SUPERPOSITION IN HOMOGENEOUS AND VECTOR VALUED SOBOLEV SPACES

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ABSTRACT. We give a sufficient condition on a function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ so that it takes by superposition the homogeneous vector valued space $\dot{W}_p^m \cap \dot{W}_{mp}^1(\mathbb{R}^n, \mathbb{R}^k)$ into the corresponding real valued space, for integers m, n, k such that $m \geq 2$, $k, n \geq 1$, and $p \in [1, +\infty[$. In case $k = 1$, this condition also turns out to be necessary. For $k > 1$, it is not proved to be necessary, but it is weaker than the conditions used till now, such as the continuity and boundedness of all derivatives up to order m .

1. INTRODUCTION

The Superposition Operator Problem for a given set E of real valued functions consists in determining the class of functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f \circ g \in E$, for all $g \in E$. This problem was solved between 1975 and 1990 in the classical Sobolev spaces $W_p^m(\mathbb{R}^n)$, for all positive integers m and n , and all $p \in [1, +\infty]$ [13, 16, 7]. In particular, Dahlberg established the following degeneracy result: for $1 + (1/p) < m < n/p$, only linear functions act by superposition from $W_p^m(\mathbb{R}^n)$ to itself. Such a triviality property does not hold if we take the intersection of W_p^m with the homogeneous space $\dot{W}_{mp}^1$. Let us consider the *admissible functions of order m* , that is, the functions $f \in C^m(\mathbb{R}^n)$ satisfying $f^{(\alpha)} \in L_\infty$ for all $\alpha \in \mathbb{N}^n$ such that $0 < |\alpha| \leq m$. D. Adams and M. Frazier proved that any admissible function $f : \mathbb{R} \rightarrow \mathbb{R}$ of order m , vanishing at zero, acts on $W_p^m \cap \dot{W}_{mp}^1$; see [2] and [20, thm. 5.2.5].¹ Taking the intersection with $\dot{W}_{mp}^1$ is the best way to obtain nontrivial superposition. It does not imply a loss of generality since, by a standard imbedding, $W_p^m \subset W_{mp}^1$ holds if $m \geq n/p$. Moreover any real valued function g such that $f \circ g \in W_p^m$, for all admissible functions f of infinite order vanishing at 0, belongs necessarily to $\dot{W}_{mp}^1$ (by taking $f(x) := \sin x$ and $f(x) := \cos x - 1$, we obtain $e^{ig} - 1 \in W_p^m \cap L_\infty$; then $e^{ig} - 1 \in W_{mp}^1$, by Gagliardo-Nirenberg, and thus $\nabla g \in L_{mp}$).

We wish to improve the Adams-Frazier theorem in various directions. Firstly we will replace the admissible functions of order m by a larger class, in such a way that the sufficient condition for superposition turns out to also be necessary.

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¹The Adams-Frazier theorem concerns the more general Bessel potential spaces $L_p^s \cap \dot{L}_{sp}^1$ for all real numbers $s > 1$. A similar result was proved for Sobolev-Slobodeckij spaces $W_p^s \cap \dot{W}_{sp}^1$ by H. Brezis and P. Mironescu [12], and by V. Maz'ya and T. Shaposhnikova [18].

Secondly we will prove that superposition works as well in the homogeneous space $\dot{W}_p^m \cap \dot{W}_{mp}^1(\mathbb{R}^n)$. Thirdly we will give a similar sufficient condition for a function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ so that it acts by superposition from $\dot{W}_p^m \cap \dot{W}_{mp}^1(\mathbb{R}^n, \mathbb{R}^k)$ to $\dot{W}_p^m \cap \dot{W}_{mp}^1(\mathbb{R}^n)$. Though it is not known to be necessary in case $k > 1$, it is sharper than admissibility of order m .

NOTATION

We denote by $\mathbb{N}$ the set of all natural numbers, including 0, and by $\mathbb{N}^*$ the set $\mathbb{N} \setminus \{0\}$. We denote by $(e_1, \dots, e_n)$ the canonical basis of $\mathbb{R}^n$. For all $j = 1, \dots, n$, we consider the symmetry σ_j which exchanges e_j and e_n and leaves invariant the other e_k . We extend σ_j to functions in the usual way, i.e. $\sigma_j f(x) := f(\sigma_j(x))$. We denote by $\mathcal{P}_m$ the set of polynomials on $\mathbb{R}^n$ of degree less than or equal to m . We denote by Q the cube $[-1/2, 1/2]^n$ and by Q^+ its positive part $[0, 1/2]^n$. All through the paper, we will use auxiliary functions, assumed to be fixed once and for all, defined as follows:

- (1) $\varphi \in C^\infty(\mathbb{R}^n)$, $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ for $x \in Q$, $\varphi(x) = 0$ for $x \notin 2Q$,
- (2) ψ , defined like φ , with Q replaced by the Euclidean unit ball of $\mathbb{R}^n$,
- (3) $\theta_k(x) := \|\psi\|_1^{-1} k^n \psi(kx)$, $k \in \mathbb{N}^*$,
- (4) $v(x_1, \dots, x_n) := x_1 \varphi(x_1, \dots, x_n)$.

It should be remarked that all the above objects clearly depend on the given integer n , though n does not appear in the notation.

We define the translation operator τ_h on distributions by the formula $(\tau_h f)(x) := f(x - h)$, and the finite difference operator by $\Delta_h f := \tau_{-h} f - f$. If E is any distribution space, and if $m \in \mathbb{N}$, we denote by $\dot{W}_E^m$ the homogeneous Sobolev space built on E , i.e. the set of distributions whose derivatives of order m belong to E . If E is endowed with a complete norm $\| \cdot \|_E$ such that the imbedding $E \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$ is continuous, then E is called a *Banach distribution space* (B.D.S.). In such a case, $\dot{W}_E^m$ is endowed with the seminorm

$$\|f\|_{\dot{W}_E^m} := \sum_{|\alpha|=m} \|f^{(\alpha)}\|_E.$$

E is said to be *translation invariant* if $\tau_h(E) \subseteq E$ for all $h \in \mathbb{R}^n$, and a $\mathcal{D}(\mathbb{R}^n)$ -*module* if any function of $\mathcal{D}(\mathbb{R}^n)$ acts by multiplication on E . We denote by E_{loc} the set of distributions f such that $uf \in E$ for all $u \in \mathcal{D}(\mathbb{R}^n)$. For $p \geq 1$, the usual homogeneous Sobolev space $\dot{W}_{L_p(\mathbb{R}^n)}^m$ is denoted by $\dot{W}_p^m(\mathbb{R}^n)$. For any closed subset F of $\mathbb{R}^n$, we denote² by $\dot{W}_p^m(F)$ the set of distributions of $\dot{W}_p^m(\mathbb{R}^n)$ with support in F . The ordinary (inhomogeneous) Sobolev space in the L_p norm is denoted by $W_p^m(\mathbb{R}^n)$. It is known that $W_p^m(\mathbb{R}^n) = L_p \cap \dot{W}_p^m(\mathbb{R}^n)$. The system of parameters (m, n, p) is said to be *subcritical* if

$$m < \frac{n}{p} \quad \text{or} : \quad m = \frac{n}{p} \quad \text{and} \quad p > 1,$$

and *supercritical* if not. It is well known that $W_p^m(\mathbb{R}^n)$ is imbedded into $L_\infty(\mathbb{R}^n)$ if and only if (m, n, p) is supercritical. All the previously defined spaces are assumed

²The same notation has other meanings in the literature; e.g., it can denote the set of the restrictions to F of the functions of $\dot{W}_p^m(\mathbb{R}^n)$.

to be spaces of real valued functions. We consider also the vector valued Sobolev space $\dot{W}_p^m(\mathbb{R}^n, \mathbb{R}^k)$, defined in an obvious way, and its inhomogeneous counterpart. We denote by p' the conjugate exponent of $p \in [1, +\infty]$, i.e. $p' := \frac{p}{p-1}$. We denote by $C_b(\mathbb{R}^n)$ the Banach space of bounded continuous functions on $\mathbb{R}^n$ and by $C_0(\mathbb{R}^n)$ the space of continuous functions on $\mathbb{R}^n$ tending to 0 at infinity. For all functions $f : \mathbb{R}^k \rightarrow \mathbb{R}$, we denote by T_f the superposition operator $g \mapsto f \circ g$, defined on $\mathbb{R}^k$ -valued mappings. As usual, the constants $c, c_1, \dots$ are strictly positive and depend only on the fixed parameters n, m, p, k , and possibly on the cutoff functions φ, ψ , unless otherwise specified; their values can change from one line to another.

PLAN

In the second section, we state the main results concerning the real valued spaces and prove the sufficiency of conditions for superposition. In the third section, we turn to the case of vector valued Sobolev spaces. The fourth section is devoted to the necessary conditions. In the last section, we discuss the existence of superposition operators on the full homogeneous space $\dot{W}_p^m$.

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2. CHARACTERIZATION OF ACTING FUNCTIONS ON $\dot{W}_p^m \cap \dot{W}_{mp}^1(\mathbb{R}^n)$

2.1. Statement of the main theorems. For all $p \in [1, +\infty[$, we consider the space $E_p(\mathbb{R})$ of functions f on $\mathbb{R}$, which belong to L_p *locally uniformly*, i.e. such that

$$\|f\|_{E_p} := \left(\sup_{a \in \mathbb{R}} \int_a^{a+1} |f(t)|^p dt \right)^{1/p} < +\infty.$$

Theorem 1. *Let m be an integer ≥ 2 , $p \in [1, +\infty[$, $n \in \mathbb{N}^*$ (with the exception of $m = n$ and $p = 1$) and let $f : \mathbb{R} \rightarrow \mathbb{R}$. Then T_f takes $\dot{W}_p^m \cap \dot{W}_{mp}^1(\mathbb{R}^n)$ to itself if and only if $f \in \dot{W}_{E_p}^m \cap \dot{W}_\infty^1(\mathbb{R})$.*

Theorem 2. *Let m be an integer ≥ 2 , $p \in [1, +\infty[$, $n \in \mathbb{N}^*$, and $f : \mathbb{R} \rightarrow \mathbb{R}$. Then T_f takes $\dot{W}_p^m \cap L_\infty(\mathbb{R}^n)$ to itself if and only if $f \in W_p^m(\mathbb{R})_{loc}$.*

Remark 1. The exceptional case of Theorem 1 is covered by Theorem 2 since $\dot{W}_1^n \cap \dot{W}_n^1(\mathbb{R}^n)$ is imbedded into $L_\infty(\mathbb{R}^n)$; see Proposition 14 in Section 5.

As announced in the Introduction, we postpone the proof of necessity of conditions on f to Section 4. The sufficiency is contained in the following statement, which gives at the same time a precise estimate of the norm of $f \circ g$.

Proposition 1. *Let m be an integer ≥ 2 , $p \in [1, +\infty[$, $n \in \mathbb{N}^*$. For all $f \in \dot{W}_{E_p}^m \cap \dot{W}_\infty^1(\mathbb{R})$ and all $g \in \dot{W}_p^m \cap \dot{W}_{mp}^1(\mathbb{R}^n)$, it follows that*

$$(5) \quad \|f \circ g\|_{\dot{W}_p^m} \leq c \left(\|f\|_{\dot{W}_{E_p}^m} + \|f\|_{\dot{W}_\infty^1} \right) \left(\|g\|_{\dot{W}_{mp}^1}^m + \|g\|_{\dot{W}_p^m} \right).$$

Remark 2. Under the assumptions of Proposition 1, the estimate

$$\|f \circ g\|_{\dot{W}_{mp}^1} \leq \|f\|_{\dot{W}_\infty^1} \|g\|_{\dot{W}_{mp}^1}$$

is classically known [16].

The remainder of Section 2 is organized as follows. First we recall the classical Gagliardo-Nirenberg inequalities, which remain the best tool to obtain superposition results; see for instance Adams [1]. Then we prove Proposition 1. Finally we discuss the sharpness of the estimate (5).

Till the end of Section 2 we set

$$\rho(f) := \|f\|_{\dot{W}_\infty^1} + \|f\|_{\dot{W}_{E_p}^m}, \quad V(g) := \|g\|_{\dot{W}_{pm}^1}^m + \|g\|_{\dot{W}_p^m}^m.$$

In all of the paper, we set $\|g\| := \|g\|_{\dot{W}_{pm}^1} + \|g\|_{\dot{W}_p^m}$ for simplicity.

