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THE DEFOCUSING ENERGY-SUPERCRITICAL NONLINEAR WAVE EQUATION IN THREE SPACE DIMENSIONS

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Abstract. We consider the defocusing nonlinear wave equation $u_{tt} - \Delta u + |u|^p u = 0$ in the energy-supercritical regime $p > 4$. For even values of the power p, we show that blowup (or failure to scatter) must be accompanied by blowup of the critical Sobolev norm. An equivalent formulation is that solutions with bounded critical Sobolev norm are global and scatter. The impetus to consider this problem comes from recent work of Kenig and Merle who treated the case of spherically-symmetric solutions.

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1. Introduction

We consider the initial value problem for the defocusing nonlinear wave equation in three space dimensions:

$$\begin{cases} u_{tt} - \Delta u + F(u) = 0, \\ u(0) = u_0, \ u_t(0) = u_1, \end{cases} \tag{1.1}$$

where the nonlinearity $F(u) = |u|^p u$ is energy-supercritical, that is, $p > 4$. For the sake of simplicity, we restrict our attention to even values of the power p only.

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The class of solutions to (1.1) is left invariant by the scaling

$$u(t,x) \mapsto \lambda^{\frac{2}{p}} u(\lambda t, \lambda x). \tag{1.2}$$

This defines a notion of *criticality*. More precisely, a quick computation shows that the only homogeneous L_x^2-based Sobolev norm left invariant by the scaling is $\dot H_x^{s_c} \times \dot H_x^{s_c-1}$, where the *critical regularity* is $s_c := \frac{3}{2} - \frac{2}{p}$. If the regularity of the initial data to (1.1) is higher/lower than the critical regularity s_c, we call the problem *subcritical/supercritical*.

We consider (1.1) for initial data belonging to the critical homogeneous Sobolev space, that is, $(u_0, u_1) \in \dot H_x^{s_c} \times \dot H_x^{s_c-1}$ in the energy-supercritical regime $s_c > 1$. We prove that any maximal-lifespan solution u with the property that (u, u_t) is uniformly bounded (throughout its lifespan) in $\dot H_x^{s_c} \times \dot H_x^{s_c-1}$ must be global and scatter.

Let us start by making the notion of a solution more precise.

Definition 1.1 (Solution)**.** A function $u : I \times \mathbb{R}^3 \to \mathbb{R}$ on a nonempty time interval $0 \in I \subset \mathbb{R}$ is a *(strong) solution* to (1.1) if $(u, u_t) \in C_t^0(K; \dot H_x^{s_c} \times \dot H_x^{s_c-1})$ and $u \in L_{t,x}^{2p}(K \times \mathbb{R}^3)$ for all compact $K \subset I$, and it obeys the Duhamel formula

$$\begin{aligned}\begin{bmatrix} u(t) \\ u_t(t) \end{bmatrix} &= \begin{bmatrix} \cos(t|\nabla|) & |\nabla|^{-1}\sin(t|\nabla|) \\ -|\nabla|\sin(t|\nabla|) & \cos(t|\nabla|) \end{bmatrix} \begin{bmatrix} u(0) \\ u_t(0) \end{bmatrix} \\ &\quad - \int_0^t \begin{bmatrix} |\nabla|^{-1}\sin\big((t-s)|\nabla|\big) \\ \cos\big((t-s)|\nabla|\big) \end{bmatrix} F(u(s))\, ds \end{aligned} \tag{1.3}$$

for all $t \in I$. We refer to the interval I as the *lifespan* of u. We say that u is a *maximal-lifespan solution* if the solution cannot be extended to any strictly larger interval. We say that u is a *global solution* if $I = \mathbb{R}$.

We define the *scattering size* of a solution to (1.1) on a time interval I by

$$S_I(u) := \int_I \int_{\mathbb{R}^d} |u(t,x)|^{2p}\, dx\, dt. \tag{1.4}$$

Associated to the notion of a solution is a corresponding notion of blowup. By the standard local theory (see Theorem 3.1), this precisely corresponds to the impossibility of continuing the solution.

Definition 1.2 (Blowup)**.** We say that a solution u to (1.1) *blows up forward in time* if there exists a time $t_1 \in I$ such that

$$S_{[t_1, \sup I)}(u) = \infty$$

and that u *blows up backward in time* if there exists a time $t_1 \in I$ such that

$$S_{(\inf I, t_1]}(u) = \infty.$$

Our main result is the following

Theorem 1.3 (Spacetime bounds)**.** *Suppose $p > 4$ is even and let $u : I \times \mathbb{R}^3 \to \mathbb{R}$ be a solution to* (1.1) *such that $(u, u_t) \in L_t^\infty(I; \dot H_x^{s_c} \times \dot H_x^{s_c-1})$. Then*

$$S_I(u) \le C\big(\|(u, u_t)\|_{L_t^\infty(I; \dot H_x^{s_c} \times \dot H_x^{s_c-1})}\big).$$

We have not considered other values of $p > 4$, for which the nonlinearity is no longer a polynomial in u; we felt that it would muddy the main thrust of the argument, without due reward.

As mentioned above, finite-time blowup of a solution to (1.1) must be accompanied by divergence of the scattering size defined in (1.4). Thus, Theorem 1.3 immediately implies

Corollary 1.4 (Spacetime bounds)**.** *If $u : I \times \mathbb{R}^3 \to \mathbb{R}$ is a maximal-lifespan solution to* (1.1) *with $(u, u_t) \in L_t^\infty(I; \dot H_x^{s_c} \times \dot H_x^{s_c-1})$, then u is global and moreover,*

$$S_{\mathbb{R}}(u) \le C\big(\|(u,u_t)\|_{L_t^\infty(\mathbb{R}; \dot H_x^{s_c} \times \dot H_x^{s_c-1})}\big).$$

This corollary takes on a more appealing form if we rephrase it in the contrapositive:

Corollary 1.5 (Nature of blowup)**.** *A solution $u : I \times \mathbb{R}^3 \to \mathbb{R}$ to* (1.1) *can only blow up in finite time or be global but fail to scatter if its $\dot H_x^{s_c} \times \dot H_x^{s_c-1}$ norm diverges.*

For spherically-symmetric initial data, Theorem 1.3 was proved by Kenig and Merle [12]. The (nonradial) analogue of Theorem 1.3 for NLS in dimensions $d \ge 5$ was proved in [17] by adapting the methods of [15]. We will discuss these papers and their relation to the results presented here more fully when we outline the proof of Theorem 1.3. Before doing this, let us briefly review some of the backstory and, in particular, the origins of some of the techniques we will be using.

When $p = 4$, or equivalently, $s_c = 1$, the critical Sobolev norm is automatically bounded in time by virtue of the conservation of energy:

$$E(u) = \int_{\mathbb{R}^3} \tfrac12 |u_t|^2 + \tfrac12 |\nabla u|^2 + \tfrac{1}{p+2} |u|^{p+2}\, dx. \tag{1.5}$$

This *energy-critical* case of (1.1) has received particular attention because of this property. Global well-posedness was proved in a series of works [5, 6, 7, 24, 29, 25, 26] with finiteness of the scattering size being added later; see [1, 4, 22, 23, 31]. Certain monotonicity formulae, the Morawetz and energy flux identities, play an important role in all these results. It is important that these monotonicity formulae also have critical scaling.

In the energy-supercritical case discussed in this paper, all conservation laws and monotonicity formulae have scaling below the critical regularity. At the present moment, there is no technology for treating large-data dispersive equations without some *a priori* control of a critical norm. Indeed, one may assert that the fundamental difficulty associated with the 3D Navier–Stokes system is controlling the possible growth of (scaling-)critical norms. This is the purpose of the $L_t^\infty(I; \dot H_x^{s_c} \times \dot H_x^{s_c-1})$ assumption in Theorem 1.3; it plays the role of the missing conservation law at the critical regularity. Nevertheless, the fact that the monotonicity formulae have noncritical scaling remains a problem.

The problem of having monotonicity formulae at a different regularity to the critical (coercive) conservation laws is a difficulty intrinsic to the nonlinear Schrödinger equation and it was in this setting that the first methods were developed for its treatment. The original breakthrough in this direction was Bourgain's paper [2]. His work introduced the induction on energy technique, which was then further developed in [3, 30]. In this paper, we will use a variant of this method that was introduced by Kenig and Merle, [9], building on work of Keraani, [13]. In this latter approach, one first shows that failure of the theorem implies the existence of minimal counterexamples. This part of the argument is based on concentration-compactness techniques and is very robust, with very little that is equation-specific. It breaks the scaling symmetry because such minimal counterexamples have an intrinsic length scale, albeit time-dependent. The second part of this approach is to use conservation laws and/or monotonicity formulae to show that such counterexamples do not exist. Like the conservation laws and monotonicity formulae themselves, this part of the argument is intrinsically equation dependent.

1.1. **Outline of the proof.** We argue by contradiction. The failure of Theorem 1.3 would imply the existence of very special types of counterexamples. Such counterexamples are then shown to have a wealth of properties not immediately apparent from their construction, so many properties, in fact, that they cannot exist.

While we will make some further reductions later, the main property of the special counterexamples is almost periodicity modulo symmetries:

Definition 1.6 (Almost periodicity modulo symmetries)**.** A solution u to (1.1) with lifespan I is said to be *almost periodic modulo symmetries* if (u, u_t) is bounded in $\dot H^{s_c}_x \times \dot H^{s_c-1}_x$ and there exist functions $N : I \to \mathbb{R}^+$, $x : I \to \mathbb{R}^3$, and $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t \in I$ and $\eta > 0$,

$$\int_{|x-x(t)|\geq C(\eta)/N(t)} \big||\nabla|^{s_c} u(t,x)\big|^2\,dx + \int_{|x-x(t)|\geq C(\eta)/N(t)} \big||\nabla|^{s_c-1} u_t(t,x)\big|^2\,dx \leq \eta$$

and

$$\int_{|\xi|\geq C(\eta)N(t)} |\xi|^{2s_c}\,|\hat u(t,\xi)|^2\,d\xi + \int_{|\xi|\geq C(\eta)N(t)} |\xi|^{2(s_c-1)}\,|\hat u_t(t,\xi)|^2\,d\xi \leq \eta.$$

We refer to the function $N(t)$ as the *frequency scale function* for the solution u, to $x(t)$ as the *spatial center function*, and to $C(\eta)$ as the *compactness modulus function*.

Remarks. 1. Given a time $t_0 \in I$ we may rescale the function $u(t_0, x)$ so as to renormalize the frequency scale to equal one. We may then perform a spatial translation to bring the spatial center of the function to the origin. Noting that these operations are symmetries of our equation and incorporating an additional time translation, this procedure yields a solution to (1.1) called the *normalization* of u associated to the time t_0:

$$u^{[t_0]}(t,x) := N(t_0)^{-\frac{2}{p}}\,u\big(t_0 + tN(t_0)^{-1}, x(t_0) + xN(t_0)^{-1}\big). \tag{1.6}$$

Note that the normalization of u is still almost periodic modulo symmetries; indeed, it admits the same compactness modulus function as u.

2. By the Ascoli–Arzela Theorem, a family of functions is precompact in $\dot H^s_x(\mathbb{R}^3)$ if and only if it is norm-bounded and there exists a compactness modulus function C so that

$$\int_{|x|\geq C(\eta)} \big||\nabla|^s f(x)\big|^2\,dx + \int_{|\xi|\geq C(\eta)} |\xi|^{2s}\,|\hat f(\xi)|^2\,d\xi \leq \eta$$

for all functions f in the family. Thus, an equivalent formulation of Definition 1.6 is as follows: u is almost periodic modulo symmetries if and only if

$$\big\{\big(u^{[t_0]}(0), \partial_t u^{[t_0]}(0)\big) : t_0 \in I\big\} \tag{1.7}$$

is a precompact subset of $\dot H^{s_c}_x \times \dot H^{s_c-1}_x$.

3. The continuous image of a compact set is compact. Thus, by Sobolev embedding, almost periodic (modulo symmetries) solutions obey the following: For each $\eta > 0$ there exists $C(\eta) > 0$ so that

(1.8)
$$\big\|u(t,x)\big\|_{L_t^\infty L_x^{\frac{3p}{2}}(\{|x-x(t)|\geq C(\eta)/N(t)\})} + \big\|\nabla_{t,x}u(t,x)\big\|_{L_t^\infty L_x^{\frac{3p}{p+2}}(\{|x-x(t)|\geq C(\eta)/N(t)\})} \leq \eta,$$

where $\nabla_{t,x}u = (u_t, \nabla u)$ denotes the space-time gradient of u.

With these preliminaries out of the way, we can now describe the first major milestone in the proof of Theorem 1.3.

Theorem 1.7 (Reduction to almost periodic solutions, [12])**.** *Assume Theorem* 1.3 *failed. Then there exists a maximal-lifespan solution* $u : I \times \mathbb{R}^3 \to \mathbb{R}$ *to* (1.1) *such that* $(u, u_t) \in L_t^\infty(I; \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1})$*, u is almost periodic modulo symmetries, and u blows up both forward and backward in time. Moreover, u is minimal among all blow-up solutions in the sense that*

$$\sup_{t\in I}\big\|\big(u(t), u_t(t)\big)\big\|_{\dot{H}_x^{s_c}\times\dot{H}_x^{s_c-1}} \leq \sup_{t\in J}\big\|\big(v(t), v_t(t)\big)\big\|_{\dot{H}_x^{s_c}\times\dot{H}_x^{s_c-1}}$$

for all maximal-lifespan solutions $v : J \times \mathbb{R}^3 \to \mathbb{R}$ that blow up in at least one time direction.

The reduction to almost periodic solutions is now a standard technique in the analysis of dispersive equations at critical regularity. Their existence was first proved by Keraani [13] in the context of the mass-critical NLS and they were first used as a tool for proving global well-posedness by Kenig and Merle [9]. As noted above, Theorem 1.7 was proved by Kenig and Merle in [12]; for other instances of the same techniques, see [10, 11, 14, 15, 16, 17, 18, 19, 32, 33].

We will also need the following further refinement of Theorem 1.7:

Theorem 1.8 (Three special scenarios for blowup, [15])**.** *Suppose that Theorem* 1.3 *failed. Then there exists a maximal-lifespan solution $u : I \times \mathbb{R}^3 \to \mathbb{R}$, which obeys* $(u, u_t) \in L_t^\infty(I; \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1})$*, is almost periodic modulo symmetries, and* $S_I(u) = \infty$*. Moreover, we can also ensure that the lifespan I and the frequency scale function $N : I \to \mathbb{R}^+$ match one of the following three scenarios:*

I. *(Finite-time blowup) We have that either* $\sup I < \infty$ *or* $|\inf I| < \infty$*.*

II. *(Soliton-like solution) We have $I = \mathbb{R}$ and*

$$N(t) = 1 \quad \textit{for all} \quad t \in \mathbb{R}.$$

III. *(Low-to-high frequency cascade) We have $I = \mathbb{R}$,*

$$\inf_{t\in\mathbb{R}} N(t) \geq 1, \quad \textit{and} \quad \limsup_{t\to+\infty} N(t) = \infty.$$

The reference given above discusses the energy-critical NLS; however, the result follows from Theorem 1.7 by the same arguments since they are essentially combinatorial and so apply to any dispersive equation. As we are treating a problem whose critical regularity lies above that of the conserved quantity, the energy-critical NLS serves as a better model than the mass-critical NLS. This is the reason for using this set of special scenarios rather than those obtained in [18].

