ENTROPY DIMENSION OF TOPOLOGICAL DYNAMICAL SYSTEMS

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ABSTRACT. We introduce the notion of topological entropy dimension to measure the complexity of entropy zero systems. It measures the superpolynomial, but subexponential, growth rate of orbits. We also introduce the dimension set, $\mathcal{D}(X, T) \subset [0, 1]$, of a topological dynamical system to study the complexity of its factors. We construct a minimal example whose dimension set consists of one number. This implies the property that every nontrivial open cover has the same entropy dimension. This notion for zero entropy systems corresponds to the K -mixing property in measurable dynamics and to the uniformly positive entropy in topological dynamics for positive entropy systems. Using the entropy dimension, we are able to discuss the disjointness between the entropy zero systems. Properties of entropy generating sequences and their dimensions have been investigated.

1. INTRODUCTION

Kolmogorov introduced the notion of entropy as an isomorphism invariant to ergodic theory from information science. Since then, entropy has been one of the central concepts in several areas of dynamical systems. Shannon's entropy is known to be the average information content in information science. In measurable dynamics, entropy measures the average growth rate of orbits, that is, the chaotic behavior of a dynamical system via independence. Entropy is also known to be a complete invariant for the Bernoulli class of measure preserving automorphisms. Recently, Ornstein and Weiss [22] have shown that any finitely observable isomorphism invariant for an ergodic system is a function of the entropy. In the topological setting, entropy as a measurement of independence is computed by the number of open sets of the iterated cover needed to cover most of the space.

We would like to investigate the properties of entropy zero systems. Although entropy zero systems make up a dense G_δ subset of all homeomorphisms, not much study has been done on the complexity of these systems. Clearly, entropy zero systems have various complexity ranging from irrational rotations to mixing transformations like horocycle flows. The second reason for motivating us to consider entropy zero systems comes from the study of general group actions. General

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groups, unlike $\mathbb{Z}$ -actions or $\mathbb{R}$ -actions, have many interesting noncocompact subgroup actions with positive finite entropies. Clearly, these general group actions have zero entropy with nontrivial complexity. We consider examples of $\mathbb{Z}^2$ -actions, each of which is generated by a full shift σ and a cellular automaton map τ . Most of these examples have the property that for each (k, l) , $\sigma^k \tau^l$ has positive entropy and the directional entropies in every direction including irrational directions are well defined, and moreover they are continuous [5, 21, 23]. It is not hard to construct examples whose directional entropies are all zero, but their complexities are very different and quite nontrivial [19]. Also there are many physical models showing intermediate chaotic behavior which have recently received a great deal of study [25, 28]. They have chaotic regions and deterministic regions, so that a typical trajectory alternates between the regions but spends most of its time in the deterministic region. Hence in a natural sense it would have “entropy” zero. The complexity of a system changes, depending on the expected time in the deterministic region or in the chaotic region.

Several notions like sequence entropy [11, 14], maximal pattern complexity [18] and maximal pattern entropy [17] have been used to analyze entropy zero systems. However these are known to be useful for systems of low complexity, such as Morse systems and Sturmian systems. We have introduced the entropy dimension in [10] to measure the complexity of entropy 0 measurable dynamics. It measures the growth rate of $H(\bigvee_{i=0}^{n-1} T^{-i} P)$. We also introduced the notion of the dimension (upper and lower) of a subset of $\mathbb{Z}$ which may have density 0. In addition, we defined an entropy generating sequence of a system and computed its dimension to measure its complexity [7]. We would like to define these notions for the topological setting and study their properties. Entropy dimension for a topological dynamical system measures the superpolynomial, but subexponential, growth rate of the number of open sets that cover the space out of the sequence of iterated open covers.

In the case of positive entropy, it is not hard to find an infinite subset $W \subset \mathbb{Z}_+$ such that along the sequence W the symbolic names are “independent” and there exists $c > 0$ with $\liminf_{n \rightarrow \infty} \frac{|W \cap [1, n]|}{n} > c$. That is, this subset W has positive lower density. We prove that if a system (X, T) has positive entropy dimension, then there exists an entropy generating sequence $S \subset \mathbb{Z}_+$ which is a union of disjoint finite sets along which the dynamics are “independent”. Given a sequence $S \subset \mathbb{Z}_+$, we define the dimension of the sequence and show that the entropy dimension of the system is the supremum of the dimensions of the entropy generating sequences. We expect that this combinatorial method of measuring the complexity of a dynamical system via the dimension of entropy generating sequence is a useful tool to analyze the complexity of general group actions of zero entropy.

Disjointness first introduced by H. Furstenberg [8] characterizes some of the properties of dynamical systems. In measurable dynamics K -mixing systems are disjoint from zero entropy systems and weak mixing systems are disjoint from group rotations. In the case of topological settings, these properties are explored in [3, 14, 16, 15]. In this paper we would like to introduce the notion of the dimension set $\mathcal{D}(X, T) \subset [0, 1]$ of a zero entropy topological system (X, T) to measure the various levels of topological complexity of subexponential growth rate. We also define the entropy dimension to be the supremum of the dimension set. If a system (X, T) has positive entropy, then $1 \in \mathcal{D}(X, T)$ and $\mathcal{D}(X, T)$ can be strictly larger than $\{1\}$ if it has nontrivial zero entropy factor. Besides the disjointness between positive

and zero entropy systems, we are able to investigate the property of disjointness within entropy zero systems with respect to the dimension set. Using the so-called localization idea introduced in [3], we prove that under some conditions on the dimension sets and minimality, two dynamical systems of disjoint dimension sets are disjoint. This can be regarded as a refinement and also a generalization of the result that uniformly positive entropy (u.p.e.) systems are disjoint from minimal and entropy zero systems. We show that there exists a minimal system whose dimension set is a singleton. That is, every open cover has the same entropy dimension. Their properties are analogous to K -mixing in measurable dynamics and u.p.e. systems in topological dynamics, which do not have nontrivial entropy zero factors.

We would like to mention a similar result in group theory. J. Milnor asked if there exists a finitely generated group of intermediate growth rate, that is, a group G generated by $\{g_1, \dots, g_l\}$ such that

$$\gamma(n) = \#\{g \in G : \text{the minimal representation of } g \text{ is } g_{i_1} g_{i_2} \cdots g_{i_n}\}$$

has subexponential growth rate. Grigorchuk [12] answered the question affirmatively by constructing a group whose generators consist of transformations on a probability space. Moreover, he has shown that there are uncountably many such groups. However the “exact” growth rates are not easy to calculate and hence are not known for many of these groups. (See also [13].)

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2. ENTROPY DIMENSION

In this paper, a *topological dynamical system* (TDS, for short) is a pair (X, T) , where X is a compact metric space and T is a continuous map from X onto itself. Before we introduce the notion of entropy dimension for a TDS, we recall some of the definitions. Given a TDS (X, T) , denote by $\mathcal{C}_X$ the set of finite covers of X and by $\mathcal{C}_X^o$ the set of finite open covers of X . Given two covers $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$, we say that $\mathcal{U}$ is finer than $\mathcal{V}$ ($\mathcal{U} \succeq \mathcal{V}$) if for every $U \in \mathcal{U}$ there is a set $V \in \mathcal{V}$ such that $U \subseteq V$. Let $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. It is clear that $\mathcal{U} \vee \mathcal{V} \succeq \mathcal{U}$ and $\mathcal{U} \vee \mathcal{V} \succeq \mathcal{V}$. Given integers $m \leq n$ and a cover $\mathcal{U} \in \mathcal{C}_X$, let $\mathcal{U}_m^n = \bigvee_{i=m}^n T^{-i}\mathcal{U}$. Given $\mathcal{U} \in \mathcal{C}_X^o$, let $\mathcal{N}(\mathcal{U})$ denote the number of the sets in a subcover of $\mathcal{U}$ with smallest cardinality.

Let (X, T) be a TDS and $\mathcal{U}$ be a finite open cover of X . For $\alpha \geq 0$, we define

$$\overline{D}(T, \alpha, \mathcal{U}) = \limsup_{n \rightarrow \infty} \frac{\log \mathcal{N}(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U})}{n^\alpha} \text{ and}$$

$$\underline{D}(T, \alpha, \mathcal{U}) = \liminf_{n \rightarrow \infty} \frac{\log \mathcal{N}(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U})}{n^\alpha}.$$

It is clear that $\overline{D}(T, \alpha, \mathcal{U}) \leq \overline{D}(T, \alpha', \mathcal{U})$ if $\alpha \geq \alpha' \geq 0$ and $\overline{D}(T, \alpha, \mathcal{U}) \notin \{0, +\infty\}$ for at most one $\alpha \geq 0$. We define the *upper entropy dimension of $\mathcal{U}$* by

$$\overline{D}(T, \mathcal{U}) = \inf\{\alpha \geq 0 : \overline{D}(T, \alpha, \mathcal{U}) = 0\} = \sup\{\alpha \geq 0 : \overline{D}(T, \alpha, \mathcal{U}) = \infty\}.$$

Similarly, $\underline{D}(T, \alpha, \mathcal{U}) \leq \underline{D}(T, \alpha', \mathcal{U})$ if $\alpha \geq \alpha' \geq 0$ and $\underline{D}(T, \alpha, \mathcal{U}) \notin \{0, +\infty\}$ for at most one $\alpha \geq 0$. We define the *lower entropy dimension of $\mathcal{U}$* by

$$\underline{D}(T, \mathcal{U}) = \inf\{\alpha \geq 0 : \underline{D}(T, \alpha, \mathcal{U}) = 0\} = \sup\{\alpha \geq 0 : \underline{D}(T, \alpha, \mathcal{U}) = \infty\}.$$

If $\overline{D}(T, \mathcal{U}) = \underline{D}(T, \mathcal{U}) = \alpha$, then we say $\mathcal{U}$ has entropy dimension α . Clearly, $0 \leq \underline{D}(T, \mathcal{U}) \leq \overline{D}(T, \mathcal{U}) \leq 1$, and if $h(T, \mathcal{U}) > 0$, then the entropy dimension of $\mathcal{U}$ is equal to 1.

Definition 2.1. Let (X, T) be a TDS. The upper (resp. lower) entropy dimension of TDS (X, T) is

$$\overline{D}(X, T) = \sup\{\overline{D}(T, \mathcal{U}) : \mathcal{U} \in \mathcal{C}_X^o\} \text{ (resp. } \underline{D}(X, T) = \sup\{\underline{D}(T, \mathcal{U}) : \mathcal{U} \in \mathcal{C}_X^o\}).$$

It is clear that $0 \leq \underline{D}(X, T) \leq \overline{D}(X, T) \leq 1$. When $\overline{D}(X, T) = \underline{D}(X, T)$, we just call it the entropy dimension of (X, T) , denoted by $D(X, T)$.

The following two propositions are the basic properties of entropy dimension in the topological setting.

Proposition 2.2. Let (X, T) be a TDS and $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X^o$.

- (1) If $\mathcal{U} \preceq \mathcal{V}$, then $\overline{D}(T, \mathcal{U}) \leq \overline{D}(T, \mathcal{V})$ and $\underline{D}(T, \mathcal{U}) \leq \underline{D}(T, \mathcal{V})$.
- (2) For any $0 \leq m \leq n$, $\overline{D}(T, \mathcal{U}) = \overline{D}(T, \mathcal{U}_m^n)$ and $\underline{D}(T, \mathcal{U}) = \underline{D}(T, \mathcal{U}_m^n)$.
- (3) $\overline{D}(T, \mathcal{U} \vee \mathcal{V}) = \max\{\overline{D}(T, \mathcal{U}), \overline{D}(T, \mathcal{V})\}$.