2.2. Gagliardo-Nirenberg inequalities. We refer to the original paper of Nirenberg [19, p. 125] for a general statement.

Proposition 2. *Let N, m be integers such that $0 < N < m$, p, q in $[1, +\infty]$, and*

$$\frac{N}{m} \leq t < 1.$$

Assume that

$$\frac{1}{r} := \frac{1}{n}(N - tm) + \frac{t}{p} + \frac{1-t}{q} > 0.$$

Then the imbedding $\dot{W}_p^m \cap L_q(\mathbb{R}^n) \hookrightarrow \dot{W}_r^N(\mathbb{R}^n)$ holds together with the estimate

$$\|u\|_{\dot{W}_r^N(\mathbb{R}^n)} \leq c \|u\|_{\dot{W}_p^m(\mathbb{R}^n)}^t \|u\|_{L_q(\mathbb{R}^n)}^{1-t}.$$

Let us note that the assumptions on N, m, p, q, t imply $1/r \leq 1$. We will exploit two particular cases of Proposition 2. By taking $N = 1 < m$, $q = \infty$ and $t = 1/m$, we obtain

$$(6) \quad \forall u \in L_\infty \cap \dot{W}_p^m(\mathbb{R}^n) \quad \|u\|_{\dot{W}_{pm}^1(\mathbb{R}^n)} \leq c \|u\|_\infty^{1-\frac{1}{m}} \|u\|_{\dot{W}_p^m(\mathbb{R}^n)}^{\frac{1}{m}}.$$

By replacing m by $m - 1$, and by taking $q = pm$ and $t = \frac{N-1}{m-1}$, we obtain

$$\forall u \in L_{pm} \cap \dot{W}_p^{m-1}(\mathbb{R}^n) \quad \|u\|_{\dot{W}_{pm/N}^{N-1}(\mathbb{R}^n)} \leq c \|u\|_{L_{pm}(\mathbb{R}^n)}^{\frac{m-N}{m-1}} \|u\|_{\dot{W}_p^{m-1}(\mathbb{R}^n)}^{\frac{N-1}{m-1}}.$$

This inequality holds as well, trivially, for $N = 1$. Then applying it to the first derivatives of u , we deduce

$$(7) \quad \forall u \in \dot{W}_{pm}^1 \cap \dot{W}_p^m(\mathbb{R}^n) \quad \|u\|_{\dot{W}_{pm/N}^N(\mathbb{R}^n)} \leq c \|u\|_{\dot{W}_{pm}^1(\mathbb{R}^n)}^{\frac{m-N}{m-1}} \|u\|_{\dot{W}_p^m(\mathbb{R}^n)}^{\frac{N-1}{m-1}}.$$

2.3. Proof of Proposition 1: The heart of the matter. We make some qualitative assumptions on f and g , which we will remove by standard approximation procedures. More precisely, we assume that f is an admissible function of order m , that g is a smooth function in $\dot{W}_p^m \cap \dot{W}_{mp}^1(\mathbb{R}^n)$, and that the first order derivatives of g tend to 0 at infinity.

We have to estimate $\|(f \circ g)^{(\alpha)}\|_p$ for $|\alpha| = m$. Let us recall the classical formula

$$(8) \quad (f \circ g)^{(\alpha)} = \sum c_{\alpha, s, \gamma} (f^{(s)} \circ g) g^{(\gamma_1)} \dots g^{(\gamma_s)},$$

where the parameters satisfy the conditions

$$(9) \quad s = 1, \dots, m, \quad |\gamma_r| > 0 \quad (r = 1, \dots, s), \quad \sum \gamma_r = \alpha,$$

and the $c_{\alpha, s, \gamma}$'s are some combinatorial constants.

In the sum (8), the estimation of lower order terms, i.e. with $s < m$, is well known; see the next subsection. Thus we concentrate on the case $s = m$. The corresponding generic term in (8) is

$$(f^{(m)} \circ g) \partial_{j_1} g \cdots \partial_{j_m} g,$$

with $(j_1, \dots, j_m) \in \{1, \dots, n\}^m$. By the Hölder inequality, we have

$$(10) \quad \|(f^{(m)} \circ g) \partial_{j_1} g \cdots \partial_{j_m} g\|_p \leq \left(\prod_{r=1}^m \int_{\mathbb{R}^n} |f^{(m)} \circ g|^p |\partial_{j_r} g|^{mp} dx \right)^{\frac{1}{mp}}.$$

Thus we are reduced to estimate

$$A_j := \left(\int_{\mathbb{R}^n} |f^{(m)} \circ g|^p |\partial_j g|^{mp} dx \right)^{1/p},$$

for any $j = 1, \dots, n$. Let $u \in \mathcal{D}(\mathbb{R})$ be such that $u \geq 0$ and

$$\forall y \in \mathbb{R} \quad \sum_{q \in \mathbb{Z}} u^2(y - q) = 1.$$

Then there exists $c = c(u) > 0$ such that

$$\forall y \in \mathbb{R} \quad \sum_{q \in \mathbb{Z}} |u'(y - q)| \leq c.$$

Let us define

$$\forall y \in \mathbb{R} \quad \Phi_q(y) := u(y - q) |f^{(m)}(y)|^p.$$

By assumption on f , it follows that $\Phi_q \in L_1(\mathbb{R})$ and $\|\Phi_q\|_1 \leq c\rho(f)^p$. Now define

$$\Psi_q(y) := \int_y^{+\infty} \Phi_q(t) dt.$$

Then $\|\Psi_q\|_\infty \leq c\rho(f)^p$. It follows that

$$\begin{aligned} A_j^p &= \sum_{q \in \mathbb{Z}} \int_{\mathbb{R}^n} |f^{(m)}(g(x))|^p |\partial_j g(x)|^{mp} u^2(g(x) - q) dx \\ &= \sum_{q \in \mathbb{Z}} \int_{\mathbb{R}^n} \Phi_q(g(x)) |\partial_j g(x)|^{mp} u(g(x) - q) dx \\ &= - \sum_{q \in \mathbb{Z}} \int_{\mathbb{R}^n} \partial_j (\Psi_q(g(x))) \partial_j g(x) |\partial_j g(x)|^{mp-2} u(g(x) - q) dx. \end{aligned}$$

Integrating by parts w.r.t the j -th variable, and using the fact that $\partial_j g \in C_0(\mathbb{R}^n)$, we obtain

$$\begin{aligned} A_j^p &\leq (mp-1) \sum_{q \in \mathbb{Z}} \int_{\mathbb{R}^n} \Psi_q(g(x)) |\partial_j^2 g(x)| |\partial_j g(x)|^{mp-2} u(g(x)-q) dx \\ &\quad + \sum_{q \in \mathbb{Z}} \int_{\mathbb{R}^n} \Psi_q(g(x)) |\partial_j g(x)|^{mp} |u'(g(x)-q)| dx \\ &\leq c\rho(f)^p \left(\int_{\mathbb{R}^n} |\partial_j g(x)|^{mp-2} |\partial_j^2 g(x)| dx + \int_{\mathbb{R}^n} |\partial_j g(x)|^{mp} dx \right). \end{aligned}$$

Since

$$\int_{\mathbb{R}^n} |\partial_j g(x)|^{mp-2} |\partial_j^2 g(x)| dx \leq \left(\int_{\mathbb{R}^n} |\partial_j g(x)|^{pm} dx \right)^{1-\frac{2}{pm}} \left(\int_{\mathbb{R}^n} |\partial_j^2 g(x)|^{\frac{pm}{2}} dx \right)^{\frac{2}{pm}}$$

holds, we deduce

$$A_j^p \leq c\rho(f)^p \left(\|g\|_{\dot{W}_{pm}^1}^{mp-2} \|g\|_{\dot{W}_{pm/2}^2} + \|g\|_{\dot{W}_{pm}^1}^{mp} \right).$$

Now applying inequality (7) with $N = 2$, and making use of (10), we deduce

$$(11) \quad \|(f^{(m)} \circ g) \partial_{j_1} g \cdots \partial_{j_m} g\|_p \leq c\rho(f) \left(\|g\|_{\dot{W}_{pm}^1}^{m-\frac{m}{p(m-1)}} \|g\|_{\dot{W}_p^m}^{\frac{1}{p(m-1)}} + \|g\|_{\dot{W}_{pm}^1}^m \right).$$

The right hand side of (11) is clearly estimated by $\rho(f) V(g)$.

Remark 3. Here we use exactly the same method (localization and integration by parts) as in [7]. The only new fact is the use of Gagliardo-Nirenberg instead of Sobolev inequalities.

2.4. End of the proof of Proposition 1. It remains to deal with lower order terms in (8) and to get rid of the smoothness assumptions on f and g .

Step 1: Estimation of terms with $s < m$.

First we observe that

$$(12) \quad \|f^{(s)}\|_\infty \leq c\rho(f) \quad \text{for } s = 1, \dots, m-1.$$

It is a variant of the Gagliardo-Nirenberg inequality; see [7, lem. 1] for an easy direct proof. Then by the Hölder inequality, we obtain

$$\|(f^{(s)} \circ g) g^{(\gamma_1)} \cdots g^{(\gamma_s)}\|_p \leq c\rho(f) \prod_{r=1}^s \|g\|_{\dot{W}_{pm/|\gamma_r|}^{|\gamma_r|}}.$$

By (7) with $u = g$, $N = |\gamma_r|$, $r = 1, \dots, s$, we see that the right hand side of the above inequality is estimated by

$$\rho(f) \left(\prod_{r=1}^s \|g\|_{\dot{W}_{pm}^1}^{\frac{m-|\gamma_r|}{m-1}} \right) \left(\prod_{r=1}^s \|g\|_{\dot{W}_p^m}^{\frac{|\gamma_r|-1}{m-1}} \right).$$

By summing up the exponents, we conclude that

$$(13) \quad \|(f^{(s)} \circ g) g^{(\gamma_1)} \cdots g^{(\gamma_s)}\|_p \leq c\rho(f) \|g\|_{\dot{W}_{pm}^1}^{\frac{m-s-1}{m-1}} \|g\|_{\dot{W}_p^m}^{\frac{m-s}{m-1}} \leq c_1 \rho(f) V(g).$$

Step 2. Let us assume again that f is admissible of order m and that g is any function in $\dot{W}_p^m \cap \dot{W}_{mp}^1(\mathbb{R}^n)$. Then $(\theta_\nu * g)_{\nu \geq 1}$ is a sequence of smooth functions such that $V(\theta_\nu * g) \leq V(g)$ for all ν . By condition $\partial_j g \in L_{pm}$, it follows that $\partial_j(\theta_\nu * g) \in C_0$, for all $j = 1, \dots, n$. Since g is locally in L_p (see Proposition 12 in Section 5) it follows that $(\theta_\nu * g)(x) \rightarrow g(x)$ a.e. and, by taking a subsequence if necessary, $\sup_\nu |\theta_\nu * g| \in L_p(\mathbb{R}^n)_{loc}$. Since f is Lipschitz continuous, we deduce that

$$f((\theta_\nu * g)(x)) \rightarrow f(g(x)) \text{ a.e.} \quad \text{and} \quad \sup_\nu |f \circ (\theta_\nu * g)| \leq \|f'\|_\infty \sup_\nu |\theta_\nu * g| + |f(0)|.$$

Hence $f \circ (\theta_\nu * g) \rightarrow f \circ g$ in the sense of distributions. Let us consider a set of parameters satisfying (9). By Subsection 2.3 and by Step 1, we have

$$(14) \quad \forall \nu \geq 1 \quad \|(f^{(s)} \circ (\theta_\nu * g)) (\theta_\nu * g)^{(\gamma_1)} \dots (\theta_\nu * g)^{(\gamma_s)}\|_p \leq c\rho(f) V(g).$$

By Proposition 2, we have $g^{(\gamma_r)} \in L_{q_r}$, with $q_r := pm/|\gamma_r|$. Then $\theta_\nu * g^{(\gamma_r)}(x) \rightarrow g^{(\gamma_r)}(x)$ a.e. and, by taking a subsequence if necessary, $\sup_\nu |\theta_\nu * g^{(\gamma_r)}| \in L_{q_r}$. Using the continuity of $f^{(s)}$, we deduce that

$$(f^{(s)} \circ (\theta_\nu * g)) (\theta_\nu * g)^{(\gamma_1)} \dots (\theta_\nu * g)^{(\gamma_s)} \rightarrow (f^{(s)} \circ g) g^{(\gamma_1)} \dots g^{(\gamma_s)} \quad \text{a.e.}$$

and that

$$\sup_{\nu \geq 1} \left| (f^{(s)} \circ (\theta_\nu * g)) (\theta_\nu * g)^{(\gamma_1)} \dots (\theta_\nu * g)^{(\gamma_s)} \right| \leq \|f^{(s)}\|_\infty \prod_{r=1}^s \sup_\nu |(\theta_\nu * g)^{(\gamma_r)}|.$$

Since the right hand side of the above inequality belongs to L_p , a consequence of $\sum q_r^{-1} = p^{-1}$, we conclude that convergence holds in the sense of distributions and that the identity (8) is still valid. Applying the Fatou Lemma to inequality (14), we obtain $(f \circ g)^{(\alpha)} \in L_p$ together with inequality (5).

