A DIAGRAMMATIC APPROACH TO CATEGORIFICATION OF QUANTUM GROUPS II

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ABSTRACT. We categorify one-half of the quantum group associated to an arbitrary Cartan datum.

Cartan data and algebras $\mathcal{A}\mathbf{f}$. A *Cartan datum* $(I, \cdot)$ consists of a finite set I and a symmetric bilinear form on $\mathbb{Z}[I]$ taking values in $\mathbb{Z}$, subject to the conditions

- $i \cdot i \in \{2, 4, 6, \dots\}$ for any $i \in I$,
- $2 \frac{i \cdot j}{i \cdot i} \in \{0, -1, -2, \dots\}$ for any $i \neq j$ in I .

We set $d_{ij} = -2 \frac{i \cdot j}{i \cdot i} \in \mathbb{N}$. To a Cartan datum assign a graph Γ with the set of vertices I and an edge between i and j if and only if $i \cdot j \neq 0$.

We recall the definition of the negative half of the quantum group associated to a Cartan datum, following [6]. Let $q_i = q^{\frac{i \cdot i}{2}}$, $[n]_i = q_i^{n-1} + q_i^{n-3} + \dots + q_i^{1-n}$, $[n]_i! = [n]_i[n-1]_i \dots [1]_i$. Let $'\mathbf{f}$ be the free associative algebra over $\mathbb{Q}(q)$ with generators θ_i , $i \in I$ and denote $\theta_i^{(n)} = \theta_i^n / [n]_i!$. We equip $'\mathbf{f}$ with an $\mathbb{N}[I]$ -grading by assigning to θ_i grading i . The tensor square $'\mathbf{f} \otimes '\mathbf{f}$ is an associative algebra with the multiplication

$$(x_1 \otimes x_2)(x'_1 \otimes x'_2) = q^{-|x_2| \cdot |x'_1|} x_1 x'_1 \otimes x_2 x'_2$$

for homogeneous x_1, x_2, x'_1, x'_2 . There is a unique algebra homomorphism $r : '\mathbf{f} \rightarrow '\mathbf{f} \otimes '\mathbf{f}$ given on generators by $r(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i$.

Proposition 1. *Algebra $'\mathbf{f}$ carries a unique $\mathbb{Q}(q)$ -bilinear form such that $(1, 1) = 1$ and*

- $(\theta_i, \theta_j) = \delta_{i,j} (1 - q_i^2)^{-1}$ for all $i, j \in I$,
- $(x, yy') = (r(x), y \otimes y')$ for $x, y, y' \in '\mathbf{f}$,
- $(xx', y) = (x \otimes x', r(y))$ for $x, x', y \in '\mathbf{f}$.

This bilinear form is symmetric.

The radical $\mathfrak{I}$ of $(,)$ is a two-sided ideal of $'\mathbf{f}$. The bilinear form descends to a non-degenerate bilinear form on the associative $\mathbb{Q}(q)$ -algebra $\mathbf{f} = '\mathbf{f} / \mathfrak{I}$. The $\mathbb{N}[I]$ -grading also descends:

$$\mathbf{f} = \bigoplus_{\nu \in \mathbb{N}[I]} \mathbf{f}_\nu.$$

The quantum version of the Gabber-Kac theorem says the following.

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Proposition 2. *The ideal $\mathfrak{J}$ is generated by the elements*

$$\sum_{a+b=d_{ij}+1} (-1)^a \theta_i^{(a)} \theta_j \theta_i^{(b)}$$

over all $i, j \in I$, $i \neq j$.

Thus, $\mathbf{f}$ is the quotient of $\mathbf{f}$ by the so-called quantum Serre relations

$$(1) \quad \sum_{a+b=d_{ij}+1} (-1)^a \theta_i^{(a)} \theta_j \theta_i^{(b)} = 0.$$

Denote by ${}_{\mathcal{A}}\mathbf{f}$ the $\mathbb{Z}[q, q^{-1}]$ -subalgebra of $\mathbf{f}$ generated by the divided powers $\theta_i^{(a)}$, over all $i \in I$ and $a \in \mathbb{N}$.

Algebras $R(\nu)$. As in [4], we consider braid-like planar diagrams, each strand labelled by an element of I , and impose the following relations:

$$(2) \quad \begin{array}{c} \text{Diagram: two strands } i \text{ and } j \text{ crossing} \\ = \left\{ \begin{array}{ll} 0 & \text{if } i = j, \\ \begin{array}{c} \text{Diagram: two parallel vertical strands } i \text{ and } j \end{array} & \text{if } i \cdot j = 0, \\ \begin{array}{c} d_{ij} \text{ (strand } i \text{ with dot)} \text{ and } j \\ + \\ \text{Diagram: } i \text{ and } d_{ji} \text{ (strand } j \text{ with dot)} \end{array} & \text{if } i \cdot j \neq 0, \end{array} \right. \end{array}$$