Proof. (1) and (2) are clear. For (3), if $\alpha > \max\{\overline{D}(T, \mathcal{U}), \overline{D}(T, \mathcal{V})\}$, then

$$\limsup_{n \rightarrow \infty} \frac{\log \mathcal{N}(\mathcal{U}_0^{n-1})}{n^\alpha} = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{\log \mathcal{N}(\mathcal{V}_0^{n-1})}{n^\alpha} = 0.$$

Hence

$$\limsup_{n \rightarrow \infty} \frac{\log \mathcal{N}((\mathcal{U} \vee \mathcal{V})_0^{n-1})}{n^\alpha} \leq \limsup_{n \rightarrow \infty} \frac{\log \mathcal{N}(\mathcal{U}_0^{n-1}) + \log \mathcal{N}(\mathcal{V}_0^{n-1})}{n^\alpha} = 0.$$

This implies $\overline{D}(T, \mathcal{U} \vee \mathcal{V}) < \alpha$. Since α is arbitrary, we have

$$\overline{D}(T, \mathcal{U} \vee \mathcal{V}) \leq \max\{\overline{D}(T, \mathcal{U}), \overline{D}(T, \mathcal{V})\}.$$

By (1) we have the result. $\square$

Remark 2.3. $\underline{D}(T, \mathcal{U} \vee \mathcal{V})$ isn't always equal to $\max\{\underline{D}(T, \mathcal{U}), \underline{D}(T, \mathcal{V})\}$ (see Example 2.8). In fact it is not hard to show that

$$\max\{\underline{D}(T, \mathcal{U}), \underline{D}(T, \mathcal{V})\} \leq \underline{D}(T, \mathcal{U} \vee \mathcal{V}) \leq \underline{D}(\mathcal{U}; \mathcal{V}),$$

where

$$\underline{D}(\mathcal{U}; \mathcal{V}) = \max\{\underline{D}(T, \mathcal{U}), \underline{D}(T, \mathcal{V}), \min\{\overline{D}(T, \mathcal{U}), \overline{D}(T, \mathcal{V})\}\}.$$

As a direct application of Proposition 2.2, we have

Proposition 2.4. Let (X, T) be a TDS. If $\{\mathcal{U}_n\}$ is a sequence of finite open covers of X with $\lim_{n \rightarrow +\infty} \text{diam}(\mathcal{U}_n) = 0$, then

$$\lim_{n \rightarrow \infty} \overline{D}(T, \mathcal{U}_n) = \overline{D}(X, T) \text{ and } \lim_{n \rightarrow \infty} \underline{D}(T, \mathcal{U}_n) = \underline{D}(X, T).$$

In particular, if $\mathcal{U}$ is a generating open cover of X (i.e. $\lim_{n \rightarrow \infty} \text{diam}(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}) = 0$), then $\overline{D}(T, \mathcal{U}) = \overline{D}(X, T)$ and $\underline{D}(T, \mathcal{U}) = \underline{D}(X, T)$.

Let (X, T) be a TDS. A *standard cover* of X is a cover $\{U, V\}$ by two non-dense open sets. Denote by $\mathcal{C}_X^s$ the set of all standard covers of X . The following proposition shows that the upper entropy dimension with respect to standard covers determines the upper entropy dimension of the system.

Proposition 2.5. *Let (X, T) be a TDS. Then*

$$\overline{D}(X, T) = \sup\{\overline{D}(T, \mathcal{W}) : \mathcal{W} \in \mathcal{C}_X^s\}.$$

Proof. We follow the argument in the proof of Proposition 1 in [3]. Since $\mathcal{C}_X^o \supseteq \mathcal{C}_X^s$, we have that $\overline{D}(X, T) \geq \sup\{\overline{D}(T, \mathcal{W}) : \mathcal{W} \in \mathcal{C}_X^s\}$. Conversely, let $\mathcal{U} \in \mathcal{C}_X^s$. Then there exist $\mathcal{U}_i = \{U_i, V_i\} \in \mathcal{C}_X^o$ ($i = 1, 2, \dots, k$) such that $\bigvee_{i=1}^k \mathcal{U}_i \succeq \mathcal{U}$ (see the proof of Proposition 1 in [3]). Note that $\overline{D}(T, \mathcal{U}) \leq \overline{D}(T, \bigvee_{i=1}^k \mathcal{U}_i) = \max\{\overline{D}(T, \mathcal{U}_i) : i = 1, 2, \dots, k\}$; hence there exists $r \in \{1, 2, \dots, k\}$ such that $\overline{D}(T, \mathcal{U}_r) \geq \overline{D}(T, \mathcal{U})$.

Now it is easy to shrink $\mathcal{U}_r = \{U_r, V_r\}$ into a standard cover $\mathcal{W}$ with $\overline{D}(T, \mathcal{W}) \geq \overline{D}(T, \mathcal{U})$. First, if U_r is dense, then V_r contains some closed ball $F = \{y : d(x, y) \leq \epsilon\}$ with $\epsilon > 0$ and $x \in U_r$. Let $W_1 = U_r \setminus F$. Since F has nonempty interior, the set W_1 is not dense, and $\{W_1, V_r\}$ is an open cover of X as V_r contains F . Since this cover is finer than $\mathcal{U}_r$, $\overline{D}(T, \{W_1, V_r\}) \geq \overline{D}(T, \mathcal{U}_r)$. By doing the same with V_r if V_r is dense, one obtains a standard cover $\mathcal{W} = \{W_1, W_2\}$ such that $\overline{D}(T, \mathcal{W}) \geq \overline{D}(T, \mathcal{U})$. This implies $\sup\{\overline{D}(T, \mathcal{W}) : \mathcal{W} \in \mathcal{C}_X^s\} \geq \overline{D}(T)$, since $\mathcal{U}$ is arbitrary. $\square$

It is easy to see the following:

Proposition 2.6. *Let (X_i, T_i) be two TDS's, $i = 1, 2$.*

- (1) $\overline{D}(X_1 \times X_2, T_1 \times T_2) = \max\{\overline{D}(X_1, T_1), \overline{D}(X_2, T_2)\}$.
- (2) *If (X_2, T_2) is a factor of (X_1, T_1) , then*

$$\overline{D}(X_2, T_2) \leq \overline{D}(X_1, T_1) \quad \text{and} \quad \underline{D}(X_2, T_2) \leq \underline{D}(X_1, T_1).$$

Remark 2.7. It is not hard to show that

$$\max\{\underline{D}(X_1, T_1), \underline{D}(X_2, T_2)\} \leq \underline{D}(X_1 \times X_2, T_1 \times T_2) \leq \underline{D}(T_1; T_2),$$

where

$$\underline{D}(T_1; T_2) = \max\{\underline{D}(X_1, T_1), \underline{D}(X_2, T_2), \min\{\overline{D}(X_1, T_1), \overline{D}(X_2, T_2)\}\}.$$

In fact, $\underline{D}(X_1 \times X_2, T_1 \times T_2)$ is not always equal to $\max\{\underline{D}(X_1, T_1), \underline{D}(X_2, T_2)\}$. We provide a simple, but not transitive, example with this property.

We denote by $[a]$ the largest integer not greater than a . For $\tau \in [0, 1]$ and $n \geq 2$, let

$$X_{n,\tau} = \{x = (x_i)_{i \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}} : x_j = 0 \text{ for } j \notin \{i[n^{1-\tau} \log n] : i = 0, 1, \dots, \lfloor \frac{n^\tau}{\log n} \rfloor\}\}.$$

Example 2.8. Let $n_0 = 0$, $n_{i+1} = 3^{n_i}$ for $i \in \mathbb{Z}_+$ and $\tau = \frac{1}{2}$. Let

$$Y_1 = \overline{\bigcup_{i \in \mathbb{Z}} \bigcup_{k=0}^{\infty} \bigcup_{j=n_{2k}}^{n_{2k+1}} \sigma^i X_{j,\tau}},$$

$$Y_2 = \overline{\bigcup_{i \in \mathbb{Z}} \bigcup_{k=0}^{\infty} \bigcup_{j=n_{2k+1}+1}^{n_{2k+2}-1} \sigma^i X_{j,\tau}}.$$

Let (Y_1, σ_1) and (Y_2, σ_2) be two TDS's, where $\sigma_i = \sigma|_{Y_i}$ for $i = 1, 2$, where σ is the left shift map on $\{0, 1\}^{\mathbb{Z}}$. Put $\mathcal{U}_1 = \{[0]_{Y_1}, [1]_{Y_1}\}$ and $\mathcal{U}_2 = \{[0]_{Y_2}, [1]_{Y_2}\}$, where $[i]_Y = \{x \in Y : x_0 = i\}$ for $i = 0, 1$ and $Y = Y_1$ or Y_2 .

We set $\widetilde{\mathcal{U}}_1 = \mathcal{U}_1 \times Y_2$ and $\widetilde{\mathcal{U}}_2 = Y_1 \times \mathcal{U}_2$. It is not hard to see that

- (1) $\underline{D}(Y_1, \sigma_1) = \underline{D}(\sigma_1, \mathcal{U}_1) = 0$ and $\overline{D}(Y_1, \sigma_1) = \overline{D}(\sigma_1, \mathcal{U}_1) = \frac{1}{2}$.

$$(2) \quad \underline{D}(Y_2, \sigma_2) = \underline{D}(\sigma_2, \mathcal{U}_2) = 0 \text{ and } \overline{D}(Y_2, \sigma_2) = \overline{D}(\sigma_2, \mathcal{U}_2) = \frac{1}{2}.$$

$$(3) \quad \underline{D}(Y_1 \times Y_2, \sigma_1 \times \sigma_2) = \underline{D}(\sigma_1 \times \sigma_2, \widetilde{\mathcal{U}_1 \vee \mathcal{U}_2}) = \frac{1}{2}.$$

Hence $0 = \max\{\underline{D}(Y_1, \sigma_1), \underline{D}(Y_2, \sigma_2)\} < \underline{D}(Y_1 \times Y_2, \sigma_1 \times \sigma_2) = \underline{D}(\sigma_1; \sigma_2) = \frac{1}{2}$.

3. SOME EQUIVALENT DEFINITIONS OF ENTROPY DIMENSION

Let $S = \{s_1 < s_2 < \cdots\}$ be a sequence of increasing integers. For $\tau \geq 0$, we define

$$\overline{D}(S, \tau) = \limsup_{n \rightarrow \infty} \frac{n}{(s_n)^\tau} \quad \text{and} \quad \underline{D}(S, \tau) = \liminf_{n \rightarrow \infty} \frac{n}{(s_n)^\tau}.$$

It is clear that $\overline{D}(S, \tau) \leq \overline{D}(S, \tau')$ if $\tau \geq \tau' \geq 0$ and $\overline{D}(S, \tau) \notin \{0, +\infty\}$ for at most one $\tau \geq 0$. We define *the upper dimension of S* by

$$\overline{D}(S) = \inf\{\tau \geq 0 : \overline{D}(S, \tau) = 0\} = \sup\{\tau \geq 0 : \overline{D}(S, \tau) = \infty\}.$$

Similarly, $\underline{D}(S, \tau) \leq \underline{D}(S, \tau')$ if $\tau \geq \tau' \geq 0$ and $\underline{D}(S, \tau) \notin \{0, +\infty\}$ for at most one $\tau \geq 0$. We define *the lower dimension of S* by

$$\underline{D}(S) = \inf\{\tau \geq 0 : \underline{D}(S, \tau) = 0\} = \sup\{\tau \geq 0 : \underline{D}(S, \tau) = \infty\}.$$

Clearly, $0 \leq \underline{D}(S) \leq \overline{D}(S) \leq 1$. When $\overline{D}(S) = \underline{D}(S) = \tau$, we say the sequence S has dimension τ .

In the following, we will investigate the dimension of a special kind of sequence, which we call the entropy generating sequence.

Let (X, T) be a TDS and $\mathcal{U} \in \mathcal{C}_X^o$. We say an increasing sequence of integers $S = \{s_1 < s_2 < \cdots\}$ is an *entropy generating sequence* of $\mathcal{U}$ if

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{N}\left(\bigvee_{i=1}^n T^{-s_i} \mathcal{U}\right) > 0.$$

Denote by $\mathcal{E}(T, \mathcal{U})$ the set of all entropy generating sequences of $\mathcal{U}$ and by $\mathcal{P}(T, \mathcal{U})$ the set of sequence $S = \{s_1 < s_2 < \cdots\}$ of $\mathbb{Z}_+$ with the property that

$$h_{\text{top}}^S(T, \mathcal{U}) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{N}\left(\bigvee_{i=1}^n T^{-s_i} \mathcal{U}\right) > 0.$$

In other words, $\mathcal{P}(T, \mathcal{U})$ is the set of an increasing sequence of integers along which $\mathcal{U}$ has positive entropy.

