THE K -THEORY OF TOEPLITZ C^* -ALGEBRAS OF RIGHT-ANGLED ARTIN GROUPS

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ABSTRACT. Toeplitz C^* -algebras of right-angled Artin groups were studied by Crisp and Laca. They are a special case of the Toeplitz C^* -algebras $\mathcal{T}(G, P)$ associated with quasi-lattice ordered groups (G, P) introduced by Nica. Crisp and Laca proved that the so-called “boundary quotients” $C_Q^*(\Gamma)$ of $C^*(\Gamma)$ are simple and purely infinite. For a certain class of finite graphs Γ we show that $C_Q^*(\Gamma)$ can be represented as a full corner of a crossed product of an appropriate C^* -subalgebra of $C_Q^*(\Gamma)$ built by using $C^*(\Gamma')$, where Γ' is a subgraph of Γ with one less vertex, by the group $\mathbb{Z}$. Using induction on the number of the vertices of Γ we show that $C_Q^*(\Gamma)$ are nuclear and moreover belong to the small bootstrap class. We also use the Pimsner-Voiculescu exact sequence to find their K -theory. Finally we use the Kirchberg-Phillips classification theorem to show that those C^* -algebras are isomorphic to tensor products of $\mathcal{O}_n$ with $1 \leq n \leq \infty$.

1. INTRODUCTION

Toeplitz C^* -algebras of right-angled Artin groups generalize both the Toeplitz algebra and the Cuntz algebras. Coburn showed in [4] that the C^* -algebra, generated by a single nonunitary isometry, is unique; i.e., every two C^* -algebras, each generated by a single nonunitary isometry, are $*$ -isomorphic. Similar uniqueness theorems about C^* -algebras generated by isometries were proved by Cuntz [7], Douglas [10], Murphy [13], and others. Laca and Raeburn in [12] and Crisp and Laca in [5] proved such uniqueness theorems for a large class of C^* -algebras, corresponding to quasi-lattice ordered groups (G, P) . One of the key points they used was to project onto the “diagonal” C^* -algebra generated by the range projections of those isometries, an idea originating from [10].

These C^* -algebras can be viewed as crossed products of commutative C^* -algebras (the C^* -algebras generated by the range projections of the isometries) by semi-groups of endomorphisms. Crisp and Laca used techniques from [11] about such crossed products together with the uniqueness theorems mentioned above to prove a structure theorem for the universal C^* -algebra $C^*(G, P)$ (which by the uniqueness theorems is isomorphic to the “reduced one” $\mathcal{T}(G, P)$) for a large class of quasi-lattice ordered groups (G, P) . We will now state [6, Corollary 8.5] and [6, Theorem 6.7] and use them throughout this paper. A graph will always mean a simple graph with a countable set of vertices.

Received by the editors November 9, 2007 and, in revised form, March 3, 2009.

2010 *Mathematics Subject Classification.* Primary 19K99, 46L80; Secondary 46L35.

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Theorem 1.1 ([6], Theorem 6.7). *Suppose that Γ is a graph with a set of vertices S (finite or infinite) such that Γ^{opp} has no isolated vertices. Then the universal C^* -algebra with generators $\{V_s | s \in S\}$ subject to the relations:*

- (1) $V_s^* V_s = I$ for each $s \in S$;
- (2) $V_s V_t = V_t V_s$ and $V_s^* V_t = V_t V_s^*$ if s and t are adjacent in Γ ;
- (3) $V_s^* V_t = 0$ if s and t are distinct and not adjacent in Γ ;
- (4) $\prod_{s \in S_\lambda} (I - V_s V_s^*) = 0$ for each $S_\lambda \subset S$ spanning a finite connected component of Γ^{opp} is purely infinite and simple.

We will denote the C^* -algebra from this theorem by $C_Q^*(\Gamma)$.

Theorem 1.2 ([6], Corollary 8.5). *Suppose that Γ is a graph with a set of vertices S (finite or infinite) such that Γ^{opp} has no isolated vertices. Let $C^*(\Gamma)$ denote the universal C^* -algebra with generators $\{V_s | s \in S\}$ subject to the relations:*

- (1) $V_s^* V_s = I$ for each $s \in S$;
- (2) $V_s V_t = V_t V_s$ and $V_s^* V_t = V_t V_s^*$ if s and t are adjacent in Γ ;
- (3) $V_s^* V_t = 0$ if s and t are distinct and not adjacent in Γ .

Then each quotient of $C^(\Gamma)$ is obtained by imposing a further collection of relations of the form*

(R) $\prod_{s \in S_\lambda} (I - V_s V_s^) = 0$, where each $S_\lambda \subset S$ spans a finite union of finite connected components of Γ^{opp} .*

We recall that by definition the opposite graph of the graph Γ is

$$\Gamma^{\text{opp}} = \{(v, w) | v, w \in S, (v, w) \notin \Gamma\}.$$

Γ^{opp} is also called the complement or the inverse of the graph Γ .

Let Γ be a finite graph with a set of vertices S such that the opposite graph Γ^{opp} is connected and has more than 1 vertex. Then $C_Q^*(\Gamma)$ is the quotient of $C^*(\Gamma)$ by the ideal generated by $\prod_{s \in S} (I - V_s V_s^*)$. Let $I_\Gamma : \langle \prod_{s \in S} (I - V_s V_s^*) \rangle_{C^*(\Gamma)} \rightarrow C^*(\Gamma)$ be the inclusion map of this ideal, and let $Q_\Gamma : C^*(\Gamma) \rightarrow C_Q^*(\Gamma)$ be the quotient map. Theorem 1.2 implicitly contains the uniqueness theorem ([5, Theorem 24]). In particular we have the following faithful representation $\pi_\Gamma : C^*(\Gamma) \rightarrow \mathcal{B}(H_\Gamma)$ which corresponds to $\mathcal{T}(A_\Gamma, A_\Gamma^+)$, where $A_\Gamma = \{s | ss' = s's \text{ if } (s, s') \in \Gamma\}$:

Let H_Γ be the Hilbert space with an orthonormal basis

$$\{\mathfrak{E}[s_1, s_2, \dots, s_n] | n \in \mathbb{N}_0, s_1, \dots, s_n \in S\} / \sim,$$

where the relation $\sim$ means $\mathfrak{E}[s_1, s_2, \dots, s_n] \sim \mathfrak{E}[s'_1, s'_2, \dots, s'_m]$ if and only if $V_{s_1} \cdots V_{s_n} = V_{s'_1} \cdots V_{s'_m}$ subject to the commutation relation (2) from Theorem 1.2.

Let π_Γ be given on a generating family of operators and vectors by

$$\pi_\Gamma(V_s)(\mathfrak{E}[\emptyset]) = \mathfrak{E}[s], \pi_\Gamma(V_s)(\mathfrak{E}[s_1, s_2, \dots, s_n]) = \mathfrak{E}[s, s_1, s_2, \dots, s_n].$$

For this representation it is true that the ideal $\langle \pi_\Gamma(\prod_{s \in S} (I - V_s V_s^*)) \rangle_{\pi_\Gamma(C^*(\Gamma))}$ coincides with $\mathcal{K}(H_\Gamma)$, the compact operators on H_Γ .

In [7], Cuntz introduced a certain type of C^* -algebras $\mathcal{O}_n$, $n = 2, 3, \dots, \infty$ generated by a set of isometries with mutually orthogonal ranges. He was able to represent $\mathcal{K} \otimes \mathcal{O}_n$ as a crossed product of an AF -algebra by $\mathbb{Z}$ ($\mathcal{K}$ stands for the C^* -algebra of the compact operators on a separable Hilbert space). There have been generalizations of these algebras that depend on the “crossed product by $\mathbb{Z}$ ” idea, for example Cuntz-Krieger algebras [9], Cuntz-Pimsner algebras [17] and others.

In our paper, for a fixed finite graph with at least three vertices Γ with Γ^{opp} connected we choose a subgraph Γ' with one less vertex such that $(\Gamma')^{\text{opp}}$ is connected. Then we represent $C_Q^*(\Gamma)$ as a full corner of a crossed product of a C^* -algebra, built by using $C^*(\Gamma')$, by the group $\mathbb{Z}$. After doing so we can use some results about C^* -algebras which are crossed products by $\mathbb{Z}$. Most importantly we use the Pimsener-Voiculescu exact sequence for the K -theory ([18]). Using induction on the number of the vertices of the graph we conclude that $C_Q^*(\Gamma)$ is nuclear and moreover belongs to the small bootstrap class (see [2, IV.3.1], [1, §22]) and thus the classification result for purely infinite simple C^* -algebras of Kirchberg-Phillips [16] applies. From this we conclude that $C^*(\Gamma)$ is isomorphic to $\mathcal{O}_{1+|\chi(\Gamma)|}$, where $\chi(\Gamma)$ is an analogue of Euler characteristic, introduced in [6]. Then we extend this result to the case when Γ is an infinite graph with countably many vertices and such that Γ^{opp} is connected, since this graph can be represented as an increasing sequence of finite subgraphs. The general case is a graph Γ with at least two and at most countably many vertices which is such that Γ^{opp} has no isolated vertex. It can be treated easily using Theorem 1.1 and the special cases described above. The conclusion is that $C_Q^*(\Gamma)$ is isomorphic to tensor products of $\mathcal{O}_n$ for $1 \leq n \leq \infty$, where we define $\mathcal{O}_1$ to be the unital Kirchberg algebra with $\mathbf{K}_0(\mathcal{O}_1) = \mathbb{Z}[1_{\mathcal{O}_1}]_0$ and $\mathbf{K}_1(\mathcal{O}_1) = \mathbb{Z}$. A Kirchberg algebra is by definition a separable, nuclear, simple, purely infinite C^* -algebra that satisfies the Universal Coefficient Theorem.

2. SOME C^* -SUBALGEBRAS OF $C_Q^*(\Gamma)$ AND THE CROSSED PRODUCT CONSTRUCTION

If Γ has two vertices and no edges, then from the construction of $C^*(\Gamma)$ it is clear that $C^*(\Gamma)$ is generated by isometries V_1 and V_2 with orthogonal ranges and such that $V_1V_1^* + V_2V_2^* < I$. This is the C^* -algebra $\mathcal{E}_2$ from [8], which is an extension of $\mathcal{O}_2$ by the compacts. Thus $C_Q^*(\Gamma) \cong \mathcal{O}_2$.

Suppose now that Γ has a set of vertices S such that $2 < \text{card}(S) < \infty$ and suppose that Γ^{opp} is connected. Since Γ^{opp} is connected if it is not a tree, we can remove an arbitrary edge from its arbitrary cycle and the graph obtained in this way (let's denote it by Γ_1^{opp}) will remain connected. Continuing in this fashion in finitely many (say l) steps we will arrive at Γ_l^{opp} , which will be a tree. Let $s \in S$ be a "leaf" for Γ_l^{opp} . Removing s and the edge that comes out of s from Γ_l^{opp} will not alter the connectedness. All this shows that if Γ' is the graph, obtained from Γ by removing the vertex s and all the edges that come out of s , then its opposite graph $(\Gamma')^{\text{opp}}$ will be connected.

Let $S' \subset S$ be the set of edges of Γ' . We can suppose that $S = \{1, \dots, n, n+1\}$ and that $S' = \{1, \dots, n\}$ for some $n \geq 2$. We want to describe the words in letters $\{V_1, \dots, V_n, V_{n+1}, V_1^*, \dots, V_n^*, V_{n+1}^*\}$.

Lemma 2.1. *Every word in letters $\{V_1, \dots, V_n, V_{n+1}, V_1^*, \dots, V_n^*, V_{n+1}^*\}$ can be written in the form $w_1w_2^*$, where w_1, w_2 are words in letters $\{V_1, \dots, V_n, V_{n+1}\}$.*

Proof. We will use induction on the length of the words. The words of length one are V_i and V_i^* and they are of such a form. Suppose that the statement of the lemma is true for all words of length $m > 1$ and shorter. Take a word w of length $m+1$. We have two cases for w :

1) $w = w'V_i^*$ and 2) $w = w'V_i$ for some $1 \leq i \leq n+1$ and some word w' of length m .

By the induction hypothesis, w' can be represented as $w' = w'_1(w'_2)^*$, where w'_1 and w'_2 are words in letters $\{V_1, \dots, V_n, V_{n+1}\}$. In case 1), $w = w'_1(w'_2)^*V_i^*$, so setting $w_1 = w'_1$ and $w_2 = V_i w'_2$ shows that w can be written in the desired form. For case 2), if the word w'_2 is empty, then setting $w_1 = w'_1 V_i$ and $w_2 = I$ shows that w has the desired form. If $w'_2 = V_j w''_2$ with w''_2 a word in letters $\{V_1, \dots, V_{n+1}\}$, then

$$w = w'_1(w''_2)^*V_j^*V_i = \begin{cases} 0, & \text{if } (i, j) \notin \Gamma, \\ w'_1(w''_2)^*, & \text{if } i = j, \\ w'_1(w''_2)^*V_iV_j^*, & \text{if } (i, j) \in \Gamma. \end{cases}$$

The first and the second cases in the above equation are words of the desired form. In the third case, we have that $w'_1(w''_2)^*V_i$ is a word of length m , so it can be represented as $\omega_1\omega_2^*$. Then $w'_1(w''_2)^*V_iV_j^* = \omega_1\omega_2^*V_j^*$ is of the desired form. This concludes the induction and proves the lemma. $\square$

Let's denote by V the isometry $V_{n+1} \in C_Q^*(\Gamma)$ and suppose without loss of generality that $V^*V_i = 0$ for $k < i \leq n$ (notice that since Γ^{opp} is connected, $k < n$). If $k > 0$, then V also commutes and $*$ -commutes with $V_1, \dots, V_k$.

Let $T_0 = C^*(V_1, \dots, V_n)$. Then from Theorem 1.2 it is easy to see that $T_0 \cong C^*(\Gamma')$. Define by induction T_m to be the closed linear span of elements of $C_Q^*(\Gamma)$ of the form $wVt_{m-1}V^*(w')^*$, where w, w' are words in letters $\{V_1, \dots, V_n\}$ and $t_{m-1} \in T_{m-1}$. The following lemma characterizes the sets T_m .

Lemma 2.2. T_m is a C^* -subalgebra of $C^*(\Gamma)$, isomorphic to $\mathcal{K}^{\otimes m} \otimes T_0$ ($\cong \mathcal{K} \otimes C^*(\Gamma')$).

