A PROPERTY OF $C_p[0, 1]$

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ABSTRACT. We prove that for every finite-dimensional compact metric space X there is an open continuous linear surjection from $C_p[0, 1]$ onto $C_p(X)$. The proof makes use of embeddings introduced by Kolmogorov and Sternfeld in connection with Hilbert's 13th problem.

1. INTRODUCTION

Maps are assumed to be continuous. A compactum is a metric compact space. For a space X , $C_p(X)$ denotes the space of continuous real-valued functions equipped with the topology of pointwise convergence. We refer to the topology of pointwise convergence as the C_p -topology.

Let $X \subset Y_1 \times \cdots \times Y_k$ be an embedding of a compactum X into the product of compacta $Y_1, \dots, Y_k$. Define the space Z as the disjoint union of $Y_1, \dots, Y_k$ and define the linear transformation $L : C(Z) \rightarrow C(X)$ by $L(g)(x) = g(y_1) + g(y_2) + \cdots + g(y_k)$ for $g \in C(Z)$ and $x = (y_1, \dots, y_k) \in X$, where $y_1, \dots, y_k$ are the coordinates of x in the product $Y_1 \times \cdots \times Y_k$. We will call L the induced transformation of the embedding $X \subset Y_1 \times \cdots \times Y_k$. It is obvious that L is continuous in both the uniform topology and the C_p -topology on the function spaces.

An embedding $X \subset Y_1 \times \cdots \times Y_k$ is called basic if the induced transformation L is surjective. Note that, in general, a surjective linear transformation of the function spaces on compacta which is continuous in both the uniform topology and the C_p -topology is not necessarily open in the C_p -topology [8]. It was shown in [8] that the transformation induced by a basic embedding in the product of two spaces ($k = 2$) is open in the C_p -topology, and it is an open problem if the similar result holds for $k > 2$. In this paper we will give a partial answer to this problem for two types of basic embeddings, namely, Kolmogorov and Sternfeld-type embeddings.

Sternfeld [6] constructed for every n -dimensional compactum X a basic embedding of X into the product of $n + 1$ one-dimensional compacta. We will call this embedding a Sternfeld-type embedding. Sternfeld-type embeddings are defined in Section 2.

Adjusting Kolmogorov's solution of Hilbert's 13th problem given in Kolmogorov's famous superposition theorem [1], Ostrand [2] defined for every n -dimensional compactum X a basic embedding $X \subset [0, 1]^{2n+1}$ which we will call a Kolmogorov-type embedding. Kolmogorov-type embeddings are described in Section 3.

In Sections 2 and 3 we will prove the following theorems.

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Theorem 1.1. *Let $X \subset Y_1 \times \cdots \times Y_{n+1}$ be a Sternfeld-type embedding of an n -dimensional compactum X into the product of one-dimensional compacta $Y_1, \dots, Y_{n+1}$. Then the induced transformation is open in the C_p -topology.*

Theorem 1.2. *Let $X \subset [0, 1]^3$ be a Kolmogorov-type embedding of a one-dimensional compactum X into the cube. Then the induced transformation is open in the C_p -topology.*

Let X be n -dimensional and compact. By Theorem 1.1 there is a one-dimensional compactum Z (= the disjoint union of $Y_1, \dots, Y_{n+1}$) for which $C_p(Z)$ admits an open continuous linear transformation onto $C_p(X)$. Let $\hat{Z}$ = the disjoint union of three copies of $[0, 1]$. By Theorem 1.2 there is an open continuous linear transformation from $C_p(\hat{Z})$ onto $C_p(Z)$. Embed $\hat{Z}$ into $[0, 1]$ and take the restriction transformation from $C_p[0, 1]$ to $C_p(\hat{Z})$ which is obviously surjective, open and continuous. Thus we obtain the main result of the paper.

Theorem 1.3. *For every finite-dimensional compactum X there is an open continuous linear transformation from $C_p[0, 1]$ onto $C_p(X)$.*

Note that $C_p[0, 1]$ and $C_p(X)$ for a compactum X are not isomorphic (= linearly homeomorphic) if $\dim X > 1$ and in many cases when $\dim X = 1$. In contrast to the uniform topology, the existence of an isomorphism between $C_p(X)$ and $C_p(Y)$ for compacta X and Y implies a great deal of similarity between X and Y ; in particular, it implies that $\dim X = \dim Y$ [3].

Theorem 1.3 generalizes some previous results by Leiderman, Pestov, Morris and the author [7], [8]. Open problems and related results are discussed in Section 4.

2. STERNFELD-TYPE EMBEDDINGS

Let X be an n -dimensional compactum. Sternfeld [6] showed that there are a decomposition $X = A_1 \cup \cdots \cup A_{n+1}$ of X into 0-dimensional subsets $A_1, \dots, A_{n+1}$ and one-dimensional compacta $Y_1, \dots, Y_{n+1}$ such that X admits an embedding $X \subset Y_1 \times \cdots \times Y_{n+1}$ having the following property: $x = p_i^{-1}(p_i(x))$ for every projection $p_i : X \rightarrow Y_i$ and $x \in A_i$. We will call such an embedding of X a Sternfeld-type embedding with respect to a decomposition $A_1, \dots, A_{n+1}$ of X . The spaces Y_i can be chosen to be dendrites [6]. A different way of constructing Sternfeld-type embeddings can be derived from [9].

