ON ASYMPTOTIC STABILITY OF GROUND STATES OF NLS WITH A FINITE BANDS PERIODIC POTENTIAL IN 1D

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ABSTRACT. We consider a nonlinear Schrödinger equation

$$iu_t - h_0 u + \beta(|u|^2)u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R},$$

with $h_0 = -\frac{d^2}{dx^2} + P(x)$ a Schrödinger operator with finitely many spectral bands. We assume the existence of an orbitally stable family of ground states. Exploiting dispersive estimates in Cuccagna (2008), Cuccagna and Visciglia (2009), and following the argument in Cuccagna (to appear) we prove that under appropriate hypotheses the ground states are asymptotically stable.

1. INTRODUCTION

Consider $h_0 = -\frac{d^2}{dx^2} + P(x)$, with $P(x) \equiv P(x+1)$ a smooth periodic potential of period 1 in $x \in \mathbb{R}$, and consider the NLS

$$(1.1) \quad iu_t - h_0 u + \beta(|u|^2)u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}.$$

In this paper we study the asymptotic stability of standing waves $u(t, x) = e^{it\omega} \phi_\omega(x)$ with $\phi_\omega(x) > 0$ for all x . We consider the following hypotheses.

- (H1) The spectrum $\sigma(h_0)$ is formed by finitely many bands. We choose h_0 so that $\inf \sigma(h_0) = 0$.
- (H2) $\beta(0) = \beta'(0) = \beta''(0) = 0$, $\beta \in C^\infty(\mathbb{R}, \mathbb{R})$.
- (H3) There exists a $p \in (1, \infty)$ such that for every $k = 0, 1$,

$$\left| \frac{d^k}{dv^k} \beta(v^2) \right| \lesssim |v|^{p-k-1} \quad \text{if } |v| \geq 1.$$

- (H4) There exists an open interval $\mathcal{O} \subseteq (0, +\infty)$ such that $h_0 u - \beta(|u|^2)u = -\omega u$ admits a C^1 -family of ground states $\phi_\omega(x)$ for $\omega \in \mathcal{O}$.
- (H5) $\frac{d}{d\omega} \|\phi_\omega\|_{L^2(\mathbb{R})}^2 > 0$ for $\omega \in \mathcal{O}$.
- (H6) Set $L_+ = h_0 + \omega - \beta(\phi_\omega^2) - 2\phi_\omega^2 \beta'(\phi_\omega^2)$. Then $\ker(L_+) = 0$ and L_+ has exactly one negative eigenvalue, which is simple.

By [ShS] the map $\omega \rightarrow \phi_\omega \in H^1(\mathbb{R})$ is C^2 , and by [We1], [GSS] (H5) yields orbital stability of the ground state $e^{i\omega t} \phi_\omega(x)$. Here we investigate asymptotic stability. We need some additional hypotheses. In §2 we introduce the linearization H_ω . By standard arguments $\sigma_e(H_\omega) = \bigcup_{\pm} \pm(\omega + \sigma(h_0))$. In particular $\sigma_e(H_\omega)$ is a, finite by (H1), union of intervals. We call thresholds of H_ω the numbers in $\mathbb{R}$ which are extremes of these intervals.

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- (H7) Let H_ω be the linearized operator around $e^{it\omega}\phi_\omega$; see Section 2. Then H_ω has a certain number of simple positive eigenvalues that for any j : $\lambda_j(\omega) \notin \sigma_e(H_\omega)$; $2\lambda_j(\omega) \in \sigma_e(H_\omega)$ and is not a threshold. H_ω does not have other eigenvalues, and each threshold of H_ω is not a resonance for H_ω .
- (H8) For multi-indexes $m = (m_1, m_2, \dots)$ and $n = (n_1, \dots)$, setting $\lambda(\omega) = (\lambda_1(\omega), \dots)$ and $(m - n) \cdot \lambda = \sum (m_j - n_j)\lambda_j$, we have the following two nonresonance hypotheses:
- (i) $(m - n) \cdot \lambda(\omega) = 0$ implies $m = n$ if $|m| \leq 3$ and $|n| \leq 3$;
 - (ii) $(m - n) \cdot \lambda(\omega)$ is not a threshold of H_ω for all (m, n) with $|m| + |n| \leq 3$.
- (H9) We assume the nondegeneracy hypothesis, Hypothesis 4.4.

Theorem 1.1. *Let $\omega_0 \in \mathcal{O}$ and $\phi_{\omega_0}(x)$ be a ground state. Let $u(t, x)$ be a solution of (1.1). Assume (H1)–(H9). Then, there exist an $\epsilon_0 > 0$ and a $C > 0$ such that if $\inf_{\gamma \in [0, 2\pi]} \|u_0 - e^{i\gamma}\phi_\omega\|_{H^1(\mathbb{R})} < \epsilon < \epsilon_0$, then there exist $\omega_+ \in \mathcal{O}$, $\theta \in C^1(\mathbb{R}; \mathbb{R})$ and $u_+ \in H^1(\mathbb{R})$ with $\|u_+\|_{H^1(\mathbb{R})} \leq C\epsilon$ such that*

$$\lim_{t \rightarrow \infty} \|u(t, \cdot) - e^{i\theta(t)}\phi_{\omega_+} - e^{-ith_0}u_+\|_{H^1} = 0.$$

Theorem 1.1 is a transposition of more general results on the NLS with $P(x) \equiv 0$ proved for 1D in [C1] and for dimensions 2 and higher in [CT], [CM]. There is now a substantial literature on asymptotic stability of ground states of the NLS, which starts with work by Soffer and Weinstein [SW1]–[SW2] and Buslaev and Perelman [BP1]–[BP2] in the early 1990s; see also [PW], [Wd1], [SW3]. In this decade we also have further work in [C4]–[C5], [TY1]–[TY3], [BS], [SW4], [GNT], [GS1]–[GS2], [M1]–[M2]. For some results on multisolitons see [P], [RSS]. We highlight in particular the breakthrough work [GS2], later refined and extended in [C1], [CT], [CM]. The present paper and the above references focus on asymptotic stability of spatially localized standing waves of equation (1.1). For a discussion of linear and orbital stability of spatially periodic or antiperiodic standing waves, see [BR], [GH] and the references therein.

The framework for Theorem 1.1 is, by now, classical. After breaking canonically the solutions $u(t, x)$ into a ground state plus a remainder; see §2, we consider an appropriate generalized NLS for the remainder R ; see (2.1). R can be decomposed using an appropriate time varying frame associated to the spectral decomposition of the linearization $H_{\omega(t)}$. Even though the spectrum of $iH_{\omega(t)}$ is on the imaginary axis, the continuous spectrum can be thought of as a stable spectrum. The discrete spectrum corresponds to central directions. As a general rule the main difficulty consists in showing that the discrete modes of R decay to 0, that is, in some sense, showing stability in the central manifold. The stabilization mechanism is known in various special situations and, in our opinion, is fundamentally well understood, although there are theorems yet to be proved. In particular, in the references listed at the end of this paper, the Hamiltonian structure of the NLS is not sufficiently exploited. We refer to [CM], [C6] for further discussion on this theme. In this respect, the present paper presents no novelties. With hypotheses (H6)–(H8) we consider a special case where the Hamiltonian structure of the NLS is exploited, but which is already known in the literature. The novelty in this paper is that, by exploiting dispersive estimates in [C2] on e^{ith_0} and extending them to e^{itH_ω} , we are able to develop the theory of radiation in the novel context involving an NLS with a linear periodic potential. So far the literature on asymptotic stability

has treated only cases $h_0 = -\partial_x^2 + P(x)$ with $P(x)$ either 0 or a short range potential, except for the special case on Lamé potentials in [C3]. We generalize [C3] considerably, in particular by considering finite energy solutions in (1.1) without imposing further decay conditions at infinity for the initial datum $u_0(x)$ and also by easing considerably the hypotheses on $\beta(|u|^2)$. Furthermore, [C3] treats only 2 band potentials, while in this paper h_0 can have any finite number of bands. To prove Theorem 1.1 we need to use [M1]–[M2] as smoothing estimates; see also the elaborations [C1], [CT]. To prove Strichartz estimates with Birman-Solomjak rather than Lebesgue spaces, we need to adapt a well-known lemma due to Christ and Kiselev [CK]; see Lemma 3.1 of [SmS].

It would be interesting to consider the case when $\sigma(h_0)$ has infinitely many bands, but in that case H_ω can have infinitely many eigenvalues, a situation we avoid by assuming that $\sigma(h_0)$ has finitely many bands. Theorem 1.1 can be generalized (see [CM]) by assuming:

- (H7') H_ω has a certain number of simple positive eigenvalues such that for any j there is an $N_j \in \mathbb{N}$ such that $\ell\lambda_j(\omega) \notin \sigma_e(H_\omega)$ for all $\ell \in \mathbb{Z}$ with $0 \leq \ell \leq N_j$ and that $(N_j + 1)\lambda_j(\omega) \in \sigma_e(H_\omega)$ but is not a threshold of H_ω . H_ω does not have other eigenvalues, and each threshold of H_ω is not a resonance. We set $N = \max_j N_j$.
- (H8') For multi-indexes $m = (m_1, m_2, \dots)$ and $n = (n_1, \dots)$, setting $\lambda(\omega) = (\lambda_1(\omega), \dots)$ and $(m - n) \cdot \lambda = \sum (m_j - n_j)\lambda_j$, we have the following two nonresonance hypotheses:
 - (i) $(m - n) \cdot \lambda(\omega) = 0$ implies $m = n$ if $|m| \leq N + 2$ and $|n| \leq N + 2$;
 - (ii) $(m - n) \cdot \lambda(\omega)$ is not a threshold of H_ω for all (m, n) with $|m| + |n| \leq N + 2$.
- (H9') There is an appropriate substitute of Hypothesis 4.4, and we replace Lemma 4.3 with an appropriate hypothesis; see Hypothesis 4.4 of [C6].

We remark that [C1] revises [C7] but is quite different from [C7]. In [C7] Lemma 5.4 is wrong. Notice that the material in §5 of [C7] can be saved proceeding as in the present paper. To shorten the paper a bit we skip some proofs which are variations of proofs already in the literature. All details can be found in [CV2], which is an extended version of the present paper.

Notation. For $s \in \mathbb{R}$, $\|u\|_{L_x^{p,s}} := \|\langle x \rangle^s u\|_{L_x^p}$. In particular $L_x^p = L_x^{p,0}$. We set $\langle f, g \rangle = \int_{\mathbb{R}} {}^t f(x) \overline{g(x)} dx$, with $f(x)$ and $g(x)$ column vectors, ${}^t A$ the transpose and $\overline{g}$ the complex conjugate of g . Sometimes we write $\langle f, g \rangle_x$ or $\langle f, g \rangle_t$ to emphasize the variable of integration. We write $\langle f, g \rangle_{sx} = \int_{\mathbb{R}^2} {}^t f(s, x) \overline{g(s, x)} ds dx$. Given $x \in \mathbb{R}$ set $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$. $R_H(z) = (H - z)^{-1}$. $W^{k,p}(\mathbb{R})$ is the space of tempered distributions $f(x)$ such that $(1 - \partial_x^2)^{k/2} f \in L^p(\mathbb{R})$. $\mathbb{R}^- = (-\infty, 0)$, $\mathbb{R}^+ = (0, \infty)$. Given two functions $f(x)$ and $g(x)$ with values in $\mathbb{C}^2$, whose elements are column vectors, their Wronskian is the scalar $W[f, g](x) = {}^t f'(x) g(x) - {}^t f(x) g'(x)$. A pair (r, p) of numbers is said to be admissible if $(r, p) \in [4, \infty] \times [2, \infty]$ with $2/r = 1/2 - 1/p$. We set $\vec{e}_1 = {}^t(1, 0)$ and $\vec{e}_2 = {}^t(0, 1)$. Set $\text{diag}(a, b)$ for the diagonal 2×2 matrix with (a, b) on the diagonal. Given a metric space X , with $k \in X$ and $S \subseteq X$, we denote the distance of k from S by $\text{dist}(k, S)$. For a pair of Banach spaces X and Y we denote by $B(X, Y)$ the Banach space of bounded linear operators from X to Y . $S(\mathbb{R}^n)$ is the space of Schwarz functions.

2. LINEARIZATION, MODULATION AND SET UP

We will use the following theorem, which follows from classical arguments [We1], [GS1]–[GS2]; see also the proof in Appendix A of [C1], which can be repeated omitting translation:

Theorem 2.1. *Suppose that $e^{i\omega t}\phi_\omega(x)$ satisfies (H1)–(H6) and $\phi_\omega(x) > 0$ for all x . Then $\exists \epsilon > 0$ and a $A_0(\omega) > 0$ such that for any $\|u(0, x) - \phi_\omega\|_{H_x^1} < \epsilon$ we have for the corresponding solution $\inf\{\|u(t, x) - e^{i\gamma}\phi_\omega(x)\|_{H_x^1} : \gamma \in \mathbb{R}\} < A_0(\omega)\epsilon$.*

Setting $u(t, x) = e^{i\Theta(t)}(\phi_{\omega(t)}(x) + r(t, x))$, $\Theta(t) = \int_0^t \omega(s)ds + \gamma(t)$ we get

$$\begin{aligned} ir_t &= h_0 r + \omega(t)r - \beta(\phi_{\omega(t)}^2)r - \beta'(\phi_{\omega(t)}^2)\phi_{\omega(t)}^2 r \\ &\quad - \beta'(\phi_{\omega(t)}^2)\phi_{\omega(t)}^2 \bar{r} + \dot{\gamma}(t)\phi_{\omega(t)} - i\dot{\omega}(t)\partial_\omega \phi_{\omega(t)} + \dot{\gamma}(t)r + n(r, \bar{r}) \end{aligned}$$

for $n(r, \bar{r}) = O(r^2)$, $\overline{n(r, \bar{r})} = n(\bar{r}, r)$. Set ${}^tR = (r, \bar{r})$, ${}^t\Phi_\omega = (\phi_\omega, \phi_\omega)$. Given $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$, $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, we write

$$H_\omega = \sigma_3 [h_0 + \omega - \beta(\phi_\omega^2) - \beta'(\phi_\omega^2)\phi_\omega^2] + i\sigma_2\beta'(\phi_\omega^2)\phi_\omega^2.$$

The equation (1.1) is Hamiltonian, because there is a real valued function $F(|u|^2)$ with $\beta(|u|^2)u = \partial_{\bar{u}}(F(|u|^2))$ and $F(0) = 0$. Then $-n(r, \bar{r}) = \partial_{\bar{r}}G(R)$ with $G(R)$ real valued, and $G(0) = 0$. For ${}^tG'(R) = (G_r, G_{\bar{r}})$ we rewrite the above equation as

$$(2.1) \quad iR_t = H_\omega R + \sigma_3 \dot{\gamma} R + \sigma_3 \dot{\gamma} \Phi - i\dot{\omega} \partial_\omega \Phi + \sigma_3 \sigma_1 G'(R).$$

Set $H_0(\omega) = \sigma_3(h_0 + \omega)$ and $V(\omega) = H_\omega - H_0(\omega)$. The essential spectrum is

$$\sigma_e = \sigma_e(H_\omega) = \sigma_e(H_0(\omega)) = \bigcup_{\pm} \pm(\sigma(h_0) + \omega).$$

Here 0 is an isolated eigenvalue. Given an operator L we set $N_g(L) = \bigcup_{j \geq 1} N(L^j)$ and $N(L) = \ker L$. [We2] implies that, if $\{\cdot\}$ means span, $N_g(H_\omega^*) = \{\Phi, \sigma_3 \partial_\omega \Phi\}$.

For each j we consider a generator $\xi_j \in \ker(H_\omega - \lambda_j)$ such that $\langle \xi_j, \sigma_3 \xi_j \rangle = 1$. We expand $R(t) \in N_g^\perp(H_{\omega(t)}^*)$ into

$$(2.2) \quad R(t) = (z \cdot \xi + \bar{z} \cdot \sigma_1 \xi) + f(t) \in \left[\sum_{j, \pm} \ker(H_{\omega(t)} \mp \lambda_j(\omega(t))) \right] \oplus L_c^2(H_{\omega(t)}).$$

Correspondingly we express (2.1) as

$$\begin{aligned} (2.3) \quad & i\dot{z}_j \xi_j - \lambda_j(\omega) z_j \xi_j = P_{\ker(H_\omega - \lambda_j)}(\dot{\gamma}(\omega, R)\sigma_3 R + \sigma_3 \sigma_1 G'(R) \\ & - iz_j \dot{\omega}(\omega, R)\partial_\omega \xi_j + i\dot{\omega}(\omega, R)\partial_\omega P_{\ker(H_\omega - \lambda_j)} R) \\ & iP_c(H_\omega) \dot{f} - H_\omega f = P_c(H_\omega)(\dot{\gamma}(\omega, R)\sigma_3 R + \sigma_3 \sigma_1 G'(R) + i\dot{\omega}(\omega, R)\partial_\omega P_c(H_\omega) R). \end{aligned}$$

3. SPACETIME ESTIMATES FOR H_ω

We list a number of linear estimates needed in the stability argument.

Lemma 3.1 (Dispersive estimates). *There exists a projection $\pi_\omega : L_x^2 \rightarrow L_x^2$ with $[\pi_\omega, H_\omega] = 0$ such that if we set $\mathcal{U}(t) = \pi_\omega e^{itH_\omega} P_c(H_\omega)$ and $\mathcal{V}(t) \equiv (1 - \pi_\omega) e^{itH_\omega} P_c(H_\omega)$, then we have for $C(\omega) > 0$ semicontinuous in ω and for all $t \in \mathbb{R}$*

$$\|\mathcal{U}(t)\|_{B(L_x^1, L_x^\infty)} \leq C\langle t \rangle^{-\frac{1}{3}} \text{ and } \|\mathcal{V}(t)\|_{B(L_x^1, L_x^\infty)} \leq C|t|^{-\frac{1}{2}}.$$

The proof is sketched in §9.

