THE L^p REGULARITY PROBLEM ON LIPSCHITZ DOMAINS

JOEL KILTY AND ZHONGWEI SHEN

ABSTRACT. This paper contains two results on the L^p regularity problem on Lipschitz domains. For second order elliptic systems and $1 < p < \infty$, we prove that the solvability of the L^p regularity problem is equivalent to that of the $L^{p'}$ Dirichlet problem. For higher order elliptic equations and systems, we show that if $p > 2$, the solvability of the L^p regularity problem is equivalent to a weak reverse Hölder condition with exponent p .

1. INTRODUCTION

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ with connected boundary. Consider the elliptic system of order 2ℓ , $\mathcal{L}(D)u = 0$ in Ω , where $u = (u^1, \dots, u^m)$,

$$(1.1) \quad (\mathcal{L}(D)u)^j = \sum_{k=1}^m \mathcal{L}^{jk}(D)u^k, \quad j = 1, \dots, m,$$

$$(1.2) \quad \mathcal{L}^{jk}(D) = \sum_{|\alpha|=|\beta|=\ell} a_{\alpha\beta}^{jk} D^\alpha D^\beta,$$

and $D = (D_1, D_2, \dots, D_d)$, $D_i = \partial/\partial x_i$ for $i = 1, 2, \dots, d$. Also, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ is a multi-index with length $|\alpha| = \alpha_1 + \dots + \alpha_d$ and $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_d^{\alpha_d}$. Let

$$(1.3) \quad \mathcal{L}^{jk}(\xi) = \sum_{|\alpha|=|\beta|=\ell} a_{\alpha\beta}^{jk} \xi^\alpha \xi^\beta \quad \text{for } \xi \in \mathbb{R}^d.$$

We will assume throughout this paper that the $a_{\alpha\beta}^{jk}$ are real constants satisfying the Legendre-Hadamard ellipticity condition

$$(1.4) \quad \mu |\xi|^{2\ell} |\eta|^2 \leq \sum_{j,k=1}^m \mathcal{L}^{jk}(\xi) \eta^j \eta^k \leq \frac{1}{\mu} |\xi|^{2\ell} |\eta|^2,$$

for some $\mu > 0$, and all $\xi \in \mathbb{R}^d$, $\eta \in \mathbb{R}^m$, as well as the symmetry condition

$$(1.5) \quad \mathcal{L}^{jk}(\xi) = \mathcal{L}^{kj}(\xi).$$

The L^p Dirichlet problem for the elliptic system $\mathcal{L}(D)u = 0$ in Ω consists of finding a solution u such that $(\nabla^{\ell-1}u)^* \in L^p(\partial\Omega)$ and $u, \nabla u, \dots, \nabla^{\ell-1}u$ take the prescribed data on $\partial\Omega$ in the sense of nontangential convergence. Here and thereafter $\nabla^k u$ denotes the tensor of all partial derivatives of order k and $(w)^*$ the

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nontangential maximal function of w . More precisely, let $WA^{k,p}(\partial\Omega, \mathbb{R}^m)$ denote the completion of the set of arrays of functions

$$(1.6) \quad \{\dot{f} = (f_\alpha)_{|\alpha| \leq k} = (D^\alpha f)_{|\alpha| \leq k} : f \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^m)\},$$

under the scale-invariant norm on $\partial\Omega$,

$$(1.7) \quad \|\dot{f}\|_{k,p} := \sum_{|\alpha| \leq k} |\partial\Omega|^{\frac{k-|\alpha|}{1-d}} \|D^\alpha f\|_p,$$

where $\|\cdot\|_p$ denotes the norm in $L^p(\partial\Omega)$. The L^p Dirichlet problem is said to be uniquely solvable if given any $\dot{f} \in WA^{\ell-1,p}(\partial\Omega, \mathbb{R}^m)$, there exists a unique function u such that

$$(1.8) \quad \begin{cases} \mathcal{L}(D)u = 0 & \text{in } \Omega, \\ D^\alpha u = f_\alpha & \text{on } \partial\Omega \text{ for } |\alpha| \leq \ell-1, \\ (\nabla^{\ell-1}u)^* \in L^p(\partial\Omega). \end{cases}$$

Moreover, the solution u satisfies the estimate

$$(1.9) \quad \|(\nabla^{\ell-1}u)^*\|_p \leq C \sum_{|\alpha|=\ell-1} \|f_\alpha\|_p.$$

If the Dirichlet data in (1.8) are taken from $WA^{\ell,p}(\partial\Omega, \mathbb{R}^m)$ instead of from $WA^{\ell-1,p}(\partial\Omega, \mathbb{R}^m)$, then they all have tangential derivatives in $L^p(\partial\Omega)$. Consequently we may expect the solution to have one order higher regularity. This is the so-called L^p regularity problem. Let $\nabla_t g$ denote the tangential derivatives of g on $\partial\Omega$. We say that the L^p regularity problem for $\mathcal{L}(D)u = 0$ in Ω is uniquely solvable if given $\dot{f} = \{f_\alpha : |\alpha| \leq \ell\} \in WA^{\ell,p}(\partial\Omega, \mathbb{R}^m)$, there exists a unique function u such that

$$(1.10) \quad \begin{cases} \mathcal{L}(D)u = 0 & \text{in } \Omega, \\ D^\alpha u = f_\alpha & \text{on } \partial\Omega \text{ for } |\alpha| \leq \ell-1, \\ (\nabla^\ell u)^* \in L^p(\partial\Omega). \end{cases}$$

Moreover, the solution u satisfies the estimate

$$(1.11) \quad \|(\nabla^\ell u)^*\|_p \leq C \sum_{|\alpha|=\ell-1} \|\nabla_t f_\alpha\|_p.$$

For p close to 2 and $d \geq 2$, the solvability of the L^p Dirichlet and regularity problems was established in [7, 8, 9, 10] for second order elliptic systems and in [6, 15, 24, 25] for higher order elliptic equations and systems. In the lower dimensional case $d = 2$ or 3, the L^p Dirichlet problem was solved for $2 - \varepsilon < p \leq \infty$ and the L^p regularity problem for $1 < p < 2 + \varepsilon$ (both ranges are sharp) in [4, 13, 14, 25]. In the higher dimensional case $d \geq 4$, the L^p Dirichlet problem for $2 < p < \frac{2(d-1)}{d-3} + \varepsilon$ was recently solved by Shen in [17, 18] for higher order elliptic equations and systems. The paper [17] also established the solvability of the L^p regularity problem for the second order elliptic systems in the case $d \geq 4$ and $\frac{2(d-1)}{d+1} - \varepsilon < p < 2$. Related results may be found in [12, 16, 26] for the Stokes system and in [19] for the biharmonic equation. We remark that the results mentioned above extend the classical work of Dahlberg, Jerison, Kenig, and Verchota in [1, 2, 3, 11, 23] on L^p boundary value problems for Laplace's equation in Lipschitz domains.

In this paper we establish two related results on the L^p regularity problem. First, for general higher order elliptic equations and systems in Ω , we show that if $p > 2$, the solvability of the L^p regularity problem is equivalent to a weak reverse Hölder

condition with exponent p on $\partial\Omega$. Let $\Delta(P, r) = B(P, r) \cap \partial\Omega$, where $P \in \partial\Omega$. The result may be formulated as follows.

Theorem 1.1. *Let $\mathcal{L}(D)$ be a system of elliptic operators of order 2ℓ satisfying conditions (1.4) and (1.5). For any bounded Lipschitz domain Ω and $p > 2$, the following are equivalent.*

- (1) *The L^p regularity problem for $\mathcal{L}(D)u = 0$ in Ω is uniquely solvable.*
- (2) *There exist $C > 0$ and $r_0 > 0$ such that for any $P \in \partial\Omega$ and $0 < r < r_0$, the weak reverse Hölder condition,*

$$(1.12) \quad \left(\frac{1}{r^{d-1}} \int_{\Delta(P, r)} |(\nabla^\ell v)^*|^p d\sigma \right)^{\frac{1}{p}} \leq C \left(\frac{1}{r^{d-1}} \int_{\Delta(P, 2r)} |(\nabla^\ell v)^*|^2 d\sigma \right)^{\frac{1}{2}},$$

holds for any solution v of $\mathcal{L}(D)v = 0$ in Ω with the properties $(\nabla^\ell v)^ \in L^2(\partial\Omega)$ and $D^\alpha v = 0$ on $\partial\Omega$ for $|\alpha| \leq \ell - 1$ on $\Delta(P, 3r)$.*

Theorem 1.1 extends the work of Shen in [18] where a similar result was established for the L^p Dirichlet problem. When combined with the main result in [18], it yields the following.

Theorem 1.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 4$. Suppose that the L^p regularity problem for $\mathcal{L}(D)u = 0$ in Ω is uniquely solvable for some $2 < p < d - 1$. Then the L^q Dirichlet problem for $\mathcal{L}(D)u = 0$ in Ω is uniquely solvable for $2 < q < q_0 + \varepsilon$, where $\frac{1}{q_0} = \frac{1}{p} - \frac{1}{d-1}$.*

In the second part of this paper we consider the case of second order elliptic systems, i.e., $\ell = 1$. In this special case we show that for any given Lipschitz domain, the L^p regularity problem and the $L^{p'}$ Dirichlet problem are in fact equivalent.

Theorem 1.3. *Let $1 < p < \infty$ and Ω be a bounded Lipschitz domain. Then, for any second order elliptic system satisfying conditions (1.4)-(1.5), the L^p regularity problem in Ω is uniquely solvable if and only if the $L^{p'}$ Dirichlet problem in Ω is uniquely solvable.*

We point out that although it is not implicitly stated, the duality between the regularity and Dirichlet problems was essentially established in the case of star-shaped Lipschitz domains for Laplace's equation in [23]. Some partial results on this duality relation may also be found in [21] for second order elliptic equations with bounded measurable coefficients. Our approach to the elliptic systems uses the basic duality argument in [23]. The main contribution here is a localization argument which allows us to treat the case of general Lipschitz domains in the absence of positivity.

It would be very interesting to see if the duality between the regularity and Dirichlet problems in Theorem 1.3 extends to higher order elliptic equations and systems. As a first step in this direction, some partial results have been obtained by the authors for the biharmonic equation $\Delta^2 u = 0$.

Note that by Theorems 1.1 and 1.3, for second order elliptic systems, as in the case $p > 2$, the solvability of the L^p Dirichlet problem for $1 < p < 2$ is also equivalent to a weak reverse Hölder condition. In particular, by the well-known self-improving property of weak reverse Hölder conditions, it follows that if the Dirichlet problem in Ω is solvable for some $1 < p < 2$, then it is also solvable in Ω for some $1 < \bar{p} < p$. This, together with Theorem 1.2 as well as results in [4], gives the following.

