DUALITY IN SPACES OF FINITE LINEAR COMBINATIONS OF ATOMS

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ABSTRACT. In this paper we describe the dual and the completion of the space of finite linear combinations of (p, ∞) -atoms, $0 < p \leq 1$. As an application, we show an extension result for operators uniformly bounded on (p, ∞) -atoms, $0 < p < 1$, whose analogue for $p = 1$ is known to be false. Let $0 < p < 1$ and let T be a linear operator defined on the space of finite linear combinations of (p, ∞) -atoms, $0 < p < 1$, which takes values in a Banach space B . If T is uniformly bounded on (p, ∞) -atoms, then T extends to a bounded operator from $H^p(\mathbb{R}^n)$ into B .

1. INTRODUCTION

For each $0 < p \leq 1$ consider the space F^p of finite linear combinations of (p, ∞) -atoms, endowed with its natural norm (or quasi-norm for $p < 1$)

$$(1) \quad \|f\|_{F^p} = \inf \left\{ \left(\sum_j' |\lambda_j|^p \right)^{\frac{1}{p}} : f = \sum_j' \lambda_j a_j, \ a_j \text{ a } (p, \infty)\text{-atom, } \lambda_j \in \mathbb{C} \right\},$$

where $\sum_j'$ denotes a finite sum. Recall that a is a (p, ∞) -atom if a is a measurable function supported on a ball B , satisfying the cancellation condition

$$\int a(x) x^\alpha dx = 0, \quad |\alpha| \leq n\left(\frac{1}{p} - 1\right),$$

and the size condition

$$|a| \leq \frac{1}{|B|^{\frac{1}{p}}}.$$

The space F^p is clearly contained in $H^p = H^p(\mathbb{R}^n)$, the standard real Hardy space on $\mathbb{R}^n$. The elements of H^p are the distributions that admit an atomic decomposition, $f = \sum_{j=1}^\infty \lambda_j a_j$, converging in the sense of distributions, for some (p, ∞) -atoms a_j and scalars λ_j with $\sum_{j=1}^\infty |\lambda_j|^p < \infty$ (for $p = 1$, $H^1 \subset L^1$ and atomic sums converge in the L^1 -norm). In [MTW] Meyer, Taibleson and Weiss observed that the F^p -norm is not comparable to the H^p -norm on F^p . Recently, it was shown in [B] that the Meyer-Taibleson-Weiss result leads to the following conclusion in the case $p = 1$: there exists a bounded linear functional on F^1 which does not extend to a bounded linear functional on H^1 . In other words, there is a linear operator which is uniformly bounded on $(1, \infty)$ -atoms but does not extend to a bounded linear operator on H^1 .

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In this paper we describe the structure of the completion $\widetilde{F}^p$ of F^p , $0 < p \leq 1$, and of its dual space. We show in particular that, when $p < 1$, F^p and H^p have the same dual, and therefore no example like the one in [B] can be exhibited for $p < 1$. An immediate consequence of this is that if $0 < p < 1$ and the linear operator

$$(2) \quad T : F^p \rightarrow B,$$

maps F^p into a Banach space B satisfying the inequality

$$\|T(a)\|_B \leq C,$$

for some positive constant C and all (p, ∞) -atoms, then T extends to a bounded linear operator from H^p into B . The argument proceeds by duality as follows. Take any u in the dual B^* of B . Since $u \circ T \in (F^p)^* = (H^p)^*$,

$$|u(T(f))| \leq C \|u\| \|f\|_{H^p},$$

and so, by the dual expression of the norm in a Banach space,

$$\|T(f)\|_B \leq C \|f\|_{H^p}.$$

We prove the following facts about $\widetilde{F}^p$, $0 < p \leq 1$.

- (i) The closed subspace $\widetilde{F}^{p,c}$ of $\widetilde{F}^p$ spanned by the continuous (p, ∞) -atoms is isomorphic to H^p as a Banach space, and $\widetilde{F}^p$ splits as the direct sum of $\widetilde{F}^{p,c}$ and a *non-trivial* complementary closed subspace N^p .
- (ii) Every element ξ of $\widetilde{F}^p$ admits an atomic decomposition

$$\xi = \sum_{j=1}^{\infty} \lambda_j a_j,$$

for (p, ∞) -atoms a_j and scalars λ_j with $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$. Moreover, the $\widetilde{F}^p$ -norm of ξ is equivalent to its atomic norm

$$\inf \left\{ \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{\frac{1}{p}} : \sum_{j=1}^{\infty} \lambda_j a_j = \xi \text{ in } \widetilde{F}^p \right\}.$$

- (iii) If an atomic sum $\sum_{j=1}^{\infty} \lambda_j a_j$, with λ_j and a_j as above, converges to 0 in $\widetilde{F}^p$, it also converges to 0 in H^p , but not vice versa. In fact, N^p consists of those elements of $\widetilde{F}^p$ that are represented by atomic sums converging to 0 in H^p .

In other words, H^p and $\widetilde{F}^p$ are both quotients of the space of “formal series” of (p, ∞) -atoms with ℓ^p coefficients, but the equivalence relation defining $\widetilde{F}^p$ is finer than that defining H^p .