A further manifestation of the minimality of u as a blow-up solution is the absence of a scattered wave at the endpoints of the lifespan I; more formally, we have the following Duhamel formulae, which play an important role in proving needed decay. This is a robust consequence of almost periodicity modulo symmetries; see, for example, [16].

Lemma 1.9 (No-waste Duhamel formulae)**.** *Let u be an almost periodic solution to* (1.1) *on its maximal-lifespan I. Then, for all $t \in I$,*

$$\begin{aligned}\begin{bmatrix} u(t) \\ u_t(t) \end{bmatrix} &= \int_t^{\sup I} \begin{bmatrix} \frac{\sin\left((t-s)|\nabla|\right)}{|\nabla|} \\ \cos\left((t-s)|\nabla|\right) \end{bmatrix} F(u(s))\, ds \\ &= -\int_{\inf I}^t \begin{bmatrix} \frac{\sin\left((t-s)|\nabla|\right)}{|\nabla|} \\ \cos\left((t-s)|\nabla|\right) \end{bmatrix} F(u(s))\, ds \end{aligned} \tag{1.9}$$

as weak limits in $\dot H^{s_c}_x \times \dot H^{s_c-1}_x$.

Representations of this type are central tools for improving the decay and/or regularity properties of the solution u. For the problem under discussion in this paper, it is better decay that is required since the key monotonicity formula (the Morawetz identity) and the key conservation law (the energy) have $\dot H^1_x \times L^2_x$ scaling. We need access to these identities in order to preclude the soliton-like and frequency-cascade solutions (described in Theorem 1.8), respectively. It is considerably easier to dispense with the finite-time blow-up solution: minimality forces the solution to lie inside a light cone, which in turn implies that the energy is zero. This argument is presented in Section 10 and is little different from the treatment in [12].

A key problem in low-dimensional situations, such as the problem under discussion in this paper, is making the integrals (1.9) converge in a better sense, for example, in some norm. In [12], an incoming/outgoing wave decomposition is used together with the weighted decay available from radial Sobolev embedding. A not dissimilar technique was used in [18, 19], which studied 2D NLS; the 3D NLW has the same poor dispersive estimate as the 2D NLS.

In this paper, we will prove that the Duhamel integrals converge by making use of the energy flux identity (cf. Lemma 3.2); the same idea (albeit with a different purpose) was used in [31, Corollary 4.3]. This is much weaker than the weighted estimates available in the radial case. Nevertheless, by expanding on the ideas in our earlier paper [15], we are able to show that u lies in $L^\infty_t L^q_x(\mathbb{R}^3)$ for some $q < \frac{3p}{2}$, the exponent given by Sobolev embedding. This is the topic of Section 5. While this does constitute better decay than the a priori bound, it is not sufficient to use the monotonicity/conservation laws; these require L^2_x-type control on $\nabla_{t,x} u$.

As in [15], we will employ the double Duhamel identity to upgrade the L^q_x estimates to the better kind of decay required. This identity was first introduced in [3] for the nonlinear Schrödinger equation and results from taking the inner product between the two formulae in (1.9). We have not seen this technique used before for the nonlinear wave equation and, in light of this, it is perhaps worth noting that in order to maintain the natural structure of the formula, one should take the inner product of the two representations of the spacetime derivative $\nabla_{t,x} u(0)$, rather than of $u(0)$ as in the Schrödinger case. This is a manifestation of the fact that (1.1) is second-order in time.

The double Duhamel integrals have very poor convergence properties. In [15], this restricted us to working in five or more (spatial) dimensions. Due to the different nature of the dispersive estimate, this would be analogous to dimensions six or higher for the wave equation. Making the double Duhamel formula converge is quite an undertaking, as we will describe. First, we localize in space. This was done already in [3] and results in an improvement equivalent to lowering the applicable dimension by two, which is still insufficient for the nonlinear wave equation in three dimensions.

In the manner as we have described it so far, the space-localized double Duhamel formula reads as follows:

(1.10)
$$\begin{aligned}\int_{\mathbb{R}^3} \chi(x)\big|\nabla_{t,x}u(0)\big|^2\,dx = &-\int_0^\infty\int_{-\infty}^0 \big\langle \nabla\tfrac{\sin(|\nabla|t)}{|\nabla|}F(t),\ \chi\nabla\tfrac{\sin(|\nabla|\tau)}{|\nabla|}F(\tau)\big\rangle\,d\tau\,dt\\ &-\int_0^\infty\int_{-\infty}^0 \big\langle \cos(|\nabla|t)F(t),\ \chi\cos(|\nabla|\tau)F(\tau)\big\rangle\,d\tau\,dt,\end{aligned}$$

where $F(t)$ is short-hand for $F(u(t))$, the nonlinearity in (1.1), χ denotes a spatial cutoff function, and the inner products are in vector- and scalar-valued $L^2_x(\mathbb{R}^3)$, respectively. When actually used in Section 7, there will be additional frequency projections and, in the first occurrence, (fractional) differential operators.

Our technique for making the integrals in (1.10) converge is inspired by a consideration of geometric optics: In a sense, (1.10) represents the nonlinearity $F(t)$ at time t looking at the nonlinearity $F(\tau)$ at time τ though a 'keyhole' whose aperture is the support of χ. Note that the main part of the nonlinearity $F(t)$ lies near the center of the wave-packet at that time, namely, $x(t)$. This indicates the path we will follow: (a) make sure that points near $x(t)$ cannot see points near $x(\tau)$, at least not directly, (b) control the amount of diffraction associated with the aperture, and (c) control the contribution from points far from $x(t)$ and/or $x(\tau)$.

For short times, part (a) of this programme is immediate from the finiteness of the speed of propagation. If the aperture is far from $x(0)$, then $x(t)$ and $x(\tau)$ do not have time to travel far enough to see one another. While this very naive picture continues to hold in the long-time regime, that is, $x(t)$ cannot catch up to a light ray emanating from $x(\tau)$ that passes through the aperture, there is no improvement with the passing of time that might allow the time integral to converge. We thus need to show that $x(t)$ and $x(\tau)$ travel strictly slower than light. This is the topic of Section 4. In previous work on critical dispersive equations, the motion of $x(t)$ has been constrained by using conservation laws, specifically, the conservation of momentum. This approach is not available in this case, since we need to control $x(t)$ first in order to obtain finiteness of the conserved quantities. In the case of radial data, $x(t) \equiv 0$.

Part (b) of the programme outlined above is encapsulated in Proposition 2.6. Thanks to the subluminality proved in Section 4, we need not consider very long times.

The proof that u (and so also $F(u)$) decays quickly away from $x(t)$ is the subject of Section 6, which handles part (c) of our programme. To the best of our knowledge, this is the first instance when power-law decay has been obtained without the benefit of radial initial data in the setting of critical dispersive equations. Moreover, we obtain this decay in a scaling-invariant space. The argument uses the energy flux identity to make the Duhamel integral converge. This places the long-time piece in L^q_x for $q > 3p/2$. To compensate for this, we interpolate with the decay estimates obtained in Section 5, which show that $u \in L^\infty_t L^q_x$ for some $q < 3p/2$.

In Section 7, we pull these three threads together to prove that not only does u have finite energy, but even that the energy decays with a power-law away from $x(t)$. This is done using an iterative procedure that takes one from s_c derivatives to $s_c - \varepsilon$ derivatives to $s_c - 2\varepsilon$ derivatives, and so forth. The last step, from $1+\varepsilon$ derivatives to finite energy, is treated separately, because this can be done much more simply — ∇ is local in space, while $|\nabla|^{1+\varepsilon}$ is not.

Sections 8 and 9 use the finiteness of the energy to show, respectively, that frequency-cascade and soliton-like solutions to (1.1) are not possible. We show that the frequency-cascade solution is inconsistent with the conservation of energy; of course, this would be meaningless had we not first proved that the energy is finite. The existence of solitons is precluded by use of the Morawetz identity (cf. [20, 21]):

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3} -a_j(x)u_t(t,x)u_j(t,x) - \tfrac12 a_{jj}(x)u(t,x)u_t(t,x)\,dx\\ &= \int_{\mathbb{R}^3} a_{jk}(x)u_j(t,x)u_k(t,x) + \tfrac{p}{2p+4}a_{jj}(x)u(t,x)^{p+2} - \tfrac12 a_{jjkk}(x)u(t,x)^2\,dx,\end{aligned}$$

where u is a solution to (1.1), subscripts indicate partial derivatives, and repeated indices are summed. More precisely, we use the special case $a(x) = |x|$, which, together with the Fundamental Theorem of Calculus and Hardy's inequality, yields

$$\int_I\int_{\mathbb{R}^3} \frac{|u(t,x)|^{p+2}}{|x|}\,dx\,dt \lesssim \|\nabla_{t,x}u\|^2_{L_t^\infty L_x^2(I\times\mathbb{R}^3)}. \tag{1.11}$$

Notice that by finite speed of propagation, the left-hand side should grow logarithmically in time.

The finite-time blow-up solution is precluded in Section 10 and does rely on Sections 4 through 7. Like the frequency-cascade, this type of solution is inconsistent with the conservation of energy; finiteness of the energy in this case follows from the fact that finite-time blow-up solutions are compactly supported at each time. The idea of using a second (noncritical) conservation law to control the growth/decay of $N(t)$ originates in the study of NLS (cf. [2, §4]); in this paper, the assumed boundedness of the critical Sobolev norm acts as a first conservation law.

2. Notation and useful lemmas

We write $X \lesssim Y$ to indicate that $X \leq CY$ for some constant C, which may change from line to line. Dependencies will be indicated with subscripts, for example, $X \lesssim_u Y$. We will write $X \sim Y$ to indicate that $X \lesssim Y \lesssim X$.

Let $\varphi(\xi)$ be a radial bump function supported in the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq \frac{11}{10}\}$ and equal to 1 on the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq 1\}$. For each number $N > 0$, we define the Fourier multipliers

$$\begin{aligned}\widehat{P_{\leq N}f}(\xi) &:= \varphi(\xi/N)\hat f(\xi),\\ \widehat{P_{>N}f}(\xi) &:= (1-\varphi(\xi/N))\hat f(\xi),\\ \widehat{P_N f}(\xi) &:= (\varphi(\xi/N) - \varphi(2\xi/N))\hat f(\xi)\end{aligned}$$

and similarly for $P_{<N}$ and $P_{\geq N}$. We also define

$$P_{M<\cdot\leq N} := P_{\leq N} - P_{\leq M} = \sum_{M<N'\leq N} P_{N'}$$

whenever $M < N$. We will only have cause to use these multipliers when M and N are *dyadic numbers* (that is, of the form 2^n for some integer n); in particular, all summations over N or M are understood to be over dyadic numbers.

Like all Fourier multipliers, the Littlewood-Paley operators commute with derivatives and the propagator. We will only need the basic properties of these operators, particularly,

Lemma 2.1 (Bernstein estimates)**.** *For* $1 \leq p \leq q \leq \infty$*,*

$$\big\| |\nabla|^{\pm s} P_N f \big\|_{L^p_x} \sim N^{\pm s} \|P_N f\|_{L^p_x},$$
$$\|P_{\leq N} f\|_{L^q_x} \lesssim N^{\frac{3}{p}-\frac{3}{q}} \|P_{\leq N} f\|_{L^p_x},$$
$$\|P_N f\|_{L^q_x} \lesssim N^{\frac{3}{p}-\frac{3}{q}} \|P_N f\|_{L^p_x}.$$

In three space dimensions, the wave equation obeys the strong form of the Huygens principle. This is most easily expressed in terms of the explicit form of the propagator:

Lemma 2.2. *For Schwartz functions* f*,*

$$\Big[\tfrac{\sin(t|\nabla|)}{|\nabla|} f\Big](x) = \tfrac{1}{4\pi t} \int_{|x-y|=t} f(y)\, dS(y), \tag{2.1}$$

where dS *denotes the usual* 2*-dimensional surface measure.*

To be absolutely clear about the normalization here, we note that if $f \equiv 1$, then RHS(2.1) $\equiv t$. A well-known consequence of (2.1) is the following:

Lemma 2.3 (Dispersive estimate)**.** *For* $2 \leq q \leq \infty$ *and* $f \in L^{q'}(\mathbb{R}^3)$*,*

$$\Big\| |\nabla|^{-1} \sin(t|\nabla|) f \Big\|_{L^q(\mathbb{R}^3)} \lesssim |t|^{-(1-\frac{2}{q})} \big\| |\nabla|^{-\frac{4}{q}} \nabla f \big\|_{L^{q'}(\mathbb{R}^3)}.$$

Proof. For $q = 2$, the result reduces to the boundedness of the Fourier multiplier $\sin(t|\xi|)$. When $q = \infty$, the desired estimate is

$$\big\| |\nabla|^{-1} \sin(t|\nabla|) f \big\|_{L^\infty(\mathbb{R}^3)} \lesssim_q |t|^{-1} \|\nabla f\|_{L^1(\mathbb{R}^3)},$$

which follows easily from (2.1) and

$$\int_{|\omega|=1} f(t\omega)\, dS(\omega) = \int_t^\infty \int_{|\omega|=1} -\omega \cdot \nabla f(r\omega)\, dS(\omega)\, dr \leq t^{-2} \|\nabla f\|_{L^1(\mathbb{R}^3)}.$$

For general q one may apply the theory of analytic interpolation. We caution the reader, however, that this requires the use of BMO and its interpolation theory; see, for example, [27, §IV.5], which uses a very closely related estimate as the motivating example. □

Note that even when $q = \infty$, this gives only t^{-1} decay. This is not integrable in time and so is insufficient to prove convergence of the Duhamel formulae (1.9).

As is now well understood, the dispersive estimate forms the basis for proving Strichartz estimates, which we record next. See [8, 23, 28] and the references therein for further information.

Lemma 2.4 (Strichartz estimates)**.** *Let* I *be a compact time interval and let* $u : I \times \mathbb{R}^3 \to \mathbb{R}$ *be a solution to the forced wave equation*

$$u_{tt} - \Delta u + F = 0.$$

Then for any $t_0 \in I$, $6 \le q < \infty$, *and* $p/2 < \tilde q \le \infty$,

$$\|u\|_{L_t^\infty \dot H_x^{s_c}} + \|u_t\|_{L_t^\infty \dot H_x^{s_c-1}} + \|u\|_{L_{t,x}^{2p}} + \|u\|_{L_t^{\tilde q} L_x^{\frac{3p\tilde q}{2\tilde q - p}}} + \|\nabla u\|_{L_t^{\frac{4p}{p-4}} L_x^{\frac{4p}{p+4}}} + \|\nabla u\|_{L_t^\infty L_x^{\frac{3p}{p+2}}}$$
$$\lesssim \|u(t_0)\|_{\dot H_x^{s_c}} + \|u_t(t_0)\|_{\dot H_x^{s_c-1}} + \big\||\nabla|^{s_c} F\big\|_{L_t^{\frac{2q}{q+6}} L_x^{\frac{q}{q-1}}},$$

where all spacetime norms are on $I \times \mathbb{R}^3$.

