd}(n, q, n)$ -elements in the type 2.8 examples, i.e., $(M/(M \cap Z)) \in \text{Lie}(q_0)$, are with $M^{(\infty)} = G_2(q_1)$, $q_0 = 2$ and $M^{(\infty)} = {}^2B_2(q_1)$, $q_0 = 2$, but these occurrences must all lie in $\mathcal{A}$. Similarly, all of the type 2.9 subgroups in [GPPS99, Tables 7 and 8] coincide with elements of $\mathcal{A}$. Since x acts irreducibly it cannot be contained in a reducible subgroup of type 2.2 and it cannot be contained in a type 2.3 example since these are only examples for $\text{ppd}(d, q, e)$ -elements, where $e + 1 \leq d$. Similarly x cannot be contained in a 2.4(a) type subgroup since these are only examples for $\text{ppd}(d, q, e)$ -elements, where $e + 1 = d$. $\square$

Suppose that x is contained in a classical example of type 2.1. By [KL90] and [GPPS99], since $n \geq 3$, all of the classical examples containing $\text{ppd}(n, q, n)$ -elements are almost simple modulo scalars. So if x is contained in a type 2.1 subgroup M , then (x, G) cannot be a minimal counterexample since $p \geq 5$. The symplectic type examples (2.5) only occur as subgroups of $GL(2^a, q)$, but it is assumed that n is an odd prime. Therefore, the only possibilities for subgroups M containing x are the extension field examples of type (2.4(b)). Since n is prime, M must be of type $GL(1, q^n).n$. Moreover, if $p \mid n$, then $p = n$ since n is prime. However, $p \nmid q^p - 1$ so $p \nmid n$. Thus, x must lie inside the Singer cycle $GL(1, q^n)$. Furthermore, $C_{GL(n, q)}(x) = GL(1, q^n)$; thus x can only lie in one such maximal subgroup and applying Theorem 4 yields that (x, G) cannot be a minimal counterexample.

Lemma 10. *If x does not lift to an element of order p in $GL(n, q)$, then (x, G) cannot be a minimal counterexample.*

Proof. Suppose that x does not lift to an element of order p in $GL(n, q)$. Now x^p is central, so x satisfies the polynomial $p(t) := t^p - \lambda$. Now $p(t)$ is irreducible over $\mathbb{F}_q$. For $p \mid (q-1)$, since x does not lift, thus any field containing a root α of $p(t)$ would be a splitting field for $p(t)$. So the degree of any irreducible factor of $p(t)$ is the degree of the splitting field extension over $\mathbb{F}_q$. However, $p(t)$ has prime degree and so it is either irreducible or it splits completely. It cannot split completely; otherwise λ would have p th roots and x would lift to an element of order p . Thus, the irreducible module for $\langle x \rangle$ has dimension p and so it suffices to deal with the case where $n = p$. So let v be a vector in V and consider the action of x on v . The vectors $v, xv, x^2v, \dots, (x^{p-1})v$ form a basis for V since x acts irreducibly. Moreover $x^pv = \lambda v$. So x is contained in a subgroup of type $GL(1, q) \wr S_p$ and x acts as a p -cycle in the S_p . So for $p \geq 5$, we have shown that (x, G) cannot be a minimal counterexample. Now suppose that $p = 3$. Then it suffices to assume that x has the form

$$\begin{pmatrix} 0 & 0 & \lambda \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now let $t^2 - \mu_2 t - \mu_1$ be an irreducible polynomial in $\mathbb{F}_q[t]$, such that $\begin{pmatrix} 0 & \mu_1 \\ 1 & \mu_2 \end{pmatrix}$ has order $q^2 - 1$. Now x is conjugate to

$$y := \begin{pmatrix} 0 & 0 & -\mu_1^{-1}\lambda \\ 0 & \mu_1 & \mu_2^{-1}(\mu_1^{-1}\lambda - \mu_1^2) \\ 1 & \mu_2 & -\mu_1 \end{pmatrix},$$

and therefore

$$x^{-1}y := \begin{pmatrix} 0 & \mu_1 & \mu_2^{-1}(\mu_1^{-1}\lambda - \mu_1^2) \\ 1 & \mu_2 & -\mu_1 \\ 0 & 0 & -\mu_1^{-1} \end{pmatrix}$$

has order a multiple of $q^2 - 1$. Thus, by [GLS98, Theorem 6.5.3], $\langle x, y \rangle$ is not solvable and hence (x, G) cannot be a minimal counterexample. The case where $p \mid q$ is considered in the next section.

8. UNIPOTENT ELEMENTS

Lemma 11. *Suppose that G_0 is a simple group of Lie type and suppose that $x \in G_0$ is unipotent of order p . If G_0 is defined over $\mathbb{F}_q$ and $q \neq 3$, then (x, G) cannot be a minimal counterexample unless $G_0 = PSU(3, q)$ or ${}^2G_2(q)$.*

Proof. The case where $G_0 = PSL(2, q)$ has already been done. Since p is an odd prime, $G_0 \neq {}^2B_2(q)$ or ${}^2F_4(q)$. In the remaining cases, by Lemma 6, for any two maximal parabolic subgroups P_1 and P_2 (containing a common Borel subgroup) there exists a conjugate of x that is contained in $P_i \setminus U_i$ for either $i = 1$ or 2 . The parabolic subgroups can be chosen so that the Levi complement has only one component, and since $q \geq 5$, it will always be almost simple. It follows that since $(xU_i, P_i/U_i)$ will be contained in $\mathcal{A}$, (x, G) cannot be a minimal counterexample. Table 3 describes the parabolic subgroups to choose and the possibilities for the Levi complement. $\square$

TABLE 3. Maximal parabolic subgroups and their Levi complements used in Lemma 11

G_0	Nodes corresponding to P_1 and P_2 (Bourbaki notation used where node is specified)	Levi complement type
$A_l(q), l \geq 2$	end nodes	$A_{l-1}(q)$
$B_2(q)$	end nodes	$A_1(q)$
$B_l(q), l \geq 3$	end nodes	$B_{l-1}(q), A_{l-1}(q)$
$C_l(q), l \geq 3$	end nodes	$C_{l-1}(q), A_{l-1}(q)$
$D_4(q)$	any 2 end nodes	$A_3(q)$
$D_l(q), l \geq 5$	any 2 end nodes	$D_{l-1}(q), A_{l-1}(q)$
${}^2A_l(q), l \geq 3, l \text{ odd}$	end and middle node	$A_{(l-1)/2}(q^2), {}^2A_{l-2}(q)$
${}^2A_l(q), l \geq 4, l \text{ even}$	end and middle node	${}^2A_{l-2}(q), A_{(l-2)/2}(q^2)$
${}^2D_4(q)$	end nodes	${}^2A_3(q), A_2(q)$
${}^2D_l(q), l \geq 5$	end nodes	${}^2D_{l-1}(q), A_{l-2}(q)$
$E_6(q)$	nodes 1 and 6	$D_5(q)$
$E_7(q)$	nodes 1 and 2	$D_6(q), A_6(q)$
$E_8(q)$	nodes 1 and 2	$D_7(q), A_7(q)$
$F_4(q)$	end nodes	$B_3(q), C_3(q)$
$G_2(q)$	end nodes	$A_1(q)$
${}^2E_6(q)$	end nodes	${}^2D_4(q), {}^2A_5(q)$

The next lemma also eliminates the possibility that $q = 3$ for classical groups.

Lemma 12. *If x is an element of order 3 in a classical group G defined over $\mathbb{F}_3$, then (x, G) cannot be a minimal counterexample.*

Proof. The aim is to show that if x is not a long root element, then, unless the dimension of the natural module V is very small, there exists a subgroup H such that (x, H) is in $\mathcal{A}$. By [Wal63, pp.34–38], if x is an element of order 3 in a classical group over $\mathbb{F}_3$, then x will nearly always fix an orthogonal decomposition unless n is very small. Suppose that x has order 3 in $SL(n, 3)$ with $n \geq 5$. Then there exist x -invariant subspaces U and W such that $V = U \oplus W$. Without loss of generality, it suffices to assume that the dimension of U , k say, is at least 3 and x acts non-trivially on U . Suppose that x does not act as a transvection on V . If x does not act as a transvection on U , then (x, G) cannot be a minimal counterexample. So assume that x acts as a transvection on U . Then x must act non-trivially on W . So the dimension of W , $n - k$, is at least 2 and it suffices to assume that x acts as a transvection on W also, but since $n \geq 5$ there is a four-dimensional subspace U' on which x acts invariantly and is not a transvection and so $x_{U'}$ is contained in a subgroup of type $GL(4, 3)$. Now suppose that x is contained in a symplectic group $Sp(n, 3)$ and that x is not a symplectic transvection. If $n \geq 8$, then x fixes an orthogonal decomposition $U \perp W$. It suffices to assume that x acts non-trivially on both U and W ; otherwise (x, G) is not a minimal counterexample. Moreover, it suffices to assume that x acts as a symplectic transvection on U and on W ; otherwise $(x_U, Sp(U))$ or $(x_W, Sp(W))$ is contained in $\mathcal{A}$. So assume that for all $u \in U$ and all $w \in W$,

$$x_U : u \rightarrow u + \lambda \kappa_U(u, a)a, \quad a \in U, \kappa(a, a) = 0;$$

$$x_W : w \rightarrow w + \lambda' \kappa_W(w, b)b, \quad b \in W, \kappa(b, b) = 0.$$

Choose $u \in U$ such that $\kappa_U(u, a) \neq 0$ and $w \in W$ such that $\kappa_W(w, b) \neq 0$. Then x acts invariantly on the non-degenerate subspace $\langle u, w, a, b \rangle$ and is not a transvection on it, so (x, G) cannot be a minimal counterexample. Now suppose that x is contained in a unitary group $SU(n, 3)$ and that x is not a unitary transvection. Suppose that $n \geq 5$. Then there exists an x -invariant orthogonal decomposition $U \perp W$ and, as before, it suffices to assume that x acts non-trivially on both subspaces. Moreover, there exists H such that (x, H) is contained in $\mathcal{A}$ unless x acts as a unitary transvection on both U and W . So for all $u \in U$ and all $w \in W$,

$$\begin{aligned} x_U : u &\rightarrow u + \lambda \kappa_U(u, a)a, \quad a \in U, \kappa(a, a) = 0; \\ x_W : w &\rightarrow w + \lambda' \kappa_W(w, b)b, \quad b \in W, \kappa(b, b) = 0. \end{aligned}$$

Choose u and w , as in the symplectic case, so that $\langle u, w, a, b \rangle$ is a 4-dimensional, non-degenerate subspace on which x acts invariantly, but not as a transvection. Then (x, G) is not a minimal counterexample in this case either. Finally, suppose that x is contained in an orthogonal group $\Omega^\epsilon(n, 3)$ and that x is not a long root element. Suppose that $n \geq 9$. Then there exists an x -invariant orthogonal decomposition $U \perp W$. It suffices to assume that the action on U and W is not trivial as in the previous cases. Since $n \geq 9$, it suffices to assume that the dimension of U , k say, is at least 5. Then if (x, G) is a minimal counterexample, x must act as a long root element on U . Now either x acts as a long root element on W , or x does not act as a long root element on W and $n - k \leq 4$. In the latter case one can add dimensions from U to W so that W has dimension at least 5 and W is still x -invariant and non-degenerate. If this is done, then x_W is contained in an orthogonal group H such that $(x_W, H) \in \mathcal{A}$. In the former case, $n - k \geq 4$ (W has Witt defect 0 if $n - k = 4$) and for all $u \in U$ and all $w \in W$,

$$\begin{aligned} x_U : u &\rightarrow u + \lambda \kappa_U(u, a)b - \lambda \kappa_U(u, b)a, \\ x_W : w &\rightarrow w + \lambda' \kappa_W(w, c)d - \lambda' \kappa_W(w, d)c, \end{aligned}$$

where $a, b \in U$; $c, d \in W$; and $Q(a) = Q(b) = \kappa_U(a, b) = 0 = Q(c) = Q(d) = \kappa(c, d)$.