Proof. Let us denote by Ω the set of all words ω in letters $\{V_1, \dots, V_n\}$ such that the letters of the word ωV cannot be commuted beyond V ; i.e., $\omega V = \omega_1 V \omega_2$ for some words ω_1, ω_2 in letters $\{V_1, \dots, V_n\}$ implies $\omega_2 = I$. It is easy to see that from the connectedness of Γ^{opp} it follows that Ω is an infinite countable set. Therefore we can enumerate its elements: $\Omega = \{\omega_0, \omega_1, \omega_2, \dots\}$, setting $\omega_0 = I$. We assume that the words in Ω don't repeat, i.e., $\omega_p \neq \omega_q$ for $p \neq q$ after using the commutation relation. Suppose by induction that $T_{m-1} \cong \mathcal{K}^{\otimes(m-1)} \otimes T_0$ for some $m \geq 1$. We want to show that $T_m \cong \mathcal{K} \otimes T_{m-1}$. Clearly $\{\omega_p V t_{m-1} V^* \omega_q^* | p, q \in \mathbb{N}_0\}$ is a $*$ -closed set. It is easy to see that each element $w' V t_{m-1} V^* w^*$ of T_m , after applying the commutation relations (2) from Theorem 1.2, can be written in the form $\omega_p V t'_{m-1} V^* \omega_q^*$ for some $p, q \in \mathbb{N}_0$ and some $t'_{m-1} \in T_{m-1}$. Therefore $\{\omega_p V t_{m-1} V^* \omega_q^* | p, q \in \mathbb{N}_0, t_{m-1} \in T_{m-1}\}$ spans a dense subset of T_m . We conclude that T_m is $*$ -closed.

We now want to show that $V^* \omega_q^* \omega_p V = \delta_{p,q} I$. Write $\omega_p = V_{j_1} \dots V_{j_s}$ and $\omega_q = V_{i_1} \dots V_{i_t}$. Then $V^* \omega_q^* \omega_p V = V^* V_{i_t}^* \dots V_{i_2}^* V_{i_1}^* V_{j_1} V_{j_2} \dots V_{j_s} V$. There are three cases:

- 1) If V_{j_1} commutes with $V_{i_1}^*, \dots, V_{i_r}^*$ ($1 \leq r < t$) and $i_{r+1} = j_1$, then $V_{i_{r+1}}^*$ will commute with $V_{i_1}^*, \dots, V_{i_r}^*$, so the word ω_q can be written in the form $\omega_q = V_{i_1} V_{i_2} \dots V_{i_t}$ with $i_1 = j_1$. Then we can write $V^* \omega_q^* \omega_p V = V^* V_{i_t}^* \dots V_{i_2}^* V_{i_1}^* V_{j_2} \dots V_{j_s} V$ and continue the argument with this word.
- 2) If V_{j_1} commutes with $V_{i_1}^*, \dots, V_{i_r}^*$ ($1 \leq r < t$) and $(j_1, i_{r+1}) \notin \Gamma$, then $V^* \omega_q^* \omega_p V = 0$. Also if $j_1 > k$ and V_{j_1} commutes with $V_{i_1}^*, \dots, V_{i_t}^*$ we also have $V^* \omega_q^* \omega_p V = 0$.

3) If V_{j_1} commutes with $V_{i_1}, \dots, V_{i_t}$ and V , then clearly $j_1 \leq k$ and from the definition of Ω it follows that V_{j_1} doesn't commute with all $V_{j_2}, \dots, V_{j_s}$. Suppose that V_{j_1} doesn't commute with V_{j_r} ($2 \leq r \leq s$) and if $r > 2$, that V_{j_1} commutes with $V_{j_2}, \dots, V_{j_{r-1}}$. Notice that $j_r \notin \{i_1, \dots, i_t\}$ since V_{j_1} commutes with $V_{i_1}^*, \dots, V_{i_t}^*$ and not with V_{j_r} .

Suppose that $V^* \omega_q^* \omega_p V \neq 0$. Then suppose that $V_{j_1}, \dots, V_{j_{r_1}}$ can be dealt with by using case 1) repeatedly. If $r_1 = s = t$, then $V^* \omega_q^* \omega_p V = \delta_{p,q} I$ is proven. If $r_1 = s < t$, then $V^* \omega_q^* \omega_p V$ reduces to $V^* V_{i_t}^* \dots V_{i_{s+1}}^* V$. If $i_t \leq k$, then $V_{i_t}^*$ would commute with V^* contradicting the fact that $\omega_q \in \Omega$. $i_t > k$ immediately implies $V^* V_{i_t}^* \dots V_{i_{s+1}}^* V = 0$ because V does not commute with all of $V_{i_t}^*, \dots, V_{i_{s+1}}^*$ so it has an orthogonal range with some of them. The case $r_1 = t < s$ is similar. If $r_1 < s$ and $r_1 < t$, then suppose that for $V_{j_{r_1+1}}$, case 3) applies. We will obtain a contradiction with the fact that $\omega_p \in \Omega$. By case 3) we can find $r_2 > r_1 + 1$ such that $V_{j_{r_1+1}}$ doesn't commute with $V_{j_{r_2}}$ and if $r_2 > r_1 + 2$, then $V_{j_{r_1+1}}$ commutes with $V_{j_{r_1+2}}, \dots, V_{j_{r_2-1}}$. Also $j_{r_2} \notin \{i_{r_1+1}, \dots, i_t\}$ ($V_{j_{r_1+1}}$ commutes with $V_{i_{r_1+1}}^*, \dots, V_{i_t}^*$ and not with $V_{j_{r_2}}$) and so case 1) cannot be applied to $V_{j_{r_2}}$. We can repeat this process finitely many times until we reach the isometry V_{j_s} for which case 3) must apply since case 1) cannot be applied as we saw above and case 2) cannot be applied by assumption. But then $j_s \leq k$ and V_{j_s} commutes with V , which contradicts $\omega_p \in \Omega$. This proves $V^* \omega_q^* \omega_p V = \delta_{p,q} I$.

It follows that $\omega_p V t_{m-1} V^* \omega_q^* \omega_{p'} V t'_{m-1} V^* \omega_{q'} = \delta_{p',q} \omega_p V t_{m-1} t'_{m-1} V^* \omega_{q'}$ and thus T_m is a C^* -algebra. The equation $V^* \omega_q^* \omega_p V = \delta_{p,q} I$ implies that

$$C^*(\{\omega_p V V^* \omega_q^* | 0 \leq p, q \leq l-1\}) \cong M_l(\mathbb{C}).$$

It is clear that $V T_{m-1} V^*$ is a C^* -algebra, isomorphic to T_{m-1} . Therefore

$$\begin{aligned} & C^*(\{\omega_p V t_{m-1} V^* \omega_q^* | 0 \leq p, q \leq l-1, t_{m-1} \in T_{m-1}\}) \\ & \cong C^*(\{\sum_{i=0}^{l-1} (\omega_i V t_{m-1} V^* \omega_i^*) | t_{m-1} \in T_{m-1}\}) \otimes C^*(\{\omega_p V V^* \omega_q^* | 0 \leq p, q \leq l-1\}) \\ & \cong T_{m-1} \otimes M_l(\mathbb{C}) = M_l(T_{m-1}), \end{aligned}$$

since $\sum_{i=0}^{l-1} (\omega_i V t_{m-1} V^* \omega_i^*)$ commutes with $\omega_p V V^* \omega_q^*$ for each $0 \leq p, q \leq l-1$ and each $t_{m-1} \in T_{m-1}$. Taking the limit $l \rightarrow \infty$ concludes the proof of the lemma. $\square$

From the proof of this lemma it easily follows that T_m is the closed linear span of

$$\{\omega_{p_m} V \dots V \omega_{p_1} V t_0 V^* \omega_{q_1}^* V^* \dots V^* \omega_{q_m}^* | \omega_{p_1}, \dots, \omega_{p_m}, \omega_{q_1}, \dots, \omega_{q_m} \in \Omega, t_0 \in T_0\}.$$

This implies that $T_m \cdot T_l \subset T_m$ and $T_l \cdot T_m \subset T_m$ for each $m \geq l \geq 0$.

Now we introduce the following C^* -subalgebras of $C_Q^*(\Gamma)$: Define $B_0 = T_0$ and $B_m = C^*(B_{m-1} \cup T_m) = C^*(T_0 \cup \dots \cup T_m)$. From what we said above it is clear that T_m is an ideal of B_m . Therefore we have an extension

$$(1) \quad 0 \longrightarrow T_m \xrightarrow{i_m} B_m \xrightarrow{p_m} B_m/T_m \longrightarrow 0,$$

where $i_m : T_m \rightarrow B_m$ is the inclusion map and $p_m : B_m \rightarrow B_m/T_m$ is the quotient map.

From [14, Theorem 3.1.7] (or [2, Corollary II.5.1.3]) it follows that $B_m = B_{m-1} + T_m$ as a linear space. From [14, Remark 3.1.3] it follows that the map $\pi_m : B_{m-1}/(B_{m-1} \cap T_m) \rightarrow B_m/T_m$ given by $b_{m-1} + B_{m-1} \cap T_m \mapsto b_{m-1} + T_m$ is an isomorphism ($b_{m-1} \in B_{m-1}$).

Define $\mathcal{I}_m \stackrel{\text{def}}{=} \langle V^m[\prod_{i=1}^n (I - V_i V_i^*)](V^*)^m \rangle_{T_m}$. Since $T_0 \cong C^*(\Gamma')$ from Theorem 1.2 it follows that $\mathcal{I}_0$ is the unique nontrivial ideal of T_0 and it is isomorphic to $\mathcal{K}$. Then from Lemma 2.2 it follows that $\mathcal{I}_m$ is the unique nontrivial ideal of T_m and it is isomorphic to $\mathcal{K}^{\otimes m} \otimes \mathcal{K}$. The ideal $\mathcal{I}_m$ can be described as the closed linear span of

$$\{\omega_{p_m} V \cdots V \omega_{p_1} V \omega_{q_1} V^* \cdots V^* \omega_{q_m}^* \mid \omega_{p_1}, \dots, \omega_{p_m}, \omega_{q_1}, \dots, \omega_{q_m} \in \Omega, \iota_0 \in \mathcal{I}_0\}.$$

Therefore it is easy to see that $V^m(V^*)^m \mathcal{I}_m V^m(V^*)^m = V^m \mathcal{I}_0 (V^*)^m$.

By the definition of $C_Q^*(\Gamma)$ we have $(I - VV^*) \prod_{i=1}^n (I - V_i V_i^*) = 0$ or $\prod_{i=1}^n (I - V_i V_i^*) = VV^* \prod_{i=1}^n (I - V_i V_i^*)$. Therefore using relations (2) and (3) from Theorem 1.1 we get

$$\begin{aligned} \prod_{i=1}^n (I - V_i V_i^*) &= VV^* \prod_{i=1}^n (I - V_i V_i^*) = V \prod_{i=1}^k (I - V_i V_i^*) V^* \prod_{i=k+1}^n (I - V_i V_i^*) \\ &= V \prod_{i=1}^k (I - V_i V_i^*) V^* \in T_1. \end{aligned}$$

It also follows that $V^m V \prod_{i=1}^k (I - V_i V_i^*) V^* (V^*)^m \in V^m T_1 (V^*)^m \subset T_{m+1}$. It is easy to see that $T_{m+1} \cdot B_m \subset T_{m+1}$ and $B_m \cdot T_{m+1} \subset T_{m+1}$. This implies that $T_m \cap T_{m+1}$ is an ideal of T_m and that $B_m \cap T_{m+1}$ is an ideal of B_m . From this we can conclude that $\mathcal{I}_m \subset (T_m \cap T_{m+1})$ for each $m \in \mathbb{N}$. The reverse inclusion is also true:

Lemma 2.3. $B_m \cap T_{m+1} = T_m \cap T_{m+1} = \mathcal{I}_m$ for each $m \in \mathbb{N}_0$.

Proof. Since $\mathcal{I}_0$ is the unique nontrivial ideal of T_0 and since $T_0 \cap T_1$ is an ideal of T_0 , then if we assume that $\mathcal{I}_0 \subsetneq T_0 \cap T_1$ it will follow that $T_0 = T_0 \cap T_1$. Then $I = 1_{T_0} = 1_{C_Q^*(\Gamma)} \in T_0 \subset T_1$. This will imply that $T_1 \cong \mathcal{K} \otimes T_0$ is a unital C^* -algebra, which is a contradiction. Therefore $\mathcal{I}_0 = T_0 \cap T_1$.

It is easy to see that for each $m \in \mathbb{N}$ we have $V^m(V^*)^m T_m V^m(V^*)^m = V^m T_0 (V^*)^m \cong T_0$ and that $V^m(V^*)^m T_{m+1} V^m(V^*)^m = V^m T_1 (V^*)^m \cong T_1$. Thus if we assume that $T_m = T_m \cap T_{m+1}$, it will follow that $V^m T_0 (V^*)^m \subset V^m T_1 (V^*)^m$ and therefore that $T_0 \subset T_1$. This is a contradiction with what we proved in the last paragraph. Therefore $T_m \cap T_{m+1} \subsetneq T_m$ and thus $T_m \cap T_{m+1} = \mathcal{I}_m$.

To conclude the proof of the lemma we have to show that $T_{m+1} \cap T_j = 0$ for each $0 \leq j < m$. In this case we have once again that $T_{m+1} \cap T_j$ is an ideal of T_j . Therefore the assumption $T_{m+1} \cap T_j \neq 0$ implies that T_{m+1} contains the minimal nonzero ideal of T_j , $\mathcal{I}_j$. In particular

$$V^j \prod_{i=1}^n (I - V_i V_i^*) (V^*)^j = V^{j+1} \prod_{i=1}^k (I - V_i V_i^*) (V^*)^{j+1} \in T_{m+1}.$$

This implies

$$\begin{aligned} V^{j+1} \prod_{i=1}^k (I - V_i V_i^*) (V^*)^{j+1} &= V^{j+1} (V^*)^{j+1} V^{j+1} \prod_{i=1}^k (I - V_i V_i^*) (V^*)^{j+1} V^{j+1} (V^*)^{j+1} \\ &\in V^{j+1} (V^*)^{j+1} T_{m+1} V^{j+1} (V^*)^{j+1} = V^{j+1} T_{m-j} (V^*)^{j+1}. \end{aligned}$$

Therefore $\prod_{i=1}^k (I - V_i V_i^*) \in T_{m-j}$. Since $\prod_{i=1}^k (I - V_i V_i^*) \in T_0$ also, then the ideal $T_0 \cap T_{m-j}$ of T_0 contains $\prod_{i=1}^k (I - V_i V_i^*)$. We will show that $\prod_{i=1}^k (I - V_i V_i^*) \notin \mathcal{I}_0$. This will imply that $T_0 \subset T_{m-j}$ for $m-j > 0$, therefore obtaining a contradiction with the fact that T_{m-j} is not unital for $m-j > 0$.