As noted in the introduction, much of the argument presented here was inspired by work on NLS. The most favourable difference between NLW and NLS is that NLW enjoys finite speed of propagation. Unfortunately, we will need to deal with a noninteger number of derivatives, which is inherently a nonlocal operator. To cope with this, we will make use of the following:

Lemma 2.5 (Mismatch estimates). *Let* ϕ_1 *and* ϕ_2 *be (smooth) functions obeying*

$$|\phi_j| \le 1 \quad \text{and} \quad \operatorname{dist}(\operatorname{supp}\phi_1, \operatorname{supp}\phi_2) \ge A,$$

for some large constant A. *Then for* $\sigma > 0$ *and* $1 \le p \le q \le \infty$,

(2.2)
$$\big\|\phi_1 |\nabla|^\sigma P_{\le 1}(\phi_2 f)\big\|_{L_x^q(\mathbb{R}^3)} + \big\|\phi_1 \nabla |\nabla|^{\sigma-1} P_{\le 1}(\phi_2 f)\big\|_{L_x^q(\mathbb{R}^3)} \lesssim A^{-\sigma - \frac{3}{p} + \frac{3}{q}} \|\phi_2 f\|_{L_x^p(\mathbb{R}^3)}.$$

Proof. Elementary computations show that if $K(x)$ denotes the convolution kernel associated to either of the Fourier multipliers $|\nabla|^\sigma P_{\le 1}$ or $\nabla|\nabla|^{\sigma-1} P_{\le 1}$, then $|K(x)| \lesssim |x|^{-3-\sigma}$.

Noting that

$$\int_A^\infty r^{-(3+\sigma)\tilde q} r^2 \, dr \lesssim_{\tilde q} A^{-\tilde q(\sigma + \frac{3}{p} - \frac{3}{q})} \quad \text{when} \quad \tfrac{1}{p} + \tfrac{1}{\tilde q} = 1 + \tfrac{1}{q},$$

the result follows from Young's inequality. □

Our next proposition shows that waves do not diffract too much through a large aperture and is an essential ingredient in justifying the geometric optics heuristics set forth in the introduction; see (1.10) and the adjacent discussion. We are content to show that there is sufficient decay in (2.4) below and have made no attempt to find the optimal bound.

Proposition 2.6 (Weak diffraction). *Let* $\phi : \mathbb{R}^3 \to [0,1]$ *be a smooth compactly supported function such that* $\phi(x) = 1$ *for* $|x| < 1$ *and* $\phi(x) = 0$ *for* $|x| > \frac{11}{10}$. *Also, let* $\theta : \mathbb{R}^3 \to [0,\infty)$ *be defined by*

$$\theta(\xi) = \prod_{j=1}^3 \Big(\frac{\sin(\xi_j)}{\xi_j}\Big)^4. \tag{2.3}$$

Then

$$\Big|\iint \big\langle \nabla \tfrac{\sin(|\nabla| t)}{|\nabla|} \theta(i\nabla) F(t),\ \phi\big(\tfrac{\cdot}{R}\big) \nabla \tfrac{\sin(|\nabla|\tau)}{|\nabla|} \theta(i\nabla) G(\tau)\big\rangle \, d\tau\, dt$$
$$+ \iint \big\langle \cos(|\nabla| t)\theta(i\nabla) F(t),\ \phi\big(\tfrac{\cdot}{R}\big) \cos(|\nabla|\tau)\theta(i\nabla) G(\tau)\big\rangle \, d\tau\, dt\Big|$$
$$\lesssim R^{-\frac{1}{10}} \|F\|_{L_t^\infty L_x^1} \|G\|_{L_\tau^\infty L_y^1}, \tag{2.4}$$

provided $F(t,x)$ and $G(\tau,y)$ are supported where

$$|t|+|\tau|+|x|+|y| \lesssim R \quad \textit{and} \quad \frac{|t-\tau|-|x-y|}{R} \gtrsim 1 \tag{2.5}$$

for any (large) constant R.

Remarks. 1. The second inequality in (2.5) shows that the spacetime points (t,x) and (τ,y) are not light-like separated. Thus if the $\phi(\cdot/R)$ cutoff were removed from (2.4), the left-hand side would equal zero by the strong Huygens principle. In this way, we see that the proposition is truly a bound on the diffraction caused by the finite-size aperture $\phi(\cdot/R)$.

2. The multiplier $\theta(i\nabla)$ plays the role of $P_{\leq 1}$. Choosing this in lieu of the regular low-frequency projection has little effect on the proof, but avoids the appearance of additional error terms when we apply it in Section 7. This is because the corresponding integral kernel has compact support.

Proof. The LHS(2.4) can be rewritten as

$$\iint\iint [K_1(t,x;\tau,y)+K_2(t,x;\tau,y)]F(t,x)G(\tau,y)\,dx\,dy\,d\tau\,dt,$$

where K_1 and K_2 are the kernels

$$K_1 := \operatorname{Re}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it|\xi|-i\tau|\eta|+ix\cdot\xi-iy\cdot\eta}R^3\check{\phi}\big(R(\xi-\eta)\big)\theta(\xi)\theta(\eta)\,d\xi\,d\eta,$$

which captures the principal behaviour of (2.4), and

$$K_2 := -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \sin(t|\xi|)\sin(\tau|\eta|)R^3\check{\phi}\big(R(\xi-\eta)\big)\theta(\xi)\theta(\eta)\big[1-\tfrac{\xi\cdot\eta}{|\xi||\eta|}\big]e^{ix\cdot\xi-iy\cdot\eta}\,d\xi\,d\eta.$$

To prove the proposition, it suffices to estimate these kernels in $L^1_{t,\tau}L^\infty_{x,y}$ on the support of F and G. In view of (2.5), this in turn can be effected by proving $R^{-21/10}$ bounds on the kernels in $L^\infty_{t,\tau,x,y}$, which is essentially what we will do. As a first step, we change variables to $\mu=\frac{\xi+\eta}{2}$ and $\nu=\frac{\xi-\eta}{2}$ and we split the integral into several pieces, first by introducing the cutoffs $\phi(R^{7/9}\mu)$ and $1-\phi(R^{7/9}\mu)$, and then in the latter case, the cutoffs $\phi(R^{8/9}\nu)$ and $1-\phi(R^{8/9}\nu)$. The job of these cutoffs is to focus attention on the dominant region of (ξ,η) space, namely, where $|\xi-\eta|\ll|\xi|+|\eta|$. While the $\check{\phi}$ term clearly concentrates $|\xi-\eta|$ near zero, the tiny tails significantly muddy the requisite computations in the dominant regime, hence the need for $\phi(R^{8/9}\nu)$. Knowing that ν is small, we see that the momentum transfer to the light on passing through the aperture is very small; however, the direction of propagation of a light ray is dictated by $\xi/|\xi|$ and so low-momentum light rays can nonetheless undergo significant changes in direction. This is the anomaly associated to the case of small μ. The temporal supports of $G(\tau)$ and $F(t)$ are too short to allow significant production/absorbtion of low-momentum light (cf. the uncertainty principle); this is the physical content of the estimate (2.6) below.

When the cutoff $\phi(R^{7/9}\mu)$ is present, we bring absolute values inside the integrals and bound the corresponding contribution (to K_1 or K_2) by

$$\lesssim \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\phi(R^{7/9}\mu)R^3|\check{\phi}(2R\nu)|\,d\mu\,d\nu \lesssim R^{-7/3}. \tag{2.6}$$

When the factor $[1-\phi(R^{7/9}\mu)][1-\phi(R^{8/9}\nu)]$ is present, we again bring the absolute values inside. We also make use of the fact that $|\check\phi(\zeta)| \lesssim_m |\zeta|^{-m}$ for all $m\in\mathbb{N}$. Thus, the contribution of this term to either kernel is

$$\begin{aligned}
&\lesssim \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \bigl[1-\phi(R^{7/9}\mu)\bigr]\bigl[1-\phi(R^{8/9}\nu)\bigr]R^3|\check\phi(2R\nu)|\theta(\mu+\nu)\theta(\mu-\nu)\,d\mu\,d\nu\\
&\lesssim \int_{\mathbb{R}^3} \bigl[1-\phi(R^{8/9}\nu)\bigr]R^3\bigl|\check\phi(2R\nu)\bigr|\,d\nu\\
&\lesssim \int_{|\nu|\gtrsim R^{-8/9}} \frac{R^3}{(R|\nu|)^{24}}\,d\nu\\
&\lesssim R^{-7/3}. \qquad (2.7)
\end{aligned}$$

It remains to estimate the contributions that are associated with the factor $[1-\phi(R^{7/9}\mu)]\phi(R^{8/9}\nu)$, which is indeed the heart of the matter. Here we must treat the two kernels separately. We begin by considering K_1, which amounts to estimating the integral

$$I_1 := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{iR\varphi(\mu,\nu)}\psi(\mu,\nu)\,d\mu\,d\nu,$$

with

$$\varphi(\mu,\nu) := \tfrac{1}{R}\bigl[t|\mu+\nu| - \tau|\mu-\nu| + (x-y)\cdot\mu + (x+y)\cdot\nu\bigr]$$

and

$$\psi(\mu,\nu) := \bigl[1-\phi(R^{7/9}\mu)\bigr]\phi(R^{8/9}\nu)R^3\check\phi(2R\nu)\theta(\mu+\nu)\theta(\mu-\nu).$$

To do this, we will employ the technique of nonstationary phase. The fact that the phase is indeed nonstationary is a consequence of (2.5).

For all multi-indices $\alpha\in\mathbb{Z}^3_{\geq 0}$ of length $|\alpha|\leq 4$, we have the following symbol-type estimates:

$$|\partial_\xi^\alpha\theta(\xi)| \lesssim_\alpha |\xi|^{-|\alpha|}, \qquad |\partial_\mu^\alpha\psi| \lesssim_\alpha R^3|\mu|^{-|\alpha|}, \qquad |\partial_\mu^\alpha\varphi| \lesssim_\alpha |\mu|^{1-|\alpha|},$$
$$\Bigl|\partial_\mu^\alpha \frac{\mu}{|\mu|}\Bigr| \lesssim_\alpha |\mu|^{-|\alpha|}, \quad\text{and}\quad \Bigl|\frac{\mu}{|\mu|}\cdot\nabla_\mu\varphi\Bigr| \gtrsim 1,$$

uniformly for $(\mu,\nu)\in\operatorname{supp}(\psi)$, which implies that $|\mu|\gtrsim R^{-7/9}$ and $|\nu|\lesssim R^{-8/9}$. To derive the last estimate, one squares both sides and uses the fact that for these μ, ν,

$$\sqrt{1-\tfrac{|\nu|^2}{|\mu+\nu|^2}} \leq \tfrac{\mu\cdot(\mu\pm\nu)}{|\mu||\mu\pm\nu|} \leq 1,$$

to obtain

$$\tfrac{\mu}{|\mu|}\cdot\nabla_\mu\varphi = \tfrac{t-\tau}{R} + \tfrac{\mu}{|\mu|}\cdot\tfrac{x-y}{R} + O\bigl(\tfrac{R^{-2/9}(|t|+|\tau|)}{R}\bigr).$$

The inequality now follows from this and (2.5).

Using these estimates and the quotient rule in the symbol calculus, we find that the vector $a := \bigl(\frac{\mu}{|\mu|}\cdot\nabla_\mu\varphi\bigr)^{-1}\frac{\mu}{|\mu|}$ obeys

$$|\partial_\mu^\alpha a| \lesssim_\alpha |\mu|^{-|\alpha|} \quad\text{uniformly in}\quad |\mu|\gtrsim R^{-7/9} \text{ and } |\nu|\lesssim R^{-8/9}$$

when $|\alpha|\leq 4$. Moreover,

$$\sup_\nu\bigl|(i\nabla_\mu\cdot a)^4\psi\bigr| \lesssim \sum_{|\alpha_1+\dots+\alpha_4+\beta|=4} |\partial_\mu^{\alpha_1}a|\cdots|\partial_\mu^{\alpha_4}a||\partial_\mu^\beta\psi| \lesssim R^3|\mu|^{-4}.$$

Thus, as $e^{iR\varphi} = R^{-4}(a \cdot i\nabla_\mu)^4 e^{iR\varphi}$, we obtain

$$\begin{aligned}
|I_1| &\lesssim R^{-\frac{8}{3}} \sup_{|\nu|\lesssim R^{-8/9}} \Big| \int_{\mathbb{R}^3} e^{iR\varphi(\mu,\nu)} \psi(\mu,\nu)\, d\mu \Big| \\
&\lesssim R^{-4-\frac{8}{3}} \sup_{|\nu|\lesssim R^{-8/9}} \Big| \int_{\mathbb{R}^3} e^{iR\varphi(\mu,\nu)} (i\nabla_\mu \cdot a)^4 \psi(\mu,\nu)\, d\mu \Big| \\
&\lesssim R^{-4-\frac{8}{3}} \int_{|\mu|\gtrsim R^{-7/9}} R^3 |\mu|^{-4}\, d\mu \\
&\lesssim R^{-7/3}. \qquad (2.8)
\end{aligned}$$

Collecting (2.6), (2.7), and (2.8), we obtain

$$|K_1(t,x;\tau,y)| \lesssim R^{-7/3}.$$

By virtue of (2.5), this settles the K_1 portion of the proposition.