If $u_1, u_2 \in U$ are such that $Q(u_i) = 0 = \kappa_U(u_1, a) = \kappa_U(u_2, b)$, and $\kappa_U(u_1, b) \neq 0$, $\kappa_U(u_2, a) \neq 0$, then x acts invariantly on the non-degenerate 4-dimensional subspace $\langle u_1, u_2, a, b \rangle$. Similarly, take $w_1, w_2 \in W$ such that x acts invariantly on the non-degenerate 4-dimensional subspace $\langle w_1, w_2, c, d \rangle$. Then x acts invariantly on the non-degenerate 8-dimensional subspace $\langle u_1, u_2, a, b, w_1, w_2, c, d \rangle$, which has Witt defect 0, and x does not act as a long root element on it. So it is enough to check the classical groups of dimension at most 8 over $\mathbb{F}_3$ in MAGMA.

If x is a transvection in $SL(n, 3)$, then one can reduce to the case where $n = 3$. Similarly if x is a transvection in $SU(n, 3)$ or $Sp(n, 3)$, then one can reduce to the case where $x \in SU(3, 3)$ or $x \in Sp(4, 3)$. If x is a long root element in an orthogonal group, then one can reduce to the six-dimensional case, but $P\Omega^+(6, 3) \cong PSL(4, 3)$ and x maps to a transvection under this isomorphism. We can therefore further reduce to $SL(3, 3)$. By [GS03], there exist three conjugates of x that generate G_0 when $G_0 = PSL(3, 3)$ or $PSU(3, 3)$, and four conjugates of x that generate $G_0 = PSp(4, 3)$. $\square$

9. CASE U

It suffices to assume that $n \geq 3$ and $(n, q) \neq (3, 2)$. By Lemmas 7, 11 and 12, if (x, G) is a minimal counterexample, with $G_0 \cong PSU(n, q)$, then x is a semisimple element in $PGU(n, q)$, or x is unipotent in $PSU(3, q)$, and $q \geq 5$.

9.1. $x \in PGU(n, q)$, $p \nmid q$ and $p \nmid (n, q + 1)$. By Lemma 3, $x \in PGU(n, q)$ lifts to an element in $GU(n, q)$ of order p , with the same sized conjugacy class. Without loss of generality, it suffices to assume that $G = PGU(n, q)$ by Lemma 4. Consider the minimal polynomial of x , $m_x(t)$ say. Observe that $m_x(t)$ divides $t^p - 1$ and $t^p - 1/(t - 1)$ factors over $\mathbb{F}_{q^2}$ as

$$q_1(t) \cdots q_s(t),$$

where the $q_i(x)$'s are polynomials of degree k (where k is the smallest positive integer such that $p \mid q^{2k} - 1$). The same argument as for the case when $G_0 = PSL(n, q)$ shows that x will leave invariant and act non-trivially and irreducibly on a k -dimensional subspace U of V . Since x acts irreducibly on U , U is either non-degenerate or totally singular (for if U is not non-degenerate, then there exists $v \in U$ such that $\kappa_U(v, u) = 0$ for all $u \in U$; but x acts irreducibly on U so $\kappa_U = 0$). If $k \geq 2$, in both cases, consider the induced isometry x_U of U . If U is totally singular, then x_U is contained in a group of type $GL(k, q^2)$ and so (x, G) cannot be a minimal counterexample. If U is non-degenerate, then x_U is contained in a subgroup of type $GU(k, q)$. Observe that if U is non-degenerate, then k is odd. For if k were even, then, since $|GU(k, q)| = q^{k(k-1)/2} \prod_{i=1}^k (q^i - (-1)^i)$, p would divide $q^k - 1 = q^{2j} - 1$, contradicting the choice of k . Thus if $k \geq 2$, then unless x acts irreducibly or $q = 2$, $(x_U, GU(k, q))$ is contained in $\mathcal{A}$ and (x, G) cannot be a minimal counterexample.

If $q = 2$, then there are exceptions in Table 1. If $k > 3$, then x_U is contained in $GU(k, 2)$, and the assumption on k implies that $p \neq 3$, so $(x_U, GU(k, 2))$ is contained in $\mathcal{A}$. If $(k, q) = (3, 2)$, then $p = 7$. Since U is non-degenerate, x also acts invariantly on $U^\perp$. For $n \geq 7$, if this action is non-scalar, then $x_{U^\perp}$ is contained in $GU(n-3, 2)$ and $(x_{U^\perp}, GU(n-3, 2))$ is contained in $\mathcal{A}$. If the action is scalar, then take a non-singular vector $w \in U^\perp$, so that x acts invariantly on $U' := U \oplus \langle w \rangle$. Therefore $x_{U'}$ is contained in $GU(4, 2)$ and, since $p \neq 3$, it follows that (x, G) is not a minimal counterexample.

If $k = 1$, then x acts invariantly on a 1-dimensional non-degenerate or singular subspace U . Observe that $q \neq 3$ since this would imply that $p = 2$. First suppose that $q \neq 2$ so that $q \geq 4$. If U is non-degenerate, then consider the action of x on $U^\perp$. Either x acts non-trivially on $U^\perp$, in which case $x_{U^\perp}$ will be contained in $GU(n-1, q)$, or x has a scalar action on $U^\perp$, in which case there exists a 2-dimensional non-degenerate subspace U' such that $x_{U'}$ is contained in $GU(2, q)$. Since $q \geq 4$, (x, G) cannot be a minimal counterexample in any case. Now suppose that $q = 2$ and $k = 1$ so that $p = 3$. If x has order 3 in $GU(n, 2)$, then a Sylow 3-subgroup is contained in a subgroup of type $GU(1, q) \wr S_n$. So it suffices to assume that x will lie in a subgroup $GU(1, 2) \wr S_n$, and if $n \geq 5$, it suffices to assume that x is contained in $GU(1, 2) \perp \cdots \perp GU(1, 2)$ since otherwise x will be non-trivial in a subgroup of type S_n . Thus for $n \geq 5$, if x is not a reflection, then there

exists an $(n-1)$ -dimensional, non-degenerate, x -invariant subspace U' such that $x_{U'} \in GU(n-1, 2)$ with $(x_{U'}, GU(n-1, 2))$ contained in $\mathcal{A}$. A MAGMA calculation shows that the only exceptions to the theorem for $G := PGU(4, 2)$ are reflections of order 3. If x is a reflection of order 3 in $GU(n, 2)$, then it suffices to treat the case where x is contained in $GU(4, 2)$. A calculation in MAGMA shows that there exist $g_1, g_2, g_3 \in G$ such that $\langle x, x^{g_1}, x^{g_2}, x^{g_3} \rangle$ is not solvable.

If $k = 1$ and there is not a 1-dimensional non-degenerate x -invariant subspace, then U is totally singular and x is contained in a parabolic subgroup. Thus by Lemma 6, it suffices to assume that x acts non-centrally on each component of the Levi complement of some maximal parabolic subgroup. The parabolic subgroups of ${}^2A_m(q)$ have Levi complements of type ${}^2A_{m-2}(q)$, $A_k(q^2){}^2A_{m-2k-2}(q)$ and, if m is odd, ${}^2A_{(m-1)/2}(q^2)$. So (x, G) cannot be a minimal counterexample unless $m = 2$, $(m, q) = (3, 2)$, or $(m, q) = (4, 2)$. If $m = 2$, then x is a reducible semisimple element in $GU(3, q)$, so $q \geq 4$ and so x leaves invariant a 2-dimensional, non-degenerate subspace U' . In this case, $x_{U'}$ is contained in $GU(2, q)$ and so $(x_{U'}, GU(2, q))$ is contained in $\mathcal{A}$. When $(m, q) = (3, 2)$, $G_0 = PSU(4, 2)$. When $(m, q) = (4, 2)$, $G_0 \cong PSU(5, 2)$. These cases can be excluded using MAGMA.

The remaining case occurs when $k = n$ and x acts irreducibly in $GU(n, q)$, where $n \geq 3$ is odd (and $(n, q) \neq (3, 2)$). Now one can use [GPPS99] to find the possibilities for a maximal subgroup M containing $\langle x, x^q \rangle$ in $GU(n, q)$. Note that x is a $\text{ppd}(n, q^2, n)$ -element, and that $n \geq 3$ is odd. So since n is not a power of 2 there are no 2.5 examples. Lemma 9 implies that M must be a type 2.1 or 2.4(b) subgroup. By [KL90], the only possible such classical maximal subgroups are of type $GU(n, q_0)$ and $O_n(q)$ (q odd). The only subgroups of this type that contain an element of order $p \geq 5$ and are not almost simple modulo scalars are those of type $O_3(3)$ when $n = q = 3$. One can treat $GU(3, 3)$ separately in MAGMA. The only other examples are the field extension examples (type 2.4(b)). By [KL90] and since n is odd these are subgroups of type $GU(n/r, q^r)$, where r is an odd prime. Now unless $n = r$ these subgroups are almost simple modulo scalars and thus (x, M) is contained in $\mathcal{A}$. If $n = r$ and $x \in M$, where M is a subgroup of type $GU(1, q^n)$, then observe that x is contained in only one such maximal subgroup and Theorem 4 implies that (x, G) cannot be a minimal counterexample.

9.2. $x \in PGU(n, q)$ and $p \mid (q+1, n)$. Observe that Lemma 4 still applies, so assume that $G = PGU(n, q)$. This time some conjugacy classes of order p could only lift to non-trivial scalars in $GU(n, q)$. If x lifts to an element of order p in $GU(n, q)$, then apply the same argument as in the previous section. If not, then x^p lifts to a non-trivial scalar in $GU(n, q)$. So x will have order $p^m j$ say, where $p \nmid j$, but since $\langle x^j \rangle \leq \langle x \rangle$ and x^j will still have order p in $PGU(n, q)$, it suffices to assume that $j = 1$. So assume that the order of x in $GU(n, q)$ is p^m . The minimal polynomial of x , $m_x(t)$, divides $t^p - \zeta_{p^{m-1}}$, where $\zeta_{p^{m-1}}$ is a primitive p^{m-1} th root of unity in $\mathbb{F}_{q^2}$, and $p^{m-1} \mid (q+1)$. Since there are p th roots of unity, either $t^p - \zeta_{p^{m-1}}$ splits, or it is irreducible over $\mathbb{F}_{q^2}$. For if a is a root of the equation $t^p - \zeta_{p^{m-1}} = 0$ contained in some field extension, then this field extension contains all of the roots, $a\omega, a\omega^2, \dots, a\omega^{p-1}$. So $t^p - \zeta_{p^{m-1}}$ will factor into irreducible polynomials of degree equal to the degree of the smallest field extension containing a . However, p is prime, so the degree of these polynomials is either 1 or

p . If $t^p - \zeta_{p^{m-1}}$ splits, then $m_x(t)|(t - \zeta_{p^m})(t - \zeta_{p^m}\omega) \dots (t - \zeta_{p^m}\omega^{p-1})$, where ζ_{p^m} is a primitive p^m th root of unity in $\mathbb{F}_{q^2}$. However this would imply that p^m divides $q+1$. For $p^{m-1} \mid (q+1)$, and since $\zeta_{p^m} \in \mathbb{F}_{q^2}$, $p^m \mid (q-1)(q+1)$, but $p \nmid q-1$ since $p \geq 3$. This would be a contradiction, since $z = \zeta_{p^m} I_n$ would lie in $Z(GU(n, q))$, so $(z^{-1}x)^p = \zeta_{p^{m-1}}^{-1} \zeta_{p^{m-1}} I_n = I_n$, and x would lift to an element of order p . So, it suffices to assume that $m_x(t) = t^p - \zeta_{p^{m-1}}$ is irreducible over $\mathbb{F}_{q^2}$. It follows that x has rational canonical form $\text{diag}[A_1, \dots, A_{n/p}]$, where

$$A_i = \begin{pmatrix} & I_{p-1} \\ \zeta_{p^{m-1}} & \end{pmatrix}.$$

Thus x acts irreducibly on a subspace W of dimension p . Now $p^m \mid q^{2p} - 1$, and in fact, $p^m \mid q^p + 1$, since if $p \mid q^p - 1$, then $q^p \equiv 1 \pmod{p}$ but also $q^p \equiv q \pmod{p}$ by Fermat's Little Theorem. Therefore $q \equiv 1 \pmod{p}$, and $p \mid q+1$, which contradicts the assumption that $p \geq 3$. So p^m divides $q^p + 1$.