For every $1 \leq p, q \leq \infty$ we introduce the Birman-Solomjak spaces

$$\ell^p(\mathbb{Z}, L_t^q[n, n+1]) \equiv \{f \in L_{loc}^q(\mathbb{R}) \text{ s.t. } \{\|f\|_{L_t^q[n, n+1]}\}_{n \in \mathbb{Z}} \in \ell^p(\mathbb{Z})\},$$

endowed with the norms

$$\begin{aligned} \|f\|_{\ell^p(\mathbb{Z}, L_t^q[n, n+1])}^p &\equiv \sum_{n \in \mathbb{Z}} \|f\|_{L_t^q[n, n+1]}^p \quad \forall \quad 1 \leq p < \infty \text{ and } 1 \leq q \leq \infty, \\ \|f\|_{\ell^\infty(\mathbb{Z}, L_t^q[n, n+1])} &\equiv \sup_{n \in \mathbb{Z}} \|f\|_{L_t^q[n, n+1]}. \end{aligned}$$

Then we have:

Lemma 3.2 (Strichartz estimates). *There exists a constant $C = C(\omega)$ upper semi-continuous in ω such that for any $k \in [0, 2]$ and for every admissible pair (r, p) we have*

$$\begin{aligned} (1) \quad & \|\mathcal{U}(t)f\|_{\ell^{\frac{3}{2}r}(\mathbb{Z}, L_t^\infty([n, n+1], W_x^{k,p}))} \leq C\|f\|_{H_x^k}, \\ (2) \quad & \|\mathcal{V}(t)f\|_{L_t^r(\mathbb{R}, W_x^{k,p})} \leq C\|f\|_{H_x^k}. \end{aligned}$$

For any two admissible pairs $(r_1, p_1), (r_2, p_2)$ we have

$$\begin{aligned} (3) \quad & \left\| \int_0^t \mathcal{U}(t-s)g(s)ds \right\|_{\ell^{\frac{3}{2}r_1}(\mathbb{Z}, L_t^\infty([n, n+1], W_x^{k,p_1}))} \\ & \leq C\|g\|_{\ell^{(\frac{3}{2}r_2)'}(\mathbb{Z}, L_t^1([n, n+1], W_x^{k,p_2'}))}, \\ (4) \quad & \left\| \int_0^t \mathcal{V}(t-s)g(s)ds \right\|_{L_t^{r_1}(\mathbb{R}, W_x^{k,p_1})} \leq C\|g\|_{L_t^{r_2'}(\mathbb{R}, W_x^{k,p_2'})}. \end{aligned}$$

Lemma 3.2 is a consequence of Lemma 3.1 and an adapted TT^* argument; see Lemma 3.1 of [C1] and our §9 for the proof.

In §7 we prove the following Kato smoothness result:

Lemma 3.3. *Fix $\tau > 3/2$. Then:*

- (1) *there exists $C = C(\tau, \omega)$, upper semicontinuous in ω , such that for any $\varepsilon \neq 0$*

$$\|R_{H_\omega}(\lambda + i\varepsilon)P_c(H_\omega)u\|_{L_\lambda^2 L_x^{2, -\tau}} \leq C\|u\|_{L_x^2};$$

- (2) *for any $u \in L_x^{2, \tau}$ the following limits exist:*

$$\lim_{\varepsilon \searrow 0} R_{H_\omega}(\lambda \pm i\varepsilon)u = R_{H_\omega}^\pm(\lambda)u \text{ in } C^0(\sigma_e(H_\omega), L_x^{2, -\tau});$$

- (3) *we have*

$$\|R_{H_\omega}^\pm(\lambda)P_c(H_\omega)\|_{B(L_x^{2, \tau}, L_x^{2, -\tau})} < C\langle \lambda \rangle^{-\frac{1}{2}};$$

- (4) *given any $u \in L_x^{2, \tau}$ we have*

$$P_c(H_\omega)u = \frac{1}{2\pi i} \int_{\sigma_e(H_\omega)} (R_{H_\omega}^+(\lambda) - R_{H_\omega}^-(\lambda))u d\lambda.$$

Claim (1) is proved as inequality (4) in Proposition 7.1. Claim (2) is proved as formula (12) in Proposition 7.1. Claim (3) is a consequence of the proof in Proposition 7.1. The limiting absorption principle Claim (4) is Lemma 7.6 below.

Lemma 3.4. *For any k and $\tau > 3/2$, $\exists C = C(\tau, k, \omega)$ upper semicontinuous in ω such that:*

(a) *for any $f \in \mathcal{S}(\mathbb{R})$,*

$$\|e^{-itH_\omega} P_c(H_\omega) f\|_{L_t^2 H_x^{k, -\tau}} \leq C \|f\|_{H_x^k};$$

(b) *for any $g(t, x) \in \mathcal{S}(\mathbb{R}^2)$,*

$$\left\| \int_{\mathbb{R}} e^{itH_\omega} P_c(H_\omega) g(t, \cdot) dt \right\|_{H_x^k} \leq C \|g\|_{L_t^2 H_x^{k, \tau}}.$$

Proof. By Proposition 7.1 below, it is enough to prove Lemma 3.4, as well as Lemmas 3.5-3.6 below, for $k = 0$. (a) implies (b) by duality:

$$\begin{aligned} |\langle f, \sigma_3 \int_{\mathbb{R}} e^{itH_\omega} P_c(H_\omega) g(t) dt \rangle_x| &= |\langle \langle x \rangle^{-\tau} e^{-itH_\omega} P_c(H_\omega) f, \sigma_3 \langle x \rangle^\tau g \rangle_{t,x}| \\ &\leq \|e^{-itH_\omega} P_c(H_\omega) f\|_{L_t^2 L_x^{2, -\tau}} \|g\|_{L_t^2 L_x^{2, \tau}} \leq \|f\|_{L_x^2} \|g\|_{L_t^2 L_x^{2, \tau}}. \end{aligned}$$

We now prove (a) for $k = 0$. Let $g(t, x) \in \mathcal{S}(\mathbb{R}^2)$ with $g(t) = P_c(H_\omega) g(t)$. Then

$$\begin{aligned} \langle e^{-itH_\omega} f, \sigma_3 g \rangle_{t,x} &= \frac{1}{\sqrt{2\pi i}} \int_{\mathbb{R}} \left\langle (R_{H_\omega}^+(\lambda) - R_{H_\omega}^-(\lambda)) f, \sigma_3 \tilde{g}(\lambda) \right\rangle_x d\lambda \\ &= \frac{1}{\sqrt{2\pi i}} \int_{\sigma_e(H_\omega)} \left\langle (R_{H_\omega}^+(\lambda) - R_{H_\omega}^-(\lambda)) f, \sigma_3 \tilde{g}(\lambda) \right\rangle_x d\lambda. \end{aligned}$$

Then from Fubini and Plancherel and by (1) of Lemma 3.3 we have

$$|\langle e^{-itH_\omega} f, \sigma_3 g \rangle_{t,x}| \leq \|(R_{H_\omega}^+(\lambda) - R_{H_\omega}^-(\lambda)) f\|_{L_x^{2, -\tau} L_\lambda^2(\sigma_e(H_\omega))} \|g\|_{L_x^{2, \tau} L_t^2}.$$

We have

$$\begin{aligned} &\|(R_{H_\omega}^+(\lambda) - R_{H_\omega}^-(\lambda)) f\|_{\ell^{2, -\tau} L_\lambda^2(\sigma_e(H_\omega))} \\ &= \lim_{\varepsilon \searrow 0} \|(R_{H_\omega}(\lambda + i\varepsilon) - R_{H_\omega}(\lambda - i\varepsilon)) f\|_{\ell^{2, -\tau} L_\lambda^2(\sigma_e(H_\omega))}. \end{aligned}$$

It is enough to prove Lemma 3.4 for $f = P_c(H_\omega) f$. Then

$$\|R_{H_\omega}(\lambda + i\varepsilon) f\|_{\ell^{2, -\tau} L_\lambda^2(\sigma_e(H_\omega))} \leq C \|f\|_{\ell^2}$$

for any $\varepsilon \neq 0$ and fixed C follows from inequality (4) in the proof of Proposition 8.1 below.

Lemma 3.5. *For any $k \in \mathbb{N}$ and $\tau > 3/2$, $\exists C = C(\tau, k, \omega)$ such that $\forall g(t, x) \in \mathcal{S}(\mathbb{R}^2)$*

$$\left\| \int_0^t e^{-i(t-s)H_\omega} P_c(H_\omega) g(s, \cdot) ds \right\|_{L_t^2 H_x^{k, -\tau}} \leq C \|g\|_{L_t^2 H_x^{k, \tau}}.$$

Proof. By Plancherel and Hölder inequalities and by (3) of Lemma 3.3 we have

$$\begin{aligned} &\left\| \int_0^t e^{-i(t-s)H_\omega} P_c(H_\omega) g(s, \cdot) ds \right\|_{L_t^2 L_x^{2, -\tau}} \\ &\leq \|R_{H_\omega}^+(\lambda) P_c(H_\omega) \hat{\chi}_{[0, +\infty)} * \hat{g}(\lambda, x)\|_{L_\lambda^2 L_x^{2, -\tau}} \\ &\leq \left\| \|R_{H_\omega}^+(\lambda) P_c(H_\omega)\|_{B(L_x^{2, \tau}, L_x^{2, -\tau})} \hat{\chi}_{[0, +\infty)} * \hat{g}(\lambda, x) \right\|_{L_x^{2, \tau}} \Big\|_{L_\lambda^2} \\ &\leq \|R_{H_\omega}^+(\lambda) P_c(H_\omega)\|_{L_\lambda^\infty(\mathbb{R}, B(L_x^{2, \tau}, L_x^{2, -\tau}))} \|g\|_{L_t^2 L_x^{2, \tau}} \leq C \|g\|_{L_t^2 L_x^{2, \tau}}. \end{aligned}$$

Lemma 3.6. *For any $k \in \mathbb{N}$ and $\tau > 3/2$, $\exists C = C(\tau, k, \omega)$ such that $\forall g(t, x) \in \mathcal{S}(\mathbb{R}^2)$*

$$(1) \quad \left\| \int_0^t \mathcal{U}(t-s)g(s, \cdot)ds \right\|_{L_t^\infty L_x^2 \cap \ell^6(\mathbb{Z}, L_t^\infty([n, n+1], W_x^{k, \infty}))} \leq C \|g\|_{L_t^2 H_x^{k, \tau}},$$

$$(2) \quad \left\| \int_0^t \mathcal{V}(t-s)g(s, \cdot)ds \right\|_{L_t^\infty L_x^2 \cap L_t^4 W_x^{k, \infty}} \leq C \|g\|_{L_t^2 H_x^{k, \tau}}.$$

Proof. For $g(t, x) \in \mathcal{S}(\mathbb{R}^2)$ set

$$T_1 g(t) = \int_0^{+\infty} \mathcal{U}(t-s)g(s)ds \text{ and } T_2 g(t) = \int_0^{+\infty} \mathcal{V}(t-s)g(s)ds.$$

Lemma 3.4 (b) implies $f := \int_0^{+\infty} e^{isH_\omega} P_c(H_\omega)g(s)ds \in L_x^2$. So in particular $\pi_\omega f$ and $(1 - \pi_\omega)f$ belong to L_x^2 . For (r, p) admissible we have

$$\begin{aligned} \|T_1 g(t)\|_{\ell^{\frac{3}{2}r}(\mathbb{Z}, L_t^\infty([n, n+1], L_x^p))} &\lesssim \|\pi_\omega f\|_{L_x^2} \lesssim \|g\|_{L_t^2 L_x^{2, \tau}}, \\ \|T_2 g(t)\|_{L_t^r L_x^p} &\lesssim \|(1 - \pi_\omega)f\|_{L_x^2} \lesssim \|g\|_{L_t^2 L_x^{2, \tau}}. \end{aligned}$$

We get as a direct consequence of [CK], Lemma 3.1 of [SmS]:

$$\left\| \int_0^t \mathcal{V}(t-s)g(s)ds \right\|_{L_t^r L_x^p} \lesssim \|g\|_{L_t^2 L_x^{2, \tau}}.$$

Our next claim is that

$$\left\| \int_0^t \mathcal{U}(t-s)g(s)ds \right\|_{\ell^{\frac{3}{2}r}(\mathbb{Z}, L_t^\infty([n, n+1], L_x^p))} \lesssim \|g\|_{L_t^2 L_x^{2, \tau}}.$$

For $(r, p) = (\infty, 2)$ this follows from [CK]. The case $(r, p) = (4, \infty)$ follows from an extension to Birman-Solomjak spaces of the result of [CK], which we prove in §10.

4. NONLINEAR ARGUMENT FOR SYSTEM (2.3)

We use the multi-index notation $z^m = \prod_j z_j^{m_j}$. We consider the Taylor expansion

$$\sigma_3 \sigma_1 G'(R) = \sum_{|m+n|=2}^3 R_{m,n}(\omega) z^m \bar{z}^n + \sum_{|m+n|=1} z^m \bar{z}^n A_{m,n}(\omega) f + O(f^2) + \dots$$

with $R_{m,n}(\omega, x)$ and $A_{m,n}(\omega, x)$ real vectors and matrices exponentially decreasing in x . We have

$$A_{m,n}(\omega) = \frac{\sigma_3 \sigma_1}{m!n!} \partial_z^m \partial_{\bar{z}}^n \partial_f G'(0), \quad R_{m,n}(\omega) = \frac{\sigma_3 \sigma_1}{m!n!} \partial_z^m \partial_{\bar{z}}^n G'(0).$$

We set $\delta_j = (\delta_{j1}, \delta_{j2}, \dots)$ with δ_{jk} the Kronecker delta. We have

$$A_{\delta_\ell, 0}(\omega) = \sigma_3 \sigma_1 \partial_{z_\ell} \partial_f G'(0) = \sigma_3 \sigma_1 \partial_{z_\ell} \partial_f G'(0) = \sigma_3 \sigma_1 G^{(3)}(0)(\cdot, \xi_\ell, P_c(H_\omega)),$$

where $G^{(3)}(0)$ is written as a symmetric trilinear form and where one of the vectors of the triple is ξ_ℓ . We have

$$\begin{aligned} P_c(H_\omega) R_{\delta_j + \delta_\ell, 0}(\omega) &= \frac{\sigma_3 \sigma_1}{(\delta_j + \delta_\ell)!} P_c(H_\omega^*) \partial_{z_j} \partial_{z_\ell} G'(0) \\ &= \frac{\sigma_3 \sigma_1}{(\delta_j + \delta_\ell)!} \partial_{z_j} \partial_{z_\ell} G'(0) \circ P_c(H_\omega) = \frac{\sigma_3 \sigma_1}{(\delta_j + \delta_\ell)!} G^{(3)}(0)(\xi_j, \xi_\ell, P_c(H_\omega)). \end{aligned}$$

For $O_{loc}(z^n) = \sum_{\ell} O_{loc}(|z_{\ell}^n|)$, (2.1) can be expressed as

$$(4.1a) \quad \begin{aligned} if_t &= (H_{\omega(t)} + P_c(H_{\omega})\sigma_3\dot{\gamma})f + \sum_{|m+n|=2} z^m \bar{z}^n P_c(H_{\omega})R_{m,n}(\omega) \\ &+ \sum_{|m+n|=1} z^m \bar{z}^n P_c(H_{\omega})A_{m,n}(\omega)f + O(f^2) + O_{loc}(z^3) \end{aligned}$$

and

$$(4.1b) \quad \begin{aligned} i\dot{z}_j \xi_j - \lambda_j(\omega)z_j \xi_j &= P_{\ker(H_{\omega}-\lambda_j)} \left(\sum_{|m+n|=2} z^m \bar{z}^n R_{m,n}(\omega) \right. \\ &+ \sum_{|m+n|=3} z^m \bar{z}^n R_{m,n}^{(1)}(\omega) + \sum_{|m+n|=1} z^m \bar{z}^n A_{m,n}(\omega)f \Big) + O(f^2) + O_{loc}(z^4) \end{aligned}$$

with the $A_{m,n}(\omega)$ and $R_{m,n}(\omega)$ the same as the expansion of $\sigma_3\sigma_1 G'(R)$ and with the $R_{m,n}^{(1)}(\omega)$ real and exponentially decreasing vectors. We have:

Lemma 4.1. *For any $C_1 > 0$, $\exists \epsilon(C_1) > 0$ and $C(C_1)$ such that if, for $0 < \epsilon < \epsilon(C_1)$, we have $\|z_j\|_{L^{\infty}(0,T)} + \|z_j\|_{L^4(0,T)}^2 \leq C_1\epsilon$ for all j , then*

$$\|f\|_{L_t^2((0,T),H_x^{1,-2})} + \|f\|_{L_t^6((0,T),L_x^{\infty}) \cap L_t^{\infty}((0,T),H_x^1)} < C(C_1)\epsilon.$$

Proof. Notice that $\sum_j \|z_j\|_{L_t^{\infty}} + \|f\|_{L_t^{\infty}H_x^1} < c_0\epsilon$ for a fixed c_0 is a consequence of orbital stability; see Theorem 2.1 above. In this proof we set $P_c(\omega) = P_c(H_{\omega})$. We split $P_c(\omega) = P_+(\omega) + P_-(\omega)$ with $P_{\pm}(\omega)$ the spectral projections in $\mathbb{R}_{\pm} \cap \sigma_e(H_{\omega})$; see Lemma 8.13 below. We now rewrite the equation for f . By orbital stability, Theorem 2.1, we can fix ω_0 such that $|\omega(t) - \omega_0| = O(\epsilon)$ for all t . Following [BP2] we set

$$\begin{aligned} if_t &= \{H_{\omega_0} + (\dot{\gamma} + \omega - \omega_0)(P_+(\omega_0) - P_-(\omega_0))\} P_c(\omega_0)f \\ &+ O_{loc}(\epsilon f) + O(f^2) + O_{loc}(z^2), \\ \text{where } O_{loc}(\epsilon f) &= (\dot{\gamma} + \omega - \omega_0)(P_c(\omega_0)\sigma_3 - (P_+(\omega_0) - P_-(\omega_0)))f \\ &+ (V(\omega) - V(\omega_0))f + (\dot{\gamma} + \omega - \omega_0)(P_c(\omega) - P_c(\omega_0))\sigma_3 f, \end{aligned}$$

with $V(\omega)$ the localized matrix potential defined under (2.1) and where the notation $O_{loc}(\epsilon f)$ is justified by the fact that $\omega - \omega_0 = O(\epsilon)$. Also, for any pair $s_1, s_2 \in \mathbb{R}$ there is $c_{s_1, s_2}(\omega)$ upper semicontinuous in ω (see Lemma 8.13) such that for $j = 0, 1$,

$$\|P_c(\omega)\sigma_3 - (P_+(\omega) - P_-(\omega))\|_{H_x^{j, s_1} \rightarrow H_x^{j, s_2}} \leq c_{s_1, s_2}(\omega) < \infty,$$

where the case $j = 1$ follows from $j = 0$ by applying H_{ω} and interpolating. We have $\|f\|_{L_t^6((0,T),L_x^{\infty})} \approx \|P_c(\omega_0)f\|_{L_t^6((0,T),L_x^{\infty})}$ by

$$\begin{aligned} \|f\|_{L_t^6((0,T),L_x^{\infty})} &= \|P_c(\omega_0)f\|_{L_t^6((0,T),L_x^{\infty})} + \|(P_c(\omega) - P_c(\omega_0))f\|_{L_t^6((0,T),L_x^{\infty})} \\ &= \|P_c(\omega_0)f\|_{L_t^6((0,T),L_x^{\infty})} + O(\epsilon)\|f\|_{L_t^6((0,T),L_x^{\infty})}. \end{aligned}$$