Sternfeld [6] proved that any Sternfeld-type embedding is basic. Sternfeld's proof is based on Borel measures, and it is not clear at all if it can be applied to prove Theorem 1.1. In this paper we use another more constructive approach which is described in Subsection 2.1 and which also shows that Sternfeld-type embeddings are basic.

2.1. An approximation procedure. Let $X \subset Y_1 \times \cdots \times Y_{n+1}$ be a Sternfeld-type embedding of an n -dimensional compactum X with respect to a decomposition $X = A_1 \cup \cdots \cup A_{n+1}$, $\dim A_i = 0$. Let us describe an approximation procedure showing that the embedding of X is basic. The case $n = 0$ is trivial. Assume that $n > 0$.

Let $f : X \rightarrow \mathbb{R}$ be continuous and $c > 0$ such that $\|f\| < c$. Fix $\epsilon > 0$ which will be determined later and which will depend only on $\|f\|$, c and n . Take a disjoint family $\mathcal{V}_i$ of open subsets of Y_i such that $\mathcal{V}_i$ covers $p_i(A_i)$ and $\text{diam} f(p_i^{-1}(V)) < \epsilon$ for every $V \in \mathcal{V}_i$. For every i choose a finite subfamily $\mathcal{U}_i \subset p_i^{-1}(\mathcal{V}_i)$ such that

$\mathcal{U} = \mathcal{U}_1 \cup \dots \cup \mathcal{U}_{n+1}$ covers X and the elements of $\mathcal{U}$ are non-empty. By V_U , $U \in \mathcal{U}_i$, we denote the set of $\mathcal{V}_i$ such that $U = p_i^{-1}(V_U)$. For every $U \in \mathcal{U}$ take a non-empty subset $F_U \subset U$ closed in X such that $\mathcal{F} = \{F_U : U \in \mathcal{U}\}$ covers X and fix a point $x_U \in F_U$. For every $U \in \mathcal{U}_i$ take a continuous function $\phi_U : Y_i \rightarrow [0, 1]$ such that $\phi_U(Y_i \setminus V_U) = 0$ and $\phi_U(p_i(F_U)) = 1$. Define $g'_i : Y_i \rightarrow \mathbb{R}$ as $g'_i = \sum_{U \in \mathcal{U}_i} \frac{1}{n+1} f(x_U) \phi_U$. Clearly $\|g'_i\| \leq \frac{1}{n+1} \|f\| < \frac{1}{n+1} c$. Let us show that for every $x \in X$ we have

$$(*) \quad |f(x) - \sum_i g'_i(y_i)| < \frac{n}{n+1} c,$$

where $y_i = p_i(x) \in Y_i$ are the coordinates of x .

Assume that $|f(x)| \leq \epsilon$. Then for every $U \in \mathcal{U}$ such that $x \in U$ we have $|f(x_U)| < 2\epsilon$, and hence $|g'_i(y_i)| < \frac{2}{n+1} \epsilon$ for every i . Then $|f(x) - \sum_i g'_i(y_i)| < \epsilon + 2\epsilon$. Thus taking $\epsilon < \frac{n}{3(n+1)} c$ we get that $(*)$ holds.

Assume that $f(x) > \epsilon$. Then for every $U \in \mathcal{U}$ such that $x \in U$ we have $0 < f(x_U) < f(x) + \epsilon$, and hence $0 < g'_i(y_i) < \frac{1}{n+1} (f(x) + \epsilon)$ for every i . Note that there is F_U , $U \in \mathcal{U}_j$, such that $x \in F_U$ and hence $\frac{1}{n+1} (f(x) - \epsilon) < g'_j(x) < \frac{1}{n+1} (f(x) + \epsilon)$. Then $0 < f(x) - g'_j(x) < \frac{n}{n+1} (f(x) + \epsilon)$ and $0 < \sum_{i \neq j} g'_i(x) < \frac{n}{n+1} (f(x) + \epsilon)$. Hence $|f(x) - \sum_i g'_i(y_i)| = |(f(x) - g'_j(x)) - \sum_{i \neq j} g'_i(x)| < \frac{n}{n+1} (f(x) + \epsilon)$. Thus taking $\epsilon < c - \|f\|$ we get that $(*)$ holds. In a similar way we check the case $f(x) < -\epsilon$ and get that in all the cases $(*)$ holds.

Recall that by Z we denote the disjoint union of $Y_1, \dots, Y_{n+1}$. Define $g' : Z \rightarrow \mathbb{R}$ by $g'|_{Y_i} = g'_i$. We have that $\|g'\| < \frac{1}{n+1} c$ and $\|f - L(g')\| < \frac{n}{n+1} c$. Applying the procedure described above iteratively, one can construct a sequence of maps $g^{(t)} : Z \rightarrow \mathbb{R}$ such that $\|g^{(t)}\| < \frac{1}{n+1} (\frac{n}{n+1})^{t-1} c$ and $\|f - L(\sum_{s=1}^t g^{(s)})\| < (\frac{n}{n+1})^t c$. Then for $g = \sum_{s=1}^\infty g^{(s)}$ we have $f = L(g)$, and hence the embedding of X is basic.