Step 3. We turn now to the general case. Taking $f_\nu := \theta_\nu * f$, we obtain a sequence of admissible functions of order m such that $\rho(f_\nu) \leq \rho(f)$ for all ν . By continuity of f , it follows that $f_\nu(x) \rightarrow f(x)$ everywhere. Moreover

$$\sup_\nu |f_\nu(g(x))| \leq \|f'\|_\infty |g(x)| + \sup_\nu |f_\nu(0)|.$$

Hence $f_\nu \circ g \rightarrow f \circ g$ in the sense of distributions. By Step 2,

$$(15) \quad \sup_{\nu \geq 1} \sup_{|\alpha|=m} \|(f_\nu \circ g)^{(\alpha)}\|_p \leq c\rho(f)V(g).$$

If $p > 1$, then L_p is the dual space of $\mathcal{D}(\mathbb{R}^n)$ endowed with the $L_{p'}$ norm. We conclude that $(f \circ g)^{(\alpha)} \in L_p$ for all $|\alpha| = m$, with the desired estimation.

In case $p = 1$, we can conclude that $(f \circ g)^{(\alpha)} \in L_1$, for all $|\alpha| = m$, by the De la Vallée-Poussin and Fatou Lemmas; see, e.g., the proof of [20, thm. 5.2.4/1, pp. 273-274]. Thus the proof of Proposition 1 is complete.

2.5. Proof of sufficiency in Theorem 2. We use the auxiliary function introduced in (2), and set $\psi_t(x) := \psi(t^{-1}x)$, for every $x \in \mathbb{R}$ and $t > 0$. Let $f \in W_p^m(\mathbb{R})_{loc}$ and $g \in \dot{W}_p^m \cap L_\infty(\mathbb{R}^n)$. By condition $m \geq 2$, and by classical Sobolev imbeddings, it follows that f is locally Lipschitz continuous. Hence $f\psi_t \in \dot{W}_\infty^1(\mathbb{R})$. By the Leibniz formula, and by inequality (12), it follows that $f\psi_t \in \dot{W}_{E_p}^m(\mathbb{R})$.

Since $f \circ g = (f\psi_{\|g\|_\infty}) \circ g$, we can use Proposition 1 and conclude that $f \circ g \in \dot{W}_p^m \cap L_\infty(\mathbb{R}^n)$.

2.6. Sharpness of the estimates. We will say that the superposition operator T_f is α -bounded on $\dot{W}_{mp}^1 \cap \dot{W}_p^m$ if there exists $c > 0$ such that

$$(16) \quad \forall g \in \mathcal{D}(\mathbb{R}^n) \quad \|f \circ g\|_{\dot{W}_p^m} \leq c \left(1 + \|g\|_{\dot{W}_{mp}^1} + \|g\|_{\dot{W}_p^m}\right)^\alpha.$$

By the estimate (5), the superposition operator associated to an admissible function of infinite order is m -bounded. Now we prove that m is the lowest possible exponent for the class of admissible functions of infinite order.

Proposition 3. *Let m be an integer ≥ 2 , $p \in [1, +\infty[$. If $\alpha < m$, then the superposition operator T_w , associated to $w(t) := \sin t$, is not α -bounded on $\dot{W}_{mp}^1 \cap \dot{W}_p^m(\mathbb{R})$.*

Proof. Arguing by contradiction, let us assume that T_w is α -bounded for some $\alpha \in]m - (1/p), m[$. Then the same holds also if we replace the sines by the cosines. Hence

$$\forall g \in \mathcal{D}(\mathbb{R}) \quad \|e^{ig}\|_{\dot{W}_p^m} \leq c_1 (1 + \|g\|)^\alpha$$

for some constant $c_1 > 0$. Using the above estimate, together with (13) for $f(t) := e^{it}$, we obtain

$$\forall g \in \mathcal{D}(\mathbb{R}) \quad \|g\|_{\dot{W}_{mp}^1}^m \leq c_2 (1 + \|g\|)^\alpha.$$

If we fix a nonzero function g , and replace g by λg in the last estimate, we derive a contradiction by taking $\lambda \rightarrow +\infty$. $\square$

There exists a natural set of nonlinear functions f , including for instance the Schwartz class, which lead to $(m - (1/p))$ -bounded superposition operators, and the last exponent is the lowest possible, with the exception of polynomial functions f .

Proposition 4. *Let m be an integer ≥ 2 , $p \in [1, +\infty[$. Let $f \in \dot{W}_\infty^1 \cap \dot{W}_p^m(\mathbb{R})$.*

1. *For all $g \in \dot{W}_{mp}^1 \cap \dot{W}_p^m(\mathbb{R}^n)$,*

$$(17) \quad \|f \circ g\|_{\dot{W}_p^m} \leq c (\|f\|_{\dot{W}_\infty^1} + \|f\|_{\dot{W}_p^m}) \|g\| (1 + \|g\|)^{m-1-(1/p)}.$$

2. *For all $g \in L_\infty \cap \dot{W}_p^m(\mathbb{R}^n)$,*

$$(18) \quad \|f \circ g\|_{\dot{W}_p^m} \leq c \|g\|_{\dot{W}_p^m} (1 + \|g\|_\infty)^{m-1-(1/p)}.$$

Proof. Step 1 : Proof of (17). We follow the same lines as in the proof of Proposition 1. The assumption on f allows us to avoid the localization procedure through the auxiliary function u . Hence (11) becomes

$$(19) \quad \|(f^{(m)} \circ g) \partial_{j_1} g \cdots \partial_{j_m} g\|_p \leq c \rho(f) \|g\|^{m-(1/p)}.$$

The desired estimate follows by (13) and (19).

Step 2 : Proof of (18). We proceed as in Step 1, and we use (6) to estimate $\|g\|_{\dot{W}_{p^m}^1}$ w.r.t. $\|g\|_{\dot{W}_p^m}$ and $\|g\|_\infty$. By computing the exponents, we obtain the desired estimate.

Proposition 5. *Let $m, n \in \mathbb{N}^*$ and $p \in [1, +\infty[$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that T_f is α -bounded for some $\alpha < m - (1/p)$. Then f is a polynomial of degree less than m .*

Proof. We use the auxiliary function v ; see (4). Since $\partial_1^m(f \circ \lambda v)(x) = \lambda^m f^{(m)}(\lambda x_1)$ on Q , (16) implies that

$$\forall \lambda \geq 1 \quad \left(\int_{-\lambda/2}^{\lambda/2} |f^{(m)}(y)|^p dy \right)^{1/p} \leq c \lambda^{\alpha-m+(1/p)}.$$

Taking $\lambda \rightarrow +\infty$, we obtain $f^{(m)} = 0$; hence f is a polynomial of degree less than m . $\square$

Remark 4. Proposition 5 should be completed by the study of acting polynomials. It is known that $L_\infty \cap \dot{W}_p^m(\mathbb{R}^n)$ is a Banach algebra; hence any polynomial acts on it. On the contrary, the affine functions are the only acting polynomials in $\dot{W}_p^m \cap \dot{W}_{mp}^1(\mathbb{R}^n)$, with the *exception* of $m = n$ and $p = 1$; see Remark 1 and Corollary 1 in Section 5.

3. A SUPERPOSITION THEOREM FOR VECTOR VALUED SPACES

The question of finding the sharp condition on $f : \mathbb{R}^k \rightarrow \mathbb{R}$, with $k > 1$, such that T_f takes $\dot{W}_p^m \cap \dot{W}_{mp}^1(\mathbb{R}^n, \mathbb{R}^k)$ to $\dot{W}_p^m \cap \dot{W}_{mp}^1(\mathbb{R}^n)$ seems far from being solved. The mere case of complex valued functions, i.e. $k = 2$, is not clear at all. As a first step towards a full solution, we propose a sufficient condition on f , directly derived from the case $k = 1$. Indeed we assume that the m -th order derivatives of f are locally uniformly in $L_p(\mathbb{R})$ w.r.t. each variable, with a strong form of uniformity w.r.t. the others.

3.1. Sobolev spaces of mixed type. Let $p \in [1, +\infty[$ and an integer $k > 1$. Let us denote by $E_p(\mathbb{R}^k)$ the space of measurable functions f on $\mathbb{R}^k$ such that

$$\|f\|_{E_p} := \sum_{j=1}^k \left(\sup_{a \in \mathbb{R}} \int_a^{a+1} \|\sigma_j f(\cdot, t)\|_{L_\infty(\mathbb{R}^{k-1})}^p dt \right)^{1/p} < +\infty.$$

The space $E_p(\mathbb{R}^k)$ is clearly related to the space $L_{\infty,p}(\mathbb{R}^k)$ of Benedek and Panzone [4]. The following continuous imbeddings hold:

$$(20) \quad L_\infty(\mathbb{R}^k) \hookrightarrow E_p(\mathbb{R}^k) \hookrightarrow E_1(\mathbb{R}^k) \hookrightarrow L_1(\mathbb{R}^k)_{loc}.$$

As a consequence, $E_p(\mathbb{R}^k)$ turns out to be a translation invariant B.D.S. in $\mathbb{R}^k$.

Proposition 6. *For any $f \in E_p(\mathbb{R}^k)$ and any $u \in C_c(\mathbb{R}^k)$, $u * f$ is a bounded continuous function.*

Proof. By (20), it suffices to assume that $p = 1$. If u is supported in $B \times [a, b]$ for some closed ball B of $\mathbb{R}^{k-1}$, then

$$\forall (x', x_k) \in \mathbb{R}^{k-1} \times \mathbb{R} \quad |(u * f)(x', x_k)| \leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{k-1}} |u(y', x_k - y_k)| dy' \right) \|f(\cdot, y_k)\|_\infty dy_k,$$

hence the existence of $c = c(u) > 0$ such that

$$|(u * f)(x', x_k)| \leq c \int_{x_k-b}^{x_k-a} \|f(\cdot, y_k)\|_\infty dy_k \leq c([b-a] + 1) \|f\|_{E_1}$$

for all $(x', x_k) \in \mathbb{R}^{k-1} \times \mathbb{R}$. The continuity of $u * f$ follows by standard arguments. $\square$

Proposition 7. *If $m > 1$ is an integer, then the following continuous imbedding holds:*

$$\dot{W}_{E_p}^m \cap L_\infty(\mathbb{R}^k) \hookrightarrow W_\infty^{m-1}(\mathbb{R}^k).$$

Proof. We assume again that $p = 1$. We start with the continuous imbedding

$$(21) \quad \dot{W}_{E_1}^m \cap L_\infty(\mathbb{R}) \hookrightarrow \dot{W}_\infty^1(\mathbb{R});$$

see (12). Now we prove, by induction on $N = |\beta| < m$, that

$$\|f^{(\beta)}\|_\infty \leq c(\|f\|_\infty + \|f\|_{\dot{W}_{E_1}^m}).$$

We deal with smooth functions, since the general case will follow by a standard approximation procedure.

