TWO-DIMENSIONAL EULER FLOWS WITH CONCENTRATED VORTICITIES

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ABSTRACT. For a planar model of Euler flows proposed by Tur and Yanovsky (2004), we construct a family of velocity fields $\mathbf{w}_\varepsilon$ for a fluid in a bounded region Ω , with concentrated vorticities ω_ε for $\varepsilon > 0$ small. More precisely, given a positive integer α and a sufficiently small complex number a , we find a family of stream functions ψ_ε which solve the Liouville equation with Dirac mass source,

$$\Delta\psi_\varepsilon + \varepsilon^2 e^{\psi_\varepsilon} = 4\pi\alpha\delta_{p_{a,\varepsilon}} \quad \text{in } \Omega, \quad \psi_\varepsilon = 0 \quad \text{on } \partial\Omega,$$

for a suitable point $p = p_{a,\varepsilon} \in \Omega$. The vorticities $\omega_\varepsilon := -\Delta\psi_\varepsilon$ concentrate in the sense that

$$\omega_\varepsilon + 4\pi\alpha\delta_{p_{a,\varepsilon}} - 8\pi \sum_{j=1}^{\alpha+1} \delta_{p_{a,\varepsilon}+a_j} \rightharpoonup 0 \quad \text{as } \varepsilon \rightarrow 0,$$

where the *satellites* $a_1, \dots, a_{\alpha+1}$ denote the complex $(\alpha+1)$ -roots of a . The point $p_{a,\varepsilon}$ lies close to a zero point of a vector field explicitly built upon derivatives of order $\leq \alpha+1$ of the regular part of Green's function of the domain.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

We are concerned with stationary Euler equations for an incompressible and homogeneous fluid,

$$(1) \quad \begin{cases} (\mathbf{w} \cdot \nabla) \mathbf{w} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{w} = 0 & \text{in } \Omega, \end{cases}$$

where $\mathbf{w}$ is the velocity field and p is the pressure. The domain Ω is either the whole $\mathbb{R}^n$, $n = 2, 3$, or a smooth, bounded domain Ω . In the latter situation the velocity field $\mathbf{w}$ is naturally required to be tangent on $\partial\Omega$, that is,

$$(2) \quad \mathbf{w} \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

ν being a unit normal vector to $\partial\Omega$. We shall restrict our investigation to the planar case $n = 2$ when Ω is bounded and introduce the vorticity $\omega = \operatorname{curl} \mathbf{w}$. By applying

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the curl operator to the first equation in (1), we are reduced to the Euler equations in vorticity form,

$$(3) \quad \begin{cases} \mathbf{w} \cdot \nabla \omega = 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{w} = 0 & \text{in } \Omega, \end{cases}$$

supplemented by (2). We refer the reader to the books [25] and [23] for an exhaustive treatment of fluid mechanics models.

Let us further rewrite Problem (3). The second equation in (3) is equivalent to rewriting the velocity field $\mathbf{w}$ as

$$\mathbf{w} = (\partial_{x_2} \psi, -\partial_{x_1} \psi).$$

In turn, the vorticity ω expresses as $\omega = -\Delta \psi$ in terms of ψ , the so-called stream function.

Now the *ansatz* $\omega = f(\psi)$ guarantees that the first equation in (3) is also automatically satisfied, and then the Euler equations reduce to solving the elliptic problem

$$(4) \quad \Delta \psi + f(\psi) = 0,$$

with Dirichlet boundary condition $\psi = 0$ on $\partial\Omega$ on a bounded domain Ω to account for (2).

Many choices of f are physically meaningful and lead to vortex-type configurations. The Stuart vortex pattern in [30] corresponds to $f(\psi) = \varepsilon^2 e^\psi$. Tur and Yanovsky have recently proposed in [35] a singular *ansatz*

$$f(\psi) = \varepsilon^2 e^\psi - 4\pi\alpha\delta_p, \quad \alpha \in \mathbb{N},$$

to describe vortex patterns of necklace type with $(\alpha+1)$ -fold symmetry in rotational shear flow. Both papers [30, 35] consider Problem (4) in the whole $\mathbb{R}^2$ and explicit solutions are easily built according to Liouville's formula below. On a bounded domain Ω , a statistical mechanics approach in [5, 6, 20] has provided a rigorous derivation of *Stuart's ansatz*.

In this paper we consider the Tur-Yanovsky problem on a bounded domain Ω , a much harder issue to pursue. In terms of the stream function ψ we are thus led to the singular Liouville equation

$$(5) \quad \begin{cases} \Delta \psi + \varepsilon^2 e^\psi = 4\pi\alpha\delta_p & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

The parameter $\varepsilon > 0$ is small and, as we will see, its smallness will yield to flows having vorticities ω concentrated on small “blobs”.

Liouville-type equations arise also in several superconductivity theories in the self-dual regime, as for the abelian Maxwell-Higgs and Chern-Simons-Higgs theories. In the latter model, a mean field form of Problem (5) on the torus arises as a limiting equation for nontopological condensates as the Chern-Simons parameter tends to zero as shown in [28, 32]. Problem (5) is a limiting model equation in this context and explains why it has attracted a lot of attention, as we describe precisely below.

In a superconducting sample Ω a Dirichlet boundary condition φ can be imposed and the homogeneous case $\varphi = 0$, discussed in [18], is especially interesting since it describes a perfectly superconductive regime on $\partial\Omega$.

The regular case $\alpha = 0$ in Problem (5), sometimes referred to as the Gelfand problem [16],

$$(6) \quad \begin{cases} \Delta\psi + \varepsilon^2 e^\psi = 0 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases}$$

has been broadly considered in the literature. When $\varepsilon > 0$ is sufficiently small, the existence of both a small and a large solution as $\varepsilon \rightarrow 0$ has long been known as first found in [11, 19]. In the language of the calculus of variations, applied to the functional

$$J(\psi) = \frac{1}{2} \int_{\Omega} |\nabla\psi|^2 - \varepsilon^2 \int_{\Omega} e^\psi, \quad \psi \in H_0^1(\Omega),$$

the small solution corresponds to a local minimizer, while the large solution is a mountain pass critical point. The blowing-up behavior of a large solution was first described in [36] when Ω is simply connected. In the general case, the analysis in the works [4, 22, 26, 27, 31] yields that, if ψ_ε is a family of blowing-up solutions of (6) with $\varepsilon^2 \int_{\Omega} e^{\psi_\varepsilon}$ uniformly bounded, then, up to subsequences, there is an integer $k \geq 1$ such that $\varepsilon^2 \int_{\Omega} e^{\psi_\varepsilon} \rightarrow 8\pi k$. Moreover, there are points $\xi_1^\varepsilon, \dots, \xi_k^\varepsilon$ in Ω , which remain away one from each other and away from $\partial\Omega$, such that

$$(7) \quad \varepsilon^2 e^{\psi_\varepsilon} - \sum_{i=1}^k 8\pi \delta_{\xi_i^\varepsilon} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

in the measure sense. Besides,

$$(8) \quad \nabla\varphi_k(\xi_1^\varepsilon, \dots, \xi_k^\varepsilon) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

where

$$\varphi_k(\xi_1, \dots, \xi_k) := \begin{cases} H(\xi_1, \xi_1) & \text{if } k = 1, \\ \sum_{i=1}^k H(\xi_i, \xi_i) + \sum_{i \neq j} G(\xi_i, \xi_j) & \text{if } k \neq 1. \end{cases}$$