For

$$(4.2) \quad U_{\pm}(t, t') = e^{-i(t-t')H_{\omega_0}} e^{\pm i \int_{t'}^t d\tau (\dot{\gamma}(\tau) + \omega(\tau) - \omega_0)} P_{\pm}(\omega_0),$$

we have

$$P_{\pm}(\omega_0)f(t) = U_{\pm}(t, 0)f(0) + \int_0^t U_{\pm}(t, t')(O_{loc}(\epsilon f) + O(f^2) + O_{loc}(z^2))dt'.$$

We set $O_{loc}(\epsilon f) + O(f^2) + O_{loc}(z^2) = \mathcal{X} + O(f^7)$. Let $\pi = \pi_{\omega_0}$ be the projection in Lemmas 3.1–3.2 and $\pi_1 = 1 - \pi$. By (1)–(2) of Lemma 3.2 and interpolation, we have for a fixed $c_0(\omega_0)$

$$\begin{aligned} \|U_{\pm}(t, 0)f(0)\|_{L_t^6((0, T), L_x^{\infty})} &\leq \|U_{\pm}(t, 0)f(0)\|_{L_t^6 L_x^{\infty}} \\ &\lesssim \|U_{\pm}(t, 0)\pi f(0)\|_{\ell^6(\mathbb{Z}, L_t^{\infty}([n, n+1], L_x^{\infty}))} + \|U_{\pm}(t, 0)\pi_1 f(0)\|_{L_t^4 L_x^{\infty}}^{\frac{2}{3}} \\ &\quad \times \|U_{\pm}(t, 0)\pi_1 f(0)\|_{L_t^{\infty} H_x^1}^{\frac{1}{3}} \leq c_0(\omega_0)\|f(0)\|_{H_x^1} \leq c_0(\omega_0)\epsilon. \end{aligned}$$

Similarly, by (3)–(4) of Lemma 3.2 and interpolation, we have

$$\begin{aligned} &\left\| \int_0^t U_{\pm}(t, t')O(f^7)dt' \right\|_{L_t^6((0, T), L_x^{\infty})} \lesssim \left\| \int_0^t U_{\pm}(t, t')\pi_1 O(f^7)dt' \right\|_{L_t^4((0, T), L_x^{\infty})}^{\frac{2}{3}} \\ &\quad \times \left\| \int_0^t U_{\pm}(t, t')\pi_1 O(f^7)dt' \right\|_{L_t^{\infty}((0, T), H_x^1)}^{\frac{1}{3}} \\ &\quad + \left\| \chi_{[0, T]} \int_0^t U_{\pm}(t, t')\pi O(f^7)dt' \right\|_{\ell^6(\mathbb{Z}, L_t^{\infty}([n, n+1], L_x^{\infty}))}. \end{aligned}$$

By Lemma 3.2 both terms on the right are bounded by $\|O(f^7)\|_{L_t^1((0, T), H_x^1)}$. Therefore, by Lemma 3.2 and Hölder inequality we have

$$\begin{aligned} &\left\| \int_0^t U_{\pm}(t, t')O(f^7)dt' \right\|_{L_t^6((0, T), L_x^{\infty})} \leq c_0\|O(f^7)\|_{L_t^1((0, T), H_x^1)} \\ &\lesssim \|f\|_{L_t^6((0, T), L_x^{\infty})}^6 \|f\|_{L_t^{\infty}(0, T)H_x^1} \lesssim \epsilon^7. \end{aligned}$$

In a similar fashion, by Lemma 3.6 and by (1) below we have

$$\left\| \int_0^t U_{\pm}(t, t')\mathcal{X}dt' \right\|_{L_t^6((0, T), L_x^{\infty})} \leq c_0(\omega)\|\mathcal{X}\|_{L_t^2((0, T), H_x^{1,2})} \lesssim \epsilon^2.$$

We claim that we have

$$(1) \quad \|\mathcal{X}\|_{L_t^2 H_x^{1,2}} \leq C(\omega) \left(\epsilon^2 + \|f\|_{L_t^2 H_x^{1,-2}}^2 \right).$$

Indeed $\mathcal{X} = O_{loc}(\epsilon f) + O_{loc}(zf) + O_{loc}(f^2) + O_{loc}(z^2)$ with

$$\begin{aligned} \|O_{loc}(\epsilon f)\|_{H_x^{1,2} L_t^2} &\lesssim \epsilon \|f\|_{L_t^2 H_x^{1,-2}}; \quad \|O_{loc}(zf)\|_{H_x^{1,2} L_t^2} \lesssim \|z\|_{\infty} \|f\|_{L_t^2 H_x^{1,-2}}; \\ \|O_{loc}(f^2)\|_{L_t^2 H_x^{1,2}} &\lesssim \|f\|_{L_t^2 H_x^{1,-2}}^2; \quad \|O_{loc}(|z|^2)\|_{H_x^{1,2} L_t^2} \lesssim \epsilon \|z^2\|_{L_t^2} \lesssim \epsilon^2. \end{aligned}$$

Hence (1) is correct. Now let $f = g + h$ with

$$\begin{aligned} ig_t &= \{H_{\omega_0} + \ell(t)(P_+(\omega_0) - P_-(\omega_0))\}g + \mathcal{X}, \quad g(0) = f(0), \\ ih_t &= \{H_{\omega_0} + \ell(t)(P_+(\omega_0) - P_-(\omega_0))\}h + O(f^7), \quad h(0) = 0. \end{aligned}$$

Then, by Lemmas 3.4 and 3.5 we get for a fixed c_0

$$\|g\|_{L_t^2((0, T), H_x^{1,-2})} \lesssim c_0(C_1 + 1)\epsilon + O(\epsilon^2) \leq C(C_1)\epsilon.$$

Finally, by Lemma 3.4

$$\begin{aligned} & \int_0^T \|e^{-i(t-s)H_{\omega_0}} e^{\pm i \int_s^t \ell(\tau) d\tau} O(f^7)(s)\|_{L_t^2((0,T), H_x^{1,-2})} \\ & \lesssim \int_0^T \|O(f^7)(s)\|_{H_x^1} ds \lesssim \epsilon^7. \end{aligned}$$

This yields $\|h\|_{L_t^2((0,T), H_x^{1,-2})} \lesssim \epsilon^7$ and completes the proof of Lemma 4.1.

Having obtained Lemma 4.1, we rewrite (4.1a) in the form

$$\begin{aligned} if_t &= \{H_{\omega_0} + (\dot{\gamma} + \omega - \omega_0)(P_+(\omega_0) - P_-(\omega_0))\} f \\ &+ \sum_{|m+n|=2} z^m \bar{z}^n P_c(\omega_0) R_{m,n}(\omega) + \sum_{|m+n|=1} z^m \bar{z}^n P_c(\omega_0) A_{m,n}(\omega) f \\ &+ (\dot{\gamma} + \omega - \omega_0) (P_c(\omega_0) \sigma_3 - (P_+(\omega_0) - P_-(\omega_0))) f + (V(\omega) - V(\omega_0)) f \\ &+ (\dot{\gamma} + \omega - \omega_0) (P_c(\omega) - P_c(\omega_0)) \sigma_3 f + O(f^2) + O_{loc}(z^3) \\ &+ (P_c(\omega) - P_c(\omega_0)) \left(\sum_{|m+n|=2} z^m \bar{z}^n R_{m,n}(\omega) + \sum_{|m+n|=1} z^m \bar{z}^n A_{m,n}(\omega) f \right). \end{aligned}$$

We then set

$$(4.3) \quad f_2 = f + \sum_{|m+n|=2} R_{H_{\omega_0}}^+((m-n) \cdot \lambda(\omega_0)) P_c(H_{\omega_0}) R_{m,n}(\omega) z^m \bar{z}^n.$$

We will need below:

Lemma 4.2. *Assume the hypotheses of Lemma 4.1. Then for $s > 1$ sufficiently large we can decompose $f_2 = h_1 + h_2 + h_3 + h_4$ with:*

- (1) *for a fixed $c_0(\omega_0)$, $\|h_1\|_{L_t^2 L_x^{2,-s}} \leq c_0(\omega_0) \|f(0)\|_{H^1} \leq c_0(\omega_0) \epsilon$;*
- (2) *for a fixed $c_1(\omega_0)$, $\|h_2\|_{L_t^2 L_x^{2,-s}} \leq c_1(\omega_0) |z(0)|^2 \leq c_1(\omega_0) \epsilon^2$;*
- (3) *$\|h_3\|_{L_t^2 L_x^{2,-s}} = O(\epsilon^2)$;*
- (4) *$\|h_4\|_{L_t^2 L_x^{2,-s}} = O(\epsilon^2)$.*

Proof. The proof is basically that of §4 in [CM]. It uses the fact that for a constant $C = C(\Lambda, \omega_0)$ upper semicontinuous in ω_0 and in $\Lambda > \omega_0$ we have

$$(5) \quad \|U_{\pm}(t, t') R_{H_{\omega}}^+(\Lambda) P_c g\|_{L_x^{2,-s}} < C \langle t - t' \rangle^{-\frac{3}{2}} \|g\|_{L_x^{2,s}}, \quad s > s_0.$$

The weighted estimate (5) can be proved proceeding along the lines of the proof of Theorem 2.4 on pp. 135–136 of [C3] by using the estimates for the Jost functions in Lemma 9.1 and the representation in Lemma 8.12.

By substitution of (4.3) in (4.1b) we get

$$\begin{aligned} i\dot{z}_j \xi_j - \lambda_j(\omega) z_j \xi_j &= P_{\ker(H_{\omega} - \lambda_j)} \left(\sum_{|m+n|=2} z^m \bar{z}^n R_{m,n}(\omega) + \sum_{|m+n|=3} z^m \bar{z}^n R_{m,n}^{(1)}(\omega) \right. \\ &- \sum_{|m'+n'|=1} \sum_{|m+n|=2} z^{m+m'} \bar{z}^{n+n'} A_{m',n'}(\omega) R_{H_{\omega_0}}^+((m-n) \cdot \lambda(\omega_0)) P_c(H_{\omega}) R_{m,n}(\omega) \\ &+ \sum_{|m+n|=1} z^m \bar{z}^n A_{m,n}(\omega) f_2 + O(f^2) + O_{loc}(z^4) \Big). \end{aligned}$$

Here we recall $P_{\ker(H_\omega - \lambda_j)} = \xi_j \langle \cdot, \sigma_3 \xi_j \rangle$. By standard normal forms arguments there exists a change of variables $\zeta_j = z_j + \sum_{|m+n|=2}^3 p_{j,m,n}(\omega) z^m \bar{z}^n$ with $\Im p_{j,m,n} = 0$ for $|m+n| = 2$ such that below we have $\Im a_{j,\ell}(\omega) = 0$ and

$$(4.4) \quad \begin{aligned} i\dot{\zeta}_j \xi_j - \lambda_j(\omega) \zeta_j \xi_j &= \xi_j \sum_{\ell} a_{j,\ell}(\omega) |\zeta_\ell|^2 \zeta_j + O_{loc}(\zeta f_2) + O(f^2) + O_{loc}(\zeta^4) \\ &- \sum_{\ell} |\zeta_\ell|^2 \zeta_j P_{\ker(H_\omega - \lambda_j)} A_{0,\delta_\ell}(\omega) R_{H_{\omega_0}}^+(\lambda_\ell(\omega_0) + \lambda_j(\omega_0)) P_c(H_\omega) R_{\delta_\ell + \delta_j, 0}(\omega), \end{aligned}$$

where $O_{loc}(\zeta^n f_2) = \sum_{\ell} O_{loc}(\zeta_\ell^n f_2)$. Applying $\langle \cdot, \sigma_3 \xi_j \rangle$ to (4.4) we get

$$(4.5) \quad \begin{aligned} i\dot{\zeta}_j - \lambda_j(\omega) \zeta_j &= \sum_{\ell} a_{j,\ell}(\omega) |\zeta_\ell|^2 \zeta_j + \langle O_{loc}(\zeta f_2) + O(f^2) + O_{loc}(\zeta^4), \sigma_3 \xi_j \rangle \\ &- \sum_{\ell} |\zeta_\ell|^2 \zeta_j \langle A_{0,\delta_\ell}(\omega) R_{H_{\omega_0}}^+(\lambda_\ell(\omega_0) + \lambda_j(\omega_0)) P_c(H_\omega) R_{\delta_\ell + \delta_j, 0}(\omega), \sigma_3 \xi_j \rangle. \end{aligned}$$

Recall that $\Im a_{j,\ell}(\omega) = 0$. Then multiplying (4.5) by $\bar{\zeta}_j$ we get

$$(4.6) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} |\zeta_j|^2 &= - \sum_{\ell} (\Gamma_{j,\ell}(\omega_0) + o(1)) |\zeta_\ell|^2 |\zeta_j|^2 \\ &+ \Im [\bar{\zeta}_j \langle O_{loc}(\zeta f_2) + O(f^2) + O_{loc}(\zeta^4), \sigma_3 \xi_j \rangle], \end{aligned}$$

where we use $\omega - \omega_0 = O(\epsilon)$,

$$\Gamma_{j,\ell}(\omega) = \Im \langle A_{0,\delta_\ell}(\omega) R_{H_\omega}^+(\lambda_\ell(\omega) + \lambda_j(\omega)) P_c(H_\omega) R_{\delta_\ell + \delta_j, 0}(\omega), \sigma_3 \xi_j(\omega) \rangle$$

and the continuous dependence in ω of $A_{0,\delta_\ell}(\omega)$, $R_{\delta_\ell + \delta_j, 0}(\omega)$ and $P_c(H_\omega)$. We have:

Lemma 4.3. *We have for $h_{j,\ell}(\omega) = \sigma_3 \sigma_1 G^{(3)}(0)(\xi_j, \xi_\ell, P_c(H_\omega))$*

$$\Gamma_{j,\ell}(\omega) = \frac{\pi}{(\delta_j + \delta_\ell)!} \langle \delta(H_\omega - \lambda_j - \lambda_\ell) h_{j,\ell}(\omega), \sigma_3 h_{j,\ell}(\omega) \rangle \geq 0.$$

The proof is in Lemma 3.4 of [C6]. We assume the following, which by $2\lambda_j(\omega) > \omega$ for any j looks like a generic condition:

Hypothesis 4.4. *We suppose that $\Gamma_{j,j}(\omega) > 0$ for any j .*

Integrating (4.6) in an interval $[0, t]$ we get

$$\begin{aligned} C_0 \sum_j \int_0^t |\zeta_j|^4 + \sum_j |\zeta_j(t)|^2 / 2 \\ \leq \sum_j |\zeta_j(0)|^2 / 2 + c_1 \left(\sum_j \int_0^t |\zeta_j|^4 \right)^{\frac{1}{2}} \|f_2\|_{L_t^2 L_x^{2,-s}} + O(\epsilon^3). \end{aligned}$$

By Lemma 4.2, $\|f_2\|_{L_t^2 L_x^{2,-s}} \leq c_2 \epsilon$ for $c_2 \approx 1$. Then for a fixed $c \approx 1$

$$\sum_j \int_0^t |z_j|^4 + \sum_j |z_j(t)|^2 \leq c \epsilon^2.$$

The proof that, for ${}^t f(t) = (h(t), \bar{h}(t))$, $h(t)$ is asymptotically free for $t \rightarrow \infty$, is similar to the analogous one in [CM]. Finally, by the same argument of [BP2], [BS], and [CM], $\lim_{t \nearrow \infty} \omega(t) = \omega_+$ for some ω_+ . Hence expressing (1) in components we obtain Theorem 1.1.

5. REVIEW ON BLOCH FUNCTIONS FOR h_0

By hypothesis for $\sigma(h_0)$ there exist $A_0^\pm = 0 < A_1^- < A_1^+ < \dots < A_{n_0}^- < A_{n_0}^+ < \infty$ such that

$$\sigma(h_0) = \bigcup_{j=0}^{n_0-2} [A_j^+, A_{j+1}^-] \cup [A_{n_0}^+, \infty).$$

The sets (A_j^-, A_j^+) for $j = 1, \dots, n_0$ are nonempty spectral gaps. For $j \geq 0$ we set $a_j^\pm = \sqrt{A_j^\pm}$. For $j \leq 0$, $a_j^\pm = -a_{-j}^\pm$. For any $E \in \mathbb{C} \setminus [0, \infty)$ there is a unique $k \in \mathbb{C}_+$, called quasimomentum, such that $h_0 u = Eu$ has a solution of the form $\tilde{\phi}_\pm(x, k) = e^{\pm i k x} \tilde{m}_\pm(x, k)$ with $\tilde{m}_\pm(x+1, k) \equiv \tilde{m}_\pm(x, k)$ and $\tilde{m}_\pm(0, k) = 1$. The correspondence $E \rightarrow k(E)$ is a conformal map from $\mathbb{C} \setminus [0, \infty)$ into a set

$$\mathcal{K} = \mathbb{C}_+ \setminus \left(\bigcup_{j=1}^{n_0} ([n(j)\pi, n(j)\pi + i h_j] \cup [-n(j)\pi, -n(j)\pi + i h_j]) \right),$$

with $n(j)$ strictly increasing for $j = 1, \dots, n_0$ and $h_j > 0$. We set $\mathcal{T} = \{\pm \pi n(j) : j = 1, \dots, n_0\}$. The inverse map $E = E(k)$ extends in an even map $\mathbb{R} \setminus \mathcal{T} \rightarrow \sigma(h_0)$. We have

$$E(\pi n(j) \pm 0) := \lim_{\varepsilon \searrow 0} E(n(j)\pi \pm \varepsilon) = \pi A_j^\pm.$$

$E(k)$ defined in $[\pi n(j), \pi n(j+1)]$ extends continuously in $[\pi n(j), \pi n(j+1)]$. The same holds for $[\pi n(n_0), \infty)$. We consider the solutions $\theta(x, k)$ and $\varphi(x, k)$ of $h_0 u = E(k)u$ which satisfy the initial conditions $\varphi(0, k) = \theta'(0, k) = 0$ and $\varphi'(0, k) = \theta(0, k) = 1$. Then

$$\tilde{\phi}_\pm(x, k) = \theta(x, k) + m^\pm(k) \varphi(x, k) \text{ with } m^\pm(k) = \frac{\varphi'(1, k) - \theta(1, k)}{2\varphi(1, k)} \pm i \frac{\sin k}{\varphi(1, k)}.$$

For real $k \in \mathbb{R} \setminus \mathcal{T}$ we have $\tilde{\phi}_\pm(x, k) = \overline{\tilde{\phi}_\mp(x, k)}$. Now set

$$N^2(k) = \int_0^1 \tilde{\phi}_+(x, k) \tilde{\phi}_-(x, k) dx.$$

Then $N^2(k) > 0$ for $k \in \mathbb{R} \setminus \mathcal{T}$, with $N^2(k)$ bounded away from 0 near $\mathcal{T}$, and $N^2(k) \neq 0$ for all $k \in \mathcal{K}$; see for instance §2 of [C2]. Then there is an $a > 0$ such that there is a well defined holomorphic square root $N(k)$ for $k \in \mathcal{K}$ with $\Im k < a$. Notice that the restriction $\Im k < a$ is due to monodromy issues. We extend $N(k) \neq 0$ for $\Im k \geq a$ as well, possibly as a discontinuous function. Then we introduce functions $m_\pm^0(x, k) = \tilde{m}_\pm^0(x, k)/N(k)$ and $\phi_\pm^0(x, k) := e^{\pm i k x} m_\pm^0(x, k)$. The following fact is well known.