We now turn to estimating the remaining portion of the integral defining K_2, namely,

$$\begin{aligned}
I_2 := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} &\sin(t|\mu+\nu|)\sin(\tau|\mu-\nu|) R^3 \check{\phi}\big(2R\nu\big) e^{i\mu(x-y)+i\nu(x+y)} \theta(\mu+\nu)\theta(\mu-\nu) \\
&\times \big[1 - \tfrac{(\mu+\nu)\cdot(\mu-\nu)}{|\mu+\nu||\mu-\nu|}\big] [1-\phi(R^{7/9}\mu)]\phi(R^{8/9}\nu)\, d\mu\, d\nu. \qquad (2.9)
\end{aligned}$$

To continue, we use the simple identity

$$\sin(t|\mu+\nu|)\sin(\tau|\mu-\nu|) = \tfrac{1}{2}\operatorname{Re} e^{it|\mu+\nu|-i\tau|\mu-\nu|} - \tfrac{1}{2}\operatorname{Re} e^{it|\mu+\nu|+i\tau|\mu-\nu|},$$

which naturally breaks I_2 into the sum of two pieces. The first summand can be estimated in a manner similar to that used to treat I_1, or by a simplified version of the technique we will use to estimate the second summand,

$$\begin{aligned}
I_2' := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} &e^{i\varphi} R^3 \check{\phi}(2R\nu)\theta(\mu+\nu)\theta(\mu-\nu)\big[1 - \tfrac{(\mu+\nu)\cdot(\mu-\nu)}{|\mu+\nu||\mu-\nu|}\big] \\
&\times [1-\phi(R^{7/9}\mu)]\phi(R^{8/9}\nu)\, d\mu\, d\nu,
\end{aligned}$$

where

$$\varphi := t|\mu+\nu| + \tau|\mu-\nu| + \mu(x-y) + \nu(x+y).$$

As a first estimate on I_2', we note that when $|\nu| \ll |\mu|$, which is where the integrand is supported,

$$1 - \tfrac{(\mu+\nu)\cdot(\mu-\nu)}{|\mu+\nu||\mu-\nu|} = O\big(\tfrac{|\nu|^2}{|\mu|^2}\big) \qquad (2.10)$$

and hence

$$\begin{aligned}
|I_2'| &\lesssim \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} R^3|\check{\phi}(2R\nu)|\theta(\mu+\nu)\theta(\mu-\nu)\tfrac{|\nu|^2}{|\mu|^2}\, d\mu\, d\nu \lesssim \int_{\mathbb{R}^3} R^3|\check{\phi}(2R\nu)||\nu|^2\, d\nu \\
&\lesssim R^{-2}. \qquad (2.11)
\end{aligned}$$

Note that this is not quite good enough: when integrated over $|t|\lesssim R$ and $|\tau|\lesssim R$, there are no powers of R left over to provide the required decay. Nevertheless, it does provide the requisite $R^{-1/10}$ bound on the $L^1_{t,\tau}L^\infty_{x,y}$ norm of K_2 in the restricted region where $|t+\tau| \le R^{9/10}$.

It remains only to estimate I_2' in the region where $|t+\tau| \ge R^{9/10}$. To do this we will use the Van der Corput Lemma, which is most cleanly done by breaking the μ

integral into six pieces, one near each coordinate semi-axis. To this end, let us take a smooth partition of unity of the unit sphere adapted to the open cover

$$\{\max(|\mu_1|,|\mu_2|)<\tfrac{9}{8}\mu_3\},\ \{\max(|\mu_1|,|\mu_2|)<-\tfrac{9}{8}\mu_3\},\ \{\max(|\mu_3|,|\mu_1|)<\tfrac{9}{8}\mu_2\},$$
$$\{\max(|\mu_3|,|\mu_1|)<-\tfrac{9}{8}\mu_2\},\ \{\max(|\mu_2|,|\mu_3|)<\tfrac{9}{8}\mu_1\},\ \{\max(|\mu_2|,|\mu_3|)<-\tfrac{9}{8}\mu_1\}.$$

We break the μ integral into pieces by introducing cutoffs $\chi(\mu/|\mu|)$, where χ denotes one of the elements of this partition of unity. By symmetry, it suffices to treat the piece associated to the first region listed above.

Recalling that we are considering only the case where $|t+\tau|\geq R^{9/10}$, $|\mu|\gtrsim R^{-7/9}$ and $|\nu|\lesssim R^{-8/9}$, we have

$$\partial^2_{\mu_1}\phi=(t+\tau)\tfrac{|\mu|^2-\mu_1^2}{|\mu|^3}+O\Big(|\mu|^{-1}R^{1-\frac{1}{9}}\Big).$$

Noting that $|\mu|-|\mu_1|\sim|\mu|\sim\mu_3$ on the support of $\chi(\mu/|\mu|)$, we deduce that on this set,

$$\big|\partial^2_{\mu_1}\phi\big|\gtrsim\frac{|t+\tau|}{\mu_3}. \tag{2.12}$$

Thus, by writing

$$\psi(\mu,\nu):=\theta(\mu+\nu)\theta(\mu-\nu)\big[1-\tfrac{(\mu+\nu)\cdot(\mu-\nu)}{|\mu+\nu||\mu-\nu|}\big][1-\phi(R^{7/9}\mu)]\chi\big(\tfrac{\mu}{|\mu|}\big)$$

and noting that

$$\|\psi\|_{L^\infty(d\mu_1)}\lesssim\langle\mu_3\rangle^{-8}\tfrac{|\nu|^2}{\mu_3^2}\quad\text{and}\quad\big\|\partial_{\mu_1}\psi\big\|_{L^1(d\mu_1)}\lesssim\langle\mu_3\rangle^{-8}\big(\tfrac{|\nu|^2}{\mu_3}+\tfrac{|\nu|^2}{\mu_3^2}\big),$$

the Van der Corput Lemma (cf. [27, p. 334]) yields

$$\begin{aligned}\iiint e^{i\varphi}\psi\,d\mu_1\,d\mu_2\,d\mu_3&\lesssim\iint_{|\mu_2|\lesssim\mu_3}\big(\tfrac{\mu_3}{|t+\tau|}\big)^{1/2}\Big\{\|\psi\|_{L^\infty(d\mu_1)}+\big\|\partial_{\mu_1}\psi\big\|_{L^1(d\mu_1)}\Big\}d\mu_2\,d\mu_3\\&\lesssim\tfrac{|\nu|^2}{|t+\tau|^{1/2}},\end{aligned}$$

uniformly for $|\nu|\lesssim R^{-8/9}$. Thus when $|t+\tau|\geq R^{9/10}$, we may bound the portion of I_2' partitioned off by $\chi(\mu/|\mu|)$ as follows:

$$\bigg|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{i\varphi(\mu,\nu)}\psi(\mu,\nu)R^3\check{\phi}(2R\nu)\phi(R^{8/9}\nu)\,d\mu\,d\nu\bigg|\lesssim R^{-2}|t+\tau|^{-1/2}.$$

As a consequence, we can bound the $L^1_{t,\tau}L^\infty_{x,y}$ norm of K_2 on this set of times by $R^{-9/20}$.

This completes the proof of the proposition. □

3. NLW background

We start by recording the standard local well-posedness theory for (1.1). All results follow from the Strichartz inequalities discussed in Lemma 2.4 and the usual contraction mapping arguments.

Theorem 3.1 (Local well-posedness)**.** *Given $(u_0,u_1)\in\dot H^{s_c}_x\times\dot H^{s_c-1}_x$ and $t_0\in\mathbb{R}$, there is a unique maximal-lifespan solution $u:I\times\mathbb{R}^3\to\mathbb{R}$ to* (1.1) *with initial data $(u(t_0),u_t(t_0))=(u_0,u_1)$. This solution also has the following properties:*

- *(Local existence) I is an open neighbourhood of t_0.*
- *(Blow-up criterion) If $\sup I$ is finite, then u blows up forward in time (in the sense of Definition* 1.2*); if $\inf I$ is finite, then u blows up backward in time.*

• *(Scattering) If* $\sup I = +\infty$ *and* u *does not blow up forward in time, then* u *scatters forward in time, that is, there exists a unique* $(u_0^+, u_1^+) \in \dot H_x^{s_c} \times \dot H_x^{s_c-1}$ *such that*

$$\lim_{t\to+\infty} \Big\| u(t) - \cos(t|\nabla|)u_0^+ - \tfrac{\sin(t|\nabla|)}{|\nabla|} u_1^+ \Big\|_{\dot H_x^{s_c}} = 0. \tag{3.1}$$

Conversely, given $(u_0^+, u_1^+) \in \dot H_x^{s_c} \times \dot H_x^{s_c-1}$ *there is a unique solution to* (1.1) *in a neighbourhood of infinity so that* (3.1) *holds.*

• *(Small data global existence) If* (u_0, u_1) *is sufficiently small in* $\dot H_x^{s_c} \times \dot H_x^{s_c-1}$, *then* u *is a global solution which does not blow up either forward or backward in time. Indeed, in this case,*

$$S_{\mathbb{R}}(u) \lesssim \big\| (u_0, u_1) \big\|_{\dot H_x^{s_c} \times \dot H_x^{s_c-1}}^{2p}.$$

Our next topic is the energy flux identity/inequality, which is a variant of the Morawetz identity/inequality discussed in the introduction and is proved in much the same way. We will use it in connection with the Duhamel formulae (1.9), in order to show that the time integrals converge.

Lemma 3.2 (Energy flux inequality)**.** *If* u *is a solution to* (1.1) *with* $(u, u_t) \in L_t^\infty(I; \dot H_x^{s_c} \times \dot H_x^{s_c-1})$, *then*

$$\int_I \int_{|x-y|=|t|} |u(t,y)|^{p+2}\, dS(y)\, dt \lesssim_u \sup_{t\in I} |t|^{1-\frac{4}{p}}$$

uniformly for $x \in \mathbb{R}^3$.

Proof. The result follows by applying the Fundamental Theorem of Calculus to

$$\mathcal{E}(t) := \int_{|x-y|\le |t|} \tfrac12 |\nabla_{t,x} u(t,y)|^2 + \tfrac{1}{p+2} |u(t,y)|^{p+2}\, dy$$

and noting that

$$\mathcal{E}(t) \lesssim \Big(\|\nabla_{t,x} u(t)\|^2_{L_x^{\frac{3p}{p+2}}} + \|u(t)\|^{p+2}_{L_x^{\frac{3p}{2}}} \Big) |t|^{1-\frac{4}{p}} \lesssim_u |t|^{1-\frac{4}{p}},$$

by Hölder's inequality and Sobolev embedding. □

The small data theory shows that the $\dot H_x^{s_c} \times \dot H_x^{s_c-1}$ norm of a blow-up solution must remain bounded from below. The fact that this norm is nonlocal in space reduces the efficacy of this statement. Our next lemma gives a lower bound in a more suitable norm:

Lemma 3.3 ($\nabla_{t,x} u$ nontriviality)**.** *Let* u *be a global solution that is almost periodic modulo symmetries. Then,*

$$\inf_{t\in\mathbb{R}} \int_{\mathbb{R}^3} |\nabla_{t,x} u(t,x)|^{\frac{3p}{p+2}}\, dx \gtrsim_u 1. \tag{3.2}$$

Proof. First we note that by the small data theory,

$$\inf_{t\in\mathbb{R}} \big\| \nabla_{t,x} u(t,x) \big\|_{\dot H_x^{s_c-1}} \gtrsim_p 1, \tag{3.3}$$

for otherwise u would have finite spacetime norm in contravention of the hypotheses of this lemma. Indeed, a solution that scatters cannot be almost periodic modulo symmetries.

Next, we note that

$$\|f\|_{L_x^{\frac{3p}{p+2}}} \div \|f\|_{\dot{H}_x^{s_c-1}} > 0$$

for any nonzero $\mathbb{R}^4$-valued $f \in \dot{H}_x^{s_c-1}$. Hence this ratio achieves a nonzero minimum on any compact set that does not contain the zero function. Indeed, since this ratio is invariant under scaling and translation, it suffices for the set to be compact modulo these symmetries. Therefore, the ratio is bounded from below on the (precompact) orbit $\nabla_{t,x}u(t)$, and so, in view of (3.3), the lemma follows. Note also that (3.3) guarantees that the orbit $\nabla_{t,x}u(t)$ does not approach the zero function. □

It is not possible to obtain lower bounds on the norm of $u(t)$ for a single time t, as it is quite conceivable that $u(t) = 0$, with all the $\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}$ norm concentrating in $u_t(t)$. Nevertheless, this phenomenon must be rather rare as our next lemma demonstrates.

Lemma 3.4 ($L_x^{3p/2}$-norm nontriviality)**.** *Let u be a global solution that is almost periodic modulo symmetries. Then, for any $A > 0$, there exists $\eta = \eta(u, A) > 0$ so that*

$$\big|\big\{t \in [t_0, t_0 + AN(t_0)^{-1}] : \|u(t)\|_{L_x^{3p/2}(\mathbb{R}^3)} \geq \eta\big\}\big| \geq \eta N(t_0)^{-1} \tag{3.4}$$

for all $t_0 \in \mathbb{R}$.

Proof. Recasting (3.4) in terms of the normalizations of u, defined in (1.6), yields

$$\big|\big\{s \in [0, A] : \|u^{[t_0]}(s)\|_{L_x^{3p/2}(\mathbb{R}^3)} \geq \eta\big\}\big| \geq \eta. \tag{3.5}$$

As the map from the initial data to the solution is continuous (a consequence of the local theory) and the set

$$\big\{\big(u^{[t_0]}(0),\ u_t^{[t_0]}(0),\ s\big) : t_0 \in \mathbb{R} \text{ and } s \in [0, A]\big\}$$

is precompact, we deduce that $\{u^{[t_0]}(s) : t_0 \in \mathbb{R} \text{ and } s \in [0, A]\}$ is precompact in $\dot{H}_x^{s_c}$. Thus by Sobolev embedding, we see that it suffices to show that for some choice of η the set appearing in (3.5) is nonempty for all $t_0 \in \mathbb{R}$. (Of course, the passage from nonemptyness to positive measure requires a reduction in η.)

To see that the set appearing in (3.5) is nonempty, we argue by contradiction. To this end, imagine that there is a sequence of times t_n so that

$$\|u^{[t_n]}(s)\|_{L_x^{3p/2}(\mathbb{R}^3)} \to 0 \quad \text{uniformly for} \quad s \in [0, A]. \tag{3.6}$$

Then, by a simple bootstrap argument using the Duhamel formula, (1.3), and the Strichartz inequality, we deduce that

$$\big\|u^{[t_n]}(s) - \cos(s|\nabla|)u^{[t_n]}(0) - |\nabla|^{-1}\sin(s|\nabla|)u_s^{[t_n]}(0)\big\|_{L_s^\infty L_x^{3p/2}([0,A]\times\mathbb{R}^3)} \to 0.$$

Thus, appealing to (3.6) once again, we obtain

$$\big\|\cos(s|\nabla|)u^{[t_n]}(0) + |\nabla|^{-1}\sin(s|\nabla|)u_s^{[t_n]}(0)\big\|_{L_s^\infty L_x^{3p/2}([0,A]\times\mathbb{R}^3)} \to 0. \tag{3.7}$$

This, we will see, contradicts the uniqueness theorem for the linear wave equation.

As noted previously (cf. the remarks after Definition 1.6), almost-periodicity of u implies that the sequence of pairs $(u^{[t_n]}(0), u_s^{[t_n]}(0))$ is precompact in $\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}$. Thus, by passing to a subsequence, we may assume that it converges and name the limit (f, g). This limit must be nonzero, for otherwise, we could apply the small-data theory to the pair $(u(t_n), u_t(t_n))$, for n large enough, and deduce that u is global and has finite spacetime norm.