Assume that W is non-degenerate since if W were totally singular, then x_W would be contained in $GL(p, q^2)$ and $(x_W, GL(p, q^2))$ would be contained in $\mathcal{A}$. Thus, if x does not lift to an element of order p , then it suffices to assume that $n = p$ and that x acts irreducibly.

Since $n = p$, and $p \mid q+1$, one can show that a maximal subgroup M of $GU(p, q)$ of type $GU(1, q) \wr S_p$ always contains a Sylow p -subgroup of $GU(p, q)$. Thus, it suffices to assume that x is contained in M , the normalizer of a maximal split torus T . Moreover, x is non-trivial in $N_G(T)/T \cong S_p$, since it acts irreducibly. So if $p \geq 5$, then x cannot be a minimal counterexample. Now suppose that $p = 3$. Then x is an irreducible element in $GU(3, q)$. The character table of $GU(3, q)$ in [Enn62] with the same argument as when $G_0 = PSU(2, q)$ implies that there exists an element z in $GU(3, q)$ of order $q^2 - 1$ such that x is conjugate to xz . So, if $x^g = xz$, then $\langle x, x^g \rangle = \langle x, z \rangle$ contains $PSU(3, q)$, since it cannot be contained in any of the maximal subgroups described in [GLS98, Theorem 6.5.3].

9.3. $x \in PSU(3, q)$ and $p \mid q$, $q \geq 5$. If x is a unipotent element in $G_0 = PSU(3, q)$, then the maximal subgroups of G_0 are described in [GLS98, Theorem 6.5.3] and Lemma 5 can be applied. By Lemma 12, there are no minimal counterexamples when $q = 3$. So assume that $q \geq 5$. If x is a transvection, then it stabilizes a non-degenerate 2-dimensional subspace, and acts non-trivially on it, so (x, G) cannot be a minimal counterexample. Thus x is not a transvection and $|C_{PSU(3, q)}(x)| = q^2$. Since (x, G) is a minimal counterexample, the only possibilities for maximal subgroups X_i containing x are of type $GU(1, q) \wr S_3$ (for $p = 3$), $GU(1, q^3)$ (for $p = 3$), and parabolic subgroups. Note that $PSU(3, 2)$ and $PGU(3, 2)$ do not contain x since they are $\{2, 3\}$ -groups that are only relevant when $3 \nmid q$. There is only one conjugacy class of each of the given subgroups and $|x^{PSU(3, q)} \cap X_i|$ is at most $6(q+1)^2$, $3(q^2 - q + 1)$, and $q^3 - 1$ in each case respectively. So

$$\begin{aligned} |G|/|C_G(x)|^2 &= q^3(q^2 - 1)(q^3 + 1)/q^4 = (q^2 - 1)(q^3 + 1)/q \\ &\geq (q^3 - 1) + (q^2 - q + 1).3 + (q + 1)^2.6 \geq \sum_i |x^G \cap X_i| \end{aligned}$$

for $q \geq 5$, and thus (x, G) cannot be a minimal counterexample by Lemma 5. $\square$

10. CASE S

If $G_0 \cong PSp(n, q)$, then the only case left to prove occurs when x is a semisimple element contained in $\text{Inndiag}(PSp(n, q)) \cong PGSp(n, q)$. Since $|PGSp(n, q) : PSp(n, q)| = (2, q-1)$, x must be contained in $PSp(n, q)$, so suppose that $G = G_0$. Furthermore, by Lemma 3, x always lifts to an element in $Sp(n, q)$ of order p .

Let e be the smallest positive integer such that $p \mid q^e - 1$. Hence the minimal polynomial of x will be a product of irreducibles of degree e , and possibly $t-1$. Also, V will have an e -dimensional x -invariant subspace U , on which x acts irreducibly. U is either totally singular or non-degenerate. This depends on e :

- e odd and $e \neq 1$. If e is odd, then U is totally singular since there are no non-degenerate subspaces of V of odd order. So, if $e \geq 3$, then it suffices to assume that x acts non-trivially on U , and x_U is contained in a subgroup H of type $GL(e, q)$. Clearly (x, H) is contained in $\mathcal{A}$ in this case.
- $e = 1$. If e is 1, then U is a 1-dimensional totally singular subspace, so x is contained in a parabolic subgroup. By Lemma 6, it suffices to assume that x acts non-centrally on all the components of the Levi complement of a maximal parabolic subgroup. This maximal parabolic subgroup can be of type $C_{m-1}(q)$ ($m \geq 3$); $A_k(q)C_{m-k-1}(q)$ ($m \geq 4, 1 \leq k \leq m-3$); $A_{m-1}(q)$; or $A_1(q)A_1(q)$ ($m = 3$). Since $p \mid q-1$, q is at least 4; thus (x, G) cannot be a minimal counterexample.
- e even, $e < n$. If U is totally singular, then x_U is contained in a subgroup H of type $GL(e, q)$, and (x, H) is in $\mathcal{A}$ unless $(e, q) = (2, 2)$ (if $(e, q) = (2, 3)$, then $p = 2$). If $(e, q) = (2, 2)$, then it suffices to assume that $n \geq 8$, since $Sp(4, 2) \cong S_6$, and the case $Sp(6, 2)$ can be excluded using MAGMA. Since U is totally singular, x is contained in a parabolic subgroup, so we can use Lemma 6 as in the previous case. It follows that (x, G) cannot be a minimal counterexample in this case either. If U is non-degenerate, then x_U is contained in a subgroup H of type $Sp(e, q)$. (x, H) is contained in $\mathcal{A}$ for $e \geq 4$ and for $e = 2, q \geq 4$. If $e = 2$, and $q \leq 3$, then $q = 2$. But it suffices to assume that $n \geq 6$, since $Sp(4, 2) \cong S_6$, so $x_{U^\perp}$ is contained in a subgroup H of type $Sp(n-e, 2)$ and (x, H) is contained in $\mathcal{A}$.

If x acts irreducibly, then [GPPS99] describes the possible maximal subgroups of $Sp(n, q)$ that could contain x . It suffices to assume that n is at least 4, since $SL(2, q) \cong Sp(2, q)$. The only M 's of concern are those that contain $\text{ppd}(n, q, n)$ -elements. By Lemma 9, it suffices to assume that M is a subgroup of type 2.1, 2.4(b) or 2.5. If M were a subgroup of type 2.1, then so long as M is almost simple modulo scalars, (x, M) is contained in $\mathcal{A}$. By [KL90], the only possible such maximal subgroups M are type 2.1(b), where M contains $Sp(n, q_0)$; and type 2.1(d), where M contains $\Omega^\epsilon(n, q_0)$ for q_0 even. In these cases, M is almost simple and (x, M) is contained in $\mathcal{A}$ unless $(n, q) = (4, 2)$. However since $Sp(4, 2) \cong S_6$, this case can be excluded. If M is of type 2.5, then by [KL90], M would be of type $PO^-(2m, 2)$, where q is an odd prime, $n = 2^m$, and P is a 2-subgroup. However since x has odd order, $xO_2(M)$ would be non-trivial in the quotient $M/O_2(M)$. Moreover, $e \geq 4$ implies that $m \geq 2$ and thus $M/O_2(M)$ is almost simple of type $O^-(2m, 2)$. The only other possibility for M is to be of type 2.4(b). In this case,

by [KL90], M would be of type $Sp(n/b, q^b)$, where b is a prime and n/b is even; or of type $GU(n/2, q)$. However, since $n \geq 4$, these are all almost simple modulo scalars unless $(n, q) = (4, 2)$, $(4, 3)$, or $(6, 2)$. These exceptions are not a problem since $Sp(4, 2) \cong S_6$, $PSp(4, 3) \cong PSU(4, 2)$ ($p \neq 3$ since x is a $\text{ppd}(4, 3, 4)$ -element) and there are no elements of prime order in $Sp(6, 2)$ that act irreducibly.

11. CASE O

It suffices to assume that $n \geq 7$ since otherwise G_0 is isomorphic to one of the classical groups that have already been considered. If $x \in \text{Inndiag}(P\Omega_n^\epsilon(q))$ has odd prime order, then $x \in P\Omega_n^\epsilon(q)$. By Lemma 3, x lifts to an element of order p in $\Omega_n^\epsilon(q)$. Lemmas 7, 11, and 12 imply that if (x, G) is a minimal counterexample, then $x \in \text{Inndiag}(G_0)$ and x is semisimple.

Let e be minimal such that $p \mid q^e - 1$, so there exists an e -dimensional subspace U on which x acts invariantly and irreducibly. Consider the different values for e :