2.2. Proof of Theorem 1.1. Suppose that $X \subset Y_1 \times \dots \times Y_{n+1}$ is a Sternfeld-type embedding of an n -dimensional compactum X with respect to a decomposition $X = A_1 \cup \dots \cup A_{n+1}$, $\dim A_i = 0$.

Let Z' be a finite subset of Z , where Z is the disjoint union of Y_i 's. Denote $Y'_i = Y_i \cap Z'$, $X'_i = p_i^{-1}(Y'_i) \cap A_i$ and $X' = X'_1 \cup \dots \cup X'_{n+1}$. It is clear that each X'_i is finite and therefore X' is finite as well.

Take any map $f : X \rightarrow \mathbb{R}$ such that $f(x) = 0$ for every $x \in X'$. We will show that there is a map $g : Z \rightarrow \mathbb{R}$ such that $g(z) = 0$ for every $z \in Z'$ and $L(g) = f$. This property, together with the fact that L is open in the uniform topology, implies that L is open in the C_p -topology at the zero-map on Z . Indeed, given $\alpha > 0$ let $W(Z', \alpha) = \{\bar{g} \in C(Z) : |\bar{g}(z)| < \alpha \text{ for } z \in Z'\}$. Since $W(Z', \alpha)$ contains all the maps on Z of norm $< \alpha$, there is $\delta > 0$ such that $L(W(Z', \alpha))$ contains all the maps on X of norm $< \delta$. Let $W(X', \delta) = \{\bar{f} \in C(X) : |\bar{f}(x)| < \delta \text{ for } x \in X'\}$. Take $\bar{f} \in W(X', \delta)$ and let $\bar{\bar{f}} \in C(X)$ be such that $\|\bar{\bar{f}}\| < \delta$ and $\bar{\bar{f}}(x) = \bar{f}(x)$ for $x \in X'$. Then there is $\bar{\bar{g}} \in W(Z', \alpha)$ such that $L(\bar{\bar{g}}) = \bar{\bar{f}}$. Set $f = \bar{f} - \bar{\bar{f}}$ and note that $f(x) = 0$ for $x \in X'$. Assuming that there is $g \in C(Z)$ such that $L(g) = f$ and $g(z) = 0$ for $z \in Z'$, we get that $\bar{g} = g + \bar{\bar{g}} \in W(Z', \alpha)$ and $L(\bar{g}) = \bar{f}$. Thus $W(X', \delta) \subset L(W(Z', \alpha))$, and hence L is open in the C_p -topology at the zero-map on Z . By the linearity of L , L is open in the C_p -topology everywhere and Theorem 1.1 follows.

The construction of g follows the procedure described in Subsection 2.1 with the following additional requirements. We assume that no point of $Y'_i \setminus p_i(A_i)$ is covered by $\mathcal{V}_i$, $p_i^{-1}(V)$ contains at most one point of X' for every $V \in \mathcal{V}_i$ and if $U \in \mathcal{U}$ contains a point of X' , then this point is also contained in F_U and $x_U = F_U \cap X'$. It is easy to see that under these assumptions we have that $g'_i(y) = 0$ for every $y \in Y'_i$ and $g'_i(p_i(x)) = 0$ for every $x \in X'$. Thus $g'(z) = 0$ for every $z \in Z'$ and $f(x) - L(g')(x) = 0$ for every $x \in X'$. Hence the approximation procedure can be repeated iteratively and we can construct a map $g : Z \rightarrow \mathbb{R}$ with the required properties. The theorem is proved.

3. KOLMOGOROV-TYPE EMBEDDINGS

Let X be an n -dimensional compactum. A cover of X is said to cover X at least $n + 1$ times if every point of X belongs to at least $n + 1$ elements of the cover. Generalizing Kolmogorov's paper [1], Ostrand [2] showed that there is a countable family Ω of closed finite covers of X such that $\inf\{\text{mesh}\mathcal{F} : \mathcal{F} \in \Omega\} = 0$, each $\mathcal{F} \in \Omega$ covers X at least $n + 1$ times and each $\mathcal{F} \in \Omega$ splits into the union $\mathcal{F} = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_{2n+1}$ of $2n + 1$ families of disjoint sets. Such a family of covers Ω with a fixed splitting $\mathcal{F} = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_{2n+1}$ for each $\mathcal{F} \in \Omega$ will be called a Kolmogorov family of covers (Kolmogorov constructed a Kolmogorov family of covers for $X = [0, 1]^n$, and this family can be easily transferred to an arbitrary n -dimensional compactum using a 0-dimensional map to $[0, 1]^n$).

In the case when a Kolmogorov family contains a cover of mesh = 0 (that may happen only if X is finite), we assume that this cover appears in the family infinitely many times. We will also assume that a Kolmogorov family contains only finitely many covers of mesh $> \epsilon$ for every $\epsilon > 0$. Thus we always assume that a Kolmogorov family is infinite and any infinite subfamily of a Kolmogorov family is Kolmogorov as well.