On the other hand, from (3.7), we see that

$$\cos(s|\nabla|)f + |\nabla|^{-1}\sin(s|\nabla|)g = 0 \quad \text{for all} \quad s \in [0, A],$$

which implies that $f \equiv g \equiv 0$, since they can be reconstructed from the behaviour of this solution of the linear wave equation as $s \to 0$. This contradicts the results of the previous paragraph and so completes the proof of the lemma. □

Corollary 3.5 (Potential energy concentration)**.** *Let u be a global solution that is almost periodic modulo symmetries. Then, there exists $C = C(u)$ so that*

$$\int_I \int_{|x-x(t)|\le C/N(t)} |u(t,x)|^{p+2}\,dx\,dt \gtrsim_u \int_I N(t)^{\frac{4}{p}-1}\,dt \tag{3.8}$$

uniformly for all intervals $I = [t_1, t_2] \subseteq \mathbb{R}$ with $t_2 \ge t_1 + N(t_1)^{-1}$.

Proof. We know that there exists $\delta = \delta(u)$ so that

$$N(t) \sim_u N(t_0) \quad \text{uniformly for} \quad t \in [t_0 - \delta N(t_0)^{-1}, t_0 + \delta N(t_0)^{-1}] \quad \text{and} \quad t_0 \in \mathbb{R}. \tag{3.9}$$

Indeed, if it were not possible to choose a δ with this property, then one could find a convergent sequence of initial data (taken from normalizations of u) whose limit blows up instantaneously, in contradiction to the local theory. For further details, see [18, Corollary 3.6] or [16, Lemma 5.18]. We note that the argument requires perturbation theory, which is an ingredient in the proof of Theorem 1.7.

In view of (3.9), it suffices to prove the result for intervals of the form $[t_0, t_0 + \delta N(t_0)^{-1}]$ for some small fixed $\delta > 0$. The simple argument that shows this requires that I contain at least one interval of this form. This is the origin of the requirement $t_2 \ge t_1 + N(t_1)^{-1}$ in the statement of the corollary; correspondingly, we require $\delta \le 1$.

As noted in (1.8), the almost-periodicity of u and Sobolev embedding imply that for any $\eta > 0$ there exists $C(\eta) > 0$ so that

$$\int_{|x-x(t)|\ge C(\eta)/N(t)} |u(t,x)|^{\frac{3p}{2}}\,dx \le \eta.$$

Combining this with Lemma 3.4 yields the following: There exists $C = C(u)$ so that the set of

$$t \in [t_0, t_0 + \delta N(t_0)^{-1}] \quad \text{such that} \quad \int_{|x-x(t)|\le C/N(t)} |u(t,x)|^{3p/2}\,dx \gtrsim_u 1$$

has measure $\gtrsim_u N(t_0)^{-1}$. In view of this, it suffices to show that given $\eta_0 > 0$ there exists $\eta_1 = \eta_1(u, \eta_0) > 0$ so that

$$\int_{|x-x(t)|\le C/N(t)} |u(t,x)|^{3p/2}\,dx \ge \eta_0$$
$$\implies N(t)^{1-\frac{4}{p}} \int_{|x-x(t)|\le C/N(t)} |u(t,x)|^{p+2}\,dx \ge \eta_1.$$

The truth of this statement follows from the almost-periodicity of u. Indeed, passing to the normalizations of $u(t)$ and recalling that these form a precompact set in $L_x^{3p/2}$, the statement reduces to the fact that if a sequence $\{f_n\}$ converges in $L_x^{3p/2}$ and converges to zero in L_x^{p+2}, then it converges to zero in $L_x^{3p/2}$. □

4. Global enemies are subluminal

The principal goal of this section is to show that for the global enemies (the soliton-like and frequency-cascade solutions) of Theorem 1.8, the center $x(t)$ of the wave packet travels strictly slower than the speed of light, at least on average, over reasonably long time intervals. Note that Definition 1.6 does not define $x(t)$ uniquely, but only up to a radius of about $N(t)^{-1}$. While this does not render the goal of this section ambiguous, it is something of a nuisance in the proof. For that reason, we first standardize $x(t)$ in some mild fashion. This is our first proposition. The main result of the section, the subluminality of global enemies, is Proposition 4.3.

Proposition 4.1 (Centering $x(t)$)**.** *Let u be a global almost periodic solution to* (1.1)*. The function $x(t)$ can be modified so that it retains all properties stated in Definition* 1.6 *(though $C(\eta)$ may need to be made larger) and in addition satisfies the following: For some large constant C_u and all $\omega \in S^2$,*

$$\int_{\omega\cdot(x-x(t))>0} |\nabla_{t,x} u(t,x)|^{\frac{3p}{p+2}}\, dx \geq \tfrac{1}{C_u}; \tag{4.1}$$

that is, each plane through $x(t)$ partitions u into two nontrivial *pieces. Moreover,*

$$|x(t_1) - x(t_2)| \leq |t_1 - t_2| + C_u N(t_1)^{-1} + C_u N(t_2)^{-1} \quad \text{for any} \quad t_1, t_2 \in \mathbb{R}; \tag{4.2}$$

indeed, this was also true for the original $x(t)$.

Before proceeding to the proof of this proposition, we pause to make the following intuition precise: Compactness (modulo scaling) prevents the solution $u(t)$ from concentrating on very narrow strips, provided the width is measured in units of $N(t)^{-1}$.

Lemma 4.2 (Small on narrow strips)**.** *Let u be a global almost periodic solution to* (1.1)*. Then for any $\eta > 0$ there exists a small constant $c(\eta) > 0$ so that*

$$\sup_{\omega\in S^2} \int_{|\omega\cdot(x-x(t))|<c(\eta)/N(t)} |\nabla_{t,x} u(t,x)|^{\frac{3p}{p+2}}\, dx \leq \eta. \tag{4.3}$$

Proof. For a single value of t this follows from the Monotone Convergence Theorem; it extends to the full orbit of $\nabla_{t,x} u(t)$ by compactness. $\square$

Our first application of this lemma is to the proof of Proposition 4.1; we will use it again in the proof of Proposition 4.3.

Proof of Proposition 4.1. We first prove (4.2). As the veracity of this equation will be deduced from the properties stated in Definition 1.6, it will be equally valid for the modified version of $x(t)$ which will be defined in due course.

Choose $\eta > 0$ to be a small number well below the $\dot H^{s_c}_x \times \dot H^{s_c-1}_x$ threshold for the small data theory. By Definition 1.6, there is a constant $C(\eta)$ so that

$$\big\|\phi\big(\tfrac{x-x(t_1)}{C(\eta)N(t_1)^{-1}}\big)u(t_1,x)\big\|_{\dot H^{s_c}} + \big\|\phi\big(\tfrac{x-x(t_1)}{C(\eta)N(t_1)^{-1}}\big)u_t(t_1,x)\big\|_{\dot H^{s_c-1}} \leq \eta \tag{4.4}$$

for some smooth cutoff $\phi : \mathbb{R}^3 \to [0,\infty)$ with $\phi(x) = 1$ for $|x| \geq 1$ and $\phi(x) = 0$ for $|x| \leq \frac{1}{2}$. Thus, by the small data theory, there is a global solution to (1.1) whose Cauchy data at time t_1 match the combination of ϕ and u given in (4.4). Moreover, per the small data theory, each critical Strichartz norm of this solution is controlled by a (p-dependent) multiple of η. By simple domain of dependence arguments, this new solution agrees with the original u on the set

$$\Omega(t) := \{x : |x - x(t_1)| \geq |t - t_1| + C(\eta)/N(t_1)\} \quad \text{for all} \quad t \in \mathbb{R}$$

and hence, by Sobolev embedding,

$$\big\|\nabla_{t,x} u(t)\big\|_{L_x^{\frac{3p}{p+2}}(\Omega(t))} \lesssim \eta \quad \text{for all} \quad t \in \mathbb{R}.$$

Now consider this estimate and (1.8) with $t = t_2$ and η much less than half the minimal $L_x^{3p/(p+2)}$ norm of $\nabla_{t,x} u(t)$, over time; this minimum is positive by virtue of Lemma 3.3. Thus we may deduce that

$$\{x : |x - x(t_1)| \leq |t_2 - t_1| + C(\eta)/N(t_1)\} \cap \{x : |x - x(t_2)| \leq C(\eta)/N(t_2)\} \neq \varnothing,$$

from which (4.2) follows.

We now turn to the proof of (4.1). First, fix $C > 0$ so that $B(t) := \{|x - x(t)| \leq C/N(t)\}$ obeys

$$\int_{B(t)} \big|\nabla_{t,x} u(t,x)\big|^{\frac{3p}{p+2}}\, dx \gtrsim_u 1 \tag{4.5}$$

uniformly for $t \in \mathbb{R}$. This is possible by virtue of Lemma 3.3 and (1.8). Now set

$$\tilde{x}(t) := x(t) + \frac{\int_{B(t)} [x - x(t)]\, |\nabla_{t,x} u(t,x)|^{\frac{3p}{p+2}}\, dx}{\int_{B(t)} |\nabla_{t,x} u(t,x)|^{\frac{3p}{p+2}}\, dx}.$$

This definition immediately implies that $|\tilde{x}(t) - x(t)| \leq C/N(t)$; thus, $\tilde{x}(t)$ maintains the properties stated in Definition 1.6, though the compactness modulus function $C(\eta)$ may need to be increased, say by the addition of C. In particular, (4.2) remains valid after a suitable increase in the constant C_u.

By construction,

$$\int_{B(t)} \omega \cdot [x - \tilde{x}(t)]\, \big|\nabla_{t,x} u(t,x)\big|^{\frac{3p}{p+2}}\, dx = 0$$

for any (unit) vector $\omega \in S^2$, while by (4.5) and Lemma 4.2,

$$\int_{B(t)} \big|\omega \cdot [x - \tilde{x}(t)]\big|\, \big|\nabla_{t,x} u(t,x)\big|^{\frac{3p}{p+2}}\, dx \gtrsim_u N(t)^{-1}.$$

Putting these two results together yields

$$\int_{B(t)} \big\{\omega \cdot [x - \tilde{x}(t)]\big\}_+ \big|\nabla_{t,x} u(t,x)\big|^{\frac{3p}{p+2}}\, dx \gtrsim_u N(t)^{-1}, \quad \text{where} \quad \{y\}_+ = \max\{0, y\}.$$

Therefore, as $x \in B(t)$ implies that $|x - \tilde{x}(t)| \leq 2CN(t)^{-1}$, we have

$$\int_{\omega\cdot(x - \tilde{x}(t))>0} \big|\nabla_{t,x} u(t,x)\big|^{\frac{3p}{p+2}}\, dx \geq \int_{B(t)} \frac{\{\omega \cdot [x - \tilde{x}(t)]\}_+}{2CN(t)^{-1}}\, \big|\nabla_{t,x} u(t,x)\big|^{\frac{3p}{p+2}}\, dx \gtrsim_u 1,$$

which proves (4.1). □

Proposition 4.3 (Global enemies are subluminal)**.** *Let u be a global almost periodic solution to* (1.1) *with $N(t) \geq 1$. Then there exists $\delta = \delta(u) > 0$ such that*

$$|x(t) - x(\tau)| \leq (1 - \delta)|t - \tau| \quad \textit{whenever} \quad |t - \tau| \geq \tfrac{1}{\delta}. \tag{4.6}$$

The proof of this proposition splits into two cases depending on whether or not $N(t)$ varies significantly over the time interval between t and τ. Before turning to the main part of the proof of Proposition 4.3, we present the key ingredient in the case of significant variation as a lemma:

Lemma 4.4. *For almost periodic solutions u to* (1.1)*, there exists $c = c(u) > 0$ so that*

$$|x(t_1) - x(t_2)| \geq |t_1 - t_2| - cN(t_1)^{-1} \quad \Longrightarrow \quad N(t_2) \leq c^{-2}N(t_1). \tag{4.7}$$

For a nonvacuous statement, we assign the names t_1 and t_2 so that $N(t_1) \leq N(t_2)$.

Proof. By time-reversal symmetry, we may assume that $t_1 < t_2$. By space-translation symmetry, we set $x(t_1) = 0$ and by rotation symmetry, we assume that $x(t_2) = (x_1(t_2), 0, 0)$ with $x_1(t_2) \geq 0$.

Assume, toward a contradiction, that $cN(t_1)^{-1} \geq c^{-1}N(t_2)^{-1}$. Then, by choosing c small enough (depending on η) and invoking the almost periodicity of u, we obtain

$$\big\|\psi\big(\tfrac{x_1 - x_1(t_2)}{cN(t_1)^{-1}}\big)u(t_2,x)\big\|_{\dot H^{s_c}_x} + \big\|\psi\big(\tfrac{x_1 - x_1(t_2)}{cN(t_1)^{-1}}\big)u_t(t_2,x)\big\|_{\dot H^{s_c-1}_x} \leq \eta \tag{4.8}$$

for some smooth cutoff $\psi : \mathbb{R} \to [0,\infty)$ with $\psi(x) = 1$ for $x \leq -1$ and $\psi(x) = 0$ for $x \geq -1/2$. Here η is chosen below the threshold for the small data theory. Using this theory and simple domain of dependence arguments, we may deduce that

$$\int_\Omega \big|\nabla_{t,x}u(t_1,x)\big|^{\frac{3p}{p+2}}\,dx \lesssim \eta^{\frac{3p}{p+2}}, \text{ where } \Omega := \big\{x : x_1 \leq x_1(t_2) - (t_2 - t_1) - cN(t_1)^{-1}\big\}. \tag{4.9}$$

Now by LHS(4.7) and the standardizations introduced at the beginning of this proof,

$$\Omega \supseteq \big\{x : (-e_1)\cdot(x - x(t_1)) \geq 2cN(t_1)^{-1}\big\},$$

with the obvious consequence for the $L^{3p/(p+2)}_x$ norm of $\nabla_{t,x}u(t_1)$ on this set. Making η small enough and then c small enough, we deduce a contradiction to the combination of Lemma 4.2 and (4.1). □

Proof of Proposition 4.3. We claim that it suffices to show that there exists $A = A(u) > 1$ so that for all $t_0 \in \mathbb{R}$ there exists $t \in [t_0, t_0 + AN(t_0)^{-1}]$ so that

$$|x(t) - x(t_0)| \leq |t - t_0| - A^{-1}N(t_0)^{-1}. \tag{4.10}$$

Indeed, with this claim in hand, we may inductively construct a sequence of times $\{t_k\}$ so that $t_0 = 0$, $0 < t_{k+1} - t_k \leq AN(t_k)^{-1}$, and

$$\begin{aligned} |x(t_m) - x(t_l)| &\leq \sum_{k=l}^{m-1} |t_{k+1} - t_k| - A^{-1}N(t_k)^{-1} \\ &\leq \sum_{k=l}^{m-1} (1 - A^{-2})|t_{k+1} - t_k| \leq (1 - A^{-2})|t_m - t_l|. \end{aligned}$$

We may deduce the result for values of t and τ lying in $[0,\infty)$ between these sample points by applying (4.2). Note that this requires choosing $\delta \leq \frac{1}{2}A^{-2}(A + 2C_u)^{-1}$, where C_u is as in (4.2). By employing time-reversal symmetry, one similarly obtains the result for $t, \tau \in (-\infty, 0]$ and thence for all pairs of times via the triangle inequality.