Theorem 5.1. *The functions $\phi_\pm^0(x, k)$ are analytic in $k \in \mathbb{R} \setminus \mathcal{T}$, and for $\mathcal{F}_{h_0} f(k) = \int_{\mathbb{R}} \phi_+^0(x, k) f(x) dx$ we have*

- (a) $\int_{\mathbb{R}} |f(y)|^2 dy = \int_{\mathbb{R}} |\mathcal{F}_{h_0} f(k)|^2 dk;$
- (b) $f(x) = \int_{\mathbb{R}} \phi_-^0(x, k) \mathcal{F}_{h_0} f(k) dk;$
- (c) $\mathcal{F}_{h_0}(h_0 f)(k) = E(k) \mathcal{F}_{h_0} f(k)$ for $k \in \mathbb{R} \setminus \mathcal{T};$
- (d) $e^{i t h_0}(x, y) = \int_{\mathbb{R}} e^{i(t E(k) - (x-y)k)} m_-^0(x, k) m_+^0(y, k) dk.$

In the sequel $\dot{g}(k) = \partial_k g(k)$. Korotyaev [K] proves:

Lemma 5.2. *Consider $E(k)$ for $k \in [\pi n(j), \pi n(j+1)]$. Then $\dot{E}(k) = 0$ for $k = n(j)\pi, n(j+1)\pi$ and $\dot{E}(k) > 0$ in $] \pi n(j), \pi n(j+1)[$. In $[\pi n(j), \pi n(j+1)]$ the equation $\ddot{E}(k) = 0$ admits exactly one solution k_j . We have $k_j \in] \pi n(j), \pi n(j+1)[$ and $\ddot{E}(k_j) \neq 0$. We have $\dot{E}(k) > 0$ in $] \pi n(n_0), \infty)$ and $\ddot{E}(k) > c_0 > 0$ in $[\pi n(n_0), \infty)$ for some $c_0 > 0$.*

Notice that $E(-k) = E(k)$.

Theorem 5.3. *Set $e^{ith_0}(x, y) = \mathcal{U}(t, x, y) + \mathcal{V}(t, x, y)$ with*

$$\begin{aligned}\mathcal{U}(t, x, y) &= \int_{|k| \leq \pi n(n_0)} e^{i(tE(k) - (x-y)k)} m_-^0(x, k) m_+^0(y, k) dk, \\ \mathcal{V}(t, x, y) &= \int_{|k| \geq \pi n(n_0)} e^{i(tE(k) - (x-y)k)} m_-^0(x, k) m_+^0(y, k) dk.\end{aligned}$$

Then $|\mathcal{U}(t, x, y)| \leq C \langle t \rangle^{-\frac{1}{3}}$ and $|\mathcal{V}(t, x, y)| \leq C |t|^{-\frac{1}{2}}$.

Assuming Lemma 5.4 below, the proof is in [F1]. In [C2] Theorem 5.3 is extended to the case of infinitely many energy bands. In the case of finitely many energy bands we have:

Lemma 5.4. *For all $\alpha \in \mathbb{N} \cup \{0\}$ there are constants $C_\alpha > 0$ such that for $(x, y) \in \mathbb{R}^2$ and for $k \neq \pm n(j)\pi$, $j = 1, \dots, n_0$, we have*

$$|\partial_k^\alpha (m_+^0(x, k) m_-^0(y, k) - 1)| \leq C_\alpha \langle k \rangle^{-1-\alpha}.$$

Proof (sketch). Cases $\alpha = 0, 1$ are stated and proved in [C2] in the case of a generic periodic potential $P(x)$, i.e. all gaps nonempty. The case with finitely many bands is much easier. In the finite bands case, there is an $M > 0$ such that the $\alpha = 0$ case is valid for all complex k with $|k| \geq M$. Here we are using the fact that $m_+^0(x, k) m_-^0(y, k)$ is holomorphic in $(\mathcal{K} \cup \mathcal{K}^*) \cup (\mathbb{R} \setminus \mathcal{T})$ (see [C2]) and where $z \in \mathcal{K}^*$ if and only if $\bar{z} \in \mathcal{K}$. The inequalities for $\alpha \geq 1$ hold by the Cauchy integral formula.

In the sequel we will set $w_0(k) = W[\phi_+^0(x, k), \phi_-^0(x, k)]$.

Lemma 5.5. *For $k \neq \pm n(j)\pi$, $j = 1, \dots, n_0$, we have $\dot{E}(k) = -iw_0(k)$.*

Proof. The formula $\dot{E}(k) = \frac{2 \sin(k)}{\varphi(1, k) N^2(k)}$ is formula (3.1) of [F2]. By direct computation $W[\tilde{\phi}_+^0(k), \tilde{\phi}_-^0(k)] = i \frac{2 \sin(k)}{\varphi(1, k)}$. So Lemma 5.5 follows by the normalization in the definition of $\phi_\pm^0(x, k)$.

6. ADDITION TO h_0 OF A SMALL POTENTIAL $q(x) \in C_0^\infty(\mathbb{R})$

We consider the operator $h = h_0 + q(x)$ with $q(x) \in C_0^\infty(\mathbb{R})$ small. We consider solutions $f_\pm(x, k) = e^{\pm i k x} m_\pm(x, k)$ of $h u = E(k) u$ with

$$(6.1) \quad \lim_{x \rightarrow +\infty} \frac{m_+(x, k)}{m_+^0(x, k)} = 1 = \lim_{x \rightarrow -\infty} \frac{m_-(x, k)}{m_-^0(x, k)}.$$

Lemma 6.1. *Assume that $q(x) \in C_0^\infty(\mathbb{R})$. Then for any $k \in \mathcal{K}$ the equation*

$$(6.2) \quad m_\pm(x, k) = m_\pm^0(x, k) - \int_x^{\pm\infty} e^{\mp i k(x-t)} A(x, t, k) q(t) m_\pm(t, k) dt,$$

$$(6.3) \quad \text{with } A(x, t, k) := \frac{\phi_+^0(x, k) \phi_-^0(t, k) - \phi_-^0(x, k) \phi_+^0(t, k)}{W[\phi_+^0(\cdot, k), \phi_-^0(\cdot, k)]},$$

has a unique solution $m_{\pm}(x, k)$ such that $f_{\pm}(x, k) = e^{\pm ikx} m_{\pm}(x, k)$ solves $hu = E(k)u$ with the asymptotic property in (6.1). There is a constant C_q such that for some $a > 0$

$$(6.4) \quad \begin{aligned} |m_{\pm}(x, k) - m_{\pm}^0(x, k)| &\leq C_q \langle k \rangle^{-1} \max(1, \mp x), \\ |\partial_k^n m_{\pm}(x, k)| &\leq C_q \langle k \rangle^{-1} \max^{n+1}(1, \mp x) \text{ for } n = 1, 2 \text{ and } \Im k < a. \end{aligned}$$

Furthermore we have continuous extensions for $k \in [\pi n(j), \pi n(j+1)]$, for $k \in [\pi n(n_0), \infty)$, for $k \in [-\pi n(j+1), -\pi n(j)]$ and for $k \in (-\infty, -\pi n(n_0)]$, where the above estimates continue to be satisfied.

Notice that estimates for $f_{\pm}(x, k)$ are also in [F2]. Set

$$D_k(x, t) = A(x, t, k) e^{-ik(x-t)} = \frac{m_+^0(x, k) m_-^0(t, k) - e^{-2ik(x-t)} m_-^0(x, k) m_+^0(t, k)}{W[\phi_+^0(x, k), \phi_-^0(x, k)]}.$$

Then Lemma 6.1 follows from standard arguments (see Lemma 1 of [DT]) by the following lemma:

Lemma 6.2. *Let $t \geq x$. Then we have the following for a fixed $C > 0$:*

- (1) $|D_k(x, t)| \leq C \langle k \rangle^{-1}$ for $|k| \geq \pi n(n_0) + 1$;
- (2) $|D_k(x, t)| \leq C \langle x - t \rangle$;
- (3) $|\dot{D}_k(x, t)| \leq C \langle x - t \rangle^2$ for $\dot{D}_k(x, t) = \partial_k D_k(x, t)$;
- (4) $|\ddot{D}_k(x, t)| \leq C \langle x - t \rangle^3$ for $\ddot{D}_k(x, t) = \partial_k^2 D_k(x, t)$.

(1) follows from Lemma 5.4, which yields a bound $|m_{\pm}^0(x, k) m_{\mp}^0(t, k)| \leq C$ for a fixed C and $w_0(k) \approx 2ik$ for $|k| \nearrow \infty$. $w_0(k) = 0$ iff $k = k_0 \pm 0$ for $k_0 \in \mathcal{T}$. By Lemmas 5.2 and 5.5 $\dot{w}_0(k_0 \pm 0) \neq 0$. Therefore, $|w_0(k)| \gtrsim \text{dist}(k, \mathcal{T})$. For fixed $t \geq x$ consider $x_1, t_1 \in [0, 2]$ with $0 \leq t_1 - x_1 \leq t - x$ and $t - x = t_1 - x_1 + L$ with $L \in \mathbb{N}$. Then we can set $D_k(x, t) = D_k^{(1)}(x, t) + D_k^{(2)}(x, t)$ with

$$\begin{aligned} D_k^{(1)}(x, t) &= \frac{m_+^0(x_1, k) m_-^0(t_1, k) - e^{2ik(t_1-x_1)} m_-^0(x_1, k) m_+^0(t_1, k)}{w_0(k)}, \\ D_k^{(2)}(x, t) &= e^{2ik(t_1-x_1)} \frac{1 - e^{2ikL}}{w_0(k)} m_-^0(x_1, k) m_+^0(t_1, k). \end{aligned}$$

$D_k^{(2)}(x, t)$ satisfies (1)–(4). Indeed Lemma 5.4 yields bounds on $m_-^0(x_1, k) m_+^0(t_1, k)$. Similarly, one can bound $\frac{1 - e^{2ikL}}{w_0(k)}$ using $|w_0(k)| \gtrsim \text{dist}(k, \mathcal{T})$ and the fact that, for k near $k_0 \in \mathcal{T}$, $w_0(k)$ for $k \geq k_0$, resp. for $k \leq k_0$, admits an analytic extension defined around k_0 ; see [C2], [F1], [F2], [K]. Finally we claim that $|\partial_k^n D_k^{(1)}(x, t)| \leq C$ for $|n| \leq 2$ and a fixed C . For k large this is a consequence of $w_0(k) = i\dot{E}(k) \approx 2ik$ and the estimates on $E(k)$ (see [K]) and Lemma 5.4. For $k_0 \in \mathcal{T}$ and $k_0^{\pm} = k_0 \pm 0$

$$e^{ik_0 x} m_+^0(x, k_0^+) \equiv e^{-ik_0 x} m_-^0(x, k_0^+) \text{ and } e^{ik_0 x} m_+^0(x, k_0^-) \equiv e^{-ik_0 x} m_-^0(x, k_0^-).$$

Then both the numerator and denominator in the fraction defining $D_k^{(1)}(x, t)$ are 0 for $k = k_0 \pm 0$ with $k_0 \in \mathcal{T}$. Since, once again, these are functions admitting analytic extensions on fixed neighborhoods around k_0 , one obtains the desired estimates $|\partial_k^n D_k^{(1)}(x, t)| \leq C$.

Lemma 6.3. *There exists $q(x) \in C_0^\infty(\mathbb{R})$ so that for all $k \in \mathcal{T}$ with $k > 0$ we have*

$$(1) \quad \int_{\mathbb{R}} q(x) |m_+^0(x, 0)|^2 dx > 0$$

$$(2) \quad \mp \int_{\mathbb{R}} q(x) |m_+^0(x, k \pm 0)|^2 dx > 0 \text{ for all } k \in \mathcal{T} \text{ with } k > 0.$$

Proof. $m_+^0(x, k \pm 0) \neq 0$ for all $k \in \mathcal{T}$. Since $\sigma(h_0)$ has finitely many bands, the potential $P(x)$ is analytic, and hence also the $m_\pm^0(x, k)$ are analytic in x . (2) can fail if and only if $|m_+^0(\cdot, k_1 + 0)| = |m_+^0(\cdot, k_2 - 0)|$ for two $k_1, k_2 \geq 0$ in $\mathcal{T}$. The latter is the same as $|\phi_+^0(\cdot, k_1 + 0)| = |\phi_+^0(\cdot, k_2 - 0)|$. Notice that $E(k_1 + 0) \neq E(k_2 - 0)$, so one of them is nonzero. It is not restrictive to assume $E(k_2 - 0) > 0$. Notice that $\phi_+^0(x, k_1 + 0)$ and $\phi_+^0(x, k_2 - 0)$ are real valued for $x \in \mathbb{R}$. Indeed, for instance, by definition $\widetilde{\phi}_+^0(x, k_1 + 0) = \widetilde{\phi}_-^0(x, k_1 + 0)$ (see (2.5) of [C2]), $w_0(k_1 + 0) = 0$ implies $\widetilde{\phi}_+^0(x, k_1 + 0) = C\widetilde{\phi}_-^0(x, k_1 + 0)$ for a fixed C and by definition $\widetilde{\phi}_+^0(0, k_1 + 0) = \widetilde{\phi}_-^0(0, k_1 + 0) = 1$. By $E(k_2 - 0) > 0$ the function $\phi_+^0(x, k_2 - 0)$ has infinitely many zeros (if it has 1, it has infinitely many since $\phi_+^0(x, k_2 - 0) = e^{ik_1 x} m_+^0(x, k_2 - 0)$ with $m_+^0(x, k_2 - 0)$ periodic in x , and if it has none, $\phi_+^0(x, k_2 - 0)$ would correspond to $E(k_2) = 0$). We choose two points $a < b$ with $\phi_+^0(a, k_2 - 0) = \phi_+^0(b, k_2 - 0) = 0$ and $\phi_+^0(x, k_2 - 0) \neq 0$ for $a < x < b$. $|\phi_+^0(\cdot, k_1 + 0)| = |\phi_+^0(\cdot, k_2 - 0)|$ implies the same statement for $\phi_+^0(x, k_1 + 0)$. But then they are both ground states for the Dirichlet problem in $[a, b]$ for the operator h . Then $\phi_+^0(x, k_1 + 0) = C\phi_+^0(x, k_2 - 0)$ for fixed C , but this is impossible by $E(k_1 + 0) \neq E(k_2 - 0)$.

We pick $q(x) \in C_0^\infty(\mathbb{R})$ small as in Lemma 6.3. By [FK] we have that $\sigma(h) = \sigma(h_0)$, and in particular h has no eigenvalues. Furthermore there are no resonances; that is, there are not nonzero solutions of any of the equations $hu = A_j^\pm u$ for $j > 0$ with $u \in L^\infty$. Combining the asymptotics in Lemma 6.1 and the analysis of the thresholds in [FK] we conclude:

Lemma 6.4. *For the Wronskian we have $W[f_+(k), f_-(k)] \neq 0$ for all k and $W[f_+(k), f_-(k)] \approx 2ik$ for $|k| \rightarrow \infty$.*

There are equalities

$$(6.5) \quad \begin{aligned} f_\mp(x, k) &= \frac{R_\pm(k)}{T(k)} f_\pm(x, k) + \frac{1}{T(k)} f_\pm(x, -k), \\ \overline{f_\pm(x, k)} &= f_\pm(x, -k), \quad \overline{T(k)} = T(-k), \quad \overline{R_\pm(k)} = R_\pm(-k), \\ |T(k)|^2 + |R_\pm(k)|^2 &= 1, \quad T(k)\overline{R_\pm(k)} + R_\mp(k)\overline{T(k)} = 0, \end{aligned}$$

with $T(k)$ and $R_\pm(k)$ smooth functions for $k \in \mathbb{R} \setminus \mathcal{T}$ with smooth extensions in $[\pi n(j), \pi n(j+1)]$, $[\pi n(n_0), \infty)$, $[-\pi n(j+1), -\pi n(j)]$ and $(-\infty, -\pi n(n_0)]$ and with $T(k) = 0$ for all $k \in \mathcal{T}$ and $T(0) = 0$. For $q(x)$ small

$$(6.6) \quad |[T(k) - 1]^{(m)}| + |R_\pm^{(m)}(k)| \leq \delta/\langle k \rangle \text{ for } |m| \leq 2.$$

Next define

$$(6.7) \quad \psi(x, k) = \begin{cases} \frac{1}{\sqrt{2\pi}} T(k) f_+(x, k) & \text{for } k \geq 0, \\ \frac{1}{\sqrt{2\pi}} T(-k) f_-(x, -k) & \text{for } k < 0. \end{cases}$$

Then by standard arguments we have the following version of Theorem 5.1 for h ; see Lemma 4.7 of [C3].

Lemma 6.5. For $\mathcal{F}_h f(k) := \int_{\mathbb{R}} \psi(x, k) f(x) dx$ we have

$$\begin{aligned} (a) \quad & \int_{\mathbb{R}} |f(y)|^2 dy = \int_{\mathbb{R}} |\mathcal{F}_h f(k)|^2 dk, \\ (b) \quad & f(x) = \int_{\mathbb{R}} \overline{\psi(x, k)} \mathcal{F}_h f(k) dk, \\ (c) \quad & \mathcal{F}_h(hf)(k) = E(k) \mathcal{F}_h f(k), \\ (d) \quad & e^{ith}(x, y) = \int_{\mathbb{R}} e^{itE(k)} \overline{\psi(x, k)} \psi(y, k) dk. \end{aligned}$$

We have:

Lemma 6.6 (Dispersive estimates). Set $e^{ith}(x, y) = \mathcal{U}(t, x, y) + \mathcal{V}(t, x, y)$ with

$$\begin{aligned} \mathcal{U}(t, x, y) &= \int_{|k| \leq \pi n(n_0)} e^{itE(k)} \overline{\psi(x, k)} \psi(y, k) dk, \\ \mathcal{V}(t, x, y) &= \int_{|k| \geq \pi n(n_0)} e^{itE(k)} \overline{\psi(x, k)} \psi(y, k) dk. \end{aligned}$$

Then $|\mathcal{U}(t, x, y)| \leq C \langle t \rangle^{-\frac{1}{3}}$ and $|\mathcal{V}(t, x, y)| \leq C |t|^{-\frac{1}{2}}$ for a fixed $C > 0$.