$$(3) \quad \begin{array}{c} \text{Diagram: } i \text{ and } j \text{ crossing with dot on } i \\ = \begin{array}{c} \text{Diagram: } i \text{ and } j \text{ crossing with dot on } j \\ \text{Diagram: } i \text{ and } j \text{ crossing with dot on } i \end{array} = \begin{array}{c} \text{Diagram: } i \text{ and } j \text{ crossing with dot on } j \\ \text{Diagram: } i \text{ and } j \text{ crossing with dot on } i \end{array} \end{array} \quad \text{for } i \neq j,$$

$$(4) \quad \begin{array}{c} \text{Diagram: } i \text{ and } i \text{ crossing with dot on } i \\ - \begin{array}{c} \text{Diagram: } i \text{ and } i \text{ crossing with dot on } i \end{array} = \begin{array}{c} \text{Diagram: two parallel vertical strands } i \end{array},$$

$$(5) \quad \begin{array}{c} \text{Diagram: } i \text{ and } i \text{ crossing with dot on } i \\ - \begin{array}{c} \text{Diagram: } i \text{ and } i \text{ crossing with dot on } i \end{array} = \begin{array}{c} \text{Diagram: two parallel vertical strands } i \end{array},$$

$$(6) \quad \begin{array}{c} \text{Diagram: } i, j, k \text{ crossing} \\ = \begin{array}{c} \text{Diagram: } i, j, k \text{ crossing} \end{array} \quad \text{unless } i = k \text{ and } i \cdot j \neq 0,$$

$$\begin{aligned}
 (7) \quad & \begin{array}{c} \text{Diagram 1} - \text{Diagram 2} \\ \text{Diagram 1: } i \text{ and } j \text{ strands cross twice, } i \text{ on top.} \\ \text{Diagram 2: } i \text{ and } j \text{ strands cross twice, } j \text{ on top.} \end{array} \\
 &= \sum_{a=0}^{d_{ij}-1} \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 3: } i \text{ strand with dot at } a, j \text{ and } i \text{ strands below.} \end{array} + \sum_{a=0}^{d_{ij}-1} \begin{array}{c} \text{Diagram 4} \\ \text{Diagram 4: } j \text{ strand with dot at } d_{ij}-1-a, i \text{ and } i \text{ strands below.} \end{array} \quad \text{if } i \cdot j \neq 0.
 \end{aligned}$$

Example 3. For the Cartan datum $\mathbf{B}_2 = \{i \cdot i = 2, j \cdot j = 4, i \cdot j = -2\}$ we have $d_{ij} = 2, d_{ji} = 1$, and the relations involving d_{ij}, d_{ji} are

$$\begin{aligned}
 & \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 5: } i \text{ and } j \text{ strands cross twice, } i \text{ on top.} \end{array} = \begin{array}{c} \text{Diagram 6} \\ \text{Diagram 6: } i \text{ strand with dot at } 1, j \text{ strand below.} \end{array} + \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 7: } j \text{ strand with dot at } 1, i \text{ strand below.} \end{array}, \\
 & \begin{array}{c} \text{Diagram 8} \\ \text{Diagram 8: } j \text{ and } i \text{ strands cross twice, } j \text{ on top.} \end{array} = \begin{array}{c} \text{Diagram 9} \\ \text{Diagram 9: } j \text{ strand with dot at } 1, i \text{ strand below.} \end{array} + \begin{array}{c} \text{Diagram 10} \\ \text{Diagram 10: } i \text{ strand with dot at } 1, j \text{ strand below.} \end{array}, \\
 & \begin{array}{c} \text{Diagram 11} - \text{Diagram 12} \\ \text{Diagram 11: } i \text{ and } j \text{ strands cross twice, } i \text{ on top.} \\ \text{Diagram 12: } i \text{ and } j \text{ strands cross twice, } j \text{ on top.} \end{array} = \begin{array}{c} \text{Diagram 13} \\ \text{Diagram 13: } i \text{ strand with dot at } 1, j \text{ and } i \text{ strands below.} \end{array} + \begin{array}{c} \text{Diagram 14} \\ \text{Diagram 14: } j \text{ strand with dot at } 1, i \text{ and } i \text{ strands below.} \end{array}, \\
 & \begin{array}{c} \text{Diagram 15} - \text{Diagram 16} \\ \text{Diagram 15: } j \text{ and } i \text{ strands cross twice, } j \text{ on top.} \\ \text{Diagram 16: } j \text{ and } i \text{ strands cross twice, } i \text{ on top.} \end{array} = \begin{array}{c} \text{Diagram 17} \\ \text{Diagram 17: } j \text{ strand with dot at } 1, i \text{ and } j \text{ strands below.} \end{array}.
 \end{aligned}$$