Here $G(z, \xi)$ denotes the Dirichlet Green's function in Ω , namely

$$G(z, \xi) = \Gamma(z - \xi) + H(z, \xi), \quad \Gamma(z) := \frac{1}{2\pi} \log \frac{1}{|z|},$$

where $H(z, \xi)$, its regular part, satisfies

$$(9) \quad \begin{cases} \Delta_z H(\cdot, \xi) = 0 & \text{in } \Omega, \\ H(\cdot, \xi) = -\Gamma(\cdot - \xi) & \text{on } \partial\Omega. \end{cases}$$

Moreover, refined asymptotic estimates hold, as established in a general setting in [7, 21], which lead to the computation of the Leray-Schauder degree of an associated nonlinear operator in [8]. In particular, ψ_ε satisfies

$$(10) \quad \psi_\varepsilon(z) = 8\pi \sum_{i=1}^k G(z, \xi_i^\varepsilon) + o(1), \quad \psi_\varepsilon(z) = \log \frac{8\mu_i^2}{(\mu_i^2 \varepsilon^2 + |z - \xi_i^\varepsilon|^2)^2} + o(1),$$

respectively, away from all points ξ_i^ε , and around each of them, for some $\mu_i > 0$. The family of functions

$$(11) \quad w_{\varepsilon, \mu, \xi}(z) := \log \frac{8\mu^2}{(\mu^2 \varepsilon^2 + |z - \xi|^2)^2}$$

corresponds to all solutions of the equation

$$\Delta w + \varepsilon^2 e^w = 0 \quad \text{in } \mathbb{R}^2,$$

such that $\int_{\mathbb{R}^2} e^w < +\infty$; see [9]. Besides,

$$\varepsilon^2 \int_{\mathbb{R}^2} e^{w_{\varepsilon, \mu, \xi}} = 8\pi, \quad \varepsilon^2 e^{w_{\varepsilon, \mu, \xi}} \rightharpoonup 8\pi \delta_\xi \quad \text{as } \varepsilon \rightarrow 0.$$

The reciprocal issue, namely the existence of solutions with the above properties, has been addressed in [1], near nondegenerate critical points of φ_k , and in [12, 15] associated to topologically nontrivial critical point situations for φ_k . In particular, it is found in [12] that ψ_ε as above exists for any $k \geq 1$ provided that the domain is not simply connected.

Important progress in the understanding of blowing-up solutions of Problem (5) with $\alpha > 0$ has been achieved in the local analysis in the works [2, 3, 34] from the point of view of quantization of blow-up levels and Harnack-type estimates. See [33] for a complete account on the topic.

Concerning construction of solutions to (5) with $\alpha > 0$, only a few results are available. The first important remark is that the functions

$$(12) \quad w_{\varepsilon, \mu}(z) = -4\pi\alpha\Gamma(z-p) + \log \frac{8\mu^2(1+\alpha)^2}{(\mu^2\varepsilon^2 + |z-p|^{2+2\alpha})^2}$$

satisfy precisely

$$(13) \quad \Delta w + \varepsilon^2 e^w = 4\pi\alpha\delta_p, \quad \varepsilon^2 \int_{\mathbb{R}^2} e^{w_{\varepsilon, \mu}} = 8\pi(1+\alpha), \quad \varepsilon^2 e^{w_{\varepsilon, \mu}} \rightharpoonup 8\pi(1+\alpha)\delta_p \quad \text{as } \varepsilon \rightarrow 0.$$

When $\alpha \in \mathbb{N}$, an integer, the family above extends to one carrying an extra parameter a which plays a similar role as ξ in (11): in complex notation, all functions

$$(14) \quad w_{\varepsilon, \mu, a}(z) = -4\pi\alpha\Gamma(z-p) + \log \frac{8\mu^2(1+\alpha)^2}{(\mu^2\varepsilon^2 + |(z-p)^{\alpha+1} - a|^2)^2}$$

satisfy (13) with the third property replaced by

$$(15) \quad \varepsilon^2 e^{w_{\varepsilon, \mu, a}} \rightharpoonup 8\pi \sum_{j=1}^{\alpha+1} \delta_{p+a_j} \quad \text{as } \varepsilon \rightarrow 0,$$

where the a_j 's are the complex $(\alpha+1)$ -roots of a .

The difference between the cases $\alpha \in \mathbb{N}$ and $\alpha \notin \mathbb{N}$ is not just cosmetics but analytically essential. In the latter case, a suitable form of nondegeneracy up to dilations holds for the solutions (12), which allows in [13, 14] the construction of solutions to (5) so that away from p ,

$$(16) \quad \psi_\varepsilon(z) = -4\pi\alpha G(z, p) + 8\pi(1+\alpha)G(z, p) + o(1) \quad \text{and} \quad \varepsilon^2 e^{\psi_\varepsilon} \rightharpoonup 8\pi(1+\alpha)\delta_p \quad \text{as } \varepsilon \rightarrow 0.$$

On the other hand, solutions with concentration points away from p have been built in [12], regardless whether or not α is an integer. Whenever $k < 1 + \alpha$, there is a solution ψ_ε of (5) and, up to a subsequence, k points $\xi_j^\varepsilon \in \Omega \setminus \{p\}$ such that away

from them,

(17)

$$\psi_\varepsilon(z) = -4\pi\alpha G(z, p) + 8\pi \sum_{j=1}^k G(z, \xi_j^\varepsilon) + o(1) \text{ and } \varepsilon^2 e^{\psi_\varepsilon} - 8\pi \sum_{j=1}^k \delta_{\xi_j^\varepsilon} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

A natural, unsettled question for which the methods in [12, 14] fail is whether in case $\alpha \in \mathbb{N}$, solutions ψ_ε with property (16), or for $k = \alpha + 1$ with property (17) exist. Expressions (14) and (15) suggest that both scenarios may be possible, using these functions, suitably corrected, as approximations of a solution of (5).