A map from X to $[0, 1]$ is said to separate a family of disjoint sets in X if the images of the sets are disjoint in $[0, 1]$. Let Ω be a Kolmogorov family of covers of X . An embedding $X \subset [0, 1]^{2n+1}$ is said to separate a cover $\mathcal{F} \in \Omega$ if the projection $p_i : X \rightarrow [0, 1]$ separates $\mathcal{F}_i$ for every i . An embedding $X \subset [0, 1]^{2n+1}$ will be called a Kolmogorov-type embedding with respect to Ω if for every $\epsilon > 0$ there is $\mathcal{F} \in \Omega$ with $\text{mesh}\mathcal{F} < \epsilon$ such that the embedding of X separates $\mathcal{F}$. Note that almost all embeddings of X into $[0, 1]^{2n+1}$ are of Kolmogorov-type with respect to Ω ; see [4]. In the next subsection we present an approximation procedure which can be derived from Kolmogorov's paper [1] and which shows that Kolmogorov-type embeddings are basic. This fact was observed by Ostrand [2]; see also [4].

Note that for a Kolmogorov-type embedding $X \subset [0, 1]^{2n+1}$ with respect to Ω we can replace Ω by any infinite subfamily of Ω and the embedding of X will remain of Kolmogorov-type with respect to the replaced Ω . Thus we can always assume that a Kolmogorov-type embedding with respect to Ω separates every cover in Ω .

3.1. An approximation procedure. Let $X \subset [0, 1]^{2n+1}$ be a Kolmogorov-type embedding with respect to a family of covers Ω . Here we describe an approximation procedure showing that the embedding of X is basic. The case $n = 0$ is trivial. Assume that $n > 0$.

Let $f : X \rightarrow \mathbb{R}$ be continuous and $c > 0$ such that $\|f\| < c$. Fix $\epsilon > 0$ which will be determined later and which will depend only on $\|f\|$, c and n . Clearly we may assume that each $\mathcal{F} \in \Omega$ consists of non-empty sets. Choose any cover $\mathcal{F} \in \Omega$ such

that mesh $f(\mathcal{F}) = \{f(F) : F \in \mathcal{F}\} < \epsilon$. For every non-empty $F \in \mathcal{F}$ fix a point $x_F \in X$ such that $f(x_F)$ is at distance $< \epsilon$ from $f(F)$. (Showing that $X \subset [0, 1]^{2n+1}$ is basic is enough to choose x_F in F ; however, in the proof of Theorem 1.2 we may need to choose x_F outside F .)

Define $g'_i : [0, 1] \rightarrow \mathbb{R}$, $1 \leq i \leq 2n+1$, such that $g'_i(p_i(F)) = \frac{1}{2n+1}f(x_F)$ for every non-empty $F \in \mathcal{F}_i$ and $\|g'_i\| \leq \frac{1}{2n+1}\|f\| < \frac{1}{2n+1}c$. Let us show that for every $x \in X$ we have

$$(*) \quad |f(x) - \sum_i g'_i(y_i)| < \frac{2n}{2n+1}c,$$

where $y_i = p_i(x) \in [0, 1]$ are the coordinates of x .

Indeed, recall that $\mathcal{F}$ covers x at least $n+1$ times. Choose a set $I_+ \subset \{1, 2, \dots, 2n+1\}$ containing exactly $n+1$ indices such that x is covered by $\mathcal{F}_i$ for every $i \in I_+$ and denote $I_- = \{1, 2, \dots, 2n+1\} \setminus I_+$. For every $i \in I_+$ there is an $F \in \mathcal{F}_i$ containing x and hence $|\frac{1}{2n+1}f(x) - g'_i(y_i)| = \frac{1}{2n+1}|f(x) - f(x_F)| < \frac{1}{2n+1}2\epsilon$. Then $|f(x) - \sum_i g'_i(y_i)| = |\sum_{i \in I_+} (\frac{1}{2n+1}f(x) - g'_i(y_i)) + \sum_{i \in I_-} (\frac{1}{2n+1}f(x) - g'_i(y_i))| < \frac{n+1}{2n+1}2\epsilon + n(\frac{1}{2n+1}\|f\| + \frac{1}{2n+1}\|f\|) = \frac{2(n+1)}{2n+1}\epsilon + \frac{2n}{2n+1}\|f\|$. Thus taking $\epsilon < \frac{n}{n+1}(c - \|f\|)$ we get that $(*)$ holds.

Denote by Z the disjoint union of $2n+1$ copies $Y_i = [0, 1]$, $1 \leq i \leq 2n+1$, of the interval $[0, 1]$. Define $g' : Z \rightarrow \mathbb{R}$ by $g'|_{Y_i} = g'_i$. We have that $\|g'\| < \frac{1}{2n+1}c$ and $\|f - L(g')\| < \frac{2n}{2n+1}c$, where L is the linear transformation induced by the embedding $X \subset [0, 1]^{2n+1}$. Applying the procedure described above iteratively, one can construct a sequence of maps $g^{(t)} : Z \rightarrow \mathbb{R}$ such that $\|g^{(t)}\| < \frac{1}{2n+1}(\frac{2n}{2n+1})^{t-1}c$ and $\|f - L(\sum_{s=1}^t g^{(s)})\| < (\frac{2n}{2n+1})^t c$. Then for $g = \sum_{s=1}^\infty g^{(s)}$ we have $f = L(g)$, and hence the embedding of X is basic.