For each $\nu \in \mathbb{N}[I]$ define the graded ring

$$(8) \quad R(\nu) \stackrel{\text{def}}{=} \bigoplus_{i, j \in \text{Seq}(\nu)} {}_j R(\nu)_i,$$

where ${}_j R(\nu)_i$ is the abelian group of all linear combinations of diagrams with $\text{bot}(D) = \mathbf{i}$ and $\text{top}(D) = \mathbf{j}$ modulo the relations (2)–(7) and $\text{Seq}(\nu)$ is the set

of weight ν sequences of elements of I . The multiplication is given by concatenation. Degrees of the generators are:

$$(9) \quad \deg \left(\begin{array}{c} | \\ \bullet \\ | \\ i \end{array} \right) = i \cdot i, \quad \deg \left(\begin{array}{cc} & \\ \diagup & \diagdown \\ i & j \end{array} \right) = -i \cdot j.$$

The rest of [4, Section 2.1] generalizes without difficulty to an arbitrary Cartan datum. To define the analogue of the module $\mathcal{Pol}_\nu$ over $R(\nu)$, we choose an orientation of each edge of Γ , then faithfully follow the exposition in Section 2.3 of [4], only changing the action of $\delta_{k,i}$ in the last of the four cases to

$$f \mapsto (x_k(s_k \mathbf{i})^d + x_{k+1}(s_k \mathbf{i})^{d'})(s_k f) \quad \text{if } i_k \longrightarrow i_{k+1},$$

where $d = d_{i_{k+1}i_k}$ and $d' = d_{i_k i_{k+1}}$. Here notation $i_k \longrightarrow i_{k+1}$ means that $i_k \cdot i_{k+1} \neq 0$ and this edge of Γ is oriented from i_k to i_{k+1} . Proposition 2.3 in [4] holds for an arbitrary $(I, \cdot)$. As in [4, Section 2.3], we define ${}_j B_i$, which might depend on minimal presentations of permutations in ${}_j S_i$ and gives a basis in ${}_j R(\nu)_i$. Corollary 2.6 in [4], showing that $\mathcal{Pol}_\nu$ is a faithful graded module over $R(\nu)$, holds for an arbitrary Cartan datum and the properties of $R(\nu)$ established in [4, Section 2.4] generalize without difficulty.

Computations in the nil-Hecke ring. In this section we slightly enhance the graphical calculus for computations in the nil-Hecke ring and record several lemmas to be used in the proof of categorified quantum Serre relations below. We use notation from Section 2.2 of [4].

A box with n incoming and n outgoing edges and $\partial(n)$ written inside denotes the longest divided difference ∂_{w_0} , the non-zero product of $\frac{n(n-1)}{2}$ divided differences from $\{\partial_1, \dots, \partial_{n-1}\}$:

When this box is part of a diagram for an element of $R(\nu)$, it denotes the corresponding element of $R(n\mathbf{i}) \subset R(\nu)$. A box labelled e_n denotes the idempotent $e_n = x_1^{n-1} x_2^{n-2} \dots x_{n-1} \partial_{w_0}$.

A box labelled $e_{i,n}$ denotes the corresponding idempotent in $R(\nu)$:

Remark 4. Similar diagrams are used in the graphical calculus of Jones-Wenzl projectors (see [3]), but the latter has no direct relation to the graphical calculus in our paper.

Lemma 5. *We have*

Proof is by induction on n :

The first equality uses that $x_1 x_2 \dots x_{n-1}$ is central in the nil-Hecke ring NH_{n-1} , allowing us to move these dots across $\partial(n-1)$. The second equality is the induction hypothesis. $\square$

The lemma implies the following graphical identities:

$$(10) \quad \begin{array}{c} \text{Diagram 1: } e_n \text{ box above } e_{n-1} \text{ box, with } n \text{ strands entering from the top and } n \text{ exiting from the bottom.} \\ \text{Diagram 2: } e_n \text{ box with } n \text{ strands entering from the top and } n \text{ exiting from the bottom.} \end{array} = \begin{array}{c} \text{Diagram 3: } e_n \text{ box above } e_{n-1} \text{ box, with } n \text{ strands entering from the top and } n \text{ exiting from the bottom.} \\ \text{Diagram 4: } e_n \text{ box with } n \text{ strands entering from the top and } n \text{ exiting from the bottom.} \end{array},$$

$$(11) \quad \begin{array}{c} \text{Diagram 1: } e_{n-1} \text{ box above } e_n \text{ box, with } n \text{ strands entering from the top and } n \text{ exiting from the bottom.} \\ \text{Diagram 2: } e_{n-1} \text{ box with } n \text{ strands entering from the top and } n \text{ exiting from the bottom.} \end{array} = \begin{array}{c} \text{Diagram 3: } e_{n-1} \text{ box above } e_n \text{ box, with } n \text{ strands entering from the top and } n \text{ exiting from the bottom.} \\ \text{Diagram 4: } e_{n-1} \text{ box with } n \text{ strands entering from the top and } n \text{ exiting from the bottom.} \end{array}.$$