Definition 3.1. Let (X, T) be a TDS and $\mathcal{U} \in \mathcal{C}_X^o$. We define

$$\overline{D}_e(T, \mathcal{U}) = \begin{cases} \sup_{S \in \mathcal{E}(T, \mathcal{U})} \overline{D}(S) & \text{if } \mathcal{E}(T, \mathcal{U}) \neq \emptyset, \\ 0 & \text{if } \mathcal{E}(T, \mathcal{U}) = \emptyset, \end{cases}$$

$$\overline{D}_p(T, \mathcal{U}) = \begin{cases} \sup_{S \in \mathcal{P}(T, \mathcal{U})} \overline{D}(S) & \text{if } \mathcal{P}(T, \mathcal{U}) \neq \emptyset, \\ 0 & \text{if } \mathcal{P}(T, \mathcal{U}) = \emptyset. \end{cases}$$

Similarly, we can define $\underline{D}_e(T, \mathcal{U})$ and $\underline{D}_p(T, \mathcal{U})$ by changing the upper dimension into the lower dimension.

Definition 3.2. Let (X, T) be a TDS. We define

$$\overline{D}_e(X, T) = \sup_{\mathcal{U} \in \mathcal{C}_X^o} \overline{D}_e(T, \mathcal{U}), \quad \overline{D}_p(X, T) = \sup_{\mathcal{U} \in \mathcal{C}_X^o} \overline{D}_p(T, \mathcal{U}).$$

Similarly, we can define $\underline{D}_e(X, T)$ and $\underline{D}_p(X, T)$.

Recall that a TDS (X, T) is called *null* if $h_{\text{top}}^S(T, \mathcal{U}) = 0$ for any increasing sequence S of $\mathbb{Z}_+$ and any $\mathcal{U} \in \mathcal{C}_X^0$. That is, $\mathcal{P}(T, \mathcal{U}) = \emptyset$ for every $\mathcal{U} \in \mathcal{C}_X^0$. The Morse system is not null [11], but its entropy dimension is zero since its complexity function has linear growth rate (see, for example, [1]).

The following proposition explains why we define the entropy generating sequence as $\liminf$ instead of $\limsup$.

Proposition 3.3. *Let (X, T) be a TDS. Then*

$$\overline{D}_p(T, \mathcal{U}) = \begin{cases} 1 & \text{if } \mathcal{P}(T, \mathcal{U}) \neq \emptyset \\ 0 & \text{if } \mathcal{P}(T, \mathcal{U}) = \emptyset \end{cases} \quad \text{for } \mathcal{U} \in \mathcal{C}_X^0.$$

Proof. We assume $\mathcal{P}(T, \mathcal{U}) \neq \emptyset$. Thus there exists $S = \{s_1 < s_2 < \cdots\} \subset \mathbb{Z}_+$ such that

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log \mathcal{N}\left(\bigvee_{i=1}^n T^{-s_i} \mathcal{U}\right) = a > 0.$$

Next we take $1 \leq n_1 < n_2 < n_3 < \cdots$ such that $n_{j+1} \geq 2s_{n_j}$ for each $j \in \mathbb{N}$. Put

$$F = S \cup \{1, 2, \dots, n_1\} \cup \bigcup_{i=1}^{\infty} \{s_{n_i} + 1, s_{n_i} + 2, \dots, n_{i+1}\}.$$

To simplify, we write $F = \{t_1 < t_2 < \cdots\}$. We have

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \mathcal{N}\left(\bigvee_{i=1}^n T^{-t_i} \mathcal{U}\right) &\geq \limsup_{j \rightarrow +\infty} \frac{\log \mathcal{N}\left(\bigvee_{i=1}^{n_j} T^{-s_i} \mathcal{U}\right)}{|F \cap [0, 1, \dots, s_{n_j}]|} \\ &\geq \limsup_{j \rightarrow +\infty} \frac{\log \mathcal{N}\left(\bigvee_{i=1}^{n_j} T^{-s_i} \mathcal{U}\right)}{2n_j} = \frac{a}{2} > 0, \end{aligned}$$

therefore $F \in \mathcal{P}(T, \mathcal{U})$. Since $n_{j+1} \geq 2s_{n_j}$ for each $j \in \mathbb{N}$, it is easy to see that the upper density of F is $\frac{1}{2}$, hence $\overline{D}(F) = 1$. This implies $\overline{D}_p(T, \mathcal{U}) = 1$. $\square$

In the following, we investigate the interrelations among these dimensions.

Proposition 3.4. *Let (X, T) be a TDS and $\mathcal{U} \in \mathcal{C}_X^0$. Then*

$$\underline{D}_e(T, \mathcal{U}) \leq \overline{D}_e(T, \mathcal{U}) \leq \underline{D}_p(T, \mathcal{U}) \leq \overline{D}(T, \mathcal{U}).$$

Proof. 1) $\underline{D}_e(T, \mathcal{U}) \leq \overline{D}_e(T, \mathcal{U})$ is obvious by Definition 3.1.

2) To show that $\overline{D}_e(T, \mathcal{U}) \leq \underline{D}_p(T, \mathcal{U})$, it is enough to assume that $\overline{D}_e(T, \mathcal{U}) > 0$.

We are given $\tau \in (0, \overline{D}_e(T, \mathcal{U}))$. There exists $S = \{s_1 < s_2 < \cdots\} \in \mathcal{E}(T, \mathcal{U})$ with $\overline{D}(S) > \tau$, i.e. $\limsup_{n \rightarrow +\infty} \frac{n}{s_n^\tau} = +\infty$. Hence

$$(3.1) \quad \limsup_{n \rightarrow +\infty} \frac{n}{n + s_n^\tau} = 1.$$

Let $F = S \cup \{\lfloor n^{\frac{1}{\tau}} \rfloor : n \in \mathbb{N}\}$.

Clearly, $\underline{D}(F) \geq \tau$. Let $F = \{t_1 < t_2 < \cdots\}$. Then for each $n \in \mathbb{N}$ there exists a unique $m(n) \in \mathbb{N}$ such that $s_n = t_{m(n)}$. Since

$$\{s_1, s_2, \dots, s_n\} \subseteq \{t_1, t_2, \dots, t_{m(n)}\} \subseteq \{s_1, s_2, \dots, s_n\} \cup \{\lfloor k^{\frac{1}{\tau}} \rfloor : k \leq s_n^\tau\},$$

we have $n \leq m(n) \leq n + s_n^\tau$. Combining this with (3.1), we get

$$(3.2) \quad \limsup_{n \rightarrow +\infty} \frac{n}{m(n)} = 1.$$

Now we have

$$\begin{aligned}
\limsup_{m \rightarrow +\infty} \frac{\log \mathcal{N}(\bigvee_{i=1}^m T^{-t_i} \mathcal{U})}{m} &\geq \limsup_{n \rightarrow +\infty} \frac{\log \mathcal{N}(\bigvee_{i=1}^{m(n)} T^{-t_i} \mathcal{U})}{m(n)} \\
&\geq \limsup_{n \rightarrow +\infty} \frac{\log \mathcal{N}(\bigvee_{i=1}^n T^{-s_i} \mathcal{U})}{n} \frac{n}{m(n)} \\
&\geq (\liminf_{n \rightarrow +\infty} \frac{\log \mathcal{N}(\bigvee_{i=1}^n T^{-s_i} \mathcal{U})}{n}) \cdot (\limsup_{n \rightarrow +\infty} \frac{n}{m(n)}) \\
&= \liminf_{n \rightarrow +\infty} \frac{\log \mathcal{N}(\bigvee_{i=1}^n T^{-s_i} \mathcal{U})}{n} \quad (\text{by (3.2)}) \\
&> 0 \quad (\text{since } S \in \mathcal{E}(T, \mathcal{U})).
\end{aligned}$$

This implies $F \in \mathcal{P}(T, \mathcal{U})$. Hence $\underline{D}_p(T, \mathcal{U}) \geq \underline{D}(F) \geq \tau$. Since τ is arbitrary, we have $\overline{D}_e(T, \mathcal{U}) \leq \underline{D}_p(T, \mathcal{U})$.

3) If $\underline{D}_p(T, \mathcal{U}) \leq \overline{D}(T, \mathcal{U})$ were not true, then there would exist $\tau \in (0, 1)$ such that $\underline{D}_p(T, \mathcal{U}) > \tau > \overline{D}(T, \mathcal{U})$. On the one hand, we have

$$(3.3) \quad \limsup_{m \rightarrow +\infty} \frac{\log \mathcal{N}(\bigvee_{i=1}^m T^{-i} \mathcal{U})}{m^\tau} = 0$$

since $\tau > \overline{D}(T, \mathcal{U})$, and on the other hand, since $\underline{D}_p(T, \mathcal{U}) > \tau$ there exists $S = \{s_1 < s_2 < \dots\} \in \mathcal{P}(T, \mathcal{U})$ with $\underline{D}(S) > \tau$, i.e. $\liminf_{n \rightarrow +\infty} \frac{n}{s_n^\tau} = +\infty$. Hence there exists $c > 0$ such that $\frac{n}{s_n^\tau} \geq c$ for all sufficiently large n . Now

$$\begin{aligned}
\limsup_{m \rightarrow +\infty} \frac{\log \mathcal{N}(\bigvee_{i=1}^m T^{-i} \mathcal{U})}{m^\tau} &\geq \limsup_{n \rightarrow +\infty} \frac{\log \mathcal{N}(\bigvee_{i=1}^{s_n} T^{-i} \mathcal{U})}{s_n^\tau} \\
&\geq \limsup_{n \rightarrow +\infty} \frac{\log \mathcal{N}(\bigvee_{i=1}^n T^{-s_i} \mathcal{U})}{n} \frac{n}{s_n^\tau} \\
&\geq \limsup_{n \rightarrow +\infty} \frac{\log \mathcal{N}(\bigvee_{i=1}^n T^{-s_i} \mathcal{U})}{n} \cdot c \\
&> 0 \quad (\text{since } S \in \mathcal{P}(T, \mathcal{U})),
\end{aligned}$$

which contradicts (3.3). □

Proposition 3.5. *Let (X, T) be a TDS and $\mathcal{U} \in \mathcal{C}_X^o$. Then*

$$\underline{D}_e(T, \mathcal{U}) \leq \underline{D}(T, \mathcal{U}) \leq \overline{D}(T, \mathcal{U}).$$

Proof. By Definition 3.1, it is clear that $\underline{D}(T, \mathcal{U}) \leq \overline{D}(T, \mathcal{U})$. Now we will show that $\underline{D}_e(T, \mathcal{U}) \leq \underline{D}(T, \mathcal{U})$. For any $\tau \in (0, \underline{D}_e(T, \mathcal{U}))$, there exists $S = \{s_1 < s_2 < \dots\} \in \mathcal{E}(T, \mathcal{U})$ such that $\underline{D}(S) > \tau$, that is, $\liminf_{n \rightarrow +\infty} \frac{n}{s_n^\tau} = +\infty$. Hence there is $c > 0$ such that $\frac{n}{s_n^\tau} \geq c^{-\tau}$, i.e. $s_n \leq cn^{\frac{1}{\tau}}$ for sufficiently large n .

For $m \in \mathbb{N}$ with $m \geq s_1$, there exists a unique $n(m) \in \mathbb{N}$ such that $s_{n(m)} \leq m-1 < s_{n(m)+1} \leq c(n(m)+1)^{\frac{1}{\tau}}$. Since $S \in \mathcal{E}(T, \mathcal{U})$, we have

$$\begin{aligned} \liminf_{m \rightarrow +\infty} \frac{1}{m^\tau} \log \mathcal{N}\left(\bigvee_{i=0}^{m-1} T^{-i} \mathcal{U}\right) &\geq \liminf_{m \rightarrow \infty} \frac{1}{c^\tau(n(m)+1)} \log \mathcal{N}\left(\bigvee_{j=1}^{n(m)} T^{-s_j} \mathcal{U}\right) \\ &\geq \frac{1}{c^\tau} \liminf_{k \rightarrow \infty} \frac{1}{k} \log \mathcal{N}\left(\bigvee_{j=1}^k T^{-s_j} \mathcal{U}\right) > 0. \end{aligned}$$

This implies $\underline{D}(T, \mathcal{U}) \geq \tau$. Finally, as τ is arbitrary, we get $\underline{D}_e(T, \mathcal{U}) \leq \underline{D}(T, \mathcal{U})$. $\square$

In most covers of positive entropy dimension, we would like to show that there exists a subsequence along which the symbolic names are “independent”.