Suppose that $\prod_{i=1}^k (I - V_i V_i^*) \in \mathcal{I}_0$. Then since $T_0 = C^*(\Gamma')$ we have $Q_{\Gamma'}(\prod_{i=1}^k (I - V_i V_i^*)) = 0$. From the connectedness of $(\Gamma')^{\text{opp}}$ it follows that we can find j , $1 \leq j \leq k$ and l , $k < l \leq n$ with $(j, l) \notin \Gamma'$. Then

$$\begin{aligned} 0 &= Q_{\Gamma'}(V_l^*) Q_{\Gamma'}(\prod_{i=1}^k (I - V_i V_i^*)) Q_{\Gamma'}(V_l) = Q_{\Gamma'}(V_l^*) Q_{\Gamma'}(\prod_{\substack{(i,l) \in \Gamma' \\ 1 \leq i \leq k}} (I - V_i V_i^*)) Q_{\Gamma'}(V_l) \\ &= Q_{\Gamma'}(V_l^* V_l) Q_{\Gamma'}(\prod_{\substack{(i,l) \in \Gamma' \\ 1 \leq i \leq k}} (I - V_i V_i^*)) = Q_{\Gamma'}(\prod_{\substack{(i,l) \in \Gamma' \\ 1 \leq i \leq k}} (I - V_i V_i^*)). \end{aligned}$$

By repeating this argument finitely many times we will arrive at the equality $Q_{\Gamma'}(I) = 0$, which is a contradiction. Therefore $\prod_{i=1}^k (I - V_i V_i^*) \notin \mathcal{I}_0$. This completes the proof of the lemma. $\square$

This lemma shows that we have an extension

$$(2) \quad 0 \rightarrow \mathcal{I}_{m-1} \xrightarrow{i'_m} B_{m-1} \xrightarrow{p'_m} B_{m-1}/\mathcal{I}_{m-1} \rightarrow 0,$$

where $i'_m : \mathcal{I}_{m-1} \rightarrow B_{m-1}$ is the inclusion map and $p'_m : B_{m-1} \rightarrow B_{m-1}/\mathcal{I}_{m-1}$ is the quotient map.

From equations (1) and (2) we have the commutative diagram with exact rows:

$$(3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I}_{m-1} & \xrightarrow{i'_m} & B_{m-1} & \xrightarrow{p'_m} & B_{m-1}/\mathcal{I}_{m-1} \longrightarrow 0 \\ & & I'_m \downarrow & & I_m \downarrow & & \cong \downarrow \pi_m \\ 0 & \longrightarrow & T_m & \xrightarrow{i_m} & B_m & \xrightarrow{p_m} & B_m/T_m \longrightarrow 0, \end{array}$$

where $I'_m : \mathcal{I}_{m-1} \rightarrow T_m$ and $I_m : B_{m-1} \rightarrow B_m$ are the inclusion maps.

Define $B \stackrel{\text{def}}{=} \bigcup_{i=0}^{\infty} B_i \subset C^*(\Gamma)$ or in other words $B \stackrel{\text{def}}{=} \varinjlim (B_m, I_m)$. Notice that if $t_m \in T_m$, then $V t_m V^* \in T_{m+1}$. Thus we have a well-defined injective endomorphism $\beta : B \rightarrow B$ given by $b \mapsto V b V^*$.

Similarly to the Cuntz' construction from [7] we define $\tilde{B} \stackrel{def}{=} \varinjlim (B^m, \alpha_m)$ as the limit of the sequence (which is also a commutative diagram)

$$(4) \quad \begin{array}{ccccccccccccccc} \dots & \xrightarrow{\alpha_{-m-1}} & B^{-m} & \xrightarrow{\alpha_{-m}} & \dots & \xrightarrow{\alpha_{-1}} & B^0 & \xrightarrow{\alpha_0} & B^1 & \xrightarrow{\alpha_1} & \dots & \xrightarrow{\alpha_{m-1}} & B^m & \xrightarrow{\alpha_m} & \dots \\ & & \downarrow j_{-m} \cong & & & & \downarrow j_0 \cong & & \downarrow j_1 \cong & & & & \downarrow j_m \cong & & \\ \dots & \xrightarrow{\beta} & B & \xrightarrow{\beta} & \dots & \xrightarrow{\beta} & B & \xrightarrow{\beta} & B & \xrightarrow{\beta} & \dots & \xrightarrow{\beta} & B & \xrightarrow{\beta} & \dots \end{array}$$

where $j_m : B^m \rightarrow B$ are $*$ -isomorphisms. Since $\tilde{B}$ is a limit C^* -algebra we have $*$ -homomorphisms $\alpha^m : B^m \rightarrow \tilde{B}$ such that $\alpha^m = \alpha^{m+1} \circ \alpha_m$ for all $m \in \mathbb{Z}$.

Now we define a $*$ -homomorphism Φ of $\tilde{B}$ to itself, which is induced by "shift to the left" on (4). In other words, if we have a stabilizing sequence $(b^m)_{m=-\infty}^{+\infty}$, where $b^m \in B^m$ for each m , then $\Phi((b^m)_{m=-\infty}^{+\infty}) = (j_m^{-1} \circ j_{m+1}(b^{m+1}))_{m=-\infty}^{+\infty}$. In particular for $b \in B$ the element $\alpha^m \circ j_m^{-1}(b)$ can be represented as the sequence $(0, \dots, 0, 0, j_m^{-1}(b), \alpha_m \circ j_m^{-1}(b), \alpha_{m+1} \circ \alpha_m \circ j_m^{-1}(b), \dots) = (0, \dots, 0, 0, j_m^{-1}(b), j_{m+1}^{-1} \circ \beta(b), j_{m+2}^{-1} \circ \beta^2(b), \dots)$. Therefore $\Phi(\alpha^m \circ j_m^{-1}(b))$ can be represented as the sequence $(0, \dots, 0, j_{m-1}^{-1}(b), j_m^{-1} \circ \beta(b), j_{m+1}^{-1} \circ \beta^2(b), \dots)$. This shows that $\Phi(\alpha^m \circ j_m^{-1}(b)) = \alpha^m \circ j_{m-1}^{-1} \circ \beta(b)$. The extension of this map to the whole of $\tilde{B}$ (we call it Φ also) is a $*$ -isomorphism, because Φ is isometric on the dense set of all stabilizing sequences (since j_m are all isomorphisms). Now let $\tilde{A}$ be the crossed product of $\tilde{B}$ by the automorphism Φ . We represent $\tilde{A}$ faithfully on a Hilbert space $\mathfrak{H}$ so that Φ is implemented by a unitary U on $\mathfrak{H}$: $\Phi(b) = UbU^*$ for $b \in \tilde{B}$. Then $\tilde{A} = C^*(\tilde{B} \cup \{U\})$.

Every element of $\tilde{A}$ is a limit of elements of the form $\tilde{a} = \sum_{i=-N}^N b_i U^i = \sum_{i=-N}^{-1} U^i \bar{b}_i + b_0 + \sum_{i=1}^N b_i U^i$, with $b_i \in \tilde{B}$, where $\bar{b}_i = U^{-i} b_i U^i \in \tilde{B}$ for $i = -N, \dots, -1$. Therefore the set of the elements of $\tilde{A}$ of the above form is dense in $\tilde{A}$.

Set $\tilde{P}_m \stackrel{def}{=} \alpha^m(1_{B^m}) \in \tilde{B}$ for each $m \in \mathbb{Z}$. Notice that $\alpha^m(1_{B^m}) = \alpha^m \circ j_m^{-1}(I) = \alpha^{m+1} \circ \alpha_m \circ j_m^{-1}(I) = \alpha^{m+1} \circ j_{m+1}^{-1}(\beta(I))$. By induction

$$\tilde{P}_m = \alpha^{m+i} \circ j_{m+i}^{-1}(\beta^i(I)), \quad m \in \mathbb{Z}, \quad i \in \mathbb{N}.$$

Therefore we can write

$$(5) \quad \tilde{P}_m = \Phi^{-m}(\tilde{P}_0), \quad m \in \mathbb{Z}.$$

Consider the C^* -algebra $\tilde{P}_0 \tilde{A} \tilde{P}_0$. Clearly $\tilde{P}_0 \tilde{B} \tilde{P}_0 \subset \tilde{P}_0 \tilde{A} \tilde{P}_0$. Since elements of the form $\tilde{a} = \sum_{i=-N}^{-1} U^i b_i + b_0 + \sum_{i=1}^N b_i U^i$ ($b_i \in \tilde{B}$) are dense in $\tilde{A}$, then elements of the form $\tilde{P}_0 \tilde{a} \tilde{P}_0 = \sum_{i=-N}^{-1} \tilde{P}_0 U^i b_i \tilde{P}_0 + \tilde{P}_0 b_0 \tilde{P}_0 + \sum_{i=1}^N \tilde{P}_0 b_i U^i \tilde{P}_0$ are dense in $\tilde{P}_0 \tilde{A} \tilde{P}_0$. It is easy to see that $U \tilde{P}_0 U^* = \Phi(\tilde{P}_0) < \tilde{P}_0$, so the range of $U \tilde{P}_0$ is contained in $\tilde{P}_0$ and therefore $\tilde{P}_0 U \tilde{P}_0 = U \tilde{P}_0$. Then

$$\begin{aligned} \tilde{P}_0 \tilde{a} \tilde{P}_0 &= \sum_{i=-N}^{-1} \tilde{P}_0 U^i b_i \tilde{P}_0 + \tilde{P}_0 b_0 \tilde{P}_0 + \sum_{i=1}^N \tilde{P}_0 b_i U^i \tilde{P}_0 \\ &= \sum_{i=-N}^{-1} (\tilde{P}_0 U^i)(\tilde{P}_0 b_i \tilde{P}_0) + \tilde{P}_0 b_0 \tilde{P}_0 + \sum_{i=1}^N (\tilde{P}_0 b_i \tilde{P}_0)(U^i \tilde{P}_0). \end{aligned}$$

This shows that if we set $S \stackrel{def}{=} U\tilde{P}_0$, then $\tilde{P}_0\tilde{A}\tilde{P}_0 = C^*(\tilde{P}_0\tilde{B}\tilde{P}_0 \cup \{S\})$. Let us also set $S_i \stackrel{def}{=} \alpha^0(j_0^{-1}(V_i))$, $i = 1, \dots, n$.

It is easy to see that $\text{Span}(\bigcup_{l=0}^{\infty} T_l)$ is dense in B . Then it follows that $\text{Span}(\bigcup_{i=0}^{\infty} \alpha^i \circ j_i^{-1}(\bigcup_{l=0}^{\infty} T_l))$ is dense in $\tilde{B}$. Therefore $\tilde{P}_0 \text{Span}(\bigcup_{i=0}^{\infty} \alpha^i \circ j_i^{-1}(\bigcup_{l=0}^{\infty} T_l)) \tilde{P}_0 = \text{Span}(\tilde{P}_0 \bigcup_{i=0}^{\infty} \alpha^i \circ j_i^{-1}(\bigcup_{l=0}^{\infty} T_l) \tilde{P}_0)$ is dense in $\tilde{P}_0\tilde{B}\tilde{P}_0$. For each $i \in \mathbb{N}$ we have

$$\begin{aligned} \tilde{P}_0 \alpha^i \circ j_i^{-1}(\bigcup_{l=0}^{\infty} T_l) \tilde{P}_0 &= \alpha^i \circ j_i^{-1}(\beta^i(I)) \alpha^i \circ j_i^{-1}(\bigcup_{l=0}^{\infty} T_l) \alpha^i \circ j_i^{-1}(\beta^i(I)) \\ &= \alpha^i \circ j_i^{-1}(\beta^i(I)(\bigcup_{l=0}^{\infty} T_l) \beta^i(I)) = \alpha^i \circ j_i^{-1}(V^i(V^*)^i(\bigcup_{l=0}^{\infty} T_l) V^i(V^*)^i) \\ &= \alpha^i \circ j_i^{-1}((V^i(V^*)^i)^2(\bigcup_{l=0}^{\infty} T_l) (V^i(V^*)^i)^2) \subset \alpha^i \circ j_i^{-1}((V^i(V^*)^i T_i)(\bigcup_{l=0}^{\infty} T_l) (T_i V^i(V^*)^i)) \\ &\subset \alpha^i \circ j_i^{-1}(V^i(V^*)^i(\bigcup_{l=i}^{\infty} T_l) V^i(V^*)^i) = \alpha^i \circ j_i^{-1}(V^i(\bigcup_{l=0}^{\infty} T_l) (V^*)^i) = \alpha^i \circ j_i^{-1}(\beta^i(\bigcup_{l=0}^{\infty} T_l)) \\ &= \alpha^i \circ \alpha_{i-1} \circ \alpha_{i-2} \circ \dots \circ \alpha_1 \circ \alpha_0 \circ j_0^{-1}(\bigcup_{l=0}^{\infty} T_l) = \alpha^0 \circ j_0^{-1}(\bigcup_{l=0}^{\infty} T_l). \end{aligned}$$

From this it follows that $\alpha_0 \circ j_0^{-1}(\text{Span}(\bigcup_{l=0}^{\infty} T_l))$ is dense in $\tilde{P}_0\tilde{B}\tilde{P}_0$ and therefore also that $\alpha^0(B^0) = \tilde{P}_0\tilde{B}\tilde{P}_0$. This shows that $\tilde{P}_0\tilde{A}\tilde{P}_0 = C^*(\alpha_0 \circ j_0^{-1}(\bigcup_{l=0}^{\infty} T_l) \cup \{S\})$.

Observe that

$$\begin{aligned} (6) \quad S\alpha^0 \circ j_0^{-1}(b)S^* &= U\tilde{P}_0\alpha^0 \circ j_0^{-1}(b)\tilde{P}_0U^* = U\alpha^0 \circ j_0^{-1}(b)U^* = \Phi(\alpha^0 \circ j_0^{-1}(b)) \\ &= \alpha^0 \circ j_0^{-1} \circ \beta(b) = \alpha^0 \circ j_0^{-1}(VbV^*). \end{aligned}$$

Since for every $m > 0$, T_m can be constructed from T_0 , the “Ad(V)” equation (6) shows that $\tilde{P}_0\tilde{A}\tilde{P}_0 = C^*(\alpha_0 \circ j_0^{-1}(\bigcup_{l=0}^{\infty} T_l) \cup \{S\}) = C^*(\alpha_0 \circ j_0^{-1}(T_0) \cup \{S\}) = C^*(\{S_1, \dots, S_n, S\})$.