The following also hold:

$$(12) \quad \begin{array}{c} \text{Diagram 1: } e_n \text{ box with } n \text{ strands entering from the top and } n \text{ exiting from the bottom. A dot is on the } a \text{th strand from the bottom.} \end{array} = \begin{cases} 0 & \text{if } a < n-1, \\ e_n & \text{if } a = n-1, \end{cases}$$

$$(13) \quad \begin{array}{c} \text{Diagram 1: } e_n \text{ box with } n \text{ strands entering from the top and } n \text{ exiting from the bottom. A dot is on the } a \text{th strand from the bottom.} \end{array} = \begin{cases} 0 & \text{if } a < n-1, \\ (-1)^{n-1} e_n & \text{if } a = n-1. \end{cases}$$

For each $i \in I$ the ring $R(mi)$ is isomorphic to the nil-Hecke ring. The grading of a dot is now $i \cdot i$, while that of a crossing is $-i \cdot i$. For this reason one needs to generalize the grading convention described in [4, Section 2.2] and define ${}_{i,m}P$ to be the right graded projective module $e_{i,m}R(mi)\{-\frac{m(m-1)i \cdot i}{4}\}$, so that the grading starts in the degree $\{-\frac{m(m-1)i \cdot i}{4}\}$. Likewise, $P_{i,m}$ is the left graded projective module $R(mi)\psi(e_{i,m})\{-\frac{m(m-1)i \cdot i}{4}\}$.

The Grothendieck group, bilinear form and projectives. We retain all notation and assumptions from [4], working over a field $\mathbb{k}$, denoting by $K_0(R(\nu))$ the Grothendieck group of the category $R(\nu)\text{-pmod}$ of graded finitely-generated projective left $R(\nu)$ -modules and forming the direct sum

$$R = \bigoplus_{\nu \in \mathbb{N}[I]} R(\nu), \quad K_0(R) = \bigoplus_{\nu \in \mathbb{N}[I]} K_0(R(\nu)).$$

$$(14) \quad (,): K_0(R(\nu)) \times K_0(R(\nu)) \longrightarrow \mathbb{Z}[q^{-1}, q] \cdot (\nu)_q,$$
$$(15) \quad (\nu)_q = \text{gdim}(\text{Sym}(\nu)) = \prod_{i \in \Gamma} \left(\prod_{a=1}^{\nu_i} \frac{1}{1 - q^{ai \cdot i}} \right),$$
$$(16) \quad ([P], [Q]) = \text{gdim}_{\mathbb{k}}(P^\psi \otimes_{R(\nu)} Q).$$
$$\langle \mathbf{i} \rangle = \sum_{k=1}^r \frac{n_k(n_k - 1)}{2} \cdot \frac{i_k \cdot i_k}{2}.$$
$$P_i = R(\nu)\psi(1_i)\{-\langle i \rangle\}, \quad {}_iP = 1_iR(\nu)\{-\langle i \rangle\}.$$
$$(17) \quad \alpha_{a,b}^+(i,j) =$$

$$i^{a+1}j^{i^b-1}R(j+(a+b)i)_{i^a j^b}.$$
$$i' i^{a+1} j i^{b-1} i'' R(\nu)_{i' i^a j i^b i''}$$

Left multiplication by $\alpha_{a,b}^+$ is a homomorphism of projective modules

$$(18) \quad \dots i^a j i^b \dots P \longrightarrow \dots i^{a+1} j i^{b-1} \dots P.$$

$$(19) \quad \dots i^{(a)} j i^{(b)} \dots P \longrightarrow \dots i^{(a+1)} j i^{(b-1)} \dots P,$$

denoted $\alpha_{(a,b)}^+$ and given by the composition

$$(20) \quad \dots i^{(a)} j i^{(b)} \dots P \hookrightarrow \dots i^a j i^b \dots P \xrightarrow{\alpha_{a,b}^+} \dots i^{(a+1)} j i^{(b-1)} \dots P.$$

It is easy to check that $\alpha_{(a,b)}^+$ is a grading-preserving homomorphism. Likewise, let

$$(21) \quad \alpha_{a,b}^-(i, j) =$$

and write $\alpha_{a,b}^-$ instead of $\alpha_{a,b}^-(i, j)$ when i and j are fixed. This element of

$$(22) \quad \dots i^{a-1} j i^{b-1} \dots R(\nu) \dots i^a j i^b \dots$$

gives rise to a grading-preserving homomorphism of projectives

$$(23) \quad \alpha_{(a,b)}^- : \dots i^{(a)} j i^{(b)} \dots P \longrightarrow \dots i^{(a-1)} j i^{(b+1)} \dots P.$$