The case $\alpha \in \mathbb{N}$ in Problem (5) is more difficult, and at the same time the most relevant to physical applications. Our main result states that both of the above situations do take place; however, the location of the vortex p does not seem possible to be prescribed in an arbitrary way as in [13, 14] in case $\alpha \notin \mathbb{N}$ but rather, as in the situation of standard type II superconductivity, it is determined by the geometry of the region and boundary conditions in an ε -dependent way.

Let us assume that Ω is simply connected and consider a holomorphic extension

$$h_p(z) = H(z, p) - H(p, p) + i\tilde{H}(z, p) \text{ in } \Omega$$

of the harmonic function $H(z, p) - H(p, p)$ so that $h_p(p) = 0$. This means $\tilde{H}(p, p) = 0$ and

$$\tilde{H}_{z_1}(z, p) = H_{z_2}(z, p), \quad \tilde{H}_{z_2}(z, p) = -H_{z_1}(z, p) \text{ for all } z \in \Omega.$$

Then we let

$$\Psi(p) = \frac{d^{\alpha+1}}{dz^{\alpha+1}} \left(e^{2\pi(\alpha+2)h_p(z)} \right) (p).$$

Our main result for problem (5) is stated as follows.

Theorem 1.1. *Assume that $\alpha \in \mathbb{N}$ and that Ω is simply connected. Then there exists $\delta > 0$ such that for each a with $|a| \leq \delta$, there is a point $p_{a,\varepsilon} \in \Omega$ away from $\partial\Omega$ and a flow described by ψ_ε , the solution of*

$$\begin{cases} \Delta\psi_\varepsilon + \varepsilon^2 e^{\psi_\varepsilon} = 4\pi\alpha\delta_{p_{a,\varepsilon}} & \text{in } \Omega, \\ \psi_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

for which the associated vorticity $\omega_\varepsilon := \varepsilon^2 e^{\psi_\varepsilon} - 4\pi\alpha\delta_{p_{a,\varepsilon}}$ is concentrated in the sense that

$$(18) \quad \omega_\varepsilon + 4\pi\alpha\delta_{p_{a,\varepsilon}} - 8\pi \sum_{j=1}^{\alpha+1} \delta_{p_{a,\varepsilon} + a_j} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

where the a_j 's are the complex $(\alpha + 1)$ -roots of a . Besides we have

$$\Psi(p_{a,\varepsilon}) \rightarrow 0 \text{ as } |a| + \varepsilon \rightarrow 0.$$

For $\alpha \geq 1$, we observe that $\alpha + 1$ vertices of any sufficiently tiny regular polygon can be allocated with a suitable center to yield these vertices as an asymptotic concentration set. The solution predicted satisfies

$$\psi_\varepsilon(z) = -4\pi\alpha G(z, p_{a,\varepsilon}) + 8\pi \sum_{j=1}^{\alpha+1} G(z, p_{a,\varepsilon} + a_j) + o(1) \text{ as } \varepsilon \rightarrow 0$$

away from the concentration points. Let us observe that this result recovers the one known for $\alpha = 0$, where $p_{a,\varepsilon} = \xi^\varepsilon - a$ and where ξ^ε , the point of concentration,

approaches a critical point of the Robin's function. Let us notice that indeed, for $\alpha = 0$,

$$\Psi(p) = 0 \iff \nabla_z H(p, p) = 0.$$

In general, the condition $\Psi(p) = 0$ involves derivatives of orders up to $\alpha + 1$ of the function $H(z, p)$ at $z = p$. Notice also that the function $\Psi(p)$ is well-defined even if Ω is not simply connected. We suspect that this requirement may be lifted but our proof uses this fact. Theorem 1.1 will be a consequence of a more general result, which states that, associated to any region $\Lambda \subset \Omega$ for which $\deg(\Psi, \Lambda, 0) \neq 0$, a solution as in Theorem 1.1 exists with concentration points inside Λ . The proof applies with just slight changes to the case of nonzero boundary data φ .

2. A MORE GENERAL RESULT

In what follows we assume that Ω is a simply connected bounded domain. Theorem 1.1 will be a consequence of a more general result concerning the Dirichlet problem

$$(19) \quad \begin{cases} \Delta u + \varepsilon^2 e^u = 4\pi N \delta_p & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where φ is a smooth function and $N \geq 1$ is an integer. Let Φ be the harmonic extension of φ to all of Ω . The substitution

$$v(z) := u(z) + 4\pi N G(z, p) - \Phi(z)$$

transforms Problem (19) into the (regular) boundary value problem

$$(20) \quad \begin{cases} \Delta v + \varepsilon^2 |z - p|^{2N} e^{2K(z)} e^v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$2K(z) = \Phi(z) - 4\pi N H(z, p).$$

The homogeneous case $\varphi = 0$ corresponds to simply having $K(z) = -2\pi N H(z, p)$.

In what follows, we identify $(z_1, z_2) \in \mathbb{R}^2$ and $z = z_1 + iz_2 \in \mathbb{C}$. We can associate to (20) a limiting problem of Liouville type, which will play a crucial role in the construction of solutions blowing up at p as $\varepsilon \rightarrow 0^+$:

$$(21) \quad \begin{cases} \Delta V + \varepsilon^2 |z - p|^{2N} e^V = 0 & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} |z - p|^{2N} e^V < \infty. \end{cases}$$

Let us recall Liouville's formula [24]. Given a holomorphic function f on $\mathbb{C}$, the function

$$(22) \quad \ln \frac{8|f'(z)|^2}{(1 + |f(z)|^2)^2} - 2 \ln \varepsilon$$

solves the equation $\Delta V + \varepsilon^2 e^V = 0$ in the set $\{z \in \mathbb{C} \mid f'(z) \neq 0\}$. We can allow $f(z)$ to have simple poles since the Liouville formula (22) still makes sense.

If now f' vanishes only at the point p with multiplicity N , the function

$$(23) \quad \ln \frac{8|f'(z)|^2}{(1 + |f(z)|^2)^2} - 2 \ln \varepsilon - \ln |z - p|^{2N}$$

solves the equation in (21) but possibly does not have energy $\int_{\mathbb{R}^2} |z-p|^{2N} e^V$ finite. The finite energy condition in the entire space forces the choice

$$f(z) = \frac{(z-p)^{N+1} - a}{\mu\varepsilon}, \quad \mu > 0, \quad a \in \mathbb{C},$$

which leads exactly to the following three-parameter family of solutions to (21):

$$(24) \quad \tilde{V}_a(z) = \ln \frac{8(N+1)^2 \mu^2}{(\mu^2 \varepsilon^2 + |(z-p)^{N+1} - a|^2)^2}, \quad z \in \mathbb{C};$$

see [10, 29]. Let us note that $\tilde{V}_a(z)$ is the regular part of $w_{\varepsilon, \mu, a}$ as defined in (14). For $a = |a|e^{i\theta} \in \mathbb{C}$, we consider the $(N+1)$ -roots of a ,

$$a_j = |a|^{\frac{1}{N+1}} e^{i\frac{\theta}{N+1} + \frac{2\pi i}{N+1}(j-1)}, \quad j = 1, \dots, N+1,$$

where i is the imaginary unit in $\mathbb{C}$.