Theorem 1.4. *Consider the second order elliptic system $\mathcal{L}(D)u = 0$ in a bounded Lipschitz domain Ω in $\mathbb{R}^d$. Let $\mathcal{A}$ denote the set of exponents $p \in (1, \infty)$ for which the Dirichlet problem in Ω is uniquely solvable. Then*

- (1) *If $d = 2$ or 3 , $\mathcal{A} = (q, \infty)$, where $1 \leq q < 2$.*
- (2) *If $d \geq 4$, $\mathcal{A} = (q, s)$, where $1 \leq q < 2 < s \leq \infty$. Furthermore, $s = \infty$ if $q \leq \frac{d-1}{d-2}$, and $s \geq \frac{q(d-1)}{qd-2q-d+1}$ if $q > \frac{d-1}{d-2}$.*

The paper is organized as follows. In Section 2 we show that the condition $(\nabla^{2\ell-1}u)^* \in L^p$ for some $p > 1$ implies that $\nabla^{2\ell-1}u$ has nontangential limits on $\partial\Omega$ and $\|(\nabla^{2\ell-1}u)^*\|_p \leq C \|\nabla^{2\ell-1}u\|_p$. The proof of Theorem 1.1 is given in Sections 3 and 4, while Theorem 1.2 is proved in Section 5. Sections 6 and 7 are devoted to the proof of Theorem 1.3.

Finally we remark that the summation convention will be used throughout this paper. Also, Ω will always be a bounded Lipschitz domain with connected boundary. We will use $\Gamma(x) = (\Gamma_{jk}(x))_{m \times m}$ to denote a matrix of fundamental solutions on $\mathbb{R}^d$ to the operator $\mathcal{L}(D)$ with pole at the origin.

2. A PRELIMINARY ESTIMATE

Theorem 2.1. *Suppose that $\mathcal{L}(D)u = 0$ in Ω and $(\nabla^{2\ell-1}u)^* \in L^p(\partial\Omega)$ for some $p > 1$. Then $\nabla^{2\ell-1}u$ has nontangential limits a.e. on $\partial\Omega$. Furthermore, $\nabla^{2\ell-1}u \in L^p(\partial\Omega)$ and*

$$(2.1) \quad \|(\nabla^{2\ell-1}u)^*\|_p \leq C \|\nabla^{2\ell-1}u\|_p,$$

where C depends only on μ, d, m, ℓ, p and the Lipschitz character of Ω .

Proof. Let $\{\Omega_r\}$ be a sequence of smooth domains such that $\Omega_r \uparrow \Omega$. Fix $x \in \Omega$. Since $\mathcal{L}(D)u = 0$ in Ω and $\mathcal{L}(D)\Gamma = \delta_0$, we may write

$$(2.2) \quad \begin{aligned} D^{\gamma+k}u^s(x) &= \int_{\Omega_r} a_{\alpha\beta}^{ij} D^{\alpha+\beta} \Gamma_{js}^x(y) \cdot D^{\gamma+k}u^i dy \\ &\quad + \int_{\Omega_r} a_{\alpha\beta}^{ij} D^{\gamma+k} \Gamma_{js}^x(y) \cdot D^{\alpha+\beta}u^i dy, \end{aligned}$$

where $\Gamma^x(y) = \Gamma(x-y)$, and γ, k are two multi-indices with $|\gamma| = \ell$ and $|k| = \ell-1$.

Next, we derive Green's representation formula by integrating by parts to switch the derivatives on Γ^x and u . This produces only boundary terms as the solid integrals cancel out. Note that we should move derivatives in such a way that no more than 2ℓ derivatives are taken on either Γ^x or u . By doing so we obtain

$$(2.3) \quad D^{\gamma+k}u^s(x) = \sum_{|\alpha|=2\ell-1} \int_{\partial\Omega_r} D^\alpha \Gamma_{js}^x \cdot \Pi_{ij}^\alpha(u^i) d\sigma,$$

where $\Pi_{ij}^\alpha(u^i)$ is a sum of derivatives of u^i of order $|\alpha|$ times various components of the unit normal to $\partial\Omega_r$.

Let $\Lambda_r : \partial\Omega \rightarrow \partial\Omega_r$ denote the homeomorphism given by Theorem 1.12 in [23]. We now rewrite (2.3) as an integral on $\partial\Omega$ to obtain

$$(2.4) \quad D^{\gamma+k}u^s(x) = \sum_{|\alpha|=2\ell-1} \int_{\partial\Omega} D^\alpha \Gamma_{js}^x(\Lambda_r(P)) \cdot \Pi_{ij}^\alpha(u^i)(\Lambda_r(P)) \omega_r d\sigma.$$

Since

$$|\Pi_{ij}^\alpha(u^i)(\Lambda_r(P))| \leq C (\nabla^{2\ell-1}u)^*(P),$$

it follows that

$$\|\Pi_{ij}^\alpha(u^i)(\Lambda_r)\|_p \leq C \|(\nabla^{2\ell-1}u)^*\|_p < \infty.$$

Consequently, there exists a subsequence, which we still denote by $\Pi_{ij}^\alpha(u^i)(\Lambda_r)$, that converges weakly in $L^p(\partial\Omega)$ to some function $g_j^\alpha \in L^p(\partial\Omega)$ as $r \rightarrow \infty$. Since $D^\alpha \Gamma_{js}^x(\Lambda_r(P)) \rightarrow D^\alpha \Gamma_{js}^x(P)$ uniformly on $\partial\Omega$, we obtain

$$D^{\gamma+k}u^s(x) = \sum_{|\alpha|=2\ell-1} \int_{\partial\Omega} D^\alpha \Gamma_{js}^x(P) \cdot g_j^\alpha d\sigma.$$

This implies that $D^{\gamma+k}u$ has nontangential limits a.e. on $\partial\Omega$. As a result, we have $|g_j^\alpha(P)| \leq C |\nabla^{2\ell-1}u(P)|$ for a.e. $P \in \partial\Omega$. It follows that

$$\|(D^{\gamma+k}u^s)^*\|_p \leq C \sum_{|\alpha|=2\ell-1} \sum_j \|g_j^\alpha\|_p \leq C \|\nabla^{2\ell-1}u\|_p.$$

This finishes the proof. $\square$

3. SUFFICIENCY OF THE WEAK REVERSE HÖLDER CONDITION

The goal of this section is to show that given any Lipschitz domain Ω and any $p > 2$, the weak reverse Hölder condition (1.12) is sufficient for the solvability of the L^p regularity problem on Ω .

Theorem 3.1. *Let $\mathcal{L}(D)$ be an elliptic operator of order 2ℓ satisfying the ellipticity condition (1.4) and the symmetry condition (1.5). Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and fix $p > 2$. Suppose that for any $\Delta(P, r) \subset \partial\Omega$ with $P \in \partial\Omega$ and $0 < r < r_0$, the reverse Hölder condition (1.12) holds for all solutions of $\mathcal{L}(D)u = 0$ in Ω with the properties $(\nabla^\ell u)^* \in L^2(\partial\Omega)$ and $D^\alpha u = 0$ on $\Delta(P, 3r)$ for $|\alpha| \leq \ell - 1$. Then the L^p regularity problem in Ω is uniquely solvable.*

The proof of the following Poincaré-type inequality may be found in [18].

Lemma 3.2. *Suppose $\ell \geq 1$. Let $\dot{f} = \{f_\alpha : |\alpha| \leq \ell\} \in \text{WA}^{2,\ell}(\partial\Omega)$ and $\Delta(P, r) \subset \partial\Omega$. Then, there exists a polynomial h of degree at most $\ell - 1$ such that*

$$(3.1) \quad \|f_\beta - D^\beta h\|_{L^2(\Delta(P, r))} \leq Cr^{\ell-|\beta|} \sum_{|\alpha|=\ell-1} \|\nabla_t f_\alpha\|_{L^2(\Delta(P, r))},$$

for any multi-index β with $|\beta| \leq \ell - 1$.

The proof of Theorem 3.1, which is similar to that of Theorem 3.1 in [18], relies on a real variable argument. The proof of the following theorem may be found in [20].

Theorem 3.3. *Let $S = \{(x', \psi(x')) : x' \in \mathbb{R}^{d-1}\}$ be a Lipschitz graph in $\mathbb{R}^d$. Let Q_0 be a surface cube in S and $F \in L^2(2Q_0)$. Let $p > 2$ and $g \in L^q(2Q_0)$ for some $2 < q < p$. Suppose that for each dyadic subcube Q of Q_0 with $|Q| \leq \beta|Q_0|$, there exist two integrable functions F_Q and R_Q on $2Q$ such that $|F| \leq |F_Q| + |R_Q|$ on*

$2Q$, and

$$(3.2) \quad \left(\frac{1}{|2Q|} \int_{2Q} |R_Q|^p d\sigma \right)^{1/p} \leq C_1 \left\{ \left(\frac{1}{|\alpha Q|} \int_{\alpha Q} |F|^2 d\sigma \right)^{1/2} + \sup_{Q' \supset Q} \left(\frac{1}{|Q'|} \int_{Q'} |g|^2 d\sigma \right)^{1/2} \right\},$$

$$(3.3) \quad \left(\frac{1}{|2Q|} \int_{2Q} |F_Q|^2 d\sigma \right)^{1/2} \leq C_2 \sup_{Q' \supset Q} \left(\frac{1}{|Q'|} \int_{Q'} |g|^2 d\sigma \right)^{1/2},$$

where $C_1, C_2 > 0$ and $0 < \beta < 1 < \alpha$. Then,

$$(3.4) \quad \left(\frac{1}{|Q_0|} \int_{Q_0} |F|^q d\sigma \right)^{1/q} \leq C_3 \left\{ \left(\frac{1}{|2Q_0|} \int_{2Q_0} |F|^2 d\sigma \right)^{1/2} + \left(\frac{1}{|2Q_0|} \int_{2Q_0} |g|^q d\sigma \right)^{1/q} \right\},$$

where C_3 depends only on $d, p, q, C_1, C_2, \alpha, \beta$, and $\|\nabla\psi\|_\infty$.

We now proceed to the proof of Theorem 3.1.

Proof of Theorem 3.1. The uniqueness for $p > 2$ follows from the uniqueness for $p = 2$. To establish the existence, we let $\dot{f} = \{f_\alpha : |\alpha| \leq \ell\} \in WA^{\ell,p}(\partial\Omega)$ and u be the solution to the L^2 regularity problem with boundary data $\{f_\alpha : |\alpha| \leq \ell - 1\}$. We will show that if $P \in \partial\Omega$ and $0 < s \leq cr_0$, then

$$(3.5) \quad \left(\frac{1}{s^{d-1}} \int_{\Delta(P,s)} |(\nabla^\ell u)^*|^p d\sigma \right)^{\frac{1}{p}} \leq C \left(\frac{1}{s^{d-1}} \int_{\Delta(P,Cs)} |(\nabla^\ell u)^*|^2 d\sigma \right)^{\frac{1}{2}} + C \left(\frac{1}{s^{d-1}} \int_{\Delta(P,Cs)} \sum_{|\alpha|=\ell-1} |\nabla_t f_\alpha|^p d\sigma \right)^{\frac{1}{p}}.$$

By covering $\partial\Omega$ with a finite number of balls of radius cr_0 , estimate (3.5) implies that

$$\begin{aligned} \|(\nabla^\ell u)^*\|_p &\leq C |\partial\Omega|^{\frac{1}{p}-\frac{1}{2}} \|(\nabla^\ell u)^*\|_2 + C \sum_{|\alpha|=\ell-1} \|\nabla_t f_\alpha\|_p \\ &\leq C |\partial\Omega|^{\frac{1}{p}-\frac{1}{2}} \sum_{|\alpha|=\ell-1} \|\nabla_t f_\alpha\|_2 + C \sum_{|\alpha|=\ell-1} \|\nabla_t f_\alpha\|_p \\ &\leq C \sum_{|\alpha|=\ell-1} \|\nabla_t f_\alpha\|_p. \end{aligned}$$

Here we have used the L^2 regularity estimate as well as Hölder's inequality. We now seek to establish estimate (3.5).