For the next few pages, denote $\dots i^{(a)} j i^{(b)} \dots P$ by ${}_{(a,b)}P$ (recall that $a + b = d + 1$, $d = d_{ij}$). We have a diagram of projective modules and grading-preserving homomorphisms

$$(24) \quad \begin{array}{ccccccc} & \xrightarrow{\quad} & {}_{(a-1,b+1)}P & \xrightleftharpoons[\alpha_{(a,b)}^-]{\alpha_{(a-1,b+1)}^+} & {}_{(a,b)}P & \xrightleftharpoons[\alpha_{(a+1,b-1)}^-]{\alpha_{(a,b)}^+} & {}_{(a+1,b-1)}P \xrightarrow{\quad} \\ & \xleftarrow{\quad} & & & & & \end{array}$$

terminating on the left at

$$(25) \quad {}_{(0,d+1)}P \xrightleftharpoons[\alpha_{(1,d)}^-]{\alpha_{(0,d+1)}^+} {}_{(1,d)}P \xrightleftharpoons{\quad} \dots$$

and on the right at

$$(26) \quad \dots \xrightleftharpoons[\alpha_{(d+1,0)}^-]{\alpha_{(d,1)}^+} {}_{(d,1)}P \xrightleftharpoons{\quad} {}_{(d+1,0)}P.$$

Relations (10), (11) imply

$$\alpha_{a+1,b-1}^- \alpha_{a,b}^+ =$$

$$\stackrel{(10)}{=} \text{Diagram 1} \stackrel{(11)}{=} \text{Diagram 2}$$

Furthermore,

$$\alpha_{a-1,b+1}^+ \alpha_{a,b}^- =$$

$$= \text{Diagram 1}$$

Therefore,

$$\alpha_{a-1,b+1}^+ \alpha_{a,b}^- - \alpha_{a+1,b-1}^- \alpha_{a,b}^+ = (-1)^{a-1} e_{i,a} \otimes 1_j \otimes e_{i,b},$$

as elements of $R(\nu)$; see below:

$$\begin{aligned}
 \alpha_{a-1,b+1}^+ \alpha_{a,b}^- - \alpha_{a+1,b-1}^- \alpha_{a,b}^+ &= \text{Diagram 1} \\
 &- \text{Diagram 2} \\
 &\stackrel{(7)}{=} \sum_{c=0}^{d-1} \text{Diagram 3} \\
 &\stackrel{(12),(13)}{=} \text{Diagram 4} \\
 &\stackrel{(12),(13)}{=} (-1)^{a-1} \text{Diagram 5},
 \end{aligned}$$

Diagram 1: Two boxes labeled $e_{i,a}$ and $e_{i,b}$. $e_{i,a}$ has 4 inputs from below labeled i, i, i, i and 4 outputs to the top. $e_{i,b}$ has 3 inputs from below labeled i, i, i and 3 outputs to the top. A crossing connects the 4th input of $e_{i,a}$ to the 1st input of $e_{i,b}$.

Diagram 2: Similar to Diagram 1, but the crossing connects the 3rd input of $e_{i,a}$ to the 1st input of $e_{i,b}$.

Diagram 3: A sum over c from 0 to $d-1$. Each term shows a vertical line between the two boxes. The left box $e_{i,a}$ has a dot on its 4th input labeled c . The right box $e_{i,b}$ has a dot on its 1st input labeled $d-1-c$.

Diagram 4: Similar to Diagram 3, but the dots are labeled $a-1$ and $b-1$ respectively.

Diagram 5: The boxes $e_{i,a}$ and $e_{i,b}$ are separated by a vertical line. $e_{i,a}$ has 4 inputs labeled i, i, i, i and 4 outputs. $e_{i,b}$ has 3 inputs labeled i, i, i and 3 outputs.

where $d = a + b - 1$. Consequently,

$$\alpha_{(a-1,b+1)}^+ \alpha_{(a,b)}^- - \alpha_{(a+1,b-1)}^- \alpha_{(a,b)}^+ = (-1)^{a-1} \cdot \text{Id},$$

as endomorphisms of the projective module ${}_{(a,b)}P$, since $e_{i,a} \otimes 1_j \otimes e_{i,b}$ acts by the identity on ${}_{(a,b)}P$.

Likewise,

$$\begin{aligned}
 \alpha_{1,d}^- \alpha_{0,d+1}^+ &= \text{diagram} \stackrel{(3)}{=} \text{diagram} + \text{diagram} \\
 &\stackrel{(12)}{=} \text{diagram} \stackrel{(12)}{=} \text{diagram} = 1_j \otimes e_{i,d+1},
 \end{aligned}$$

The diagrams are as follows:

- Diagram 1: A box labeled $e_{i,d+1}$ and a box labeled $e_{i,d}$ are stacked. Four strands enter from the bottom, labeled j, i, i, i . The strands labeled i cross each other and then enter the $e_{i,d}$ box. The strand labeled j enters the $e_{i,d+1}$ box.
- Diagram 2: A box labeled $e_{i,d+1}$ with four strands entering from the bottom, labeled j, i, i, i . A dot labeled d' is on the strand labeled j .
- Diagram 3: A box labeled $e_{i,d+1}$ with four strands entering from the bottom, labeled j, i, i, i . A dot labeled d is on the strand labeled j .
- Diagram 4: A box labeled $e_{i,d+1}$ with four strands entering from the bottom, labeled j, i, i, i . A dot labeled d is on the strand labeled j .
- Diagram 5: A box labeled $e_{i,d+1}$ with four strands entering from the bottom, labeled j, i, i, i .