We now turn to verifying the claim made at the beginning of this proof. Let c be as in Lemma 4.4. If $N(t) > c^{-2}N(t_0)$ for some $t \in [t_0, t_0 + AN(t_0)^{-1}]$, then by that lemma,

$$|x(t) - x(t_0)| \leq |t - t_0| - cN(t_0)^{-1} \leq |t - t_0| - A^{-1}N(t_0)^{-1},$$

provided we ensure $A \geq c^{-1}$. This settles this case. Suppose now that $N(t) < c^2 N(t_0)$ for some $t \in [t_0, t_0 + AN(t_0)^{-1}]$. Then by Lemma 4.4,

$$|x(t) - x(t_0)| \leq |t - t_0| - cN(t)^{-1} \leq |t - t_0| - c^{-1}N(t_0)^{-1} \leq |t - t_0| - A^{-1}N(t_0)^{-1},$$

provided A is chosen so that $A \geq c$. This settles this case.

It remains to verify our claim in the case

$$c^2 \leq \frac{N(t)}{N(t_0)} \leq c^{-2} \quad \text{for all} \quad t \in [t_0, t_0 + AN(t_0)^{-1}], \tag{4.11}$$

for which we will argue by contradiction. For notational convenience, we translate so that $t_0 = 0$ and $x(t_0) = 0$. Now by assuming that (4.10) fails and making use of (4.2) and (4.11), we deduce that

$$t \in [0, AN(0)^{-1}] \implies \big||x(t)| - t\big| \leq BN(0)^{-1}$$

for some $B = B(u) \geq C_u(1 + c^{-2}) + 1$, where C_u is as in (4.2). By enlarging B and using (4.11), we can ensure that

$$\big\{\big||x| - t\big| \leq B/N(0)\big\} \supseteq \big\{|x - x(t)| \leq C/N(t)\big\},$$

with C as in Corollary 3.5. Using this corollary, it follows that

$$\int_{B/N(0)}^{A/N(0)} \int_{||x|-t| \leq B/N(0)} |u(t,x)|^{p+2}\, dx\, dt \gtrsim_u (A - B)N(0)^{\frac{4}{p}-2}, \tag{4.12}$$

whenever $A \geq B + c^{-2}$.

On the other hand, we can obtain an upper bound on LHS(4.12) from the energy flux identity. As a first step, we observe that by Lemma 3.2 we have

$$\int_{\mathbb{R}^3} \chi_{\{|x| \leq 2B/N(0)\}} \int_{B/N(0)}^{A/N(0)} \int_{|x-y|=t} |u(t,y)|^{p+2}\, dS(y)\, dt\, dx \lesssim_u A^{1-\frac{4}{p}} B^3 N(0)^{\frac{4}{p}-4}.$$

To continue, we change variables via $y = x + z$ and then $x = x' - z$ to obtain

$$\int_{B/N(0)}^{A/N(0)} \int_{\mathbb{R}^3} \int_{|z|=t} |u(t,x')|^{p+2} \chi_{\{|x'-z| \leq 2B/N(0)\}}\, dS(z)\, dx'\, dt \lesssim_u A^{1-\frac{4}{p}} B^3 N(0)^{\frac{4}{p}-4}.$$

Noting that

$$\int_{|z|=t} \chi_{\{|x'-z| \leq 2L\}}\, dS(z) \gtrsim L^2 \quad \text{when} \quad \big||x'| - t\big| \leq L \quad \text{and} \quad |t| \geq L$$

for any $L > 0$ and hence for $L = B/N(0)$, we are led to

$$\int_{B/N(0)}^{A/N(0)} \int_{||x'|-t| \leq B/N(0)} |u(t,x')|^{p+2}\, dx'\, dt \lesssim_u A^{1-\frac{4}{p}} B N(0)^{\frac{4}{p}-2}.$$

To finish the proof, we merely note that this contradicts (4.12) once A is chosen sufficiently large. □

5. Additional decay

In this section we prove additional decay for the soliton-like and frequency-cascade solutions described in Theorem 1.8.

Proposition 5.1 (L^q breach of scaling)**.** *Let u be a global solution to* (1.1) *that is almost periodic modulo symmetries. In particular,*

$$\big\|(u,u_t)\big\|_{L_t^\infty(\mathbb{R};\dot H_x^{s_c}\times \dot H_x^{s_c-1})}<\infty. \tag{5.1}$$

Also assume that

$$\inf_{t\in\mathbb{R}} N(t)\geq 1. \tag{5.2}$$

Then $u\in L_t^\infty L_x^q$ for $\frac{3p^2+20p-16}{6p}<q\leq\frac{3p}{2}$. In particular, $u\in L_t^\infty L_x^p$ (as $p\geq 6$) and by Hölder's inequality, $F(u)\in L_t^\infty L_x^1$.

The remainder of this section is dedicated to the proof of Proposition 5.1.

Let u be a solution to (1.1) that obeys the hypotheses of Proposition 5.1. Let $\eta>0$ be a small constant to be chosen later. Then by almost periodicity modulo symmetries combined with (5.2), there exists $N_0=N_0(\eta)$ such that

$$\big\||\nabla|^{s_c}P_{\leq N_0}u\big\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^3)}+\big\||\nabla|^{s_c-1}P_{\leq N_0}u_t\big\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^3)}\leq\eta. \tag{5.3}$$

Now for $\frac{3p}{2}<r<\infty$ define

$$A_r(N):=N^{\frac{3}{r}-\frac{2}{p}}\sup_{t\in\mathbb{R}}\|u_N(t)\|_{L_x^r}$$

for frequencies $N\leq 10pN_0$. The key to proving Proposition 5.1 is to show that $A_r(N)$ decays well as $N\to 0$; as one can interpolate with the trivial bound $A_{3p/2}(N)\lesssim_u 1$, one need only consider the case when r is close to infinity. Note that by Bernstein's inequality combined with Sobolev embedding and (5.1),

$$A_r(N)\lesssim\|u_N\|_{L_t^\infty L_x^{\frac{3p}{2}}}\lesssim\big\||\nabla|^{s_c}u\big\|_{L_t^\infty L_x^2}<\infty \tag{5.4}$$

for all $N\leq 10pN_0$.

We next prove a recurrence formula for $A_r(N)$.

Lemma 5.2 (Recurrence)**.** *For $\frac{3p}{2}<r<\infty$ we have*

$$\begin{aligned}A_r(N)\lesssim_u\Big\{\big(\tfrac{N}{N_0}\big)^{1-\frac{2}{p}-\frac{3}{r}}&+\eta^2\sum_{\frac{N}{10p}\leq M\leq N_0}\big(\tfrac{N}{M}\big)^{1-\frac{2}{p}-\frac{3}{r}-}A_r(M)^{p-1}\\ &+\eta^2\sum_{M<\frac{N}{10p}}\big(\tfrac{M}{N}\big)^{\frac{1}{2}+\frac{3}{r}-}A_r(M)^{p-1}\Big\}^{\frac{4r-6p}{r(p+4)}},\end{aligned} \tag{5.5}$$

for all $N\leq 10pN_0$.

Proof. Fix $N\leq 10pN_0$. By time-translation symmetry, it suffices to prove

$$\begin{aligned}N^{\frac{3}{r}-\frac{2}{p}}\|u_N(0)\|_{L_x^r}\lesssim_u\Big\{\big(\tfrac{N}{N_0}\big)^{1-\frac{2}{p}-\frac{3}{r}}&+\eta^2\sum_{\frac{N}{10p}\leq M\leq N_0}\big(\tfrac{N}{M}\big)^{1-\frac{2}{p}-\frac{3}{r}-}A_r(M)^{p-1}\\ &+\eta^2\sum_{M<\frac{N}{10p}}\big(\tfrac{M}{N}\big)^{\frac{1}{2}+\frac{3}{r}-}A_r(M)^{p-1}\Big\}^{\frac{4r-6p}{r(p+4)}}.\end{aligned} \tag{5.6}$$

Using the Duhamel formula (1.9) into the future we write

$$u_N(0) = \int_0^\infty -\frac{\sin\bigl(t|\nabla|\bigr)}{|\nabla|} F_N(u(t))\,dt.$$

Now let $T > 0$, to be chosen later. Using the explicit form of the propagator (cf. Lemma 2.2), Hölder's inequality, and the energy flux inequality Lemma 3.2, we estimate the long-time contribution (without the Littlewood–Paley projection) as follows:

$$\begin{aligned}
\Bigl\|\int_T^\infty -\frac{\sin\bigl(t|\nabla|\bigr)}{|\nabla|} F(u(t))\,dt\Bigr\|_{L_x^\infty} &\lesssim \Bigl\|\int_T^\infty \frac{1}{t}\int_{|x-y|=t} F(u(t,y))\,dS(y)\,dt\Bigr\|_{L_x^\infty}\\
&\lesssim \sum_{R\geq T} \frac{1}{R}\Bigl\|\int_R^{2R}\int_{|x-y|=t} F(u(t,y))\,dS(y)\,dt\Bigr\|_{L_x^\infty}\\
&\lesssim \sum_{R\geq T} \frac{1}{R} R^{\frac{3}{p+2}}\Bigl\|\int_R^{2R}\int_{|x-y|=t} |u(t,y)|^{p+2} dS(y) dt\Bigr\|_{L_x^\infty}^{\frac{p+1}{p+2}}\\
&\lesssim_u \sum_{R\geq T} \frac{1}{R} R^{\frac{3}{p+2}} R^{(1-\frac{4}{p})\frac{p+1}{p+2}}\\
&\lesssim_u T^{-\frac{2}{p}}. \qquad (5.7)
\end{aligned}$$

On the other hand, by (1.9),

$$\int_T^\infty -\frac{\sin\bigl(t|\nabla|\bigr)}{|\nabla|} F(u(t))\,dt = \cos\bigl(T|\nabla|\bigr)u(T) - \frac{\sin\bigl(T|\nabla|\bigr)}{|\nabla|} u_t(T),$$

and so, using Sobolev embedding and (5.1) we get

$$\Bigl\|\int_T^\infty -\frac{\sin\bigl(t|\nabla|\bigr)}{|\nabla|} F(u(t))\,dt\Bigr\|_{L_x^{\frac{3p}{2}}} \lesssim \bigl\||\nabla|^{s_c} u\bigr\|_{L_t^\infty L_x^2} + \bigl\||\nabla|^{s_c-1} u_t\bigr\|_{L_t^\infty L_x^2} \lesssim_u 1.$$

Therefore, interpolation (and L^r-boundedness of the Littlewood–Paley projection) yields the following estimate for the long-time contribution:

$$\Bigl\|\int_T^\infty -\frac{\sin\bigl(t|\nabla|\bigr)}{|\nabla|} F_N(u(t))\,dt\Bigr\|_{L_x^r} \lesssim \Bigl\|\int_T^\infty -\frac{\sin\bigl(t|\nabla|\bigr)}{|\nabla|} F(u(t))\,dt\Bigr\|_{L_x^r} \lesssim_u T^{\frac{3}{r}-\frac{2}{p}}, \qquad (5.8)$$

valid for any $\frac{3p}{2} \leq r \leq \infty$. We will make use of this inequality without the frequency projection in the next section; see (6.6).

We now turn to the short-time contribution. By the Bernstein and Strichartz inequalities,

$$\begin{aligned}
\Bigl\|\int_0^T -\frac{\sin\bigl(t|\nabla|\bigr)}{|\nabla|} F_N(u(t))\,dt\Bigr\|_{L_x^r} &\lesssim N^{\frac{3}{2}-\frac{3}{r}}\Bigl\|\int_0^T -\frac{\sin\bigl(t|\nabla|\bigr)}{|\nabla|} F_N(u(t))\,dt\Bigr\|_{L_x^2}\\
&\lesssim N^{\frac{3}{2}-\frac{3}{r}} \|F_N(u)\|_{L_t^{\frac{2r}{r+6}} L_x^{\frac{r}{r-1}}([0,T]\times\mathbb{R}^3)}\\
&\lesssim N^{\frac{3}{2}-\frac{3}{r}} T^{\frac{r+6}{2r}} \|F_N(u)\|_{L_t^\infty L_x^{\frac{r}{r-1}}}. \qquad (5.9)
\end{aligned}$$

Collecting (5.8) and (5.9) we obtain

$$N^{\frac{3}{r}-\frac{2}{p}} \|u_N(0)\|_{L_x^r} \lesssim_u (NT)^{\frac{3}{r}-\frac{2}{p}} + (NT)^{\frac{1}{2}+\frac{3}{r}} N^{1-\frac{2}{p}-\frac{3}{r}} \|F_N(u)\|_{L_t^\infty L_x^{\frac{r}{r-1}}}. \qquad (5.10)$$

To estimate the right-hand side of the inequality above we decompose

$$u = u_{>N_0} + u_{\leq N_0} = u_{>N_0} + u_{\frac{N}{10p}\leq\cdot\leq N_0} + u_{<\frac{N}{10p}}$$

and thus, taking advantage of our assumption that the power p is even, we write

$$F_N(u) = P_N\Big(u_{>N_0}\sum_{k=0}^{p}\tbinom{p+1}{k+1}u_{>N_0}^k u_{\leq N_0}^{p-k} + \sum_{k=0}^{p}\tbinom{p+1}{k}u_{<\frac{N}{10p}}^k u_{\frac{N}{10p}\leq\cdot\leq N_0}^{p+1-k}\Big). \tag{5.11}$$

To estimate the contribution of the first term in this decomposition to (5.10), we use Hölder, Sobolev embedding, Bernstein, and (5.1):

$$\begin{aligned}\sum_{k=0}^{p}\tbinom{p+1}{k+1}\Big\|P_N\big(u_{>N_0}u_{>N_0}^k u_{\leq N_0}^{p-k}\big)\Big\|_{L_t^\infty L_x^{\frac{r}{r-1}}} &\lesssim \|u\|^p_{L_t^\infty L_x^{\frac{3p}{2}}}\|u_{>N_0}\|_{L_t^\infty L_x^{\frac{3r}{r-3}}}\\ &\lesssim \|u\|^p_{L_t^\infty \dot H_x^{s_c}}\Big\||\nabla|^{\frac12+\frac3r}u_{>N_0}\Big\|_{L_t^\infty L_x^2}\\ &\lesssim_u N_0^{-1+\frac2p+\frac3r}\|u_{>N_0}\|_{L_t^\infty \dot H_x^{s_c}}\\ &\lesssim_u N_0^{-1+\frac2p+\frac3r}.\end{aligned}$$