The property holds trivially for $\beta = 0$. Let us assume that it holds for every β such that $|\beta| = N - 1$, and prove that, for such a β and any $j = 1, \dots, k$, the same holds for $\partial_j(f^{(\beta)})$. Let us consider $g_x := \sigma_j(f^{(\beta)})(x, \cdot)$ for all $x \in \mathbb{R}^{k-1}$. Applying imbedding (21) to g_x , with m replaced by $m - N + 1$, we obtain

$$\|g'_x\|_{L_\infty(\mathbb{R})} \leq c \left(\|g_x\|_{L_\infty(\mathbb{R})} + \sup_{a \in \mathbb{R}} \int_a^{a+1} |g_x^{(m-N+1)}(t)| dt \right),$$

for all $x \in \mathbb{R}^{k-1}$. The above right hand side is clearly estimated by

$$\|f^{(\beta)}\|_\infty + \|f\|_{\dot{W}_{E_1}^m}.$$

Now applying the induction assumption, and taking the supremum w.r.t. x , we obtain

$$\|\partial_j f^{(\beta)}\|_\infty \leq c(\|f\|_\infty + \|f\|_{\dot{W}_{E_1}^m}).$$

This ends the proof. $\square$

Then combining the two preceding propositions, we deduce the following.

Proposition 8. *For any $f \in \dot{W}_{E_p}^m \cap \dot{W}_\infty^1(\mathbb{R}^k)$ and any $u \in C_c(\mathbb{R}^k)$, $u * f$ is an admissible function of order m .*

Proposition 9. *Let $f \in \dot{W}_{E_p}^m(\mathbb{R}^k)$.*

- *If $m = 1$, f is locally essentially bounded.*
- *If $m = 2$, f is locally Lipschitz continuous.*
- *If $m > 2$, f is continuously differentiable.*

Proof. Only the first statement needs a proof. We assume again that $p = 1$. By imbedding (20), and by classical results, see e.g. Maz'ja [17, 1.1.1, 1.1.2], we know that f is locally integrable and that

$$(22) \quad (\tau_{he_j} f - f)(x_1, \dots, x_k) = \int_{x_j}^{x_j+h} \partial_j f(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_k) dt,$$

for a.e. $x \in \mathbb{R}^k$, for all $h \in \mathbb{R}$, for all $j = 1, \dots, k$. To any $h = (h_1, \dots, h_k) \in \mathbb{R}^k$, we associate the sequence $(\bar{h}_j)_{0 \leq j \leq k}$ of vectors defined by $\bar{h}_0 = 0$ and $\bar{h}_j = \bar{h}_{j-1} + h_j e_j$ for $j > 0$. By (22), we deduce

$$\|\tau_h f - f\|_\infty \leq \sum_{j=1}^k \|\tau_{\bar{h}_j} f - \tau_{\bar{h}_{j-1}} f\|_\infty \leq (|h| + 1) \sum_{j=1}^k \|\partial_j f\|_{E_1}.$$

To conclude that f is locally essentially bounded, we combine the above inequality with the following lemma, whose easy proof is left to the reader. $\square$

Lemma 1. *Let f be a locally integrable function on $\mathbb{R}^k$ such that $\sup_{|h| \leq R} \|\tau_h f - f\|_\infty$ is finite for all $R > 0$. Then f is locally essentially bounded.*

Remark 5. By exploiting the same ideas as in the above proof, we could show that any function of $\dot{W}_{E_p}^1(\mathbb{R}^k)$ is locally Hölder continuous of exponent $1/p'$ if $p > 1$, and a.e. continuous if $p = 1$.

3.2. Superposition theorem.

Theorem 3. *Let m, k be integers ≥ 2 , $1 < p < +\infty$, $n \in \mathbb{N}^*$.*

- (1) *If $f \in \dot{W}_\infty^1 \cap \dot{W}_{E_p}^m(\mathbb{R}^k)$, then T_f takes $\dot{W}_p^m \cap \dot{W}_{mp}^1(\mathbb{R}^n, \mathbb{R}^k)$ to $\dot{W}_p^m \cap \dot{W}_{mp}^1(\mathbb{R}^n)$.*
- (2) *If f belongs locally to $\dot{W}_{E_p}^m(\mathbb{R}^k)$, then T_f takes $\dot{W}_p^m \cap L_\infty(\mathbb{R}^n, \mathbb{R}^k)$ to $\dot{W}_p^m \cap L_\infty(\mathbb{R}^n)$.*

Proof. It is essentially the same as the corresponding proof for $k = 1$. Thus we just outline a few changes.

In Subsection 2.3, we start from the formula

$$(f \circ g)^{(\alpha)} = \sum c_{\alpha, \beta, \gamma, l} (f^{(\beta)} \circ g) g_{l_1}^{(\gamma_1)} \dots g_{l_s}^{(\gamma_s)},$$

where the parameters satisfy the conditions

$$0 < s = |\beta| \leq m, |\gamma_r| > 0 (r = 1, \dots, s), \sum \gamma_r = \alpha, (l_1, \dots, l_s) \in \{1, \dots, k\}^s,$$

and $g = (g_1, \dots, g_k)$, with g_l real valued for $l = 1, \dots, k$. Thus we are reduced to estimating

$$\left(\int_{\mathbb{R}^n} |f^{(\beta)} \circ g|^p |\partial_j g_l|^{mp} dx \right)^{1/p},$$

for $|\beta| = m$, $j = 1, \dots, n$, $l = 1, \dots, k$. The functions Φ_q will be replaced by

$$\forall y \in \mathbb{R} \quad \Phi_{l, \beta, q}(y) := u(y - q) \|(\sigma_l f^{(\beta)})(\cdot, y)\|_{L_\infty(\mathbb{R}^{k-1})}^p.$$

The rest is unchanged, up to replacement of g by its coordinates g_l in some places, and redefinition of $\rho(f)$ using the space $E_p(\mathbb{R}^k)$.

We turn to Subsection 2.4. In Step 1, the estimate (12) is replaced by Proposition 7. In Step 3, we use Proposition 8 to ensure that the approximating functions f_ν are admissible.

By Propositions 7 and 9, if f belongs locally to $\dot{W}_{E_p}^m(\mathbb{R}^k)$, then $uf \in \dot{W}_\infty^1 \cap \dot{W}_{E_p}^m(\mathbb{R}^k)$ for all $u \in \mathcal{D}(\mathbb{R}^k)$. Thus the proof of the second statement of Theorem 3 is similar to that of Theorem 2. $\square$

Remark 6. The validity of Theorem 3 in case $p = 1$ is an open question. Indeed it is not clear that the techniques used in Step 3 of Subsection 2.4 (De La Vallée-Poussin Lemma) work for $k > 1$. From estimate (15) we can only deduce that $(f \circ g)^{(\alpha)}$ is a bounded Borel measure for all $|\alpha| = m$. In case f is a limit of admissible functions of order m in $\dot{W}_\infty^1 \cap \dot{W}_{E_1}^m(\mathbb{R}^k)$, we recover $(f \circ g)^{(\alpha)} \in L_1$ since L_1 is closed into the space of bounded Borel measures on $\mathbb{R}^n$.

Remark 7. Proposition 4 has a counterpart for $k > 1$: the assumption $f \in \dot{W}_p^m(\mathbb{R})$ has to be replaced by

$$\sum_{|\alpha|=m} \sum_{j=1}^k \int_{\mathbb{R}} \|\sigma_j f^{(\alpha)}(\cdot, t)\|_{L_\infty(\mathbb{R}^{k-1})}^p dt < +\infty.$$

Proposition 5 works as well, with the conclusion that f is a polynomial of degree less than m , in *each* variable.

4. NECESSARY CONDITIONS FOR SUPERPOSITION

All along Section 4, “closed ball” means nondegenerate closed ball, i.e. closed ball of nonzero radius.

4.1. Statement of the main results.

Theorem 4. *Let m, k, n in $\mathbb{N}^*$, $p \in [1, +\infty[$, and $f : \mathbb{R}^k \rightarrow \mathbb{R}$.*

- (1) *In the subcritical case, if T_f takes $\dot{W}_{mp}^1 \cap \dot{W}_p^m(B, \mathbb{R}^k)$ to $\dot{W}_p^m(\mathbb{R}^n)$ for a closed ball B of $\mathbb{R}^n$, then f is uniformly Lipschitz continuous.*
- (2) *In case $m > n/p$, if T_f takes $\dot{W}_{mp}^1 \cap \dot{W}_p^m(\mathbb{R}^n, \mathbb{R}^k)$ to $\dot{W}_p^m(\mathbb{R}^n)$, then f is uniformly Lipschitz continuous.*
- (3) *If T_f takes $\dot{W}_n^1 \cap \dot{W}_1^n(\mathbb{R}^n, \mathbb{R}^k)$ to $\dot{W}_1^n(\mathbb{R}^n)$, then f is locally Lipschitz continuous.*

Theorem 5. *Let m, k, n in $\mathbb{N}^*$, $p \in [1, +\infty[$, and $f : \mathbb{R}^k \rightarrow \mathbb{R}$.*

- (1) *In the subcritical case, if T_f takes $\dot{W}_{mp}^1 \cap \dot{W}_p^m(B, \mathbb{R}^k)$ to $\dot{W}_{mp}^1(\mathbb{R}^n)$ for a closed ball B of $\mathbb{R}^n$, then f is uniformly Hölder continuous of exponent $1/m$, in the sense that*

$$\sup \left\{ \frac{|f(b) - f(b')|}{|b - b'|^{1/m}} : b, b' \in \mathbb{R}^k, 0 < |b - b'| \leq 1 \right\} < +\infty.$$

- (2) *In case $m > n/p$, if T_f takes $\dot{W}_{mp}^1 \cap \dot{W}_p^m(\mathbb{R}^n, \mathbb{R}^k)$ to $\dot{W}_{mp}^1(\mathbb{R}^n)$, then f is uniformly Hölder continuous of exponent $1/m$.*
- (3) *If T_f takes $\dot{W}_n^1 \cap \dot{W}_1^n(\mathbb{R}^n, \mathbb{R}^k)$ to $\dot{W}_n^1(\mathbb{R}^n)$, then f is locally Hölder continuous of exponent $1/n$.*

Remark 8. The space $\dot{W}_n^1 \cap \dot{W}_1^n(\mathbb{R}^n)$ is imbedded in $C_b(\mathbb{R}^n)$; see Proposition 14 in Section 5. Thus we cannot expect that the uniform Lipschitz or Hölder continuity will be necessary.

Corollary 1. *Let m, k, n in $\mathbb{N}^*$, $p \in [1, +\infty[$, with the exception of $m = n$ and $p = 1$. Let $f : \mathbb{R}^k \rightarrow \mathbb{R}$ be a polynomial.*

- (1) *If T_f takes $\dot{W}_{mp}^1 \cap \dot{W}_p^m(\mathbb{R}^n, \mathbb{R}^k)$ to $\dot{W}_p^m(\mathbb{R}^n)$ or $\dot{W}_{mp}^1(\mathbb{R}^n)$, then f is an affine function.*
- (2) *In the subcritical case, if T_f takes $\dot{W}_{mp}^1 \cap \dot{W}_p^m(B, \mathbb{R}^k)$ to $\dot{W}_p^m(\mathbb{R}^n)$ or $\dot{W}_{mp}^1(\mathbb{R}^n)$, for a closed ball B of $\mathbb{R}^n$, then f is an affine function.*

Theorem 6. *Let m, k, n in $\mathbb{N}^*$, $p \in [1, +\infty[$, and $f : \mathbb{R} \rightarrow \mathbb{R}$. If one of the following conditions holds:*

- *(m, n, p) is subcritical and T_f takes $\dot{W}_{mp}^1 \cap \dot{W}_p^m(B)$ to $\dot{W}_p^m(\mathbb{R}^n)$, for a closed ball B of $\mathbb{R}^n$,*
- *$m > n/p$ and T_f takes $\dot{W}_{mp}^1 \cap \dot{W}_p^m(\mathbb{R}^n)$ to $\dot{W}_p^m(\mathbb{R}^n)$,*

then $f \in \dot{W}_{E_p}^m(\mathbb{R})$.

The remainder of Section 4 is organized as follows. We first state the fundamental lemmas, which say that all superposition operators are bounded in some sense. Then we recall some more technical lemmas. Finally we give the proof of the main results.

4.2. Basic lemmas. The following heuristic principle is a convenient tool to obtain necessary conditions: *if the superposition operator T_f is defined as a mapping from one function space to another, then T_f is a bounded mapping.* Applications of this principle appear in Katznelson [15, chap. VIII], Igari [14], and in our former papers [7, 8, 9]. In the case of the Adams-Frazier spaces, we have the following results.