So, the reason why $(F^1)^*$ is strictly larger than $(H^1)^*$ is that it is the direct sum of $(\widetilde{F}^{1,c})^* = (H^1)^*$ and $(N^1)^*$. Notice that $(N^1)^*$ is non-trivial, as the dual of the non-trivial Banach space N^1 . On the other hand, it turns out that $(N^p)^*$ is trivial for $p < 1$.

To describe our results we need to introduce some notation and recall some basic classical facts in the theory of Banach algebras (see Section 3 for details).

Denote by $L_0^\infty(\mathbb{R}^n)$ the space of bounded measurable functions on $\mathbb{R}^n$ vanishing at infinity. Then $L_0^\infty(\mathbb{R}^n)$ is a commutative C^* -algebra without unit, and its maximal ideal space is a locally compact, non-compact space, which we call $\widehat{\mathbb{R}^n}$ (cf. [F]).

By the Gelfand-Naimark theorem, the Gelfand transform $f \rightarrow \hat{f}$ establishes an isometric isomorphism between $L_0^\infty(\mathbb{R}^n)$ and the algebra $C_0(\widehat{\mathbb{R}^n})$ of all continuous functions on $\widehat{\mathbb{R}^n}$ vanishing at ∞ . On the other hand, $C_0(\mathbb{R}^n)$ is a closed subalgebra of $L_0^\infty(\mathbb{R}^n)$, and its maximal ideal space is $\mathbb{R}^n$. This embedding induces a continuous projection π from $\widehat{\mathbb{R}^n}$ onto $\mathbb{R}^n$. Clearly, if $f \in C_0(\mathbb{R}^n)$, then $\hat{f} = f \circ \pi$.

In a similar way, given any ball B in $\mathbb{R}^n$, the maximal ideal space of $L^\infty(B)$ is a compact space $\widehat{B}$, endowed with a projection π_B onto $\bar{B}$ induced by the inclusion of $C(\bar{B})$ in $L^\infty(B)$. Moreover, $L^\infty(B) \cong C(\widehat{B})$, again by the Gelfand-Naimark theorem.

The restriction map $f \mapsto f|_B$ from $L_0^\infty(\mathbb{R}^n)$ to $L^\infty(B)$ induces a natural embedding $\iota_B : \widehat{B} \rightarrow \widehat{\mathbb{R}^n}$, which is compatible with the projections π and π_B , in the sense that

$$\pi_B = \pi \circ \iota_B .$$

Similar embeddings $\iota_{B,B'} : \widehat{B'} \rightarrow \widehat{B}$ exist for pairs of balls B, B' with $B' \subset B$, with the same compatibility with respect to the corresponding projections.

Denote by m the Lebesgue measure on $\mathbb{R}^n$. The continuous linear functional $f \mapsto \int f dm$ on $L^\infty(B)$ is represented by a positive Borel measure $\widehat{m}_B$ on $\widehat{B}$, that is,

$$(3) \quad \int f dm = \int \hat{f} d\widehat{m}_B, \quad f \in L^\infty(B) .$$

If B is contained in a second ball B' , then the restriction of $\widehat{m}_{B'}$ to $\widehat{B}$ is precisely $\widehat{m}_B$ and thus we can define a positive Borel measure $\widehat{m}$ globally on $\widehat{\mathbb{R}^n}$ by requiring that its restriction to $\widehat{B}$ be $\widehat{m}_B$ for each ball B .

We can now state our main result.

Theorem. (A) *Let ℓ be a bounded linear functional on F^1 . Then there exist a function $b \in BMO(\mathbb{R}^n)$ and a Radon measure μ on $\widehat{\mathbb{R}^n}$, singular with respect to $\widehat{m}$, satisfying*

$$(4) \quad |\mu|(\widehat{B}) \leq C m(B), \quad \text{for each ball } B$$

such that

$$(5) \quad \ell(f) = \int f b dm + \int \hat{f} d\mu, \quad f \in F^1 .$$

Conversely, if b and μ are as above, then the identity (5) defines a bounded linear functional on F^1 and

$$\|\ell\|_{(F^1)^*} \cong \|b\|_{BMO} + \sup_B \frac{|\mu|(\widehat{B})}{m(B)} .$$

(B) *Each bounded linear functional on F^p , $0 < p < 1$, extends uniquely to a bounded linear functional on $H^p(\mathbb{R}^n)$. Thus $(F^p)^* = H^p(\mathbb{R}^n)^*$, $0 < p < 1$.*

It is clear that relation (5) determines the function b and the measure μ uniquely. Therefore $(F^1)^*$ differs from $(H^1)^* = BMO$ by the presence of the complementary subspace S of singular measures satisfying (4). We will show that S is non-trivial; in fact, the Meyer, Taibleson and Weiss argument may be interpreted as the construction of a non-zero measure in S . The decomposition of $(F^1)^*$ as $BMO \oplus S$

is the dual counterpart of the decomposition of $\widetilde{F^1}$ as $\widetilde{F^{1,c}} \oplus N^1$, although S and BMO do not coincide with the annihilators of $\widetilde{F^{1,c}}$ and N^1 , respectively.