Let $k \geq 2$ and B be a nonempty finite subset of $\mathbb{Z}_+$. Assume $\mathcal{U}$ is the cover of $\{0, 1, \dots, k\}^B = \prod_{z \in B} \{0, 1, \dots, k\}$ consisting of subsets of the form $\prod_{z \in B} \{i_z\}^c$, where $1 \leq i_z \leq k$ and $\{i_z\}^c = \{0, 1, \dots, k\} \setminus \{i_z\}$ for each $z \in B$. For $S \subseteq \{0, 1, \dots, k\}^B$ we let C_S denote the minimal cardinality of subcovers of $\mathcal{U}$ one needs to cover S . Note that we shall use natural logarithms unless we explicitly indicate otherwise.

For any $t > 0$, by Stirling’s formula there exist $c(t) > 0$ and $N(t) \in \mathbb{N}$ such that

$$(3.4) \quad \sum_{j=0}^{[c(t)n]+1} \binom{n}{j} < 2^{tn}, \text{ for any } n \geq N(t).$$

We let

$$(3.5) \quad D_k = k^5 N\left(\frac{\log \frac{k+1}{k}}{4}\right) \quad \text{and} \quad C_k = \frac{1}{c\left(\frac{\log \frac{k+1}{k}}{4}\right)}.$$

For $\ell \geq 2$ and $n \geq \max\{\ell C_k, D_k\}$, we put

$$N_k(n, \ell) = k^2 2^m \left(\frac{n}{m}\right)^{2m},$$

where $m = \left\lceil \frac{\log(\sum_{i=0}^{\ell-1} \binom{n}{i})}{\log \frac{k+1}{k}} \right\rceil + 1$ and $[*]$ is the integer part of a real number $*$.

Since $\ell \leq c\left(\frac{\log \frac{k+1}{k}}{4}\right)n$ following (3.4), we have $m \leq \frac{n}{3} - 1$. As $n \geq D_k \geq k^5$ and $\ell \geq 2$, it is not hard to see that

$$(3.6) \quad 3 \leq 5 \log k \leq \log n \leq m \leq \min\left\{\frac{n}{3} - 1, \frac{\ell \log n}{\log\left(\frac{k+1}{k}\right)}\right\}.$$

Since $3 \leq 5 \log k \leq \log n \leq m$ we have $m^m \geq 3^{5 \log k} \geq k^2$, and thus

$$N_k(n, \ell) = k^2 2^m \left(\frac{n}{m}\right)^{2m} \leq \frac{k^2}{m^m} n^{2m} \leq n^{2m} \leq n^{\frac{2\ell \log n}{\log\left(\frac{k+1}{k}\right)}} = e^{\frac{2\ell}{\log\left(\frac{k+1}{k}\right)} \log^2 n}.$$

That is,

$$(3.7) \quad N_k(n, \ell) \leq e^{\frac{2\ell}{\log\left(\frac{k+1}{k}\right)} \log^2 n}.$$

The following combinatorial result was proved in [17].

Lemma 3.6. *Let $\ell \geq 2$. For every finite set $B \subset \mathbb{Z}_+$ with $|B| \geq \max\{\ell C_k, D_k\}$ and $S \subseteq \{0, 1, 2, \dots, k\}^B$ with $C_S \geq N_k(|B|, \ell)$, there exists a subset $W \subseteq B$ with $|W| \geq \ell$ and $S|_W \supseteq \{1, 2, \dots, k\}^W$.*

Let (X, T) be a TDS. Let $A_1, A_2, \dots, A_k$ be k -subsets of X and let $W \subseteq \mathbb{Z}_+$. We say $\{A_1, A_2, \dots, A_k\}$ is *independent* along W if for any $s \in \{1, 2, \dots, k\}^W$ we have $\bigcap_{w \in W} T^{-w} A_{s(w)} \neq \emptyset$ (see [20]). A direct application of Lemma 3.6 and equation (3.7) gives

Lemma 3.7. *Let (X, T) be a TDS, and let $A_1, A_2, \dots, A_k$ be k -pairwise disjoint nonempty closed subsets of X ($k \geq 2$), $\mathcal{U} = \{A_1^c, A_2^c, \dots, A_k^c\}$. For $\tau \in (0, 1]$, $0 < \eta < \tau$ and $c > 0$ there exists $N \in \mathbb{N}$ (depending on k, τ, η, c) such that if we have a finite subset B of $\mathbb{Z}_+$ satisfying $|B| \geq N$ and $\mathcal{N}(\bigvee_{i \in B} T^{-i} \mathcal{U}) \geq e^{c|B|^\tau}$, then there exists $W \subseteq B$ with $|W| \geq |B|^\eta$ such that $\{A_1, A_2, \dots, A_k\}$ is independent along W .*

Proof. Let D_k and C_k be the value in (3.5). Take $N \geq \max\{C_k^{\frac{1}{1-\eta}}, D_k\}$ such that for any $n \geq N$ we have $n^{\tau-\eta} \geq \frac{2}{c \log(\frac{k+1}{k})} \log^2 n$. It is clear that N depends on k, τ, η and c .

Let B be a finite subset of $\mathbb{Z}_+$ with $|B| \geq N$ and $\mathcal{N}(\bigvee_{i \in B} T^{-i} \mathcal{U}) \geq e^{c|B|^\tau}$.

Consider the map $\phi_B : X \rightarrow \{0, 1, \dots, k\}^B$ defined by

$$(\phi_B(x))(z) = \begin{cases} i & \text{if } T^z(x) \in A_i \text{ for some } 1 \leq i \leq k \\ 0 & \text{otherwise} \end{cases} \quad \text{for } z \in B.$$

Then $C_{\phi_B(X)} = \mathcal{N}(\bigvee_{z \in B} T^{-z} \mathcal{U}) \geq e^{c|B|^\tau}$. Note that (see (3.7))

$$N_k(|B|, |B|^\eta) \leq e^{\frac{2|B|^\eta}{\log(\frac{k+1}{k})} \log^2 |B|} \leq e^{c|B|^\tau}.$$

Hence by Lemma 3.6 there exists $W \subseteq B$ such that $|W| \geq |B|^\eta$ and $\phi_B(X)|_W = \{1, 2, \dots, k\}^W$. Thus for any $s \in \{1, 2, \dots, k\}^W$ there exists $x_s \in X$ such that $\phi_B(x_s)|_W = s$, which implies $x_s \in \bigcap_{w \in W} T^{-w} A_{s(w)}$. This shows that $\{A_1, A_2, \dots, A_k\}$ is independent along W . $\square$

With the help of the above lemma, we have the following theorem.

Theorem 3.8. *Let $A_1, A_2, \dots, A_k$ be k -pairwise disjoint nonempty closed subsets of a TDS (X, T) ($k \geq 2$) and $\mathcal{U} = \{A_1^c, A_2^c, \dots, A_k^c\}$. Then there exists a sequence $F \in \mathcal{E}(T, \mathcal{U})$ such that $\overline{D}(F) = \overline{D}(T, \mathcal{U})$ when $\mathcal{E}(T, \mathcal{U})$ is nonempty. Hence by Proposition 3.4, $\overline{D}(T, \mathcal{U}) = \overline{D}_e(T, \mathcal{U})$.*

Proof. If $\overline{D}(T, \mathcal{U}) = 0$ and $\mathcal{E}(T, \mathcal{U})$ is nonempty, then any sequence in $\mathcal{E}(T, \mathcal{U})$ will have upper dimension zero.

Now assume $\overline{D}(T, \mathcal{U}) > 0$ and let $\{\tau_j\} \subset (0, \overline{D}(T, \mathcal{U}))$ be a sequence of strictly increasing real numbers such that $\lim_{j \rightarrow \infty} \tau_j = \overline{D}(T, \mathcal{U})$. Then we can choose $a > 0$ so that

$$\limsup_{n \rightarrow +\infty} \frac{1}{n^{\tau_j}} \log \mathcal{N}\left(\bigvee_{i=1}^n T^{-i} \mathcal{U}\right) > a \text{ for } j \in \mathbb{N}.$$

Let $\tau_{j-1} < \eta_j < \tau_j$ for $j \in \mathbb{N}$. By Lemma 3.7, there exists $N_j \in \mathbb{N}$ such that for every finite set B with $|B| \geq N_j$ and $\mathcal{N}(\bigvee_{i \in B} T^{-i} \mathcal{U}) \geq e^{\frac{a}{2}|B|^{\tau_j}}$, we can find $W \subseteq B$ with $|W| \geq |B|^{\eta_j}$ and $\{A_1, A_2, \dots, A_k\}$ is independent along W .

Take $1 = n_1 < n_2 < \dots$ such that $(n_{j+1} - n_j)^{\eta_j} \geq j n_j + N_j$ and

$$\mathcal{N}\left(\bigvee_{i=n_j+1}^{n_{j+1}} T^{-i} \mathcal{U}\right) \geq e^{\frac{a}{2}(n_{j+1}-n_j)^{\tau_j}}$$

for each $j \in \mathbb{N}$. Now for each $j \in \mathbb{N}$ there exist $W_j \subseteq \{n_j + 1, n_j + 2, \dots, n_{j+1}\}$ with $|W_j| \geq (n_{j+1} - n_j)^{\eta_j}$ and $\{A_1, A_2, \dots, A_k\}$ is independent along W_j .

For any nonempty set $B \subseteq W_j$ and $s = (s(z))_{z \in B} \in \{1, 2, \dots, k\}^B$, we can find $x_s \in \bigcap_{z \in B} T^{-z} A_{s(z)}$. Let $X_B = \{x_s : s \in \{1, 2, \dots, k\}^B\}$. It is clear that for any $t \in \{1, 2, \dots, k\}^B$ we have $|\bigcap_{z \in B} T^{-z} A_{t(z)}^c \cap X_B| \leq (k-1)^{|B|}$. Combining this fact with $|X_B| = k^{|B|}$, we get

$$(3.8) \quad \mathcal{N}\left(\bigvee_{z \in B} T^{-z} \mathcal{U}\right) \geq \frac{k^{|B|}}{(k-1)^{|B|}} \text{ for any } B \subseteq W_j.$$

Put $F = \bigcup_{i=1}^{\infty} W_j$ and write $F = \{t_1 < t_2 < \dots\}$. First for $n \in \mathbb{N}$ with $n \geq |W_1|$ there exists a unique $k(n) \in \mathbb{N}$ such that $\sum_{i=1}^{k(n)} |W_i| \leq n < \sum_{i=1}^{k(n)+1} |W_i|$. Now

$$\begin{aligned} \mathcal{N}\left(\bigvee_{j=1}^n T^{-t_j} \mathcal{U}\right) &\geq \max\left\{\mathcal{N}\left(\bigvee_{w \in W_{k(n)}} T^{-w} \mathcal{U}\right), \mathcal{N}\left(\bigvee_{w \in W_{k(n)+1} \cap \{t_1, \dots, t_n\}} T^{-w} \mathcal{U}\right)\right\} \\ &\geq \max\left\{\left(\frac{k}{k-1}\right)^{|W_{k(n)}|}, \left(\frac{k}{k-1}\right)^{n - \sum_{i=1}^{k(n)} |W_i|}\right\} \quad (\text{by (3.8)}) \\ &\geq \left(\frac{k}{k-1}\right)^{\frac{n - \sum_{i=1}^{k(n)-1} |W_i|}{2}}. \end{aligned}$$

Hence

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \frac{1}{n} \log \mathcal{N}\left(\bigvee_{j=1}^n T^{-t_j} \mathcal{U}\right) &\geq \liminf_{n \rightarrow +\infty} \frac{n - \sum_{i=1}^{k(n)-1} |W_i|}{2n} \log\left(\frac{k}{k-1}\right) \\ &\geq \liminf_{n \rightarrow +\infty} \left(\frac{1}{2} - \frac{\sum_{i=1}^{k(n)-1} |W_i|}{2|W_{k(n)}|}\right) \cdot \log\left(\frac{k}{k-1}\right) \\ &\geq \liminf_{n \rightarrow +\infty} \left(\frac{1}{2} - \frac{\sum_{i=1}^{k(n)-1} (n_{i+1} - n_i)}{2|W_{k(n)}|}\right) \cdot \log\left(\frac{k}{k-1}\right) \\ &\geq \liminf_{n \rightarrow +\infty} \left(\frac{1}{2} - \frac{n_{k(n)}}{2(n_{k(n)+1} - n_{k(n)})^{\eta_{k(n)}}}\right) \cdot \log\left(\frac{k}{k-1}\right) \\ &\geq \liminf_{n \rightarrow +\infty} \left(\frac{1}{2} - \frac{1}{2k(n)}\right) \cdot \log\left(\frac{k}{k-1}\right) = \frac{1}{2} \log\left(\frac{k}{k-1}\right) > 0. \end{aligned}$$

This shows $F \in \mathcal{E}(T, \mathcal{U})$.