Lemma 2. *Let m, k, n in $\mathbb{N}^*$ and $p \in [1, +\infty[$. Let $f : \mathbb{R}^k \rightarrow \mathbb{R}$ be such that*

$$T_f \left(\dot{W}_{mp}^1 \cap \dot{W}_p^m(\mathbb{R}^n, \mathbb{R}^k) \right) \subset \dot{W}_p^m(\mathbb{R}^n).$$

Then there exists $r > 0$ such that

$$\sup\{\|f \circ g\|_{\dot{W}_p^m} : g \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^k), \|g\| \leq r\} < +\infty.$$

Proof. Since the conclusion of the lemma holds trivially for a constant function f , we can assume that $f(0) = 0$. Arguing by contradiction, we obtain a sequence $(h_j)_{j \geq 0}$ in $\mathcal{D}(\mathbb{R}^n, \mathbb{R}^k)$ such that $\|h_j\| \leq 2^{-j}$ and

$$\lim_{j \rightarrow +\infty} \|f \circ h_j\|_{\dot{W}_p^m} = +\infty.$$

Let us denote by B_j a closed ball which includes the support of h_j , and introduce a sequence $(a_j)_{j \geq 0}$ in $\mathbb{R}^n$ in such a way that the balls $B_j + a_j$ are pairwise disjoint. Then $g := \sum_{j \geq 0} \tau_{a_j} h_j$ belongs to $\dot{W}_{mp}^1 \cap \dot{W}_p^m(\mathbb{R}^n, \mathbb{R}^k)$. By construction of g and by the assumption that $f(0) = 0$, we have $(f \circ g)(x) = (f \circ \tau_{a_j} h_j)(x)$ for all $x \in B_j + a_j$. Hence

$$\|f \circ g\|_{\dot{W}_p^m} \geq \sum_{|\alpha|=m} \left(\int_{B_j + a_j} |(f \circ g)^{(\alpha)}(x)|^p dx \right)^{1/p} = \|f \circ h_j\|_{\dot{W}_p^m},$$

for all $j \geq 0$, a contradiction. $\square$

Lemma 3. *Let m, k, n in $\mathbb{N}^*$, $p \in [1, +\infty[$, and $f : \mathbb{R}^k \rightarrow \mathbb{R}$. For all closed balls B of $\mathbb{R}^n$, we consider the following two properties:*

$$(\mathcal{H}_B) \quad T_f \left(\dot{W}_{mp}^1 \cap \dot{W}_p^m(B, \mathbb{R}^k) \right) \subset \dot{W}_p^m(\mathbb{R}^n),$$

$$(\mathcal{Q}_B) \quad \exists r > 0 \quad \sup\{\|f \circ g\|_{\dot{W}_p^m} : g \in \mathcal{D}(B, \mathbb{R}^k), \|g\| \leq r\} < +\infty.$$

If there exists a closed ball B_0 such that $(\mathcal{H}_{B_0})$ holds, then $(\mathcal{Q}_B)$ holds for all closed balls B .

Proof. We assume again that $f(0) = 0$. Using affine transformations, it is easily seen that $(\mathcal{H}_{B_0}) \Leftrightarrow (\mathcal{H}_{B_1})$ and $(\mathcal{Q}_{B_0}) \Leftrightarrow (\mathcal{Q}_{B_1})$ hold for any closed balls B_0, B_1 . So we assume that $(\mathcal{H}_B)$ holds, where $B = \overline{B}(0, 1)$, and we will prove the existence of some closed ball $K \subset B$ enjoying property $(\mathcal{Q}_K)$.

We consider a family of closed balls $K_j := \overline{B}(a_j, r_j)$ ($j \in \mathbb{N}$) such that the balls $\overline{B}(a_j, 2r_j)$ are pairwise disjoint and included into B . We take

$$\psi_j(x) := \psi(r_j^{-1}(x - a_j)),$$

where ψ is the auxiliary function introduced in (2) and denote by M_j the norm of the operator $g \mapsto \psi_j g$ on the space $\dot{W}_p^m(B)$; see Proposition 13 in Section 5. Arguing by contradiction, we obtain a sequence $(g_j)_{j \geq 0}$ such that $g_j \in \mathcal{D}(K_j, \mathbb{R}^k)$, $\|g_j\| \leq 2^{-j}$ and $\|f \circ g_j\|_{\dot{W}_p^m} \geq j M_j$, for all $j \geq 0$. Let us define $g := \sum_{j \geq 0} g_j$. Then

$g \in \dot{W}_{mp}^1 \cap \dot{W}_p^m(B, \mathbb{R}^k)$ and $\psi_j(f \circ g) = f \circ g_j$. Hence $j \leq \|f \circ g\|_{\dot{W}_p^m}$ for all j , a contradiction. $\square$

Lemma 4. *Let m, k, n in $\mathbb{N}^*$, $p \in [1, +\infty[$, and $f : \mathbb{R}^k \rightarrow \mathbb{R}$. If there exists a closed ball B_0 such that $(\mathcal{H}_{B_0})$ holds, then, for all $a \in \mathbb{R}^k$ and all closed balls B , there exist a nonlinear operator $U_a : \mathcal{D}(\mathbb{R}^n, \mathbb{R}^k) \rightarrow \dot{W}_p^m(\mathbb{R}^n)$ such that*

$$\forall x \in B, \forall g \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^k) \quad U_a(g)(x) = f(a + g(x)) - f(a),$$

and a number $r > 0$ such that

$$(23) \quad \sup\{\|U_a(g)\|_{\dot{W}_p^m} : g \in \mathcal{D}(B, \mathbb{R}^k), \|g\| \leq r\} < +\infty.$$

Proof. Let $a \in \mathbb{R}^k$. Arguing as in the preceding proof, it suffices to assume that $(\mathcal{H}_{B_0})$ holds for $B_0 := \overline{B}(0, 2)$ and to obtain some closed ball $B \subset \overline{B}(0, 1)$ with the desired properties. Let us define

$$U_a(g)(x) := f(\psi(x)(a + g(x))) - f(\psi(x)a).$$

By property $(\mathcal{H}_{B_0})$, and by definition of ψ , it follows that U_a takes $\mathcal{D}(\mathbb{R}^n, \mathbb{R}^k)$ to $\dot{W}_p^m(\mathbb{R}^n)$ and that

$$\forall x \in \overline{B}(0, 1), \forall g \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^k) \quad U_a(g)(x) = f(a + g(x)) - f(a).$$

Now arguing by contradiction, exactly as in the preceding proof, we obtain a ball $B \subset \overline{B}(0, 1)$ and a number $r > 0$ such that (23) holds. $\square$

4.3. Miscellaneous estimates.

4.3.1. *Estimates of orthogonal sums.* We will sometimes use the following result, whose proof is immediate.

Proposition 10. *There exist $c_1, c_2 > 0$, depending only on n, m, p such that*

$$c_1 \left(\sum_j \|g_j\|_{\dot{W}_p^m(\mathbb{R}^n)}^p \right)^{1/p} \leq \left\| \sum_j g_j \right\|_{\dot{W}_p^m(\mathbb{R}^n)} \leq c_2 \left(\sum_j \|g_j\|_{\dot{W}_p^m(\mathbb{R}^n)}^p \right)^{1/p},$$

for all families (g_j) of distributions such that $\text{supp } g_j \cap \text{supp } g_{j'}$ is negligible for $j \neq j'$. The same property holds in $W_p^m(\mathbb{R}^n)$.

4.3.2. *Sequences of cutoff functions.* The equality

$$(24) \quad \forall \lambda > 0 \quad \|f(\lambda(\cdot))\|_{\dot{W}_p^m(\mathbb{R}^n)} = \lambda^{m-(n/p)} \|f\|_{\dot{W}_p^m(\mathbb{R}^n)}$$

results in the existence of cutoff functions whose norms tend to 0. More precisely we define the sequence $(\varphi_\nu)_{\nu \geq 0}$ in $\mathcal{D}(\mathbb{R}^n)$ by

- $\varphi_\nu(x) := \varphi(2^\nu x)$ in case $m < n/p$,
- $\varphi_\nu(x) := \varphi(2^{-\nu} x)$ in case $m > n/p$,
- $\varphi_\nu(x) := \frac{1}{\nu+1} \sum_{j=0}^{\nu} \varphi(2^j x)$ in case $m = n/p$ and $p > 1$,

where φ is the auxiliary function introduced in (1). We claim that $\lim_{\nu \rightarrow +\infty} \|\varphi_\nu\|_{\dot{W}_p^m} = 0$. In case $m \neq n/p$, this follows immediately by (24). Now assume that $m = n/p$

and $p > 1$. Let us introduce $S_l := 2^{-l+1}Q \setminus 2^{-l}Q$, $l = 0, \dots, \nu-1$, and $S_\nu := 2^{1-\nu}Q$. For all $|\alpha| = m$, it follows that

$$\forall x \in S_l \quad |\varphi_\nu^{(\alpha)}(x)| = \frac{1}{\nu+1} \left| \sum_{j=0}^l 2^{jm} \varphi^{(\alpha)}(2^j x) \right| \leq \frac{c_1}{\nu+1} 2^{lm}.$$

Hence

$$\|\varphi_\nu^{(\alpha)}\|_p^p = \sum_{l=0}^{\nu} \int_{S_l} |\varphi_\nu^{(\alpha)}(x)|^p dx \leq \frac{c_2}{(\nu+1)^p} \sum_{l=0}^{\nu} 2^{lmp} 2^{-ln} = \frac{c_2}{(\nu+1)^{p-1}},$$

the desired property.

Remark 9. In case $m = n/p$, the existence of a sequence such as (φ_ν) could be deduced from a capacity estimate of Maz'ya [17, prop. 2, p. 393]. The sequence (φ_ν) has similar properties in Besov and Lizorkin-Triebel spaces; see [20, lem. 2.3.1/7, p. 57]. Such a sequence does not exist in $\dot{W}_1^n(\mathbb{R}^n)$; see Lemma 6 in Section 5.

4.3.3. A convenient Besov space. Together with the space $\dot{W}_p^m$, it is useful to consider the homogeneous Besov space defined as follows.

Definition 1. Let $m \in \mathbb{N}^*$. A distribution u belongs to $\dot{B}_{p,\infty}^m(\mathbb{R}^n)$ if

$$\|u\|_{\dot{B}_{p,\infty}^m} := \sup_{h \neq 0} \frac{1}{|h|^m} \left(\int_{\mathbb{R}^n} |\Delta_h^{m+1} u(x)|^p dx \right)^{1/p} < +\infty.$$

The continuous imbedding $\dot{W}_p^m \hookrightarrow \dot{B}_{p,\infty}^m$ holds; see e.g. [5, formula 4.16, p. 336].

4.4. Proofs of the main results. We follow the same method as in [7, 8]; see also [20, thm. 5.3.1/1, p. 292]. According to an idea of S. Igari [14], we construct a comb-shaped function. By testing T_f on this function, and adjusting the parameters, we obtain the various desired properties for f .