The nature of the elements of N^p , including $p = 1$, is somehow mysterious. It is not clear at all to us if they can be represented by concrete analytic objects.

Section 2 contains the discussion of the completion of F^p and a constructive argument which proves the non-triviality of N^p . In Section 3 we prove the Theorem. We also give an example of a non-zero singular measure satisfying (4).

We remark here that a variation of the main argument in the proof of the Theorem provides an alternative proof of some results in [MSV] and [YZ] on the equivalence of the finite and infinite atomic norms of $(1, q)$ -atoms, $q < \infty$, and on extension of bounded operators defined on finite linear combinations of (p, q) -atoms with $1 < q < \infty$.

2. THE COMPLETION OF F^p

Let $F^{p,c}$ stand for the subspace of H^p consisting of finite linear combinations of continuous (p, ∞) atoms. A surprising recent result in [MSV] states that the H^p and the F^p norms are equivalent on $F^{p,c}$, $0 < p \leq 1$. Indeed, the result is proved in [MSV] only for $p = 1$, but, as suggested in Remark 3.2 there, the same argument extends to the case $0 < p < 1$. A complete proof may be found in [MSV2, section 3].

More precisely, we can quote Lemma 3.1 and Remark 3.2 in [MSV] as follows.

Lemma 1. *The following norms are equivalent on $F^{p,c}$:*

- (a) *the H^p -norm;*
- (b) *the F^p -norm (1);*
- (c) *the $F^{p,c}$ -norm*

$$\|f\|_{F^{p,c}} = \inf \left\{ \sum_j' |\lambda_j| : f = \sum_j' \lambda_j a_j, \ a_j \text{ a continuous } (p, \infty)\text{-atom, } \lambda_j \in \mathbb{C} \right\}.$$

Since $F^{p,c}$ is dense in H^p , the natural inclusion of $F^{p,c}$ in F^p extends uniquely to a continuous linear operator T from H^p to $\widetilde{F^p}$. By Lemma 1, T maps H^p isomorphically onto the closure $\widetilde{F^{p,c}}$ of $F^{p,c}$ in $\widetilde{F^p}$. Notice that, again by Lemma 1, $\widetilde{F^{p,c}}$ is the completion of $F^{p,c}$ endowed either with the norm $\|\cdot\|_{F^{p,c}}$ or with the norm inherited from F^p .

On the other hand, the inclusion of F^p (endowed with its natural norm) into H^p is continuous, and it extends to a continuous linear operator U from $\widetilde{F^p}$ to H^p . We then have the diagram

$$H^p \xrightarrow{T} \widetilde{F^p} \xrightarrow{U} H^p,$$

with $U \circ T$ being the identity map. In particular, U is surjective. Set $P = T \circ U$, so that P is a projection, that is, $P^2 = P$. The kernel of P is the kernel of U , which we denote by N^p , and the kernel of $I - P$ is $T(H^p) = \widetilde{F^{p,c}}$. Hence we get the topological direct sum decomposition

$$\widetilde{F^p} = \widetilde{F^{p,c}} \oplus N^p.$$

Notice that N^p is non-trivial, since otherwise the H^p and the F^p norms would be comparable on F^p .

To better understand the space $\widetilde{F^p}$ we now prove the following.

Proposition. *Given any sequence of (p, ∞) -atoms a_j and any ℓ^p -sequence of scalars λ_j , the series $\sum_{j=1}^{\infty} \lambda_j a_j$ converges in $\widetilde{F^p}$ to an element ξ such that $\|\xi\|_{\widetilde{F^p}}^p \leq \sum_{j=1}^{\infty} |\lambda_j|^p$. Conversely, each $\xi \in \widetilde{F^p}$ can be written as*

$$(6) \quad \xi = \sum_{j=1}^{\infty} \lambda_j a_j ,$$

where each a_j is a (p, ∞) -atom and the sum is convergent in $\widetilde{F^p}$. Moreover,

$$(7) \quad \|\xi\|_{\widetilde{F^p}}^p = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j|^p \right\} ,$$

where the infimum is taken over all decompositions (6) of ξ .

Proof. Let ξ be an element of $\widetilde{F^p}$. To prove (6), express ξ as the limit in $\widetilde{F^p}$ of a sequence S_k of elements of F^p . Given $\epsilon > 0$, we may assume that $\|S_1\|_{F^p}^p < (1 + \epsilon)\|\xi\|_{\widetilde{F^p}}^p$ and that $\|S_k - S_{k+1}\|_{F^p}^p < \epsilon^k \|\xi\|_{\widetilde{F^p}}^p$. Thus

$$\xi = \lim_{k \rightarrow \infty} S_1 + (S_2 - S_1) + \cdots + (S_k - S_{k-1}) .$$

Set

$$S_1 = \sum_{j=1}^{N_1} \lambda_j a_j ,$$

where the above expression has been chosen so that

$$\sum_{j=1}^{N_1} |\lambda_j|^p < (1 + \epsilon) \|\xi\|_{\widetilde{F^p}}^p .$$

Similarly, set

$$S_\ell - S_{\ell-1} = \sum_{j=N_{\ell-1}+1}^{N_\ell} \lambda_j a_j , \quad \ell \geq 2 ,$$

with

$$\sum_{j=N_{\ell-1}+1}^{N_\ell} |\lambda_j|^p < \epsilon^\ell \|\xi\|_{\widetilde{F^p}}^p .$$

Then $\sum_{j=1}^{\infty} |\lambda_j|^p < (1 + \epsilon)^{-1} \|\xi\|_{\widetilde{F^p}}^p$ and the partial sums $\xi_m = \sum_{j=1}^m \lambda_j a_j$ form a Cauchy sequence in F^p . This shows that (6) holds.