Note that

$$\begin{aligned} \limsup_{m \rightarrow +\infty} \frac{m}{t_m^{\eta_j}} &\geq \limsup_{k \rightarrow +\infty} \frac{|W_1| + |W_2| + \dots + |W_k|}{t_{|W_1|+|W_2|+\dots+|W_k|}^{\eta_j}} \\ &\geq \limsup_{k \rightarrow +\infty} \frac{|W_1| + |W_2| + \dots + |W_k|}{n_{k+1}^{\eta_j}} \geq \limsup_{k \rightarrow +\infty} \frac{|W_k|}{n_{k+1}^{\eta_j}} \\ &\geq \limsup_{k \rightarrow +\infty} \frac{(n_{k+1} - n_k)^{\eta_k}}{n_{k+1}^{\eta_j}} \geq 1. \end{aligned}$$

We have $\overline{D}(F) \geq \eta_j$; hence $\overline{D}(F) = \overline{D}(T, \mathcal{U})$. This finishes the proof. $\square$

Remark 3.9. Let $A_1, A_2, \dots, A_k$ be k -pairwise disjoint nonempty closed subsets of a TDS (X, T) ($k \geq 2$) and $\mathcal{U} = \{A_1^c, A_2^c, \dots, A_k^c\}$.

- (1) Following the proof of Theorem 3.8, it is not hard to show that

$$\underline{D}(T, \mathcal{U}) = \underline{D}_e(T, \mathcal{U}).$$

- (2) Notice that $\overline{D}_e(T, \mathcal{U}) \leq \overline{D}_p(T, \mathcal{U})$. Hence we have $\overline{D}(T, \mathcal{U}) = \overline{D}_e(T, \mathcal{U}) \leq \overline{D}_p(T, \mathcal{U})$.

Theorem 3.10. *Let (X, T) be a TDS. Then*

- (1) $\overline{D}_e(X, T) = \underline{D}_p(X, T) = \overline{D}(X, T)$.
 (2) $\underline{D}_e(X, T) \leq \underline{D}(X, T) \leq \overline{D}(X, T) \leq \overline{D}_p(X, T)$.

Proof. (1) By Proposition 3.4, we have $\overline{D}_e(X, T) \leq \underline{D}_p(X, T) \leq \overline{D}(X, T)$. Now it is sufficient to show that $\overline{D}(X, T) \leq \overline{D}_e(X, T)$. By Proposition 2.5 and Theorem 3.8, we have

$$\overline{D}(X, T) = \sup\{\overline{D}(T, \mathcal{W}) : \mathcal{W} \in \mathcal{C}_X^s\} = \sup\{\overline{D}_e(T, \mathcal{W}) : \mathcal{W} \in \mathcal{C}_X^s\} \leq \overline{D}_e(X, T).$$

(2) By Proposition 3.5, we have $\underline{D}_e(X, T) \leq \underline{D}(X, T) \leq \overline{D}(X, T)$. Then by (1), $\overline{D}(X, T) = \underline{D}_p(X, T) \leq \overline{D}_p(X, T)$. This finishes the proof of (2). $\square$

Remark 3.11. Let (X, T) be a TDS.

- (1) $D_e(X, T) = D(X, T)$ if one of the two values exists.
 (2) If (X, T) has a generating open cover, then there exists an entropy generating sequence (of this cover) F such that $\overline{D}(F) = \overline{D}_e(X, T) = \overline{D}(X, T)$. In particular, if (X, T) is a symbolic system, then it has an entropy generating sequence F whose (upper) dimension is the (upper) entropy dimension.

4. DIMENSION PAIRS AND UNIFORM DIMENSION SYSTEMS

In this section, we will localize the notion of entropy dimension.

Definition 4.1. Let $(x_1, x_2) \in X \times X \setminus \Delta_X$. The entropy dimension of (x_1, x_2) is $\overline{D}(x_1, x_2) = \lim_{n \rightarrow \infty} \overline{D}(\mathcal{U}_n) \in [0, 1]$, where $\mathcal{U}_n = \{X \setminus \overline{B(x_1, \frac{1}{n})}, X \setminus \overline{B(x_2, \frac{1}{n})}\}$. When $\overline{D}(x_1, x_2) = \alpha$, we will say (x_1, x_2) is an α -dimension pair, or an α -pair for short. The set of α -pairs of (X, T) is denoted by $E^\alpha(X, T)$. Denote $G_\alpha(X, T) = \bigcup_{\beta \geq \alpha} E^\beta(X, T)$.

Lemma 4.2. *Let (X, T) be a TDS and $\mathcal{U} = \{U_1, U_2\} \in \mathcal{C}_X^o$. Then there exists $(x_1, x_2) \in U_1^c \times U_2^c$ such that $\overline{D}(x_1, x_2) \geq \overline{D}(T, \mathcal{U})$.*

Proof. We follow the argument in the proof of Proposition 2 in [3]. We shall show that one can find a coarser open cover $\mathcal{U}_1 = \{U_1^1, U_2^1\}$ than $\mathcal{U}$, having the property that $\overline{D}(T, \mathcal{U}_1) \geq \overline{D}(T, \mathcal{U})$ and $\text{diam}((U_i^1)^c) \leq \frac{1}{2} \text{diam}(U_i^c)$ for $i = 1, 2$. By induction one gets two decreasing sequences of nonempty closed sets, $\{(U_1^j)^c\}$ and $\{(U_2^j)^c\}$, converging respectively to the points x_1 and x_2 such that $\overline{D}(x_1, x_2) \geq \overline{D}(T, \mathcal{U})$.

If U_1^c is a singleton, then we put $U_1^1 = U_1$. If U_1^c has at least two different points y and y' , fix $\epsilon_1 > 0$ with $\epsilon_1 \leq \frac{d(y, y')}{4}$, and construct a cover of U_1^c by open balls with radius ϵ_1 centered in U_1^c ; call it $\mathcal{A}$. Since U_1^c is compact, there exist $A_1, A_2, \dots, A_u \in \mathcal{A}$ such that $\bigcup_{i=1}^u A_i \supseteq U_1^c$. Write $F_i = U_1^c \cap \overline{A_i}$, $i = 1, 2, \dots, u$. By the choice of ϵ_1 , each closed set F_i is a proper subset of U_1^c with $\text{diam}(F_i) \leq \frac{1}{2} \text{diam}(U_1^c)$. Since $\{U_1, U_2\}$ is coarser than $\bigvee_{i=1}^u \{F_i^c, U_2\}$, we have

$$\overline{D}(T, \mathcal{U}) \leq \overline{D}(T, \bigvee_{i=1}^u \{F_i^c, U_2\}) = \max\{\overline{D}(T, \{F_i^c, U_2\}), i = 1, 2, \dots, u\},$$

where the last equality comes from Proposition 2.2(3). Thus there exists $i_* \in \{1, 2, \dots, u\}$ such that $\overline{D}(T, \{F_{i_*}^c, U_2\}) \geq \overline{D}(T, \mathcal{U})$. We denote the set $F_{i_*}^c$ by U_1^1 . Apply the same argument for U_2 , and obtain U_2^1 . Now we have found a coarser open cover $\mathcal{U}_1 = \{U_1^1, U_2^1\}$ than $\mathcal{U}$, having the property that $\overline{D}(T, \mathcal{U}_1) \geq \overline{D}(T, \mathcal{U})$ and $\text{diam}((U_i^1)^c) \leq \frac{1}{2} \text{diam}(U_i^c)$ for $i = 1, 2$.

Repeating the arguments, one gets two decreasing sequences of nonempty closed sets $\{(U_1^j)^c\}, \{(U_2^j)^c\}$ such that

- a) $\text{diam}((U_i^{j+1})^c) \leq \frac{1}{2} \text{diam}((U_i^j)^c)$ for $i = 1, 2$ and $j = 0, 1, 2, \dots$ (here let $U_i^0 = U_i$, $i = 1, 2$).
- b) $\overline{D}(T, \mathcal{U}_j) \geq \overline{D}(T, \mathcal{U})$ for each $j = 1, 2, \dots$.

By a) above, we have $\bigcap_{j=1}^{\infty} (U_1^j)^c = \{x_1\}$ and $\bigcap_{j=1}^{\infty} (U_2^j)^c = \{x_2\}$ for some $x_1, x_2 \in X$. Moreover, since $x_1 \in U_1^c$, $x_2 \in U_2^c$ and $U_1^c \cap U_2^c = \emptyset$, we have $x_1 \neq x_2$.

Finally, we show $\overline{D}(x_1, x_2) \geq \overline{D}(T, \mathcal{U})$. Since $x_1 \neq x_2$, there exists $n_0 \in \mathbb{N}$ such that $\overline{B}(x_1, \frac{1}{n_0}) \cap \overline{B}(x_2, \frac{1}{n_0}) = \emptyset$. For any $n \geq n_0$, there exists $j(n) \in \mathbb{N}$ large enough such that $(U_i^{j(n)})^c \subseteq B(x_i, \frac{1}{n})$ for $i = 1, 2$, and thus

$$\overline{D}(T, \{X \setminus \overline{B}(x_1, \frac{1}{n}), X \setminus \overline{B}(x_2, \frac{1}{n})\}) \geq \overline{D}(T, \mathcal{U}_{j(n)}) \geq \overline{D}(T, \mathcal{U}),$$

where the last inequality comes from b). Hence we have

$$\overline{D}(x_1, x_2) = \lim_{n \rightarrow \infty} \overline{D}(T, \{X \setminus \overline{B}(x_1, \frac{1}{n}), X \setminus \overline{B}(x_2, \frac{1}{n})\}) \geq \overline{D}(T, \mathcal{U}).$$

This finishes the proof of the lemma. $\square$

Proposition 4.3. *Let (X, T) be a TDS. Then $G_\alpha(X, T) \cup \Delta_X$ is a closed and invariant subset of $X \times X$ for any $\alpha \geq 0$.*

Proof. Let $\{(x_i, y_i)\}_{i=1}^\infty \subseteq G_\alpha(X, T)$ and $\lim_{i \rightarrow +\infty} (x_i, y_i) = (x, y)$. If $x = y$, then $(x, y) \in \Delta_X$. Now assume that $x \neq y$. For any $n \in \mathbb{N}$ with $\frac{2}{n} < d(x, y)$ let $\mathcal{U}_n = \{X \setminus \overline{B}(x, \frac{1}{n}), X \setminus \overline{B}(y, \frac{1}{n})\}$. Then there exists $i \in \mathbb{N}$ and $n_i \in \mathbb{N}$ such that $\mathcal{U}_n \succeq \{X \setminus \overline{B}(x_i, \frac{1}{n_i}), X \setminus \overline{B}(y_i, \frac{1}{n_i})\}$. Hence

$$\overline{D}(T, \mathcal{U}_n) \geq \overline{D}(T, \{X \setminus \overline{B}(x_i, \frac{1}{n_i}), X \setminus \overline{B}(y_i, \frac{1}{n_i})\}) \geq \overline{D}(x_i, y_i) \geq \alpha.$$

Let $n \rightarrow +\infty$; we get $\overline{D}(x, y) \geq \alpha$. So $(x, y) \in G_\alpha(X, T)$. The invariance is clear. $\square$

Remark 4.4. It is also easy to see that for any closed T -invariant subset W of (X, T) , if $(x_1, x_2) \in G_\alpha(W, T|_W)$, then $(x_1, x_2) \in G_\alpha(X, T)$.