4.4.1. Construction of the comb-shaped function. Let $\varkappa := \frac{1}{2m+3}$ and $A_N := \mathbb{Z}^n \cap [-N, N]^n$, for all $N \in \mathbb{N}^*$. We consider distinct elements b, b' of $\mathbb{R}^k$, and parameters $N \in \mathbb{N}^*$, $\nu \in \mathbb{N}$, $r > 0$, whose values will be established below depending on b, b' . Our test function will be defined by

$$(25) \quad \forall x \in \mathbb{R}^n \quad g(x) := \sum_{\mu \in A_N} \varphi \left(\frac{1}{\varkappa} \left(\frac{x}{r} - \mu \right) \right) (b' - b) + \varphi_\nu(x) b,$$

where φ is the auxiliary function introduced in (1), and φ_ν was introduced in Subsection 4.3.2. The first condition on the parameters will be

$$(26) \quad r(2N+1) \leq 2^{\varepsilon\nu},$$

where $\varepsilon = -1$ if $m \leq n/p$ and $\varepsilon = +1$ if $m > n/p$. By inequality $\varkappa < 1/2$ and by condition (26), we deduce that the cubes $r(2\varkappa Q + \mu)$, $\mu \in \mathbb{Z}^n$, are pairwise disjoint, and that $r(2\varkappa Q + \mu) \subset r(Q + \mu) \subset 2^{\varepsilon\nu}Q$, for all $\mu \in A_N$. Hence

$$(27) \quad g(x) = b' \quad \text{if } x \in r(\varkappa Q + \mu) \text{ for some } \mu \in A_N,$$

$$(28) \quad g(x) = b \quad \text{if } x \in 2^{\varepsilon\nu}Q \setminus \bigcup_{\mu \in A_N} r(2\varkappa Q + \mu).$$

By Proposition 10 and equality (24), we obtain

$$(29) \quad \left\| \sum_{\mu \in A_N} \varphi\left(\frac{1}{\mathcal{X}}\left(\frac{\cdot}{r} - \mu\right)\right) \right\| \leq c_1 \left(r^{(n/p)-m} N^{n/p} + r^{\frac{n}{pm}-1} N^{\frac{n}{pm}} \right).$$

4.4.2. *Uniform Lipschitz continuity.* Now we assume that f satisfies the acting properties of statements 1 and 2 in Theorem 4. By Lemmas 2 and 3, we can find constants δ_1, δ_2 such that $\|f \circ h\|_{\dot{W}_p^m} \leq \delta_2$ for all $h \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^k)$ such that $\|h\| \leq \delta_1$, and h supported in $2Q$ if $m \leq n/p$. In order to apply this property to $h = g$, we need the following inequalities:

$$(30) \quad |b| \|\varphi_\nu\| \leq \frac{\delta_1}{2},$$

$$(31) \quad 1 \leq \frac{\delta_1}{5c_1|b-b'|} \leq r^{(n/p)-m} N^{n/p} \leq \frac{\delta_1}{4c_1|b-b'|}.$$

Now we discuss the choice of ν, N, r with respect to b, b' , such that conditions (26), (30) and (31) hold.

Step 1: Case $m < n/p$.

By definition of φ_ν , see Section 4.3.2, we can choose $\nu = \nu(b) \in \mathbb{N}^*$ such that (30) holds. Then we take

$$r := \left(\frac{\delta_1}{4c_1|b-b'|} N^{-n/p} \right)^{\frac{p}{n-mp}},$$

which ensures condition (31). Since

$$rN = \left(\frac{\delta_1}{4c_1|b-b'|} \right)^{\frac{p}{n-mp}} N^{\frac{mp}{pm-n}},$$

it is possible to choose $N = N(b, b') \in \mathbb{N}^*$ such that (26) holds.

Step 2: Case $m = n/p$.

We choose ν as in the preceding step. Then we take $N \in \mathbb{N}^*$ such that (31) holds. Such a choice is possible if $|b-b'| \leq c_2$, where $c_2 > 0$ depends only on p, n, δ_1 . Then we put

$$r := \frac{2^{-\nu}}{2N+1}.$$

Step 3: Case $m > n/p$.

We take $N := 1$,

$$r := \left(\frac{\delta_1}{4c_1|b-b'|} \right)^{\frac{p}{n-mp}},$$

and ν sufficiently large to obtain both (30) and (26).

In the three cases, by combining inequalities (29), (30) and (31), we deduce $\|g\| \leq \delta_1$. Using the imbedding of $\dot{W}_p^m$ in the Besov space, we deduce

$$\|f \circ g\|_{\dot{B}_{p,\infty}^m} \leq \delta_3$$

for some constant δ_3 depending only on δ_2, m, n, p . For any $x \in r(\varkappa Q^+ + \mu)$, we have

$$\begin{aligned} \forall j = 0, \dots, m+1 \quad & x + jr\varkappa e_1 \in r(Q + \mu) \subset 2^{\varepsilon\nu}Q, \\ \forall j = 1, \dots, m+1 \quad & x + jr\varkappa e_1 \notin \bigcup_{\mu' \in A_N} r(2\varkappa Q + \mu'). \end{aligned}$$

If $x \in r(\varkappa Q^+ + \mu)$, equalities (27) and (28), and formula (4.1) of Bennett and Sharpley [5, p. 332] for iterated difference operators imply that

$$|\Delta_{r\varkappa e_1}^{m+1}(f \circ g)(x)| = |f(b') - f(b)|.$$

Then

$$\begin{aligned} \delta_3 &\geq c_3 r^{-m} \left(\sum_{\mu \in A_N} \int_{r(\varkappa Q^+ + \mu)} |\Delta_{r\varkappa e_1}^{m+1}(f \circ g)(x)|^p dx \right)^{1/p} \\ &\geq c_4 |f(b') - f(b)| N^{n/p} r^{(n/p)-m}. \end{aligned}$$

By (31) we obtain the existence of two constants c_5, c_6 , depending only on f and on the various parameters, such that $|f(b') - f(b)| \leq c_6 |b - b'|$ for all $b, b' \in \mathbb{R}^k$ satisfying $|b' - b| \leq c_5$. Thus we have proved that f is uniformly Lipschitz continuous.

4.4.3. Local Lipschitz continuity. Here we assume that $n = m, p = 1$, and that f satisfies the acting properties of statement 3 in Theorem 4. Let us fix $a \in \mathbb{R}^k$. By Lemma 4, we can find constants δ_1, δ_2 such that $\|U_a(h)\|_{\dot{W}_1^1} \leq \delta_2$ for all $h \in \mathcal{D}(2Q, \mathbb{R}^k)$ such that $\|h\| \leq \delta_1$. We test U_a on the function g , defined by (25), with $\nu := 1$ and $r := 1/(3N)$. Then condition (31) is fulfilled by some $N \in \mathbb{N}^*$, as soon as $|b - b'| \leq c_2$, where $c_2 > 0$ depends only on f and on the various parameters. If $x \in r(\varkappa Q^+ + \mu)$, we have

$$|\Delta_{r\varkappa e_1}^{n+1}(U_a(g))(x)| = |f(a + b') - f(a + b)|,$$

exactly as in the preceding subsection. Then

$$\delta_3 \geq c_4 |f(a + b') - f(a + b)| N^n.$$

By (30) and (31), we obtain the existence of constants c_5, c_6, c_7 , depending only on f and on the various parameters, such that $|f(a + b') - f(a + b)| \leq c_6 |b - b'|$ for all $b, b' \in \mathbb{R}^k$ satisfying $|b' - b| \leq c_5$ and $|b| \leq c_7$. Thus we have proved that f is Lipschitz continuous in a neighborhood of a .

4.4.4. Proof of Theorem 5. We proceed exactly as in the proof of Theorem 4. The end of Subsection 4.4.2 becomes

$$\delta_3 \geq \|f \circ g\|_{\dot{B}_{pm, \infty}^1} \geq c_4 |f(b') - f(b)| N^{n/pm} r^{(n/pm)-1}.$$

The desired uniform Hölder continuity follows by (31). A similar modification holds in Subsection 4.4.3.

4.4.5. *Proof of Theorem 6.* We use the same notation and ideas as in the proof of Theorem 4. In particular, the estimate (29) holds with φ replaced by the function v introduced in (4), and $\kappa := 1/2$. If b is any real number, we set

$$\forall x \in \mathbb{R}^n \quad g_b(x) := 2\lambda \sum_{\mu \in A_N} v\left(2\left(\frac{x}{r} - \mu\right)\right) + \varphi_\nu(x)b,$$

where $\lambda := \delta_1/(8c_1)$ and $\nu = \nu(b) \in \mathbb{N}^*$ is chosen such that (30) holds. It remains to choose r and N satisfying (26) and

$$(32) \quad r^{(n/p)-m} N^{n/p} = 1.$$

In case $m < n/p$, (32) gives r as a function of N . Then rN appears as a negative power of N ; hence (26) is true if N is sufficiently large. In case $m = n/p$, the condition (32) reads as $N = 1$; then

$$r := \frac{2^{-\nu}}{3}$$

implies (26). In case $m > n/p$, we take $r := 1$, $N := 1$, and $\nu \geq 2$. In all cases, we conclude that $\|g_b\| \leq \delta_1$. Hence we have

$$(33) \quad \|f \circ g_b\|_{\dot{W}_p^m} \leq \delta_2.$$

If $x \in r(\frac{1}{2}Q + \mu)$, with $\mu \in A_N$, it follows that

$$(f \circ g_b)(x) = f(4\lambda(r^{-1}x_1 - \mu_1) + b).$$

Then using (32), (33), and a change of variables, we find a constant $c > 0$ such that

$$\forall b \in \mathbb{R} \quad \int_{b-\lambda}^{b+\lambda} |f^{(m)}(y)|^p dy \leq c,$$

which means that $f^{(m)} \in E_p(\mathbb{R})$.

5. SUPERPOSITION IN $\dot{W}_p^m$

Except in some very particular cases, there is no nontrivial superposition operator in $\dot{W}_p^m(\mathbb{R}^n)$. This can be proved by using polynomials which belong to $\dot{W}_p^m$; see Proposition 11. Of course, this is not a fair way of doing things. We have reasons to think that *degeneracy holds even if we exclude polynomials*. To make this idea more clear, let us recall that a *realization* of $\dot{W}_p^m(\mathbb{R}^n)$ is a linear subspace E such that $\dot{W}_p^m(\mathbb{R}^n) = \mathcal{P}_{m-1} \oplus E$; see [6] for generalities on realizations. Then, we can formulate the following:

Conjecture. *Assume that*

$$(34) \quad m > 1 + (1/p) \quad \text{and} \quad m \neq n/p.$$

If E is any realization of $\dot{W}_p^m(\mathbb{R}^n)$, then the only functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that T_f takes E to $\dot{W}_p^m(\mathbb{R}^n)$ are the affine functions.

Of course $m = 1 + (1/p)$ means: $m = 2$ and $p = 1$. But we prefer the above formulation, which prepares the groundwork for an extension to Sobolev spaces of fractional smoothness order.

5.1. A first degeneracy result.

Proposition 11. *Let m, n, k be integers such that $m > 1$, $1 \leq k < n$, and $p \in [1, +\infty[$. If $f : \mathbb{R}^k \rightarrow \mathbb{R}$ acts by superposition from $\dot{W}_p^m(\mathbb{R}^n, \mathbb{R}^k)$ to $\dot{W}_p^m(\mathbb{R}^n)$, then f is an affine function.*

Proof. Step 1 : Reduction to polynomials. Since $n \geq k$, we can consider the function $g(x_1, \dots, x_n) := f(x_1, \dots, x_k)$. By assumption $m > 1$, it follows that all polynomials of degree one belong trivially to $\dot{W}_p^m(\mathbb{R}^n)$. Hence $g \in \dot{W}_p^m(\mathbb{R}^n)$. By $n > k$, it follows that $\partial_j^m f = 0$, for $j = 1, \dots, k$, and that f is a polynomial.

Step 2 : Reduction to affine functions. Assume that a polynomial $f : \mathbb{R}^k \rightarrow \mathbb{R}$, of degree > 1 , acts from $\dot{W}_p^m(\mathbb{R}^n, \mathbb{R}^k)$ to $\dot{W}_p^m(\mathbb{R}^n)$. We may lower the degree of f by observing that $\Delta_a f$ acts as well as f , for all $a \in \mathbb{R}^k$. Thus we are reduced to the case where f is a nonzero quadratic form. We choose $\gamma \in \mathbb{R}^k$ such that $f(\gamma) = \pm 1$, and we define $g(x) := x_1^{m-1} \gamma$. Then $g \in \dot{W}_p^m(\mathbb{R}^n, \mathbb{R}^k)$; hence the function $x \mapsto x_1^{2(m-1)}$ belongs to $\dot{W}_p^m(\mathbb{R}^n)$, a contradiction. $\square$

Remark 10. It is well known [16] that any Lipschitz continuous function on $\mathbb{R}^k$ acts by superposition from $\dot{W}_p^1(\mathbb{R}^n, \mathbb{R}^k)$ to $\dot{W}_p^1(\mathbb{R}^n)$.

5.2. Homogeneous Sobolev spaces and their realizations.

5.2.1. *Homogeneous versus inhomogeneous spaces.* If we want to consider $\dot{W}_p^m$ as a normed space, we clearly need to take the quotient by the subspace $\mathcal{P}_{m-1}$. Then the following properties are well known; see for instance [17, 1.1.13, p. 24].