Notice also that, for each $\xi \in \widetilde{F^p}$, the inequality $\|\xi\|_{\widetilde{F^p}}^p \leq \inf \{ \sum_{j=1}^{\infty} |\lambda_j|^p \}$, where the infimum is taken over all possible expressions (6), is due to the fact that $\|\cdot\|_{\widetilde{F^p}}^p$ satisfies the triangle inequality. $\square$

The atomic decomposition of elements of $\widetilde{F^p}$ given above provides an explicit description of the operator U .

Corollary. *Let $\xi \in \widetilde{F^p}$ be represented by the sum (6). Then $U(\xi)$ is the sum of the same series in H^p .*

We end this section by providing a constructive proof of the non-triviality of N^p . Let us first describe the Meyer-Taibleson-Weiss construction as presented in [B]. Let B denote the open ball centered at the origin with radius 1. Take a sequence of open disjoint balls $B_j, j \geq 1$, such that $\bigcup_j B_j$ is dense in B . Notice that we may also choose the B_j so that the Lebesgue measure of their union $\sum_{j \geq 1} |B_j|$ is as small as we wish. As shown in [B], for each j there exists a (non-continuous) (p, ∞) -atom a_j supported on B_j with the property that $|a_j| \geq c |B_j|^{-\frac{1}{p}}$, where c is a small positive constant depending only on n . Thus, setting

$$(8) \quad f = \sum_{j \geq 1} |B_j|^{\frac{1}{p}} a_j,$$

we get $|f| \geq c$ on $\bigcup B_j$. From that it is not difficult to conclude (see [B]) that

$$\|f\|_{F^p} \geq c |B|^{\frac{1}{p}}.$$

On the other hand, we clearly have $\|f\|_{H^p}^p \leq \sum_{j \geq 1} |B_j|$, so that the ratio between H^p -norm and F^p -norm can be made as small as we wish.

We can now construct a sequence $\{f_m\}$ in F^p satisfying

$$\begin{aligned} \|f_m\|_{F^p}^p &\geq c^p |B|, \\ \|f_m - f_{m+1}\|_{F^p}^p &\leq 2^p \frac{|B|}{2^m}, \\ \|f_m\|_{H^p}^p &\leq \frac{|B|}{2^m}. \end{aligned}$$

The first two conditions imply that $\{f_m\}$ has a non-zero limit $\xi \in \widetilde{F^p}$, whereas the third implies that $Uf_m = f_m$ tends to 0 in H^p . Hence $\xi \in N^p$.

The functions f_m have the form (8); precisely,

$$(9) \quad f_m = \sum_{j \geq 1} |B_j^m|^{\frac{1}{p}} a_j^m,$$

where, for each m , $\{B_j^m\}_j$ is a disjoint family of balls contained in B with dense union and small total measure, and each a_j^m is a (p, ∞) -atom with $|a_j^m| \geq c |B_j^m|^{-\frac{1}{p}}$.

The first function f_1 can be any function as in (8) with, say, $\sum_{j \geq 1} |B_j^1| < |B|/2$. We then construct inductively f_{m+1} from f_m as follows.

Take N so large that $\sum_{j > N} |B_j^m| < (1/4) \sum_{j \geq 1} |B_j^m|$. Inside each $B_j^m, 1 \leq j \leq N$, we take open disjoint balls $B'_{jl}, l \geq 1$, such that $\bigcup_{l \geq 1} B'_{jl}$ is dense in B_j^m and $\sum_{l \geq 1} |B'_{jl}| < |B_j^m|/4$.

Then

$$(10) \quad \sum_{j=1}^N \sum_{l \geq 1} |B'_{jl}| + \sum_{j > N} |B_j^m| \leq \frac{1}{2} \sum_{j \geq 1} |B_j^m|.$$

Let a'_{jl} be a (p, ∞) -atom supported on B'_{jl} with $|a'_{jl}| \geq c |B'_{jl}|^{-\frac{1}{p}}$. Set

$$f_{m+1} = \sum_{j=1}^N \sum_{l \geq 1} |B'_{jl}|^{\frac{1}{p}} a'_{jl} + \sum_{j > N} |B_j^m|^{\frac{1}{p}} a_j^m.$$

Since $|f_{m+1}| \geq c$ on an open dense subset of B , $\|f_{m+1}\|_{F^p}^p \geq c^p |B|$. Moreover,

$$f_m - f_{m+1} = \sum_{j=1}^N \left(|B_j^m|^{\frac{1}{p}} a_j^m - \sum_{l \geq 1} |B'_{jl}|^{\frac{1}{p}} a'_{jl} \right).$$