3.2. Embeddings of one-dimensional compacta. Let X be a one-dimensional compactum and $X \subset [0, 1]^3$ a Kolmogorov-type embedding with respect to a Kolmogorov family Ω of covers of X . Denote $Y_i = [0, 1]$, $1 \leq i \leq 3$, and let Z be the disjoint union of Y_i 's. Recall that by p_i we denote the projection $p_i : X \rightarrow Y_i$.

Reserved and free points. We say that $z \in Y_i \subset Z$ is a reserved point of Z with respect to Ω if for all but finitely many covers $\mathcal{F}$ in Ω , $\mathcal{F}_i$ intersects $p_i^{-1}(z)$. That is, there is $F \in \mathcal{F}_i$ such that F intersects $p_i^{-1}(z)$. Note that since p_i separates $\mathcal{F}_i$, at most one element of $\mathcal{F}_i$ can intersect $p_i^{-1}(z)$.

A point $z \in Y_i \subset Z$ is said to be strongly reserved with respect to Ω if $\mathcal{F}_i$ intersects $p_i^{-1}(z)$ for every $\mathcal{F} \in \Omega$ and the collection $\{F : F \cap p_i^{-1}(z) \neq \emptyset, F \in \mathcal{F}_i, \mathcal{F} \in \Omega\}$ converges to a point $x \in X$. That is, every neighborhood of x in X contains all but finitely many elements of the collection. We will say that the point x witnesses the reservation of z or that z is reserved by x .

A point $z \in Y_i \subset Z$ which is not reserved with respect to Ω is said to be free with respect to Ω . A point $z \in Y_i \subset Z$ is said to be fully free with respect to Ω if $\mathcal{F}_i$ does not intersect $p_i^{-1}(z)$ for every $\mathcal{F} \in \Omega$.

It is obvious that if $z \in Z$ is reserved (strongly reserved, reserved by $x \in X$, fully free) with respect to Ω , then z remains to be reserved (strongly reserved, reserved by x , fully free, respectively) with respect to any infinite subfamily of Ω . Note that if $z \in Z$ is reserved (free) with respect to Ω , then replacing Ω by its infinite subfamily we can get that z is strongly reserved (fully free) with respect to Ω . Also

note that since each $\mathcal{F} \in \Omega$ covers X at least twice, we get that if $x \in X$ is a point such that every coordinate of x is either strongly reserved or fully free with respect to Ω , then at least two coordinates of x are reserved by x .

Chains. A chain χ of length m with respect to Ω is a couple $\chi = (A, B)$ such that $A = \{z_0, z_1, \dots, z_{2m}\}$ is a sequence of $2m + 1$ elements of Z , $B = \{x_1, x_2, \dots, x_m\}$ is a sequence of m elements of X such that z_{2j-2}, z_{2j-1} and z_{2j} are the coordinates of x_j , the points $\{z_0, \dots, z_{2m-1}\}$ are strongly reserved with respect to Ω , z_{2j-2} and z_{2j-1} are reserved by x_j and the point z_{2m} is strongly reserved or fully free with respect to Ω . Note the coordinates z_{2j-2}, z_{2j-1} and z_{2j} of x_j do not necessarily go in the order corresponding to the order of the coordinates $x_j = (y_1, y_2, y_3)$ of x_j in the product of Y_1, Y_2 and Y_3 (for example it may happen that $z_{2j-2} = y_2$). The only thing that we assume is that $\{z_{2j-2}, z_{2j-1}, z_{2j}\} = \{y_1, y_2, y_3\}$ as subsets of Z . The chain χ is of length 0 if A contains only one point z_0 and $B = \emptyset$. The points z_0 and z_{2m} (x_1 and x_m if $m > 0$) are called the initial and the terminal Z -points (X -points, respectively) of the chain χ . The sequences A and B are called the Z -sequence and the X -sequence of χ , respectively. A chain $\chi' = (A', B')$ is said to be an extension of the chain χ if the sequences A' and B' start with A and B , respectively. A chain χ' is said to be a continuation of the chain χ if the terminal Z -point of χ is the initial Z -point of χ' . If $\chi' = (A', B')$ is a continuation of length m' of the chain $\chi(A, B)$, then we can define the chain $\chi'' = \chi + \chi' = (A'', B'')$ of length $m + m'$ by letting B'' be the sequence B followed by the sequence B' and A'' be the sequence A followed by the sequence A' when the last element of A is identified with the first element of A' . We will call χ and χ' the head of length m and the tail of length m' , respectively, of the chain χ'' and also write $\chi = \chi'' - \chi'$ and $\chi' = \chi'' - \chi$.

Almost free, periodic and non-periodic points. If the terminal Z -point of the chain χ is not free, then χ can be extended in the following way. Define x_{m+1} to be the point of X witnessing the reservation of z_{2m} , replace Ω by its infinite subfamily in order to obtain that every coordinate of x_{m+1} is either strongly reserved or fully free and define z_{2m+1} and z_{2m+2} as the other two coordinates of x_{m+1} (in addition to z_{2m}) such that z_{2m+1} is reserved by x_{m+1} (recall that x_{m+1} has at least two coordinates reserved by x_{m+1}). Note that when constructing chains we will constantly replace Ω by its infinite subfamily. Thus when talking about two (or finitely many) chains built one after another we will always refer to the smallest subfamily obtained in the last replacement.