The function $\tilde{V}_a(z)$ solves the PDE in (20) with $K = 0$ but does not have the right boundary condition. By the Liouville formula, we will choose more carefully the function $f(z)$ to achieve the Dirichlet boundary condition and include the potential $e^{2K(z)}$ in the equation. Our approach is related to that by Weston in [36] for $N = 0$. Nonetheless, the assumptions in [36] on the Riemann conformal map of Ω onto the unit disc are shown here to be unnecessary, and our improved approach is also essential to deal with a nonzero $N \in \mathbb{N}$.

We explain below how to choose $f(z)$, and the simply connectedness of Ω will be crucial. The details of our construction will be presented in Section 3.

Let $Q \in C(\Omega, \mathbb{C})$ be a holomorphic function in Ω so that

$$Q(p) = 1, \quad Q(z) - \frac{z-p}{N+1} Q'(z) \neq 0 \quad \text{in } \Omega.$$

Since, as already observed, f can have simple poles, let us take

$$f(z) = \frac{(z-p)^{N+1} Q^{-1}(z) - a}{\mu\varepsilon}$$

in Liouville's formula (23) in order to get a solution

$$V_a(z) = \ln \frac{8(N+1)^2 \mu^2 |Q(z) - \frac{z-p}{N+1} Q'(z)|^2}{(\mu^2 \varepsilon^2 |Q(z)|^2 + |(z-p)^{N+1} - aQ(z)|^2)^2}$$

of the equation

$$-\Delta v = \varepsilon^2 |z-p|^{2N} e^v \quad \text{in } \Omega.$$

Given the projection operator $P : H^1(\Omega) \rightarrow H_0^1(\Omega)$, we will search for Q and μ (depending on a and $\varepsilon > 0$ small) such that $PV_a = V_a - 2K$. The function PV_a will then be a solution of (20).

Since $Q - \frac{z-p}{N+1} Q' \neq 0$ in Ω , the harmonic function

$$R_{\mu, Q, a}(z) = \frac{1}{2} \left[PV_a - V_a + \ln(8(N+1)^2 \mu^2) + \ln |Q(z) - \frac{z-p}{N+1} Q'(z)|^2 - 8\pi H_{Q, a}(z) \right]$$

satisfies by the Maximum Principle $R_{\mu, Q, a} = O(\varepsilon^2)$ uniformly in Ω , where $H_{Q, a}(z)$ is the solution of

$$\begin{cases} \Delta H = 0 & \text{in } \Omega, \\ H(z) = \frac{1}{2\pi} \ln |(z-p)^{N+1} - aQ(z)| & \text{on } \partial\Omega. \end{cases}$$

Observe that

$$H_{Q,0}(z) = (N+1)H(z, p),$$

where $H(z, p)$ is the regular part of the Green's function at p . Since Ω is a simply connected domain, let $h_{Q,a}(z)$, $r_{\mu,Q,a}(z)$ and $k(z)$ be the holomorphic extensions in Ω of the harmonic functions $H_{Q,a}(z) - H_{Q,a}(p)$, $R_{\mu,Q,a}(z) - R_{\mu,Q,a}(p)$ and $K(z) - K(p)$, respectively, so that $h_{Q,a}(p) = r_{\mu,Q,a}(p) = k(p) = 0$. Set

$$c_{\mu,Q,a} = \frac{1}{(N+1)!} \frac{d^{N+1}}{dz^{N+1}} \left(e^{4\pi h_{Q,a}(z) + r_{\mu,Q,a}(z) + k(z)} \right) (p).$$

For a and $\varepsilon > 0$ small, in Section 3 we will find a solution $Q_{\mu,a}$ of the equation:

$$(25) \quad \begin{aligned} & -\ln |Q(z) - \frac{z-p}{N+1} Q'(z)| + 4\pi H_{Q,a}(z) + R_{\mu,Q,a}(z) + K(z) \\ & = 4\pi H_{Q,a}(p) + R_{\mu,Q,a}(p) + K(p) + \operatorname{Re} (c_{\mu,Q,a}(z-p)^{N+1}) \end{aligned}$$

in Ω , where Re stands for the real part of a complex number. Next, we will solve

$$(26) \quad \mu = \frac{e^{4\pi H_{Q,a}(p) + R_{\mu,Q,a}(p) + K(p)}}{\sqrt{8}(N+1)} \Big|_{Q=Q_{\mu,a}}$$

for some μ_a , where a and $\varepsilon > 0$ are small parameters. Setting $Q_a = Q_{\mu_a,a}$, the function

$$V_a(z) = \ln \frac{8(N+1)^2 \mu_a^2 |Q_a(z) - \frac{z-p}{N+1} Q'_a(z)|^2}{(\mu_a^2 \varepsilon^2 |Q_a(z)|^2 + |(z-p)^{N+1} - a Q_a(z)|^2)^2}$$

satisfies:

$$(27) \quad PV_a = V_a - 2K(z) + 2\operatorname{Re} (c_a(z-p)^{N+1}) \quad \text{in } \Omega,$$

where $c_a := c_{\mu_a,Q_a,a}$.

As a conclusion, by means of (27), Problem (20) reduces to solving the equation

$$c_a(p) = \frac{1}{(N+1)!} \frac{d^{N+1}}{dz^{N+1}} \left(e^{4\pi h_{Q_a,a}(z) + r_{\mu_a,Q_a,a}(z) + k(z)} \right) (p) = 0.$$

This implies that every “stable” isolated zero point $p_0 \in \Omega$ for the map

$$\Psi(p) := \frac{d^{N+1}}{dz^{N+1}} \left(e^{4\pi(N+1)h_p(z) + k(z)} \right) (p),$$

in the sense that Ψ has a nonzero local index at p_0 , will provide us with points $p_{a,\varepsilon}$, for a and $\varepsilon > 0$ small, so that $c_a = 0$. More generally, we have the validity of the following result:

Theorem 2.1. *Let $\Lambda \subseteq \Omega$ be a region such that $\deg(\Psi, \Lambda, 0) \neq 0$. Then there exists $\delta > 0$ such that for each a with $|a| \leq \delta$, there is a point $p_{a,\varepsilon} \in \Lambda$ so that Problem (20) for $p = p_{a,\varepsilon}$ has a solution v_ε with*

$$(28) \quad \varepsilon^2 |z - p_{a,\varepsilon}|^{2N} e^{2K(z)} e^{v_\varepsilon} - 8\pi \sum_{j=1}^{N+1} \delta_{p_{a,\varepsilon} + a_j} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

where the a_j 's are the complex $(N+1)$ -roots of a . Besides we have

$$\Psi(p_{a,\varepsilon}) \rightarrow 0 \quad \text{as } |a| + \varepsilon \rightarrow 0.$$

Let us stress that if p_0 is a stable isolated zero of Ψ , then we can obtain solutions as above with $p_{a,\varepsilon} \rightarrow p_0$ as $|a| + \varepsilon \rightarrow 0$.