Now we want to apply Theorem 1.1 to the C^* -algebra $A \stackrel{def}{=} \tilde{P}_0\tilde{A}\tilde{P}_0$. $S_i = \alpha^0 \circ j_0^{-1}(V_i)$ are clearly isometries ($i = 1, \dots, n$). $S^*S = \tilde{P}_0U^*U\tilde{P}_0 = \tilde{P}_0$ and therefore S is also an isometry. Thus condition (1) holds.

It is clear from (6) that $SS^* = \alpha^0 \circ j_0^{-1}(VV^*)$. Therefore

$$\begin{aligned} 0 &= \alpha^0 \circ j_0^{-1}(0) = \alpha^0 \circ j_0^{-1}((I - VV^*) \prod_{i=1}^n (I - V_i V_i^*)) \\ &= (\tilde{P}_0 - \alpha^0 \circ j_0^{-1}(VV^*)) \prod_{i=1}^n (\tilde{P}_0 - \alpha^0 \circ j_0^{-1}(V_i V_i^*)) = (\tilde{P}_0 - SS^*) \prod_{i=1}^n (\tilde{P}_0 - S_i S_i^*). \end{aligned}$$

This proves that condition (4) holds.

Conditions (2) and (3) obviously hold for all pairs of isometries from $\{S_1, \dots, S_n\}$. If $n \geq i > k$, then $S_i S_i^* S S^* = \alpha^0 \circ j_0^{-1}(V_i V_i^* V V^*) = 0$, so condition (3) holds also

for all pairs (S_i, S) with $k < i \leq n$. For $1 \leq i \leq k$ one has

$$\begin{aligned} SS_i &= S\alpha^0 \circ j_0^{-1}(V_i) = S\alpha^0 \circ j_0^{-1}(V_i)S^*S = \Phi(\alpha^0 \circ j_0^{-1}(V_i))S = \alpha^0 \circ j_0^{-1}(VV_iV^*)S \\ &= \alpha^0 \circ j_0^{-1}(V_iVV^*)S = \alpha^0 \circ j_0^{-1}(V_i)\alpha^0 \circ j_0^{-1}(VV^*)S = S_iSS^*S = S_iS. \end{aligned}$$

This shows that $SS_i = S_iS$. In the same way one can show that $SS_i^* = S_i^*S$. Therefore condition (4) holds for all pairs (S, S_i) with $1 \leq i \leq k$.

Applying Theorem 1.1 we get $A \cong C_Q^*(\Gamma)$. Obviously we also have $C_Q^*(\Gamma) \cong \tilde{P}_m \tilde{A} \tilde{P}_m$ for each $m \in \mathbb{Z}$.

We recall here (see [2, IV.3.1], [1, §22]) that each C^* -algebra in the small bootstrap class $\mathfrak{N}$ satisfies the Universal Coefficient Theorem. The small bootstrap class $\mathfrak{N}$ is the smallest class of C^* -algebras that satisfies:

- (i) $\mathbb{C} \in \mathfrak{N}$.
- (ii) $\mathfrak{N}$ is closed under stable isomorphism.
- (iii) $\mathfrak{N}$ is closed under inductive limits.
- (iv) $\mathfrak{N}$ is closed under crossed products by $\mathbb{Z}$.
- (v) If $0 \rightarrow \mathfrak{J} \rightarrow \mathfrak{A} \rightarrow \mathfrak{A}/\mathfrak{J} \rightarrow 0$ is an exact sequence, and two of $\mathfrak{J}, \mathfrak{A}, \mathfrak{A}/\mathfrak{J}$ are in $\mathfrak{N}$, so is the third.

The C^* -algebras in this class are all nuclear.

The following theorem holds:

Theorem 2.4. *In the above settings: $\tilde{A} \cong \tilde{B} \rtimes_{\Phi} \mathbb{Z}$ and $A \cong C_Q^*(\Gamma)$ is Morita equivalent to $\tilde{A}$. Both of the C^* -algebras A and $\tilde{A}$ are simple, belong to $\mathfrak{N}$ and $\mathbf{K}_*(\tilde{A}) = \mathbf{K}_*(A)$. Also if we suppose that $[\tilde{P}_0]_0$ generates $\mathbf{K}_0(\tilde{A})$, then it follows that $[\tilde{P}_0]_0$ generates $\mathbf{K}_0(A)$.*

Proof. We showed above that $\tilde{P}_m \tilde{A} \tilde{P}_m \cong C_Q^*(\Gamma)$ for each $m \in \mathbb{Z}$. It is easy to see that $\tilde{A} = \overline{\bigcup_{m=0}^{\infty} \tilde{P}_m \tilde{A} \tilde{P}_m}$ and since each $\tilde{P}_m \tilde{A} \tilde{P}_m$ is simple it follows that $\tilde{A}$ is

simple too. Therefore every nonzero projection in $\tilde{A}$ is full. In particular $\tilde{P}_0$ is a full projection and therefore $A = \tilde{P}_0 \tilde{A} \tilde{P}_0$ is a full corner of $\tilde{A}$ and is therefore Morita equivalent to $\tilde{A}$. It follows that A and $\tilde{A}$ are stably isomorphic (by Brown's Theorem [3]) and therefore $\mathbf{K}_*(A) = \mathbf{K}_*(\tilde{A})$.

Because A is a subalgebra of $\tilde{A}$ any equivalence between projections from a matrix algebra over A will carry on to a relation between the same projections thought of as elements of a matrix algebra over $\tilde{A}$. Now if $[\tilde{P}_0]_0$ generates $\mathbf{K}_0(\tilde{A})$, then $\mathbf{K}_0(\tilde{A})$ is a cyclic group. If $\mathbf{K}_0(\tilde{A}) = \mathbb{Z}_n$, then $[\tilde{P}_0]_0$ has order n in $\mathbf{K}_0(\tilde{A})$ and therefore $[\tilde{P}_0]_0$ must have order n in $\mathbf{K}_0(A)$ by the observation above. If $\mathbf{K}_0(\tilde{A}) = \mathbb{Z}$ and $[\tilde{P}_0]_0$ does not generate $\mathbf{K}_0(A)$, then it is easy to see that we get a contradiction again by considering the observation above. Therefore $[\tilde{P}_0]_0$ generates $\mathbf{K}_0(A)$.

If $\tilde{A}$ belongs to $\mathfrak{N}$, then from the definition it follows that A also does since it is stably isomorphic to $\tilde{A}$.

To conclude the proof of the lemma it remains to show that starting from any finite graph G with G^{opp} connected and going through the above construction the C^* -algebra (let us denote it by $\tilde{A}_G$, the analogue of $\tilde{A}$ for G) belongs to $\mathfrak{N}$. We will do this by using induction on the number of the vertices of G . If G has only two vertices and no edges, then $C_Q^*(G) \cong \mathcal{O}_2$ and $C^*(G) \cong \mathcal{E}_2$, so the statement is true in this case. Suppose that the statement is true for any graph G with at most $n \geq 2$ vertices such that its opposite graph G^{opp} is connected. In particular $C_Q^*(\Gamma')$

(and therefore also $C^*(\Gamma')$) belong to $\mathfrak{N}$. Then $B_0 = T_0 \cong C^*(\Gamma')$ as constructed above also does. This implies that $T_m \cong T_0 \otimes \mathcal{K}$ also belongs to $\mathfrak{N}$, since $\mathfrak{N}$ is closed under stabilization. We now use induction on m to show that B_m belongs to $\mathfrak{N}$. We found out that $B_0 \in \mathfrak{N}$. Suppose that $B_{m-1} \in \mathfrak{N}$ for some $m \in \mathbb{N}$, $m \geq 1$. In equation (3), B_{m-1} and $\mathcal{I}_{m-1} (\cong \mathcal{K})$ belong to $\mathfrak{N}$ and therefore from condition (v) of the definition of the bootstrap category and the exactness of the top row in (3), it follows that $B_{m-1}/\mathcal{I}_{m-1} \in \mathfrak{N}$. Since π_m is an isomorphism, it follows that $B_m/T_m \in \mathfrak{N}$. Since T_m also belongs to $\mathfrak{N}$, the exactness of the bottom row in (3) and condition (v) from the definition of the bootstrap category imply that $B_m \in \mathfrak{N}$. By definition $B \stackrel{\text{def}}{=} \varinjlim B_m$ and therefore $B \in \mathfrak{N}$, since $\mathfrak{N}$ is closed under taking inductive limits. In equation (4) all the C^* -algebras B^m are isomorphic to B and therefore $\tilde{B} \stackrel{\text{def}}{=} \varinjlim B^m$ belongs to $\mathfrak{N}$. Finally since the bootstrap category is closed under crossed products by $\mathbb{Z}$ we conclude using induction that the C^* -algebra $\tilde{A} \cong \tilde{B} \rtimes_{\Phi} \mathbb{Z}$ belongs to the bootstrap class. Finally, as we showed in the last paragraph, this implies that A belongs to $\mathfrak{N}$. This concludes the inductive step because $A \cong C_Q^*(\Gamma)$ and Γ is an arbitrary graph with $n+1$ vertices such that Γ^{opp} is connected.

The theorem is proved. $\square$

3. THE COMPUTATION OF THE $\mathbf{K}$ -THEORY

For a finite graph G with G^{opp} connected, Crisp and Laca conjectured in [6] that the order of $[1_{C_Q^*(G)}]_0$ in $\mathbf{K}_0(C_Q^*(G))$ is $|\chi(G)|$, where $\chi(G)$ is the Euler characteristic of G . $\chi(G)$ is defined as

$$\chi(G) = 1 - \sum_{j=1}^{\infty} (-1)^{j-1} \times \{\text{number of complete subgraphs of } G \text{ on } j \text{ vertices}\}.$$

We will use the settings from the previous section. Denote $P_m \stackrel{\text{def}}{=} V^m(V^*)^m$, $m \in \mathbb{N}_0$. Denote also $Q \stackrel{\text{def}}{=} \prod_{i=1}^k (I - V_i V_i^*)$. Let $\Gamma_k = \{(i, j) \mid 1 \leq i, j \leq k, (i, j) \in \Gamma'\} \subset \Gamma'$.

Since the vertex $n+1$ of Γ is connected with each of the vertices $1, \dots, k$ and none of the others we have

$$\begin{aligned} \chi(\Gamma) &= 1 - \sum_{j=1}^n (-1)^{j-1} \times \{\text{number of complete subgraphs of } \Gamma' \text{ on } j \text{ vertices}\} \\ &\quad - (1 - \sum_{j=1}^k (-1)^{j-1} \times \{\text{number of complete subgraphs of } \Gamma_k \text{ on } j \text{ vertices}\}). \end{aligned}$$

Therefore

$$(7) \quad \chi(\Gamma) = \chi(\Gamma') - \chi(\Gamma_k).$$

The following lemma is based on the “Euler characteristics idea” and is essentially due to Crisp and Laca:

Lemma 3.1. *If E is a C^* -subalgebra of B that contains T_m (for $m \in \mathbb{N}_0$) we have*

$$(8) \quad \chi(\Gamma')[P_m]_0 = [P_{m+1}Q]_0 \text{ (in } \mathbf{K}_0(E)).$$

If E is a C^* -subalgebra of B that contains T_m and T_{m+1} (for $m \in \mathbb{N}_0$) we have

$$(9) \quad \chi(\Gamma')[P_m]_0 = \chi(\Gamma_k)[P_{m+1}]_0 \text{ (in } \mathbf{K}_0(E)).$$

If E is a C^* -subalgebra of B that contains T_{m+1} (for $m \in \mathbb{N}_0$) we have

$$(10) \quad [P_{m+1}Q]_0 = \chi(\Gamma_k)[P_{m+1}]_0 \text{ (in } \mathbf{K}_0(E)).$$

Proof. In the last section we showed that

$$(11) \quad \prod_{i=1}^n (I - V_i V_i^*) = V \prod_{i=1}^k (I - V_i V_i^*) V^*.$$

Since $V^m \prod_{i=1}^n (I - V_i V_i^*) (V^*)^m = \prod_{i=1}^n (V^m (V^*)^m - V^m V_i V_i^* (V^*)^m)$, then by multiplying equation (11) by V^m on the left and by $(V^*)^m$ on the right we get

$$\begin{aligned} \prod_{i=1}^n (V^m (V^*)^m - V^m V_i V_i^* (V^*)^m) &= V^{m+1} \prod_{i=1}^k (I - V_i V_i^*) (V^*)^{m+1} \\ &= V^{m+1} (V^*)^{m+1} Q. \end{aligned}$$

This implies the following:

If E is a C^* -subalgebra of B that contains T_m (for $m \in \mathbb{N}_0$) we have

$$(12) \quad \prod_{i=1}^n (V^m (V^*)^m - V^m V_i V_i^* (V^*)^m) = V^{m+1} (V^*)^{m+1} Q.$$

If E is a C^* -subalgebra of B that contains T_{m+1} (for $m \in \mathbb{N}_0$) we have

$$(13) \quad V^{m+1} \prod_{i=1}^k (I - V_i V_i^*) (V^*)^{m+1} = V^{m+1} (V^*)^{m+1} Q.$$

Note that if E is an appropriate C^* -subalgebra of B , then for each projection P that commutes with $V_1 V_1^*$ we have $[V^m P (V^*)^m - V^m P V_1 V_1^* (V^*)^m]_0 = [V^m P (V^*)^m]_0 - [V^m P V_1 V_1^* (V^*)^m]_0$. Suppose by induction that for some $n > l \geq 1$, if P is a projection that commutes with $V_1 V_1^*, \dots, V_l V_l^*$ we have

$$\begin{aligned} (14) \quad & [V^m \prod_{i=1}^l (P - P V_i V_i^*) (V^*)^m]_0 \\ &= [V^m P (V^*)^m]_0 - \sum_{i=1}^l [V^m P V_i V_i^* (V^*)^m]_0 \\ &+ \sum_{j=2}^l (-1)^j \left(\sum_{\substack{1 \leq i_1 < \dots < i_j \leq l \\ (i_s, i_t) \in \Gamma', 1 \leq s < t \leq j}} [V^m P V_{i_1} \dots V_{i_j} V_{i_j}^* \dots V_{i_1}^* (V^*)^m]_0 \right). \end{aligned}$$