Let Z' be a finite subset of Z . Replace Ω by its infinite subfamily in order that every point of Z' is either strongly reserved or fully free. Enumerate the points of Z' , and for every $z \in Z'$ define the chain $\chi(z, 0)$ as the chain of length 0 with the initial Z -point z . For every m we will replace Ω by its infinite subfamily and construct for every $z \in Z'$ a chain $\chi(z, m)$ proceeding from m to $m + 1$ as follows. Define $\chi(z, m + 1) = \chi(z, m)$ if the terminal Z -point of $\chi(z, m)$ is free. According to our enumeration of Z' , go over the points z of Z' such that the terminal Z -point of $\chi(z, m)$ is not free, and replacing (if necessary) each time Ω by its infinite subfamily, extend $\chi(z, m)$ to a chain of $\chi(z, m + 1)$ as described above (by adding one element to the X -sequence and two elements to Z -sequence of $\chi(z, m)$). Thus the length of $\chi(z, m) \leq m$ and the length of $\chi(z, m) = m$ if the terminal Z -point of $\chi(z, m)$ is not free.

Let us call $z \in Z'$ almost free if there is an m such that the terminal Z -point of $\chi(z, m)$ is free. If $z \in Z'$ is not almost free, then we will say that z is periodic if there is an m such that the X -sequence of $\chi(z, m)$ contains two equal elements and z is non-periodic otherwise (that is, for every m all the elements of the X -sequence of $\chi(z, m)$ are distinct).

Define the X -support (Z -support) of $z \in Z'$ as the subset of X (Z) consisting of all the elements of the X -sequences (Z -sequences) of $\chi(z, m)$ for all m . The Z -support of z is the union of the coordinates of the points of the X -support of z . It is obvious that the X -support and the Z -support of an almost free point in Z' are finite.

Note that if for two chains χ and χ' of length m and m' , respectively, we have that $x_j = x'_{j'}$ for the elements $x_j, x'_{j'}$, $j < m$ and $j' < m'$ in the X -sequences of χ and χ' , respectively, then for the elements x_{j+1} and $x'_{j'+1}$ following x_j and $x'_{j'}$ in the X -sequences we also have $x_{j+1} = x'_{j'+1}$. Indeed, if $x_j = x_{j+1}$, then all the coordinates of x_j are reserved by x_j , and therefore $x'_{j'+1} = x_j$. If $x_{j+1} \neq x_j$, then one of the coordinates of x_j is reserved by x_{j+1} , and therefore $x'_{j'+1} = x_{j+1}$. Also note that if we assume in addition that $j + 1 < m$, $j' + 1 < m'$ and the elements of the X -sequence of χ are distinct, then not only $x_{j+1} = x'_{j'+1}$ but also $z_{2j} = z'_{2j'}$, $z_{2j+1} = z'_{2j'+1}$ and $z_{2j+2} = z'_{2j'+2}$ for the corresponding elements of the Z -sequences of χ and χ' , respectively. Indeed, $z_{2j}, z_{2j+1}, z_{2j+2}$ are the coordinates of x_{j+1} which are characterized by the properties: z_{2j+2} is the only coordinate of x_{j+1} not reserved by x_{j+1} , and z_{2j} is the only coordinate of x_{j+1} which is also a coordinate of x_j (since otherwise either z_{2j+1} or z_{2j+2} would be reserved by x_j and by x_{j+1} or x_{j+2} which contradicts the assumption that x_j, x_{j+1} and x_{j+2} are distinct). Then the required equalities follow because the similar characterization holds for the coordinates of $x'_{j'+1}$ and $x_j = x'_{j'}$, $x_{j+1} = x'_{j'+1}$, $x_{j+2} = x'_{j'+2}$.

Thus we get the following facts for the points in Z' . The X -support and the Z -support of a periodic point are finite. The X -support of a non-periodic point z is infinite, and the elements of the X -sequence of $\chi(z, m)$ are distinct for every m . The X -supports of a non-periodic point and a periodic point are disjoint. The X -supports of a non-periodic point z and an almost free point are disjoint because otherwise the X -sequence of $\chi(z, m)$ would contain an element with a fully free coordinate for some m . Since each element of the Z -sequence of $\chi(z, m)$ of a non-periodic or periodic point z is reserved by an element of the X -sequence of $\chi(z, m+1)$, we get that the elements of the Z -sequence of $\chi(z, m)$ of a non-periodic point z are distinct for every m and the Z -supports of a non-periodic point and a periodic point are disjoint. Because every point of the Z -support of an almost free point in Z' is either fully free or reserved by a point in its X -support, we get that the Z -supports of a non-periodic point and an almost free point are disjoint.