- e odd, $e \geq 3$. If e is odd, then $p \nmid |O(e, q)|$, so U must be totally singular. It follows that x_U is contained in a subgroup H of type $GL(e, q)$ and $(x_U, H) \in \mathcal{A}$.
- $e = 1$. If $e = 1$, then $q \geq 4$ since $p \mid q - 1$. If x acts invariantly on a non-degenerate 1-dimensional subspace U , then consider the action of x on $U^\perp$. If this action is non-scalar, then (x, G) is not a minimal counterexample since $x_{U^\perp}$ is contained in a subgroup H of type $O^\epsilon(n-1, q)$ and $(x_{U^\perp}, H)$ is contained in $\mathcal{A}$ since $n \geq 7$. If the action is scalar, then there exists a 3-dimensional subspace Y of $U^\perp$ such that $U' := U \oplus Y$ is non-degenerate and x -invariant. In this case, $x_{U'}$ will be contained in a subgroup H of type $O^\epsilon(4, q)$. In particular, $(x_{U'}, H)$ would be contained in $\mathcal{A}$. If x acts invariantly on a singular, 1-dimensional subspace, then x is contained in a parabolic subgroup. Thus, by Lemma 6, it suffices to assume that x acts non-centrally on each component of the Levi complement of some maximal parabolic subgroup. The possible types of maximal parabolic subgroups are: $A_{m-1}(q)$, or $B_{m-1}(q)$ if $G_0 = B_m(q)$; $D_{m-1}(q)$, $A_{m-1}(q)$, $A_{m-3}(q)A_1(q)A_1(q)$, or $A_k(q)D_{m-k-1}(q)$ if $G_0 = D_m(q)$; or ${}^2D_{m-1}(q)$, $A_{m-2}(q)$, $A_k(q)D_{m-k-1}(q)$, or $A_{m-3}(q)A_1(q^2)$ if $G_0 = {}^2D_m(q)$. Since if $G_0 = B_m(q)$, then $m \geq 3$ and in the other cases $m \geq 4$, it follows that (x, G) cannot be a minimal counterexample.
- $e = 2$. If $e = 2$, then $p \mid q + 1$. If U is totally singular, then x is contained in a parabolic subgroup and Lemma 6 is applied as above. If $G_0 = B_m(q)$, then $m \geq 3$ and $q \geq 5$ since $B_m(2^a) \cong C_m(2^a)$. The only complication is that if $G_0 = D_4(2)$, then all of the components of a parabolic subgroup of type $A_1(q)A_1(q)A_1(q)$ are solvable. One can verify in MAGMA that there are no counterexamples when $G_0 = D_4(2)$. Now suppose that x acts invariantly on a 2-dimensional non-degenerate subspace U . Then U will be anisotropic because of the order of x . If the action of x on $U^\perp$ is non-scalar, then $x_{U^\perp}$ will be contained in a subgroup H of type $O^{-\epsilon}(n-2, q)$ (since U has Witt defect 1, [KL90, 4.1.6]) and $(x_{U^\perp}, H)$ will be contained in $\mathcal{A}$. Suppose that x acts as a scalar on $U^\perp$. In this case, let W be a 4-dimensional non-degenerate subspace of $U^\perp$ (of Witt defect 0). Then x will act invariantly on the non-degenerate space $U' = U \oplus W$. So $x_{U'}$ will be

contained in a subgroup H of type $O^-(6, q)$, and $(x_{U'}, H)$ will be contained in $\mathcal{A}$.

- e even, $e \geq 4$. If e is even, then $p \mid q^{e/2} + 1$. Suppose that U is totally singular. Then x_U will be contained in a subgroup H of type $GL(e, q)$, and (x_U, H) will be contained in $\mathcal{A}$ since $e \geq 4$. So assume that U is non-degenerate. If $e \neq n$, then x_U will lie in a subgroup H of type $O^-(e, q)$, with (x_U, H) contained in $\mathcal{A}$. The only case left to consider is where x acts irreducibly on $O^-(e, q)$.

Since $e = n$ is even it suffices to assume that $n \geq 8$. One can use [KL90] and [GPPS99] to find the possible maximal overgroups of x . Lemma 9 implies that M must be a subgroup of type 2.1, 2.4(b) or 2.5. The only subgroups M of type 2.1 are of type $O^-(n, q_0)$, and if M were such a subgroup, then (x, M) would be contained in $\mathcal{A}$. There are no symplectic type normalizer maximal subgroups in $O^-(n, q)$, so there are no 2.5 type maximal subgroups. This leaves field extension examples of type 2.4. The possibilities are subgroups of type $GU(n/2, q)$, $O^-(n/2, q^2)$, and $O^-(n/r, q^r)$ for r a prime and $e/r \geq 4$. All of these are almost simple modulo scalars and (x, M) would be contained in $\mathcal{A}$. Thus, (x, G) cannot be a minimal counterexample.

12. $E_l(q)$

Now suppose that G_0 is an exceptional group of type $E_l(q)$, for $l = 6, 7$, or 8 . If (x, G) is a minimal counterexample, then by Lemmas 7 and 11 either $x \in G_0$ and $p = q = 3$, or $x \in \text{Inndiag}(G_0)$ and $p \nmid q$.

First suppose that $p = q = 3$. If x is a long root element, then $\langle x, x^g \rangle$ is either a 3-group or a fundamental $SL(2, 3)$ subgroup, by [GLS98, Proposition 3.2.9]. The unipotent conjugacy classes are described in [Miz77, Miz80]. Tables 4, 5, and 6 list the representatives for the unipotent classes of order 3 in $E_l(3)$ and describe a subsystem subgroup H containing each representative. The tables show that there are no minimal counterexamples when x is unipotent.

TABLE 4. Conjugacy classes in $E_6(3)$ of elements of order 3

Representative in $E_6(3)$	Roots generating subsystem	Subsystem type
$x_{100000}(1)^a$	$\{100000, 001000, 000100, 000010, 000001\}$	$A_5(q)$
$x_{100000}(1)x_{001000}(1)$	$\{100000, 001000, 000100, 000010, 000001\}$	$A_5(q)$
$x_{100000}(1)x_{000100}(1)$	$\{100000, 001000, 000100, 000010, 000001\}$	$A_5(q)$
$x_{100000}(1)x_{001000}(1)x_{000010}(1)$	$\{100000, 001000, 000100, 000010, 000001\}$	$A_5(q)$
$x_{100000}(1)x_{000100}(1)x_{000001}(1)$	$\{100000, 001000, 000100, 000010, 000001\}$	$A_5(q)$
$x_{100000}(1)x_{001000}(1)x_{000010}(1)x_{000001}(1)$	$\{100000, 001000, 000100, 000010, 000001\}$	$A_5(q)$
$x_{100000}(1)x_{001000}(1)x_{000100}(1)x_{000010}(1)$	$\{100000, 010000, 000100, 000010, 010000\}$	$D_5(q)$
$x_{100000}(1)x_{001000}(1)x_{000010}(1)x_{000001}(1)x_{010000}(1)$	$\{100000, 010000, 001000, 000010, 000001\}$	$A_2(q)A_2(q)A_1(q)$
$x_{100000}(1)x_{000100}(1)x_{000001}(1)x_{122321}(1)$	$\{100000, 000100, 000010, 000001, 122321\}$	$A_1(q)A_3(q)A_1(q)$

^aIn this case, x is a long root element in $A_5(3)$ and so we can find g_1, g_2 such that $\langle x, x^{g_1}, x^{g_2} \rangle$ is not solvable.

TABLE 5. Conjugacy classes in $E_7(3)$ of elements of order 3

Representative in $E_7(3)$	Roots generating subsystem	Subsystem type
$x_{34}(1)x_{36}(1)x_{37}(1)x_{38}(1)x_{40}(1)$	$\alpha_{34}, \alpha_{40}, \alpha_{36}, \alpha_{38}, \alpha_{37}$	$A_2(q)A_2(q)A_1(q)$
$x_{34}(1)x_{36}(1)x_{38}(1)x_{40}(1)$	$\alpha_{34}, \alpha_{40}, \alpha_{36}, \alpha_{38}$	$A_2(q)A_2(q)$
$x_{37}(1)x_{38}(1)x_{39}(1)x_{40}(1)x_{41}(1)$	$\alpha_{37}, \alpha_{38}, \alpha_{39}, \alpha_{40}, \alpha_{41}$	$A_1(q)^2A_2(q)A_1(q)$
$x_{42}(1)x_{43}(1)x_{44}(1)x_{45}(1)$	$\alpha_{42}, \alpha_{45}, \alpha_{43}, \alpha_{44}$	$A_2(q)A_1(q)A_1(q)$
$x_{44}(1)x_{46}(1)x_{49}(1)$	$\alpha_{44}, \alpha_{46}, \alpha_{49}$	$A_2(q)A_1(q)$
$x_{42}(1)x_{43}(1)x_{44}(1)x_{51}(\zeta)x_{49}(1)$	$\alpha_3, \alpha_5, \alpha_7, \alpha_{38}, \alpha_{49}$	$D_4(q)A_1(q)$
$x_{44}(1)x_{46}(1)$	α_{44}, α_{46}	$A_2(q)$
$x_{42}(1)x_{43}(1)x_{44}(1)x_{51}(\zeta)$	$\alpha_3, \alpha_5, \alpha_7, \alpha_{38}$	$D_4(q)$
$x_{47}(1)x_{48}(1)x_{49}(1)x_{53}(1)$	$\alpha_3, \alpha_5, \alpha_{44}, \alpha_{53}, \alpha_{49}$	$A_3(q)A_1(q)A_1(q)$
$x_{47}(\zeta)x_{48}(1)x_{49}(1)x_{53}(1)$	$\alpha_3, \alpha_5, \alpha_{44}, \alpha_{53}, \alpha_{49}$	$A_3(q)A_1(q)A_1(q)$
$x_{47}(1)x_{48}(1)x_{49}(1)$	$\alpha_3, \alpha_5, \alpha_{44}, \alpha_{49}$	$A_3(q)A_1(q)$
$x_{47}(\zeta)x_{48}(1)x_{49}(1)$	$\alpha_3, \alpha_5, \alpha_{44}, \alpha_{49}$	$A_3(q)A_1(q)$
$x_{53}(1)x_{54}(1)x_{55}(1)$	$\alpha_2, \alpha_7, \alpha_{50}, \alpha_{55}$	$A_3(q)A_1(q)$
$x_{58}(1)x_{59}(1)$	$\alpha_2, \alpha_5, \alpha_{57}$	$A_3(q)$
$x_{63}(1)^a$	α_1, α_{62}	$A_2(q)$

^aIn this case, x is a long root element in $A_2(3)$ and so we can find g_1, g_2 such that $\langle x, x^{g_1}, x^{g_2} \rangle$ is not solvable.

TABLE 6. Conjugacy classes in $E_8(3)$ of elements of order 3

Representative in $E_8(3)$	Roots generating subsystem	Subsystem type
$x_{53}(1)x_{54}(1)x_{55}(1)x_{117}(1)x_{118}(1)x_{119}(1)$	$\alpha_{53}, \alpha_{119}, \alpha_{54}, \alpha_{55}, \alpha_{117}, \alpha_{118}$	$A_2(q)^2 A_1(q)^2$
$x_{56}(1)x_{57}(1)x_{117}(1)x_{118}(1)x_{119}(1)$	$\alpha_{56}, \alpha_{57}, \alpha_{117}, \alpha_{118}, \alpha_{119}$	$A_2(q)A_2(q)A_1(q)$
$x_{56}(1)x_{57}(1)x_{117}(1)x_{118}(1)$	$\alpha_{56}, \alpha_{57}, \alpha_{117}, \alpha_{118}$	$A_2(q)A_2(q)$
$x_{53}(1)x_{54}(1)x_{55}(1)x_{117}(1)x_{124}(\zeta)x_{122}(1)$	$\alpha_{53}, \alpha_{122}, \alpha_{54}, \alpha_{55}, \alpha_{117}, \alpha_{124}$	$A_2(q)A_1(q)^4$
$x_{58}(1)x_{59}(1)x_{123}(1)x_{124}(1)x_{125}(1)$	$\alpha_{58}, \alpha_{59}, \alpha_{123}, \alpha_{124}, \alpha_{125}$	$A_1(q)A_2(q)A_1(q)A_1(q)$
$x_{60}(1)x_{126}(1)x_{127}(1)x_{128}(1)$	$\alpha_{60}, \alpha_{126}, \alpha_{127}, \alpha_{128}$	$A_2(q)A_1(q)A_1(q)$
$x_{63}(1)x_{127}(1)x_{130}(1)$	$\alpha_{63}, \alpha_{127}, \alpha_{130}$	$A_1(q)A_2(q)$
$x_{63}(1)x_{126}(1)x_{127}(1)x_{128}(1)x_{133}(\zeta)$	$\alpha_2, \alpha_5, \alpha_7, \alpha_{124}, \alpha_{63}$	$D_4(q)A_1(q)$
$x_{63}(1)x_{135}(1)x_{136}(1)x_{137}(1)$	$\alpha_1, \alpha_{101}, \alpha_{62}, \alpha_{136}, \alpha_{137}$	$A_3(q)A_1(q)A_1(q)$
$x_{127}(1)x_{130}(1)$	$\alpha_{124}, \alpha_2, \alpha_5, \alpha_7$	$D_4(q)$
$x_{126}(1)x_{127}(1)x_{128}(1)x_{133}(\zeta)$	$\alpha_2, \alpha_5, \alpha_7, \alpha_{124}$	$D_4(q)$
$x_{141}(1)x_{142}(1)x_{143}(1)$	$\alpha_1, \alpha_6, \alpha_{135}, \alpha_{143}$	$A_3(q)A_1(q)$
$x_{150}(1)x_{151}(1)$	$\alpha_3, \alpha_2, \alpha_{148}$	$A_3(q)$
$x_{157}(1)^a$	α_8, α_{156}	$A_2(q)$

^aIn this case, x is a long root element in $A_2(3)$ and so there exist g_1, g_2 such that $\langle x, x^{g_1}, x^{g_2} \rangle$ is not solvable.