Proposition 12. *Let $m, n \in \mathbb{N}^*$, $p \in [1, +\infty]$.*

- (1) $\dot{W}_p^m(\mathbb{R}^n)$ is imbedded into $L_{p,loc} \cap \mathcal{S}'(\mathbb{R}^n)$.
- (2) The quotient space of $\dot{W}_p^m(\mathbb{R}^n)$ by $\mathcal{P}_{m-1}$ is a Banach space for the norm $\| - \|_{\dot{W}_p^m}$.
- (3) If $(u_k)_{k \geq 0}$ is a sequence in $\dot{W}_p^m(\mathbb{R}^n)$ such that

$$\lim_{k \rightarrow +\infty} \|u_k - u\|_{\dot{W}_p^m} = 0,$$

for some $u \in \dot{W}_p^m(\mathbb{R}^n)$, there exists a sequence $(q_k)_{k \geq 0}$ in $\mathcal{P}_{m-1}$ such that $\lim_{k \rightarrow +\infty} (u_k - q_k) = u$ in $L_p(\mathbb{R}^n)_{loc}$.

The difficulty with polynomials can be avoided by dealing with functions supported by proper subsets of $\mathbb{R}^n$.

Proposition 13. *If F is a proper closed subset of $\mathbb{R}^n$, then $\dot{W}_p^m(F)$ is a B.D.S., and a $\mathcal{D}(\mathbb{R}^n)$ -module, for the norm $\| - \|_{\dot{W}_p^m}$. If F is compact, then $\dot{W}_p^m(F)$ coincides with the space of distributions of $W_p^m(\mathbb{R}^n)$ with support in F , and the norms $\| - \|_{\dot{W}_p^m}$ and $\| - \|_{W_p^m}$ are equivalent on $\dot{W}_p^m(F)$.*

Proof. Let (u_k) be a Cauchy sequence in $\dot{W}_p^m(F)$, for the norm $\| - \|_{\dot{W}_p^m}$. By Proposition 12, there exist $u \in \dot{W}_p^m(\mathbb{R}^n)$ and a sequence (q_k) in $\mathcal{P}_{m-1}$ such that $\lim_{k \rightarrow +\infty} (u_k - q_k) = u$ in $L_p(\mathbb{R}^n)_{loc}$ and $\lim_{k \rightarrow +\infty} \|u_k - u\|_{\dot{W}_p^m} = 0$. Then we have $\lim_{k \rightarrow +\infty} q_k = -u$ in $L_p(\mathbb{R}^n \setminus F)_{loc}$. Since all Fréchet topologies coincide on $\mathcal{P}_{m-1}$, we conclude that the sequence (q_k) converges to a polynomial $q \in \mathcal{P}_{m-1}$ in $\mathcal{D}'(\mathbb{R}^n)$. Then the sequence (u_k) converges to $u + q$ in $\mathcal{D}'(\mathbb{R}^n)$. We conclude that

$u + q$ is supported in F , and that it is the limit of (u_k) in $\dot{W}_p^m(F)$. The module property follows by Proposition 12 and an induction on m . In case F is compact, the imbedding of $\dot{W}_p^m(F)$ into $L_p(\mathbb{R}^n)$ follows by Proposition 12, hence the imbedding into $W_p^m(\mathbb{R}^n)$. The equivalence of norms follows by the Closed Graph Theorem. $\square$

5.2.2. Realizations of homogeneous Sobolev spaces. A homogeneous space has generally an infinite number of realizations. Since $\dot{W}_p^m(\mathbb{R}^n)$ is translation invariant, it is natural to search for a realization with the same property. If it exists, such a translation invariant realization has some chance to be unique, see [6, prop. 2], and then it can be viewed as a canonical realization. Now we construct this canonical realization for $m \leq n/p$ (the situation becomes more difficult for $m > n/p$; see [6, thm. 1]). Two preliminary results will be useful.

Lemma 5. *Let $f \in \dot{W}_p^m(\mathbb{R}^n)$ be such that its Fourier transform has a compact support in $\mathbb{R}^n \setminus \{0\}$. Then f belongs to the inhomogeneous Sobolev space $W_p^m(\mathbb{R}^n)$.*

Proof. Let us introduce a function $u \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ such that $u(\xi) = 1$ on the support of $\hat{f}$. It is classically known that there exists a family $(\omega_\alpha)_{\alpha \in \mathbb{N}^n, |\alpha|=m}$ of C^∞ functions on the unit sphere of $\mathbb{R}^n$ such that

$$\sum_{|\alpha|=m} (i\xi)^\alpha \omega_\alpha(\xi) = 1$$

for all ξ of norm 1. If we define $\eta_\alpha \in \mathcal{S}(\mathbb{R}^n)$ by

$$\widehat{\eta}_\alpha(\xi) := \frac{1}{|\xi|^m} u(\xi) \omega_\alpha\left(\frac{\xi}{|\xi|}\right),$$

it follows that $f = \sum_{|\alpha|=m} \eta_\alpha * f^{(\alpha)}$. Hence $f \in L_p(\mathbb{R}^n)$, the desired property. $\square$

Lemma 6. *Let m, m_1, n be integers, and $p \in [1, +\infty[$, such that $0 \leq m_1 < m \leq n/p$. Let us define $p_1 \in]p, +\infty]$ by the equality*

$$(35) \quad \frac{n}{p} - m = \frac{n}{p_1} - m_1.$$

(1) *Under the assumptions*

$$(36) \quad m_1 > 0 \quad \text{or} \quad m < \frac{n}{p},$$

the imbedding $W_p^m(\mathbb{R}^n) \hookrightarrow W_{p_1}^{m_1}(\mathbb{R}^n)$ holds, together with the homogeneous estimate

$$(37) \quad \forall f \in W_p^m(\mathbb{R}^n) \quad \|f\|_{\dot{W}_{p_1}^{m_1}(\mathbb{R}^n)} \leq c \|f\|_{\dot{W}_p^m(\mathbb{R}^n)}.$$

(2) *In case $m_1 = 0$ and $m = n/p$, the above imbeddings and estimates hold with $W_{p_1}^{m_1}(\mathbb{R}^n)$ and $\dot{W}_{p_1}^{m_1}(\mathbb{R}^n)$ replaced by $BMO(\mathbb{R}^n)$ if $p > 1$, and by $C_0(\mathbb{R}^n)$ if $p = 1$.*

Proof. All the imbeddings are well known. Applying them to the functions $x \mapsto f(\lambda x)$ and taking λ to infinity yield the desired homogeneous estimate. $\square$

Motivated by Lemma 6, we define

$$(38) \quad \begin{aligned} &\bullet \dot{W}_p^m(\mathbb{R}^n) \text{ by } \dot{W}_p^m \cap L_q(\mathbb{R}^n), \text{ with} \\ &\frac{1}{q} := \frac{1}{p} - \frac{m}{n}, \end{aligned}$$

for m an integer, $0 < m < n/p$,

- $\dot{W}_1^n(\mathbb{R}^n)$ by $\dot{W}_1^n \cap C_b(\mathbb{R}^n)$,
- $\dot{W}_{n/m}^m(\mathbb{R}^n)$ by $\dot{W}_{n/m}^m \cap BMO(\mathbb{R}^n)$, for m an integer, $0 < m < n$.

All these spaces will be endowed with the norm (or seminorm) $\|\cdot\|_{\dot{W}_p^m}$.

Proposition 14. *Let m, n be positive integers, and $p \in [1, +\infty[$.*

(1) *If $m < n/p$, then*

$$\dot{W}_p^m(\mathbb{R}^n) = \dot{W}_p^m(\mathbb{R}^n) \oplus \mathcal{P}_{m-1}.$$

(2) *Let $m = n/p$. For all $f \in \dot{W}_p^m(\mathbb{R}^n)$ there exists $g \in \dot{W}_p^m(\mathbb{R}^n)$, uniquely defined up to the addition of a constant, such that $f - g \in \mathcal{P}_{m-1}$.*

Proof. Let us define a family of functions $(u_t)_{t>0}$ in $\mathcal{S}'(\mathbb{R}^n)$ by $\hat{u}_t(\xi) := \psi(t\xi)$, where ψ is the auxiliary function introduced in (2). The following properties are well known:

$$(39) \quad \forall f \in \dot{W}_p^m(\mathbb{R}^n) \quad \|u_t * f\|_{\dot{W}_p^m} \leq \|u_1\|_1 \|f\|_{\dot{W}_p^m},$$

$$(40) \quad \forall f \in L_p(\mathbb{R}^n) \quad \lim_{t \rightarrow 0} \|u_t * f - f\|_p = 0,$$

$$(41) \quad \text{if } p > 1: \quad \forall f \in L_p(\mathbb{R}^n) \quad \lim_{t \rightarrow +\infty} \|u_t * f\|_p = 0.$$

Now we define a sequence of operators $(\Lambda_k)_{k>1}$ on $\mathcal{S}'(\mathbb{R}^n)$ by $\Lambda_k f := (u_{k-1} - u_k) * f$.

Let $f \in \dot{W}_p^m(\mathbb{R}^n)$. Since the Fourier transform of $\Lambda_k f$ is supported in the annulus $k^{-1} \leq |\xi| \leq 2k$, it follows by Lemmas 5 and 6, and by inequality (39), that

$$(42) \quad \forall k > 1 \quad \|\Lambda_k f\|_E \leq c \|\Lambda_k f\|_{\dot{W}_p^m},$$

where E denotes L_q , BMO or C_0 , according to the values of the parameters m, n, p . Then we have the following discussion.

Step 1 : Case $p > 1$. By (40), (41), and (42), it follows that the sequence $(\Lambda_k f)_{k>1}$ has a strong limit g in L_q and BMO , respectively. It follows *a fortiori* that

- $\Lambda_k f \rightarrow g$ in $\mathcal{S}'(\mathbb{R}^n)$ if $m < n/p$,
- $\Lambda_k f + c_k \rightarrow g$ in $\mathcal{S}'(\mathbb{R}^n)$, for some number sequence $(c_k)_{k>1}$, if $m = n/p$.

In both cases, we deduce that

$$\lim_{k \rightarrow +\infty} \widehat{\Lambda_k f} = \widehat{g} \quad \text{in } \mathcal{D}'(\mathbb{R}^n \setminus \{0\}).$$

Then $\widehat{f} = \widehat{g}$ on $\mathbb{R}^n \setminus \{0\}$. We conclude that $f - g$ is a polynomial. By differentiating m times, we obtain $f - g \in \mathcal{P}_{m-1}$.

Substep 1.1 : Case $p > 1$, $m < n/p$. According to (38) it follows that $q < +\infty$; hence $\dot{W}_p^m(\mathbb{R}^n) \cap \mathcal{P}_{m-1} = \{0\}$. The proof is complete.

Substep 1.2 : Case $p > 1$, $m = n/p$. Since $BMO(\mathbb{R}^n) \cap \mathcal{P}_{m-1} = \mathcal{P}_0$, we conclude that f has a representative modulo $\mathcal{P}_{m-1}$ in BMO , unique up to the addition of a constant.

Step 2 : Case $p = 1$. By (42), the sequence $(\Lambda_k f)_{k>1}$ is bounded in $L_q(\mathbb{R}^n)$, where $q = \frac{n}{n-m}$. Since $q > 1$, there exists a subsequence $(\Lambda_{k_j} f)_{j \geq 0}$ converging to some

$g \in L_q$ in the weak topology $\sigma(L_q, L_{q'})$. Arguing as in the preceding step, we obtain $f - g \in \mathcal{P}_{m-1}$.

Substep 2.1 : Case $p = 1, m < n$. Since $q < +\infty$, we conclude exactly as in Substep 1.1.

Substep 2.2 : Case $p = 1, m = n$. The continuity of g follows by the imbeddings $\dot{W}_1^n(\mathbb{R}^n) \hookrightarrow W_1^n(\mathbb{R}^n)_{loc}$ and $W_1^n(\mathbb{R}^n) \hookrightarrow C_0(\mathbb{R}^n)$. Since $C_b(\mathbb{R}^n) \cap \mathcal{P}_{n-1} = \mathcal{P}_0$, we conclude that f has a representative modulo $\mathcal{P}_{n-1}$ in $C_b(\mathbb{R}^n)$, unique up to the addition of a constant. $\square$

We conclude by stating the Sobolev type imbeddings between the canonical realizations. The proof follows the same lines as that of Proposition 14 and is left to the reader.