It also follows from what we said before that if for two non-periodic points z_1 and z_2 in Z' the chains $\chi(z_1, m_1)$ and $\chi(z_2, m_2)$, $m_1, m_2 > 0$, have the same terminal Z -points, then $\chi(z_1, m_1 + k) - \chi(z_1, m_1) = \chi(z_2, m_2 + k) - \chi(z_2, m_2)$ for every k (the tails of length k of $\chi(z_1, m_1 + k)$ and $\chi(z_2, m_2 + k)$ are equal). Let us say that two non-periodic points in Z' are equivalent if their X -supports intersect. Then for every non-periodic point $z \in Z'$ we can find $m(z) > 0$ such that if z_1, z_2 are equivalent non-periodic points in Z' we have that the chains $\chi(z_1, m(z_1))$ and $\chi(z_2, m(z_2))$ have the same terminal Z -points. Denote by Z'_- a collection of non-periodic points having exactly one representative in each equivalence class. For

every $z \in Z'_-$ denote $\chi_-(z, k) = \chi(z, m(z) + k) - \chi(z, m(z))$ and call $\chi_-(z, k)$ the reduced chain of length k generated by z . Note that for every k all the elements of both the X -sequence and the Z -sequence of $\chi_-(z, k)$, $z \in Z'_-$, are distinct, and for $z_1, z_2 \in Z'_-$, $z_1 \neq z_2$, we have that the elements of the X -sequence and Z -sequence of $\chi_-(z_1, k)$ are distinct from the elements of the X -sequence and the Z -sequence, respectively, of $\chi_-(z_2, k)$.

Define $X' \subset X$ as the set consisting of the X -supports of almost free and periodic points of Z' and the elements of the X -sequences of the chains $\chi(z, m(z))$ for the non-periodic points $z \in Z'$. Similarly, define $Z'_+ \subset Z$ as the set consisting of the Z -supports of almost free and periodic points of Z' and the elements of the Z -sequences of the chains $\chi(z, m(z))$ for the non-periodic points $z \in Z'$.

Clearly, X' and Z'_+ are finite and $Z'_- \subset Z' \subset Z'_+$. Fix an integer k and let $0 \leq j \leq k$. Define $X^{(j)} \subset X$ as the union of X' and the elements of the X -sequences of $\chi_-(z, j)$ for all $z \in Z'_-$ and define $Z^{(j)} \subset Z$ as the union of Z'_+ and the elements of the Z -sequences of $\chi_-(z, j)$ for all $z \in Z'_-$.

It follows from our construction that the elements of the X -sequence of $\chi_-(z, k)$, $z \in Z'_-$, do not lie in $X^{(0)} = X'$, the initial Z -point of the chain $\chi_-(z, k)$, $z \in Z'_-$, is the only element of the Z -sequence of $\chi_-(z, k)$ lying in $Z^{(0)} = Z'_+$, the coordinates of the points of $X^{(j)}$ are contained in $Z^{(j)}$ and every point of $Z^{(j)}$ is either fully free or reserved by a point in $X^{(j+1)}$.

3.3. Proof of Theorem 1.2. Let $X \subset [0, 1]^3$ be a Kolmogorov-type embedding with respect to Ω . Take a finite subset Z' of Z . Following Subsection 3.2 define the finite sets $X' \subset X$ and $Z'_- \subset Z' \subset Z'_+ \subset Z$.

Let a map $\phi : X \rightarrow \mathbb{R}$ be such that $\phi(x) = 0$ for every $x \in X'$ and let $\delta > 0$. We will construct a map $g : Z \rightarrow \mathbb{R}$ such that $\|\phi - L(g)\| < \delta$ and $g(z) = 0$ for every $z \in Z'$. This property, together with the fact that L is open in the uniform topology, implies that L is open in the C_p -topology (apply a reasoning similar to the one used in the proof of Theorem 1.1). The case $\|\phi\| = 0$ is trivial, so we can assume that $\|\phi\| > 0$.

Fix an integer $k \geq 0$ which will be determined later and which will depend only on $\|\phi\|$ and δ . Following Subsection 3.2 construct the chains $\chi_-(z, k)$, $z \in Z'_-$, and the sets $X^{(j)}$ and $Z^{(j)}$, $0 \leq j \leq k$, and replace Ω by its infinite subfamily used in the last step of the construction.

Define $h : Z^{(k)} \rightarrow \mathbb{R}$ as follows: for every $z \in Z'_+$ set $h(z) = 0$, and for every chain $\chi_-(z, k) = (A, B)$ with $z \in Z'_-$, $A = \{x_1, \dots, x_k\}$ and $B = \{z_0, \dots, z_{2k}\}$ set $h(z_{2j-2}) = h(z_{2j}) = 0$ and $h(z_{2j-1}) = \phi(x_j)$ in order to get $\phi(x_j) = h(z_{2j-2}) + h(z_{2j-1}) + h(z_{2j})$ for $1 \leq j \leq k$. Extend h over Z such that $\|h\| \leq \|\phi\|$. Then $\|L(h)\| \leq 3\|\phi\|$ and $\phi(x) = L(h)(x)$ for every $x \in X^{(k)}$.