Fix $P \in \partial\Omega$ and $0 < s < cr_0$. By rotation and translation we may assume that $P = 0$ and

$$(3.6) \quad B(P, r_0) \cap \Omega = B(P, r_0) \cap \{(x', x_d) \in \mathbb{R}^d : x_d > \psi(x')\},$$

$$(3.7) \quad B(P, r_0) \cap \partial\Omega = B(P, r_0) \cap \{(x', \psi(x')) \in \mathbb{R}^d : x' \in \mathbb{R}^{d-1}\},$$

where $\psi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ is a Lipschitz function. Consider the surface cube

$$Q_0 = \{(x', \psi(x')) \in \mathbb{R}^{d-1} \times \mathbb{R} : |x_1| < s, \dots, |x_{d-1}| < s\}.$$

Let Q be a small subcube of Q_0 with diameter r . Choose $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $0 \leq \varphi \leq 1$, $\varphi = 1$ on $8Q$, $\varphi = 0$ on $\mathbb{R}^d \setminus 16Q$, and $|D^\alpha \varphi| \leq \frac{C}{r^{|\alpha|}}$ for $|\alpha| \leq 2\ell$. Let h be the polynomial of degree at most $\ell - 1$, given by Lemma 3.2, but with $\Delta(P, r)$ replaced with $16Q$. Write $u = v + w + h$, where v is the solution to the L^2 regularity problem with boundary data $(u - h)\varphi$ and w is the solution to the L^2 regularity problem with boundary data $(1 - \varphi)(u - h)$. Note that for $|\alpha| \leq \ell - 1$,

$$(3.8) \quad D^\alpha v = D^\alpha((u - h)\varphi) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} (f_\beta - D^\beta h) D^{\alpha - \beta} \varphi.$$

Now, let

$$\begin{aligned} F &= |(\nabla^\ell u)^*|, \\ g &= \sum_{|\alpha|=\ell-1} |\nabla_t f_\alpha|, \\ F_Q &= 2|(\nabla^\ell v)^*|, \\ R_Q &= 2|(\nabla^\ell w)^*|. \end{aligned}$$

Using the fact that v is the solution of the L^2 regularity problem, we obtain

$$\begin{aligned} \frac{1}{|2Q|} \int_{2Q} |F_Q|^2 d\sigma &\leq \frac{C}{|2Q|} \int_{\partial\Omega} |(\nabla^\ell v)^*|^2 d\sigma \\ (3.9) \quad &\leq \frac{C}{|2Q|} \sum_{|\alpha|=\ell-1} \int_{\partial\Omega} |\nabla_t D^\alpha v|^2 d\sigma. \end{aligned}$$

Now, note that

$$\begin{aligned} &\sum_{|\alpha|=\ell-1} \int_{\partial\Omega} |\nabla_t D^\alpha v|^2 d\sigma \\ &\leq C \sum_{|\beta| \leq \ell-1} \int_{16Q} |\nabla_t (f_\beta - D^\beta h)|^2 \frac{d\sigma}{r^{2(\ell-1-|\beta|)}} \\ &\quad + C \sum_{|\beta| \leq \ell-1} \int_{16Q} |f_\beta - D^\beta h|^2 \frac{d\sigma}{r^{2(\ell-|\beta|)}} \\ &\leq C \sum_{|\alpha|=\ell-1} \int_{16Q} |\nabla_t f_\alpha|^2 d\sigma \\ &\quad + C \sum_i \sum_{|\beta| \leq \ell-2} \int_{16Q} |D_{x_i} (f_\beta - D^\beta h)|^2 \frac{d\sigma}{r^{2(\ell-1-|\beta|)}} \\ &\quad + C \sum_{|\beta| \leq \ell-1} \frac{1}{r^{2(\ell-|\beta|)}} \int_{16Q} |f_\beta - D^\beta h|^2 d\sigma \\ &\leq C \sum_{|\alpha|=\ell-1} \int_{16Q} |\nabla_t f_\alpha|^2 d\sigma + C \sum_{|\beta| \leq \ell-1} \frac{1}{r^{2(\ell-|\beta|)}} \int_{16Q} |f_\beta - D^\beta h|^2 d\sigma \\ (3.10) \quad &\leq C \sum_{|\alpha|=\ell-1} \int_{16Q} |\nabla_t f_\alpha|^2 d\sigma, \end{aligned}$$

where we have used Lemma 3.2 in the last step. By combining estimates (3.9) and (3.10) we obtain

$$(3.11) \quad \frac{1}{|2Q|} \int_{2Q} |F_Q|^2 d\sigma \leq \frac{C}{|16Q|} \sum_{|\alpha|=\ell-1} \int_{16Q} |\nabla_t f_\alpha|^2 d\sigma \leq \frac{C}{|16Q|} \int_{16Q} |g|^2 d\sigma.$$

This implies that

$$(3.12) \quad \left(\frac{1}{|2Q|} \int_{2Q} |F_Q|^2 d\sigma \right)^{\frac{1}{2}} \leq C \sup_{Q' \supset Q} \left(\frac{1}{|Q'|} \int_{Q'} |g|^2 d\sigma \right)^{\frac{1}{2}}.$$

Note that w is a solution of the L^2 regularity problem with $(\nabla^\ell w)^* \in L^2(\partial\Omega)$ and $D^\alpha w = 0$ on $16Q$ for $|\alpha| \leq \ell - 1$. Thus, we may use the weak reverse Hölder inequality (1.12) and the above estimates on v to obtain

$$\begin{aligned} & \left(\frac{1}{|2Q|} \int_{2Q} |R_Q|^p d\sigma \right)^{\frac{1}{p}} = 2 \left(\frac{1}{|2Q|} \int_{2Q} |(\nabla^\ell w)^*|^p d\sigma \right)^{\frac{1}{p}} \\ & \leq C \left(\frac{1}{|4Q|} \int_{4Q} |(\nabla^\ell w)^*|^2 d\sigma \right)^{\frac{1}{2}} \\ & \leq C \left(\frac{1}{|4Q|} \int_{4Q} |(\nabla^\ell u)^*|^2 d\sigma \right)^{\frac{1}{2}} + C \left(\frac{1}{|4Q|} \int_{4Q} |(\nabla^\ell v)^*|^2 d\sigma \right)^{\frac{1}{2}} \\ (3.13) \quad & \leq C \left(\frac{1}{|4Q|} \int_{4Q} |F|^2 d\sigma \right)^{\frac{1}{2}} + C \sup_{Q' \supset Q} \left(\frac{1}{|Q'|} \int_{Q'} |g|^2 d\sigma \right)^{\frac{1}{2}}. \end{aligned}$$

We should point out that the weak reverse Hölder condition on surface balls is equivalent to the weak reverse Hölder condition on surface cubes. This is because we may cover a surface cube by sufficiently small surface balls with finite overlap and vice versa. Thus, both conditions of Theorem 3.3 are satisfied and estimate (3.5) follows by covering $\Delta(P, s)$ with a finite number of sufficiently small surface cubes. This establishes the solvability of the L^q regularity problem for any $2 < q < p$. Finally, since the weak reverse Hölder condition (1.12) has the self-improving property, the argument above also gives the solvability of the L^q regularity problem for $2 < q < p + \varepsilon$ and, in particular, for $q = p$. $\square$

4. NECESSITY OF THE WEAK REVERSE HÖLDER CONDITION

In this section we show that the reverse Hölder condition (1.12) with exponent $p > 2$ is also necessary for the solvability of the L^p regularity problem.

Theorem 4.1. *Let $\mathcal{L}(D)$ be an elliptic operator of order 2ℓ satisfying the ellipticity condition (1.4) and the symmetry condition (1.5). Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and fix $p > 2$. Suppose that the L^p regularity problem for $\mathcal{L}(D)u = 0$ in Ω is uniquely solvable. Then the weak reverse Hölder inequality (1.12) holds for solutions of $\mathcal{L}(D)u = 0$ in Ω with the properties $(\nabla^\ell u)^* \in L^2(\partial\Omega)$ and $D^\alpha u = 0$ on $\Delta(P, 3r)$ for $|\alpha| \leq \ell - 1$.*

Proof. We begin by choosing $r_0 > 0$ so that for any $P \in \partial\Omega$, (3.6)-(3.7) hold after a possible rotation of the coordinate system. Fix $P_0 \in \partial\Omega$ and $0 < r < cr_0$. Let u be a solution of $\mathcal{L}(D)u = 0$ in Ω such that $(\nabla^\ell u)^* \in L^2(\partial\Omega)$ and $D^\alpha u = 0$ on

$\Delta(P_0, 10r)$ for $|\alpha| \leq \ell - 1$. For a function v on Ω and $P \in \partial\Omega$ define

$$\begin{aligned}\mathcal{M}_1(v)(P) &= \sup\{|v(x)| : x \in \gamma(P) \text{ and } |x - P| < c_0 r\}, \\ \mathcal{M}_2(v)(P) &= \sup\{|v(x)| : x \in \gamma(P) \text{ and } |x - P| \geq c_0 r\},\end{aligned}$$

where

$$\gamma(P) = \gamma_a(P) = \{x \in \Omega : |x - P| < (1 + a)\text{dist}(x, \partial\Omega)\}$$

and $a > 1$ is sufficiently large.