We know that $V_{l+1} V_{l+1}^*$ commutes with each of $V_1 V_1^*, \dots, V_l V_l^*$. If P commutes with $V_1 V_1^*, \dots, V_{l+1} V_{l+1}^*$, then we can apply (14) to the family $V_1 V_1^*, \dots, V_l V_l^*$ and

the projection $PV_{l+1}V_{l+1}^*$ to obtain the following equation:

$$\begin{aligned}
& [V^m V_{l+1} V_{l+1}^* \prod_{i=1}^l (P - PV_i V_i^*) (V^*)^m]_0 \\
&= [V^m \prod_{i=1}^l (PV_{l+1} V_{l+1}^* - PV_{l+1} V_{l+1}^* V_i V_i^*) (V^*)^m]_0 \\
&= [V^m PV_{l+1} V_{l+1}^* (V^*)^m]_0 - \sum_{i=1}^l [V^m PV_{l+1} V_{l+1}^* V_i V_i^* (V^*)^m]_0 \\
&\quad + \sum_{j=2}^l (-1)^j \left(\sum_{\substack{1 \leq i_1 < \dots < i_j \leq l \\ (i_s, i_t) \in \Gamma', 1 \leq s < t \leq j}} [V^m PV_{l+1} V_{l+1}^* V_{i_1} \dots V_{i_j} V_{i_j}^* \dots V_{i_1}^* (V^*)^m]_0 \right).
\end{aligned}$$

Now since $V^m V_{l+1} V_{l+1}^* \prod_{i=1}^l (P - PV_i V_i^*) (V^*)^m < V^m \prod_{i=1}^l (P - PV_i V_i^*) (V^*)^m$ it is easy to see that we have

$$\begin{aligned}
& [V^m (P - PV_{l+1} V_{l+1}^*) \prod_{i=1}^l (P - PV_i V_i^*) (V^*)^m]_0 \\
&= [V^m \prod_{i=1}^l (P - PV_i V_i^*) (V^*)^m - V^m V_{l+1} V_{l+1}^* \prod_{i=1}^l (P - PV_i V_i^*) (V^*)^m]_0 \\
&= [V^m \prod_{i=1}^l (P - PV_i V_i^*) (V^*)^m]_0 - [V^m V_{l+1} V_{l+1}^* \prod_{i=1}^l (P - PV_i V_i^*) (V^*)^m]_0 \\
&= [V^m P (V^*)^m]_0 - \sum_{i=1}^l [V^m PV_i V_i^* (V^*)^m]_0 \\
&\quad + \sum_{j=2}^l (-1)^j \sum_{\substack{1 \leq i_1 < \dots < i_j \leq l \\ (i_s, i_t) \in \Gamma', 1 \leq s < t \leq j}} [V^m PV_{i_1} \dots V_{i_j} V_{i_j}^* \dots V_{i_1}^* (V^*)^m]_0 - [V^m PV_{l+1} V_{l+1}^* (V^*)^m]_0 \\
&\quad + \sum_{i=1}^l [V^m PV_{l+1} V_{l+1}^* V_i V_i^* (V^*)^m]_0 \\
&\quad - \sum_{j=2}^l (-1)^j \sum_{\substack{1 \leq i_1 < \dots < i_j \leq l \\ (i_s, i_t) \in \Gamma', 1 \leq s < t \leq j}} [V^m PV_{l+1} V_{l+1}^* V_{i_1} \dots V_{i_j} V_{i_j}^* \dots V_{i_1}^* (V^*)^m]_0 \\
&= [V^m P (V^*)^m]_0 - \sum_{i=1}^{l+1} [V^m PV_i V_i^* (V^*)^m]_0 \\
&\quad + \sum_{j=2}^{l+1} (-1)^j \sum_{\substack{1 \leq i_1 < \dots < i_j \leq l+1 \\ (i_s, i_t) \in \Gamma', 1 \leq s < t \leq j}} [V^m PV_{i_1} \dots V_{i_j} V_{i_j}^* \dots V_{i_1}^* (V^*)^m]_0.
\end{aligned}$$

Then by induction it follows that for $l = k$ or $l = n$ we get

$$\begin{aligned} & [\prod_{i=1}^l (V^m(V^*)^m - V^m V_i V_i^*(V^*)^m)]_0 \\ &= [I]_0 - \sum_{i=1}^l [V^m V_i V_i^*(V^*)^m]_0 \\ &+ \sum_{j=2}^l (-1)^j \sum_{\substack{1 \leq i_1 < \dots < i_j \leq l \\ (i_s, i_t) \in \Gamma', 1 \leq s < t \leq j}} [V^m V_{i_1} \dots V_{i_j} V_{i_j}^* \dots V_{i_1}^*(V^*)^m]_0. \end{aligned}$$

Combining the last equation with equations (12) and (13) we obtain the following equations:

If E is a C^* -subalgebra of B that contains T_m (for $m \in \mathbb{N}_0$) we have

$$\begin{aligned} (15) \quad & [V^m(V^*)^m]_0 - \sum_{i=1}^n [V^m V_i V_i^*(V^*)^m]_0 \\ &+ \sum_{j=2}^n (-1)^j \left(\sum_{\substack{1 \leq i_1 < \dots < i_j \leq n \\ (i_s, i_t) \in \Gamma', 1 \leq s < t \leq j}} [V^m V_{i_1} \dots V_{i_j} V_{i_j}^* \dots V_{i_1}^*(V^*)^m]_0 \right) \\ &= [V^{m+1}(V^*)^{m+1}Q]_0. \end{aligned}$$

If E is a C^* -subalgebra of B that contains T_{m+1} (for $m \in \mathbb{N}_0$) we have

$$\begin{aligned} (16) \quad & [V^{m+1}(V^*)^{m+1}]_0 - \sum_{i=1}^k [V^{m+1} V_i V_i^*(V^*)^{m+1}]_0 \\ &+ \sum_{j=2}^k (-1)^j \left(\sum_{\substack{1 \leq i_1 < \dots < i_j \leq k \\ (i_s, i_t) \in \Gamma_k, 1 \leq s < t \leq j}} [V^{m+1} V_{i_1} \dots V_{i_j} V_{i_j}^* \dots V_{i_1}^*(V^*)^{m+1}]_0 \right) \\ &= [V^{m+1}(V^*)^{m+1}Q]_0. \end{aligned}$$

It is easy to see that in each C^* -subalgebra of B that contains T_m the projection $V^m V_{i_1} \dots V_{i_j} V_{i_j}^* \dots V_{i_1}^*(V^*)^m$ is Murray - von Neumann equivalent to $V^m(V^*)^m$ via the partial isometry $V^m V_{i_1} \dots V_{i_j}(V^*)^m \in T_m$, where $\{i_1, \dots, i_j\} \subset \{1, \dots, n\}$.

This observation together with equations (15) and (16) gives:

If E is a C^* -subalgebra of B that contains T_m we have

$$(17) \quad [P_m]_0 - \sum_{i=1}^n [P_m]_0 + \sum_{j=2}^n (-1)^j \left(\sum_{\substack{1 \leq i_1 < \dots < i_j \leq n \\ (i_s, i_t) \in \Gamma', 1 \leq s < t \leq j}} [P_m]_0 \right) = [P_{m+1}Q]_0.$$

If E is a C^* -subalgebra of B that contains T_{m+1} we have

$$(18) \quad [P_{m+1}]_0 - \sum_{i=1}^k [P_{m+1}]_0 + \sum_{j=2}^k (-1)^j \left(\sum_{\substack{1 \leq i_1 < \dots < i_j \leq k \\ (i_s, i_t) \in \Gamma_k, 1 \leq s < t \leq j}} [P_{m+1}]_0 \right) = [P_{m+1}Q]_0.$$

The last two equations prove formulas (8) and (10). Formula (9) follows from (8) and (10). $\square$

Remark 3.2. It also follows from this lemma that if we denote the isometries that generate $C^*(\Gamma)$ by $\tilde{V}, \tilde{V}_1, \dots, \tilde{V}_n$, then

$$[(I - \tilde{V}\tilde{V}^*) \prod_{i=1}^n (I - \tilde{V}_i \tilde{V}_i^*)]_0 = \chi(\Gamma)[I]_0 \text{ (in } K_0(C^*(\Gamma))\text{)}.$$

Therefore in the extension

$$(19) \quad 0 \rightarrow \langle (I - \tilde{V}\tilde{V}^*) \prod_{i=1}^n (I - \tilde{V}_i \tilde{V}_i^*) \rangle \xrightarrow{I_\Gamma} C^*(\Gamma) \xrightarrow{Q_\Gamma} C_Q^*(\Gamma) \rightarrow 0$$

the map $I_{\Gamma*}$ on $\mathbf{K}_0$ is given by

$$[(I - \tilde{V}\tilde{V}^*) \prod_{i=1}^n (I - \tilde{V}_i \tilde{V}_i^*)]_0 \mapsto \chi(\Gamma)[I]_0.$$

Now we can state and prove the following theorem:

Theorem 3.3. *Suppose that G is a finite graph with at least two vertices and suppose that G^{opp} is connected. Then*

$$(20) \quad \mathbf{K}_0(C_Q^*(G)) = \begin{cases} \mathbb{Z}_{|\chi(G)|}, & \text{if } \chi(G) \neq 0, \\ \mathbb{Z}, & \text{if } \chi(G) = 0, \end{cases} \quad \mathbf{K}_1(C_Q^*(G)) = \begin{cases} 0, & \text{if } \chi(G) \neq 0, \\ \mathbb{Z}, & \text{if } \chi(G) = 0, \end{cases}$$

and $[1_{C_Q^*(G)}]_0$ generates $\mathbf{K}_0(C_Q^*(G))$ in all cases.

Moreover $\mathbf{K}_0(C^*(G)) = \mathbb{Z}$, $\mathbf{K}_1(C^*(G)) = 0$ and $[1_{C^*(G)}]_0$ generates $\mathbf{K}_0(C^*(G))$ in all cases.

Proof. We will use induction on the number of vertices of G . If G has two vertices (and no edges), then $C_Q^*(G) = \mathcal{O}_2$ and $C^*(G) = \mathcal{E}_2$ and in this case certainly the statement is true. Suppose that the statement is true for all graphs G with at most $n \geq 2$ vertices and with G^{opp} connected. The graph Γ considered above was a randomly chosen graph with $n+1$ vertices and with the property that Γ^{opp} is connected. If we show that the statement holds for Γ , then this will prove the statement by induction.

We note that from Lemma 2.2 and the assumption it follows that $\mathbf{K}_0(T_m) = \mathbb{Z}[P_m]_0$ and $\mathbf{K}_1(T_m) = 0$ for all $m \in \mathbb{N}_0$. Also since $\mathcal{I}_m \cong \mathcal{K}$ we have $\mathbf{K}_0(\mathcal{I}_m) = \mathbb{Z}[P_m Q]_0$ and $\mathbf{K}_1(\mathcal{I}_m) = 0$ for all $m \in \mathbb{N}_0$. Finally we recall that π_m is an isomorphism for all $m \in \mathbb{N}_0$.

From the $\mathbf{K}$ -theory six-term exact sequences for the two exact rows of (3) we have the following commutative diagram:

$$\begin{array}{ccccccc}
 \mathbf{K}_0(\mathcal{I}_{m-1}) & & \xrightarrow{i'_{m*}} & \mathbf{K}_0(B_{m-1}) & & \xrightarrow{p'_{m*}} & \mathbf{K}_0\left(\frac{B_{m-1}}{\mathcal{I}_{m-1}}\right) \\
 & \searrow I'_{m*} & & \downarrow I_{m*} & & & \cong \swarrow \pi_{m*} \\
 & & \mathbf{K}_0(T_m) & \xrightarrow{i_{m*}} & \mathbf{K}_0(B_m) & \xrightarrow{p_{m*}} & \mathbf{K}_0\left(\frac{B_m}{T_m}\right) \\
 \uparrow \gamma_m^{\text{ind}} & & \uparrow \delta_m^{\text{ind}} & & & & \downarrow \\
 & & \mathbf{K}_1\left(\frac{B_m}{T_m}\right) & \xleftarrow{p_{m*}} & \mathbf{K}_1(B_m) & \leftarrow & 0 \\
 & \cong \nearrow \pi_{m*} & & \uparrow I_{m*} & & & \nwarrow \\
 \mathbf{K}_1\left(\frac{B_{m-1}}{\mathcal{I}_{m-1}}\right) & & \xleftarrow{p'_{m*}} & \mathbf{K}_1(B_{m-1}) & & \leftarrow & 0,
 \end{array}$$

where γ_m^{ind} and δ_m^{ind} are the index maps for the corresponding six-term exact sequences.

Since $\mathcal{I}_{m-1}$ is generated by $P_m Q$ from Lemma 3.1 it follows that the map $i_{m*} : \mathbf{K}_0(\mathcal{I}_{m-1}) \rightarrow \mathbf{K}_0(B_{m-1})$ is induced by $[P_m Q]_{\mathbf{K}_0(\mathcal{I}_{m-1})} \mapsto \chi(\Gamma') [P_{m-1}]_{\mathbf{K}_0(B_{m-1})}$. Also the map $I'_{m*} : \mathbf{K}_0(\mathcal{I}_{m-1}) \rightarrow \mathbf{K}_0(T_m)$ is induced by $[P_m Q]_{\mathbf{K}_0(\mathcal{I}_{m-1})} \mapsto \chi(\Gamma_k) [P_m]_{\mathbf{K}_0(T_m)}$.

When we “apply” β to equations (1) and (2) we obtain the following commutative diagrams with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{I}_{m-1} & \xrightarrow{i'_m} & B_{m-1} & \xrightarrow{p'_m} & B_{m-1}/\mathcal{I}_{m-1} \longrightarrow 0 \\
 (22) & & \beta \downarrow & & \beta \downarrow & & \downarrow \bar{\beta} \\
 0 & \longrightarrow & \mathcal{I}_m & \xrightarrow{i'_{m+1}} & B_m & \xrightarrow{p'_{m+1}} & B_m/\mathcal{I}_m \longrightarrow 0
 \end{array}$$

and

$$\begin{array}{ccccccc}
 0 & \longrightarrow & T_m & \xrightarrow{i_m} & B_m & \xrightarrow{p_m} & B_m/T_m \longrightarrow 0 \\
 (23) & & \beta \downarrow & & \beta \downarrow & & \downarrow \bar{\beta} \\
 0 & \longrightarrow & T_{m+1} & \xrightarrow{i_{m+1}} & B_{m+1} & \xrightarrow{p_{m+1}} & B_{m+1}/T_{m+1} \longrightarrow 0,
 \end{array}$$

where $\bar{\beta}$ and $\bar{\bar{\beta}}$ are induced by β on the above quotients.