TABLE 7. Bounds on $|x^G \cap X_i|$ for subgroups X_i of $E_6(q)$, $p \geq 5$.
 $d = (2, q - 1)$, $e = (3, q - 1)$, $f = (3, q + 1)$, $g = (3, q^2 - 1)$,
 $h = (4, q - 1)$

X_i	Bound on $ x^G \cap X_i $	Cruder bound
$d.(L(2, q) \times L(6, q)).de$	0	0
$e.L(3, q)^3.e^2.S_3$	0	q^9
$f.(L(3, q^2) \times U(3, q)).g.2$	0	0
$L(3, q^3).(e \times 3)$	0	0
$d^2.(P\Omega^+(8, q) \times (q - 1/d)^2).d^2.S_3$	q^2	
$({}^3D_4(q) \times (q^2 + q + 1)).3$	$q^2 + q + 1$	q^3
$h.(P\Omega^+(10, q) \times (q - 1/h)).h$	$h(q - 1)$	q^3
$(q - 1)^6.W(E_6), q \geq 5$	$(q - 1)^6.51840$	q^{13}
$(q^2 + q + 1)^3.3^{1+2}SL(2, 3)$	$(q^2 + q + 1)^3$	q^9
$3^{3+3}.SL(3, 3)$	0	

The only other possibility is that $p \nmid q$ and $x \in \text{Inndiag}(G_0)$. By Lemma 4, it suffices to assume that $G = \text{Inndiag}(G_0)$. If x is semisimple, then consider the case where x is contained in a parabolic subgroup. By Lemma 6, there is a conjugate of x that is contained in a maximal parabolic subgroup that does not centralize any component of the Levi complement. For $l = 6$, P will be of type $D_5(q)$, $A_1(q)A_4(q)$, or $A_5(q)$, and so (x, G) cannot be a minimal counterexample. Similarly, for $l = 7$ and 8 one can reduce to a case where x is acting non-centrally on a component of a Levi complement. So it suffices to assume that x is not contained in any parabolic subgroups. In this case, x is semisimple, and $C_G(x)$ is a reductive group containing no unipotent elements. Thus, $C_G(x)$ is a torus, and by [Sei83], for example, it follows that $|C_G(x)| \leq (q + 1)^l$. The conjugacy classes of semisimple elements of order 3 are described in [GLS98, Table 4.7.3A]. So if x is not contained in a parabolic subgroup, then it suffices to assume that $p \geq 5$, since $|C_G(x)| > (q + 1)^l$ for any $x \in E_l(q)$ of order 3. This observation is useful since it implies that $x \in O_\infty(M)$ for all maximal subgroups M containing x ; otherwise (x, G) could not be a minimal counterexample. If $l = 6$, then

$$|G|/|C_G(x)|^2 \geq \frac{q^{36}(q^{12} - 1)(q^9 - 1)(q^8 - 1)(q^6 - 1)(q^5 - 1)(q^2 - 1)}{3(q + 1)^{12}},$$

which is at least q^{55} , for $q \geq 2$. The maximal subgroups of $E_6(q)$ are described in [LS03] and [LSS92]. The possible maximal subgroups X_i containing x are listed in Table 7 together with a crude bound on $|x^G \cap X_i|$. Clearly the hypotheses of Lemma 5 are satisfied and there is no minimal counterexample when $l = 6$.

Now suppose that $l = 7$. Then

$$\frac{|G|}{|C_G(x)|^2} \geq \frac{q^{63}(q^{18} - 1)(q^{14} - 1)(q^{12} - 1)(q^{10} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)}{2(q + 1)^{14}},$$

which is at least q^{111} for $q \geq 2$. Table 8 implies that the hypotheses of Lemma 5 are satisfied, and there is no minimal counterexample when $l = 7$. If $l = 8$, then

$$\frac{|G|}{|C_G(x)|^2} \geq \frac{q^{120}(q^{30} - 1)(q^{24} - 1)(q^{20} - 1)(q^{18} - 1)(q^{14} - 1)(q^{12} - 1)(q^8 - 1)(q^2 - 1)}{2(q + 1)^{16}},$$

which is at least q^{239} for $q \geq 2$. Table 9 shows that the hypotheses of Lemma 5 are

TABLE 8. Bounds on $|x^G \cap X_i|$ for subgroups X_i of $E_7(q)$, $p \geq 5$.
 $d = (2, q-1)$, $e = (3, q-\epsilon)$, $f = (4, q-\epsilon)/d$, $g = (8, q-\epsilon)/d$

$X_i \leq E_7(q)$	Bound on $ x^G \cap X_i $
$d.(L(2, q) \times P\Omega^+(12, q)).d$	0
$f.L^\epsilon(8, q).g.(2 \times (2/f), \epsilon = +1)$	0
$f.L^\epsilon(8, q).g.(2 \times (2/f), \epsilon = -1)$	0
$e.L^\epsilon(3, q) \times L^\epsilon(6, q).de.2, \epsilon = +1$	0
$e.L^\epsilon(3, q) \times L^\epsilon(6, q).de.2, \epsilon = -1$	0
$d^2.(L(2, q)^3 \times P\Omega^+(8, q)).d^3.S_3$	0
$(L(2, q^3) \times {}^3D_4(q)).3d$	0
$d^3.(L(2, q)^7.d^4.L(3, 2))$	0
$L(2, q^7).7d$	0
$e.(E_6(q) \times (q-1)/e).e.2, \epsilon = 1$	q
$e.({}^2E_6(q) \times (q+1)/e).e.2, \epsilon = -1$	q^2
$(q-1)^7.W(E_7)$	q^{30}
$(q+1)^7.W(E_7)$	q^{37}
$(2^2 \times P\Omega^+(8, q).2^2).S_3$	0
${}^3D_4(q).3$	0

satisfied, and (x, G) cannot be a minimal counterexample.

13. ${}^2E_6(q)$

If x is unipotent, then Lemma 11 implies that $p = 3$. For $q = 3$, the unipotent class representatives were obtained from Frank Lübeck, using CHEVIE ([GHL96]). From the class representatives, one can deduce that (x, G) cannot be a minimal counterexample. If x is semisimple, and contained in a maximal parabolic subgroup, then, by Lemma 6, it suffices to assume that x acts non-centrally on all of the components of the Levi complement. If this parabolic subgroup is an end-node parabolic subgroup, then the Levi complement is of type ${}^2D_4(q)$ or ${}^2A_5(q)$. If P is not an end-node parabolic subgroup, then it can be either of type $A_1(q^2)A_2(q)$ or $A_1(q)A_2(q^2)$. Thus, (x, G) cannot be a minimal counterexample if x is contained in a parabolic subgroup.

So suppose that x is semisimple and does not lie in any parabolic subgroups. Then $C_G(x)$ is a torus, and as in the previous section, note that if x has order 3, then $|C_G(x)| > (q+1)^6$ (by [GLS98, Table 4.7.3A]). Thus, by [Sei83], it suffices to assume that $p \geq 5$. Moreover,

$$|G|/|C_G(x)|^2 \geq \frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)}{3(q+1)^{12}},$$

which is at least q^{55} for $q \geq 2$. The possible maximal subgroups containing x are given in Table 10. Again, the hypothesis of Lemma 5 holds and (x, G) cannot be a minimal counterexample.

14. $F_4(q)$

Observe that $\text{Inndiag}(G_0) = G_0$. If x is unipotent, then $q = 3$, and [Law95] and [Sho74] contain representatives for the classes of elements of order 3. They are listed in Table 11 together with subsystem overgroups of x that show that (x, G)

TABLE 9. Bounds on $|x^G \cap X_i|$ for X_i a subgroup of $E_8(q)$, $p \geq 5$.
 $d = (2, q - 1)$, $e = (3, q - \epsilon)$, $f = (9, q - \epsilon)/e$, $g = (5, q - \epsilon)$

$X_i \leq E_8(q)$	Bound on $ x^G \cap X_i $
$d.P\Omega^+(16, q).d$	0
$d.(L(2, q) \times E_7(q)).d$	0
$f.(L^\epsilon(9, q)).e.2, \epsilon = +1$	0
$f.(L^\epsilon(9, q)).e.2, \epsilon = -1$	0
$e.(L^\epsilon(3, q) \times E_6^\epsilon(q)).e.2, \epsilon = +1$	0
$e.(L^\epsilon(3, q) \times E_6^\epsilon(q)).e.2, \epsilon = -1$	0
$g.(L^\epsilon(5, q))^2.g.4, \epsilon = +1$	5
$g.(L^\epsilon(5, q))^2.g.4, \epsilon = -1$	5
$SU(5, q^2).4$	0
$PGU(5, q^2).4$	0
$d^2.(P\Omega^+(8, q))^2.d^2.(S_3 \times 2)$	0
$d^2.(P\Omega^+(8, q^2)).(S_3 \times 2)$	0
$({}^3D_4(q))^2.6$	0
$({}^3D_4(q^2)).6$	0
$e^2.L^\epsilon(3, q)^4.e^2.GL(2, 3), \epsilon = +1$	0
$e^2.L^\epsilon(3, q)^4.e^2.GL(2, 3), \epsilon = -1$	0
$U(3, q^2)^2.8$	0
$U(3, q^4).8$	0
$d^4.L(2, q)^8.d^4.AGL(3, 2), q > 2$	0
$(q - 1)^8.W(E_8)$	q^{46}
$(q + 1)^8.W(E_8)$	q^{46}
$(q^4 + q^3 + q^2 + q + 1)^2.(5 \times SL(2, 5))$	$5q^{10}$
$(q^2 + q + 1)^4.2.(3 \times U(4, 2))$	q^{15}
$(q^2 + 1)^4.(4 \circ 2^{1+4}).A_6.2$	q^{12}
$q^8 + q^7 - q^5 - q^4 - q^3 + q + 1.\mathbb{Z}_{30}$	q^{15}
$(q^4 - q^2 + 1)^2.(\mathbb{Z}_{12} \circ GL(2, 3))$	q^{10}
$(q^8 - q^7 + q^5 - q^4 + q^3 - q + 1).\mathbb{Z}_{30}$	q^{15}
$(q^4 - q^3 + q^2 - q + 1)^2.(5 \times SL(2, 5))$	$5q^{10}$
$(q^2 - q + 1)^4.2.(3 \times U(4, 2))$	q^{12}
$2^{5+10}.SL(5, 2)$ (exotic)	0
$5^3.SL(3, 5)$ (exotic)	q^7

cannot be a minimal counterexample. If x is semisimple and contained in a parabolic subgroup, then Lemma 6 implies that x acts non-trivially on all of the components of the Levi complement of some parabolic subgroup P . If P is an end-node parabolic subgroup, then the Levi complement is of type $B_3(q)$ or $C_3(q)$. If P is not an end-node parabolic subgroup, then P is of type $A_1(q)A_2(q)$. It suffices to assume that x does not centralize the $A_1(q)$ or $A_2(q)$ components, so (x, G) cannot be a minimal counterexample. Now suppose that x does not lie in any parabolic subgroups. Then $|C_G(x)| \leq (q+1)^4$ by [Sei83]. However, by [GLS98, Table 4.7.3A], this condition implies that $p \neq 3$. So suppose that $p \geq 5$ and note that

$$|G|/|C_G(x)|^2 \geq q^{24}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)/(q + 1)^8,$$