For each j , the function

$$|B_j^m|^{\frac{1}{p}} a_j^m - \sum_{l \geq 1} |B'_{jl}|^{\frac{1}{p}} a'_{jl}$$

is supported on B_j^m and its absolute value is not greater than 2. Hence

$$\|f_m - f_{m+1}\|_{F^p}^p \leq \sum_{j=1}^N 2^p |B_j^m|.$$

Now, we relabel the balls in such a way that

$$\{B_j^{m+1}\}_{j \geq 1} = \{B_j^m\}_{j > N} \cup \{B'_{jl}\}_{j \leq N, l \geq 1},$$

and rename the atoms in f_{m+1} as a_j^{m+1} accordingly. Then, inductively from (10),

$$\sum_{j \geq 1} |B_j^m| \leq 2^{-m} |B|$$

for every m , and the required estimates can be easily verified.

3. PROOF OF THE THEOREM

We start by proving, for the reader's sake, a few statements made (explicitly or not) in the last part of the introduction concerning the Gelfand spectrum $\widehat{\mathbb{R}^n}$ and its projection π on $\mathbb{R}^n$.

The first statement we want to prove is that π is, in fact, well defined. Given ϕ in $\widehat{\mathbb{R}^n}$, i.e., a non-trivial multiplicative functional on $L_0^\infty(\mathbb{R}^n)$, it is clear that its restriction to $C_0(\mathbb{R}^n)$ is also multiplicative. We must show that this restriction is “evaluation” at some point $x = \pi(\phi)$ of $\mathbb{R}^n$, or, equivalently, that it is not identically zero.

Since $L_0^\infty(\mathbb{R}^n)$ is a C^* -algebra, it is symmetric, so that $\phi(\bar{f}) = \overline{\phi(f)}$ for every f . Therefore, $f \geq 0$ implies that $\phi(f) \geq 0$, so that ϕ is monotonic on real-valued functions. If ϕ vanishes identically on $C_0(\mathbb{R}^n)$, it also vanishes on characteristic functions of compact sets. By linearity and continuity, this would be a contradiction.

The second statement is that the mapping π is surjective. We know that to each $\phi \in \widehat{\mathbb{R}^n}$ we can associate a point $\pi(\phi)$ in $\mathbb{R}^n$. Given $y \in \mathbb{R}^n$, we can define a translate $\tau_y \phi \in \widehat{\mathbb{R}^n}$ by

$$(11) \quad \tau_y \phi(f) = \phi(f(\cdot + y)).$$

It is quite clear that $\pi(\tau_y \phi) = \pi(\phi) + y$. Since $\widehat{\mathbb{R}^n}$ is non-empty, π is surjective.

The last statement which remained unproved in the introduction is that $\widehat{\mathbb{R}^n}$ is the union of the $\widehat{B}$ over all balls B . This is a direct consequence of (ii) in the following lemma.

Lemma 2. *Let B be an open ball in $\mathbb{R}^n$. Then*

(i)

$$(12) \quad \widehat{B} = \{\phi \in \widehat{\mathbb{R}^n} : \phi(\chi_B) = 1\} = \text{supp } \widehat{\chi_B},$$

where $\widehat{f}$ stands for the Gelfand transform of $f \in L_0^\infty(\mathbb{R}^n)$.

(ii)

$$\pi^{-1}(B) \subset \widehat{B} \subset \pi^{-1}(\overline{B}).$$

Proof. To prove (12) notice that $\phi(\chi_B)$ is either 0 or 1 by the multiplicative property.

If $\phi(\chi_B) = 1$, then $\phi(f) = \phi(f\chi_B)$, $f \in L_0^\infty(\mathbb{R}^n)$, which means that ϕ factors through a character of $L^\infty(B)$. Thus $\phi \in \widehat{B}$. The argument can be reversed, so (12) is proved.

Assume now that for some $\phi \in \widehat{\mathbb{R}^n}$ we have $\pi(\phi) \in B$. Let f be a continuous function on $\mathbb{R}^n$, with $f(\pi(\phi)) = 1$ and compact support contained in B . Then $f\chi_B = f$ and so

$$1 = \phi(f) = \phi(f)\phi(\chi_B) = \phi(\chi_B).$$

Then $\phi \in \widehat{B}$ because of (12).

If $\pi(\phi)$ is not in $\overline{B}$, then there is a continuous function f on $\mathbb{R}^n$, with $f(\pi(\phi)) = 1$ and compact support in $\mathbb{R}^n \setminus \overline{B}$. Thus $f\chi_B = 0$ and so $\phi(\chi_B) = 0$, that is, ϕ is not in $\widehat{B}$. $\square$

We now turn to the proof of the Theorem. We begin by discussing the converse statement in part (A) of the Theorem. Obviously, given $b \in BMO$, the linear functional $f \mapsto \int f b dm$ is bounded on $\widetilde{F^1}$ with a norm controlled from above by the BMO -norm of b . On the other hand, restriction of the functional to $\widetilde{F^{1,c}}$ gives a control from below by the same BMO -norm.