For $\mathcal{U}(t, x, y)$ the argument is the same as Theorem 5.3. For $\mathcal{V}(t, x, y)$ the proof follows along the lines of Lemmas 2.4 and 2.5 in [Wd2] using [C2].

7. WAVE OPERATORS AND PARTIAL DIAGONALIZATION FOR H_ω

We write $H_\omega = \sigma_3(h + \omega) + B^*(\omega)A$ with $A(x) = \langle x \rangle^{-\tau}$ with $\tau > 3/2$ and with $B^*(x, \omega)$ a C^2 function in (x, ω) with values in the space of 2×2 real valued matrices. For any ω in some compact set K there is a constant $c(K) > 0$ and $\alpha > 0$ such that $|e^{\alpha|x|} B^*(x, \omega)| \leq c_m(K) \forall x \in \mathbb{R}$. We have:

Proposition 7.1. Assume that H_ω does not have resonances at $\pm\omega$ and $\sigma_e(H_\omega)$ does not contain eigenvalues. Then there are isomorphism inverses of each other, $W(\omega): L_x^2 \rightarrow L_c^2(H_\omega)$ and $Z(\omega): L_c^2(H_\omega) \rightarrow L_x^2$, defined as follows: for $u \in L_x^2$, and v such that $\sigma_3 v \in L_c^2(H_\omega)$,

$$\langle Wu, v \rangle = \langle u, v \rangle + \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \langle AR_{\sigma_3(h+\omega)}(\lambda + i\varepsilon)u, BR_{H_\omega^*}(\lambda + i\varepsilon)v \rangle d\lambda;$$

for $u \in L_c^2(H_\omega)$, $v \in L_x^2$,

$$\langle Zu, v \rangle = \langle u, v \rangle + \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \langle AR_{H_\omega}(\lambda + i\varepsilon)u, BR_{\sigma_3(h+\omega)}(\lambda + i\varepsilon)v \rangle d\lambda.$$

Then $P_c(H_\omega)H_\omega = W\sigma_3(h+\omega)Z$. $\|W(\omega)\|_{B(L_x^2, L_c^2(H_\omega))}$ and $\|Z(\omega)\|_{B(L_c^2(H_\omega), L_x^2)}$ are uniformly locally bounded in ω .

We need to show that there is a fixed $c > 0$ such that $\forall \epsilon \neq 0$

- (1) $\int \|\langle x \rangle^{-\tau} R_{\sigma_3(h+\omega)}(\lambda + i\epsilon)u\|_{L_x^2}^2 d\lambda \leq c\|u\|_{L_x^2}^2$ for all $u \in L_x^2$,
- (2) $\int \|BR_{\sigma_3(h+\omega)}(\lambda + i\epsilon)u\|_{L_x^2}^2 d\lambda \leq c\|u\|_2^2$ for all $u \in L_x^2$,
- (3) $\int \|BR_{H_\omega^*}(\lambda + i\epsilon)u\|_{L_x^2}^2 d\lambda \leq c\|u\|_{L_x^2}^2$ for all $u \in L^2(H_\omega^*) := \sigma_3 L^2(H_\omega)$,
- (4) $\int \|\langle x \rangle^{-\tau} R_{H_\omega}(\lambda + i\epsilon)u\|_{L_x^2}^2 d\lambda \leq c\|u\|_{L_x^2}^2$ for all $u \in L_c^2(H_\omega)$.

Let us first prove (1), (2). They are consequences of (5) for $\tau > 3/2$:

$$(5) \quad \int \|\langle x \rangle^{-\tau} R_h(\lambda + i\epsilon)u\|_{L_x^2}^2 d\lambda \leq c\|u\|_{L_x^2}^2 \text{ for all } u \in L_x^2.$$

By (5.3) in Theorem 5.1 of [K], (5) will follow from

$$(6) \quad \|\langle x \rangle^{-\tau} R_h(z)\langle x \rangle^{-\tau}\|_{B(L_x^2, L_x^2)} < C \text{ for all } z \text{ with } 0 < |\Im z|.$$

Observe that for $\Im z \geq 0$, $\pm(x-y) \geq 0$ and setting $w(k) = W[f_+(k), f_-(k)]$,

$$\langle x \rangle^{-\tau} R_h(E(k), x, y)\langle y \rangle^{-\tau} = e^{ik|x-y|}\langle x \rangle^{-\tau} m_\pm(x, k)m_\mp(y, k)\langle y \rangle^{-\tau} w^{-1}(k).$$

By Lemma 6.1, $|m_\pm(x, k)m_\mp(y, k)| \lesssim \langle x \rangle \langle y \rangle$. By Lemma 6.4, $|w(k)| \gtrsim \langle k \rangle$. Hence

$$(7) \quad \langle x \rangle^{-\tau} |R_h(z, x, y)| \langle y \rangle^{-\tau} \leq C \langle z \rangle^{-\frac{1}{2}} \langle x \rangle^{1-\tau} \langle y \rangle^{1-\tau}$$

for all z with $\Im z \geq 0$, resp. $\Im z \leq 0$ (for $\Im z = 0$ there are two different continuations). Then for all z with $\Im z \geq 0$, resp. $\Im z \leq 0$,

$$(8) \quad \|\langle x \rangle^{-\tau} R_h(z, x, y)\langle y \rangle^{-\tau}\|_{L_{x,y}^2} \leq C \langle z \rangle^{-\frac{1}{2}}.$$

(8) implies (6). We now consider (3) and (4), and for definiteness we focus on (4). We have for $A = \langle x \rangle^{-\tau}$,

$$(9) \quad AR_{H_\omega}(z)u = (1 + AR_{\sigma_3(h+\omega)}(z)B^*)^{-1}AR_{\sigma_3(h+\omega)}(z)u.$$

(8) implies $\|AR_{\sigma_3(h+\omega)}(z)B^*\|_{B(L^2, L^2)} \lesssim \langle z \rangle^{-1/2}$. By Fredholm theory there is a bounded 0 measure set $X \subset \mathbb{R}$ such that in $\mathbb{R} \setminus X$ the following limits exist:

$$\lim_{\epsilon \searrow 0} R_{H_\omega}(\lambda + i\epsilon) = R_{H_\omega}^\pm(\lambda) \text{ in } C_{loc}^0(\mathbb{R} \setminus X, B(L_x^{2,\tau}, L_x^{2,-\tau})).$$

A point λ belongs to X exactly if $\ker(1 + AR_{\sigma_3(h+\omega)}^\alpha(\lambda)B^*) \neq 0$ in L^2 for $\alpha = +$ or $-$. The points in $X \setminus \sigma_e(H_\omega)$ are eigenvalues of H_ω . This follows from (9). Furthermore, the hypothesis that $\sigma_e(H_\omega)$ contains no eigenvalues of H_ω implies that $X \setminus \sigma_e(H_\omega)$ is exactly the set of eigenvalues of H_ω . The exponential decay of $H_\omega - \sigma_3(h_0 + \omega)$ implies, by standard arguments, that H_ω has finitely many eigenvalues. Therefore for z close to an eigenvalue of H_ω and for $u \in L_c^2(H_\omega)$ we have that $\|AR_{H_\omega}(z)u\|_{L_x^2} \leq c\|R_{H_\omega}(z)u\|_{L_x^2} \leq c'\|u\|_{L_x^2}$ for fixed constants. We will show below that $X \cap \sigma_e(H_\omega)$ is empty. Notice that this yields (4). To see this let $N \subset \mathbb{R}$ be the neighborhood of $\sigma_e(H_\omega)$ formed by the points with distance $< \delta$ from $\sigma_e(H_\omega)$, with $\delta > 0$ a small number. We split the integral in (4) into two integrals with domain $\mathbb{R} \setminus N$ and N . The integral with domain $\mathbb{R} \setminus N$ is bounded, since for $\lambda \in \mathbb{R} \setminus N$ we have $\|\langle x \rangle^{-\tau} R_{H_\omega}(\lambda + i\epsilon)u\|_2 \leq \|R_{H_\omega}(\lambda + i\epsilon)u\|_2 \leq C\|u\|_2$ for

a fixed C and $\mathbb{R} \setminus N$ is a bounded set. To bound the integral with domain N we use formula (9) above where $H_0 = \sigma_3(h + \omega)$:

$$\langle x \rangle^{-\tau} R_{H_\omega}(\lambda + i\varepsilon)u = (1 + \langle x \rangle^{-\tau} R_{H_0}(\lambda + i\varepsilon)B^*)^{-1} \langle x \rangle^{-\tau} R_{H_0}(\lambda + i\varepsilon)u.$$

Then for any $u \in L^2$

$$(10) \quad \begin{aligned} \|\langle x \rangle^{-\tau} R_{H_\omega}(\lambda + i\varepsilon)u\|_{L_\lambda^2(N, L_x^2)} &\leq \|(1 + \langle x \rangle^{-\tau} R_{H_0}(\lambda + i\varepsilon)B^*)^{-1}\|_{L_\lambda^\infty(N, B(L_x^2, L_x^2))} \\ &\quad \|\langle x \rangle^{-\tau} R_{H_0}(\lambda + i\varepsilon)u\|_{L_\lambda^2(N, L_x^2)}. \end{aligned}$$

The first factor on the right-hand side of (10) is uniformly bounded by the hypothesis that $X \cap \sigma_e(H_\omega)$ is empty, while the second factor on the right-hand side of (10) is bounded for any $u \in \ell^2$ by (1). This yields (4), with the proof of (3) being similar.

Now we return to the set X . Our aim is to show that X is formed exactly by the eigenvalues of H_ω . It is enough to show that $X \cap \sigma_e(\mathcal{H}_\omega)$ is empty. We will proceed in two steps. We will first show that by (H7) X does not contain any of these extremes thresholds of $\sigma_3(h + \omega)$, Lemma 7.2. We will then show that the points of X in the interior of the spectral bands are necessarily eigenvalues of H_ω . Since such “embedded” eigenvalues do not exist by (H7), we can conclude that $X \cap \sigma_e(\mathcal{H}_\omega)$ is empty.

Lemma 7.2. *X does not contain extremes of the spectral bands of $\sigma_3(h + \omega)$.*

Proof. Suppose the claim is wrong and that $\lambda \in X$ is an extremum of one of the spectral bands of $\sigma_3(h + \omega)$, and pick $w \neq 0$ in L^2 such that

$$(1 + AR_{\sigma_3(h+\omega)}(\lambda)B^*)w = 0.$$

Then $(1 + B^*AR_{\sigma_3(h+\omega)}(\lambda))B^*w = 0$. Then $\psi = R_{\sigma_3(h+\omega)}(\lambda)B^*w$ is $\psi \neq 0$ and by standard arguments is a nontrivial distributional solution of $(H_\omega - \lambda)u = 0$. We claim that

$$(10) \quad \psi \in L_x^\infty.$$

If (10) is true, we get a contradiction with hypothesis (H7). The proof of (10) reduces to show that for a rapidly decreasing bounded function $g(y)$ we have

$$\|f\|_{L_x^\infty} < \infty \text{ for } f(x) = \int_{\mathbb{R}} R_h(x, y, \lambda)g(y)dy.$$

This is a consequence of Lemmas 6.1 and 6.4 and, for $w(k) = W[f_+(k), f_-(k)]$, of

$$f(x) = f_+(x, k) \int_{-\infty}^x f_-(y, k)g(y)dy/w(k) + f_-(x, k) \int_x^\infty f_-(y, k)g(y)dy/w(k).$$

This proves Lemma 7.2.

Lemma 7.3. *For any λ_0 in the interior of $\sigma_e(H_\omega)$ there are an open interval $I \subset \sigma_e(H_\omega)$ with $\lambda_0 \in I$ and a constant C such that $\|AR(\lambda \pm i\varepsilon)u\|_{L_\lambda^2(I)L_x^2} \leq C\|u\|_{L_x^2}$ for any u .*

Proof. Let $AB_1^* = AB^* - \sigma_3 q$. Then $H_\omega = \sigma_3(h_0 + \omega) + AB_1^*$. The points $\lambda_0 \in \sigma(\sigma_3(h_0 + \omega))$, where $\|AR(\lambda \pm i\varepsilon)u\|_{L_\lambda^2(I)L_x^2}$ is unbounded for I a small neighborhood of λ_0 , are such that if we set

$$(11) \quad AR_{H_\omega}^\pm(\lambda)u = (1 + AR_{\sigma_3(h_0+\omega)}^\pm(\lambda)B_1^*)^{-1}AR_{\sigma_3(h_0+\omega)}(\lambda)u,$$

we have for one of the two signs

$$\ker(1 + AR_{\sigma_3(h_0+\omega)}^\pm(\lambda_0)B_1^*) \neq 0.$$

By a standard argument (see §2 of [CPV]), if we assume Lemma 7.4 below we conclude that such a point λ_0 is an eigenvalue of H_ω . Since we are excluding eigenvalues inside the spectral bands of $\sigma_e(H_\omega)$, we are done with the proof Lemma 7.3.

Lemmas 7.2 and 7.3 imply that X coincides with $\sigma_d(H_\omega) \cap \mathbb{R}$. Moreover the above arguments imply that the following limits exist:

$$(12) \quad \lim_{\epsilon \searrow 0} R_{H_\omega}(\lambda \pm i\epsilon) = R_{H_\omega}^\pm(\lambda) \text{ in } C^0(\sigma_e(H_\omega), B(L_x^{2,\tau}, L_x^{2,-\tau})).$$

Lemma 7.4. *Fix E_0 in the interior of a spectral band of h_0 . Suppose that there is a function $\psi(x)$ such that $\psi \in L_x^{2,s}$ for all $s > 0$ and $\mathcal{F}_{h_0}(\psi)(\pm k_0) = 0$, where $E_0 = E(k_0)$. Then $R_{h_0}^\pm(E_0)\psi \in L_x^2$.*

Proof. Consider a cutoff $\chi(E) \in C_0^\infty(\mathbb{R})$ with $\chi(E) = 1$ for E close to E_0 and $\chi(E) = 0$ if $|E - E_0| \geq \varepsilon_0 > 0$ for ε_0 a small fixed number. Then

$$R_{h_0}^\pm(E_0)\psi = R_{h_0}^\pm(E_0)\chi(h_0)\psi + R_{h_0}^\pm(E_0)(1 - \chi(h_0))\psi.$$

The second term on the right is in L_x^2 . With the notation in §5 we have

$$\mathcal{F}_{h_0}(\chi(h_0)\psi)(k) = \int_{\mathbb{R}} e^{ikx} m_+^0(x, k) \chi(E(k)) \psi(x) dx.$$

Notice that $m_+^0(x, k) \chi(E(k))$ is the symbol of a smoothing pseudodifferential operator. This implies that $\partial_k^n \mathcal{F}_{h_0}(\chi(h_0)\psi) \in L_k^2$ for all n by Lemma 5.4. Since $k_0 \notin \mathcal{T}$, $\dot{E}(k_0) \neq 0$ by Lemma 5.2. Then also

$$\partial_k^n \left[\frac{\mathcal{F}_{h_0}(\psi)(k) \chi(E(k))}{E(k) - E_0} \right] \in C^0 \cap L_k^1 \text{ for all } n.$$

We have

$$R_{h_0}^\pm(E_0)\chi(h_0)\psi(x) = \int_{\mathbb{R}} e^{-ikx} m_-^0(x, k) \frac{\mathcal{F}_{h_0}(\psi)(k)}{E(k) - E_0} \chi(E(k)) dk.$$

Integrating by parts we conclude $x^N R_{h_0}^\pm(E_0)\chi(h_0)\psi(x) \in L_x^\infty$ for all N .

By Proposition 7.1 and by the spectral theorem we conclude:

Lemma 7.5. *For any $u \in L_x^2$ we have*

$$P_c(H_\omega)u = \lim_{M \rightarrow \infty} \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\sigma_e(H_\omega)} \chi_{[-M, M]}(\lambda) [R_{H_\omega}(\lambda + i\epsilon) - R_{H_\omega}(\lambda - i\epsilon)] u d\lambda.$$

Finally, we obtain the limiting absorption principle:

Lemma 7.6. *For any $u \in L_x^{2,\tau}$ with $\tau > 3/2$ we have*

$$P_c(H_\omega)u = \lim_{M \rightarrow \infty} \frac{1}{2\pi i} \int_{\sigma_e(H_\omega)} \chi_{[-M, M]}(\lambda) [R_{H_\omega}^+(\lambda) - R_{H_\omega}^-(\lambda)] u d\lambda.$$

Proof. For $u \in L_x^{2,\tau}$ and for fixed M the $\epsilon \rightarrow 0^+$ limit in Lemma 7.5 converges in $L_x^{2,-\tau}$ to the integral in Lemma 7.6.

8. PLANE WAVES FOR H_ω

We recall $H_\omega = \sigma_3(h_0 + \omega) + V(x)$ with:

- $V(x)$ a real entries square 2 matrix s.t. $\sigma_1 V(x) = -V(x)\sigma_1$ and $\sigma_3 V(x) = {}^t V(x)\sigma_3$;
- $|V(x)| \leq C e^{-\sqrt{2\omega}|x|}$;
- H_ω has no eigenvalues in $\sigma_e(H_\omega)$, and the points and the thresholds of the spectral bands of $\sigma_e(H_\omega)$ are not resonances. This last statement means that for any $k \in \mathcal{T}$ and for $\lambda = E(k \pm 0) + \omega$, as well as for $\lambda = -E(k \pm 0) - \omega$, if $g \in L_x^\infty$ satisfies $H_\omega g = \lambda g$, then $g = 0$.

We set

$$\eta(x) = \int_x^\infty |V(t)| dt, \quad \gamma(x) = \int_x^\infty \langle t \rangle |V(t)| dt.$$

We have $\sigma_e(H_\omega) = (\omega + \sigma(h_0)) \cup (-\omega - \sigma(h_0))$. Because of the symmetries of H_ω we look at Jost functions $F_\pm(x, k)$ close to $\omega + \sigma(h_0)$.