Proposition 4.5. *Let $\pi : (X, T) \rightarrow (Y, S)$ be a factor map between two TDS's.*

- (1) *If $(x_1, x_2) \in E^\alpha(X, T)$ and $y_i = \pi(x_i)$, $y_1 \neq y_2$, then $\overline{D}(y_1, y_2) \geq \alpha$.*
- (2) *If $(y_1, y_2) \in E^\alpha(Y, S)$, then there exists $(x_1, x_2) \in E^\alpha(X, T)$ and $\pi(x_i) = y_i$, $i = 1, 2$.*

Proof. (1) If $(x_1, x_2) \in E^\alpha(X, T)$ and $y_i = \pi(x_i)$, $y_1 \neq y_2$, then

$$\begin{aligned} \overline{D}(y_1, y_2) &= \lim_{n \rightarrow +\infty} \overline{D}(S, \overline{\{Y \setminus B(y_1, \frac{1}{n}), Y \setminus B(y_2, \frac{1}{n})\}}) \\ &= \lim_{n \rightarrow +\infty} \overline{D}(T, \pi^{-1} \overline{\{Y \setminus B(y_1, \frac{1}{n}), Y \setminus B(y_2, \frac{1}{n})\}}) \\ &\geq \lim_{m \rightarrow +\infty} \overline{D}(T, \overline{\{X \setminus B(x_1, \frac{1}{m}), X \setminus B(x_2, \frac{1}{m})\}}) = \overline{D}(x_1, x_2). \end{aligned}$$

(2) Let $(y_1, y_2) \in E^\alpha(Y, S)$. For any $n \in \mathbb{N}$ with $\frac{2}{n} < d(y_1, y_2)$, let $\mathcal{U}_n = \{Y \setminus \overline{B(y_1, \frac{1}{n})}, Y \setminus \overline{B(y_2, \frac{1}{n})}\}$. Then there exists $(x_1^n, x_2^n) \in \pi^{-1}(\overline{B(y_1, \frac{1}{n})}) \times \pi^{-1}(\overline{B(y_2, \frac{1}{n})})$ with $\overline{D}(x_1^n, x_2^n) \geq \overline{D}(T, \pi^{-1}(\mathcal{U}_n)) = \overline{D}(S, \mathcal{U}_n) \geq \alpha$. Hence $(x_1^n, x_2^n) \in G_\alpha(X, T)$. Take a subsequence $\{n_i\}$ such that

$$\lim_{i \rightarrow +\infty} (x_1^{n_i}, x_2^{n_i}) = (x_1, x_2)$$

for some $(x_1, x_2) \in X \times X$. Clearly, $\pi(x_1) = y_1, \pi(x_2) = y_2$, so $x_1 \neq x_2$. Now on the one hand, by Proposition 4.3 we have $(x_1, x_2) \in G_\alpha(X, T)$. On the other hand, by (1) $\overline{D}(x_1, x_2) \leq \overline{D}(y_1, y_2) = \alpha$. Therefore $\overline{D}(x_1, x_2) = \alpha$; that is, $(x_1, x_2) \in E^\alpha(X, T)$. $\square$

Remark 4.6. Let (X_i, T_i) be two TDS's and (x_i, y_i) be an α_i -pair of (X_i, T_i) , where $i = 1, 2$ and $0 < \alpha_1 < \alpha_2$. Then it is easy to see that the dimension of $((x_1, x_2), (y_1, y_2))$ is no more than α_1 . This shows that (x_2, y_2) has a preimage whose entropy dimension is strictly smaller than its own entropy dimension.

Definition 4.7. For a TDS (X, T) , we call the subset $\{\alpha \geq 0 : E^\alpha(X, T) \neq \emptyset\}$ of $[0, 1]$ the *dimension set* of (X, T) and denote it by $\mathcal{D}(X, T)$ or just $\mathcal{D}$. If $0 \notin \mathcal{D}(X, T)$, we will say (X, T) has *strictly positive entropy dimension*. Let $\alpha \in (0, 1]$; we call (X, T) an α -uniform entropy dimension system (α -u.d. system for short) if $\mathcal{D} = \{\alpha\}$ and call (X, T) an α^+ -dimension system (α^+ -d. system for short) if $\mathcal{D}(X, T) \subset [\alpha, 1]$. If (X, T) is the trivial system, we let $\mathcal{D}(X, T) = \emptyset$.

Now let us recall some concepts in topological dynamics. For a dynamical system (X, T) , we say (X, T) is *transitive* if for each pair of nonempty open subsets U and V of X , $N(U, V) = \{n \in \mathbb{N} : U \cap T^{-n}V \neq \emptyset\} \neq \emptyset$; (X, T) is *weakly mixing* if $(X \times X, T \times T)$ is transitive. The orbit of $x \in X$, $\{x, T(x), T^2(x), \dots\}$, is denoted by $\text{orb}(x, T)$. For a transitive system, $x \in X$ is a *transitive point* if the orbit of x is dense, and it is known that the set of transitive points forms a dense G_δ subset. If the set of the transitive points is the whole space X , we then say that (X, T) is *minimal*, and each point in X is a *minimal point*.

It is known ([24] and [2]) that (X, T) is weakly mixing if and only if for each pair of two nonempty open subsets U and V of X , $N(U, U) \cap N(U, V) \neq \emptyset$.

Lemma 4.8. *If (X, T) has strictly positive entropy dimension, then it is weakly mixing.*

Proof. If (X, T) is not weakly mixing, then there are two nonempty open subsets U'_i of X , $i = 1, 2$, such that there exists no $n \in \mathbb{N}$ with that $U'_1 \cap T^{-n}U'_1 \neq \emptyset$ and $U'_1 \cap T^{-n}U'_2 \neq \emptyset$. Take $x_i \in U'_i$ and closed neighborhoods U_i of x_i with $U_i \subset U'_i$, $i = 1, 2$ and $U_1 \cap U_2 = \emptyset$. Clearly, $U_1 \cap T^{-n}U_1 = \emptyset$ or $U_1 \cap T^{-n}U_2 = \emptyset$ for any $n \in \mathbb{Z}_+$. Moreover, for any $n \in \mathbb{Z}_+$ we can take $W_n = U_1^c$ or U_2^c such that $U_1 \subset T^{-n}W_n$.

Let $\mathcal{U} = \{U_1^c, U_2^c\}$. On the one hand, $\overline{D}(T, \mathcal{U}) > 0$ since $\mathcal{U}$ is a standard cover of X . On the other hand, for $n \in \mathbb{N}$ consider for each $x \in X$ the first $i \in \{0, 1, 2, \dots, n-1\}$ such that $T^i x \in U_1$, when there exists one. We get that the $\bigvee_{i=0}^{n-1} T^{-i} \mathcal{U}$ admits a subcover of the sets of the form

$$U_1^c \cap T^{-1} U_1^c \cap \dots \cap T^{-(i-1)} U_1^c \cap T^{-i} W_0 \cap T^{-(i+1)} W_1 \cap \dots \cap T^{-(n-1)} W_{n-i-1},$$

$i = 0, 1, \dots, n-1$. Hence for all $n \in \mathbb{N}$, $\mathcal{N}(\bigvee_{i=0}^{n-1} T^{-i} \mathcal{U}) \leq n+1$, and therefore $\overline{D}(T, \mathcal{U}) = 0$, a contradiction. $\square$

Proposition 4.9. *Let $\alpha \in (0, 1]$. Then*

- (1) *A nontrivial TDS (X, T) is an α -u.d. system if and only if $\overline{D}(T, \mathcal{U}) = \alpha$ for any standard open cover $\mathcal{U}$ of X .*
- (2) *A nontrivial factor of an α -u.d. system is also an α -u.d. system.*
- (3) *A nontrivial factor of an α^+ -d. system is also an α^+ -d. system.*

Proof. (2) and (3) come from Proposition 4.5. Now we will prove (1). Assume that $\overline{D}(T, \mathcal{U}) = \alpha$ for any standard open cover $\mathcal{U}$ of X . Then it is clear that for any $x \neq y \in X$, $\overline{D}(x, y) = \alpha$; that is, (X, T) is an α -u.d. system.

Conversely, suppose that (X, T) is an α -u.d. system. Let $\mathcal{U} = \{U, V\}$ be a standard open cover of X . Then, on the one hand, by Lemma 4.2 there exists $(x, y) \in U^c \times V^c$ such that $\overline{D}(T, \mathcal{U}) \leq \overline{D}(x, y) = \alpha$. On the other hand, take $(x', y') \in \text{int}(U^c) \times \text{int}(V^c)$; then $x' \neq y'$ and $\overline{D}(T, \mathcal{U}) \geq \overline{D}(x', y') = \alpha$. Hence $\overline{D}(T, \mathcal{U}) = \alpha$. $\square$

For $\alpha \in [0, 1]$, we say a TDS (X, T) is a *zero α -dimension system* (for short an α -z.d. system) if $\bigcup_{\beta > \alpha} E^\beta(X, T) = \emptyset$. For a dynamical system, the topological Pinsker factor, i.e. the maximal factor with zero topological entropy, exists [4]. Similarly we have (following [4])

Theorem 4.10. *For a dynamical system (X, T) and $\alpha \in [0, 1]$, the smallest closed invariant equivalence relation containing $\bigcup_{\beta > \alpha} E^\beta(X, T)$ induces the maximal α -z.d. factor.*

Theorem 4.11. *Let $\pi : (X, T) \rightarrow (Y, S)$ be a factor map between two TDS's. Then $\mathcal{D}(X, T) \supseteq \mathcal{D}(Y, S)$. In particular, the dimension set is an invariant under topological conjugacy.*

Proof. This follows from Proposition 4.5(2). $\square$

Let (X, T) and (Y, S) be two TDS's, and $\pi_1 : X \times Y \rightarrow X$, $\pi_2 : X \times Y \rightarrow Y$ be the projections. $J \subseteq X \times Y$ is called a *joining* of (X, T) and (Y, S) if J is a closed, $T \times S$ -invariant (i.e. $T \times S(J) \subseteq J$) with $\pi_1(J) = X$, $\pi_2(J) = Y$. Clearly, $X \times Y$ is a joining of (X, T) and (Y, S) . A joining J of (X, T) and (Y, S) is said to be *proper* if $J \neq X \times Y$. A joining J of (X, T) and (Y, S) is said to be *minimal* if J contains no strictly smaller joining of (X, T) and (Y, S) . We say that (X, T) and (Y, S) are *disjoint* if $X \times Y$ contains no proper joining of (X, T) and (Y, S) . Recall that when (X, T) and (Y, S) are disjoint, at least one of them is minimal.

The following is a generalization of the theorem that u.p.e. systems are disjoint from minimal entropy zero systems [3].

Theorem 4.12. *Let (X, T) be a TDS and (Y, S) be a minimal TDS. If $\mathcal{D}(X, T) > \mathcal{D}(Y, S)$ (i.e. for any $\alpha \in \mathcal{D}(X, T)$ and $\beta \in \mathcal{D}(Y, S)$, $\alpha > \beta$), then (X, T) is disjoint from (Y, S) .*

Proof. For the completeness of the paper, we include the proof which is similar to the argument of Proposition 6 in [3]. If (Y, S) or (X, T) is a trivial system, then it is obvious that (X, T) is disjoint from (Y, S) . In the following we assume that (Y, S) and (X, T) are both nontrivial.

Let J be any given joining of (X, T) and (Y, S) . We assume that J is minimal, i.e., J contains no strictly smaller joining of (X, T) and (Y, S) . Indeed, the intersection of a decreasing family of joinings is closed invariant and, by compactness, has projections X and Y , so it is a joining. Applying Zorn's lemma, we therefore obtain the existence of a minimal joining inside any joining.

Let $J(x) = \{y \in Y : (x, y) \in J\}$ for each $x \in X$. First we claim $J(x) \cap J(Tx) \neq \emptyset$ for $x \in X$. Assume to the contrary that there exists $x \in X$ with $J(x) \cap J(Tx) = \emptyset$. Clearly, $x \neq Tx$. Let $\alpha = \overline{D}(x, Tx)$. Then $\alpha \in \mathcal{D}(X, T)$. By Proposition 4.5(2), there exist $y_1, y_2 \in Y$ such that $(x, y_1), (Tx, y_2) \in J$ and $\overline{D}((x, y_1), (Tx, y_2)) = \alpha$. Since $J(x) \cap J(Tx) = \emptyset$, $y_1 \neq y_2$. Then by Proposition 4.5(1), we have $\overline{D}(y_1, y_2) \geq \alpha$, which is a contradiction to the assumption of $\mathcal{D}(X, T) > \mathcal{D}(Y, S)$.

Next we consider the subset of $X \times Y$:

$$J' = \bigcup_{x \in X} \{x\} \times (J(x) \cap J(Tx)).$$

Since $SJ(x) \subseteq J(Tx)$ and the map $x \mapsto J(x)$ is upper semi-continuous, J' is closed and $T \times S$ -invariant. Now by the above claim $J(x) \cap J(Tx) \neq \emptyset$ for $x \in X$. Hence $\pi_1(J') = X$, where $\pi_1 : X \times Y \rightarrow X$, $\pi_2 : X \times Y \rightarrow Y$ are the projections.