TABLE 10. Bounds on $|x^G \cap X_i|$ for subgroups X_i of ${}^2E_6(q)$, $p \geq 5$.
 $d = (2, q - 1)$, $e = (3, q + 1)$, $f = (3, q - 1)$, $g = (3, q^2 - 1)$,
 $h = (4, q + 1)$

X_i	Bound on $ x^G \cap X_i $	Cruder Bound
$d.(L(2, q) \times U(6, q).de$	0	0
$e.(U(3, q)^3.e^2.S_3$	0	
$f.L(3, q^2) \times L(3, q).g.2$	0	
$U(3, q^3).(e \times 3)$	0	
$d^2.(P\Omega^+(8, q) \times (q + 1/d)^2).d^2.S_3$	$(q + 1)^2$	q^2
$({}^3D_4(q) \times (q^2 - q + 1)).3$	$(q^2 - q + 1)$	
$h.(P\Omega^-(10, q) \times (q + 1/h)).h$	$(q + 1)$	
$(q + 1)^6.W(E_6)$, $q \geq 5$	$(q + 1)^6.51840$	
$(q^2 - q + 1)^3.3^{1+2}SL(2, 3)$	$(q^2 - q + 1)^3$	q^{14}
$3^{3+3}.SL(3, 3)$	0	q^6

TABLE 11. Conjugacy class representatives in $F_4(3)$

Representative in $F_4(3)$	Roots generating subsystem	Subsystem type
$x_1 = x_{1+2}(1)^a$	1, 1 + 3	$A_2(3)$
$x_2 = x_{1-2}(1)x_{1+2}(-1)$	1 - 2, 2 - 3, 3 - 4, 4	$B_4(3)$
$x_3 = x_{1-2}(1)x_{1+2}(-\eta)$	1 - 2, 2 - 3, 3 - 4, 4	$B_4(3)$
$x_4 = x_2(1)x_{3+4}(1)$	2 - 3, 3 - 4, 4	$B_3(3)$
$x_5 = x_{2-3}(1)x_4(1)x_{2+3}(1)$	2 - 3, 3 - 4, 4	$B_3(3)$
$x_6 = x_{2-3}(1)x_4(1)x_{2+3}(\eta)$	2 - 3, 3 - 4, 4	$B_3(3)$
$x_7 = x_2(1)x_{1-2+3+4}(1)$	2, 1 - 2 + 3 + 4	$A_2(3)$
$x_8 = x_{2-3}(1)x_4(1)x_{1-2}(1)$	2 - 3, 1 - 2, 4	$A_2(3)A_1(3)$
$x_{11} = x_{2+3}(1)x_{1+2-3-4}(1)x_{1-2+3+4}(1)$	2 + 3, 1 + 2 - 3 - 4, 1 - 2 + 3 + 4	$A_1(3)A_2(3)$

^aIn this case, x is a long root element in $A_2(3)$ and so there exist g_1, g_2 such that $\langle x, x^{g_1}, x^{g_2} \rangle$ is not solvable.

which is at least q^{38} for $q \geq 2$. It is clear from Table 12 that (x, G) cannot be a minimal counterexample in this case either.

15. ${}^2F_4(2^a)'$, WHERE a IS ODD

Suppose that $a > 1$. Since $p \neq 2$, x is semisimple. If x is contained in a parabolic subgroup, then Lemma 6 can be applied. If the resulting subgroup is an end-node parabolic subgroup, then it will be of type ${}^2B_2(2^a)$, in which case (x, G) cannot be a minimal counterexample. If P is not an end-node parabolic subgroup, then the Levi complement will be of type $A_1(2^{2a})$, so (x, G) cannot be a minimal counterexample in this case either. So suppose that x is not contained in any parabolic subgroups. Then $p \geq 5$, by the same argument as for $F_4(q)$, and

$$|G|/|C_G(x)|^2 \geq q^{12}(q^6 + 1)(q^4 - 1)(q^3 + 1)(q - 1)/2(q + 1)^8.$$

TABLE 12. Bounds on $|x^G \cap X_i|$ for subgroups X_i of $F_4(q)$, $p \geq 5$.
 $d = (2, q - 1)$, $e = (3, q - \epsilon)$

Type of X_i in G with $G_0 = F_4(q)$	Bound on $ x^G \cap X_i $
$2.(L(2, q) \times PSp(6, q)).2$, q odd	0
$d.\Omega(9, q)$, $(2, q_0)$ classes	0
$d^2.P\Omega^+(8, q).S_3$, $(2, q_0)$ classes	0
${}^3D_4(q).3$ $(2, p)$ classes	0
$e.(L^\epsilon(3, q) \times L^\epsilon(3, q)).e.2$, $\epsilon = +1$	e^2
$e.(L^\epsilon(3, q) \times L^\epsilon(3, q)).e.2$, $\epsilon = -1$	e^2
$(Sp(4, q) \times Sp(4, q)).2$	0
$Sp(4, q^2).2$	0
$(q - 1)^4.W(F_4)$, $q = 2^a$, $a > 2$	q^8
$(q + 1)^4.W(F_4)$, $q = 2^a$, $a > 1$	q^{11}
$(q^2 + q + 1)^2.(3 \times SL(2, 3))$, $q = 2^a$	q^6
$(q^2 - q + 1)^2.(3 \times SL(2, 3))$, $q = 2^a$, $a > 1$	q^6
$(q^2 + 1)^2.(\mathbb{Z}_{30} \circ GL(2, 3))$, $q = 2^a$, $a > 1$	q^9
$(q^4 - q^2 + 1).\mathbb{Z}_{30}$, $q = 2^a$, $a > 1$	q^7
$3^3.SL(3, 3)$, $q_0 \geq 5$	0

TABLE 13. Bounds on $|x^G \cap X_i|$ for subgroups X_i of ${}^2F_4(q)$, $p \geq 5$.
 $d = (2, q - 1)$, $e = (3, q - \epsilon)$

Type of X_i in G with $G_0 = {}^2F_4(q)$	Bound on $ x^G \cap X_i $
$SU(3, q).2$	0
$PGU(3, q).2$	0
$({}^2B_2(q) \times {}^2B_2(q)).2$	0
$Sp(4, q).2$	0
$B_2(q) : 2$	0
${}^2F_4(q_0)$	0
$(q + 1)^2.GL(2, 3)$	q^4
$(q + \sqrt{2q} + 1)^2.(\mathbb{Z}_4 \circ GL(2, 3))$	q^4
$(q - \sqrt{2q} + 1)^2.(\mathbb{Z}_4 \circ GL(2, 3))$, $q > 8$	q^2
$(q^2 + \sqrt{2q^3} + q + \sqrt{2q} + 1).\mathbb{Z}_{12}$	q^5
$(q^2 - \sqrt{2q^3} + q - \sqrt{2q} + 1).\mathbb{Z}_{12}$	q^2

This is at least q^{15} for $q \geq 8$. The maximal subgroups are given in [Mal91] and include the calculations in Table 13.

Thus (x, G) cannot be a minimal counterexample for $q \geq 8$. If $a = 1$, then $q = 2$ and the possibilities for the order of x are 3, 5, and 13. There are unique classes of cyclic subgroups of orders 3, 5, and 13, by [CCN85]; thus a conjugate of x is contained in a subgroup isomorphic to $PSL(2, 25)$. So (x, G) cannot be a minimal counterexample in this case either.

The only outer automorphisms are field automorphisms. If x is a field automorphism, then x normalizes an end-node parabolic subgroup and acts non-trivially on the Levi complement. Therefore, (x, G) cannot be a minimal counterexample.

16. $G_2(q)$

Observe that, since $G_2(2)' \cong PSU(3, 3)$, it suffices to assume that $q \neq 2$. First consider the case where q is a power of 2; so in particular, x is semisimple. The algebraic group G_2 fixes a non-degenerate quadratic form by [SS97, 4.1] and [LSS96] for example. It follows that any element of G_0 is conjugate to an element of either $SL_3(q) : 2$ or $SU(3, q) : 2$. Either x is non-central in one of these groups, in which case (x, G) is not a minimal counterexample, or x is central in $SL^\epsilon(3, q)$, and therefore is contained in a parabolic subgroup P . In the latter case, applying Lemma 6 implies that it suffices to assume that x acts non-centrally on the Levi complement, which is of type $A_1(q)$. So (x, G) is not a minimal counterexample in this case either.

Now suppose that q is odd. If x is semisimple, then, since it has odd order, it must be contained in $SL^\epsilon(3, q)$ for either $\epsilon = +$ or $\epsilon = -$. Thus if x is not a central element in this subgroup, then (x, G) is not a minimal counterexample. If x is central in the $SL^\epsilon(3, q)$, then x is contained in a parabolic subgroup P . So by Lemma 6, it suffices to assume that x acts non-centrally on the Levi complement, which is of type $A_1(q)$. Since $p \mid q - \epsilon$ and q is odd, it follows that $q \geq 5$ and (x, G) cannot be a minimal counterexample in this case either. Similarly, if x is unipotent and $q \neq 3$, then Lemma 6 implies that x acts non-trivially on an $A_1(q)$ Levi component of a parabolic subgroup.

Suppose that $q = 3 = p$ and that x is not a root element. It is easily verified using MAGMA that there are two conjugacy classes of elements of order 3 (long root elements and short root elements) that belong in Table 1. Moreover, in these cases, there exist $g_1, g_2 \in G_2(3)$ such that $\langle x, x^{g_1}, x^{g_2} \rangle$ is not solvable.

17. ${}^3D_4(q)$

One can use MAGMA for the cases $q = 2$ and $q = 3$, so assume from now on that $q \geq 4$. If x is unipotent, then, by Lemma 6, it suffices to assume that x acts non-centrally on a Levi component of a parabolic subgroup of type $A_1(q)$, or $A_1(q^3)$. So, since $q \geq 4$, (x, G) cannot be a minimal counterexample. Similarly, if x is semisimple and is contained in a parabolic subgroup, then Lemma 6 applies, as in the unipotent case. So it suffices to assume that x is not contained in any parabolic subgroups. It follows that $C_G(x)$ is a torus and $|C_G(x)| \leq (q + 1)^4$ by [Sei83]. Thus,

$$\frac{|G|}{|C_G(x)|^2} \geq \frac{q^{12}(q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1)}{(q + 1)^8},$$

which is at least q^{18} for $q \geq 4$. As usual, observe that $p \geq 5$ by [GLS98]. The possible maximal subgroups containing x can be deduced from [Kle88b] and are listed in Table 14, and Lemma 5 shows that (x, G) cannot be a minimal counterexample.