We first remark that (12) clearly implies that, given $f \in L_0^\infty(\mathbb{R}^n)$, the support of f is contained in B if and only if the support of $\widehat{f}$ is contained in $\widehat{B}$.

Let μ be a Radon measure on $\widehat{\mathbb{R}^n}$ satisfying (4). For each $(1, \infty)$ -atom a supported on a ball B one has

$$\left| \int \widehat{a} d\mu \right| \leq \|a\|_\infty |\mu|(\widehat{B}) \leq \frac{|\mu|(\widehat{B})}{m(B)} < C.$$

Hence μ determines a bounded linear functional on $\widetilde{F^1}$.

Assume now that ℓ is a bounded linear functional on F^1 . Fix a ball B and let $L_0^\infty(B)$ stand for the set of functions in $L^\infty(B)$ with zero integral. Given $f \in L_0^\infty(B)$,

$$\frac{1}{m(B)} \frac{f}{\|f\|_\infty}$$

is a $(1, \infty)$ -atom. Thus

$$(13) \quad |\ell(f)| \leq \|\ell\| \|f\|_\infty m(B), \quad f \in L_0^\infty(B).$$

The restriction of ℓ to $L_0^\infty(B)$ extends to a bounded linear functional on $L^\infty(B) = C(\widehat{B})$. Thus there exists a measure ν_B on $\widehat{B}$ such that

$$(14) \quad \ell(f) = \int \widehat{f} d\nu_B, \quad f \in L_0^\infty(B).$$

If $f \in L^\infty(B)$, then clearly $\widehat{f}_{\widehat{B}} = f_B$, where g_E stands for the mean of the function g on the set E with respect to the underlying measure ($\widehat{m}$ or m in the case at hand). Then

$$\begin{aligned}
 \ell(f - f_B) &= \int_{\widehat{B}} (\widehat{f} - \widehat{f}_{\widehat{B}}) d\nu_B \\
 (15) \qquad &= \int_{\widehat{B}} (\widehat{f} - \widehat{f}_{\widehat{B}}) \left(d\nu_B - \nu_B(\widehat{B}) \frac{d\widehat{m}}{\widehat{m}(\widehat{B})} \right) \\
 &= \int_{\widehat{B}} \widehat{f} \left(d\nu_B - \frac{\nu_B(\widehat{B})}{\widehat{m}(\widehat{B})} d\widehat{m} \right),
 \end{aligned}$$

for each $f \in L^\infty(B)$. Therefore, if ν_B represents ℓ on $L_0^\infty(B)$, that is, if (14) holds, then $d\nu_B - \nu_B(\widehat{B}) \frac{d\widehat{m}}{\widehat{m}(\widehat{B})}$ is uniquely determined.

Let B_N stand for the open ball with center at the origin and radius N , $N = 1, 2, \dots$. Take any measure ν_1 on $\widehat{B}_1$ that represents ℓ on $L_0^\infty(B_1)$. Every other such measure differs from ν_1 by a constant multiple of $\chi_{\widehat{B}_1} \widehat{m}$. By the preceding remark applied to B_N there exists a unique measure ν_N on $\widehat{B}_N$ which represents ℓ on $L_0^\infty(B_N)$ and $\nu_N(B_1) = \nu_1(B_1)$. Clearly ν_N restricted to $\widehat{B}_{N-1}$ is precisely ν_{N-1} . Therefore we can define a measure ν on $\mathbb{R}^n$ by requiring that ν restricted to $\widehat{B}_N$ be ν_N .

Given any ball B take N such that $B \subset B_N$. Since the restriction of ν to $\widehat{B}_N$ represents ℓ on $L_0^\infty(B_N)$, which contains $L_0^\infty(B)$, the restriction of ν to $\widehat{B}$ represents ℓ on $L_0^\infty(B)$ as well. By (15),

$$\left| \int_{\widehat{B}} \widehat{f} \left(d\nu - \nu(\widehat{B}) \frac{d\widehat{m}}{\widehat{m}(\widehat{B})} \right) \right| \leq 2 \|\ell\| \|f\|_\infty m(B), \quad f \in L^\infty(B)$$

or

$$(16) \qquad \left\| \nu - \nu(\widehat{B}) \frac{d\widehat{m}}{\widehat{m}(\widehat{B})} \right\|_{\widehat{B}} \leq 2 \|\ell\| m(B).$$

Let us now consider the Radon-Nikodym decomposition of ν

$$\nu = g \widehat{m} + \mu,$$

where $g \in L_{\text{loc}}^1(\widehat{m})$ and μ is singular with respect to $\widehat{m}$. By (16),

$$(17) \qquad |\mu|(\widehat{B}) \leq 2 \|\ell\| m(B)$$

and

$$(18) \qquad \int_{\widehat{B}} \left| g - g_{\widehat{B}} - \frac{\mu(\widehat{B})}{\widehat{m}(\widehat{B})} \right| d\widehat{m} \leq 2 \|\ell\| m(B).$$

We are left with the task of finding the *BMO*-function b .

Combining (17) and (18) we readily get

$$(19) \qquad \int_{\widehat{B}} |g - g_{\widehat{B}}| d\widehat{m} \leq 4 \|\ell\| m(B).$$

We need a lemma.