Thus for $f = \phi - L(h)$ we have $\|f\| \leq 4\|\phi\|$ and $f(x) = 0$ for every $x \in X^{(k)}$. Now we will apply the approximation procedure, Subsection 3.1, for f and $c = 5\|\phi\|$ to construct the map $g' : Z \rightarrow \mathbb{R}$. The approximation procedure will be applied with the following additional requirements. We assume that the cover $\mathcal{F} \in \Omega$ is chosen so that each element of $\mathcal{F}$ contains at most one point of $X^{(k)}$, and we assume that for every $1 \leq i \leq 3$ we have that the set $p_i(F)$ contains at most one point of $Z^{(k)} \cap Y_i$ for every $F \in \mathcal{F}_i$, $g'_i(z) = 0$ for every fully free $z \in Z^{(k)} \cap Y_i$ and, finally, if $p_i(F)$, $F \in \mathcal{F}_i$, contains a strongly reserved point $z \in Z^{(k)} \cap Y_i$, then F is so close to the point $x \in X$ witnessing the reservation of z that we can set $x_F = x$ (recall that

every point of $Z^{(k)}$ is either fully free or strongly reserved). One can easily verify that such a cover $\mathcal{F} \in \Omega$ satisfying the requirements of Subsection 3.1 along with those we just listed can be chosen indeed. The conditions we impose on $\mathcal{F}$ imply that if a point $z \in Y_i \cap Z^{(k)}$ is reserved by a point $x \in X$, then $g'_i(z) = \frac{1}{2n+1}f(x)$. Thus we get that $g'(z) = 0$ if $z \in Z^{(k)}$ is fully free and $g'(z) = 0$ if $z \in Z^{(k)}$ is reserved by a point $x \in X$ with $f(x) = 0$.

Recall that every point of $Z^{(k-1)}$ is either fully free or reserved by a point of $X^{(k)}$. Then, since $f(X^{(k)}) = 0$ (that is, $f(x) = 0$ for every $x \in X^{(k)}$) we get that $g'(Z^{(k-1)}) = 0$. Since the coordinates of the points in $X^{(k-1)}$ are contained in $Z^{(k-1)}$ we also get that $L(g')(X^{(k-1)}) = 0$. Thus applying the approximation procedure iteratively we can construct the maps $g^{(t)} : Z \rightarrow \mathbb{R}$, $1 \leq t \leq k$, such that $L(g^{(t)})(X^{(k-t)}) = 0$, $g^{(t)}(Z^{(k-t)}) = 0$, $\|g^{(t)}\| < \frac{1}{2n+1}(\frac{2n}{2n+1})^{t-1}c$ and $\|f - L(\sum_{s=1}^t g^{(s)})\| < (\frac{2n}{2n+1})^t c$. Then for $g = h + \sum_{t=1}^k g^{(t)}$ we have that $g(Z^{(0)}) = 0$ and $\|\phi - L(g)\| < (\frac{2n}{2n+1})^k c$. Now assume that k is taken such that $(\frac{2n}{2n+1})^k c = (\frac{2n}{2n+1})^k (5\|\phi\|) < \delta$. Thus we get that $g(Z') = 0$ and $\|\phi - L(g)\| < \delta$, and the theorem follows.

4. PROBLEMS

As we already mentioned in Section 1, Theorems 1.1 and 1.2 are partial positive solutions of the following open problem which was posed in [8].

Problem 4.1. Let $X \subset Y_1 \times \cdots \times Y_k$ be a basic embedding of a compactum X into the product of compacta $Y_1, \dots, Y_k$. Is the induced transformation always open in the C_p -topology?

Problem 4.1 in its full generality seems to be difficult. Therefore it makes sense to discuss some cases of this problem related to the types of basic embedding considered in this paper.

It was already mentioned in Section 1 that Problem 4.1 has the affirmative answer if $k = 2$ [8]. The case $k = 2$ and Theorem 1.1 can be considered in the following generalizing context.

Let X and $Y_1, \dots, Y_k$ be compact, $X \subset Y_1 \times \cdots \times Y_k$ and let $p_i : X \rightarrow Y_i$ be the projections. Define $S_i(X) = \{x \in X : p_i^{-1}(p_i(x)) = x\}$, $E_i(X) = X \setminus S_i(X)$, $E(X) = E_1(X) \cap \cdots \cap E_k(X)$, $E^1(X) = E(X)$, and by induction $E^t(X) = E(E^{t-1}(X))$. Let us call the embedding of X a Sternfeld embedding of general type if there is t such that $E^t(X) = \emptyset$. Sternfeld showed that any basic embedding into the product of two spaces is of Sternfeld's general type and that any Sternfeld embedding of general type is basic [5]. Theorem 1.1 admits a relatively simple generalization for embeddings with $E(X) = \emptyset$. It would be interesting to answer

Problem 4.2. Is the induced transformation of any Sternfeld embedding of general type open in the C_p -topology?

Note that not every basic embedding is of Sternfeld's general type (for example no embedding of a circle S^1 into $[0, 1]^k$ is of Sternfeld's general type).

In connection to Theorem 1.2 it seems very interesting to address the case of Kolmogorov-type embeddings of compacta of $\dim > 1$.

Problem 4.3. Is the induced transformation of any Kolmogorov-type embedding open in the C_p -topology?

Note that the $[0, 1]$ interval in Theorem 1.3 cannot be replaced by a 0-dimensional compactum [8]. Also note Theorem 1.3 does not hold if X is not strongly countable dimensional [7]. This suggests the following problem.

Problem 4.4. Characterize compacta X admitting a linear open continuous transformation from $C_p[0, 1]$ onto $C_p(X)$.

Problem 4.4 is also unsettled for not necessarily open transformations [7].

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