In what remains of this paper we will prove Theorem 2.1 with the scheme described above and deduce Theorem 1.1 as a corollary.

3. THE REDUCTION TO $c_a = 0$

Let $Q \in C(\bar{\Omega}, \mathbb{C})$ be a holomorphic function in Ω so that $Q(p) = 1$ and

$$Q(z) - \frac{z-p}{N+1}Q'(z) \neq 0 \quad \text{for all } z \in \Omega.$$

Let

$$(29) \quad V_a(z) = \ln \frac{8(N+1)^2 \mu^2 |Q(z) - \frac{z-p}{N+1}Q'(z)|^2}{(\mu^2 \varepsilon^2 |Q(z)|^2 + |(z-p)^{N+1} - aQ(z)|^2)^2},$$

and denote by PV_a the projection of V_a onto the space $H_0^1(\Omega)$. Namely PV_a is the unique solution of

$$\begin{cases} -\Delta PV_a = -\Delta V_a = \varepsilon^2 |z-p|^{2N} e^{V_a} & \text{in } \Omega, \\ PV_a = 0 & \text{on } \partial\Omega. \end{cases}$$

For a small, let $H_{Q,a}(z)$ be the solution of

$$(30) \quad \begin{cases} \Delta H = 0 & \text{in } \Omega, \\ H(z) = \frac{1}{2\pi} \ln |(z-p)^{N+1} - aQ(z)| & \text{on } \partial\Omega. \end{cases}$$

Since we assume $Q - \frac{z-p}{N+1}Q' \neq 0$ in Ω , the function $\ln |Q(z) - \frac{z-p}{N+1}Q'(z)|$ is harmonic in Ω . Since the harmonic function

$$R_{\mu,Q,a}(z) = \frac{1}{2} \left[PV_a(z) - V_a(z) + \ln(8(N+1)^2 \mu^2) + \ln |Q(z) - \frac{z-p}{N+1}Q'(z)|^2 - 8\pi H_{Q,a}(z) \right]$$

satisfies $R_{\mu,Q,a} = O(\varepsilon^2)$ uniformly on $\partial\Omega$ as $\varepsilon \rightarrow 0$ (together with any boundary derivatives), by elliptic estimates (see [17]) we get that

$$(31) \quad \begin{aligned} PV_a &= V_a - \ln(8(N+1)^2 \mu^2) - \ln |Q(z) - \frac{z-p}{N+1}Q'(z)|^2 + 8\pi H_{Q,a}(z) + 2R_{\mu,Q,a}(z), \\ R_{\mu,Q,a} &= O(\varepsilon^2) \end{aligned}$$

in $C^1(\bar{\Omega})$. Let $h_{Q,a}(z)$, $r_{\mu,Q,a}(z)$ and $k(z)$ be the holomorphic extensions in Ω of the harmonic functions $H_{Q,a}(z) - H_{Q,a}(p)$, $R_{\mu,Q,a}(z) - R_{\mu,Q,a}(p)$ and $K(z) - K(p)$ so that $h_{Q,a}(p) = r_{\mu,Q,a}(p) = k(p) = 0$. Set

$$c_{\mu,Q,a} = \frac{1}{(N+1)!} \frac{d^{N+1}}{dz^{N+1}} \left(e^{4\pi h_{Q,a}(z) + r_{\mu,Q,a}(z) + k(z)} \right) (p).$$

It suffices for equation (25) to hold true that $Q(z)$ satisfies the equation

$$Q(z) - \frac{z-p}{N+1}Q'(z) = \exp(4\pi h_{Q,a}(z) + r_{\mu,Q,a}(z) + k(z) - c_{\mu,Q,a}(z-p)^{N+1}),$$

or equivalently

$$(32) \quad \begin{aligned} Q(z) &= -(N+1)(z-p)^{N+1} \int \frac{dz}{(z-p)^{N+2}} \exp(4\pi h_{Q,a}(z) + r_{\mu,Q,a}(z) \\ &\quad + k(z) - c_{\mu,Q,a}(z-p)^{N+1}), \end{aligned}$$

where the symbol $\int$ designates a primitive in Ω of the argument function. The choice of the complex constant $c = c_{\mu, Q, a}$ guarantees that the right hand side in expression (32) is a well-defined single-valued function. Indeed, let us set

$$W(z) := \exp(4\pi h_{Q, a}(z) + r_{\mu, Q, a}(z) + k(z))$$

and

$$\begin{aligned}\Sigma(z) &:= \exp(4\pi h_{Q, a}(z) + r_{\mu, Q, a}(z) + k(z) - c(z-p)^{N+1}) \\ &= W(z) \exp(-c(z-p)^{N+1}).\end{aligned}$$

We expand near $z = p$, using $W(p) = 1$,

$$\begin{aligned}\Sigma(z) &= \left(1 + W'(p)(z-p) + \cdots + \frac{1}{(N+1)!} \frac{d^{N+1}W}{dz^{N+1}}(p)(z-p)^{N+1} + \cdots\right) \\ &\quad \times \left(1 - c(z-p)^{N+1} + \frac{c^2}{2}(z-p)^{2N+2} + \cdots\right),\end{aligned}$$

so that our choice of $c = c_{\mu, Q, a} = \frac{1}{(N+1)!} \frac{d^{N+1}W}{dz^{N+1}}(p)$ guarantees that the Taylor expansion of $\Sigma(z)$ does not contain a term of order $(z-p)^{N+1}$, say

$$\Sigma(z) = \sum_{j=0}^{\infty} b_j(z-p)^j$$

with $b_{N+1} = 0$. Thus we make sense of the whole R.H.S. in expression (32) as

$$\begin{aligned}& -(N+1)(z-p)^{N+1} \int \frac{\Sigma(z)}{(z-p)^{N+2}} \\ &= -(N+1)(z-p)^{N+1} \sum_{j=0}^N b_j \int \frac{dz}{(z-p)^{N+2-j}} \\ &= -(N+1)(z-p)^{N+1} \int \frac{dz}{(z-p)^{N+2}} \left(\Sigma(z) - \sum_{j=0}^{N+1} b_j(z-p)^j \right) \\ &= \sum_{j=0}^N b_j \frac{N+1}{N+1-j} (z-p)^j \\ &= -(N+1)(z-p)^{N+1} \int_p^z \frac{dw}{(w-p)^{N+2}} \left(\Sigma(w) - \sum_{j=0}^{N+1} b_j(w-p)^j \right).\end{aligned}$$

The latter integral is well-defined since its argument function is holomorphic and Ω is simply connected. Let us note that for $b_{N+1} \neq 0$, an additional term would arise of the form $-(N+1)b_{N+1}(z-p)^{N+1} \ln(z-p)$ making the R.H.S. in (32) a multi-valued function.