where $d' = d_{ji}$, and

$$\alpha_{(1,d)}^- \alpha_{(0,d+1)}^+ = \text{Id},$$

as endomorphisms of ${}_{(0,d+1)}P$. A similar computation shows that

$$\alpha_{d,1}^+ \alpha_{d+1,0}^- = (-1)^d e_{i,d+1} \otimes 1_j,$$

as elements of $R(\nu)$, and

$$\alpha_{(d,1)}^+ \alpha_{(d+1,0)}^- = (-1)^d \cdot \text{Id},$$

as endomorphisms of ${}_{(d+1,0)}P$. Moreover,

$$\alpha_{a+1,b-1}^+ \alpha_{a,b}^+ = 0 \quad \text{and} \quad \alpha_{a-1,b+1}^- \alpha_{a,b}^- = 0.$$

Proposition 6. *For each $i, j \in I, i \neq j$ there are isomorphisms of graded right projective modules*

$$\bigoplus_{a=0}^{\lfloor \frac{d+1}{2} \rfloor} \dots i^{(2a)} j i^{(d+1-2a)} \dots P \cong \bigoplus_{a=0}^{\lfloor \frac{d}{2} \rfloor} \dots i^{(2a+1)} j i^{(d-2a)} \dots P.$$

Proof. When $i \cdot j = 0$, the isomorphism reads

$$\dots j i \dots P \cong \dots i j \dots P$$

and is given by left multiplication by the ij intersection. When $i \cdot j < 0$, earlier computations show that the maps

$$\bigoplus_{a=0}^{\lfloor \frac{d+1}{2} \rfloor} (2a, d+1-2a) P \xrightleftharpoons[\alpha'']{\alpha'} \bigoplus_{a=0}^{\lfloor \frac{d}{2} \rfloor} (2a+1, d-2a) P$$

given by

$$\begin{aligned}\alpha' &= \sum_{a=0}^{\lfloor \frac{d}{2} \rfloor} \alpha_{(2a, d+1-2a)}^+ + \sum_{a=0}^{\lfloor \frac{d+1}{2} \rfloor} \alpha_{(2a, d+1-2a)}^-, \\ \alpha'' &= \sum_{a=0}^{\lfloor \frac{d+1}{2} \rfloor} \alpha_{(2a+1, d-2a)}^- - \sum_{a=0}^{\lfloor \frac{d}{2} \rfloor} \alpha_{(2a+1, d-2a)}^+, \end{aligned}$$

are mutually-inverse isomorphisms, implying the proposition. Maps α', α'' together are given by summing over all arrows in the diagram (24), with every fourth arrow appearing with the minus sign. $\square$

Corollary 7. *For each $i, j \in I, i \neq j$ there are isomorphisms of graded left projective modules*

$$\bigoplus_{a=0}^{\lfloor \frac{d+1}{2} \rfloor} P_{\dots i(2a) j i^{(d+1-2a)} \dots} \cong \bigoplus_{a=0}^{\lfloor \frac{d}{2} \rfloor} P_{\dots i(2a+1) j i^{(d-2a)} \dots}.$$

Proposition 6 and Corollary 7 generalize Proposition 2.13 in [4] and can be considered a categorification of the quantum Serre relations. Corollaries 2.14 and 2.15 of [4], establishing quantum Serre relations for the characters of any $M \in R(\nu)\text{-mod}$, generalize to an arbitrary Cartan datum in the same way.

Grothendieck group as the quantum group. Induction and restriction functors for inclusions $R(\nu) \otimes R(\nu') \subset R(\nu + \nu')$ turn $K_0(R)$ into a twisted bialgebra, and all results of [4, Section 2.6] remain valid for an arbitrary Cartan datum. As in [4, Section 3.1] we define a homomorphism of twisted bialgebras

$$\gamma : \mathcal{A}\mathbf{f} \longrightarrow K_0(R)$$

which takes the product of divided powers $\theta_{\mathbf{i}} = \theta_{i_1}^{(n_1)} \dots \theta_{i_r}^{(n_r)}$ to $[P_{\mathbf{i}}]$, where $\mathbf{i} = i_{i_1}^{(n_1)} \dots i_{i_r}^{(n_r)}$. Homomorphism γ intertwines the bilinear forms on $\mathcal{A}\mathbf{f}$ and $K_0(R)$,

$$(x, y) = (\gamma(x), \gamma(y)), \quad x, y \in \mathcal{A}\mathbf{f}.$$

Due to the quantum Gabber-Kac theorem, this homomorphism is injective. Surjectivity of γ follows from the arguments identical to those given in [4, Section 3.2], which, in turn, were adopted from [5, Section 5]. Alternatively, the arguments could be adopted from [2] and [9]; we settled on using a single source. We obtain

Theorem 8. $\gamma : \mathcal{A}\mathbf{f} \longrightarrow K_0(R)$ is an isomorphism of $\mathbb{N}[I]$ -graded twisted bialgebras.