For $\lambda \notin \sigma_e(H_\omega)$ let $k \in \mathbb{C}$ with $\Im k \geq 0$ such that $\lambda = E(k) + \omega$ (in the sequel, for λ and k in the same sentence, we will always have the relation $\lambda = E(k) + \omega$). We now consider solutions of $H_\omega u = \lambda u$ of the following form, where $A(x, t, k)$ is given by (6.4):

$$(8.1) \quad \begin{aligned} F_\pm(x, k) = & \phi_\pm^0(x, k) \vec{e}_1 - \int_x^{\pm\infty} A(x, t, k) \text{diag}(1, 0) V(t) F_\pm(t, k) dt \\ & - \int_{\mathbb{R}} R_{h_0}(-2\omega - E(k), x, t) \text{diag}(0, 1) V(t) F_\pm(t, k) dt. \end{aligned}$$

Lemma 8.1. *Assume (H7) and (H9). Then there is a small $\delta > 0$ and a finite set $S \subset \overline{\mathbb{C}}_+$ such that for $k \notin S$ with $0 \leq \Im k \leq \delta$, (8.1) has for any choice of sign a a unique solution satisfying the estimates listed below for $M_\pm(x, k) := e^{\mp i k x} F_\pm(x, k)$. These solutions solve $H_\omega u = \lambda u$ with $\lambda = \omega + E(k)$ and with the asymptotic property $F_\pm(x, k) = \phi_\pm^0(x, k) \vec{e}_1 + o(e^{\pm i k x})$ for $x \rightarrow \pm\infty$. For any fixed $0 < a < \sqrt{2\omega}$ and any $\varepsilon > 0$ there is C_ε such that $\forall x \in \mathbb{R}$ and $\forall k \in \mathbb{C}$ with $0 \leq \Im k \leq \delta$ and $\text{dist}(k, S) > \varepsilon > 0$ we have:*

- (1) $|M_\pm(x, k) - m_\pm^0(x, k) \vec{e}_1| \leq C_\varepsilon e^{-a \max(1, \mp x)} \langle k \rangle^{-1} (1 + \max(1, \mp x));$
- (2) $|\partial_k [M_\pm(x, k) - m_\pm^0(x, k) \vec{e}_1]| \leq C_\varepsilon \max^2(1, \mp x);$
- (3) $|\partial_k^2 [M_\pm(x, k) - m_\pm^0(x, k) \vec{e}_1]| \leq C_\varepsilon \max^3(1, \mp x).$

Proof. We sketch the proof for the $+$ case and drop the index. Using the notation in §6,

$$(8.2) \quad \begin{aligned} M(x, k) = & m_+^0(x, k) \vec{e}_1 - \int_x^\infty D_k(x, t) \text{diag}(1, 0) V(t) M(t, k) dt \\ & - \int_{\mathbb{R}} R_{h_0}(-2\omega - E(k), x, t) e^{-i k(x-t)} \text{diag}(0, 1) V(t) M(t, k) dt. \end{aligned}$$

We have $|R_{h_0}(-2\omega - E(k), x, t)e^{-ik(x-t)}| \leq Ce^{-\sqrt{2\omega}|x-t|}\langle k \rangle^{-1}$. For $V = \{V_{\ell,j}\}$ let

$$\begin{aligned} {}^tM &= (M_1, M_2), \quad M_1^{(1)}(x, k) := - \int_x^\infty D_k(x, t)V_{12}(t)M_2(t, k)dt, \\ M_1^{(0)}(x, k) &:= m_+^0(x, k) - \int_x^\infty D_k(x, t)V_{11}(t)M_1(t, k)dt. \end{aligned}$$

Then $M_1(x, k) = M_1^{(0)}(x, k) + M_1^{(1)}(x, k)$. By standard arguments:

Lemma 8.2. *For given $M_2 \in L_x^\infty$ for some fixed $C = C(V)$ we have*

$$\begin{aligned} (1) \quad & |M_1^{(0)}(x, k) - m_+^0(x, k)| \leq C \max(1, -x) \gamma(x) \langle k \rangle^{-1}; \\ (2) \quad & |M_1^{(1)}(x, k)| \leq C \max(1, -x) \gamma(x) \langle k \rangle^{-1} \|M_2(\cdot, k)\|_{L_x^\infty}. \end{aligned}$$

Continuation of proof of Lemma 8.1. We have $M_1^{(1)}(x, k) = L(k)M_2(x, k)$ with $L(k)$ a linear operator such that for a fixed C

$$(3) \quad |L(k)M_2(x)| \leq C \langle k \rangle^{-1} \max(1, -x) \|M_2(\cdot, k)\|_{L_x^\infty}.$$

Eliminate M_1 from (8.2) to get the following system for $K(k)$ defined below:

$$\begin{aligned} (8.3) \quad & (1 + K(k))M_2(x, k) = - \int_{\mathbb{R}} R_{h_0}(-2\omega - E(k), x, t)e^{-ik(x-t)}V_{21}(t)M_1^{(0)}(t, k)dt \\ & K(k)g(x) = \int_{\mathbb{R}} R_{h_0}(-2\omega - E(k), x, t)e^{-ik(x-t)}(V_{21}(t)L(k)[g](t) + V_{22}(t)g(t))dt. \end{aligned}$$

Lemma 8.3. *Let $0 \leq \Im k < \delta < \sqrt{2\omega} - a$ for some $a > 0$. Then $K(k)$ maps L_x^∞ in $e^{-a|x|}W_x^{1,\infty}$. There is a finite set $S \subset \overline{\mathbb{T}}_+$ such that for k as above with $k \notin S$, equation (8.3) admits a unique solution $M_2(x, k) \in L_x^\infty$. Furthermore, for any $\varepsilon > 0$ there is a C_ε such that if $\text{dist}(k, S) > \varepsilon$, then*

$$(1) \quad \left| e^{a|x|}M_2(x, k) \right| \leq C_\varepsilon \langle k \rangle^{-1}.$$

Proof. The fact that $K(k) : L_x^\infty \rightarrow e^{-a|x|}W_x^{1,\infty}$ if $0 \leq \Im k < \delta < \sqrt{2\omega} - a$ is an elementary computation. For $|k| \rightarrow \infty$ we have $\|K(k)\|_{B(L_x^\infty, L_x^\infty)} \rightarrow 0$, so we can solve (8.3) by obtaining (1). $K(k) : L_x^\infty \rightarrow L_x^\infty$ is a compact operator. For $|k| \lesssim 1$ by the Fredholm alternative, S will be formed by the k with $\ker(1 + K(k)) \neq 0$. The set S is necessarily discrete and contained in $|k| \lesssim 1$. We start by considering $k \in S \cap \mathbb{R}$. If we have such a k , then we have a nonzero solution of the equation obtained from (8.3) by replacing $M_1^{(0)}(t, k)$ with 0. Going backwards in the above argument, we obtain the existence of a nontrivial $g \in L_x^\infty$ satisfying

$$\begin{aligned} (8.4) \quad & g(x) = - \int_x^\infty A(x, t, k) \text{diag}(1, 0)V(t)g(t)dt \\ & - \int_{\mathbb{R}} R_{h_0}(-2\omega - E(k), x, t) \text{diag}(0, 1)V(t)g(t)dt. \end{aligned}$$

Such $g \in L_x^\infty$ is a distributional solution of $H_\omega g = \lambda g$ with $\lambda = E(k) + \omega$. By hypothesis we are assuming such solutions do not exist for $\lambda = E(k \pm 0) + \omega$ with $k \in \mathcal{T}$, that is, for λ a threshold in $\sigma_e(H_\omega)$. By continuity, $\ker(1 + K(k)) = 0$ near such thresholds. Since the points in $\mathcal{T}$ are the only possible accumulation points of S , we conclude that S is finite.

Continuation of proof of Lemma 8.1. Lemmas 8.2-8.3 imply (1) in Lemma 8.1. To prove (2) we differentiate (8.2) in k obtaining for the dot representing ∂_k , and for $E_k(x, t) = R_{h_0}(-2\omega - E(k), x, t)$,

$$(4) \quad \begin{aligned} \dot{M}(x, k) &= \vec{e}_1 \dot{m}_+^0(x, k) \\ &- \left(\int_x^\infty D_k(x, t) \text{diag} \begin{pmatrix} 1 & 0 \end{pmatrix} + \int_{\mathbb{R}} E_k(x, t) \text{diag} \begin{pmatrix} 0 & 1 \end{pmatrix} \right) V(t) \dot{M}(t, k) dt \\ &- \left(\int_x^\infty \dot{D}_k(x, t) \text{diag} \begin{pmatrix} 1 & 0 \end{pmatrix} + \int_{\mathbb{R}} \dot{E}_k(x, t) \text{diag} \begin{pmatrix} 0 & 1 \end{pmatrix} \right) V(t) M(t, k) dt. \end{aligned}$$

We have $|\partial_k^j E_k(x, t)| \leq C e^{-\alpha \langle k \rangle |x-t|}$ for $j = 0, 1$ for fixed $C > 0$ and $\alpha > 0$. We write $\dot{M}_1(x, k) = \dot{M}_1^{(0)}(x, k) + \dot{M}_1^{(1)}(x, k)$, with $\dot{M}_1^{(1)}(x, k) = L(k) \dot{M}_2(x, k)$ and with

$$\begin{aligned} \dot{M}_1^{(0)}(x, k) \vec{e}_1 &= \vec{e}_1 \dot{m}_+^0(x, k) \\ &+ (\text{third line of (4)}) - \vec{e}_1 \int_x^\infty D_k(x, t) V_{12}(t) \dot{M}_2(t, k) dt. \end{aligned}$$

Then

$$\begin{aligned} |\dot{M}_1^{(0)}(x, k)| &\leq C \max^2(1, -x) \text{ and} \\ |L(k) \dot{M}_2(x, k)| &\leq C \langle k \rangle^{-1} \max(1, -x) \|\dot{M}_2(\cdot, k)\|_{L_x^\infty}. \end{aligned}$$

We obtain an analogue of system (8.3), with $M_1^{(0)}(x, k)$ replaced by $\dot{M}_1^{(0)}(x, k)$. Then we conclude $|e^{a|x|} \dot{M}_2(x, k)| \leq \frac{C}{1+|k|}$. Repeating the argument in a similar way, we get claim (3) in Lemma 8.1.

If $S \cap \mathbb{R} = \emptyset$, by taking δ small enough we can neglect the set S . However, in Lemma 5.4 of [C1] the proof that $S \cap \mathbb{R} = \emptyset$ is wrong. In fact, the discussion in [KS] allows for the existence of S . So let us now assume $S \cap \mathbb{R} \neq \emptyset$. For k near $S \cap \mathbb{R}$ we consider the system

$$(8.5) \quad (1 + R_{\sigma_3(h_0+\omega)}^+(\lambda)V) \Phi_\pm(\cdot, k) = \phi_\pm^0(x, k) \vec{e}_1.$$

Lemma 8.4 . *Let $\lambda = E(k) + \omega$ with $E(k) \in \sigma(h_0)$, with $k \neq \pm \pi n(j)$ for all j .*

- (1) *For any choice of signs, system (8.5) admits exactly one solution in L_x^∞ .*
- (2) *For $k \notin S$ we have $\Phi_\pm(x, k) = c_\pm(k) F_\pm(x, k)$ for some constants $c_\pm(k)$.*
- (3) *Fix a decomposition $V = B^* A$ with A and $B^* \in C^2$ and exponentially decreasing. Then $k \rightarrow A \Phi_\pm(x, k) \in L_x^2$ is a real analytic map for k near S .*
- (4) *$c_\pm(k)$ are real analytic functions in k for k near S .*

Proof. For definiteness pick $+$. For $\lambda = E + \omega$, by standard arguments

$$(5) \quad R_{\sigma_3(h_0+\omega)}^+(\lambda)V \text{ is a } L_x^\infty \rightarrow L_x^\infty \text{ compact operator.}$$

By (5) and the Fredholm theory we know that (8.5) has a unique solution unless there is a nonzero solution $g \in L_x^\infty$ of

$$(6) \quad (1 + R_{\sigma_3(h_0+\omega)}^+(\lambda)V)g = 0.$$

Notice that ${}^t g = (g_1, g_2)$,

$$(7) \quad H_\omega g = \lambda g \text{ in the sense of distributions,}$$

and that $|g_2(x)| \leq C e^{-\sqrt{2\omega}|x|}$ for some $C > 0$. By (6) we obtain

$$\langle (1 + R_{\sigma_3(h_0+\omega)}^+(\lambda)V)g, \sigma_3 Vg \rangle = 0.$$

Then

$$(8) \quad \langle \delta(h_0 - E(k))(V_{11}g_1 + V_{12}g_2), V_{11}g_1 + V_{12}g_2 \rangle = 0.$$

Notice that $V_{11}g_1 + V_{12}g_2 \in L_x^{2,s}$ for any $s \in \mathbb{R}$. For $\psi \in L_x^{2,s}$ with $s > 1/2$ we have

$$\langle \delta(h_0 - E(k))\psi, \psi \rangle = |\dot{E}(k)|^{-1} (|\mathcal{F}_{h_0}\psi(k)|^2 + |\mathcal{F}_{h_0}\psi(-k)|^2).$$

So if we set $\psi = V_{11}g_1 + V_{12}g_2$, then $\mathcal{F}_{h_0}\psi(\pm k) = 0$. We can apply Lemma 7.4 and conclude that

$$g_1 = R_{h_0}^+(E(k))\psi \in L^2(\mathbb{R}).$$

So $g \in L_x^2$ and (7) imply that g is an eigenfunction and that λ is an eigenvalue. Since we are excluding this possibility, we conclude that there are non nonzero solutions of (6) and that claim (1) is correct.

From (8.5) we get for $w_0(k) = W[\phi_+^0(k), \phi_-^0(k)]$,

$$(8.6) \quad \begin{aligned} \Phi_{\pm}(x, k) &= \phi_{\pm}^0(x, k) \vec{e}_1, \\ &+ \phi_+^0(x, k) \int_{-\infty}^x \phi_-^0(y, k) \text{diag}(1, 0) V(y) \Phi_{\pm}(y, k) dy / w_0(k) \\ &- \phi_-^0(x, k) \int_x^{\infty} \phi_+^0(y, k) \text{diag}(1, 0) V(y) \Phi_{\pm}(y, k) dy / w_0(k) \\ &- \int_{\mathbb{R}} R_{h_0}(\omega - E(k), x, y) \text{diag}(0, 1) V(y) \Phi_{\pm}(y, k) dy. \end{aligned}$$

Then, for

$$(9) \quad \begin{aligned} c_+(k) &= 1 + \int_{\mathbb{R}} \phi_-^0(y, k) (V_{11}(y) \Phi_{1+}(y, k) + V_{12}(y) \Phi_{2+}(y, k)) dy / w_0(k), \\ c_-(k) &= 1 - \int_{\mathbb{R}} \phi_+^0(y, k) (V_{11}(y) \Phi_{1-}(y, k) + V_{12}(y) \Phi_{2-}(y, k)) dy / w_0(k) \end{aligned}$$

and for $k \notin S$ we have for $g_{\pm}(x) := \Phi_{\pm}(x, k) - c_{\pm}(k)F_{\pm}(x, k)$,

$$g \in L_x^{\infty} \text{ with } \lim_{x \rightarrow \pm\infty} g_{\pm}(x) = 0 \text{ and } H_{\omega}g_{\pm} = \lambda g_{\pm}.$$

Since $k \notin S$ this yields $g_{\pm} \equiv 0$.

To prove claims (3) and (4) we write (8.5) as

$$(1 + AR_{\sigma_3(h_0+\omega)}^+(\lambda)B^*)A\Phi_{\pm}(\cdot, k) = A\phi_{\pm}^0(x, k) \vec{e}_1.$$

Since $AR_{\sigma_3(h_0+\omega)}^+(\lambda)B^* \in B(L_x^2, L_x^2)$ is real analytic in k , and similarly $k \rightarrow A\phi_{\pm}^0(x, k) \in L_x^2$ is real analytic in k , we obtain claims (3) and (4) by the chain rule.

Lemma 8.1 yields Jost functions $F(x, k)$ for H_{ω} for energy λ close to $\omega + \sigma(h)$ and for $k \notin S$. By $\sigma_1 H_{\omega} = -H_{\omega} \sigma_1$ we conclude that $\sigma_1 F(x, k)$ are Jost functions for H_{ω} for energy λ close to $-\omega - \sigma(h)$. From $\sigma_3 H_{\omega} = H_{\omega}^* \sigma_3$ we get that $\sigma_3 F(x, k)$ are Jost functions for H_{ω}^* for energy λ close to $\omega + \sigma(h)$ and that $\sigma_3 \sigma_1 F(x, k)$ are Jost functions for H_{ω}^* for energy λ close to $-\omega - \sigma(h)$. By elementary arguments we get:

Lemma 8.5. *For $k \in \mathbb{R} \setminus S$ we have $\overline{F_{\pm}(x, k)} = F_{\pm}(x, -k)$, and for $k \neq 0$ we have*

$$(1) \quad F_{\mp}(x, k) = \frac{1}{T(k)} \overline{F_{\pm}(x, k)} + \frac{R_{\pm}(k)}{T(k)} F_{\pm}(x, k),$$

where $T(k)$ and $R_{\pm}(k)$ are defined by the above formula.

Lemma 8.6. *All the Wronskians below are constant. We have for any $k \in \mathbb{R} \setminus S$,*

$$\begin{aligned} (1) \quad & W[\overline{F_{\pm}(k)}, F_{\pm}(k)] = W[\phi_{\mp}^0(\cdot, k), \phi_{\pm}^0(\cdot, k)], \\ (2) \quad & T(k) = \frac{W[\phi_{\mp}^0(\cdot, k), \phi_{\pm}^0(\cdot, k)]}{W[F_{\mp}(k), F_{\pm}(k)]}, \quad R_{\pm}(k) = \frac{W[F_{\mp}(k), \overline{F_{\pm}(k)}]}{W[F_{\pm}(k), F_{\mp}(k)]}, \\ (3) \quad & \overline{T(k)} = T(-k), \quad \overline{R_{\pm}(k)} = R_{\pm}(-k), \\ (4) \quad & |T(k)|^2 + |R_{\pm}(k)|^2 = 1, \quad T(k)\overline{R_{\pm}(k)} + R_{\mp}(k)\overline{T(k)} = 0. \end{aligned}$$

Proof. For the fact that the Wronskians are constant see Lemma 5.8 [KS]. The rest of the proof is the same as Lemma 5.6 of [C1].

Lemma 8.7. *For $k \in \mathbb{R}$ we have*

$$(1) \quad W[\Phi_+(k), \Phi_-(k)] = c_+(k)W[\phi_+^0(\cdot, k), \phi_-^0(\cdot, k)] = c_-(k)W[\phi_+^0(\cdot, k), \phi_-^0(\cdot, k)].$$

In particular, $c_+(k) = c_-(k) = T(k)$.