Since $b_0 = 1$, let us stress that any solution Q of (32) automatically satisfies $Q(p) = 1$ and $Q - \frac{z-p}{N+1}Q' \neq 0$ in Ω , as required. For $a = 0$ and $\varepsilon = 0$, the constant $c_{\mu, Q, a}$ reduces to

$$c_0 = \frac{1}{(N+1)!} \frac{d^{N+1}}{dz^{N+1}} \left(e^{4\pi(N+1)h_p(z) + k(z)} \right) (p),$$

where $h_p(z)$ is the holomorphic extension of $H(z, p) - H(p, p)$ so that $h_p(p) = 0$. Correspondingly, (25) has the solution Q_0 given by:

$$Q_0(z) = -(N+1)(z-p)^{N+1} \int \frac{dz}{(z-p)^{N+2}} \exp \left(4\pi(N+1)h_p(z) + k(z) - c_0(z-p)^{N+1} \right).$$

By the Implicit Function Theorem, we have the following existence result.

Lemma 3.1. *For a and $\varepsilon > 0$ small, in a small neighborhood of Q_0 there exists a unique holomorphic function $Q_{\mu,a} \in C(\bar{\Omega}, \mathbb{C})$ (depending smoothly on μ and a) satisfying (32):*

$$Q(z) = -(N+1)(z-p)^{N+1} \int \frac{dz}{(z-p)^{N+2}} \exp \left(4\pi h_{Q,a}(z) + r_{\mu,Q,a}(z) + k(z) - c_{\mu,Q,a}(z-p)^{N+1} \right).$$

Proof. Let $H = \{Q \in C(\bar{\Omega}, \mathbb{C}): Q \text{ is holomorphic in } \Omega\}$. For given $\mu > 0$, define the map

$$\begin{aligned} P: \mathbb{R}^+ \times H \times \mathbb{C} &\rightarrow H \\ (\varepsilon, Q, a) &\rightarrow Q(z) + (N+1)(z-p)^{N+1} \int \frac{\Sigma(z)}{(z-p)^{N+2}}, \end{aligned}$$

where $\Sigma(z) = \exp \left(4\pi h_{Q,a}(z) + r_{\mu,Q,a}(z) + k(z) - c_{\mu,Q,a}(z-p)^{N+1} \right)$. We have that

$$\begin{aligned} P(0, Q, 0) &= Q + (N+1)(z-p)^{N+1} \int \frac{1}{(z-p)^{N+2}} \exp \left(4\pi(N+1)h_p(z) \right. \\ &\quad \left. + k(z) - c_0(z-p)^{N+1} \right). \end{aligned}$$

Since $\partial_Q P(0, Q_0, 0) = \text{Id}$ and $P(0, Q_0, 0) = 0$, by the Implicit Function Theorem we find $\varepsilon_0 > 0$ small and a smooth map $(\varepsilon, a) \in (0, \varepsilon_0) \times B_{\varepsilon_0}(0) \rightarrow Q_{\varepsilon,\mu,a} \in H$ so that $Q_{0,\mu,0} = Q_0$ and $P(\varepsilon, Q_{\varepsilon,\mu,a}, a) = 0$. The required smallness of ε_0 is easily shown to be independent of μ , and $Q_{\varepsilon,\mu,a}$ depends smoothly also on μ , provided μ remains bounded and bounded away from zero. To keep light notation, we will omit the explicit dependence of $Q_{\varepsilon,\mu,a}$ on ε and simply write $Q_{\mu,a}$. $\square$

Next, for $a = 0$ and $\varepsilon = 0$, equation (26) has a unique solution

$$\mu_0 = \frac{e^{4\pi(N+1)H(p,p)+K(p)}}{\sqrt{8}(N+1)}.$$

By perturbation, we get

Lemma 3.2. *For a and $\varepsilon > 0$ small, in a small neighborhood of μ_0 there exists a unique solution μ_a (depending smoothly on a) to (26).*

Proof. Set

$$\begin{aligned} T: \mathbb{R}^+ \times \mathbb{R} \times \mathbb{C} &\rightarrow \mathbb{R} \\ (\varepsilon, \mu, a) &\rightarrow \mu - \frac{e^{4\pi H_{Q_{\mu,a},a}(p)+R_{\mu,Q_{\mu,a},a}(p)+K(p)}}{\sqrt{8}(N+1)}. \end{aligned}$$

Since $Q_{\mu,a}$ depends smoothly on μ and a , the same regularity holds for T . We have that

$$T(0, \mu, 0) = \mu - \frac{e^{4\pi(N+1)H(p,p)+K(p)}}{\sqrt{8}(N+1)}.$$

Since $T(0, \mu_0, 0) = 0$ and $\partial_\mu T(0, \mu_0, 0) = 1$, we find $\varepsilon_0 > 0$ small and a smooth map

$$(\varepsilon, a) \in (0, \varepsilon_0) \times B_{\varepsilon_0}(0) \mapsto \mu_{\varepsilon,a}$$

so that $\mu_{0,0} = \mu_0$ and $T(\varepsilon, \mu_{\varepsilon,a}, a) = 0$, by means of the Implicit Function Theorem. As before, we simply write μ_a instead of $\mu_{\varepsilon,a}$, and the proof is complete. $\square$

4. PROOFS OF THE MAIN RESULTS

Proof of Theorem 2.1. Let $\Lambda \Subset \Omega$ be a region as in the statement of the theorem and set

$$\Psi(p) := \frac{d^{N+1}}{dz^{N+1}} \left(e^{4\pi(N+1)h_p(z)+k(z)} \right) (p).$$

Finding a solution of Problem (20) as desired reduces then to finding $p \in \Lambda$ such that

$$\Psi_{a,\varepsilon}(p) = \frac{d^{N+1}}{dz^{N+1}} \left(e^{4\pi h_{Q_a,a}(z)+r_{\mu_a,Q_a,a}(z)+k(z)} \right) (p) = 0,$$

where $Q_a = Q_{\mu_a,a}$, with $Q_{\mu,a}$ and μ_a given by Lemmas 3.1 and 3.2, respectively. Since $\Psi_{a,\varepsilon}(p)$ depends continuously on a, ε, p and $\Psi_{0,0}(p) = \Psi(p)$, we get that the degree of $\Psi_{a,\varepsilon}(p)$ on Λ w.r.t. 0 is well-defined, for $|a| + \varepsilon$ small, and coincides with $\deg(\Psi, \Lambda, 0)$. The latter one being nontrivial by assumption, for a and $\varepsilon > 0$ small we find a solution $p_{a,\varepsilon} \in \Lambda$ of $\Psi_{a,\varepsilon}(p) = 0$.