This theorem holds without any restrictions on the Cartan datum and on the ground field $\mathbb{k}$ over which $R(\nu)$ is defined. All other results and observations of Sections 3.2 and 3.3 of [4] extend to the general case as well. The cyclotomic quotients of $R(\nu)$, described in [4, Section 3.4], generalize to an arbitrary Cartan datum.

It would be interesting to relate our construction to Lusztig's geometric realization of U^- in the non-simply laced case [6] and to Brundan-Kleshchev's categorification [1], [5] of $U_{q=1}^-$ in the affine Dynkin case $A_n^{(2)}$.

A multi-grading. For every pair (i, j) of distinct vertices of Γ , algebras $R(\nu)$ can be equipped with an additional grading, by assigning degrees -1 and 1 to the ij and ji crossings, respectively,

$$\deg \left(\begin{array}{c} \diagup \quad \diagdown \\ i \quad j \end{array} \right) = -1, \quad \deg \left(\begin{array}{c} \diagdown \quad \diagup \\ j \quad i \end{array} \right) = 1,$$

and degree 0 to all other diagrammatic generators of $R(\nu)$. These gradings are independent, and together with the principal grading, introduced above, make $R(\nu)$ into a multi-graded ring (with $\frac{n(n-1)}{2} + 1$ independent gradings where $n = |\text{Supp}(\nu)|$). The direct sum of the categories of multi-graded finitely-generated projective left $R(\nu)$ -modules, over all $\nu \in \mathbb{N}[I]$, categorifies a multi-parameter deformation [7], [8] of the quantum universal enveloping algebra U^- , the quotient of the free associative algebra on $\theta_i, i \in I$, by the relations

$$(27) \quad \sum_{a+b=d_{ij}+1} (-1)^a q_{ij}^a \theta_i^{(a)} \theta_j \theta_i^{(b)} = 0,$$

where q_{ij} are formal variables subject to conditions $q_{ij}q_{ji} = 1$.

Modifications in the simply-laced case. This section explains how to deform algebras $R(\nu)$ in the simply-laced case so that the main results of [4] will hold for the modified algebras. These deformations can be non-trivial only when the graph has cycles. As in [4], we start with an unoriented graph Γ without loops and multiple edges. Next, fix an orientation of each edge of Γ , work over a base field $\mathbb{k}$, and, for each oriented edge $i \rightarrow j$, choose two invertible elements τ_{ij} and τ_{ji} in $\mathbb{k}$. Denote such a datum {orientations, invertible elements} by τ .

For each $\nu \in \mathbb{N}[I]$ consider $\mathbb{k}$ -vector space $\mathcal{Pol}_\nu$ defined as in [4]. This space is the sum of polynomial rings in $|\nu|$ variables, over all sequences in $\text{Seq}(\nu)$. Define $R_\tau(\nu)$ to be the endomorphism algebra of $\mathcal{Pol}_\nu$ generated by the endomorphisms $1_i, x_{k,i}, \delta_{k,i}$, over all possible k and i , with the action as in [4, Section 2.3], with the only difference being the action of $\delta_{k,i}$ in the last of the four cases:

$$f \mapsto (\tau_{i_k i_{k+1}} x_{k+1}(s_k i) - \tau_{i_{k+1} i_k} x_k(s_k i))(s_k f) \quad \text{if } i_k \rightarrow i_{k+1},$$

instead of

$$f \mapsto (x_k(s_k i) + x_{k+1}(s_k i))(s_k f) \quad \text{if } i_k \rightarrow i_{k+1}.$$

The algebra $R_\tau(\nu)$ has a diagrammatic description similar to that of $R(\nu)$, with the following defining relations:

$$(28) \quad \begin{array}{c} \text{Diagram of two strands } i \text{ and } j \text{ crossing} \end{array} = \begin{cases} 0 & \text{if } i = j, \\ \begin{array}{c} \text{Diagram of two parallel vertical strands } i \text{ and } j \end{array} & \text{if } i \cdot j = 0, \\ \tau_{ij} \begin{array}{c} \text{Diagram of strand } i \text{ with a dot} \end{array} \begin{array}{c} \text{Diagram of strand } j \end{array} - \tau_{ji} \begin{array}{c} \text{Diagram of strand } i \end{array} \begin{array}{c} \text{Diagram of strand } j \text{ with a dot} \end{array} & \text{if } i \longrightarrow j, \\ \tau_{ji} \begin{array}{c} \text{Diagram of strand } i \end{array} \begin{array}{c} \text{Diagram of strand } j \text{ with a dot} \end{array} - \tau_{ij} \begin{array}{c} \text{Diagram of strand } i \text{ with a dot} \end{array} \begin{array}{c} \text{Diagram of strand } j \end{array} & \text{if } i \longleftarrow j, \end{cases}$$