Then, $(\nabla^\ell u)^* = \max\{\mathcal{M}_1(\nabla^\ell u), \mathcal{M}_2(\nabla^\ell u)\}$. If $x \in \gamma(P)$ for some $P \in \Delta(P_0, r)$ and $|x - P| \geq c_0 r$, then by the interior estimates we have

$$|\nabla^\ell u(x)| \leq \frac{C}{r^d} \int_{B(x, cr)} |\nabla^\ell u(y)| dy \leq \frac{C}{r^{d-1}} \int_{\Delta(P_0, 2r)} |(\nabla^\ell u)^*| d\sigma.$$

It follows that for any $p > 2$,

$$(4.1) \quad \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, r)} |\mathcal{M}_2(\nabla^\ell u)|^p d\sigma \right)^{\frac{1}{p}} \leq C \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, 2r)} |(\nabla^\ell u)^*|^2 d\sigma \right)^{\frac{1}{2}}.$$

We now estimate $\mathcal{M}_1(\nabla^\ell u)$ on $\Delta(P_0, r)$. First, choose $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi = 1$ on $B(P_0, 2r)$, $\varphi = 0$ on $\mathbb{R}^d \setminus B(P_0, 3r)$, and $|D^\alpha \varphi| \leq \frac{C}{r^{|\alpha|}}$ for $|\alpha| \leq 2\ell$. Note that

$$\begin{aligned}(4.2) \quad & [\mathcal{L}(D)(u\varphi)]^j \sum_{k=1}^m \sum_{|\alpha|=|\beta|=\ell} a_{\alpha\beta}^{jk} D^\alpha D^\beta (u^k \varphi) \\ &= \sum_{k=1}^m \sum_{|\alpha|=|\beta|=\ell} a_{\alpha\beta}^{jk} D^\alpha \left\{ \varphi D^\beta u^k + \sum_{\gamma \leq \beta} \frac{\beta!}{\gamma!(\beta-\gamma)!} D^\gamma u^k D^{\beta-\gamma} \varphi \right\} \\ &= \sum_{k=1}^m \sum_{|\alpha|=|\beta|=\ell} \sum_{\gamma < \alpha} a_{\alpha\beta}^{jk} \frac{\alpha!}{\gamma!(\alpha-\gamma)!} D^{\beta+\gamma} u^k D^{\alpha-\gamma} \varphi \\ &\quad + \sum_{k=1}^m \sum_{|\alpha|=|\beta|=\ell} \sum_{\gamma < \beta} a_{\alpha\beta}^{jk} \frac{\beta!}{\gamma!(\beta-\gamma)!} \sum_{\delta \leq \alpha} \frac{\alpha!}{\delta!(\alpha-\delta)!} D^{\beta-\gamma+\alpha-\delta} \varphi D^{\delta+\gamma} u^k,\end{aligned}$$

where we used the fact that $\mathcal{L}(D)u = 0$ in Ω .

Recall that $\Gamma(x) = (\Gamma_{ij}(x))_{m \times m}$ denotes a matrix of fundamental solutions on $\mathbb{R}^d$ to the operator $\mathcal{L}(D)$ with pole at the origin. We remark that if d is odd or $2\ell < d$, $\Gamma_{ij}(x)$ is homogeneous of degree $2\ell - d$ and smooth away from the origin. If d is even and $2\ell > d$, then $\Gamma_{ij}(x) = \Gamma_{ij}^{(1)}(x) + \ln|x| \cdot \Gamma_{ij}^{(2)}(x)$, where $\Gamma_{ij}^{(1)}(x)$ is homogeneous of degree $2\ell - d$ and $\Gamma_{ij}^{(2)}(x)$ is a polynomial of degree $2\ell - d$. In this case we replace $\ln|x|$ with $\ln(|x|/r)$. This can be done since $\Gamma_{ij}^{(2)}(x)$ is a polynomial of degree $2\ell - d$. In either case we have the estimate

$$(4.3) \quad |D^\alpha \Gamma(x)| \leq \frac{C_\alpha}{|x|^{d-2\ell+|\alpha|}} \quad \text{for } |\alpha| \geq 2\ell - d + 1,$$

since the derivatives D^α eliminate the logarithmic singularity if $|\alpha| > 2\ell - d$.

Fix $y_0 \in \mathbb{R}^d \setminus \overline{\Omega}$ so that $|y_0 - P_0| = r \sim \text{dist}(y_0, \partial\Omega)$. As in [18], let $\Gamma(x, y) = \Gamma(x - y)$ and define

$$(4.4) \quad F_{ij}(x, y) = \Gamma(x, y) - \sum_{|\gamma| \leq 2\ell-1} \frac{(y - y_0)^\gamma}{\gamma!} D_y^\gamma \Gamma_{ij}(x, y_0).$$

The summation term in (4.4) is a solution of $\mathcal{L}(D)u = 0$ in Ω in both the x and y variables. It is subtracted from $\Gamma(x, y)$ to create the desired decay when $|x - P_0| \geq 5r$ and $|y - P_0| \leq 4r$. Let $T(P, s) = \Omega \cap B(P, s)$. By the Taylor remainder theorem and (4.3), if $x \in \Omega \setminus T(P_0, 5r)$ and $y \in T(P_0, 3r)$, then

$$(4.5) \quad |\nabla_x^\ell D_y^\alpha F_{ij}(x, y)| \leq \frac{Cr^{2\ell-|\alpha|}}{|x - y|^{d+\ell}} \text{ for } |\alpha| \leq 2\ell.$$

Also, if $x \in T(P_0, 5r)$ and $y \in T(P_0, 3r)$, we have

$$(4.6) \quad |\nabla_x^\ell D_y^\alpha F_{ij}(x, y)| \leq \frac{Cr^{\ell-|\alpha|-1}}{|x - y|^{d-1}} \text{ for } |\alpha| \leq \ell - 1.$$

To establish (4.6) we consider two cases: $|\alpha| > \ell - d$ and $|\alpha| \leq \ell - d$. For the first case we use estimate (4.3). The second case is handled by noting that the term involving the possible logarithmic function in $\nabla_x^\ell D_y^\alpha \Gamma_{ij}(x, y)$ is bounded by $C|x - y|^{\ell-d-|\alpha|} \ln \left| \frac{|x-y|}{r} \right|$. Since $|x - y| \leq Cr$ and $\ell - |\alpha| - 1 > 0$, it is bounded by the right-hand side of (4.6).

Next, define $w(x) = (w^1(x), \dots, w^m(x))$ by

$$\begin{aligned} w^i(x) &= \sum_{|\alpha|=|\beta|=\ell} \sum_{\gamma < \alpha} (-1)^\gamma a_{\alpha\beta}^{jk} \frac{\alpha!}{\gamma!(\alpha-\gamma)!} \int_{\Omega} D_y^\gamma (F_{ij}(x, y) D^{\alpha-\gamma} \varphi) D^\beta u^k dy \\ &\quad + \sum_{|\alpha|=|\beta|=\ell} \sum_{\gamma < \beta} \sum_{\delta \leq \alpha} (-1)^\gamma a_{\alpha\beta}^{jk} \frac{\beta!}{\gamma!(\beta-\gamma)!} \frac{\alpha!}{\delta!(\alpha-\delta)!} \\ &\quad \times \int_{\Omega} D_y^\gamma (F_{ij}(x, y) D^{\alpha-\delta+\beta-\gamma} \varphi) D^\delta u^k dy. \end{aligned}$$

Note that $\mathcal{L}(D)(w) = \mathcal{L}(D)(u\varphi)$ in Ω . This follows from integration by parts and (4.2).

Now, on $\Delta(P_0, r)$ we have

$$\mathcal{M}_1(\nabla^\ell u) = \mathcal{M}_1(\nabla^\ell(u\varphi)) \leq \mathcal{M}_1(\nabla^\ell w) + \mathcal{M}_1(\nabla^\ell(u\varphi - w)).$$

For $x \in T(P_0, 5r)$, we use (4.6) to obtain

$$(4.7) \quad |\nabla^\ell w(x)| \leq C \sum_{|\gamma| \leq \ell} \frac{1}{r^{\ell-|\gamma|+1}} \int_{T(P_0, 3r) \setminus T(P_0, 2r)} \frac{|D^\gamma u(y)|}{|x - y|^{d-1}} dy.$$

Thus, if $P \in \Delta(P_0, r)$, we have

$$\begin{aligned}
 \mathcal{M}_1(\nabla^\ell w)(P) &\leq C \sum_{|\gamma| \leq \ell} r^{|\gamma| - \ell - d} \int_{T(P_0, 3r)} |D^\gamma u| dy \\
 &\leq C \left(\frac{1}{r^d} \int_{T(P_0, 3r)} |\nabla^\ell u|^2 dy \right)^{\frac{1}{2}} \\
 &\quad + C \sum_{|\gamma| \leq \ell-1} r^{|\gamma| - \ell} \left(\frac{1}{r^d} \int_{T(P_0, 3r)} |D^\gamma u|^2 dy \right)^{\frac{1}{2}} \\
 &\leq C \left(\frac{1}{r^d} \int_{T(P_0, 3r)} |\nabla^\ell u|^2 dy \right)^{\frac{1}{2}} \\
 &\leq C \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, 3r)} |(\nabla^\ell u)^*|^2 d\sigma \right)^{\frac{1}{2}},
 \end{aligned}$$

where we have used the assumption $D^\gamma u = 0$ on $\Delta(P_0, 10r)$ for $|\gamma| \leq \ell - 1$ and the Poincaré inequality in the third inequality. This gives

$$(4.8) \quad \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, r)} |\mathcal{M}_1(\nabla^\ell w)|^p d\sigma \right)^{\frac{1}{p}} \leq C \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, 3r)} |(\nabla^\ell u)^*|^2 d\sigma \right)^{\frac{1}{2}}.$$

We still need to estimate $\mathcal{M}_1(\nabla^\ell(u\varphi - w))$. This is where the assumption that the L^p regularity problem on Ω is uniquely solvable is used. Recall that $\mathcal{L}(D)(u\varphi - w) = 0$ in Ω . As in [18], we also have $(\nabla^{\ell-1}(u\varphi - w))^* \in L^2(\partial\Omega)$. Thus, we may apply the uniqueness of the L^2 Dirichlet problem and the L^p regularity estimate to obtain

$$\begin{aligned}
 \int_{\Delta(P_0, r)} |\mathcal{M}_1(\nabla^\ell(u\varphi - w))|^p d\sigma &\leq \int_{\partial\Omega} |(\nabla^\ell(u\varphi - w))^*|^p d\sigma \\
 &\leq C \int_{\partial\Omega} |\nabla_t \nabla^{\ell-1}(u\varphi - w)|^p d\sigma \\
 &\leq C \int_{\partial\Omega} |\nabla_t \nabla^{\ell-1}(u\varphi)|^p d\sigma \\
 &\quad + C \int_{\partial\Omega} |\nabla_t \nabla^{\ell-1}w|^p d\sigma \\
 &= C \int_{\partial\Omega} |\nabla_t \nabla^{\ell-1}w|^p d\sigma,
 \end{aligned}$$

where the last step follows from the observation that $\nabla^{\ell-1}(u\varphi) = 0$ on $\partial\Omega$.