In this way, the function

$$v_\varepsilon = \ln \frac{8(N+1)^2 \mu_a^2 |Q_a(z) - \frac{z-p}{N+1} Q'_a(z)|^2}{(\mu_a^2 \varepsilon^2 |Q_a(z)|^2 + |(z-p_{a,\varepsilon})^{N+1} - a Q_a(z)|^2)^2}$$

is a solution to Problem (20) with $p = p_{a,\varepsilon}$ for which the concentration property (28) easily follows. Since $\Psi_{a,\varepsilon}(p_{a,\varepsilon}) = 0$, we finally have that $\Psi(p_{a,\varepsilon}) \rightarrow 0$ as $|a| + \varepsilon \rightarrow 0$. $\square$

Proof of Theorem 1.1. To establish Theorem 1.1, we observe that in the homogeneous case $\varphi = 0$, $K(z) = -2\pi N H(z, p)$ and the function Ψ becomes

$$\Psi(p) = \frac{d^{N+1}}{dz^{N+1}} \left(e^{2\pi(N+2)h_p(z)} \right) (p).$$

We will compute the total degree of this map by estimating its behavior near $\partial\Omega$. Let us observe that

$$(33) \quad H(z, p) - \frac{1}{2\pi} \ln |z - \hat{p}| \rightarrow 0 \quad \text{in } C^{N+1}(\bar{\Omega}) \text{ as } p \rightarrow \partial\Omega,$$

where $\hat{p} \in \mathbb{R}^2 \setminus \bar{\Omega}$ is the reflection of p with respect to $\partial\Omega$. Let us stress that the reflection map is well-defined for points in Ω that are near $\partial\Omega$. The C^0 -validity of (33) follows from the Maximum Principle applied to the harmonic function $H(z, p) - \frac{1}{2\pi} \ln |z - \hat{p}|$ by means of the asymptotic behavior

$$H(z, p) - \frac{1}{2\pi} \ln |z - \hat{p}| = \frac{1}{2\pi} \ln \frac{|z - p|}{|z - \hat{p}|} \rightarrow 0 \text{ unif. on } \partial\Omega, \text{ as } p \rightarrow \partial\Omega.$$

Elliptic regularity (see [17]) then implies the validity of (33).

Let us denote $d = \text{dist}(p, \partial\Omega)$. Then, from (33), we obtain

$$\begin{aligned} H(p, p) &= \frac{1}{2\pi} \ln |p - \hat{p}| + o(1) = \frac{1}{2\pi} \ln(2d) + o(1), \\ H(z, p) - H(p, p) &= \frac{1}{2\pi} \ln \frac{|z - \hat{p}|}{2d} + o(1) \quad \text{in } C^{N+1}(\bar{\Omega}) \end{aligned}$$

as $d \rightarrow 0$. We extend $H(z, p) - H(p, p)$ holomorphically in Ω by $h_p(z)$ with $h_p(p) = 0$, and as $d \rightarrow 0$, the expansion

$$h_p(z) = \frac{1}{2\pi} \ln \frac{z - \hat{p}}{2d} + o(1) \quad \text{in } C^{N+1}(\bar{\Omega})$$

holds. Since as $d \rightarrow 0$,

$$e^{2\pi(N+2)h_p(z)} = \left(\frac{z - \hat{p}}{2d}\right)^{N+2} (1 + o(1))$$

in $C^{N+1}(\bar{\Omega})$, we finally get that in the homogeneous case,

$$\Psi(p) = (N+2)! \frac{p - \hat{p}}{2d} (1 + o(1))$$

as $d \rightarrow 0$. But $\frac{p - \hat{p}}{2d} = \nu_\Omega(p + \hat{p}/2)$, where $\nu_\Omega(x)$ is the inward unit normal vector at $x \in \partial\Omega$ and $p + \hat{p}/2$ is the projection of p onto the boundary. The winding number of ν_Ω along $\partial\Omega$ is ± 1 and, by stability, we get that

$$\deg(\Psi, 0, \Omega_\delta) = \pm 1 \neq 0$$

for $\delta > 0$ small. Here, $\Omega_\delta = \{x \in \Omega : d > \delta\}$.

Theorem 2.1 thus applies with $\Lambda = \Omega_\delta$ to provide, for a and $\varepsilon > 0$ small, $p_{a,\varepsilon} \in \Omega_\delta$ and solutions v_ε to Problem (20) with $p = p_{a,\varepsilon}$ for which the concentration property (28) holds. Setting $\psi_\varepsilon(z) = v_\varepsilon(z) - 4\pi NG(z, p)$, the function ψ_ε is a solution to Problem (5) with $\alpha = N$, $p = p_{a,\varepsilon}$ and (28) rewritten in terms of ω_ε as (18). The proof is concluded. $\square$

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REFERENCES

- [1] S. Baraket, F. Pacard, *Construction of singular limits for a semilinear elliptic equation in dimension 2*. Calc. Var. Partial Differential Equations **6** (1998), no. 1, 1–38. MR1488492 (98j:35057)
- [2] D. Bartolucci, C.-C. Chen, C.-S. Lin, G. Tarantello, *Profile of blow-up solutions to mean field equations with singular data*. Comm. Partial Differential Equations **29** (2004), no. 7-8, 1241–1265. MR2097983 (2005i:58022)
- [3] D. Bartolucci, G. Tarantello, *Liouville type equations with singular data and their applications to periodic multivortices for the electroweak theory*. Comm. Math. Phys. **229** (2002), no. 1, 3–47. MR1917672 (2003e:58026)
- [4] H. Brezis, F. Merle, *Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions*. Comm. Partial Differential Equations **16** (1991), no. 8-9, 1223–1253. MR1132783 (92m:35084)
- [5] E. Caglioti, P.L. Lions, C. Marchioro, M. Pulvirenti, *A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description*. Comm. Math. Phys. **143** (1992), 501–525. MR1145596 (93i:82060)
- [6] E. Caglioti, P.L. Lions, C. Marchioro, M. Pulvirenti, *A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description Part II*. Comm. Math. Phys. **174** (1995), 229–260. MR1362165 (96k:82059)
- [7] C.-C. Chen, C.-S. Lin, *Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces*. Comm. Pure Appl. Math. **55** (2002), no. 6, 728–771. MR1885666 (2003d:53056)
- [8] C.-C. Chen, C.-S. Lin, *Topological degree for a mean field equation on Riemann surfaces*. Comm. Pure Appl. Math. **56** (2003), no. 12, 1667–1727. MR2001443 (2004h:35065)