18. ${}^2B_2(2^a)$, $a \neq 1$ ODD

If $a = 1$, then ${}^2B_2(2^a)$ is solvable, so it suffices to assume that $a \neq 1$. The maximal subgroups are described in [Suz62] and are listed in Table 15. Note that $|G| = q^2(q^2 + 1)(q - 1)$, where $q = 2^a$. Also, observe that $p \nmid q$ since p is odd and it suffices to assume that the only subfield subgroup that can contain x is ${}^2B_2(2)$, since otherwise (x, G) would not be a minimal counterexample. By [Suz62, Theorem 4], for example, any element of odd order in ${}^2B_2(q)$ has its centralizer

TABLE 14. Bounds on $|x^G \cap X_i|$ for subgroups X_i of ${}^3D_4(q)$, $q \geq 4$, and $p \geq 5$. $d = (2, q-1)$, $e = (3, q-\epsilon)$, $f = (3, q^2 + \epsilon q + 1)$

Type of X_i in G with $G_0 = {}^3D_4(q)$	Bound on $ x^G \cap X_i $
$G_2(q)$	0
$PGL^\epsilon(3, q)$, $q \equiv \epsilon \pmod{3}$	0
${}^3D_4(q_1)$, $q_1^\alpha = q$, $\alpha \neq 3$ prime	0
$L(2, q^3) \times L(2, q)$, $q_0 = 2$	0
$(SL(2, q^3) \circ SL(2, q)).2$, $q_0 \neq 2$	0
$((q^2 + q + 1) \circ SL(3, q)).(3, q^2 + q + 1).2$	q^3
$((q^2 - q + 1) \circ SU(3, q)).(3, q^2 - q + 1).2$	q^2
$(q^2 + q + 1)^2.SL(2, 3)$	$(q+1)^4$
$(q^2 - q + 1)^2.SL(2, 3)$	q^4
$(q^4 - q^2 + 1).4$	q^4

TABLE 15. Maximal subgroups of ${}^2B_2(2^a)$

Subgroup	Bound on $ x^G \cap M $	Comments
H	$q^2(q-1)$	Borel subgroup
$D_{2(q-1)}$	$2(q-1)$	maximal rank
$N(A_1)$	$4(q + \sqrt{2q} + 1)$	maximal rank
$N(A_2)$	$4(q - \sqrt{2q} + 1)$	maximal rank
${}^2B_2(2^{a/b}), b \mid a$	$q^{2/b}(q^{2/b} + 1)(q^{1/b} - 1)$	One class [Suz62, Theorem 10]

contained in one of the cyclic groups of order $q-1$, $q + \sqrt{2q} + 1$ or $q - \sqrt{2q} + 1$. So there are three mutually exclusive possibilities for p : $p \mid q-1$, $p \mid q + \sqrt{2q} + 1$, and $p \mid q - \sqrt{2q} + 1$. If $p \mid q-1$, then

$$|G|/|C_G(x)|^2 \geq q^2(q^2 + 1)(q-1)/(q-1)^2$$

and

$$\sum_i |x^G \cap X_i| \leq q^2(q-1) + 2(q-1) + |{}^2B_2(2)|.$$

An elementary calculation shows that since $q \geq 8$,

$$|G|/|C_G(x)|^2 > \sum_i |x^G \cap X_i|$$

and Lemma 5 applies. Similarly if $p \mid q \pm \sqrt{2q} + 1$, then

$$\sum_i |x^G \cap X_i| \leq 4(q^2 \pm \sqrt{2q} + 1) + |{}^2B_2(2)|$$

and the hypotheses of Lemma 5 are satisfied. Thus, (x, G) cannot be a minimal counterexample.

If x is an outer automorphism, then it must be a field automorphism and the same counting argument as for $PSL(2, q)$ applies. Observe that it suffices to assume that there are no subfield subgroups among the H_i 's except ${}^2B_2(q_0) = C_{G_0}(x)$. If there were, then x would be contained in $\text{Aut}({}^2B_2(q^{1/r}))$ for some prime $r \neq p$ and (x, G) would not be a minimal counterexample. So

$$|G_0|/|C_{G_0}(x)|^2 = q^2(q^2 + 1)(q-1)/q_0^4(q_0^2 + 1)^2(q_0 - 1)^2$$

TABLE 16. Maximal subgroups of ${}^2G_2(3^a)$

Subgroup	Comments
$P = [q^3] \cdot (q-1)$	Borel subgroup, only one class
$2 \times L(2, q), q \geq 27$	maximal rank
$(2^2 \times D_{(q+1)/2}) : 3, q \geq 27$	maximal rank
$\mathbb{Z}_{q+\sqrt{3q}+1} : \mathbb{Z}_6$	maximal rank
$\mathbb{Z}_{q-\sqrt{3q}+1} : \mathbb{Z}_6, q \geq 27$	maximal rank
${}^2G_2(q_0), q = q_0^\alpha, \alpha \text{ prime}$	

and

$$1 + \sum_{i=1}^m \frac{|H_i|}{|C_{H_i}(x)|^2} \leq 1 + \frac{q^2(q-1)}{q_0^4(q_0-1)^2} + \frac{2(q-1)}{(q_0-1)^2} \\ + \frac{4(q+\sqrt{2q}+1)}{(q_0+\sqrt{2q_0}+1)^2} + \frac{4(q-\sqrt{2q}+1)}{(q_0-\sqrt{2q_0}+1)^2}.$$

A computation shows that the required inequality holds for all $q \geq 2^3$ and all $p \geq 3$.

19. ${}^2G_2(3^a)$, $a \neq 1$ ODD

Observe that if $a = 1$, then ${}^2G'_2(3) \cong L(2, 8)$, so suppose that $a \neq 1$. Also, $|G| = q^3(q^3+1)(q-1)$ and the maximal subgroups are given in [Kle88a], which are listed in Table 16. If $p \nmid q = 3^a$, then there are three mutually exclusive possibilities: $p \mid (q^2-1)$, $p \mid q - \sqrt{3q} + 1$, and $p \mid q + \sqrt{3q} + 1$. First suppose that $p \mid q^2-1$. Then a Sylow p -subgroup is contained inside a maximal subgroup $2 \times PSL(2, q)$, so some conjugate of x is contained in $PSL(2, q)$. Thus, (x, G) cannot be a minimal counterexample.

If $p \mid q^2 - q + 1$, then a Sylow p -subgroup is contained in one of the abelian Hall subgroups of order $q \pm \sqrt{3q} + 1$, so it suffices to assume that x is contained in one of these Hall subgroups and that $|C_G(x)| = q \pm \sqrt{3q} + 1$ (see part (4) of the main theorem in [War66]). Then an easy count shows that the hypotheses of Lemma 5 are satisfied. If $p \mid q$, then [War66] shows that there are three conjugacy classes of elements of order $p = 3$. One class contains elements in the center of a Sylow 3-subgroup and these elements have centralizers of order q^3 . The other two conjugacy classes have centralizers of order $2q^2$. Elements in these classes centralize an involution w , so they are contained in $C_G(w) \cong L(2, q) \times 2$ and so (x, G) cannot be a minimal counterexample in this case. Now [Law95] gives a representative $x_{2a+b}(1)x_{3a+2b}(1)$ for the conjugacy class of elements t with $|C_G(t)| = q^3$. This is contained in ${}^2G_2(3) \cong L(2, 8) : 3$, so (x, G) cannot be a minimal counterexample in this case either. If x is an outer automorphism, then it must be a field automorphism. The same method as for ${}^2B_2(2^a)$ applies here. As before, it suffices to assume that there are no subfield subgroups among the H_i 's, other than ${}^2B_2(2^{a/p})$. So

$$|G_0|/|C_{G_0}(x)|^2 = q^3(q^3+1)(q-1)/q_0^6(q_0^3+1)^2(q_0-1)^2$$

and

$$1 + \sum_{i=1}^m \frac{|H_i|}{|C_{H_i}(x)|^2} \leq 1 + \frac{q^3(q-1)}{q_0^6(q_0-1)^2} + \frac{6(q+1)}{(q_0+1)^2} + \frac{6(q+\sqrt{3q}+1)}{(q_0+\sqrt{3q_0}+1)^2} \\ + \frac{6(q-\sqrt{3q}+1)}{(q_0-\sqrt{3q_0}+1)^2} + \frac{2q(q^2-1)}{q_0^2(q_0^2-1)^2}.$$

A computation now shows that (x, G) cannot be a minimal counterexample for any prime power q .

20. SPORADIC GROUPS

If G_0 is one of the following sporadic groups, then a MAGMA calculation shows that there exists $g \in G$ such that $\langle x, x^g \rangle$ is not solvable:

$$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_2, J_3, Co_2, \\ Co_3, McL, HS, Suz, He, Fi_{22}, Fi_{23}, Fi_{24}.$$

There are 9 remaining sporadic groups, which are a little more awkward. One can use [CCN85], which describes the conjugacy classes and maximal subgroups. In certain circumstances, one can show that some element of a conjugacy class is contained inside some smaller almost simple group. In particular, one can do this if there is a unique conjugacy class of elements of order p , or a multiple of p that powers up to the conjugacy class in question. Then any almost simple subgroup containing elements of this order will contain an element of x^G , and thus (x, G) cannot be a minimal counterexample. Clearly, this also applies if all of the conjugacy classes of elements of order p are powers of each other. In the remaining cases, one can use MAGMA with a little more care. The details are listed in Tables 17, 18, 19, 20, 21, 22, 23, 24, and 25.

This completes the proof of Theorem A*.

TABLE 17. Janko group, J_4

Class(es)	MAGMA	Subgroup containing x and class that powers up to x
3		$M_{22} : 2, 3$
5		$M_{22} : 2, 5$
7		$M_{22} : 2, 7$ ^a
11A		$PSU(3, 11) : 2, 44$
11B		$\langle x, a \rangle$ generates ^b
23		$2^{11} : M_{24}, 23$
29		$\langle x, a \rangle$ generates ^c
31		$L(2, 32), 31$
37		$U(3, 11), 37$
43		$\langle x, a \rangle$ generates ^d

^a $7A = (7B)^3$, $7B = 7A^3$.

^bIn this case, a is a standard generator in class 2A; x is a standard representative for class 11B; x^3a has order 43 and x^2a has order 35, so $\langle x, a \rangle$ cannot be contained in any maximal subgroups; thus $\langle x, x^a \rangle$ is not solvable since its index in $\langle x, a \rangle$ is at most 2.

^cIn this case, x is a standard representative for class 29A. We can show in MAGMA that the group order is a multiple of 29.44.

^dIn this case, x is a standard representative for class 43A, but a calculation in MAGMA shows that 43.23 divides the order of $\langle x, a \rangle$.

TABLE 18. Conway group, Co_1

Class(es)	MAGMA	Subgroup containing x and class that powers up to x
3	done	
5	done	
7A		$A_9, 42$
7B	done	
11		$Co_3, 11$
13		$3 : Suz : 2, 13$
23		$Co_2, 23$

TABLE 19. Rudvalis group, Ru

Class(es)	MAGMA	Subgroup containing x and class that powers up to x
3	done	
5	done	
7		$A_8, 7$
13		$PSL(2, 13), 13$
29		$PSL(2, 29), 29$

TABLE 20. O’Nan group, $O'N$

Class(es)	MAGMA	Subgroup containing x and class that powers up to x
3	done	
5		$A_7, 5$
7A		$PSL(3, 7), 14$
7B	done	
11		$J_1, 11$
19		$J_1, 19$
31		$PSL(2, 31), 31$

TABLE 21. Harada–Norton group, HN

Class(es)	MAGMA	Subgroup containing x and class that powers up to x
3A		$A_{12}, 21A$
3B		$A_{12}, 9$
5A		$A_{12}, 35$
5B-E	done ^a	
7		$A_{12}, 7$
11		$A_{12}, 11$
19		$PSU(3, 8), 19$

^aMAGMA calculation performed using permutation representation in [ABN].