We can now start examining the five different cases depending on $\chi(\Gamma')$ and $\chi(\Gamma_k)$:

(case **I**): $\chi(\Gamma') = 0$ and $\chi(\Gamma_k) = 0$.

By assumption $i'_{m*} = 0 = I'_{m*}$. From (21) it is easy to see that $\delta_m^{\text{ind}} = 0$. Therefore (21) splits into two:

$$(24) \quad \begin{array}{ccccccc} \dots & \xrightarrow{i'_{m*}=0} & \mathbf{K}_0(B_{m-1}) & \xrightarrow[p_{m*}]{p'_{m*}} & \mathbf{K}_0(B_{m-1}/\mathcal{I}_{m-1}) & \longrightarrow & 0 \\ & & \downarrow I_{m*} & & \downarrow \pi_{m*} & \cong & \\ & & & & & & \end{array}$$

$$(25) \quad \begin{array}{ccccccc} \xrightarrow{\delta_m^{\text{ind}}=0} & \mathbf{K}_0(T_m) & \xrightarrow{i_{m*}} & \mathbf{K}_0(B_m) & \xrightarrow{p_{m*}} & \mathbf{K}_0(B_m/T_m) & \longrightarrow 0, \\ & & & & & & \end{array}$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{K}_1(B_{m-1}) & \xrightarrow{p'_{m*}} & \mathbf{K}_1(B_{m-1}/\mathcal{I}_{m-1}) & \xrightarrow{\gamma_m^{\text{ind}}} & \mathbf{K}_0(\mathcal{I}_{m-1}) \xrightarrow{i'_{m*}=0} \dots \\ & & \downarrow I_{m*} & & \downarrow \pi_{m*} & \cong & \\ 0 & \longrightarrow & \mathbf{K}_1(B_m) & \xrightarrow[p_{m*}]{p_{m*}} & \mathbf{K}_1(B_m/T_m) & \xrightarrow{\delta_m^{\text{ind}}=0} & \dots \end{array}$$

By assumption we have $\mathbf{K}_0(B_0) = \mathbb{Z}[P_0]_0$ and $\mathbf{K}_1(B_0) = 0$. Suppose by induction that $\mathbf{K}_0(B_{m-1}) = \mathbb{Z}[P_0]_0 \oplus \dots \oplus \mathbb{Z}[P_{m-1}]_0$.

Then from (24) it follows that $\mathbf{K}_0(B_m) = I_{m*}(\mathbf{K}_0(B_{m-1})) \oplus i_{m*}(\mathbf{K}_0(T_m))$ since all extensions of free abelian groups are trivial. Finally $\mathbf{K}_0(T_m) = \mathbb{Z}[P_m]_0$ concludes the induction. Therefore $\mathbf{K}_0(B_m) = \mathbb{Z}[P_0]_0 \oplus \dots \oplus \mathbb{Z}[P_m]_0$ for each $m \in \mathbb{N}$. Notice that we can write $\mathbf{K}_0(B_m) = \mathbb{Z}[P_0]_0 \oplus \dots \oplus \mathbb{Z}\beta_*^m([P_0]_0)$.

Suppose by induction that

$$\begin{aligned} \mathbf{K}_1(B_{m-1}) &= \mathbb{Z}(p_{1*})^{-1} \circ \pi_{1*} \circ (\gamma_1^{\text{ind}})^{-1}([P_1Q]_0) \oplus \dots \\ &\quad \dots \oplus \mathbb{Z}(p_{m-1*})^{-1} \circ \pi_{m-1*} \circ (\gamma_{m-1}^{\text{ind}})^{-1}([P_{m-1}Q]_0). \end{aligned}$$

This is true for $m = 1$ by assumption. From (25) we see that $\mathbf{K}_1(B_m) = (p_{m*})^{-1} \circ \pi_{m*}(\mathbf{K}_1(B_{m-1}/\mathcal{I}_{m-1}))$. Since all groups are free abelian, all the extensions are trivial and therefore $\mathbf{K}_1(B_m) = I_{m*}(\mathbf{K}_1(B_{m-1})) \oplus \mathbb{Z}(p_{m*})^{-1} \circ \pi_{m*} \circ (\gamma_m^{\text{ind}})^{-1}([P_mQ]_0)$. This concludes the induction. From the functoriality of the index map and from equations (22) and (23) it follows that $(p_{m*})^{-1} \circ \pi_{m*} \circ (\gamma_m^{\text{ind}})^{-1}([P_mQ]_0) = (p_{m*})^{-1} \circ \pi_{m*} \circ (\gamma_m^{\text{ind}})^{-1} \circ \beta_*([P_{m-1}Q]_0) = \beta_* \circ (p_{m-1*})^{-1} \circ \pi_{m-1*} \circ (\gamma_{m-1}^{\text{ind}})^{-1}([P_{m-1}Q]_0)$. Therefore we can write

$$\begin{aligned} \mathbf{K}_1(B_m) &= \mathbb{Z}(p_{1*})^{-1} \circ \pi_{1*} \circ (\gamma_1^{\text{ind}})^{-1}([P_1Q]_0) \oplus \dots \\ &\quad \dots \oplus \mathbb{Z}\beta_*^{m-1} \circ (p_{1*})^{-1} \circ \pi_{1*} \circ (\gamma_1^{\text{ind}})^{-1}([P_1Q]_0). \end{aligned}$$

If $u \in B_1$ is a unitary with $[u]_1 = (p_{1*})^{-1} \circ \pi_{1*} \circ (\gamma_1^{\text{ind}})^{-1}([P_1Q]_0)$, then we can write

$$\mathbf{K}_1(B_m) = \mathbb{Z}[u]_1 \oplus \dots \oplus \mathbb{Z}\beta_*^{m-1}([u]_1).$$

From this we get $\mathbf{K}_0(B) = \bigoplus_{i=0}^{\infty} \mathbb{Z}\beta_*^i([I]_0)$ and $\mathbf{K}_1(B) = \bigoplus_{i=1}^{\infty} \mathbb{Z}\beta_*^{i-1}([u]_1)$.

Let $\tilde{u} = \alpha^0 \circ j_0(u) \in \tilde{B}$. Then it is easy to see that $\mathbf{K}_0(\tilde{B}) = \bigoplus_{i=-\infty}^{\infty} \mathbb{Z}\Phi_*^i([\tilde{P}_0]_0)$

and $\mathbf{K}_1(\tilde{B}) = \bigoplus_{i=-\infty}^{\infty} \mathbb{Z}\Phi_*^{i-1}([\tilde{u}]_1)$.

The Pimsner-Voiculescu exact sequence gives

$$\begin{array}{ccccc}
 \bigoplus_{i=-\infty}^{\infty} \mathbb{Z}\Phi_*^i([\tilde{P}_0]_0) & \xrightarrow{\text{id}_* - \Phi_*} & \bigoplus_{i=-\infty}^{\infty} \mathbb{Z}\Phi_*^i([\tilde{P}_0]_0) & \longrightarrow & \mathbf{K}_0(\tilde{A}) \\
 \uparrow & & & & \downarrow \\
 \mathbf{K}_1(\tilde{A}) & \longleftarrow & \bigoplus_{i=-\infty}^{\infty} \mathbb{Z}\Phi_*^{i-1}([\tilde{u}]_1) & \xleftarrow{\text{id}_* - \Phi_*} & \bigoplus_{i=-\infty}^{\infty} \mathbb{Z}\Phi_*^{i-1}([\tilde{u}]_1).
 \end{array}$$

From this we can conclude that $\mathbf{K}_0(\tilde{A}) = \mathbb{Z}[\tilde{P}_0]_0$, $\mathbf{K}_1(\tilde{A}) = \mathbb{Z}$. From Theorem 2.4 it follows that $\mathbf{K}_0(\tilde{A}) \cong \mathbf{K}_0(C_Q^*(\Gamma)) = \mathbb{Z}$ and $\mathbf{K}_1(\tilde{A}) \cong \mathbf{K}_1(C_Q^*(\Gamma)) = \mathbb{Z}$ and that $[1_{C_Q^*(\Gamma)}]_0$ generates $\mathbf{K}_0(C_Q^*(\Gamma))$.

From Remark 3.2 it follows that in the extension (19) the map $I_{\Gamma*}$ on $\mathbf{K}_0$ is zero. This shows that $\mathbf{K}_0(C^*(\Gamma)) = \mathbb{Z}[1_{C^*(\Gamma)}]_0$ and $\mathbf{K}_1(C^*(\Gamma)) = 0$.

This concludes the proof of (case I).

(case II): $\chi(\Gamma') \neq 0$ and $\chi(\Gamma_k) = 0$.

By assumption $\mathbf{K}_0(B_0) = \mathbb{Z}[P_0]_0$, $\mathbf{K}_1(B_0) = 0$ and $\mathbf{K}_0(B_0/\mathcal{I}_0) = \mathbb{Z}_{|\chi(\Gamma')|}p'_{1*}([P_0]_0)$, $\mathbf{K}_1(B_0/\mathcal{I}_0) = 0$.

Suppose by induction that

$$\mathbf{K}_0(B_{m-1}) = \mathbb{Z}[P_{m-1}]_0 \oplus \mathbb{Z}_{|\chi(\Gamma')|}[P_{m-2}]_0 \oplus \cdots \oplus \mathbb{Z}_{|\chi(\Gamma')|}[P_0]_0,$$

$\mathbf{K}_0(B_{m-1}/\mathcal{I}_{m-1}) = \mathbb{Z}_{|\chi(\Gamma')|}p'_{m*}([P_{m-1}]_0) \oplus \mathbb{Z}_{|\chi(\Gamma')|}p'_{m*}([P_{m-2}]_0) \oplus \cdots \oplus \mathbb{Z}_{|\chi(\Gamma')|}p'_{m*}([P_0]_0)$, $\mathbf{K}_1(B_{m-1}) = 0$ and $\mathbf{K}_1(B_{m-1}/\mathcal{I}_{m-1}) = 0$.

Then from diagram (21), it immediately follows that $\mathbf{K}_1(B_m) = 0$ and $\mathbf{K}_1(B_m/\mathcal{I}_m) = 0$.

Then (21) reduces to the following commutative diagram with exact rows:

(26)

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbf{K}_0(\mathcal{I}_{m-1}) & \xrightarrow{i'_{m*}} & \mathbf{K}_0(B_{m-1}) & \xrightarrow{p'_{m*}} & \mathbf{K}_0(B_{m-1}/\mathcal{I}_{m-1}) \longrightarrow 0 \\
 & & \downarrow I'_{m*} = 0 & & \downarrow I_{m*} & & \cong \downarrow \pi_{m*} \\
 0 & \longrightarrow & \mathbf{K}_0(\mathcal{I}_m) & \xrightarrow{i_{m*}} & \mathbf{K}_0(B_m) & \xrightarrow{p_{m*}} & \mathbf{K}_0(B_m/\mathcal{I}_m) \longrightarrow 0.
 \end{array}$$

From Lemma 3.1 we have that $\chi(\Gamma')[P_l]_0 = 0$ in $\mathbf{K}_0(B_m)$ for $l = 0, \dots, m-1$. Since π_{m*} is an isomorphism then by the induction hypothesis and (26) it is easy to see that p_{m*} restricted to $\mathcal{G} = \langle [P_0]_0, \dots, [P_{m-1}]_0 \rangle_{\mathbf{K}_0(B_m)}$ is an isomorphism. This fact also implies that there are no relations between $[P_m]_0$ and $\mathcal{G}$ (since the bottom row of (26) is exact). Since i_{m*} is injective, then $[P_m]_0$ is of infinite order in $\mathbf{K}_0(B_m)$. Clearly $\mathbf{K}_0(B_m)$ is generated by $[P_m]_0$ and $\mathcal{G}$. Therefore

$$\mathbf{K}_0(B_m) = \mathbb{Z}[P_m]_0 \oplus \mathbb{Z}_{|\chi(\Gamma')|}[P_{m-1}]_0 \oplus \cdots \oplus \mathbb{Z}_{|\chi(\Gamma')|}[P_0]_0.$$

From the following six-term exact sequence

$$\begin{array}{ccccc}
 \mathbf{K}_0(\mathcal{I}_m) & \xrightarrow{i'_{m*}} & \mathbf{K}_0(B_m) & \xrightarrow{p'_{m*}} & \mathbf{K}_0(B_m/\mathcal{I}_m) \\
 \uparrow & & & & \downarrow \\
 \mathbf{K}_1(B_m/\mathcal{I}_m) & \xleftarrow{p'_{m*}} & \mathbf{K}_1(B_m) & \xleftarrow{i'_{m*}} & \mathbf{K}_1(\mathcal{I}_m)
 \end{array}$$

and the fact that i'_{m*} is given by $[P_{m+1}Q]_0 \mapsto \chi(\Gamma')[P_m]_0$ we easily get that $\mathbf{K}_1(B_m/\mathcal{I}_m) = 0$ and that

$$\mathbf{K}_0(B_m/\mathcal{I}_m) = \mathbb{Z}_{|\chi(\Gamma')|} p'_{m+1*}([P_m]_0) \oplus \mathbb{Z}_{|\chi(\Gamma')|} p'_{m+1*}([P_{m-1}]_0) \oplus \dots \oplus \mathbb{Z}_{|\chi(\Gamma')|} p'_{m+1*}([P_0]_0).$$

This completes the induction.

We have $\mathbf{K}_0(B) = \bigoplus_{i=0}^{\infty} \mathbb{Z}_{|\chi(\Gamma')|} [P_i]_0$ and $\mathbf{K}_1(B) = 0$. We can write $\mathbf{K}_0(B) = \bigoplus_{i=0}^{\infty} \mathbb{Z}_{|\chi(\Gamma')|} \beta_*^i([P_0]_0)$. Therefore $\mathbf{K}_0(\tilde{B}) = \bigoplus_{i=-\infty}^{\infty} \mathbb{Z}_{|\chi(\Gamma')|} \Phi_*^i([\tilde{P}_0]_0)$ and $\mathbf{K}_1(\tilde{B}) = 0$.