Note that $\pi_2(J')$ is a closed S -invariant subset of Y ; we deduce $\pi_2(J') = Y$ from the minimality of (Y, S) . So J' is a joining of (X, T) and (Y, S) . Thus $J' = J$, since J is a minimal joining and $J' \subseteq J$. This implies that $J(x) \cap J(Tx) = J(x)$, i.e.,

$$(4.1) \quad J(x) \subseteq J(Tx).$$

Since (Y, S) and (X, T) are both nontrivial, we know that (X, T) has strictly positive entropy dimension by the assumption of $\mathcal{D}(X, T) > \mathcal{D}(Y, S)$. Thus (X, T) is weakly mixing by Lemma 4.8. Hence there exists $x_0 \in X$ such that

$$\omega(x_0, T) = \{\text{all limit points of the sequence } \{T^n x_0\}_{n=1}^\infty \text{ in } X\}$$

is the whole space X . Now note that $x_0 \in \omega(x_0, T)$; hence there exists a sequence of natural numbers $n_1 < n_2 < \dots$ such that

$$(4.2) \quad \lim_{i \rightarrow +\infty} T^{n_i} x_0 = x_0.$$

By (4.1), we have

$$(4.3) \quad J(x_0) \subseteq J(Tx_0) \subseteq J(T^{n_1} x_0) \subseteq J(T^{n_2} x_0) \subseteq \dots$$

Combining (4.2), (4.3) and the fact that the map $x \mapsto J(x)$ is upper semi-continuous, we have

$$J(x_0) \subseteq J(Tx_0) \subseteq J(T^{n_1} x_0) \subseteq J(T^{n_2} x_0) \subseteq \dots \subseteq J(x_0).$$

This shows that $J(x_0) = J(Tx_0) \supseteq SJ(x_0)$. So $J(x_0) = Y$, as (Y, S) is minimal. Moreover, by (4.1), we have $J(T^n x_0) = Y$ for any $n \in \mathbb{N}$. Combining this with the fact that $\omega(x_0, T) = X$, we have $J(x) = Y$ for any $x \in X$ since the map $x \mapsto J(x)$ is upper semi-continuous. This implies

$$J = \bigcup_{x \in X} \{x\} \times J(x) = X \times Y,$$

i.e., (X, T) is disjoint from (Y, S) . □

We provide an example to show that the minimality of the system of lower entropy dimension is required.

Example 4.13. For any nontrivial minimal system (X, T) , there exists a transitive system (Y, S) with $\overline{D}(S) = 0$, such that (X, T) and (Y, S) are not disjoint.

Proof. Take $x_1 \neq x_2 \in X$ and open neighborhoods U_i of x_i , $i = 1, 2$, such that $U_1 \cap U_2 = \emptyset$. By Furstenberg's result (see Theorem 2.17 of [9]) there exists $\{p_i\}_{i=1}^{+\infty}$ such that $FS(\{p_i\}_{i=1}^{+\infty}) \subseteq N(x_1, U_1)$, where $FS(\{p_i\}_{i=1}^{+\infty})$ is the set consistent with all the finite sums of the sequence $\{p_i\}_{i=1}^{+\infty}$ and $N(x_1, U_1) = \{n \in \mathbb{N} : T^n(x_1) \in U_1\}$. Now take $1 = i_1 < i_2 < \dots < i_k$ such that if $q_k = \sum_{j=i_k}^{i_{k+1}-1} p_j$ for $k \in \mathbb{N}$, then $q_{k+1} \geq 2^{k+1}(\sum_{\ell=1}^k q_\ell)$ for $k \in \mathbb{N}$.

Let $A = \{0\} \cup FS(\{q_k\}_{k=1}^{+\infty})$ and $y = 1_A \in \{0, 1\}^{\mathbb{Z}_+}$ be the indicator function of A ; that is, $y(k) = 1$ iff $k \in A$. Let $\sigma : \{0, 1\}^{\mathbb{Z}_+} \rightarrow \{0, 1\}^{\mathbb{Z}_+}$ be the left shift map. Let $Y = \{\sigma^i y : i \in \mathbb{Z}_+\}$. It is not hard to see that (Y, σ) is a transitive system with transitive point y by the choice of A . Clearly (Y, σ) is not minimal. By a simple calculation, the number of the subwords of length n of y is no more than $4n$, and hence $\overline{D}(Y, \sigma) = 0$.

Let $J = \{(T^i x_1, \sigma^i y) : i \in \mathbb{Z}_+\}$. Then clearly J is a joining of (X, T) and (Y, S) . Let $V_1 = \{z \in Y : z(0) = 1\}$. In the following we will show that $(x_2, y) \notin J$. In fact, if $(x_2, y) \in J$, then there exists $i \in \mathbb{N}$ such that $(T^i x_1, \sigma^i y) \in U_2 \times V_1$. That is, $i \in N(x_1, U_2) \cap A$ since $\sigma^i y \in V_1$ iff $i \in A$. This implies that $i \in N(x_1, U_2) \cap (\{0\} \cup N(x_1, U_1))$. But $N(x_1, U_2) \cap (\{0\} \cup N(x_1, U_1)) = N(x_1, U_1) \cap N(x_1, U_2) = \emptyset$ since $U_1 \cap U_2 = \emptyset$ and $x_1 \notin U_2$, a contradiction.

This shows $J \neq X \times Y$, and so (X, T) and (Y, S) are not disjoint. $\square$

As a direct application of Theorem 4.12, we have

Corollary 4.14. 1. α -u.d. systems are disjoint from minimal β -u.d. systems when $0 \leq \beta < \alpha \leq 1$.

2. A strictly positive entropy dimension system is disjoint from all minimal systems with dimension set $\{0\}$.

In Section 5 we will construct a nontrivial minimal β -u.d. system for any $\beta \in (0, 1)$. This allows us to construct the following example.

Example 4.15. For any $\alpha \in (0, 1]$, there exists a minimal system (X, T) such that $\overline{D}(X, T) = \alpha$, but $E^\alpha(X, T) = \emptyset$.

Proof. Given $\alpha \in (0, 1]$, let (X_i, T_i) be a nontrivial minimal $\frac{i}{i+1}\alpha$ -u.d. system and let $(X, T) = (\prod_{i=1}^\infty X_i, \prod_{i=1}^\infty T_i)$. By Corollary 4.14, $(X_1 \times X_2, T_1 \times T_2)$ is minimal (otherwise, any minimal set of it will be a joining of (X_1, T_1) and (X_2, T_2)). Suppose $(\prod_{i=1}^k X_i, \prod_{i=1}^k T_i)$ is minimal for some $k \in \mathbb{N}$. Since $\overline{D}(\prod_{i=1}^k X_i, \prod_{i=1}^k T_i) = \max_{1 \leq i \leq k} \{\overline{D}(X_i, T_i)\} = \overline{D}(X_k, T_k) = \frac{k}{k+1}\alpha$, which is smaller than $\overline{D}(X_{k+1}, T_{k+1})$, by Theorem 4.12, $(\prod_{i=1}^{k+1} X_i, \prod_{i=1}^{k+1} T_i)$ is also minimal. Hence the infinite product system (X, T) is minimal.

Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ be two different points in X . Then we can find $k \in \mathbb{N}$ such that $x_k \neq y_k$. By Proposition 4.5(1), $\overline{D}(x, y) \leq \overline{D}(x_k, y_k) < \alpha$. So $E^\alpha(X, T) = \emptyset$ and $\overline{D}(X, T) \leq \alpha$. Since $\overline{D}(X, T) > \overline{D}(X_i, T_i)$ for any $i \in \mathbb{N}$, we have $\overline{D}(X, T) \geq \alpha$. Hence $\overline{D}(X, T) = \alpha$. $\square$

It is clear that $h_{top}(X, T) = 0$. By taking $\alpha = 1$, we have a minimal example with zero entropy while its entropy dimension is 1.

5. REALIZATION OF DIMENSION SETS

Recall that the dimension set $\mathcal{D}(X, T) = \{\alpha \geq 0 : E^\alpha(X, T) \neq \emptyset\}$ of a TDS (X, T) . It is clear that $\emptyset \neq \mathcal{D}(X, T) \subseteq [0, 1]$ for any nontrivial TDS (X, T) . In this section, we investigate which subsets of $[0, 1]$ may be realized as the dimension set of a TDS.

Let $\mathbb{D} = \{A \subseteq [0, 1] : \text{there is TDS } (X, T) \text{ s.t. } \mathcal{D}(X, T) = A\}$; that is, $\mathbb{D}$ is the set of all subsets of $[0, 1]$ which can be realized as a dimension set of a TDS.

Lemma 5.1. *Let $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathbb{D}$. Then $\{0\} \cup \bigcup_{i \in \mathbb{N}} A_i \in \mathbb{D}$.*

Proof. Since $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathbb{D}$, there exists a sequence of TDS's (X_i, T_i) such that $\mathcal{D}(X_i, T_i) = A_i$ for each $i \in \mathbb{N}$. We may assume that all X_i 's are a closed subset of some one compact metric space Z and $\lim_{i \rightarrow +\infty} X_i = p \in Z$ under the Hausdorff metric.

Now let $X = \{(0, p)\} \cup \bigcup_{i=1}^{\infty} \{\frac{1}{i}\} \times X_i$. Then X is a compact subset of the product $[0, 1] \times Z$; hence it is a compact metric space. Then we define $T : X \rightarrow X$ with $T(\frac{1}{i}, x) = (\frac{1}{i}, T_i(x))$ for $x \in X_i$ and $T(0, p) = (0, p)$. Clearly, T is a continuous map on X . So let (X, T) be a TDS. It is not hard to see that $\mathcal{D}(X, T) = \{0\} \cup \bigcup_{i \in \mathbb{N}} A_i$, so $\{0\} \cup \bigcup_{i \in \mathbb{N}} A_i \in \mathbb{D}$. $\square$

Remark 5.2. By Lemma 5.1 and Theorem 5.3, we know that if A is a countable subset of $[0, 1]$ with $0 \in A$, then $A \in \mathbb{D}$.

In [6], J. Cassaigne provided a method of construction of uniformly recurrent infinite words which have different subexponential growth rates of n -names. A uniformly recurrent word is clearly a minimal point of the system of its orbit closure. Now, using this method, we will construct a minimal α -u.d. system for any given $\alpha \in (0, 1)$.

Theorem 5.3. *For any $\alpha \in [0, 1]$, there exists a minimal TDS (X, T) such that $\mathcal{D}(X, T) = \{\alpha\}$.*

Proof. Irrational rotation on a circle has the property of minimality and $\mathcal{D}(X, T) = \{0\}$. Also any minimal u.p.e. system has $\mathcal{D}(X, T) = \{1\}$.

Now let $0 < \alpha < 1$ be fixed and let $\varphi(n) = n^\alpha$. Let $z_1 = 0$ and $z_{j+1} = z_j j z_j$. Then let the dyadic valuation word $\mathbf{v}$ be the limit of the sequence of words (z_j) .

Denote by A^* the collection of finite or infinite words whose alphabets are from the set A . We define inductively the substitution $\psi : \mathbb{N}^* \rightarrow A_2^*$, where $A_2 = \{0, 1\}$, and the family $(x_k)_{k \in \mathbb{N}}$ of prefixes of the dyadic valuation word $\mathbf{v}$ as follows:

- $\psi(0) = 0, \psi(1) = 1$;
- x_k is the longest prefix of $\mathbf{v}$ such that

$$|\psi(x_k)| \leq \varphi^{-1}(k+1) - \varphi^{-1}(k);$$

- for all $j \geq 1$, $\psi(2j) = \psi(x_{\lfloor \log j \rfloor})\psi(j)$ and $\psi(2j+1) = \psi(x_{\lfloor \log j \rfloor})1\psi(j)$.

Let $\mathbf{u} = \psi(\mathbf{v})$ and $X = \text{orb}(\mathbf{u}, \sigma)$, where σ is the left shift map on $\{0, 1\}^{\mathbb{N}}$. The difference of our construction from the irreducible example in [6] is that $|\psi(2j)|$ has length $|\psi(2j+1)| - 1$.

First, we will show that $\mathbf{u}$ is a minimal point, hence (X, σ) is a minimal TDS.