To estimate the contribution of the second term on the right-hand side of (5.11) to (5.10), we first note that

$$\begin{aligned}&\Big\|P_N\big(\sum_{k=0}^{p}\tbinom{p+1}{k}u_{<\frac{N}{10p}}^k u_{\frac{N}{10p}\leq\cdot\leq N_0}^{p+1-k}\big)\Big\|_{L_t^\infty L_x^{\frac{r}{r-1}}}\\ &\qquad\lesssim \Big\|u_{\frac{N}{10p}\leq\cdot\leq N_0}^{p+1}\Big\|_{L_t^\infty L_x^{\frac{r}{r-1}}} + \Big\|u^p_{<\frac{N}{10p}}u_{\frac{N}{10p}\leq\cdot\leq N_0}\Big\|_{L_t^\infty L_x^{\frac{r}{r-1}}}.\end{aligned}$$

By Hölder, Bernstein, and (5.3) we estimate

$$\begin{aligned}&\Big\|u_{\frac{N}{10p}\leq\cdot\leq N_0}^{p+1}\Big\|_{L_t^\infty L_x^{\frac{r}{r-1}}}\\ &\lesssim \sum_{\frac{N}{10p}\leq N_1\leq\cdots\leq N_{p+1}\leq N_0}\|u_{N_1}\|_{L_t^\infty L_x^r}\cdots\|u_{N_{p-1}}\|_{L_t^\infty L_x^r}\|u_{N_p}\|_{L_t^\infty L_x^{\frac{2r}{r-p}}}\|u_{N_{p+1}}\|_{L_t^\infty L_x^{\frac{2r}{r-p}}}\\ &\lesssim \sum_{\frac{N}{10p}\leq N_1\leq\cdots\leq N_{p+1}\leq N_0}\|u_{N_1}\|_{L_t^\infty L_x^r}\cdots\|u_{N_{p-1}}\|_{L_t^\infty L_x^r}N_p^{\frac2p-\frac{3(r-p)}{2r}}N_{p+1}^{\frac2p-\frac{3(r-p)}{2r}}\|u_{\leq N_0}\|^2_{L_t^\infty \dot H_x^{s_c}}\\ &\lesssim_u \eta^2\sum_{\frac{N}{10p}\leq N_1\leq\cdots\leq N_{p-1}\leq N_0}A_r(N_1)\cdots A_r(N_{p-1})N_{p-1}^{-1+\frac2p+\frac3r}\\ &\lesssim_u \eta^2 N^{-1+\frac2p+\frac3r}\sum_{\frac{N}{10p}\leq N_1\leq\cdots\leq N_{p-1}\leq N_0}\Big(\frac{N}{N_{p-1}}\Big)^{1-\frac2p-\frac3r}\big[A_r(N_1)^{p-1}+\cdots+A_r(N_{p-1})^{p-1}\big]\\ &\lesssim_u \eta^2 N^{-1+\frac2p+\frac3r}\sum_{\frac{N}{10p}\leq M\leq N_0}\Big(\frac{N}{M}\Big)^{1-\frac2p-\frac3r-}A_r(M)^{p-1}.\end{aligned}$$

Similarly, we estimate

$$\begin{aligned}
&\big\| u^p_{<\frac{N}{10p}} u_{\frac{N}{10p}\le\cdot\le N_0} \big\|_{L_t^\infty L_x^{\frac{r}{r-1}}} \\
&\lesssim \| u_{\frac{N}{10p}\le\cdot\le N_0} \|_{L_t^\infty L_x^2} \sum_{N_1\le\cdots\le N_p<\frac{N}{10p}} \| u_{N_1}\|_{L^\infty_{t,x}} \cdots \| u_{N_{p-1}}\|_{L^\infty_{t,x}} \| u_{N_p}\|_{L_t^\infty L_x^{\frac{2r}{r-2}}} \\
&\lesssim_u \eta^2 N^{-\frac32+\frac2p} \sum_{N_1\le\cdots\le N_p<\frac{N}{10p}} N_1^{\frac2p} A_r(N_1)\cdots N_{p-1}^{\frac2p} A_r(N_{p-1}) N_p^{-\frac32+\frac2p+\frac3r} \\
&\lesssim_u \eta^2 N^{-\frac32+\frac2p} \sum_{N_1\le\cdots\le N_{p-1}<\frac{N}{10p}} N_1^{\varepsilon} N_{p-1}^{-\frac32+\frac2p+\frac3r} \Big[N_1^{(\frac2p-\varepsilon)(p-1)} A_r(N_1)^{p-1} + \cdots \\
&\qquad\qquad + N_{p-1}^{\frac{2(p-1)}{p}} A_r(N_{p-1})^{p-1} \Big] \\
&\lesssim_u \eta^2 N^{-1+\frac2p+\frac3r} \sum_{M<\frac{N}{10p}} \Big(\frac{M}{N}\Big)^{\frac12+\frac3r-} A_r(M)^{p-1}.
\end{aligned}$$

Putting everything together, we obtain

$$\begin{aligned}
N^{\frac3r-\frac2p} \|u_N(0)\|_{L_x^r} \lesssim_u (NT)^{\frac3r-\frac2p} + (NT)^{\frac12+\frac3r} \Big\{ \big(\tfrac{N}{N_0}\big)^{1-\frac2p-\frac3r} \\
+ \eta^2 \sum_{\frac{N}{10p}\le M\le N_0} \big(\tfrac{N}{M}\big)^{1-\frac2p-\frac3r-} A_r(M)^{p-1} \\
+ \eta^2 \sum_{M<\frac{N}{10p}} \big(\tfrac{M}{N}\big)^{\frac12+\frac3r-} A_r(M)^{p-1} \Big\}.
\end{aligned}$$

Setting

$$\begin{aligned}
T := \tfrac1N \Big\{ \big(\tfrac{N}{N_0}\big)^{1-\frac2p-\frac3r} + \eta^2 \sum_{\frac{N}{10p}\le M\le N_0} \big(\tfrac{N}{M}\big)^{1-\frac2p-\frac3r-} A_r(M)^{p-1} \\
+ \eta^2 \sum_{M<\frac{N}{10p}} \big(\tfrac{M}{N}\big)^{\frac12+\frac3r-} A_r(M)^{p-1} \Big\}^{-\frac{2p}{p+4}},
\end{aligned}$$

we deduce (5.6). This completes the proof of the lemma. □

To resolve the recurrence in Lemma 5.2 and so prove Proposition 5.1, we need the following simple lemma. The recurrence (5.12) in the lemma involves both the past and the future (that is, $l<k$ and $l>k$), which makes it an acausal variant of the usual Gronwall inequality.

Lemma 5.3 (Acausal Gronwall inequality)**.** *Given $\eta, C, \gamma, \gamma' > 0$, let $\{x_k\}_{k\ge0}$ be a bounded nonnegative sequence obeying*

$$x_k \le C2^{-\gamma k} + \eta \sum_{l<k} 2^{-\gamma|k-l|} x_l + \eta \sum_{l\ge k} 2^{-\gamma'|k-l|} x_l \quad \textit{for all} \quad k\ge 0. \tag{5.12}$$

If $\eta \le \frac14 \min\{1-2^{-\gamma}, 1-2^{-\gamma'}, 1-2^{\rho-\gamma}\}$ for some $0<\rho<\gamma$, then $x_k \le (4C + \|x\|_{\ell^\infty})\, 2^{-\rho k}$.

Proof. Let $X_k := \sup\{x_m : m \geq k\}$, so that (5.12) implies

$$\begin{aligned} X_k &\leq C\,2^{-\gamma k} + \eta \sum_{l<k} 2^{-\gamma|k-l|} X_l + \eta \sum_{p=0}^{\infty} \bigl[2^{-\gamma p} + 2^{-\gamma' p}\bigr] X_k \\ &\leq C\,2^{-\gamma k} + \eta \sum_{l<k} 2^{-\gamma|k-l|} X_l + \tfrac{1}{2} X_k. \end{aligned}$$

The result now follows by a simple inductive argument. □

Using this lemma we can now complete the

Proof of Proposition 5.1. For any positive r, the power appearing outside the braces in (5.5) is less than one. Thus, by concavity, Lemma 5.2 implies that

$$\begin{aligned} A_r(N) \lesssim_u \bigl(\tfrac{N}{N_0}\bigr)^{\gamma} + \eta^2 \sum_{\frac{N}{10p} \leq M \leq N_0} \bigl(\tfrac{N}{M}\bigr)^{\gamma} A_r(M)^{(p-1)\frac{4r-6p}{r(p+4)}} \\ + \eta^2 \sum_{M < \frac{N}{10p}} \bigl(\tfrac{M}{N}\bigr)^{\gamma'} A_r(M)^{(p-1)\frac{4r-6p}{r(p+4)}}, \end{aligned}$$

for all

$$N \leq 10pN_0, \quad \tfrac{3p}{2} < r < \infty, \quad \gamma < \bigl(1 - \tfrac{2}{p} - \tfrac{3}{r}\bigr)\tfrac{4r-6p}{r(p+4)}, \quad \text{and} \quad \gamma' < \bigl(\tfrac{1}{2} + \tfrac{3}{r}\bigr)\tfrac{4r-6p}{r(p+4)}.$$

When $\frac{6p(p-1)}{3p-8} \leq r < \infty$, the power atop $A_r(M)$ on the right-hand side of the inequality above is ≥ 1. Discarding surplus powers by invoking (5.4), we can apply Lemma 5.3 and deduce that

$$\|u_N\|_{L_t^\infty L_x^r} \lesssim_u N^{\frac{2}{p} - \frac{3}{r} + (1 - \frac{2}{p} - \frac{3}{r})\frac{4r-6p}{r(p+4)} -} \tag{5.13}$$

for all $N \leq 10pN_0$. (In applying Lemma 5.3, we set $N = 10p \cdot 2^{-k} N_0$, $x_k = A_r(10p \cdot 2^{-k} N_0)$, and take η sufficiently small.)

To continue, we use interpolation followed by (5.13) and (5.1):

$$\begin{aligned} \|u_N\|_{L_t^\infty L_x^q} &\leq \|u_N\|_{L_t^\infty L_x^r}^{\frac{r(q-2)}{q(r-2)}} \|u_N\|_{L_t^\infty L_x^2}^{\frac{2(r-q)}{q(r-2)}} \\ &\lesssim_u N^{\frac{r(q-2)}{q(r-2)}[\frac{2}{p} - \frac{3}{r} + (1 - \frac{2}{p} - \frac{3}{r})\frac{4r-6p}{r(p+4)}] -} N^{-\frac{2(r-q)}{q(r-2)}(\frac{3}{2} - \frac{2}{p})} \end{aligned}$$

for all $N \leq 10pN_0$. Thus, letting $r \to \infty$, we get

$$\|u_N\|_{L_t^\infty L_x^q} \lesssim_u N^{\frac{6(q-2)}{q(p+4)} - \frac{3p-4}{pq} -}$$

for all $N \leq 10pN_0$. Therefore, using Bernstein together with (5.1), for $\frac{3p^2+20p-16}{6p} < q < \frac{3p}{2}$ we obtain

$$\begin{aligned} \|u\|_{L_t^\infty L_x^q} &\leq \|u_{\leq N_0}\|_{L_t^\infty L_x^q} + \|u_{>N_0}\|_{L_t^\infty L_x^q} \\ &\lesssim_u \sum_{N \leq N_0} N^{\frac{6(q-2)}{q(p+4)} - \frac{3p-4}{pq} -} + \sum_{N > N_0} N^{\frac{2}{p} - \frac{3}{q}} \lesssim_u 1, \end{aligned}$$

which completes the proof of Proposition 5.1. □

6. Quantitative decay

In this section we consider the soliton-like and frequency-cascade solutions (in the sense of Theorem 1.8) and obtain a quantitative bound for how such solutions decay away from $x(t)$ in a critical space, specifically, $L_x^{3p/2}$. Note that compactness merely gives a nonquantitative decay.

Proposition 6.1 (Spatial decay)**.** *Let u be a global solution to* (1.1) *that is almost periodic modulo symmetries. Also assume that*

$$\big\|(u,u_t)\big\|_{L_t^\infty(\mathbb{R};\dot H_x^{s_c}\times \dot H_x^{s_c-1})}<\infty \quad \text{and} \quad \inf_{t\in\mathbb{R}} N(t)\geq 1. \tag{6.1}$$

Then

$$\sup_{t\in\mathbb{R}}\int_{|x-x(t)|\geq R}|u(t,x)|^{\frac{3p}{2}}\,dx \lesssim_u R^{-\gamma} \tag{6.2}$$

for any $\gamma<\frac{6p^2-20p+16}{3p^2}$ and, in particular, for some $\gamma>1$ when $p\geq 6$.

Proof. We prove this by bootstrap; the requisite smallness comes from compactness. We elaborate upon this, before launching into the main part of the argument.

Let $\phi:\mathbb{R}^3\to[0,1]$ be a smooth function with $\phi(x)=1$ when $|x|\geq 1$ and $\phi(x)=0$ when $|x|<\frac12$. As u is almost periodic modulo symmetries and (6.1) holds, for any $\eta>0$ we may choose R_0 so that

$$\sup_{t\in\mathbb{R}}\Big\{\big\|\phi\big(\tfrac{10}{R_0}[x-x(t)]\big)u\big\|_{\dot H^{s_c}}+\big\|\phi\big(\tfrac{10}{R_0}[x-x(t)]\big)u_t\big\|_{\dot H^{s_c-1}}\Big\}\leq\eta. \tag{6.3}$$

Requiring η to be small enough that the small-data global well-posedness theory applies and making use of simple domain of dependence arguments, we deduce that

$$\sup_{T\in\mathbb{R}}\Big\{\|u\|_{L_t^{\frac{4p(p-1)}{3p+2}}L_x^{\frac{12p(p-1)}{5(p-2)}}(\{|x-x(T)|\geq\frac{R_0}{10}+|t-T|\})} + \|\nabla u\|_{L_t^\infty L^{\frac{3p}{p+2}}(\{|x-x(T)|\geq\frac{R_0}{10}+|t-T|\})}\Big\}\lesssim\eta. \tag{6.4}$$

We now turn to the main part of the proof of Proposition 6.1. By the time-translation symmetry of the problem, it suffices to consider a single time, say $t=0$. By space-translation symmetry, we may set $x(0)=0$. Using Lemma 1.9 we may represent $u(0)$ as an integral over $[0,\infty)$, which we choose to break here into two pieces: $[0,\delta R]\cup[\delta R,\infty)$. Thus $u(0)=f+g$ with

$$f:=\int_{\delta R}^\infty\frac{\sin\big(t|\nabla|\big)}{|\nabla|}F(u(t))\,dt \quad\text{and}\quad g:=\int_0^{\delta R}\frac{\sin\big(t|\nabla|\big)}{|\nabla|}F(u(t))\,dt. \tag{6.5}$$

Here $\delta>0$ is a small number that will be chosen in due course.