Proof. We have the asymptotic behaviors, with $\partial_x o(1) = o(1)$,

$$\begin{aligned} \Phi_+(x, k) &= c_+(k)\phi_+^0(x, k)\vec{e}_1 + o(1) \text{ for } x \rightarrow +\infty, \\ \Phi_-(x, k) &= \phi_-^0(x, k)\vec{e}_1 + C_1(k)\phi_+^0(x, k)\vec{e}_1 + o(1) \text{ for } x \rightarrow +\infty, \\ \Phi_-(x, k) &= c_-(k)\phi_-^0(x, k)\vec{e}_1 + o(1) \text{ for } x \rightarrow -\infty, \\ \Phi_+(x, k) &= \phi_+^0(x, k)\vec{e}_1 + C_2(k)\phi_-^0(x, k)\vec{e}_1 + o(1) \text{ for } x \rightarrow -\infty, \end{aligned}$$

with $c_{\pm}(k)$ defined in Lemma 8.4 and for some constants $C_j(k)$ with $j = 1, 2$. In particular, for later use, $C_1(k)$ is smooth for k near S and given by

$$(2) \quad C_1(k) = \int_{\mathbb{R}} \phi_-^0(y, k)(V_{11}(y)\Phi_{1-}(y, k) + V_{12}(y)\Phi_{2-}(y, k))dy/w_0(k).$$

(1) follows by using these asymptotic expansions and the fact that the Wronskian $W[\Phi_+(k), \Phi_-(k)]$ is a constant. We have

$$T(k) = \frac{W[\phi_-^0, \phi_+^0]}{W[F_-, F_+]} = c_+c_- \frac{W[\phi_-^0, \phi_+^0]}{W[\Phi_-, \Phi_+]} = \frac{c_+c_-}{c_{\pm}} = c_{\pm}.$$

Lemma 8.8. *$T(k)$ and $R_{\pm}(k)$ are in $C^1(\mathbb{R})$, and there is $C > 0$ such that for $n \leq 1$,*

$$|d^n [T(k) - 1]/dk^n| + |d^n R_{\pm}(k)/dk^n| \leq C/\langle k \rangle.$$

Proof. Notice that for $k \notin T$ we have $W[\phi_-^0(k), \phi_+^0(k)] \neq 0$. So (4) of Lemma 8.6 and the nonresonance hypothesis at the thresholds imply $W[F_-, F_+] \neq 0$ in $\mathbb{R} \setminus S$. There $T(k)$ and $R_{\pm}(k)$ are C^1 with the desired asymptotic estimates by Lemma 8.1. Near S we use $T(k) = c_{\pm}(k)$ and (3) of Lemma 8.4 to conclude that $T \in C^1(\mathbb{R})$. Near S

$$R_+(k) = \frac{W[F_-, \overline{F_+}]}{W[F_+, F_-]} = \frac{|c_+|^2}{c_+^2} \frac{W[\Phi_-, \overline{\Phi_+}]}{W[\Phi_+, \Phi_-]} = C_1(k) \frac{\overline{c_+}^2}{c_+^2}.$$

Here $C_1(k)$ is in (2) of Lemma 8.7 and is smooth. If $k_0 \in S$, then there is some $m \geq 1$ such that $c_+^{(m)}(k_0) \neq 0$; then also $\overline{c_+}/c_+$ is smooth near k_0 . So $R_+ \in C^1(\mathbb{R})$. The argument for R_- is similar.

We consider the following system:

$$(8.7) \quad \begin{aligned} G_{\pm}(x, k) = & \phi_{\pm}^0(x, k_1) \vec{e}_2 - \int_x^{\pm\infty} A(x, t, k) \text{diag} (1, 0) V(t) G_{\pm}(t, k) dt \\ & - \int_x^{\pm\infty} A(x, t, k_1) \text{diag} (0, 1) V(t) g_{\pm}(t, k) dt, \end{aligned}$$

where $E(k_1) = -E(k) - 2\omega$. The proof of Lemmas 8.9 and 8.10 below follow by routine arguments.

Lemma 8.9. *For $\lambda = \omega + E(k)$ with $0 \leq \Im k < \delta$, system (8.7) admits exactly one solution $G_{\pm}(x, k)$ which satisfies $H_{\omega} G_{\pm} = \lambda G_{\pm}$; $G_{\pm}(x, k)$ is real for $k \in \mathbb{R}$. There is C such that*

$$\begin{aligned} |G_{\pm}(x, k) - \phi_{\pm}^0(x, k_1) \vec{e}_2| & \leq C \langle k \rangle^{-1} \left| \int_x^{\pm\infty} \langle t \rangle |V(t)| dt \right| e^{\mp x \Re \sqrt{k^2 + 2\omega}}, \\ |\partial_x G_{\pm} - \partial_x \phi_{\pm}^0(x, k_1) \vec{e}_2| & \leq C \left| \int_x^{\pm\infty} \langle t \rangle |V(t)| dt \right| e^{\mp x \Re \sqrt{k^2 + 2\omega}}. \end{aligned}$$

We have that $w(k) = [G_{-}(k), G_{+}(k)]$ is a continuous function with $w(k) = 0$ if and only if $k \in S$. We have $w(k) = 2\sqrt{k^2 + 2\omega}(1 + o(1))$ for $|k| \rightarrow \infty$.

We have $[F_{\pm}(x, k), G_{\pm}(x, k)] = 0$ for any $k \notin S$. For $\Im k > 0$ and $g \in L^{\infty}(b, +\infty)$, for some $b \in \mathbb{R}$ and with $H_{\omega} g = \lambda g$, $g(x) = \mu F_{+}(x, k) + \nu G_{+}(x, k)$ for constants μ and ν . If $g \in L^{\infty}(-\infty, b)$ solves $H_{\omega} g = \lambda g$, then $g(x) = \mu F_{-}(x, k) + \nu G_{-}(x, k)$.

Lemma 8.10. *Let $\lambda = \omega + E(k)$ with $0 < \Im k < \delta$ and $k \notin S$. The resolvent $R_{H_{\omega}}(\lambda)$ has an integral kernel given by $R_{H_{\omega}}(x, y, \lambda) = R_1(x, y) + R_2(x, y)$ where for $x < y$*

$$R_1(x, y) = -\frac{F_{-}(x, k)^t (\sigma_3 F_{+}(y, k))}{W[F_{+}(\cdot, k), F_{-}(\cdot, k)]} \text{ and } R_2(x, y) = -\frac{G_{-}(x, k)^t (\sigma_3 G_{+}(y, k))}{W[G_{+}(\cdot, k), G_{-}(\cdot, k)]},$$

and for $x > y$

$$R_1(x, y) = -\frac{F_{+}(x, k)^t (\sigma_3 F_{-}(y, k))}{W[F_{+}(\cdot, k), F_{-}(\cdot, k)]} \text{ and } R_2(x, y) = -\frac{G_{+}(x, k)^t (\sigma_3 G_{-}(y, k))}{W[G_{+}(\cdot, k), G_{-}(\cdot, k)]}.$$

We have $R_{H_{\omega}}(x, y, \bar{\lambda}) = \overline{R_{H_{\omega}}(x, y, \lambda)}$ and $R_{H_{\omega}}(x, y, -\lambda) = -\sigma_1 R_{H_{\omega}}(x, y, \lambda) \sigma_1$.

We define:

Definition 8.11 (Plane waves). For $k \notin S$ set $\Psi(x, k) := \frac{1}{\sqrt{2\pi}} T(k) F_{+}(x, k)$ for $k \geq 0$ and $\Psi(x, k) := \frac{1}{\sqrt{2\pi}} T(-k) F_{-}(x, -k)$ for $k < 0$. Near S set $\Psi(x, k) = \frac{1}{\sqrt{2\pi}} \Phi_{+}(x, k)$ for $k > 0$ and $\Psi(x, k) = \frac{1}{\sqrt{2\pi}} \Phi_{-}(x, -k)$ for $k < 0$.

The interpretation of the $\Psi(x, k)$ is justified by the following lemma:

Lemma 8.12. *The operator $P_c(H_{\omega})$ has, for ${}^t A$ the transpose of A , the kernel*

$$P_c(H_{\omega})(x, y) = \int_{\mathbb{R}} \left[\overline{\Psi(x, k)} {}^t (\sigma_3 \Psi(y, k)) + \sigma_1 \overline{\Psi(x, k)} {}^t (\sigma_3 \sigma_1 \Psi(y, k)) \right] dk.$$

In particular we have $(e^{itH_{\omega}} P_c(H_{\omega}))(x, y) =$

$$\int_{\mathbb{R}} \left[e^{it(E(k)+\omega)} \overline{\Psi(x, k)} {}^t (\sigma_3 \Psi(y, k)) + e^{-it(E(k)+\omega)} \sigma_1 \overline{\Psi(x, k)} {}^t (\sigma_3 \sigma_1 \Psi(y, k)) \right] dk.$$

Lemma 8.13. *The following operators $P_{\pm}(\omega)$ are well defined in L_x^2 :*

$$P_+(\omega)u = \lim_{M \nearrow \infty} \lim_{\epsilon \searrow 0} \frac{1}{2\pi i} \int_{[\omega, M] \cap (\omega + \sigma(h_0))} [R_{H_\omega}(\lambda + i\epsilon) - R_{H_\omega}(\lambda - i\epsilon)] u d\lambda,$$

$$P_-(\omega)u = \lim_{M \nearrow \infty} \lim_{\epsilon \searrow 0} \frac{1}{2\pi i} \int_{[-M, -\omega] \cap (-\omega - \sigma(h_0))} [R_{H_\omega}(\lambda + i\epsilon) - R_{H_\omega}(\lambda - i\epsilon)] u d\lambda$$

and have kernel

$$P_+(\omega)(x, y) = \int_{\mathbb{R}} \overline{\Psi(x, k)}^t (\sigma_3 \Psi(y, k)) dk,$$

$$P_-(\omega)(x, y) = \int_{\mathbb{R}} \sigma_1 \overline{\Psi(x, k)}^t (\sigma_3 \sigma_1 \Psi(y, k)) dk.$$

For any $M > 0$ and $N > 0$ and for $C = C(N, M, \omega)$ upper semicontinuous in ω , we have

$$(1) \quad \|\langle x \rangle^M (P_+(\omega) - P_-(\omega) - P_c(H_\omega) \sigma_3) f\|_{L_x^2} \leq C \|\langle x \rangle^{-N} f\|_{L_x^2}.$$

9. PROOF OF LEMMAS 3.1-3.2

Lemma 3.1 is a consequence of the following statement:

Lemma 9.1. *Set $P_c(H_\omega) e^{itH_\omega}(x, y) = \mathcal{U}(t, x, y) + \mathcal{V}(t, x, y)$ with*

$$\mathcal{U}(t, x, y) = \int_{|k| \leq \pi n(n_0)} e^{itE(k)} \phi(x, y, k) dk,$$

$$\mathcal{V}(t, x, y) = \int_{|k| \geq \pi n(n_0)} e^{itE(k)} \phi(x, y, k) dk$$

with $\phi(x, y, k) = \overline{\Psi(x, k)}^t (\sigma_3 \Psi(y, k)) + \sigma_1 \overline{\Psi(x, k)}^t (\sigma_3 \sigma_1 \Psi(y, k)).$

Then $|\mathcal{U}(t, x, y)| \leq C \langle t \rangle^{-\frac{1}{3}}$ and $|\mathcal{V}(t, x, y)| \leq C |t|^{-\frac{1}{2}}$.

Proof. Using cutoff functions we distinguish between k close to S and k away from S . In the latter case the proof is the same as Lemma 6.5. So for $\chi(k)$ a smooth cutoff function with small support I with $I \cap S = \{\widehat{k}\}$ and with $I \subset \mathbb{R}_+$ (the above choices are not restrictive) contained either in $|k| < \pi n(n_0)$ or in $|k| > \pi n(n_0)$, we consider the integral

$$(1) \quad \int_I e^{itE(k)} \chi(k) \overline{\Phi}_+(x, k)^t \Phi_-(y, k) dk.$$

Notice that by (8.6)

$$\Phi_+(x, k) = \phi_{\pm}^0(x, k) \overrightarrow{e}_1 (1 + A(x, k)) + \phi_-^0(x, k) B(x, k) + C(x, y, k)$$

with $|\partial_k^a A(x, k)| + |\partial_k^a B(x, k)| + |\partial_k^a C(x, y, k)| < c_0$ for fixed c_0 for $a = 0, 1$. Then by stationary phase $|(1)| \leq \langle t \rangle^{-\alpha}$ with α either $1/2$ if I is in $|k| < \pi n(n_0)$ or $1/3$ if I is in $|k| > \pi n(n_0)$. Before proving Lemma 3.2 we need:

Lemma 9.2. *Let $I_{\alpha, \beta}[f] = K_{\alpha, \beta}(t) * f$ with $K_{\alpha, \beta} \equiv \chi_{[0, 1)} t^{-\alpha} + \chi_{[1, \infty)} t^{-\beta}$ for given $0 < \alpha < 1$ and $0 < \beta < 1$. Then we have the following estimates:*

$$\|I_{\alpha, \beta}[f]\|_{\ell^p(\mathbb{Z}, L_t^r[n, n+1])} \leq C \|f\|_{\ell^q(\mathbb{Z}, L_t^s[n, n+1])},$$

where $C \equiv C(p, q, r, s) > 0$ and

$$\begin{aligned} (1) \quad & 1 + \frac{1}{r} \geq \alpha + \frac{1}{s}, \\ (2) \quad & 1 + \frac{1}{p} \leq \beta + \frac{1}{q}, \\ (3) \quad & (r, s) \neq \left(\infty, \frac{1}{1-\alpha} \right), \left(\frac{1}{\alpha}, 1 \right), \\ (4) \quad & (p, q) \neq \left(\infty, \frac{1}{1-\beta} \right), \left(\frac{1}{\beta}, 1 \right). \end{aligned}$$

If moreover we assume $\alpha = 0$ and $0 < \beta < 1$, then

$$\|I_{0,\beta} f\|_{\ell^p(\mathbb{Z}, L_t^r[n, n+1])} \leq C \|f\|_{\ell^q(\mathbb{Z}, L_t^s[n, n+1])},$$

where $1 \leq r, s \leq \infty$ and $1 \leq p, q \leq \infty$ satisfy (2) and (4).

The elementary proof is in Lemma 2.2 of [GV].

Proof of Lemma 3.2. Using the definition of $\mathcal{U}(t)$ and $\mathcal{V}(t)$, the proof of the Strichartz estimates for (2) and (4) of Lemma 3.2 are the same as that in Lemma 3.1 of [C1]. We turn now to the estimate for $\mathcal{U}(t)$. In the sequel the pairs (r, p) , (r_1, p_1) and (r_2, p_2) are always admissible. First of all, by $[H_\omega, \mathcal{U}(t)] = 0$, it is not restrictive to consider only case $k = 0$. We set $P_\pm = P_\pm(H_\omega)$. We have $[\mathcal{U}, P_\pm] = 0$, $\mathcal{U}^* \sigma_3 = \sigma_3 \mathcal{U}$ and $P_\pm^* \sigma_3 = \sigma_3 P_\pm$.

First step: Proof of (1) Lemma 3.2. The case $(r, p) = (\infty, 2)$ is trivial; hence by interpolation it suffices to prove (1) in the case $(r, p) = (4, \infty)$. By Lemma 3.1

$$\left\| \int_0^\infty \mathcal{U}(t-s) F(s) ds \right\|_{L_x^\infty} \leq C \int_0^\infty \|F(s)\|_{L_x^1} \frac{ds}{\langle t-s \rangle^{\frac{1}{3}}}.$$

By Lemma 9.2 with $\alpha = 0$ and $\beta = \frac{1}{3}$,

$$(9.1) \quad \left\| \int_0^\infty \mathcal{U}(t-s) F(s) ds \right\|_{\ell^6(\mathbb{Z}, L_t^\infty([n, n+1], L_x^\infty))} \leq C \|F\|_{\ell^{\frac{6}{5}}(\mathbb{Z}, L_t^1([n, n+1], L_x^1))}.$$

By Fubini's theorem we have

$$\begin{aligned} & \left\langle \int_0^\infty \mathcal{U}(s) P_\pm F(s) ds, \sigma_3 \int_0^\infty \mathcal{U}(t) P_\pm F(t) dt \right\rangle_x \\ &= \left\langle \int_0^\infty \mathcal{U}(t-s) P_\pm F(s) ds, \sigma_3 P_\pm F(t) \right\rangle_{t,x}. \end{aligned}$$

This implies

$$\begin{aligned} (9.2) \quad & \left\| \int_0^\infty \mathcal{U}(s) F(s) ds \right\|_{L_x^2}^2 \lesssim \left\| \int_0^\infty \mathcal{U}(t-s) F(s) ds \right\|_{\ell^6(\mathbb{Z}, L_t^\infty([n, n+1], L_x^\infty))} \\ & \times \|F\|_{\ell^{\frac{6}{5}}(\mathbb{Z}, L_t^1([n, n+1], L_x^1))}^2 \leq C \|F\|_{\ell^{\frac{6}{5}}(\mathbb{Z}, L_t^1([n, n+1], L_x^1))}^2 \end{aligned}$$

by (9.1) and by $\|P_\pm u\|_{L_x^2}^2 \approx |\langle P_\pm u, \sigma_3 P_\pm u \rangle|$. The latter follows from the following facts, for W and Z the operators in Proposition 7.1: for $u \in L_c^2(H_\omega)$ and $u = Wv$, we have $\|P_+ u\|_{L_x^2} \approx \|\text{diag}(1, 0)v\|_{L_x^2}$, $\|P_- u\|_{L_x^2} \approx \|\text{diag}(0, 1)v\|_{L_x^2}$; for any pair $\tilde{u} = W\tilde{v}$ we have $\langle \tilde{u}, \sigma_3 \tilde{u} \rangle = \langle \tilde{v}, \sigma_3 \tilde{v} \rangle$.

To finally deduce (1) of Lemma 3.2, notice that by combining the Fubini theorem, a duality argument, the Hölder inequality and (9.2), we get for $f \in L_c^2(H_\omega)$

$$\begin{aligned} \|\mathcal{U}(t)f\|_{\ell^6(\mathbb{Z}, L_t^\infty([n, n+1], L_x^\infty))} &= \sup_{G \in \mathcal{B}_{\frac{6}{5}, 1, 1}} \langle \mathcal{U}(t)f, \sigma_3 G(t, x) \rangle_{t, x} \\ &= \sup_{G \in \mathcal{B}_{\frac{6}{5}, 1, 1}} \left\langle f, \sigma_3 \int_0^\infty \mathcal{U}(t) P_c(H_\omega) G(t) dt \right\rangle_x \\ &\leq \|f\|_{L_x^2} \sup_{G \in \mathcal{B}_{\frac{6}{5}, 1, 1}} \left\| \int_0^\infty \mathcal{U}(t) P_c(H_\omega) G(t) dt \right\|_{L_x^2} \leq C \|f\|_{L_x^2}, \end{aligned}$$

where

$$\mathcal{B}_{\frac{6}{5}, 1, 1} := \left\{ G \in \ell^{\frac{6}{5}}(\mathbb{Z}, L_t^1([n, n+1], L_x^1)) \text{ s.t. } \|G\|_{\ell^{\frac{6}{5}}(\mathbb{Z}, L_t^1([n, n+1], L_x^1))} = 1 \right\}.$$

Second step: Proof of (3) Lemma 3.2. We split the proof in 7 subcases.