The Pimsner-Voiculescu exact sequence gives

$$\begin{array}{ccccc} \bigoplus_{i=-\infty}^{\infty} \mathbb{Z}_{|\chi(\Gamma')|} \Phi_*^i([\tilde{P}_0]_0) & \xrightarrow{\text{id}_* - \Phi_*} & \bigoplus_{i=-\infty}^{\infty} \mathbb{Z}_{|\chi(\Gamma')|} \Phi_*^i([\tilde{P}_0]_0) & \longrightarrow & \mathbf{K}_0(\tilde{A}) \\ \uparrow & & & & \downarrow \\ \mathbf{K}_1(\tilde{A}) & \longleftarrow & 0 & \longleftarrow & 0. \end{array}$$

We conclude that $\mathbf{K}_0(\tilde{A}) = \mathbb{Z}_{|\chi(\Gamma')|} [\tilde{P}_0]_0$ and $\mathbf{K}_1(\tilde{A}) = 0$. From Theorem 2.4 we get $\mathbf{K}_0(C_Q^*(\Gamma)) = \mathbb{Z}_{|\chi(\Gamma)|} [1_{C_Q^*(\Gamma)}]_0$ and $\mathbf{K}_1(C_Q^*(\Gamma)) = 0$ (notice that $\chi(\Gamma) = \chi(\Gamma') - \chi(\Gamma_k) = \chi(\Gamma') - 0$).

From Remark 3.2 it follows that $I_{\Gamma*}$ is “multiplication by $\chi(\Gamma)$ ”, so $\mathbf{K}_0(C^*(\Gamma)) = \mathbb{Z}[1_{C^*(\Gamma)}]_0$ and $\mathbf{K}_1(C^*(\Gamma)) = 0$.

This concludes the proof of (case II).

(case III): $\chi(\Gamma') = 0$ and $\chi(\Gamma_k) \neq 0$.

By assumption we have that $\mathbf{K}_0(B_0) = \mathbb{Z}[P_0]_0$, $\mathbf{K}_0(B_0/\mathcal{I}_0) = \mathbb{Z}p'_{1*}([P_0]_0)$, $\mathbf{K}_1(B_0) = 0$ and $\mathbf{K}_1(B_0/\mathcal{I}_0) = \mathbb{Z}$.

Suppose by induction that

$$\mathbf{K}_0(B_{m-1}) = \mathbb{Z}[P_0]_0 \oplus \mathbb{Z}_{|\chi(\Gamma_k)|} [P_1]_0 \oplus \dots \oplus \mathbb{Z}_{|\chi(\Gamma_k)|} [P_{m-1}]_0$$

and that $\mathbf{K}_1(B_{m-1}) = 0$. Then from diagram (21) we see that the maps γ_m^{ind} and $p'_{m*} : \mathbf{K}_0(B_{m-1}) \rightarrow \mathbf{K}_0(B_{m-1}/\mathcal{I}_{m-1})$ are isomorphisms. Also since π_m is an isomorphism this implies that $I_{m*} : \mathbf{K}_0(B_{m-1}) \rightarrow \mathbf{K}_0(B_m)$ is an isomorphism into and that p_{m*} restricted to $\mathcal{G} = \langle [P_0]_0, \dots, [P_{m-1}]_0 \rangle_{\mathbf{K}_0(B_m)}$ is also an isomorphism. From the fact that $p_{m*}|_{\mathcal{G}}$ is injective it follows that there are no relations between $[P_m]_0$ and $\mathcal{G}$.

The commutativity of

$$\begin{array}{ccc} \mathbf{K}_1(B_{m-1}/\mathcal{I}_{m-1}) & \xrightarrow[\cong]{\gamma_m^{\text{ind}}} & \mathbf{K}_0(\mathcal{I}_{m-1}) \\ \pi_{m*} \downarrow \cong & & I_{m*} \downarrow [= \times \chi(\Gamma_k)] \\ \mathbf{K}_1(B_m/\mathcal{I}_m) & \xrightarrow{\delta_m^{\text{ind}}} & \mathbf{K}_0(T_m) \end{array}$$

implies that δ_m^{ind} is “multiplication by $\chi(\Gamma_k)$ ”. Thus $[P_m]_0$ in $\mathbf{K}_0(B_m)$ is of order $\chi(\Gamma_k)$ (as it should be by Lemma 3.1). Therefore

$$\mathbf{K}_0(B_m) = \mathbb{Z}[P_0]_0 \oplus \mathbb{Z}_{|\chi(\Gamma_k)|} [P_1]_0 \oplus \dots \oplus \mathbb{Z}_{|\chi(\Gamma_k)|} [P_m]_0.$$

We also showed that δ_m^{ind} is injective and therefore $\mathbf{K}_1(B_m) = 0$.

Now we easily get $\mathbf{K}_0(B) = \mathbb{Z}[P_0]_0 \oplus \bigoplus_{i=1}^{\infty} \mathbb{Z}_{|\chi(\Gamma_k)|} \beta_*^i([P_0]_0)$ and $\mathbf{K}_1(B) = 0$. From this it follows that $\mathbf{K}_0(\tilde{B}) = \bigoplus_{i=-\infty}^{\infty} \mathbb{Z}_{|\chi(\Gamma_k)|} \Phi_*^i([\tilde{P}_0]_0)$ and $\mathbf{K}_1(\tilde{B}) = 0$.

The Pimsner-Voiculescu exact sequence gives

$$\begin{array}{ccccc} \bigoplus_{i=-\infty}^{\infty} \mathbb{Z}_{|\chi(\Gamma_k)|} \Phi_*^i([\tilde{P}_0]_0) & \xrightarrow{\text{id}_* - \Phi_*} & \bigoplus_{i=-\infty}^{\infty} \mathbb{Z}_{|\chi(\Gamma_k)|} \Phi_*^i([\tilde{P}_0]_0) & \longrightarrow & \mathbf{K}_0(\tilde{A}) \\ \uparrow & & & & \downarrow \\ \mathbf{K}_1(\tilde{A}) & \longleftarrow & 0 & \longleftarrow & 0. \end{array}$$

We conclude that $\mathbf{K}_0(\tilde{A}) = \mathbb{Z}_{|\chi(\Gamma_k)|} [\tilde{P}_0]_0$ and $\mathbf{K}_1(\tilde{A}) = 0$. From Theorem 2.4 we get $\mathbf{K}_0(C_Q^*(\Gamma)) = \mathbb{Z}_{|\chi(\Gamma)|} [1_{C_Q^*(\Gamma)}]_0$ and $\mathbf{K}_1(C_Q^*(\Gamma)) = 0$ (notice that $\chi(\Gamma) = \chi(\Gamma') - \chi(\Gamma_k) = 0 - \chi(\Gamma_k)$).

From Remark 3.2 it follows that $I_{\Gamma*}$ is “multiplication by $\chi(\Gamma)$ ”, so $\mathbf{K}_0(C^*(\Gamma)) = \mathbb{Z}[1_{C^*(\Gamma)}]_0$ and $\mathbf{K}_1(C^*(\Gamma)) = 0$.

This concludes the proof of (case **III**).

(cases **IV** and **V**): $\chi(\Gamma') \neq 0$, $\chi(\Gamma_k) \neq 0$.

Let's denote $x = \chi(\Gamma')$, $y = \chi(\Gamma_k)$ and let $\text{GCD}(x, y) = d > 0$ be the greatest common divisor of x and y . Then by Bézout's identity there exist $a, b \in \mathbb{Z}$ such that $ax + by = d$. Denote also $x' = x/d$ and $y' = y/d$. Then $ax' + by' = 1$.

By assumption we have that $\mathbf{K}_0(B_0) = \mathbb{Z}[P_0]_0$, $\mathbf{K}_1(B_0/\mathcal{I}_0) = 0$, $\mathbf{K}_1(B_0) = 0$ and $\mathbf{K}_0(B_0/\mathcal{I}_0) = \mathbb{Z}_{|x|} p'_{1*}([P_0]_0)$.

Then $0 = \mathbf{K}_1(B_0/\mathcal{I}_0) \cong \mathbf{K}_1(B_1/T_1)$, which implies $\mathbf{K}_1(B_1) = 0$. Diagram (21) for $m = 1$ reduces to

$$(27) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{K}_0(\mathcal{I}_0) & \xrightarrow{i'_{1*}} & \mathbf{K}_0(B_0) & \xrightarrow{p'_{1*}} & \mathbf{K}_0(B_0/\mathcal{I}_0) \longrightarrow 0 \\ & & I'_{1*} \downarrow & & I_{1*} \downarrow & & \cong \downarrow \pi_{1*} \\ 0 & \longrightarrow & \mathbf{K}_0(T_1) & \xrightarrow{i_{1*}} & \mathbf{K}_0(B_1) & \xrightarrow{p_{1*}} & \mathbf{K}_0(B_1/T_1) \longrightarrow 0. \end{array}$$

Clearly in $\mathbf{K}_0(B_1)$ we have $x[P_0]_0 - y[P_1]_0 = 0$. Consider $g = b[P_0]_0 + a[P_1]_0$, $g' = x'[P_0]_0 - y'[P_1]_0 \in \mathbf{K}_0(B_1)$. Since $ag' + y'g = (ax' + y'b)[P_0]_0 = [P_0]_0$ and $x'g - bg' = (x'a + by')[P_1]_0 = [P_1]_0$, it follows that g and g' generate $\mathbf{K}_0(B_1)$. Since i_{1*} is injective on $\mathbf{K}_0$ it follows that $\mathbf{K}_0(B_1)$ is an infinite group. Clearly $dg' = dx'[P_0]_0 - dy'[P_1]_0 = 0$. Therefore g is of infinite order in $\mathbf{K}_0(B_1)$ and moreover g and g' are not related (or otherwise g would be of finite order). If we suppose that $0 < d' \mid d$ and $d'g' = 0$, then it will follow that $d'x'[P_0]_0 = d'y'[P_1]_0 \in \ker(p_{1*})$. But the order of $p_{1*}([P_0]_0)$ in $\mathbf{K}_0(B_1/T_1)$ is $|x|$, so $d'x' \geq x$ or $d' \geq d$. Therefore $d' = d$ and $\mathbf{K}_0(B_1) = \mathbb{Z}(b[P_0]_0 + a[P_1]_0) \oplus \mathbb{Z}_d(x'[P_0]_0 - y'[P_1]_0)$.

Therefore we showed that $\mathbf{K}_0(B_1) = \{\mathbb{Z}[P_0]_0 \oplus \mathbb{Z}[P_1]_0 \mid x[P_0]_0 - y[P_1]_0 = 0\}$. Suppose by induction that for $m \geq 2$, $\mathbf{K}_1(B_{m-1}) = 0$ and that

$$\begin{aligned} \mathbf{K}_0(B_{m-1}) = \{ & \mathbb{Z}[P_0]_0 \oplus \cdots \oplus \mathbb{Z}[P_{m-1}]_0 \mid x[P_0]_0 - y[P_1]_0 = 0, \dots, \\ & x[P_{m-2}]_0 - y[P_{m-1}]_0 = 0 \}. \end{aligned}$$

From the induction hypothesis it follows that $[P_{m-1}]_0$ is of infinite order in $\mathbf{K}_0(B_{m-1})$ and therefore that $i'_{m*} : \mathbf{K}_0(\mathcal{I}_{m-1}) \rightarrow \mathbf{K}_0(B_{m-1})$ is injective and therefore $0 = \mathbf{K}_1(B_{m-1}/\mathcal{I}_{m-1}) \cong \mathbf{K}_1(B_m/T_m)$. This shows that $\mathbf{K}_1(B_m) = 0$ and that

(21) reduces to

$$(28) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{K}_0(\mathcal{I}_{m-1}) & \xrightarrow{i'_{m*}} & \mathbf{K}_0(B_{m-1}) & \xrightarrow{p'_{m*}} & \mathbf{K}_0(B_{m-1}/\mathcal{I}_{m-1}) \longrightarrow 0 \\ & & \downarrow I'_{m*} & & \downarrow I_{m*} & & \cong \downarrow \pi_{m*} \\ 0 & \longrightarrow & \mathbf{K}_0(T_m) & \xrightarrow{i_{m*}} & \mathbf{K}_0(B_m) & \xrightarrow{p_{m*}} & \mathbf{K}_0(B_m/T_m) \longrightarrow 0. \end{array}$$

It is easy to see that

$$\begin{aligned} \mathbf{K}_0(B_{m-1}/\mathcal{I}_{m-1}) &= \{\mathbb{Z}p'_{m*}([P_{m-1}]_0) \oplus \cdots \oplus \mathbb{Z}p'_{m*}([P_0]_0) | xp'_{m*}([P_{m-1}]_0) = 0, \\ &\quad xp'_{m*}([P_{m-2}]_0) - yp'_{m*}([P_{m-1}]_0) = 0, \dots, xp'_{m*}([P_0]_0) - yp'_{m*}([P_1]_0) = 0\}. \end{aligned}$$

Since I'_{m*} is “multiplication by $\chi(\Gamma_k)$ ” and therefore injective, then by the Five Lemma it follows that I_{m*} is also injective. Therefore if we denote $\mathcal{G} = I_{m*}(\mathbf{K}_0(B_{m-1}))$, then $\mathbf{K}_0(B_m) = \langle [P_m]_0, \mathcal{G} \rangle$. One obvious relation in $\mathbf{K}_0(B_m)$ besides the relations that come from $\mathbf{K}_0(B_{m-1})$ is $x[P_{m-1}]_0 - y[P_m]_0 = 0$, and this relation follows from Lemma 3.1.

Therefore $\mathbf{K}_0(B_m)$ is a quotient of the group

$$F = \{\mathbb{Z}\rho_0 \oplus \cdots \oplus \mathbb{Z}\rho_m | x\rho_{m-1} - y\rho_m = 0, \dots, x\rho_0 - y\rho_1 = 0\},$$

where the quotient map $f : F \rightarrow \mathbf{K}_0(B_m)$ is defined on the generators as $\rho_l \mapsto [P_l]_0$, $l = 0, \dots, m$. Then if $F' = \mathbb{Z}\rho_m$, the quotient $F_q = F/F'$ is isomorphic to

$$\begin{aligned} F_q &= \{\mathbb{Z}\rho_0 \oplus \cdots \oplus \mathbb{Z}\rho_m | x\rho_{m-1} - y\rho_m = 0, \dots, x\rho_0 - y\rho_1 = 0, \rho_m = 0\} \\ &= \{\mathbb{Z}\rho_0 \oplus \cdots \oplus \mathbb{Z}\rho_{m-1} | x\rho_{m-1} = 0, x\rho_{m-2} - y\rho_{m-1} = 0, \dots, x\rho_0 - y\rho_1 = 0\}. \end{aligned}$$

Obviously we have the commutative diagram of abelian groups with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F_q \longrightarrow 0 \\ & & \downarrow f' & & \downarrow f & & \downarrow f_q \\ 0 & \longrightarrow & \mathbf{K}_0(T_m) & \xrightarrow{i_{m*}} & \mathbf{K}_0(B_m) & \xrightarrow{p_{m*}} & \mathbf{K}_0(B_m/T_m) \longrightarrow 0, \end{array}$$

where f_q is the homomorphism induced by f , and f' is the restriction of f to F' . Then obviously f' and f_q are isomorphisms (since π_{m*} is an isomorphism). Therefore by the Five Lemma it follows that f is also an isomorphism.