Lemma 3. *For each function $g \in L^1_{\text{loc}}(\widehat{m})$ there exists a unique function $f \in L^1_{\text{loc}}(m)$ with the property that for each ball B ,*

$$\int g \widehat{\varphi} d\widehat{m} = \int f \varphi dm, \quad \varphi \in L^\infty(B).$$

Such f satisfies

$$\int_{\widehat{B}} |g - g_{\widehat{B}}| d\widehat{m} = \int_B |f - f_B| dm,$$

for each ball B .

Once the lemma is proved we complete the proof of part (A) of the Theorem by just calling b the function f associated with g in Lemma 3. Inequality (19) tells us that $b \in BMO(\mathbb{R}^n)$ and that its $BMO(\mathbb{R}^n)$ norm is not greater than $4\|\ell\|$.

Proof of Lemma 3. We will show that for each ball B the Gelfand transform, which is an isometry between $L^\infty(B)$ and $C(\widehat{B})$, extends to an isometry between $L^1(B, m)$ and $L^1(\widehat{B}, \widehat{m})$. This immediately provides a further extension of the Gelfand transform to a topological isomorphism between $L^1_{\text{loc}}(m)$ and $L^1_{\text{loc}}(\widehat{m})$.

We begin by showing that, for each ball B in $\mathbb{R}^n$ and every $f \geq 0$ in $L^\infty(B)$,

$$(20) \quad \int_{\widehat{B}} \widehat{f} d\widehat{m} = \int_B f dm.$$

This follows from

$$\begin{aligned} \int_B f dm &= \sup_{\varphi} \int_B f \varphi dm \\ &= \sup_{\varphi} \int_{\widehat{B}} \widehat{f} \widehat{\varphi} d\widehat{m} \\ &= \int_{\widehat{B}} \widehat{f} d\widehat{m}, \end{aligned}$$

where the supremum is taken on the closed unit ball of $L^\infty(B)$.

By linearity, (20) provides an extension of the Gelfand transform to a topological isomorphism $f \rightarrow \widehat{f}$ of $L^1_{\text{loc}}(m)$ onto $L^1_{\text{loc}}(\widehat{m})$. Given $g \in L^1_{\text{loc}}(\widehat{m})$ take $f \in L^1_{\text{loc}}(m)$ with $g = \widehat{f}$. The first identity in the statement of Lemma 3 follows by approximating $f \in L^1(B, m)$ by functions in $L^\infty(B)$ and the second follows from (20). $\square$

Before proving part (B) of the Theorem we give an explicit example, modeled on the Meyer-Taibleson-Weiss argument, of a non-zero measure which is singular with respect to $\widehat{m}$ and satisfies (4).

Take an open set U of $\mathbb{R}^n$, $U \subset B_0 = \{x : |x| \leq 1\}$, such that U is dense in B_0 and $m(U) < m(B_0)$. Then the compact set $E = B_0 \setminus U$ has positive Lebesgue measure. Set $V = \pi^{-1}(U)$, so that $V \subset \widehat{B_0}$ by Lemma 1. Then $U \subset \pi(\overline{V})$ and so $\pi(\overline{V}) = B_0$, because U is dense in B_0 . Hence $\pi(\partial V) = E$. Now, the boundary of each open set in $\widehat{B_0}$ has zero $\widehat{m}$ measure ([R, p. 286]). Therefore $\widehat{m}(\partial V) = 0$ but $m(\pi(\partial V)) = m(E) > 0$. Identify $C(E)$ to the subspace S of continuous functions on ∂V of the form $f \circ \pi$, $f \in C(E)$. The bounded linear functional on S defined

by $f \rightarrow \int f dm$ extends by Hahn-Banach to a bounded linear functional on $C(\partial V)$ with the same norm. Thus there exists a positive measure μ on ∂V such that

$$\int (f \circ \pi) d\mu = \int f dm, \quad f \in C(E).$$

If B is an open ball, then by Lemma 2,

$$\mu(\widehat{B}) \leq \mu(\pi^{-1}(\overline{B})) = m(\overline{B} \cap E) \leq m(B),$$

and condition (4) is satisfied.

Proof of part (B) of the Theorem. The argument is analogous to the proof of part (A), except for minor technical details. If $0 < p < 1$, then, as we will see, the singular measure μ vanishes and so we will conclude that $(F^p)^* = H^p(\mathbb{R}^n)^*$.

Let ℓ be a bounded linear functional on F^p , $0 < p < 1$. Let d be the integer part of $n(\frac{1}{p} - 1)$. Given a ball B let $L_d^\infty(B)$ stand for the set of functions $f \in L^\infty(B)$ such that

$$\int f(x) x^\alpha dx = 0, \quad |\alpha| \leq d.$$

For each $f \in L_d^\infty(B)$,

$$\frac{1}{m(B)^{\frac{1}{p}}} \frac{f}{\|f\|_\infty}$$

is a (p, ∞) -atom and so

$$(21) \quad |\ell(f)| \leq \|\ell\| \|f\|_\infty m(B)^{\frac{1}{p}}, \quad f \in L_d^\infty(B).$$