Let w be a factor of $\mathbf{u}$. Then there exists an index k_0 such that the prefix $\psi(x_{k_0})$ of $\mathbf{u}$ is long enough to contain the first occurrence of w . Let $j_0 = 2^{k_0+1}$. Then for all $j \geq j_0$, $\psi(x_{k_0})$ is a prefix of $\psi(x_{\lfloor \log j \rfloor - 1})$, which in turn, by construction of ψ , is a prefix of $\psi(j)$. So w is a factor of $\psi(j)$ for all $j \geq j_0$. Since the occurrence of $j \geq j_0$ has bounded gaps in $\mathbf{v}$ and $\psi(x_{k_0})$ also appears in $\psi(x_k)$ with bounded gaps for $k \geq k_0$, we can deduce that w occurs in $\mathbf{u}$ with bounded gaps. We let $L = |\psi(j_0 z_{j_0} j_0)| + 2\lfloor \log j_0 \rfloor$. By a similar discussion in Lemma 4 of [6], every factor of $\mathbf{u}$ of length L contains an occurrence of w . So w occurs in $\mathbf{u}$ with bounded gaps, i.e., $\mathbf{u}$ is a minimal point.

Now let us compute the number of n -blocks of $\mathbf{u}$. We denote by $p_{\mathbf{u}}(n)$ the complexity function of $\mathbf{u}$. Again by the similar discussion in [6], we have

$$2^{\underline{k}(n)} \leq p_{\mathbf{u}}(n) \leq 12n^2 m(n) 2^{\overline{k}(n+2)},$$

where $m(n) = \min\{k : |\psi(x_{k-1})| \geq n\}$ and $m(n) \ll n^\beta$ for some $\beta > 0$, $\overline{k}(n) = \min\{k : |\psi(2^k)| \geq n\}$, $\underline{k}(n) = \min\{k : |\psi(2^{k+1}-1)| \geq n\}$ and $\overline{k}(n) \sim \underline{k}(n) \sim \varphi(n) = n^\alpha$.

Then $\log p_{\mathbf{u}}(n) \sim n^\alpha$. So for the open cover $\mathcal{U} = \{[0]_X, [1]_X\}$ of X ,

$$\log \mathcal{N}\left(\bigvee_{i=0}^{n-1} \sigma^{-i}\mathcal{U}\right) = \log p_{\mathbf{u}}(n) \sim n^\alpha.$$

Hence $\overline{D}(\sigma, \mathcal{U}) = \alpha$.

Because $\mathcal{U}$ is a generating open cover of (X, σ) , by Proposition 2.4, for any standard cover $\mathcal{V}$ of X ,

$$\overline{D}(\sigma, \mathcal{V}) \leq \overline{D}(\sigma, \mathcal{U}) \leq \alpha.$$

Let x and y be any two different points in X . Then there exist i_1, i_2 and m such that $x \in [\sigma^{i_1}\psi(x_m)]$, $y \in [\sigma^{i_2}\psi(x_m)]$ (for a finite word a , $[a]$ denotes the cylinder $\{x \in \{0, 1\}^{\mathbb{N}} : x_0 x_1 \cdots x_{|a|-1} = a\}$) and these two cylinders are disjoint. We assume that $i_2 < i_1$ and set $i = i_1 - i_2$. Let $\mathcal{V} = \{V_0^c, V_1^c\}$, where $V_0 = [\sigma^{i_1}\psi(x_m)]$, $V_1 = [\sigma^{i_2}\psi(x_m)]$. Now we will prove $\overline{D}(\sigma, \mathcal{V}) \geq \alpha$.

For any sufficiently large integer k , we write it as $k = 2il + m_0$, where $l = \lfloor \frac{k-m}{2i} \rfloor$ and $m_0 \geq m$. For any word $s = s_1 s_2 \cdots s_k \in \{0, 1\}^k$, we associate an integer $2^k \leq j_s < 2^{k+1}$ by the formula $j_s = 2^k + \sum_{h=1}^k s_h 2^{k-h}$. Let $S = \{s \in \{0, 1\}^k : s = r 0^{m_0}, r \in \{0^i 1^i, 1^i 0^i\}^l\}$. Then for any $s \in S$, the word $\psi(j_s)$ has the same length $|\psi(2^k)| + il$.

Let $k_h = i_1 + (h-1)i + \sum_{t=1}^{(2h-1)i} |\psi(x_{k-t})|$. We can examine that

$$[\psi(j_s)] \subset \bigcap_{h=1}^l \sigma^{-k_h} V_{s_{(2h-1)i}}.$$

This means $\mathcal{N}(\bigvee_{h=1}^l \sigma^{-k_h} \mathcal{V}) = 2^l$.

Let $\underline{k}(n) = \min\{k : |\psi(2^{k+1}-1)| \geq n\}$. Then by the similar argument of [6], $\underline{k}(n) \sim n^\alpha$. For large n , take $k = \underline{k}(n) - 1$; then $|\psi(j)| < n$ for $2^{\underline{k}(n)-1} \leq j < 2^{\underline{k}(n)}$. Hence, $k_l < n$. So we have

$$\mathcal{N}\left(\bigvee_{h=1}^n \sigma^{-k_h} \mathcal{V}\right) \geq \mathcal{N}\left(\bigvee_{h=1}^l \sigma^{-k_h} \mathcal{V}\right) = 2^l.$$

Notice that

$$l = \lfloor \frac{k(n) - 1 - m}{2i} \rfloor \sim k(n) \sim \varphi(n) = n^\alpha.$$

We get $\overline{D}(\sigma, \mathcal{V}) \geq \alpha$.

Hence (x, y) is an α -pair, and it follows that (X, σ) is a minimal α -u.d. system. $\square$

Theorem 5.4. *The cardinality of $\mathbb{D}$ is $\aleph$, i.e. the cardinality of $\mathbb{R}$.*

Proof. Let $Z = [0, 1]^\mathbb{N}$ be the Hilbert cube with the product metric ρ . Then (Z, ρ) is a compact metric space. Let $\mathbb{Q}_0 = \mathbb{Q} \cap (-1, 2]$ and $\mathcal{C}$ be the set of all $\{C_\tau\}_{\tau \in \mathbb{Q}_0}$ satisfying

- 1) C_τ is a nonempty closed set of Z for each $\tau \in \mathbb{Q}_0$.
- 2) $C_\tau \subseteq C_{\tau'}$ when $\tau, \tau' \in \mathbb{Q}_0$ with $2 \geq \tau \geq \tau' > -1$.
- 3) For each $\tau \in \mathbb{Q}_0$, $C_\tau = \bigcap_{\tau' \in \mathbb{Q}_0: \tau' < \tau} C_{\tau'}$.

Since $\mathbb{Q}_0$ is countable and the cardinality of the sets of all nonempty closed sets of Z is $\aleph$, we know that the cardinality of $\mathcal{C}$ is $\aleph$. For each $\{C_\tau\}_{\tau \in \mathbb{Q}_0} \in \mathcal{C}$, we define

$$\mathcal{D}(\{C_\tau\}_{\tau \in \mathbb{Q}_0}) = \{\tau \in [0, 1] : (\bigcap_{\tau' \in \mathbb{Q}_0: \tau' \leq \tau} C_{\tau'}) \setminus (\bigcup_{\tau'' \in \mathbb{Q}_0: \tau'' > \tau} C_{\tau''}) \neq \emptyset\}.$$

Let $\mathbb{D}(\mathcal{C}) = \{\mathcal{D}(\{C_\tau\}_{\tau \in \mathbb{Q}_0}) : \{C_\tau\}_{\tau \in \mathbb{Q}_0} \in \mathcal{C}\}$. Since the cardinality of $\mathcal{C}$ is $\aleph$, the cardinality of $\mathbb{D}(\mathcal{C})$ is $\aleph$. Next we will show $\mathbb{D} \subseteq \mathbb{D}(\mathcal{C})$; thus the cardinality of $\mathbb{D}$ is $\aleph$ since $\{\{\alpha\} : \alpha \in [0, 1]\} \subset \mathbb{D}$.

For any $A \in \mathbb{D}$, there exists a TDS (X, T) such that $A = \mathcal{D}(X, T)$; that is, $A = \{\tau \in [0, 1] : E^\tau(X, T) \neq \emptyset\}$. Let $\phi : X \times X \hookrightarrow Z$ be the topological embedding, i.e. $\phi : X \times X \rightarrow \phi(X \times X)$ is a homeomorphism. Now let

$$A_\tau = \begin{cases} \phi(G_0(X, T) \cup \Delta_X) & \text{if } \tau \in (-1, 0), \\ \phi(G_\tau(X, T) \cup \Delta_X) & \text{if } \tau \in [0, 1], \\ \phi(\Delta_X) & \text{if } \tau \in (1, 2]. \end{cases}$$

Since each $G_\tau(X, T) \cup \Delta_X$ is a closed subset of $X \times X$, A_τ is a nonempty closed subset of Z . Note that for $\tau \in (0, 1]$ we have $G_\tau(X, T) = \bigcap_{\tau' < \tau} G_{\tau'}(X, T)$ and $G_\tau(X, T) \subseteq G_{\tau'}(X, T)$ when $1 \geq \tau \geq \tau' \geq 0$. It is not hard to see that $\{A_\tau\}_{\tau \in \mathbb{Q}_0} \in \mathcal{C}$ and $A_\tau = \bigcap_{\tau' \in \mathbb{Q}_0: \tau' < \tau} A_{\tau'}$ for each $\tau \in (-1, 2]$.

Now it is clear that

$$\begin{aligned} A &= \{\tau \in [0, 1] : E^\tau(X, T) \neq \emptyset\} = \{\tau \in [0, 1] : A_\tau \setminus (\bigcup_{\tau'' \in (-1, 2]: \tau'' > \tau} A_{\tau''}) \neq \emptyset\} \\ &= \{\tau \in [0, 1] : (\bigcap_{\tau' \in \mathbb{Q}_0: \tau' < \tau} A_{\tau'}) \setminus (\bigcup_{\tau'' \in \mathbb{Q}_0: \tau'' > \tau} A_{\tau''}) \neq \emptyset\} \\ &= \mathcal{D}(\{A_\tau\}_{\tau \in \mathbb{Q}_0}) \in \mathbb{D}(\mathcal{C}). \end{aligned}$$

Since A is arbitrary, we get $\mathbb{D} \subseteq \mathbb{D}(\mathcal{C})$. $\square$

Theorem 5.4 tells us that there exist many nonempty subsets of $[0, 1]$ which cannot be realized as a dimension set of a TDS. The following result gives us some examples of such subsets.

Theorem 5.5. *Let A be a nonempty subset of $(0, 1]$ with $\inf\{t : t \in A\} \notin A$. Then $A \notin \mathbb{D}$.*

Proof. If this is not true, then there exists a TDS (X, T) with $\mathcal{D}(X, T) = A$. Since $0 \notin A$, (X, T) has strictly positive entropy dimension, and so (X, T) is weakly mixing by Lemma 4.8. Let (x, y) be a transitive point of $(X \times X, T \times T)$ and $\alpha = \overline{\mathcal{D}}(x, y)$. Then $\alpha \in A$. By Proposition 4.5, $\overline{\mathcal{D}}(T^i x, T^i y) \geq \alpha$ for $i \in \mathbb{Z}_+$. Therefore, $\{(T^i x, T^i y)\}_{i \in \mathbb{Z}_+} \subseteq G_\alpha(X, T) \cup \Delta_X$. Since $G_\alpha(X, T) \cup \Delta_X$ is a $T \times T$ -invariant closed subset of $X \times X$ we have $X \times X = G_\alpha(X, T) \cup \Delta_X$, as (x, y) is a transitive point of $(X \times X, T \times T)$. This implies $A \subseteq [\alpha, 1]$ and $\alpha = \inf\{t : t \in A\} \notin A$, a contradiction with $\alpha \in A$. $\square$

Finally, we list a few open questions on the properties of dimension sets.

Question 5.6. Let A be a subset of $[0, 1]$ with $0 \in A$. Is $A \in \mathbb{D}$ iff A is an F_σ -set?

Question 5.7. For $0 < \alpha < \beta \leq 1$, does there exist a minimal TDS (X, T) such that $\mathcal{D}(X, T) = [\alpha, \beta]$? In general, which subsets of $[0, 1]$ can be realized as a dimension set of a minimal TDS?

Question 5.8. How should we characterize $\mathbb{D}$?

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