This shows that

$$\mathbf{K}_0(B_m) = \{\mathbb{Z}[P_0]_0 \oplus \cdots \oplus \mathbb{Z}[P_m]_0 | x[P_0]_0 - y[P_1]_0 = 0, \dots, x[P_{m-1}]_0 - y[P_m]_0 = 0\}.$$

We also showed above that $\mathbf{K}_1(B_m) = 0$ and this concludes the induction.

Now it is easy to see that $\mathbf{K}_1(B) = 0$ and that

$$\mathbf{K}_0(B) = \left\{ \bigoplus_{i=0}^{\infty} \mathbb{Z}\beta_*^i([P_0]_0) | \chi(\Gamma')\beta_*^i([P_0]_0) - \chi(\Gamma_k)\beta_*^{i+1}([P_0]_0) = 0, i \in \mathbb{N}_0 \right\}.$$

Then $\mathbf{K}_1(\tilde{B}) = 0$ and

$$\mathbf{K}_0(\tilde{B}) = \left\{ \bigoplus_{i=-\infty}^{\infty} \mathbb{Z}\Phi_*^i([\tilde{P}_0]_0) | \chi(\Gamma')\Phi_*^i([\tilde{P}_0]_0) - \chi(\Gamma_k)\Phi_*^{i+1}([\tilde{P}_0]_0) = 0, i \in \mathbb{Z} \right\}.$$

The Pimsner-Voiculescu exact sequence gives

$$(29) \quad \begin{array}{ccccc} \mathbf{K}_0(\tilde{B}) & \xrightarrow{\text{id}_* - \Phi_*} & \mathbf{K}_0(\tilde{B}) & \longrightarrow & \mathbf{K}_0(\tilde{A}) \\ \uparrow & & & & \downarrow \\ \mathbf{K}_1(\tilde{A}) & \longleftarrow & 0 & \longleftarrow & 0. \end{array}$$

(case **IV**): $\chi(\Gamma') \neq 0$, $\chi(\Gamma_k) \neq 0$ and $\chi(\Gamma') = \chi(\Gamma_k)$.

In this case,

$$\begin{aligned} \mathbf{K}_0(\tilde{A}) &= \left\{ \bigoplus_{i=-\infty}^{\infty} \mathbb{Z} \Phi_*^i([\tilde{P}_0]_0) | \chi(\Gamma') \Phi_*^i([\tilde{P}_0]_0) - \chi(\Gamma') \Phi_*^{i+1}([\tilde{P}_0]_0) = 0, \right. \\ &\quad \left. \Phi_*^i([\tilde{P}_0]_0) - \Phi_*^{i+1}([\tilde{P}_0]_0) = 0, i \in \mathbb{Z} \right\} \\ &= \left\{ \bigoplus_{i=-\infty}^{\infty} \mathbb{Z} \Phi_*^i([\tilde{P}_0]_0) | \Phi_*^i([\tilde{P}_0]_0) - \Phi_*^{i+1}([\tilde{P}_0]_0) = 0, i \in \mathbb{Z} \right\} = \mathbb{Z}[\tilde{P}_0]_0. \end{aligned}$$

To examine $\ker(\text{id}_* - \Phi_*)$ take $\omega = \sum_{i=-j}^j t_i \Phi_*^i([\tilde{P}_0]_0) \in \ker(\text{id}_* - \Phi_*)$, where $t_i \in \mathbb{Z}$.

Then

$$0 = (\text{id}_* - \Phi_*)(\omega) = \sum_{i=-j}^j t_i (\text{id}_* - \Phi_*)(\Phi_*^i([\tilde{P}_0]_0)) = \sum_{i=-j}^j t_i (\Phi_*^i([\tilde{P}_0]_0) - \Phi_*^{i+1}([\tilde{P}_0]_0)).$$

Therefore $t_i = s_i |\chi(\Gamma')|$ for some integers s_i , $i = -j, \dots, j$. From this it easily follows that $\omega = \sum_{i=-j}^j s_i |\chi(\Gamma')| \Phi_*^i([\tilde{P}_0]_0)$. Thus $\ker(\text{id}_* - \Phi_*) = |\chi(\Gamma')| \mathbb{Z}[\tilde{P}_0]_0$. This shows that $\mathbf{K}_1(\tilde{A}) = \mathbb{Z}$.

From Theorem 2.4 it follows that $\mathbf{K}_0(\tilde{A}) \cong \mathbf{K}_0(C_Q^*(\Gamma)) = \mathbb{Z}$, $\mathbf{K}_1(\tilde{A}) \cong \mathbf{K}_1(C_Q^*(\Gamma)) = \mathbb{Z}$ and that $[1_{C_Q^*(\Gamma)}]_0$ generates $\mathbf{K}_0(C_Q^*(\Gamma))$.

From Remark 3.2 it follows that in the extension (19) the map I_{Γ_*} on $\mathbf{K}_0$ is zero (since $\chi(\Gamma) = \chi(\Gamma') - \chi(\Gamma_k) = 0$). Therefore $\mathbf{K}_0(C^*(\Gamma)) = \mathbb{Z}[1_{C^*(\Gamma)}]_0$ and $\mathbf{K}_1(C^*(\Gamma)) = 0$.

This concludes the proof of (case **IV**).

(case **V**): $\chi(\Gamma') \neq 0$, $\chi(\Gamma_k) \neq 0$ and $\chi(\Gamma') \neq \chi(\Gamma_k)$.

In this case,

$$\begin{aligned} \mathbf{K}_0(\tilde{A}) &= \left\{ \bigoplus_{i=-\infty}^{\infty} \mathbb{Z} \Phi_*^i([\tilde{P}_0]_0) | \chi(\Gamma') \Phi_*^i([\tilde{P}_0]_0) - \chi(\Gamma_k) \Phi_*^{i+1}([\tilde{P}_0]_0) = 0, \right. \\ &\quad \left. \Phi_*^i([\tilde{P}_0]_0) - \Phi_*^{i+1}([\tilde{P}_0]_0) = 0, i \in \mathbb{Z} \right\} \\ &= \{ \mathbb{Z}[\tilde{P}_0]_0 | \chi(\Gamma')[\tilde{P}_0]_0 - \chi(\Gamma_k)[\tilde{P}_0]_0 = 0 \} = \mathbb{Z}_{|\chi(\Gamma') - \chi(\Gamma_k)|}[\tilde{P}_0]_0 = \mathbb{Z}_{|\chi(\Gamma)|}[\tilde{P}_0]_0. \end{aligned}$$

We only need to show that $\mathbf{K}_1(\tilde{A}) = 0$ or that $\text{id}_* - \Phi_*$ is injective.

Take $\omega = \sum_{i=-j}^j t_j \Phi_*^i([\tilde{P}_0]_0)$, $t_i \in \mathbb{Z}$ and suppose that $(\text{id}_* - \Phi_*)(\omega) = 0$. Then

$$\begin{aligned} 0 &= (\text{id}_* - \Phi_*)(\omega) = \sum_{i=-j}^j t_j (\Phi_*^i([\tilde{P}_0]_0) - \Phi_*^{i+1}([\tilde{P}_0]_0)) \\ &= t_{-j} \Phi_*^{-j}([\tilde{P}_0]_0) + \sum_{i=-j+1}^j (t_i - t_{i-1}) \Phi_*^i([\tilde{P}_0]_0) - t_j \Phi_*^{j+1}([\tilde{P}_0]_0). \end{aligned}$$

If $\chi(\Gamma')$ doesn't divide t_{-j} , then the equality $-t_{-j} \Phi_*^{-j}([\tilde{P}_0]_0) = (t_i - t_{i-1}) \Phi_*^i([\tilde{P}_0]_0) - t_j \Phi_*^{j+1}([\tilde{P}_0]_0)$ is impossible. If $\chi(\Gamma')$ divides t_{-j} , then ω can be expressed in terms of $\Phi_*^{-j+1}([\tilde{P}_0]_0), \dots, \Phi_*^j([\tilde{P}_0]_0)$. By induction we see that we can write $\omega = t[\tilde{P}_0]_0$ for some $t \in \mathbb{Z}$. But then clearly $(\text{id}_* - \Phi_*)(\omega) = 0$ is possible if and only if $t = 0$. This shows that $\text{id}_* - \Phi_*$ is injective and therefore that $\mathbf{K}_1(\tilde{A}) = 0$.

From Theorem 2.4 we get $\mathbf{K}_0(C_Q^*(\Gamma)) = \mathbb{Z}_{|\chi(\Gamma)|} [1_{C_Q^*(\Gamma)}]_0$ and $\mathbf{K}_1(C_Q^*(\Gamma)) = 0$.

From Remark 3.2 it follows that I_{Γ^*} is “multiplication by $\chi(\Gamma)$ ”, so $\mathbf{K}_0(C^*(\Gamma)) = \mathbb{Z}[1_{C^*(\Gamma)}]_0$ and $\mathbf{K}_1(C^*(\Gamma)) = 0$.

This concludes the proof of (case **V**).

The theorem is proved. $\square$

Now we can apply the Kirchberg-Phillips Classification Theorem ([16]) to $C_Q^*(G)$ for a finite graph G such that G^{opp} is connected and with at least two vertices, using Theorem 1.1, Proposition 2.4 and Proposition 3.3. We obtain

$$(30) \quad C_Q^*(G) \cong \mathcal{O}_{1+|\chi(G)|}.$$

For infinite graphs with connected opposite graphs we can argue similarly as in [8, Corollary 3.11] to prove the following:

Proposition 3.4. *Let G be an infinite graph with countably many vertices and such that G^{opp} is connected. Then $C^*(G)$ ($= C_Q^*(G)$) belongs to the small bootstrap class. Moreover $\mathbf{K}_0(C^*(G)) = \mathbb{Z}[1_{C^*(G)}]_0$ and $\mathbf{K}_1(C^*(G)) = 0$.*

Proof. By induction we will find an increasing sequence G_n of subgraphs of G with n vertices, $n \geq 2$, which are such that G_n^{opp} is connected for each $n \geq 2$ and also $G_n \xrightarrow{n \rightarrow \infty} G$. Obviously we can find two vertices v_1 and v_2 that are not connected (since G^{opp} is connected). Then we choose G_2 to be the graph with vertices v_1 and v_2 and no edges. Suppose we have defined the subgraph G_n for some $n \geq 2$. Let $v_1, \dots, v_n$ be the vertices of G_n . Since G^{opp} is connected we can find a vertex v_{n+1} of G different from $v_1, \dots, v_n$ such that v_{n+1} is not connected with all of the vertices $v_1, \dots, v_n$. Then obviously the subgraph G_{n+1} of G on vertices $v_1, \dots, v_{n+1}$ and edges coming from G is such that G_{n+1}^{opp} is connected. This completes the induction.

From Proposition 3.3 we have $\mathbf{K}_0(C^*(G_n)) = \mathbb{Z}[1_{C^*(G_n)}]_0$ and $\mathbf{K}_1(C^*(G_n)) = 0$. It is easy to see that $C^*(G) = \varinjlim C^*(G_n)$. Therefore from Proposition 2.4 we get that $C_Q^*(G)$ belongs to the small bootstrap category $\mathfrak{N}$. Also $\mathbf{K}_0(C^*(G)) = \varinjlim_{n \rightarrow \infty} \mathbf{K}_0(C^*(G_n)) = \mathbb{Z}[1_{C^*(G)}]_0$ and $\mathbf{K}_1(C^*(G)) = \varinjlim_{n \rightarrow \infty} \mathbf{K}_1(C^*(G_n)) = 0$.

This proves the proposition. $\square$

From Theorem 1.1 we know that $C^*(G) = C_Q^*(G)$ is purely infinite and simple. Again using the Kirchberg-Phillips Theorem we get that if G is an infinite graph on countably many vertices such that G^{opp} is connected, then $C^*(G) = C_Q^*(G) \cong \mathcal{O}_\infty$.

If we define for an infinite countable graph G with G^{opp} connected $\chi(G) \stackrel{\text{def}}{=} \infty$, then we can once again write $C_Q^*(G) \cong \mathcal{O}_{1+|\chi(G)|}$.

Remark 3.5. Let G_1 and G_2 be two disjoint graphs. Then by $G_1 * G_2$ we denote their join, which is the graph obtained from G_1 and G_2 by connecting each vertex of G_1 with each vertex of G_2 . Then if we start with a graph G on countably many vertices which is such that G^{opp} doesn't have any isolated vertices, then we can find a sequence of subgraphs G_n , $n \in \mathbb{N}$ (some of the G_n 's can have zero vertices) such that G_n^{opp} are all connected and such that $G = \bigstar_{n=1}^{\infty} G_n$. For a graph F with zero vertices we write $C_Q^*(F) = \mathbb{C}$.

Then from Theorem 1.1 it easily follows that $C_Q^*(G) = \bigotimes_{n=1}^{\infty} C_Q^*(G_n)$.

We record our main result:

Theorem 3.6. *Let G be a graph with at least two and at most countably many vertices such that G^{opp} has no isolated vertices. Write $G = \bigstar_{n=1}^{\infty} G_n$ as in Remark 3.5 with G_n being a subgraph of G such that G_n^{opp} is connected.*

Then

$$(31) \quad C_Q^*(G) = \bigotimes_{n=1}^{\infty} C_Q^*(G_n) \cong \bigotimes_{n=1}^{\infty} \mathcal{O}_{1+|\chi(G_n)|}.$$

ACKNOWLEDGEMENTS

Most of the work on this paper was done when the author was a graduate student at Texas A&M University. He would like to thank Ron Douglas for choosing this research topic and for the many useful conversations the author had with him. The author wants to thank Marcelo Laca for sending hard copies of some of his papers on the subject. Finally the author thanks Ken Dykema, Marcelo Laca and George Elliott for some discussions.

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