Now, let $p = \frac{q(d-1)}{d-q}$. Then, $\frac{d-1}{p} = \frac{d}{q} - 1$. Using (4.7) and Lemma 4.2 in [18] we obtain

$$\begin{aligned} \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, 5r)} |\nabla^\ell w|^p d\sigma \right)^{\frac{1}{p}} &\leq \frac{C}{r^{\frac{d-1}{p}}} \left\| r^{|\gamma|-\ell-1} \sum_{|\gamma| \leq \ell} |D^\gamma u| \right\|_{L^q(T(P_0, 3r))} \\ &\leq C \frac{1}{r^{\frac{d}{q}}} \left(\int_{T(P_0, 3r)} |\nabla^\ell u|^q dy \right)^{\frac{1}{q}} \\ &\leq \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, 3r)} |(\nabla^\ell u)^*|^{\bar{q}} d\sigma \right)^{\frac{1}{\bar{q}}}, \end{aligned}$$

where $\bar{q} = \max\{q, 2\}$. This gives

$$(4.9) \quad \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, 5r)} |\nabla^\ell w|^p d\sigma \right)^{\frac{1}{p}} \leq C \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, 3r)} |(\nabla^\ell u)^*|^{\bar{q}} d\sigma \right)^{\frac{1}{\bar{q}}}.$$

Finally, if $P \in \partial\Omega \setminus \Delta(P_0, 5r)$, we use estimate (4.5) to obtain

$$\begin{aligned} |\nabla^\ell w(P)| &\leq \frac{Cr^{d+\ell-1}}{|P-P_0|^{d+\ell-1}} \sum_{|\gamma| \leq \ell} r^{|\gamma|-\ell-d} \int_{T(P_0, 3r)} |D^\gamma u| dy \\ &\leq C \frac{r^{d+\ell-1}}{|P-P_0|^{d+\ell-1}} \left(\frac{1}{r^d} \int_{T(P_0, 3r)} |\nabla^\ell u|^2 dy \right)^{\frac{1}{2}} \\ &\leq \frac{Cr^{d+\ell-1}}{|P-P_0|^{d+\ell-1}} \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, 3r)} |(\nabla^\ell u)^*|^2 d\sigma \right)^{\frac{1}{2}}. \end{aligned}$$

This implies that

$$(4.10) \quad \left(\frac{1}{r^{d-1}} \int_{\partial\Omega \setminus \Delta(P_0, 5r)} |\nabla^\ell w|^p d\sigma \right)^{\frac{1}{p}} \leq C \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, 3r)} |(\nabla^\ell u)^*|^{\bar{q}} d\sigma \right)^{\frac{1}{\bar{q}}}.$$

Combining estimates (4.1), (4.8), (4.9), and (4.10), we have proved that

$$(4.11) \quad \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, r)} |(\nabla^\ell u)^*|^p d\sigma \right)^{\frac{1}{p}} \leq C \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, 3r)} |(\nabla^\ell u)^*|^{\bar{q}} d\sigma \right)^{\frac{1}{\bar{q}}},$$

where $\bar{q} = \max(q, 2)$ and $\frac{d-1}{p} = \frac{d}{q} - 1$. Note that since $p \geq 2$,

$$\frac{1}{q} - \frac{1}{p} = \frac{1}{d} \left(1 - \frac{1}{p} \right) \geq \frac{1}{2d}.$$

Thus we may iterate estimate (4.11) to obtain

$$(4.12) \quad \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, cr)} |(\nabla^\ell u)^*|^p d\sigma \right)^{\frac{1}{p}} \leq C \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, r)} |(\nabla^\ell u)^*|^2 d\sigma \right)^{\frac{1}{2}},$$

starting with $q = 2$. We may do this since the L^p solvability of the regularity problem implies the L^s solvability for $2 < s < p$.

By covering $\Delta(P_0, r)$ with sufficiently small surface balls $\{\Delta(P_j, cr)\}$, we obtain the weak reverse Hölder condition (1.12). $\square$

Remark 4.2. Under the same assumptions as in Theorem 4.1, we have

$$(4.13) \quad \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, r)} |\mathcal{M}_1(\nabla^\ell u)|^p d\sigma \right)^{\frac{1}{p}} \leq \frac{C}{r} \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, 6r)} |(\nabla^{\ell-1} u)^*|^2 d\sigma \right)^{\frac{1}{2}},$$

where $\mathcal{L}(D)u = 0$ in $T(P_0, 10r)$, $\mathcal{M}_1(\nabla^{\ell-1} u) \in L^2(\Delta(P_0, 10r))$, and $D^\alpha u = 0$ on $\Delta(P_0, 10r)$ for $|\alpha| \leq \ell - 1$. Indeed, a careful inspection of the proof of Theorem 4.1 shows that

$$(4.14) \quad \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, r)} |\mathcal{M}_1(\nabla^\ell u)|^p d\sigma \right)^{\frac{1}{p}} \leq C \left(\frac{1}{r^d} \int_{T(P_0, 3r)} |\nabla^\ell u|^2 d\sigma \right)^{\frac{1}{2}}.$$

Using $D^\alpha u = 0$ on $\Delta(P_0, 10r)$ for $|\alpha| \leq \ell - 1$ and the fact that the L^2 regularity problem is uniquely solvable on every bounded Lipschitz domain, one may deduce that the right-hand side of (4.14) is bounded by the right-hand side of (4.13). We leave the details to the reader.

5. PROOF OF THEOREM 1.2

Let $\frac{1}{q_0} = \frac{1}{p} - \frac{1}{d-1}$. By Theorem 1.1 in [18], it suffices to establish the weak reverse Hölder condition

$$(5.1) \quad \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, r)} |(\nabla^{\ell-1} u)^*|^{q_0} d\sigma \right)^{\frac{1}{q_0}} \leq C \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, 20r)} |(\nabla^{\ell-1} u)^*|^2 d\sigma \right)^{\frac{1}{2}},$$

where $\mathcal{L}(D)u = 0$ in Ω , $(\nabla^{\ell-1} u)^* \in L^2(\partial\Omega)$ and $D^\alpha u = 0$ on $\Delta(P_0, 100r)$ for $|\alpha| \leq \ell - 1$. Clearly we may assume that $\Omega \cap B(P_0, Cr)$ is given by the region above a Lipschitz graph in the sense of (3.6)-(3.7).

Let $P \in \Delta(P_0, r)$ and $x \in \gamma(P)$. It follows from the interior estimates that

$$(5.2) \quad |\nabla^\ell u(x)| \leq \frac{C}{s^{d-1}} \int_{|y-P| < cs} |(\nabla^\ell u)^*(y)| d\sigma,$$

where $s = \text{dist}(x, \partial\Omega)$. Next, write

$$D^\alpha u(x', x_d) - D^\alpha u(x', \tilde{x}_d) = - \int_{x_d}^{\tilde{x}_d} D^{\alpha+e_d} u(x', s) ds,$$

where $|\alpha| = \ell - 1$. This, together with (5.2), gives that

$$(5.3) \quad \mathcal{M}_1(\nabla^{\ell-1} u)(P) \leq C \int_{\Delta(P_0, 3r)} \frac{\widetilde{\mathcal{M}}_1(\nabla^\ell u)}{|P-y|^{d-2}} d\sigma(y) + \widetilde{\mathcal{M}}_2(\nabla^{\ell-1} u)(P),$$

where $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ are defined in the same fashion as $\mathcal{M}_1$ and $\mathcal{M}_2$, but using a family of slightly larger nontangential approach regions $\{\gamma_b(P) : P \in \partial\Omega\}$, where

$b > a$. Thus, by the fractional integral estimates as well as the obvious estimate for $\widetilde{\mathcal{M}}_2(\nabla^{\ell-1}u)$, we have

$$\begin{aligned}
 & \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, r)} |(\nabla^{\ell-1}u)^*|^{q_0} d\sigma \right)^{\frac{1}{q_0}} \\
 (5.4) \quad & \leq C \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, 3r)} |(\nabla^{\ell-1}u)^*|^2 d\sigma \right)^{1/2} \\
 & \quad + Cr \left(\frac{1}{r^{d-1}} \int_{\Delta(P_0, 3r)} |\widetilde{\mathcal{M}}_1(\nabla^\ell u)|^p d\sigma \right)^{\frac{1}{p}}.
 \end{aligned}$$

The desired estimate (5.1) now follows from (5.4) and (4.13). $\square$

6. DUALITY BETWEEN THE REGULARITY AND DIRICHLET PROBLEMS, PART I

The remaining two sections of this paper are devoted to the proof of Theorem 1.3. In this section we show that for any second order elliptic system, the solvability of the L^p regularity problem implies that of the $L^{p'}$ Dirichlet problem.

To simplify the notation in the case $\ell = 1$, we write the $m \times m$ system as $\mathcal{L}(u) = 0$, where $(\mathcal{L}(u))^\alpha = a_{ij}^{\alpha\beta} D_i D_j u^\beta$ for $\alpha = 1, \dots, m$, and

$$(6.1) \quad \mu |\xi|^2 |\eta|^2 \leq a_{ij}^{\alpha\beta} \xi_i \xi_j \eta^\alpha \eta^\beta \leq \frac{1}{\mu} |\xi|^2 |\eta|^2$$

for some $\mu > 0$, and all $\xi \in \mathbb{R}^d$ and $\eta \in \mathbb{R}^m$. Without loss of generality, we may assume that $a_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha}$ in the place of (1.4).

Theorem 6.1. *Let $1 < p < \infty$ and Ω be a bounded Lipschitz domain. Suppose that the L^p regularity problem for $\mathcal{L}(u) = 0$ in Ω is uniquely solvable. Then, the $L^{p'}$ Dirichlet problem for $\mathcal{L}(u) = 0$ in Ω is uniquely solvable.*

Proof. We begin with the existence. Let $f \in C_0^\infty(\mathbb{R}^d)$ and u be the solution of the L^p regularity problem in Ω with boundary data f ; that is, $\mathcal{L}(u) = 0$ in Ω , $u = f$ on $\partial\Omega$ and $\|(\nabla u)^*\|_p \leq C \|\nabla_t f\|_p$. Since $(\nabla u)^* \in L^p(\partial\Omega)$, it follows from Theorem 2.1 that ∇u exists a.e. on $\partial\Omega$ in the sense of nontangential convergence. Therefore, we have Green's representation formula

$$(6.2) \quad u(x) = \int_{\partial\Omega} \Gamma^x \cdot \frac{\partial u}{\partial \nu} d\sigma - \int_{\partial\Omega} \frac{\partial \Gamma^x}{\partial \nu} \cdot u d\sigma,$$

where $\Gamma^x(y) = \Gamma(x-y)$ and $\frac{\partial u}{\partial \nu}$ denotes the conormal derivative of u on $\partial\Omega$, defined by

$$(6.3) \quad \left(\frac{\partial u}{\partial \nu} \right)^\alpha = a_{ij}^{\alpha\beta} n_i D_j u^\beta,$$

where $n = (n_1, \dots, n_d)$ is the outward unit normal to $\partial\Omega$. Thus, by the well-known singular integral estimates on Lipschitz surfaces,

$$(6.4) \quad \|(u)^*\|_{p'} \leq C \left\| \frac{\partial u}{\partial \nu} \right\|_{-1, p'} + C \|f\|_{p'},$$

where $\|\cdot\|_{-1,p'}$ denotes the norm in $W^{-1,p'}(\partial\Omega)$, the dual of the Sobolev space $W^{1,p}(\partial\Omega)$ equipped with the scale-invariant norm

$$\|h\|_{W^{1,p}(\partial\Omega)} = \|\nabla_t h\|_p + |\partial\Omega|^{\frac{1}{1-d}} \|h\|_p.$$

To estimate the first term on the right-hand side of (6.4), we let $g \in C_0^\infty(\mathbb{R}^d)$ and w be the solution of the L^p regularity problem with data g . Using integration by parts, we have

$$\left| \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot g \, d\sigma \right| = \left| \int_{\partial\Omega} u \cdot \frac{\partial w}{\partial \nu} \, d\sigma \right| \leq \|f\|_{p'} \left\| \frac{\partial w}{\partial \nu} \right\|_p.$$

Now, since

$$\left\| \frac{\partial w}{\partial \nu} \right\|_p \leq C \|(\nabla w)^*\|_p \leq C \|\nabla_t g\|_p \leq C \|g\|_{W^{1,p}(\partial\Omega)},$$

by duality we obtain

$$(6.5) \quad \left\| \frac{\partial u}{\partial \nu} \right\|_{-1,p'} \leq C \|f\|_{p'}.$$

So, by combining estimates (6.4) and (6.5), we obtain $\|(u)^*\|_{p'} \leq C \|f\|_{p'}$.