- [9] W. Chen, C. Li, *Classification of solutions of some nonlinear elliptic equations*. Duke Math. J. **63** (1991), 615–623. MR1121147 (93e:35009)
- [10] K.S. Chou, T.Y.H. Wan, *Asymptotic radial symmetry for solutions of $\Delta u + e^u$ in a punctured disc*. Pacific J. of Math. **163** (1994), 269–276. MR1262297 (95a:35038)
- [11] M. Crandall, P.H. Rabinowitz, *Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems*. Arch. Rational Mech. Anal. **58** (1975), no. 3, 207–218. MR0382848 (52:3730)
- [12] M. del Pino, M. Kowalczyk, M. Musso, *Singular limits in Liouville-type equations*. Calc. Var. Partial Differential Equations **24** (2005), no. 1, 47–81. MR2157850 (2006h:35089)
- [13] P. Esposito, *Blow up solutions for a Liouville equation with singular data*. Recent advances in elliptic and parabolic problems (Taiwan, 2004), 39–57, edited by Chiun-Chuan Chen, Michel Chipot and Chang-Shou Lin. MR2172566 (2006e:35098)
- [14] P. Esposito, *Blow up solutions for a Liouville equation with singular data*. SIAM J. Math. Anal. **36** (2005), no. 4, 1310–1345. MR2139452 (2005m:35088)
- [15] P. Esposito, M. Grossi, A. Pistoia, *On the existence of blowing-up solutions for a mean field equation*. Ann. Inst. Henri Poincaré Analyse Non Linéaire **22**, no. 2 (2005), 227–257. MR2124164 (2005k:35112)
- [16] I.M. Gelfand, *Some problems in the theory of quasilinear equations*. Amer. Math. Soc. Transl. **29** (1963), no. 2, 295–381. MR0153960 (27:3921)
- [17] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*. 2nd Edition, Springer-Verlag, 1983. MR737190 (86c:35035)
- [18] J. Han, J. Jang, *Self-dual Chern-Simons vortices on bounded domains*. Lett. Math. Phys. **64** (2003), 45–56. MR1997718 (2004e:58020)
- [19] D.D. Joseph, T.S. Lundgren, *Quasilinear problems driven by positive sources*. Arch. Rat. Mech. Anal. **49** (1973), 241–269. MR0340701 (49:5452)
- [20] M.H.K. Kiessling, *Statistical mechanics of classical particles with logarithmic interaction*. Comm. Pure Appl. Math. **46** (1993), 27–56. MR1193342 (93k:82003)
- [21] Y.-Y. Li, *Harnack type inequality: The method of moving planes*. Comm. Math. Phys. **200** (1999), 421–444. MR1673972 (2000c:58024)
- [22] Y.-Y. Li, I. Shafrir, *Blow-up analysis for solutions of $-\Delta u = V e^u$ in dimension two*. Indiana Univ. Math. J. **43** (1994), no. 4, 1255–1270. MR1322618 (96c:35048)
- [23] P.-L. Lions, *Mathematical topics in fluid mechanics. Vol. 1. Incompressible models*. Oxford Lecture Series in Mathematics and its Applications, 3. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1996. MR1422251 (98b:76001)
- [24] J. Liouville, *Sur l'Equation aux Difference Partielles $\frac{d^2 \log \lambda}{du dv} \pm \frac{\lambda}{2a^2} = 0$* . C.R. Acad. Sci. Paris **36** (1853), 71–72.
- [25] A. Majda, A. Bertozzi, *Vorticity and incompressible flow*. Cambridge Texts in Applied Mathematics, **27**. Cambridge University Press, Cambridge, 2002. MR1867882 (2003a:76002)
- [26] L. Ma, J. Wei, *Convergence for a Liouville equation*. Comment. Math. Helv. **76** (2001), 506–514. MR1854696 (2002h:35097)
- [27] K. Nagasaki, T. Suzuki, *Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities*. Asymptotic Anal. **3** (1990), no. 2, 173–188. MR1061665 (91f:35053)
- [28] M. Nolasco, G. Tarantello, *Double vortex condensates in the Chern-Simons-Higgs theory*. Calc. Var. Partial Differential Equations **9** (1999), 31–91. MR1710938 (2001e:58022)
- [29] J. Prajapat, G. Tarantello, *On a class of elliptic problem in $\mathbb{R}^2$: Symmetry and uniqueness results*. Proc. Roy. Soc. Edinburgh Sect. A **131** (2001), 967–985. MR1855007 (2002j:35108)
- [30] J.T. Stuart, *On finite amplitude oscillations in laminar mixing layers*. J. Fluid. Mech. **29** (1967), no. 3, 417–440.
- [31] T. Suzuki, *Two-dimensional Emden-Fowler equation with exponential nonlinearity*. Nonlinear diffusion equations and their equilibrium states, 3 (Gregynog, 1989), 493–512. Progr. Nonlinear Differential Equations Appl., 7, Birkhäuser Boston, Boston, MA, 1992. MR1167859 (93e:35041)
- [32] G. Tarantello, *Multiple condensate solutions for the Chern-Simons-Higgs theory*. J. Math. Phys. **37** (1996), 3769–3796. MR1400816 (97f:58045)
- [33] G. Tarantello, *Analytical aspects of Liouville-type equations with singular sources*. Stationary partial differential equations. Vol. I, 491–592, Handbook Differ. Equ., North-Holland, Amsterdam, 2004. MR2103693 (2006e:35124)

- [34] G. Tarantello, *A quantization property for blow up solutions of singular Liouville-type equations*. J. Funct. Anal. **219** (2005), no. 2, 368–399. MR2109257 (2005i:35073)
- [35] A. Tur, V. Yanovsky, *Point vortices with a rational necklace: New exact stationary solutions of the two-dimensional Euler equation*. Phys. Fluids **16** (2004), no. 8, 2877–2885. MR2075552 (2005b:76029)
- [36] V.H. Weston, *On the asymptotic solution of a partial differential equation with an exponential nonlinearity*. SIAM J. Math. Anal. **9** (1978), no. 6, 1030–1053. MR512508 (80a:35022)

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