TABLE 22. Lyons group, Ly

Class(es)	MAGMA	Subgroup containing x and class that powers up to x
3A		$2.A_{11}, 21A$
3B		$2.A_{11}, 9$
5A		$2.A_{11}, 20$
5B	done ^a	
7		$2.A_{11}, 7$
11		$2.A_{11}, 11$
31		$5^3.PSL(3, 5), 31$
37	done ^b	
67	done ^c	

^aIf a, b , and x are standard representatives for classes 2A, 5A, and 5B, respectively, then ax^b has order 67, so $\langle a, x^b \rangle$ cannot be contained in any maximal subgroup.

^bIf a and x are standard representatives for classes 2A and 37A, respectively, then ax has order 67, so $\langle a, x \rangle$ cannot be contained in any maximal subgroup.

^cIf a and x are standard representatives for classes 2A and 67A, respectively, then ax has order 14, so $\langle a, x \rangle$ cannot be contained in any maximal subgroup.

TABLE 23. Thompson Group, Th

Class(es)	MAGMA	Subgroup containing x and class that powers up to x
3A		$PSU(3, 8), 21$
3B		$A_9, 9$
3C		$A_9, 15$
5		$A_9, 5$
7		$A_9, 7$
13		$PSL(3, 3), 13$
19		$PSL(2, 19), 19$
31		$2^5.PSL(5, 2), 31$

TABLE 24. Baby Monster, B

Class(es)	MAGMA	Subgroup containing x and class that powers up to x
3A		$HN, 21$
3B		$HN, 9$
5A		$HN, 35$
5B		$HN, {}^a$
7		$PSL(2, 49), 7$
11		$PSL(2, 11), 11$
13		$PSL(3, 3), 13$
17		$PSL(2, 17), 17$
19		$HN, 19$
23		$FI_{23}, 23$
31		$PSL(2, 31), 31$
47	done ^b	

^aThe order of $C_B(x)$ is a multiple of 5^6 , but $5^5 \nmid C_B(y)$ if $y \in 5A$, so any member of the class 5B in HN (centralizer order 500,000) must be in the Baby Monster class 5B.

^bSince ax has order 9, where a and x are standard representatives for classes 2A and 47A, respectively, $\langle a, x \rangle$ generates since the only maximal subgroup with order a multiple of 47 is $47 : 23$.

TABLE 25. Monster Group, M

Class(es)	$ C_G(x) $	Subgroup containing x and class that powers up to x
3A		$B, 48$
3B		$A_{12}, 9$
3C		$PSU(3, 8) \times A_5, 57$
5A		$PSL(2, 11) \times M_{12}, 55$
5B		$PSL(2, 25), 25$
7A		$(A_5 \times A_{12}), 105$
7B		contained in $PSL(2, 71)$ group by [NW02, pg 596]
11		$2.B, 11$
13A	73008	$S_3 \times Th^a$
13B	52728	Lies in $6.Suz$ by [NW02, pg 593]
17		$2.B, 17$
19		$2.B, 19$
23		$2.B, 23$
29		$3.Fi_{24}, 29$
31		$2.B, 31$
41		$3^8.O^-(8, 3), 41$
47		$2.B, 47$
59		$PSL(2, 59), 59$
71		$PSL(2, 71), 71$

^aSince an element of order 13 in Th has centralizer order 39, it follows that any such element is in 13A since 39.6 divides $|C_G(x)|$ but does not divide 52728.

REFERENCES

- [ABN] Rachel Abbott, John Bray, Simon Nickerson, Steve Linton, Simon Norton, Richard Parker, Ibrahim Suleiman, Jonathan Tripp, Peter Walsh, and Robert Wilson, *A www-atlas of finite group representations*.
- [AG84] M. Aschbacher and R. Guralnick, *Some applications of the first cohomology group*, J. Algebra **90** (1984), no. 2, 446–460. MR760022 (86m:20060)
- [Bos] Wieb Bosma, John Cannon, and Catherine Playoust. *The Magma algebra system. I. The user language*. J. Symbolic Comput. **24** (3-4): 235–265, 1997. MR1484478
- [Bur04] Timothy C. Burness, *Fixed point spaces in actions of classical algebraic groups*, J. Group Theory **7** (2004), no. 3, 311–346. MR2063000 (2005c:14054)
- [CCN85] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups*, Oxford University Press, Eynsham, 1985, Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray. MR827219 (88g:20025)
- [Enn62] Veikko Ennola, *On the conjugacy classes of the finite unitary groups*, Ann. Acad. Sci. Fenn. Ser. A I No. **313** (1962), 13pp. MR0139651 (25:3082)
- [FGG] Paul Flavell, Robert Guralnick, and Simon Guest, *Characterizations of the solvable radical*, Proc. Amer. Math. Soc. **138** (2010), no. 4, 1161–1170. MR2578510.
- [GGKP08a] Nikolai Gordeev, Fritz Grunewald, Boris Kunyavskii, and Eugene Plotkin, *A commutator description of the solvable radical of a finite group*, Groups Geom. Dyn. **2** (2008), no. 1, 85–120. MR2367209 (2008j:20057)
- [GGKP08b] ———, *A description of Baer–Suzuki type of the solvable radical of a finite group*, J. Pure and Applied Algebra **213** (2009), 250–258. MR2467402 (2009i:20045)
- [GHL96] Meinolf Geck, Gerhard Hiss, Frank Lübeck, Gunter Malle, and Götz Pfeiffer, *CHEVIE—a system for computing and processing generic character tables*, Appl. Algebra Engrg. Comm. Comput. **7** (1996), no. 3, 175–210, Computational methods in Lie theory (Essen, 1994). MR1486215 (99m:20017)
- [GK00] Robert M. Guralnick and William M. Kantor, *Probabilistic generation of finite simple groups*, J. Algebra **234** (2000), no. 2, 743–792, Special issue in honor of Helmut Wielandt. MR1800754 (2002f:20038)
- [GL83] Daniel Gorenstein and Richard Lyons, *The local structure of finite groups of characteristic 2 type*, Mem. Amer. Math. Soc. **42** (1983), no. 276, vii+731pp. MR690900 (84g:20025)
- [GLS98] Daniel Gorenstein, Richard Lyons, and Ronald Solomon, *The classification of the finite simple groups. Number 3*, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1998. MR1490581 (98j:20011)
- [GPPS99] Robert Guralnick, Tim Penttilä, Cheryl E. Praeger, and Jan Saxl, *Linear groups with orders having certain large prime divisors*, Proc. London Math. Soc. (3) **78** (1999), no. 1, 167–214. MR1658168 (99m:20113)
- [GPS07] Robert Guralnick, Eugene Plotkin, and Aner Shalev, *Burnside-type problems related to solvability*, Internat. J. Algebra Comput. **17** (2007), no. 5-6, 1033–1048. MR2355682 (2008j:20110)
- [GS03] Robert M. Guralnick and Jan Saxl, *Generation of finite almost simple groups by conjugates*, J. Algebra **268** (2003), no. 2, 519–571. MR2009321 (2005f:20057)
- [Gur98] Robert M. Guralnick, *Some applications of subgroup structure to probabilistic generation and covers of curves*, Algebraic groups and their representations (Cambridge, 1997), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 517, Kluwer Acad. Publ., Dordrecht, 1998, pp. 301–320. MR1670777 (2000d:20062)
- [KL90] Peter Kleidman and Martin Liebeck, *The subgroup structure of the finite classical groups*, London Mathematical Society Lecture Note Series, vol. 129, Cambridge University Press, Cambridge, 1990. MR1057341 (91g:20001)
- [Kle88a] Peter B. Kleidman, *The maximal subgroups of the Chevalley groups $G_2(q)$ with q odd, the Ree groups ${}^2G_2(q)$, and their automorphism groups*, J. Algebra **117** (1988), no. 1, 30–71. MR955589 (89j:20055)
- [Kle88b] ———, *The maximal subgroups of the Steinberg triality groups ${}^3D_4(q)$ and of their automorphism groups*, J. Algebra **115** (1988), no. 1, 182–199. MR937609 (89f:20024)

- [Law95] R. Lawther, *Jordan block sizes of unipotent elements in exceptional algebraic groups*, Comm. Algebra **23** (1995), no. 11, 4125–4156. MR1351124 (96h:20084)
- [LS03] Martin W. Liebeck and Gary M. Seitz, *A survey of maximal subgroups of exceptional groups of Lie type*, Groups, combinatorics & geometry (Durham, 2001), World Sci. Publ., River Edge, NJ, 2003, pp. 139–146. MR1994964 (2004f:20089)
- [LSS92] Martin W. Liebeck, Jan Saxl, and Gary M. Seitz, *Subgroups of maximal rank in finite exceptional groups of Lie type*, Proc. London Math. Soc. (3) **65** (1992), no. 2, 297–325. MR1168190 (93e:20026)
- [LSS96] ———, *Factorizations of simple algebraic groups*, Trans. Amer. Math. Soc. **348** (1996), no. 2, 799–822. MR1316858 (96g:20064)
- [Mal91] Gunter Malle, *The maximal subgroups of ${}^2F_4(q^2)$* , J. Algebra **139** (1991), no. 1, 52–69. MR1106340 (92d:20068)
- [Miz77] Kenzo Mizuno, *The conjugate classes of Chevalley groups of type E_6* , J. Fac. Sci. Univ. Tokyo Sect. IA Math. **24** (1977), no. 3, 525–563. MR0486170 (58:5951)
- [Miz80] ———, *The conjugate classes of unipotent elements of the Chevalley groups E_7 and E_8* , Tokyo J. Math. **3** (1980), no. 2, 391–461. MR605099 (82m:20046)
- [NW02] Simon P. Norton and Robert A. Wilson, *Anatomy of the Monster. II*, Proc. London Math. Soc. (3) **84** (2002), no. 3, 581–598. MR1888424 (2003b:20023)
- [Sei83] Gary M. Seitz, *The root subgroups for maximal tori in finite groups of Lie type*, Pacific J. Math. **106** (1983), no. 1, 153–244. MR694680 (84g:20085)
- [Sho74] Toshiaki Shoji, *The conjugacy classes of Chevalley groups of type (F_4) over finite fields of characteristic $p \neq 2$* , J. Fac. Sci. Univ. Tokyo Sect. IA Math. **21** (1974), 1–17. MR0357641 (50:10109)
- [SS97] Jan Saxl and Gary M. Seitz, *Subgroups of algebraic groups containing regular unipotent elements*, J. London Math. Soc. (2) **55** (1997), no. 2, 370–386. MR1438641 (98m:20057)
- [Suz62] Michio Suzuki, *On a class of doubly transitive groups*, Ann. of Math. (2) **75** (1962), 105–145. MR0136646 (25:112)
- [Tho68] John G. Thompson, *Nonsolvable finite groups all of whose local subgroups are solvable*, Bull. Amer. Math. Soc. **74** (1968), 383–437. MR0230809 (37:6367)
- [Wal63] G. E. Wall, *On the conjugacy classes of classical groups*, J. Austral. Math. Soc. **3** (1963), 1–62. MR0150210 (27:212)
- [War66] Harold N. Ward, *On Ree's series of simple groups*, Trans. Amer. Math. Soc. **121** (1966), 62–89. MR0197587 (33:5752)
- [Wie64] Helmut Wielandt, *Finite permutation groups*, Translated from the German by R. Bercov, Academic Press, New York, 1964. MR0183775 (32:1252)

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