Next, we establish the existence in the general case. Let $f \in L^{p'}(\partial\Omega)$ and choose $f_k \in C_0^\infty(\mathbb{R}^d)$ such that $f_k \rightarrow f$ in $L^{p'}(\partial\Omega)$ as $k \rightarrow \infty$. Let u_k be the unique solution of the L^p regularity problem in Ω with data f_k . Since $u_j - u_k$ is the unique solution of the L^p regularity problem with data $f_j - f_k$, we have

$$\|(u_j - u_k)^*\|_{p'} \leq C \|f_j - f_k\|_{p'} \rightarrow 0.$$

Using

$$\sup_K |u_j - u_k| \leq C_K \|(u_j - u_k)^*\|_{p'},$$

where $K \Subset \Omega$ is compact, we see that $u_j \rightarrow u$ uniformly on every compact subset of Ω . This implies that $\mathcal{L}(u) = 0$ and $\|(u)^*\|_{p'} \leq C \|f\|_{p'}$. To see this, note that $\|(u_j)^*\|_{p'} \leq C \|f_j\|_{p'}$, where

$$(u_j)^*(P) = \sup\{|u_j(x)| : x \in \gamma(P) \text{ and } \text{dist}(x, \partial\Omega) \geq \varepsilon\}.$$

Letting $j \rightarrow \infty$ and then $\varepsilon \rightarrow 0$ we obtain $\|(u)^*\|_{p'} \leq C \|f\|_{p'}$, as desired.

To complete the existence part, we need to show that $u \rightarrow f$ nontangentially almost everywhere. First, note that $\|(u_j - u_k)^*\|_{p'} \leq C \|f_j - f_k\|_{p'}$. Letting $k \rightarrow \infty$ and then $\varepsilon \rightarrow 0$, we obtain

$$\|(u_j - u)^*\|_{p'} \leq C \|f_j - f\|_{p'}.$$

To show that u has nontangential limits, we define

$$\Lambda(P) = \limsup_{\substack{x \rightarrow P \\ x \in \gamma(P)}} u(x) - \liminf_{\substack{x \rightarrow P \\ x \in \gamma(P)}} u(x).$$

Now, fix j and note that

$$\begin{aligned} \Lambda &= \limsup(u - u_j + u_j) - \liminf(u - u_j + u_j) \\ &\leq \limsup(u - u_j) + \limsup u_j - \liminf(u - u_j) - \liminf u_j \\ &= \limsup(u - u_j) - \liminf(u - u_j) \\ &\leq 2(u - u_j)^*. \end{aligned}$$

Thus, $\|\Lambda\|_{p'} \leq C\|(u - u_j)^*\|_{p'} \rightarrow 0$ as $j \rightarrow \infty$. This implies that $\Lambda = 0$ a.e. and hence u has nontangential limits a.e. on $\partial\Omega$.

It remains to show that $u \rightarrow f$ a.e. on $\partial\Omega$. To do this, let

$$\tilde{\Lambda}(P) = \left| \lim_{\substack{x \rightarrow P \\ x \in \gamma(P)}} u(x) - f(P) \right|.$$

Note that for any j ,

$$\tilde{\Lambda} = \left| \lim_{\substack{x \rightarrow P \\ x \in \gamma(P)}} (u - u_j) + (f_j - f)(P) \right| \leq |(u - u_j)^*(P)| + |(f_j - f)(P)|.$$

This implies that

$$\|\tilde{\Lambda}\|_{p'} \leq \|(u - u_j)^*\|_{p'} + \|f_j - f\|_{p'} \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Thus, $\tilde{\Lambda} = 0$ a.e. and $u = f$ a.e. on $\partial\Omega$ in the sense of nontangential convergence.

In the second part of this proof we establish the uniqueness of the solution. To this end, suppose that

$$\begin{cases} \mathcal{L}(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{a.e. on } \partial\Omega, \\ (u)^* \in L^{p'}(\partial\Omega). \end{cases}$$

We need to show that $u \equiv 0$ in Ω .

Fix $x \in \Omega$ and let $G^x = G(x, y) = \Gamma(x - y) - v^x(y)$, where v^x is the solution of the L^p regularity problem with data $\Gamma(x - y)$; i.e.,

$$\begin{cases} \mathcal{L}(v^x) = 0 & \text{in } \Omega, \\ v^x(y) = \Gamma(x - y) & \text{on } \partial\Omega, \\ (\nabla v^x)^* \in L^p(\partial\Omega). \end{cases}$$

Note that $(\nabla v^x)^* \in L^p(\partial\Omega)$ implies that $(\nabla_y G^x)^{*,\varepsilon} \in L^p(\partial\Omega)$ if $2\varepsilon < \text{dist}(x, \partial\Omega)$, where we have used the notation

$$(w)^{*,\varepsilon}(P) = \sup \{ |w(x)| : x \in \gamma(P) \text{ and } \text{dist}(x, \partial\Omega) < \varepsilon \}.$$

Choose $\varphi_\varepsilon \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi_\varepsilon = 1$ on $\{y \in \Omega : \text{dist}(y, \partial\Omega) \geq \varepsilon\}$, $\varphi_\varepsilon = 0$ on $\Omega \setminus \{y \in \Omega : \text{dist}(y, \partial\Omega) \geq \varepsilon/2\}$, $|\nabla \varphi_\varepsilon| \leq \frac{C}{\varepsilon}$, and $|\nabla^2 \varphi_\varepsilon| \leq \frac{C}{\varepsilon^2}$. Note that

$$(6.6) \quad u(x) = u\varphi_\varepsilon(x) = \int_{\Omega} G(x, y) \mathcal{L}(u\varphi_\varepsilon) dy$$

and

$$\begin{aligned} (\mathcal{L}(u\varphi_\varepsilon))^\alpha &= a_{ij}^{\alpha\beta} D_i D_j \{u^\beta \varphi_\varepsilon\} \\ &= a_{ij}^{\alpha\beta} \{D_i D_j u^\beta \cdot \varphi_\varepsilon + D_j u^\beta \cdot D_i \varphi_\varepsilon + D_i u^\beta \cdot D_j \varphi_\varepsilon + u^\beta \cdot D_i D_j \varphi_\varepsilon\} \\ &= a_{ij}^{\alpha\beta} D_j u^\beta \cdot D_i \varphi_\varepsilon + a_{ij}^{\alpha\beta} D_i u^\beta \cdot D_j \varphi_\varepsilon + a_{ij}^{\alpha\beta} u^\beta D_i D_j \varphi_\varepsilon, \end{aligned}$$

where we used the fact that $\mathcal{L}(u) = 0$ in Ω . This implies that

$$\begin{aligned} u^\alpha(x) &= \int_{\Omega} G_{\alpha\gamma}(x, y) a_{ij}^{\gamma\beta} \{D_j u^\beta \cdot D_i \varphi_\varepsilon + D_i u^\beta \cdot D_j \varphi_\varepsilon + u^\beta \cdot D_i D_j \varphi_\varepsilon\} dy \\ (6.7) \quad &= - \int_{\Omega} \left\{ D_j G_{\alpha\gamma}(x, y) a_{ij}^{\gamma\beta} u^\beta D_i \varphi_\varepsilon + G_{\alpha\gamma}(x, y) a_{ij}^{\gamma\beta} u^\beta D_i D_j \varphi_\varepsilon \right. \\ &\quad \left. + D_i G_{\alpha\gamma}(x, y) a_{ij}^{\gamma\beta} u^\beta D_j \varphi_\varepsilon \right\} dy. \end{aligned}$$

It follows that

$$\begin{aligned}
 |u^\alpha(x)| &\leq \frac{C}{\varepsilon} \int_{E_\varepsilon} |\nabla_y G(x, y)| |u(y)| dy + \frac{C}{\varepsilon^2} \int_{E_\varepsilon} |G(x, y)| |u(y)| dy \\
 &\leq \frac{C}{\varepsilon} \left\{ \int_{E_\varepsilon} |\nabla_y G(x, y)|^p dy \right\}^{\frac{1}{p}} \left\{ \int_{E_\varepsilon} |u|^{p'} dy \right\}^{\frac{1}{p'}} \\
 (6.8) \quad &\quad + \frac{C}{\varepsilon^2} \left\{ \int_{E_\varepsilon} |G(x, y)|^p dy \right\}^{\frac{1}{p}} \left\{ \int_{E_\varepsilon} |u|^{p'} dy \right\}^{\frac{1}{p'}},
 \end{aligned}$$

where $E_\varepsilon = \{x \in \Omega : (\varepsilon/2) \leq \text{dist}(x, \partial\Omega) \leq \varepsilon\}$.

Using the mean value theorem and $G(x, y) = 0$ for $y \in \partial\Omega$, it is easy to see that $|G(x, y)| \leq C\varepsilon(\nabla_y G^x)^{*,\varepsilon}(P)$ if $y \in E_\varepsilon \cap \gamma(P)$. It then follows from (6.8) that

$$\begin{aligned}
 |u^\alpha(x)| &\leq \frac{C}{\varepsilon^{1-\frac{1}{p}}} \left\{ \int_{\partial\Omega} |(\nabla_y G^x)^{*,\varepsilon}|^p d\sigma \right\}^{\frac{1}{p}} \left\{ \int_{E_\varepsilon} |u|^{p'} dy \right\}^{\frac{1}{p'}} \\
 &\leq C_x \left\{ \frac{1}{\varepsilon} \int_{E_\varepsilon} |u|^{p'} dy \right\}^{\frac{1}{p'}} \\
 &\leq C_x \left\{ \int_{\partial\Omega} |(u)^{*,\varepsilon}|^{p'} d\sigma \right\}^{\frac{1}{p'}}.
 \end{aligned}$$

Finally, since $u = 0$ a.e. on $\partial\Omega$ in the sense of nontangential convergence, it follows that $(u)^{*,\varepsilon} \rightarrow 0$ a.e. as $\varepsilon \rightarrow 0$. Using $(u)^{*,\varepsilon} \leq (u)^* \in L^{p'}(\partial\Omega)$ and the dominated convergence theorem, we obtain $\|(u)^{*,\varepsilon}\|_{p'} \rightarrow 0$ as $\varepsilon \rightarrow 0$. This implies that $u \equiv 0$ in Ω , and the uniqueness of the solution is established. $\square$

7. DUALITY BETWEEN THE REGULARITY AND DIRICHLET PROBLEMS, PART II

In this final section we prove the other implication in Theorem 1.3.