For each $f \in L^\infty(B)$ let $P_B(f)$ be (the restriction to B of) the unique polynomial of degree not greater than d such that

$$\int f(x) x^\alpha dx = \int_B P_B(f)(x) x^\alpha dx, \quad |\alpha| \leq d.$$

Since $P_B(f)$ is the orthogonal projection (in $L^2(B)$) of f into the subspace of polynomials of degree not greater than d ,

$$\|P_B(f)\|_2 \leq \|f\|_2 \leq \|f\|_\infty,$$

where the L^2 norms are taken with respect to the normalized Lebesgue measure on B . Now, we want to compare the norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ on the space $P_d(B)$ of restrictions to B of polynomials of degree not greater than d . After appropriate translation and dilation we may assume that B has center 0 and radius 1. Since $P_d(B)$ is finite dimensional, there is a constant $C(d, n)$, depending only on d and n , such that

$$\|P\|_\infty \leq C(d, n) \|P\|_2, \quad P \in P_d(B),$$

and so

$$\|P_B(f)\|_\infty \leq C(d, n) \|f\|_\infty, \quad f \in L^\infty(B).$$

Therefore by (21),

$$(22) \quad |\ell(f - P_B(f))| \leq (1 + C(d, n)) \|\ell\| \|f\|_\infty m(B)^{\frac{1}{p}}, \quad f \in L^\infty(B).$$

By (21) there is a measure ν_B on $\widehat{B}$ such that

$$(23) \quad \ell(f) = \int \widehat{f} d\nu_B, \quad f \in L_d^\infty(B).$$

Given a measure λ on $\widehat{B}$ there is a unique polynomial $P_B(\lambda) \in P_d(B)$ such that

$$\int_{\widehat{B}} (\pi(\phi))^\alpha d\lambda(\phi) = \int_B P_B(\lambda)(x) x^\alpha dx, \quad |\alpha| \leq d.$$

Hence, for every polynomial Q of degree $\leq d$,

$$\int_{\widehat{B}} \widehat{Q} d\lambda = \int_B P_B(\lambda) Q dm = \int_{\widehat{B}} \widehat{P_B(\lambda)} \widehat{Q} d\widehat{m}.$$

Therefore, by (23),

$$\begin{aligned} \ell(f - P_B(f)) &= \int_{\widehat{B}} (\widehat{f} - \widehat{P_B(f)}) d\nu_B \\ (24) \qquad &= \int_{\widehat{B}} (\widehat{f} - \widehat{P_B(f)}) (d\nu_B - \widehat{P_B(\nu_B)} d\widehat{m}) \\ &= \int_{\widehat{B}} \widehat{f} (d\nu_B - \widehat{P_B(\nu_B)} d\widehat{m}), \end{aligned}$$

for each $f \in L^\infty(B)$. Hence the measure $\nu_B - \widehat{P_B(\nu_B)} \widehat{m}$ is determined by ℓ .

As before, with B_N denoting the ball of radius N centered at the origin, we fix a measure ν_1 on $\widehat{B}_1$ that represents ℓ on $L_d^\infty(B_1)$ and then take the unique measure ν_N on $\widehat{B}_N$ which represents ℓ on $L_d^\infty(B_N)$ and such that $P_{B_1}(\nu_N) = P_{B_1}(\nu_1)$. Then ν_N restricted to B_{N-1} is ν_{N-1} , so we can define a measure ν on $\widehat{\mathbb{R}^n}$ by requiring that ν restricted to B_N be ν_N .

Given any ball B , take N such that $B \subset B_N$. Then the restriction of ν to $L_d^\infty(B)$ is ℓ , so, by (22) and (24),

$$\left| \int_{\widehat{B}} \widehat{f} (d\nu - \widehat{P_B(\nu)} d\widehat{m}) \right| \leq C \|f\|_\infty m(B)^{\frac{1}{p}}, \quad f \in L^\infty(B).$$

Hence

$$(25) \qquad |\nu - \widehat{P_B(\nu)} \widehat{m}|(\widehat{B}) \leq C m(B)^{\frac{1}{p}}.$$

Now consider the Radon-Nikodym decomposition of ν ,

$$\nu = g \widehat{m} + \mu,$$

with μ singular with respect to $\widehat{m}$. We get, by (25) and Lemma 2,

$$|\mu|(\pi^{-1}(B)) \leq |\mu|(\widehat{B}) \leq C m(B)^{\frac{1}{p}},$$

for each open ball B . Since $0 < p < 1$, we readily conclude that $\mu = 0$. Indeed, let r be the radius of B . Covering B by $A_n k^n$ balls of radius r/k , we see that the constant C in the right-hand side of the above inequality can be replaced by $C A_n k^{n(1-\frac{1}{p})}$. Letting k tend to ∞ , we obtain the conclusion.

Now take $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ with $g = \widehat{f}$. Then

$$\int_B |f - P_B(f)| dm \leq C m(B)^{\frac{1}{p}},$$

which is precisely the condition that guarantees that f determines a bounded linear functional on $H^p(\mathbb{R}^n)$ ([TW]). Thus ℓ is a bounded linear functional on $H^p(\mathbb{R}^n)$ and the proof is complete. $\square$

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