$$(29) \quad \begin{array}{c} \text{Diagram of two strands } i \text{ and } j \text{ crossing with a dot on strand } i \end{array} = \begin{array}{c} \text{Diagram of two strands } i \text{ and } j \text{ crossing with a dot on strand } j \end{array} = \begin{array}{c} \text{Diagram of two strands } i \text{ and } j \text{ crossing with a dot on strand } i \end{array} \quad \text{for } i \neq j,$$

$$(30) \quad \begin{array}{c} \text{Diagram of two strands } i \text{ and } i \text{ crossing with a dot on strand } i \end{array} - \begin{array}{c} \text{Diagram of two strands } i \text{ and } i \text{ crossing with a dot on strand } i \end{array} = \begin{array}{c} \text{Diagram of two parallel vertical strands } i \text{ and } i \end{array},$$

$$(31) \quad \begin{array}{c} \text{Diagram of two strands } i \text{ and } i \text{ crossing with a dot on strand } i \end{array} - \begin{array}{c} \text{Diagram of two strands } i \text{ and } i \text{ crossing with a dot on strand } i \end{array} = \begin{array}{c} \text{Diagram of two parallel vertical strands } i \text{ and } i \end{array},$$

$$(32) \quad \begin{array}{c} \text{Diagram of three strands } i, j, k \text{ crossing} \end{array} = \begin{array}{c} \text{Diagram of three strands } i, j, k \text{ crossing} \end{array} \quad \text{unless } i = k \text{ and } i \cdot j = -1,$$

$$(33) \quad \begin{array}{c} \text{Diagram of three strands } i, j, i \text{ crossing} \end{array} - \begin{array}{c} \text{Diagram of three strands } i, j, i \text{ crossing} \end{array} = \tau_{ij} \begin{array}{c} \text{Diagram of three parallel vertical strands } i, j, i \end{array} \quad \text{if } i \longrightarrow j,$$

$$(34) \quad \begin{array}{c} \text{Diagram of three strands } i, j, i \text{ crossing} \end{array} - \begin{array}{c} \text{Diagram of three strands } i, j, i \text{ crossing} \end{array} = -\tau_{ij} \begin{array}{c} \text{Diagram of three parallel vertical strands } i, j, i \end{array} \quad \text{if } i \longleftarrow j.$$

Reverse the orientation of a single edge $i - j$ and change τ_{ij} to $-\tau_{ij}$ and τ_{ji} to $-\tau_{ji}$. Denote the new datum by τ' . Algebras $R_\tau(\nu)$ and $R_{\tau'}(\nu)$ are isomorphic via a map which is the identity on diagrams. This way, the study of $R_\tau(\nu)$ reduces to the case of any preferred orientation of Γ . Rescaling one of the two possible types of the

ij crossing by $\lambda \in \mathbb{k}$ changes τ_{ij} to $\lambda\tau_{ij}$ and τ_{ji} to $\lambda\tau_{ji}$ while keeping the rest of the data fixed. We see that $R_\tau(\nu)$ depends only on products $\tau_{ij}\tau_{ji}^{-1}$, over all edges of Γ , via non-canonical isomorphisms. Rescalings of ii crossings and dots further reduce the number of parameters to the rank of the first homology group of Γ . When graph Γ is a forest (has no cycles), algebras $R_\tau(\nu)$ are all isomorphic to $R(\nu)$ via rescaling of generators. When Γ has a single cycle, rescaling of generators reduces this family of algebras to a one-parameter family, with the parameter taking values in $\mathbb{k}^*$. It is likely that $R_\tau(\nu)$ has a description via equivariant convolution algebras in Lusztig's geometrization [6] of U^- when all $\tau_{ij} = 1$ (compare with Conjecture 1.2 in [4]).

Form

$$R_\tau = \bigoplus_{\nu \in \mathbb{N}[I]} R_\tau(\nu).$$

The Grothendieck group $K_0(R_\tau)$ of the category of finitely-generated graded left projective modules can be naturally identified with the integral version $\mathcal{A}\mathbf{f}$ of U^- . All other essential constructions and results of [4] generalize from $R(\nu)$ to algebras $R_\tau(\nu)$ in a straightforward fashion.

Modifications in the general case. Rings $R(\nu)$ associated to an arbitrary Cartan datum admit similar modifications that depend on choosing an orientation of Γ and invertible elements τ_{ij}, τ_{ji} of the ground field $\mathbb{k}$ for each oriented edge $i \rightarrow j$. The key point is the change in the definition of the endomorphism algebra, making $\delta_{k,i}$ act by

$$f \mapsto (\tau_{i_k i_{k+1}} x_{k+1} (s_k \mathbf{i})^{d'} - \tau_{i_{k+1} i_k} x_k (s_k \mathbf{i})^d) (s_k f) \quad \text{if } i_k \rightarrow i_{k+1}$$

in the last of the four cases, with $d = d_{i_{k+1} i_k}$ and $d' = d_{i_k i_{k+1}}$. Our proof of categorified quantum Serre relations for $R(\nu)$ requires only minor changes in the general case of $R_\tau(\nu)$. Everything else generalizes as well.

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