Theorem 7.1. *Let Ω be a bounded Lipschitz domain and $1 < p < \infty$. If the $L^{p'}$ Dirichlet problem for $\mathcal{L}(u) = 0$ in Ω is uniquely solvable, then the L^p regularity problem for $\mathcal{L}(u) = 0$ in Ω is uniquely solvable.*

The proof of this theorem relies on the following lemma. It also uses the solvability of the L^2 regularity problem for second order elliptic systems, established in [7, 8, 9, 10].

Lemma 7.2. *Under the same assumptions as in Theorem 7.1, we have*

$$(7.1) \quad \|(\nabla u)^*\|_p + |\partial\Omega|^{\frac{1}{1-d}} \|(u)^*\|_p \leq C \left\{ \|\nabla_t u\|_p + |\partial\Omega|^{\frac{1}{1-d}} \|u\|_p \right\},$$

where u is the solution of the L^2 regularity problem with data $f \in C_0^\infty(\mathbb{R}^d)$.

We first demonstrate how to deduce Theorem 7.1 from Lemma 7.2.

Proof of Theorem 7.1. We begin with the existence. By dilation we may assume that $|\partial\Omega| = 1$. Let $f \in W^{1,p}(\partial\Omega)$. Choose $f_k \in C_0^\infty(\mathbb{R}^d)$ such that $f_k \rightarrow f$ in $W^{1,p}(\partial\Omega)$. Let u_k be the solution of the L^2 regularity problem with data f_k . Then $u_j - u_k$ is the solution of the L^2 regularity problem with data $f_j - f_k$. Thus, using estimate (7.1), we obtain

$$(7.2) \quad \|(u_j)^*\|_p + \|(\nabla u_j)^*\|_p \leq C \|f_j\|_{W^{1,p}(\partial\Omega)},$$

$$(7.3) \quad \|(u_j - u_k)^*\|_p + \|(\nabla u_j - \nabla u_k)^*\|_p \leq C \|f_j - f_k\|_{W^{1,p}(\partial\Omega)}.$$

It follows from estimate (7.3) that u_j converges uniformly on any compact subset of Ω . By interior estimates, this implies that $u_j \rightarrow u$ and $\nabla u_j \rightarrow \nabla u$ uniformly on any compact subset of Ω and $\mathcal{L}(u) = 0$ in Ω . By letting $j \rightarrow \infty$, as in the proof of Theorem 6.1, we obtain

$$(7.4) \quad \|(u)^*\|_p + \|(\nabla u)^*\|_p \leq C\|f\|_{W^{1,p}(\partial\Omega)},$$

$$(7.5) \quad \|(u_k - u)^*\|_p + \|(\nabla u_k - \nabla u)^*\|_p \leq C\|f_k - f\|_{W^{1,p}(\partial\Omega)}.$$

We point out that estimate (7.5) implies that $u = f$ on $\partial\Omega$ in the sense of nontangential convergence. This follows from the same argument used in the proof of Theorem 6.1 for the existence of nontangential limits.

To demonstrate the uniqueness, we fix $x \in \Omega$ and suppose that

$$\begin{cases} \mathcal{L}(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ (\nabla u)^* \in L^p(\partial\Omega). \end{cases}$$

We need to show that $u \equiv 0$ in Ω . To do this, let $G^x = G(x, y) = \Gamma(x - y) - w^x(y)$ in Ω , where w^x is the unique solution of the $L^{p'}$ Dirichlet problem in Ω with data $\Gamma(x - y)$; i.e.,

$$(7.6) \quad \begin{cases} \mathcal{L}(w^x) = 0 & \text{in } \Omega, \\ w^x = \Gamma^x & \text{on } \partial\Omega, \\ (w^x)^* \in L^{p'}(\partial\Omega). \end{cases}$$

Note that since $(w^x)^* \in L^{p'}(\partial\Omega)$, we have $(G^x)^{*,\varepsilon} \in L^{p'}(\partial\Omega)$ if $2\varepsilon < \text{dist}(x, \partial\Omega)$.

We now proceed as in the proof of Theorem 6.1. It follows from (6.7) that

$$\begin{aligned} |u(x)| &\leq \frac{C}{\varepsilon} \int_{E_\varepsilon} |G^x| |\nabla u| dy + \frac{C}{\varepsilon^2} \int_{E_\varepsilon} |G^x| |u| dy \\ &\leq C \int_{\partial\Omega} (G^x)^{*,\varepsilon} \cdot (\nabla u)^* d\sigma \\ (7.7) \quad &\leq C \|(G^x)^{*,\varepsilon}\|_{p'} \|(\nabla u)^*\|_p, \end{aligned}$$

where we have used the observation that $|u(y)| \leq C\varepsilon |(\nabla u)^*(P)|$ if $y \in \gamma(P) \cap E_\varepsilon$. Note that $\|(G^x)^{*,\varepsilon}\|_{p'} \rightarrow 0$ as $\varepsilon \rightarrow 0$. This follows easily from the dominated convergence theorem, since $(G^x)^{*,\varepsilon} \rightarrow 0$ a.e. as $\varepsilon \rightarrow 0$ and $(G^x)^{*,\varepsilon} \leq (G^x)^{*,\varepsilon_0} \in L^{p'}(\partial\Omega)$ for $\varepsilon \leq \varepsilon_0$. Thus we may conclude that $u \equiv 0$ in Ω . This completes the proof. $\square$

The rest of this section is devoted to the proof of Lemma 7.2.

Lemma 7.3. *Let Ω be a bounded Lipschitz domain with $|\partial\Omega| = 1$ and $1 < p < \infty$. Suppose that the $L^{p'}$ Dirichlet problem for $\mathcal{L}$ in Ω is uniquely solvable. Also suppose that $u \in C^\infty(\Omega)$, ∇u exists a.e. on $\partial\Omega$, and $u = 0$ on $\Omega \setminus B(P, r)$ for some $P \in \partial\Omega$. We further assume that for some $C_0 > C_1 > 10$, $\Omega \subset B(P, C_0 r)$,*

$$B(P, C_1 r) \cap \Omega = B(P, C_1 r) \cap \{(x', x_d) : x_d > \eta(x')\},$$

$\mathcal{L}(u) \in L^{p'}(\Omega)$ and $\left(\frac{\partial u}{\partial x_d}\right)^* \in L^{p'}(\partial\Omega)$. Then,

$$\int_{\partial\Omega} |\nabla u|^{p'} d\sigma \leq C \int_{\partial\Omega} \left| \frac{\partial u}{\partial x_d} \right|^{p'} d\sigma + C \int_{\Omega} |\mathcal{L}(u)|^{p'} dx + C \int_K |\nabla u|^{p'} dx,$$

where K is a compact subset of Ω and C is a constant which may depend on r and K .

Proof. We may assume that $P = 0$. Let $\tilde{u} = u - \Gamma * \mathcal{L}(u)$ in Ω . Then $\mathcal{L}(\tilde{u}) = 0$ in Ω and

$$(7.8) \quad \int_{\partial\Omega} |\nabla u|^{p'} d\sigma \leq C \int_{\partial\Omega} |\nabla \tilde{u}|^{p'} d\sigma + C \int_{\partial\Omega} |\nabla \Gamma * \mathcal{L}(u)|^{p'} d\sigma.$$

Next, we estimate each of the terms on the right-hand side of (7.8). We begin with the second term. Note that

$$\begin{aligned} \int_{\partial\Omega} |\nabla \Gamma * \mathcal{L}(u)|^{p'} d\sigma &\leq C \int_{\Omega} |\nabla \Gamma * \mathcal{L}(u)|^{p'} dx + C \int_{\Omega} |\nabla^2 \Gamma * \mathcal{L}(u)|^{p'} dx \\ &\leq C \int_{\Omega} |\mathcal{L}(u)|^{p'} dx, \end{aligned}$$

where we've used the Calderón-Zygmund estimates on the second term and the fractional integral estimates on the first term. To estimate the first term on the right-hand side of (7.8), we observe that by the square function estimates (e.g. see [5]),

$$(7.9) \quad \int_{\partial\Omega} |\nabla \tilde{u}|^{p'} d\sigma \leq \int_{\partial\Omega} |(\nabla \tilde{u})^*|^{p'} d\sigma \leq C \int_{\partial\Omega} |S(\nabla \tilde{u})|^{p'} d\sigma + C \sup_{K_1} |\nabla \tilde{u}|^{p'},$$

where K_1 is a compact subset of Ω and $S(w)$ denotes the usual square function of w , defined by using a regular family of nontangential cones. Also note that by interior estimates,

$$\sup_{K_1} |\nabla \tilde{u}|^{p'} \leq \int_{K_2} |\nabla \tilde{u}|^{p'} dx \leq \int_{K_2} |\nabla u|^{p'} dx + C \int_{\Omega} |\mathcal{L}(u)|^{p'} dx,$$

where $K_2 \supset K_1$ is a compact subset of Ω .

It remains to estimate the term involving the square function in (7.9). The key observation here is that

$$(7.10) \quad S(\nabla \tilde{u}) = S(\nabla \Gamma * \mathcal{L}(u)) \leq C \int_{\Omega} |\mathcal{L}(u)| dx \quad \text{on } \partial\Omega \setminus B(0, 2r),$$

$$(7.11) \quad S(\nabla \tilde{u}) \leq C \tilde{S}\left(\frac{\partial \tilde{u}}{\partial x_d}\right) + C \sup_{K_3} |\nabla \tilde{u}| \quad \text{on } B(0, 2r) \cap \partial\Omega,$$

where $K_3 \supset K_2$ is a compact subset of Ω and $\tilde{S}(w)$ denotes the square function of w , defined by using a regular family of nontangential cones which are slightly larger than the ones used for $S(w)$. Estimate (7.10) follows easily from the assumption that $u = 0$ in $\Omega \setminus B(0, r)$. The proof of (7.11) in the case of harmonic functions on upper-half spaces may be found in [22]. It extends easily to the case of general second order elliptic systems in Lipschitz domains.

With estimates (7.10)-(7.11) at our disposal, we have

$$\begin{aligned}
 \int_{\partial\Omega} |S(\nabla\tilde{u})|^{p'} d\sigma &\leq \int_{B(0,2r)\cap\partial\Omega} |S(\nabla\tilde{u})|^{p'} d\sigma + \int_{\partial\Omega\setminus B(0,2r)} |S(\nabla\tilde{u})|^{p'} d\sigma \\
 &\leq C \int_{B(0,2r)\cap\partial\Omega} \left| \tilde{S} \left(\frac{\partial\tilde{u}}{\partial x_d} \right) \right|^{p'} d\sigma + C \int_{\Omega} |\mathcal{L}(u)|^{p'} dx + C \sup_{K_3} |\nabla\tilde{u}|^{p'} \\
 &\leq C \int_{\partial\Omega} \left| \left(\frac{\partial\tilde{u}}{\partial x_d} \right)^* \right|^{p'} d\sigma + C \int_{\Omega} |\mathcal{L}(u)|^{p'} dx + C \sup_{K_3} |\nabla\tilde{u}|^{p'} \\
 (7.12) \quad &\leq C \int_{\partial\Omega} \left| \frac{\partial\tilde{u}}{\partial x_d} \right|^{p'} d\sigma + C \int_{\Omega} |\mathcal{L}(u)|^{p'} dx + C \int_K |\nabla u|^{p'} dx \\
 &\leq C \int_{\partial\Omega} \left| \frac{\partial u}{\partial x_d} \right|^{p'} d\sigma + C \int_{\partial\Omega} |\nabla\Gamma * \mathcal{L}(u)|^{p'} d\sigma \\
 &\quad + C \int_{\Omega} |\mathcal{L}(u)|^{p'} dx + C \int_K |\nabla u|^{p'} dx \\
 &\leq C \int_{\partial\Omega} \left| \frac{\partial u}{\partial x_d} \right|^{p'} d\sigma + C \int_{\Omega} |\mathcal{L}(u)|^{p'} dx + C \int_K |\nabla u|^{p'} dx,
 \end{aligned}$$

where $K \supset K_3$ is a compact subset of Ω .