Proposition 15. *Let m, p, m_1, p_1 be parameters satisfying the assumptions of Lemma 6. Under assumption (36) the imbedding $\dot{W}_p^m(\mathbb{R}^n) \hookrightarrow \dot{W}_{p_1}^{m_1}(\mathbb{R}^n)$ holds, together with the estimate*

$$\forall f \in \dot{W}_p^m(\mathbb{R}^n) \quad \|f\|_{\dot{W}_{p_1}^{m_1}(\mathbb{R}^n)} \leq c \|f\|_{\dot{W}_p^m(\mathbb{R}^n)}.$$

In the critical case $m = n/p$, the canonical realization turns out to coincide with the Adams-Frazier space:

Proposition 16. *For every integer $m, 0 < m \leq n$, it follows that*

$$\dot{W}_{n/m}^m(\mathbb{R}^n) = \dot{W}_{n/m}^m \cap \dot{W}_n^1(\mathbb{R}^n).$$

Proof. The equality $\dot{W}_n^1(\mathbb{R}^n) = \dot{W}_n^1(\mathbb{R}^n)$ is clear. Hence $\dot{W}_n^1(\mathbb{R}^n)$ is imbedded in C_b in case $n = 1$, and in BMO in case $n > 1$. Thus the desired equality follows by Proposition 15. $\square$

Remark 11. There exist other ways to prove Propositions 14 and 15, e.g. by using the classical Sobolev representation formulas, see [17, chap. 1], or by general facts on Besov spaces [6]. The method used here seems to be the simplest way to transfer the imbedding properties of classical Sobolev spaces to their homogeneous counterparts.

5.3. A further degeneracy result.

Proposition 17. *Let $n, k \in \mathbb{N}^*$, $p \in [1, +\infty[$, and let m be an integer satisfying (34). Let Q' be a nonempty open cube in $\mathbb{R}^n$. Let us define F as the closure of Q' if $m < n/p$, and as the complementary of Q' if $m > n/p$. If $f : \mathbb{R}^k \rightarrow \mathbb{R}$ acts by superposition from $\dot{W}_p^m(F, \mathbb{R}^k)$ to $\dot{W}_p^m(\mathbb{R}^n)$, then f is an affine function.*

Proof. Without loss of generality, we can assume that Q' is the interior of the unit cube Q . We will use the auxiliary functions introduced in (1) and (4).

Step 1 : Reduction to polynomials. By easy modifications of Lemmas 2 and 3, there exist $c_1, c_2 > 0$ such that $\|f \circ g\|_{\dot{W}_p^m} \leq c_2$ for all $g \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^k)$ satisfying

$$(43) \quad \|g\|_{\dot{W}_p^m} \leq c_1 \quad \text{and} \quad \text{supp } g \subseteq F.$$

We fix $a \in \mathbb{R}^{k-1}$, $j \in \{1, \dots, k\}$, and define $f_{j,a}(x) := f(\sigma_j(a, x))$. (In case $k = 1$, we replace $f_{j,a}$ by f itself.) For an arbitrary large positive number λ , we define $r = r(\lambda) > 0$ by the equality

$$\lambda r^{m-(n/p)} \|v\|_{\dot{W}_p^m} = c_1.$$

We also fix a constant c_3 such that

$$c_3^{m-(n/p)} |a| \|\varphi\|_{\dot{W}_p^m} \leq c_1,$$

and $c_3 \leq 1$ if $m > n/p$, $c_3 \geq 2$ if $m < n/p$. We put

$$\begin{aligned} g(x) &:= \sigma_j(\varphi(c_3 x) a, \lambda v(rx)) \quad \text{for } m < n/p, \\ g(x) &:= \sigma_j((1 - \varphi(c_3 x)) a, \lambda v(r(x - z^r))) \quad \text{for } m > n/p. \end{aligned}$$

It is easily seen that g satisfies condition (43) if $r \geq 2$ in case $m < n/p$, and if $|z^r|$ is sufficiently large in case $m > n/p$. In the same way, we have

- $f(g(x)) = f_{j,a}(\lambda r(x_1 - z_1^r))$ on the cube $r^{-1}Q + z^r$, if $|z^r|$ is sufficiently large, in case $m > n/p$,
- $f(g(x)) = f_{j,a}(\lambda r x_1)$ on the cube $r^{-1}Q$, if $r \geq c_3$, in case $m < n/p$.

By integrating $|\partial_1^m(f \circ g)|^p$ on the cube $r^{-1}Q + z^r$ or $r^{-1}Q$ and by using inequality $\|f \circ g\|_{\dot{W}_p^m} \leq c_2$, we obtain

$$\left(\int_{-\lambda/2}^{\lambda/2} |f_{j,a}^{(m)}(y)|^p dy \right)^{1/p} \leq c_2 \lambda^{(1/p)+1-m}.$$

By taking λ to $+\infty$, we conclude that $f_{j,a}^{(m)} = 0$. Hence f is a polynomial.

Step 2 : An auxiliary operator. Let us define

$$\begin{aligned} S_f(g)(x) &:= \varphi(4x) f(g(x)) \quad \text{for } m < n/p, \\ S_f(g)(x) &:= (1 - \varphi(x/2)) f(g(x)) \quad \text{for } m > n/p. \end{aligned}$$

In case $m < n/p$, we have

$$S_{\Delta_a f}(g) = S_f(\varphi(2(\cdot))(a + g)) - S_f(g)$$

for all $a \in \mathbb{R}^k$, and a similar identity holds in case $m > n/p$. We conclude the following: *if the function f satisfies the property*

$$(44) \quad S_f(\dot{W}_p^m(F, \mathbb{R}^k)) \subset \dot{W}_p^m(\mathbb{R}^n),$$

then the function $\Delta_a f$ has the same property, for all $a \in \mathbb{R}^k$.

Step 3 : Reduction to affine functions. Let us assume that f is a polynomial of degree greater than 2 satisfying (44). By iterating the last remark we are reduced to the case in which f is a nonzero quadratic form. Then we choose $\gamma \in \mathbb{R}^k$ such that $f(\gamma) = \pm 1$, and α such that

$$\frac{1}{2} \left| m - \frac{n}{p} \right| < \alpha < \left| m - \frac{n}{p} \right|.$$

Let us define

$$\begin{aligned} g(x) &:= \varphi(2x) |x|^{-\alpha\gamma} \quad \text{for } m < n/p, \\ g(x) &:= (1 - \varphi(x)) |x|^{\alpha\gamma} \quad \text{for } m > n/p. \end{aligned}$$

Then $g \in \dot{W}_p^m(F, \mathbb{R}^k)$, but $S_f(g) \notin \dot{W}_p^m(\mathbb{R}^n)$, a contradiction. $\square$

In case $m < n/p$, the above lines constitute an alternative proof of the classical result of Dahlberg [13], with a generalization to vector valued Sobolev spaces.

Since the compactly supported functions in $\dot{W}_p^m(\mathbb{R}^n)$ belong clearly to the canonical realization introduced in Section 5.2.2, the degeneracy holds *a fortiori* in $\dot{W}_p^m(\mathbb{R}^n)$:

Corollary 2. *Let $n, k \in \mathbb{N}^*$, $p \in [1, +\infty[$, m an integer such that $1 + (1/p) < m < n/p$. If $f : \mathbb{R}^k \rightarrow \mathbb{R}$ acts by superposition from $\dot{W}_p^m(\mathbb{R}^n, \mathbb{R}^k)$ to $\dot{W}_p^m(\mathbb{R}^n)$, then f is an affine function.*

5.4. The exceptional cases. There exist nontrivial superposition operators from $\dot{W}_p^m(\mathbb{R}^n)$ to $\dot{W}_p^m(\mathbb{R}^n)$:

- In case $m = n/p$. If $m \geq 2$, this follows by Proposition 16, and by the results of Section 2. If $m = 1$, see Remark 10.
- In case $m = 2$, $p = 1$, as explained in the following statement.

Proposition 18. *Let f be a function of $\mathbb{R}$ to itself.*

- (1) *f acts by superposition on $\dot{W}_1^2(\mathbb{R})$ if and only if $f \in \dot{W}_1^2(\mathbb{R})$.*
- (2) *T_f takes $\dot{W}_1^2(\mathbb{R}^2)$ to $\dot{W}_1^2(\mathbb{R}^2)$ if and only if $f \in W_1^2(\mathbb{R})_{loc}$.*
- (3) *In case $n > 2$, T_f takes $\dot{W}_1^2(\mathbb{R}^n)$ to $\dot{W}_1^2(\mathbb{R}^n)$ if and only if $f \in \dot{W}_1^2(\mathbb{R})$.*

Proof. Step 1 : Case $n = 1$. Let us recall a classical result on $\dot{W}_1^1(\mathbb{R})$: □

Lemma 7. *Any function of $\dot{W}_1^1(\mathbb{R})$ is absolutely continuous and admits finite limits at $\pm\infty$.*

Now we take functions f, g in $\dot{W}_1^2(\mathbb{R})$. Using the chain rule, and generalized integration by parts, for absolutely continuous functions, we derive

$$\int_{\mathbb{R}} |(f'' \circ g)(x)| g'(x)^2 dx \leq \|f''\|_1 (\|g''\|_1 + |g'(+\infty)| + |g'(-\infty)|).$$

Applying the Leibniz formula, we obtain $(f \circ g)'' \in L_1(\mathbb{R})$, together with the estimate

$$\|(f \circ g)''\|_1 \leq \|f''\|_1 (2\|g''\|_1 + |g'(+\infty)| + |g'(-\infty)|) + |f'(-\infty)| \|g''\|_1.$$

The necessity of $f \in \dot{W}_1^2(\mathbb{R})$, for an acting function f , follows by taking $g = \text{id}_{\mathbb{R}}$.

Step 2 : Case $n > 2$.

Substep 2.1. Let us assume that f, g are smooth functions, with $f \in \dot{W}_1^2(\mathbb{R})$, $g \in \dot{W}_1^2(\mathbb{R}^n)$, and $\partial_j g \in C_0(\mathbb{R}^n)$ for $j = 1, \dots, n$. Let $k, j \in \{1, \dots, n\}$. Using an integration by parts w.r.t the k -th variable, we obtain

$$\int_{\mathbb{R}^n} |f''(g(x))| \partial_k g(x)^2 dx \leq \|f''\|_1 \|\partial_k^2 g\|_1.$$

Hence

$$\begin{aligned} \|\partial_k \partial_j (f \circ g)\|_1 &\leq \int_{\mathbb{R}^n} |f'(g(x))| |\partial_k \partial_j g(x)| dx + \int_{\mathbb{R}^n} |f''(g(x))| |\partial_k g(x)| |\partial_j g(x)| dx \\ &\leq \|f'\|_{\infty} \|\partial_k \partial_j g\|_1 + \|f''\|_1 \|\partial_k^2 g\|_1^{1/2} \|\partial_j^2 g\|_1^{1/2}. \end{aligned}$$

We conclude that

$$\|f \circ g\|_{\dot{W}_1^2} \leq c (\|f'\|_{\infty} + \|f''\|_1) \|g\|_{\dot{W}_1^2}.$$

Substep 2.2. In case f is any function in $\dot{W}_1^2(\mathbb{R})$, and g any function in $\dot{W}_1^2(\mathbb{R}^n)$, we argue by approximation as in Steps 2 and 3 of Section 2.4. Notice that the assumption $g \in L_q$, with $q = \frac{n}{n-2}$, plays here a fundamental role to insure that the approximating functions $\theta_{\nu} * g$ have derivatives in C_0 .

Substep 2.3. To prove the necessity of $f'' \in L_1$, we restrict T_f to the inhomogeneous space $W_1^2(\mathbb{R}^n)$, and apply the result of [7].

Step 3 : Case $n = 2$. This follows by Proposition 16 and Theorem 2.

OPEN QUESTIONS

Firstly we give a list of unsolved questions in the paper itself:

- We underline that our results are not fully satisfactory in case $p = 1$ and $k > 1$; see Remark 6.
- The *necessity* of the conditions on f in Theorem 3 is not clear at all.
- In Section 5, we restricted partly to $k = 1$, or $k < n$. An investigation of the general case would be of interest. We could also introduce realizations in case $m > n/p$ and discuss degeneracy in this context.

The main open question is the extension of our results to the fractional Sobolev spaces (Besov or Lizorkin-Triebel spaces), that is, obtaining the sharp version of the Adams-Frazier, Brezis-Mironescu and Maz'ya-Shaposhnikova theorems [2, 12, 18]. We refer to our preceding papers [10, 11] and to the work of Allaoui [3] for the first steps in this direction.

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