First subcase: $(r_i, p_i) = (\infty, 2)$ for $i = 1, 2$. In this case the estimate (3) of Lemma 3.2 is equivalent to the following one:

$$\left\| \int_0^t \mathcal{U}(t-s) F(s) ds \right\|_{L_t^\infty L_x^2} \leq C \|F\|_{L_t^1 L_x^2},$$

whose proof is elementary,

$$\left\| \int_0^t \mathcal{U}(t-s) F(s) ds \right\|_{L_x^2} \leq \int_0^t \|\mathcal{U}(t-s) F(s)\|_{L_x^2} ds \leq C \int_0^t \|F(s)\|_{L_x^2} ds,$$

where we have used the fact that $\|\mathcal{U}(t)f\|_{L_x^2} \leq C \|f\|_{L_x^2}$ for a fixed C .

Second subcase: $(r_i, p_i) = (4, \infty)$ for $i = 1, 2$. It is similar to the proof of (9.1).

Third subcase: $(r_1, p_1) = (4, \infty)$ and $(r_2, p_2) = (\infty, 2)$. In this case (3) of Lemma 3.2 reduces to

$$(9.3) \quad \left\| \int_0^t \mathcal{U}(t-s) F(s) ds \right\|_{\ell^6(\mathbb{Z}, L_t^\infty([n, n+1], L_x^\infty))} \leq C \|F\|_{L_t^1 L_x^2}.$$

Notice that we have the identity

$$\int_0^t \mathcal{U}(t-s) F(s) ds = \int_0^\infty \chi(t-s) \mathcal{U}(t-s) F(s) ds,$$

with χ the characteristic function of the half line $(0, \infty)$. By the Minkowski inequality we get

$$\begin{aligned} &\left\| \int_0^t \mathcal{U}(t-s) F(s) ds \right\|_{\ell^6(\mathbb{Z}, L_t^\infty([n, n+1], L_x^\infty))} \\ &\leq \int_0^\infty \|\chi(t-s) \mathcal{U}(t-s) F(s)\|_{\ell^6(\mathbb{Z}, L_t^\infty([n, n+1], L_x^\infty))} ds \\ &\leq \int_0^\infty \|\mathcal{U}(t-s) F(s)\|_{\ell^6(\mathbb{Z}, L_t^\infty([n, n+1], L_x^\infty))} ds \leq C \int_0^\infty \|F(s)\|_{L_x^2} ds, \end{aligned}$$

where we have used (1) of Lemma 3.2 in the last step.

Fourth subcase: (r_1, p_1) any admissible pair and $(r_2, p_2) = (\infty, 2)$. This follows by interpolation between the first and the third subcases.

Fifth subcase: $(r_1, p_1) = (\infty, 2)$, $(r_2, p_2) = (4, \infty)$. This follows by a duality argument. In fact we have

$$\begin{aligned} \left\| \int_0^t \mathcal{U}(t-s)F(s)ds \right\|_{L_t^\infty L_x^2} &= \sup_{G \in \mathcal{B}_{1,2}} \left\langle \int_0^t \mathcal{U}(t-s)F(s)ds, \sigma_3 G \right\rangle_{t,x} \\ &= \sup_{G \in \mathcal{B}_{1,2}} \left\langle F, \sigma_3 \int_t^\infty \mathcal{U}(t-s)G(s)ds \right\rangle_{t,x}, \end{aligned}$$

where

$$\mathcal{B}_{1,2} := \left\{ G \in L_t^1 L_x^2 \text{ s.t. } \|G\|_{L_t^1 L_x^2} = 1 \right\}.$$

By Hölder inequality

$$\begin{aligned} \left\| \int_0^t \mathcal{U}(t-s)F(s)ds \right\|_{L_t^\infty L_x^2} &\leq \|F\|_{\ell^{\frac{6}{5}}(\mathbb{Z}, L_t^1([n, n+1], L_x^1))} \\ &\times \sup_{G \in \mathcal{B}_{1,2}} \left\| \int_t^\infty \mathcal{U}(t-s)G(s)ds \right\|_{\ell^6(\mathbb{Z}, L_t^\infty([n, n+1], L_x^\infty))} \leq C \|F\|_{\ell^{\frac{6}{5}}(\mathbb{Z}, L_t^1([n, n+1], L_x^1))}, \end{aligned}$$

where in the last step we used the estimate

$$\left\| \int_t^\infty \mathcal{U}(t-s)G(s)ds \right\|_{\ell^6(\mathbb{Z}, L_t^\infty([n, n+1], L_x^\infty))} \leq C \|G\|_{L_t^1 L_x^2},$$

whose proof is similar to (9.3).

Sixth subcase: (r_1, p_1) any admissible pair and $(r_2, p_2) = (4, \infty)$. This follows by interpolation of the second and fifth cases.

Seventh subcase: Remaining cases. This follows by interpolation of the fourth and sixth cases.

10. EXTENSION OF A RESULT BY CHRIST AND KISELEV TO BIRMAN-SOLOMJAK SPACES

Given two Banach spaces X and Y let $K(s, t)$ be a continuous function with values in $B(X, Y)$. Let us introduce the operators

$$T_K f(t) = \int_{-\infty}^{\infty} K(t, s) f(s) ds$$

and

$$\tilde{T}_K f(t) = \int_{-\infty}^t K(t, s) f(s) ds.$$

In this section we shall prove the following modified version of Lemma 3.1 in [SmS].

Proposition 10.1. *Let $1 \leq p, q, r \leq \infty$ be such that $1 \leq r < \min\{p, q\} \leq \infty$. Assume that there exist $C > 0$ such that*

$$(10.1) \quad \|T_K f\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))} \leq C \|f\|_{L_t^r(X)}.$$

Then

$$(10.2) \quad \|\tilde{T}_K f\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))} \leq C' \|f\|_{L_t^r(X)},$$

where $C' = C'(C, p, q, r) > 0$ is another suitable constant.

Remark. In the case $p = q$ the previous proposition follows from [CK].

We shall need the following lemma.

Lemma 10.2. *Let $k \in \mathbb{N}$ and $1 \leq p, q, r \leq \infty$ be as in Proposition 10.1. Assume that $\{I_j\}_{j=1, \dots, 2^k} \subset \mathbb{R}$ is a family of intervals (eventually unbounded) such that*

$$\mathbb{R} = \bigcup_j I_j \text{ and } \text{int} I_j \cap \text{int} I_k = \emptyset \text{ for } j \neq k.$$

Assume also that $g_j \in \ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))$ for $j = 1, \dots, 2^k$ is a family of functions such that

$$(10.3) \quad \|g_j\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))} \leq C 2^{-\frac{k}{r}} \quad \forall j = 1, \dots, 2^k.$$

Then the following estimate holds:

$$(10.4) \quad \left\| \sum_{j=1}^{2^k} \chi_{I_j} g_j \right\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))} \leq C' 2^{k(\frac{1}{q} - \frac{M \inf\{p, q\}}{qr})},$$

where $C' = C'(C, p, q, r) > 0$ and in particular C' does not depend on g_j, I_j, k .

Proof. Let us fix the following notation:

$$(10.5) \quad G(x) \equiv \sum_{j=1}^{2^k} \chi_{I_j} g_j$$

and

$$\mathcal{I} \equiv \{I_j \text{ for } j = 0, 1, \dots, 2^k\},$$

where I_j are the intervals given in the statement and $I_0 \equiv \emptyset$. To every $I_j \in \mathcal{I}$ we associate a new segment $\tilde{I}_j \subset I_j$ defined as follows:

$$\tilde{I}_j \equiv (z_j, z_{j+1}),$$

where $z_j, z_{j+1} \in \mathbb{Z} \cup \{\pm\infty\}$, $z_j \equiv \inf\{z \in \mathbb{Z} \cup \{-\infty\} | z \in I_j\}$ and $z_{j+1} \equiv \sup\{z \in \mathbb{Z} \cup \{+\infty\} | z \in I_j\}$. Notice that the possibility $\tilde{I}_j = \emptyset$ is allowed. Next we introduce

$$\tilde{\mathcal{I}} \equiv \{I_j \in \mathcal{I} | \tilde{I}_j \neq \emptyset\}$$

and also

$$(10.6) \quad \tilde{\mathbb{Z}} \equiv \{z \in \mathbb{Z} | (z, z+1) \cap (\bigcup_{j=1}^{2^k} \tilde{I}_j) = \emptyset\}.$$

Notice that in a more explicit way we can write

$$\tilde{\mathcal{I}} = \{I_{i_0}, I_{i_1}, \dots, I_{i_h}\}$$

for suitable $0 \leq h \leq 2^k$ and $0 = i_0 < i_1 < \dots < i_h \leq 2^k$. For every function $F \in \ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))$ we get

$$\|\chi_{(\bigcup_{I_j \in \tilde{\mathcal{I}}} \tilde{I}_j)} F\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))} = \left(\sum_{j=i_0, \dots, i_h} \|\chi_{\tilde{I}_j} F\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))}^q \right)^{\frac{1}{q}}.$$

In particular, if we choose in this identity $F = G$, with G as in (10.5), then we get

$$\begin{aligned}
 & \|\chi_{(\cup_{I_j \in \bar{I}} \bar{I}_j)} G\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))}^q = \|\chi_{(\cup_{I_j \in \bar{I}} \bar{I}_j)} (\sum_{l=1}^{2^k} \chi_{I_l} g_l)\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))}^q \\
 (10.7) \quad &= \sum_{j=i_0, i_1, \dots, i_h} \|\chi_{\bar{I}_j} g_j\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))}^q \\
 &\leq \sum_{j=i_0, i_1, \dots, i_h} \|g_j\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))}^q \leq C^q h 2^{-\frac{kq}{r}},
 \end{aligned}$$

where we have used the assumption (10.3). Next we shall estimate

$$\|\chi_{\mathbb{R} \setminus (\cup_{I_j \in \bar{\mathcal{I}}} \bar{I}_j)} G\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))},$$

with G as in (10.5). In order to do that let us associate to any integer $n_0 \in \tilde{\mathbb{Z}}$, defined in (10.6), the following set:

$$\mathcal{I}_{n_0} \equiv \{I \in \mathcal{I} \mid I \cap (n_0, n_0 + 1) \neq \emptyset\}.$$

In particular, we can write in a more explicit way:

$$\mathcal{I}_{n_0} \equiv \{I_{i(n_0)}, I_{i(n_0)+1}, \dots, I_{i(n_0)+h(n_0)}\},$$

where

$$0 \leq h(n_0) \leq 2^k, i(n_0) \in 1, \dots, 2^k.$$

It is easy to deduce that the assumptions on $\{I_j\}_{j=1, \dots, 2^k}$ imply

$$(10.8) \quad 0 \leq h \leq 2^k \text{ and } \sum_{n_0 \in \tilde{\mathbb{Z}}} h(n_0) \leq 2^{k+1}.$$

Next notice that for every $n_0 \in \tilde{\mathbb{Z}}$ we have

$$\begin{aligned}
 & \|\chi_{(n_0, n_0+1)} G\|_{L^p([n_0, n_0+1], Y)}^p = \sum_{J \in \mathcal{I}_{n_0}} \|\chi_{J \cap [n_0, n_0+1]} G\|_{L^p([n_0, n_0+1], Y)}^p \\
 &= \sum_{h=0}^{h(n_0)} \|\chi_{I_{i(n_0)+h} \cap [n_0, n_0+1]} g_{i(n_0)+h}\|_{L^p([n_0, n_0+1], Y)}^p \\
 &\leq \sum_{h=0}^{h(n_0)} \|g_{i(n_0)+h}\|_{L^p([n_0, n_0+1], Y)}^p \leq C^p (h(n_0) + 1) 2^{-\frac{kp}{r}},
 \end{aligned}$$

and this implies

$$(10.9) \quad \|\chi_{(n_0, n_0+1)} G\|_{L^p([n_0, n_0+1], Y)}^q \leq C^q 2^{\frac{q}{p}} 2^{-\frac{kq}{r}} h(n_0)^{\frac{q}{p}} = C^q 2^{\frac{q}{p}} (2^{-\frac{kp}{r}} h(n_0))^{\frac{q}{p}}.$$

Next we split the proof of (10.4) into two cases.

First case: $q \geq p$

In this case we have the following estimate:

$$(10.10) \quad C^q 2^{\frac{q}{p}} (2^{-\frac{kp}{r}} h(n_0))^{\frac{q}{p}} \leq C^q 2^{\frac{q}{p}} 2^{-\frac{kp}{r}} h(n_0),$$

where at the last step we have used the assumption $q \geq p$ and the following fact:

$$2^{-\frac{kp}{r}} h(n_0) \leq 2^{-k} h(n_0) \leq 1,$$

that in turn follows from the hypothesis $p > r$ and $0 \leq h(n_0) \leq 2^k$.

By combining (10.7) with (10.9) and (10.10) we get

$$\begin{aligned} \|G\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))}^q &= \|\chi_{(\cup_{I_j \in \tilde{I}} \tilde{I}_j)} G\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))}^q + \sum_{n_0 \in \tilde{\mathbb{R}}} \|G\|_{L^p([n_0, n_0+1], Y)}^q \\ &\leq C^q 2^{-\frac{kq}{r}} h + \sum_{n_0 \in \tilde{\mathbb{Z}}} C^q 2^{\frac{q}{p}} 2^{-\frac{kp}{r}} h(n_0), \end{aligned}$$

that due to (10.8) implies

$$\|G\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))}^q \leq C^q 2^{k(1-\frac{q}{r})} + 2C^q 2^{\frac{q}{p}} 2^{k(1-\frac{p}{r})}.$$

Hence, due to the hypothesis $q \geq p > r$, we get

$$\|G\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))}^q \leq 3C^q 2^{\frac{q}{p}} 2^{k(1-\frac{p}{r})}.$$

Second case: $q < p$

In this case we have

$$(10.11) \quad C^q 2^{\frac{q}{p}} (2^{-\frac{kp}{r}} h(n_0))^{\frac{q}{p}} = C^q 2^{\frac{q}{p}} 2^{-\frac{kq}{r}} h(n_0)^{\frac{q}{p}} \leq C^q 2^{\frac{q}{p}} 2^{-\frac{kq}{r}} h(n_0),$$

where at the last step we have used $h(n_0) \geq 1$ and $q < p$. By combining (10.7) with (10.9) and (10.11) we get

$$\begin{aligned} \|G\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))}^q &= \|\chi_{(\cup_{I_j \in \tilde{I}} \tilde{I}_j)} G\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))}^q + \sum_{n_0 \in \tilde{\mathbb{Z}}} \|G\|_{L^p([n_0, n_0+1], Y)}^q \\ &\leq C^q 2^{-\frac{kq}{r}} h + \sum_{n_0 \in \tilde{\mathbb{Z}}} C^q 2^{\frac{q}{p}} 2^{-\frac{kq}{r}} h(n_0) \leq 3C^q 2^{\frac{q}{p}} 2^{k(1-\frac{q}{r})}, \end{aligned}$$

where at the last step we have used (10.8).

Proof of Proposition 10.1. We shall follow the notation used in [SmS]. Let $f \in L_t^r(X)$ satisfy the following conditions:

- (1) $\|f\|_{L_t^r(X)} = 1$;
- (2) the function $F(t)$ defined below is a bijection between $(-\infty, \infty)$ and $(0, 1)$,

$$(10.12) \quad F(t) = \int_{-\infty}^t \|f(s)\|_X^r ds.$$

By an elementary density argument we have that (10.2) will follow once we show that $\|\tilde{T}_K f\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))} \leq C'$ for any $f \in L_t^r(X)$ that satisfies the conditions above. Next we consider the set of all dyadic subintervals of $[0, 1]$. If I and J are two such subintervals, we say that $I \sim J$ if the following conditions hold:

- (1) I and J have the same length;
- (2) I must lie on the left of J ;
- (3) there exist two dyadic cubes I_0 and J_0 whose length is twice the length of I and J , and moreover I_0 and J_0 are adjacent.

Notice that if J is fixed, then there are two dyadic intervals such that $I \sim J$. Following [CK], [SmS] we can write the identity

$$\tilde{T}_K f = \sum_{\{I, J: I \sim J\}} \chi_{F^{-1}J} T_K(\chi_{F^{-1}I} f),$$

where χ_A denotes the characteristic function of the set A and $F(t)$ is given in (10.12). Due to the Minkowski inequality we get

$$(10.13) \quad \begin{aligned} & \|\tilde{T}_K f\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))} \\ & \leq \sum_{k=2}^{\infty} \left\| \sum_{\{I, J: I \sim J, |I|=2^{-k}\}} \chi_{F^{-1}J} T(\chi_{F^{-1}I} f) \right\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))}. \end{aligned}$$

Next for every $k \geq 2$ we fix the following notation:

$$\begin{aligned} I_j^k &\equiv F^{-1} \left(\frac{j}{2^k}, \frac{j+1}{2^k} \right) \text{ for } j = 0, 1, \dots, 2^k - 1, \\ f_j^k &= \chi_{I_j^k} f \text{ and } g_j^k = T_K(f_j^k). \end{aligned}$$

Notice that

$$\|f_j^k\|_{L_t^r(X)} = 2^{-\frac{k}{r}} \quad \forall j = 0, 1, \dots, 2^k - 1,$$

and due to the assumption (10.2) we deduce that

$$\begin{aligned} & \|g_j^k\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))} \\ & = \|T_K(f_j^k)\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))} \leq C 2^{-\frac{k}{r}} \quad \forall j = 0, \dots, 2^k - 1. \end{aligned}$$

We are then in a position to use Lemma 10.2 in order to deduce

$$\begin{aligned} & \left\| \sum_{\{I, J: I \sim J, |I|=2^{-k}\}} \chi_{F^{-1}J} T_K(\chi_{F^{-1}I} f) \right\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))} \\ & = \left\| \sum_{j=2, \dots, 2^k-1} \chi_{I_{j-1}^k} g_j^k(x) + \sum_{j=2, \dots, 2^k-1} \chi_{I_{j-2}^k} g_j^k(x) \right\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))} \\ & \leq \left\| \sum_{j=2, \dots, 2^k} \chi_{I_{j-1}^k} g_j^k(x) \right\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))} + \left\| \sum_{j=2, \dots, 2^k} \chi_{I_{j-2}^k} g_j^k(x) \right\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))} \\ & \leq 2C' 2^{k(\frac{1}{q} - \frac{\text{Min}\{p, q\}}{qr})}. \end{aligned}$$

By (10.13) we finally get

$$\|\tilde{T}_K f\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))} \leq 2C' \sum_{k=2}^{\infty} 2^{k(\frac{1}{q} - \frac{\text{Min}\{p, q\}}{qr})},$$

and since $\frac{1}{q} - \frac{\text{Min}\{p, q\}}{qr} < 0$ we have the desired result.

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