We point out that it is in estimate (7.12) where we used the solvability of the $L^{p'}$ Dirichlet problem in Ω . This is possible since $\mathcal{L}(\frac{\partial\tilde{u}}{\partial x_d}) = 0$ in Ω and

$$(7.13) \quad \left(\frac{\partial\tilde{u}}{\partial x_d} \right)^* \in L^{p'}(\partial\Omega).$$

To see (7.13), we only need to show that the radial maximal function

$$(7.14) \quad \mathcal{M} \left(\frac{\partial\tilde{u}}{\partial x_d} \right) (P) = \sup_{0 < t < c} \left| \frac{\partial\tilde{u}}{\partial x_d}(P + te_d) \right| \in L^{p'}(B(0,2r) \cap \partial\Omega).$$

Since

$$\mathcal{M} \left(\frac{\partial\tilde{u}}{\partial x_d} \right) \leq \mathcal{M} \left(\frac{\partial u}{\partial x_d} \right) + \mathcal{M}(\nabla\Gamma * \mathcal{L}(u))$$

and $\mathcal{M} \left(\frac{\partial u}{\partial x_d} \right) \leq \left(\frac{\partial u}{\partial x_d} \right)^* \in L^{p'}(\partial\Omega)$, estimate (7.13) follows from

$$(7.15) \quad \int_{B(0,2r)\cap\partial\Omega} |\mathcal{M}(\nabla\Gamma * \mathcal{L}(u))|^{p'} d\sigma \leq C \int_{\Omega} |\mathcal{L}(u)|^{p'} dx.$$

Finally we remark that the desired estimate (7.15) is a consequence of the inequality

$$\int_{B(0,2r)\cap\partial\Omega} |\mathcal{M}(w)|^{p'} d\sigma \leq C \int_{\Omega} |\nabla w|^{p'} dx + C \int_{\Omega} |w|^{p'} dx$$

for any $w \in C^1(\Omega)$. This completes the proof of Lemma 7.3. $\square$

We are now in a position to give

Proof of Lemma 7.2. We may assume that $|\partial\Omega| = 1$. Suppose that $f \in C_0^\infty(\mathbb{R}^d)$ and

$$\begin{cases} \mathcal{L}(u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \\ (\nabla u)^* \in L^2(\partial\Omega). \end{cases}$$

By Theorem 2.1, ∇u has nontangential limits a.e. on $\partial\Omega$. We will show that

$$(7.16) \quad \left\| \frac{\partial u}{\partial \nu} \right\|_p \leq C \{ \|\nabla_t f\|_p + \|f\|_p \}.$$

The nontangential maximal function estimate (7.1) follows from (7.16) and Theorem 2.1 as well as Green's representation formula. To establish (7.16), we first note that it suffices to consider the case $\text{supp}(f) \subset B(P_0, r)$, where $P_0 \in \partial\Omega$ and $B(P_0, C_1 r) \cap \partial\Omega$ is given by the graph of a Lipschitz function after a possible rotation. For otherwise write $f = \sum_{j=1}^M f \varphi_j$, where $\varphi_j \in C_0^\infty(\mathbb{R}^d)$ and $\sum \varphi_j = 1$ on $\partial\Omega$. Then, $u = \sum u_j$ where u_j is the solution with data $f \varphi_j$ and we have

$$\left\| \frac{\partial u}{\partial \nu} \right\|_p \leq \sum_j \left\| \frac{\partial u_j}{\partial \nu} \right\|_p \leq C \sum_j \|f \varphi_j\|_{W^{1,p}(\partial\Omega)} \leq C \|f\|_{W^{1,p}(\partial\Omega)}.$$

Let $g \in C_0^\infty(\mathbb{R}^d)$ and

$$\begin{cases} \mathcal{L}(w) = 0 & \text{in } \Omega, \\ w = g & \text{on } \partial\Omega, \\ (\nabla w)^* \in L^2(\partial\Omega). \end{cases}$$

Using integration by parts we obtain

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot g \, d\sigma = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot w \, d\sigma = \int_{\partial\Omega} u \cdot \frac{\partial w}{\partial \nu} \, d\sigma = \int_{\partial\Omega} f \cdot \frac{\partial w}{\partial \nu} \, d\sigma.$$

By duality it suffices to show that

$$(7.17) \quad \left| \int_{\partial\Omega} f \cdot \frac{\partial w}{\partial \nu} \, d\sigma \right| \leq C \|f\|_{W^{1,p}(\partial\Omega)} \|g\|_{p'}.$$

To this end we assume that $P_0 = 0$ and that

$$B(0, C_1 r) \cap \Omega = B(0, C_1 r) \cap \{(x', x_d) : x_d > \eta(x')\},$$

where $\eta : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ is a Lipschitz function. Next, choose $\psi \in C_0^\infty(B(0, 4r))$ such that $\psi = 1$ on $B(0, 3r)$. Now define

$$\tilde{w}(x', x_d) = - \int_{x_d}^\infty \psi(x', t) w(x', t) \, dt.$$

Then $\frac{\partial \tilde{w}}{\partial x_d} = \psi w$ in Ω and, in particular, $\frac{\partial \tilde{w}}{\partial x_d} = w$ in $B(0, 3r) \cap \Omega$. Also note that

$$\begin{aligned} (\mathcal{L}(\tilde{w}))^\alpha &= - \int_{x_d}^\infty (\mathcal{L}(\psi w))^\alpha \, dt \\ &= - \int_{x_d}^\infty a_{ij}^{\alpha\beta} \{ w^\beta D_i D_j \psi + D_i w^\beta D_j \psi + D_j w^\beta D_i \psi \} \, dt. \end{aligned}$$

It follows that on $B(0, 2r) \cap \Omega$,

$$|\mathcal{L}(\tilde{w})| \leq C \sup_K |w|,$$

where $K \Subset \Omega$ is compact. We now observe that

$$\begin{aligned} \int_{\partial\Omega} f \cdot \frac{\partial w}{\partial \nu} d\sigma &= \int_{\partial\Omega} f^\alpha a_{ij}^{\alpha\beta} n_i D_j w^\beta d\sigma = \int_{\partial\Omega} f^\alpha a_{ij}^{\alpha\beta} n_i D_j D_d \tilde{w}^\beta d\sigma \\ &= \int_{\partial\Omega} f^\alpha a_{ij}^{\alpha\beta} (n_i D_d - n_d D_i) D_j \tilde{w}^\beta d\sigma + \int_{\partial\Omega} f^\alpha n_d [\mathcal{L}(\tilde{w})]^\alpha d\sigma \\ &= - \int_{\partial\Omega} (n_i D_d - n_d D_i) f^\alpha \cdot a_{ij}^{\alpha\beta} D_j \tilde{w}^\beta d\sigma + \int_{\partial\Omega} f^\alpha n_d [\mathcal{L}(\tilde{w})]^\alpha d\sigma. \end{aligned}$$

This implies that

$$\begin{aligned} \left| \int_{\partial\Omega} f \cdot \frac{\partial w}{\partial \nu} d\sigma \right| &\leq C \|\nabla_t f\|_p \left\{ \int_{B(0,r) \cap \partial\Omega} |\nabla \tilde{w}|^{p'} d\sigma \right\}^{\frac{1}{p'}} + C \|f\|_p \sup_K |w| \\ &\leq C \|f\|_{W^{1,p}(\partial\Omega)} \left\{ \left\{ \int_{B(0,r) \cap \partial\Omega} |\nabla \tilde{w}|^{p'} d\sigma \right\}^{\frac{1}{p'}} + \sup_K |w| \right\}. \end{aligned}$$

Note that since the $L^{p'}$ Dirichlet problem in Ω is solvable, we have that

$$\sup_K |w| \leq C \|(w)^*\|_{p'} \leq C \|g\|_{p'}.$$

Thus, we only need to show that

$$(7.18) \quad \int_{B(0,r) \cap \partial\Omega} |\nabla \tilde{w}|^{p'} d\sigma \leq C \|g\|_{p'}^{p'}.$$

Note that $\tilde{w} = 0$ in $\Omega \setminus B(0, 4r)$. This allows us to apply Lemma 7.3 to obtain

$$\begin{aligned} \int_{B(0,r) \cap \partial\Omega} |\nabla \tilde{w}|^{p'} d\sigma &\leq C \int_{B(0,4r) \cap \partial\Omega} |w|^{p'} d\sigma \\ (7.19) \quad &+ C \int_{\Omega} |\mathcal{L}(\tilde{w})|^{p'} dx + C \int_K |\nabla \tilde{w}|^{p'} dx. \end{aligned}$$

To handle the second term of (7.19), we use Hardy's inequality (see e.g. [22], p.272) to obtain

$$\begin{aligned} \int_{\Omega} |\mathcal{L}(\tilde{w})|^{p'} dx &= \int_{B(0,4r) \cap \Omega} \left| \int_{x_d}^{\infty} \mathcal{L}(\psi w) dt \right|^{p'} dx \\ &\leq C \int_{B(0,4r) \cap \Omega} |\mathcal{L}(\psi w)|^{p'} \{\text{dist}(x, \partial\Omega)\}^{p'} dx \\ &\leq C \int_{\Omega} \{|\nabla w|^{p'} + |w|^{p'}\} \{\text{dist}(x, \partial\Omega)\}^{p'} dx \\ &\leq C \int_{\partial\Omega} |(w)^*|^{p'} d\sigma \\ &\leq C \int_{\partial\Omega} |g|^{p'} d\sigma. \end{aligned}$$

By interior estimates, the last term in (7.19) is bounded by

$$C \sup_{\tilde{K}} |w|^{p'} \leq C \int_{\partial\Omega} |g|^{p'} d\sigma,$$

where $\tilde{K} \supset K$ is a compact subset of Ω . With this last estimate we complete the proof of estimate (7.18) and hence the proof of Lemma 7.3. $\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KENTUCKY 40506

E-mail address: jkilty@ms.uky.edu

Current address: Department of Mathematics, Centre College, Danville, Kentucky 40422

E-mail address: joel.kilty@centre.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KENTUCKY 40506

E-mail address: zshen2@email.uky.edu