The estimate we need for the long-time piece f was already obtained in (5.7):

$$\|f\|_{L_x^\infty(\mathbb{R}^3)}\lesssim_u(\delta R)^{-\frac{2}{p}}. \tag{6.6}$$

By contrast, we estimate g in a more natural (scale-invariant) space. Note that by finite speed of propagation, both for the propagator $|\nabla|^{-1}\sin(t|\nabla|)$ (cf. Lemma 2.2) as well as for the center $x(t)$ of the wave packet (cf. Proposition 4.1), we see that for $|x|\geq R$ the value of $g(x)$ depends only on the values of u in the set

$$\Omega_R:=\big\{(t,x):t\in[0,\delta R]\text{ and }|x-x(t)|\geq(1-2\delta)R-2C_u\big\},$$

where C_u is as in (4.2). With this in hand, we now estimate g using Sobolev embedding (which is valid on the complement of a ball) together with the Strichartz and Hölder inequalities:

$$\begin{aligned}
\|g\|_{L_x^{3p/2}(|x|\geq R)} &\lesssim \Big\| \int_0^{\delta R} \frac{\sin\big(t|\nabla|\big)}{|\nabla|} \nabla F(u(t))\, dt \Big\|_{\dot H^{s_c-1}(|x|\geq R)} \\
&\lesssim \big\| \nabla F(u) \big\|_{L_{t,x}^{\frac{4p}{3p+2}}(\Omega_R)} \\
&\lesssim \big\| u \big\|_{L_t^\infty L_x^{3p/2}(\Omega_R)} \big\| u \big\|^{p-1}_{L_t^{\frac{4p(p-1)}{3p+2}} L_x^{\frac{12p(p-1)}{5(p-2)}}(\Omega_R)} \big\| \nabla u \big\|_{L_t^\infty L^{\frac{3p}{p+2}}(\Omega_R)}. \qquad (6.7)
\end{aligned}$$

Now requiring $R > R_0(u) \geq 8C_u$ and $\delta \leq 1/8$, we see that Ω_R is included in the region where $|x - x(t)| \geq R/2$, which we apply to the first copy of u. If $R \geq R_0$, then Ω_R also is included in the region covered by (6.4), which we apply to the next two factors. In this way we obtain

$$\|g\|_{L_x^{3p/2}(|x|\geq R)} \lesssim \eta^p \big\| u \big\|_{L_t^\infty L_x^{3p/2}(\mathbb{R}\times\{|x-x(t)|\geq R/2\})} \qquad (6.8)$$

for a fixed small δ and $R \geq R_0(\eta, u)$.

Next we put the two pieces, f and g, together to bound LHS(6.2). This is a simple application of standard tricks from real interpolation: Fix $A > 0$ so that $\|f\|_{L_x^\infty} \leq A/2$; note that by (6.6), $A \lesssim_u (\delta R)^{-2/p}$. Then

$$|u(0,x)|^{3p/2} \leq A^{\frac{3p-2q}{2}} |u(0,x)|^q + |2g(x)|^{3p/2}$$

and so, using (6.8),

$$\begin{aligned}
\int_{|x|\geq R} |u(0,x)|^{3p/2}\, dx \lesssim_{u,\delta} R^{-\frac{3p-2q}{p}} \|u(0,x)\|^q_{L_x^q(\mathbb{R}^3)} \\
+ \eta^p \sup_t \int_{|x-x(t)|\geq R/2} |u(t,x)|^{3p/2}\, dx
\end{aligned}$$

for $R \geq R_0(\eta, u)$.

We now have our basic inductive step. Defining

$$B(R) := \sup_{t\in\mathbb{R}} \int_{|x-x(t)|\geq R} |u(t,x)|^{3p/2}\, dx,$$

restoring space- and time-translation invariance, and invoking Proposition 5.1, we have

$$B(R) \lesssim_{u,\delta} R^{-\frac{3p-2q}{p}} + \eta^p B(\tfrac{1}{2}R) \quad \text{for any} \quad q > \tfrac{3p^2+20p-16}{6p}$$

and $R \geq R_0(\eta, u)$. On the other hand, by (6.1) and Sobolev embedding, $B(R) \lesssim_u 1$ for $R \leq R_0(\eta, u)$. The desired estimate now follows by choosing η sufficiently small and performing a simple induction. □

7. Global enemies have finite energy

In this section, we prove that the soliton-like and frequency-cascade solutions described in Theorem 1.8 have finite energy, that is, $\nabla_{t,x} u$ is square integrable. The first and main step is the following:

Theorem 7.1. *Let u be a global solution to* (1.1) *that is almost periodic modulo symmetries. Assume also that*

$$\inf_{t\in\mathbb{R}} N(t) \geq 1$$

and

$$\big\| |\nabla|^{s-1}\nabla_{t,x} u\big\|_{L_t^\infty L_x^2} < \infty \tag{7.1}$$

for some $1 < s \le s_c$*. Then for all* $0 < \varepsilon \le \varepsilon_0(p)$,

$$\big\| |\nabla|^{s-1-\varepsilon}\nabla_{t,x} u\big\|_{L_t^\infty L_x^2} < \infty, \tag{7.2}$$

provided $s - 1 - \varepsilon > 0$.

Proof. By time-translation symmetry, it suffices to prove the claim for $t = 0$. By space-translation symmetry, we may also assume $x(0) = 0$.

By Bernstein's inequality and (7.1), it suffices to prove the following space-localized low-frequency bound: For some $\beta = \beta(\varepsilon) > 0$,

$$\big\| \theta(i\nabla) P_{\le 1} |\nabla|^{s-1-\varepsilon}\nabla_{t,x} u(0)\big\|^2_{L_x^2(B_R)} \lesssim_{u,\varepsilon} R^{-3\beta} \tag{7.3}$$

uniformly for $R \ge R_0(u)$ and 'Whitney' balls $B_R = \{x \in \mathbb{R}^3 : |x - x_0| \le R\}$ with $|x_0| = 3R$. Here θ is as defined in (2.3) and plays the role of a low-frequency projection, but one whose convolution kernel has compact support; the utility of this fact will be apparent in due course and is responsible for the appearance of θ in Proposition 2.6.

To see that (7.3) really does suffice, we note that

$$\big\| \theta(i\nabla) P_{\le 1} f\big\|_{L_x^2} \lesssim 1 \quad \Longrightarrow \quad \big\| P_{\le 1} f\big\|_{L_x^2} \lesssim 1 \tag{7.4}$$

because $|\theta(\xi)| \gtrsim 1$ for $|\xi| \le \frac{11}{10}$, which is the Fourier support of $P_{\le 1}$.

To obtain (7.3) we use both Duhamel formulae in (1.9) to write:

$$\begin{aligned}
&\big\| \theta(i\nabla) P_{\le 1} |\nabla|^{s-1-\varepsilon}\nabla_{t,x} u(0)\big\|^2_{L_x^2(B_R)} \\
&\quad = -\int_0^\infty \int_{-\infty}^0 \Big\langle \nabla \tfrac{\sin(|\nabla| t)}{|\nabla|} \theta(i\nabla) |\nabla|^{s-1-\varepsilon} F_{\le 1}(t), \\
&\qquad\qquad \chi_R \nabla \tfrac{\sin(|\nabla|\tau)}{|\nabla|} \theta(i\nabla) |\nabla|^{s-1-\varepsilon} F_{\le 1}(\tau)\Big\rangle \, d\tau \, dt \\
&\qquad - \int_0^\infty \int_{-\infty}^0 \Big\langle \cos(|\nabla| t) \theta(i\nabla) |\nabla|^{s-1-\varepsilon} F_{\le 1}(t), \\
&\qquad\qquad \chi_R \cos(|\nabla|\tau) \theta(i\nabla) |\nabla|^{s-1-\varepsilon} F_{\le 1}(\tau)\Big\rangle \, d\tau \, dt,
\end{aligned} \tag{7.5}$$

where χ_R is a smooth cutoff function associated to the ball B_R. More precisely, we set

$$\chi_R(x) = \phi\big(\tfrac{x - x_0}{R}\big),$$

where $\phi : \mathbb{R}^3 \to [0,1]$ is a smooth function obeying $\phi(x) = 1$ for $|x| \le 1$ and $\phi(x) = 0$ for $|x| > \frac{11}{10}$. Recall that x_0 denotes the center of B_R and obeys $|x_0| = 3R$.

In order to bound the time integrals, we need to use the fact that we can bound Strichartz norms of u far from $x(t)$. We used this argument already in Section 6 but will repeat the details here. By choosing R_0 sufficiently large, compactness of our solution guarantees that

$$\sup_{t \in \mathbb{R}} \Big\{ \big\| \phi^c\big(\tfrac{22}{\delta R}[x - x(t)]\big) u\big\|_{\dot H_x^{s_c}} + \big\| \phi^c\big(\tfrac{22}{\delta R}[x - x(t)]\big) u_t\big\|_{\dot H_x^{s_c - 1}} \Big\} \le \eta, \tag{7.6}$$

where $\phi^c = 1 - \phi$ and $\delta = \delta(u) > 0$ denotes the subluminality constant from Proposition 4.3. Requiring η to be small enough that the small data global well-posedness theory applies and making use of simple domain of dependence arguments, we deduce that

$$\sup_{T\in\mathbb{R}} \big\| u \big\|_{L_t^{\tilde q} L_x^{\frac{3p\tilde q}{2\tilde q - p}}(\{|x-x(T)|\geq \frac{\delta R}{20}+|t-T|\})} \lesssim_{\tilde q} \eta \quad \text{for all} \quad \tfrac{p}{2} < \tilde q \leq \infty. \tag{7.7}$$

Returning to (7.5) and using the strong Huygens principle and the fact that $\operatorname{supp}\check\theta \subseteq [-4,4]^3$, we see that we can insert a smooth cutoff χ_{DoD} to the appropriate domain of dependence in the middle of each of the four products $\theta(i\nabla)|\nabla|^{s-1-\varepsilon}$, specifically to the set of spacetime points that have a light ray connecting them to a point of the form (t,x) with $t=0$ and $\operatorname{dist}(x, \operatorname{supp}\chi_R) \leq 4\sqrt{3}$. In particular, we can choose χ_{DoD} so that

$$\begin{gathered} \operatorname{supp}\chi_{DoD} \subseteq \{(t,x)\in\mathbb{R}\times\mathbb{R}^3 : (1-\tfrac{\delta}{10^6})|t| - \tfrac65 R \leq |x-x_0| \leq (1+\tfrac{\delta}{10^6})|t| + \tfrac65 R\}, \\ \chi_{DoD}(t,x) = 1 \quad \text{when} \quad \big||x-x_0| - |t|\big| \leq \tfrac65 R \\ \text{and} \quad |\partial^\alpha \chi_{DoD}| \lesssim_{\delta,\alpha} (|t|+R)^{-|\alpha|} \end{gathered}$$

for all multi-indices α. We will also need a slightly fattened version of χ_{DoD}, which we call $\tilde\chi_{DoD}$. It is defined so that

$$\begin{gathered} \tilde\chi_{DoD}(t,x) = 1 \quad \text{when} \quad \operatorname{dist}(x, \operatorname{supp}\chi_{DoD}(t)) \leq \tfrac{1}{10}R + \tfrac{\delta}{10^6}|t|, \\ \operatorname{supp}\tilde\chi_{DoD} \subseteq \{(t,x) : (1-\tfrac{3\delta}{10^6})|t| - \tfrac85 R \leq |x-x_0| \leq (1+\tfrac{3\delta}{10^6})|t| + \tfrac85 R\}, \\ \text{and} \quad |\partial_x^\alpha \tilde\chi_{DoD}| \lesssim_{\alpha,\delta} (|t|+R)^{-|\alpha|} \end{gathered}$$

for all multi-indices α.

By (4.2), we have $|x(t)| \leq |t| + 2C_u$ for all $t\in\mathbb{R}$ and thus, for R large enough (to defeat C_u),

$$\operatorname{dist}(x(t), \operatorname{supp}\tilde\chi_{DoD}(t)) \geq \tfrac{\delta}{20}(|t|+R) \quad \text{for} \quad 0 \leq |t| \leq \tfrac{R}{2}. \tag{7.8}$$

On the other hand, by the subluminality bound (4.6), we know that $|x(t)| \leq (1-\delta)|t|$ for $|t| \geq 1/\delta$. Thus,

$$|x(t) - x_0| \leq 3R + (1-\delta)|t| \quad \text{for} \quad |t| \geq \tfrac{1}{\delta}$$

and hence, for $|t| \geq \frac{10}{\delta}$R,

(7.9)

$$\operatorname{dist}(x(t), \operatorname{supp}\tilde\chi_{DoD}(t)) \geq \big((1-\tfrac{3\delta}{10^6})|t| - \tfrac85 R\big) - \big(3R + (1-\delta)|t|\big) \geq \tfrac{\delta}{20}(|t|+R).$$

The most dangerous regime is when $|t| \in [\frac R2, \frac{10}{\delta}R]$, for then $x(t)$ may lie near (indeed inside) the support of χ_{DoD}. Here we make use of a further smooth partition of unity, namely, $1 = \chi_{near} + \chi_{far}$ with

$$\operatorname{supp}\chi_{near} \subseteq \{(t,x) : \tfrac R2 \leq |t| \leq \tfrac{10}{\delta}R \text{ and } |x-x(t)| \leq \tfrac{\delta}{5}(|t|+R)\}$$

and

$$\chi_{far}(t,x) = 0 \quad \text{when} \quad \tfrac R2 \leq |t| \leq \tfrac{10}{\delta}R \quad \text{and} \quad |x-x(t)| \leq \tfrac{\delta}{10}(|t|+R).$$

Note that this can be done in a manner such that

$$|\partial_x^\alpha \chi_{near}| + |\partial_x^\alpha \chi_{far}| \lesssim_{\alpha,\delta} (|t|+R)^{-|\alpha|}.$$

for all multi-indices α. It is for the sake of notational convenience that we have defined χ_{near} to be identically equal to zero for t outside the region $\frac{R}{2} \leq |t| \leq \frac{10}{\delta}R$ and, correspondingly, $\chi_{far} \equiv 1$ there.

We will also need a slightly fattened version $\tilde{\chi}_{far}$ of χ_{far}, chosen so that

$$\tilde{\chi}_{far}(t,x) = 1 \quad \text{when} \quad \operatorname{dist}(x, \operatorname{supp}\chi_{far}(t)) \leq \tfrac{\delta}{40}(|t|+R),$$
$$\operatorname{supp}\tilde{\chi}_{far}(t) \subseteq \big\{x : |x - x(t)| \geq \tfrac{\delta}{20}(|t|+R)\big\} \quad \text{when} \quad \tfrac{1}{2}R \leq |t| \leq \tfrac{10}{\delta}R,$$
$$\text{and} \quad |\partial_x^\alpha \tilde{\chi}_{far}| \lesssim_{\alpha,\delta} (|t|+R)^{-|\alpha|},$$

for all multi-indices α.

Collecting (7.8), (7.9), and the definition of $\tilde{\chi}_{far}$, we note that

$$\operatorname{dist}(x(t), \operatorname{supp}[\tilde{\chi}_{DoD}\tilde{\chi}_{far}](t)) \geq \tfrac{\delta}{20}(|t|+R). \tag{7.10}$$

On the other hand, for $(t,x), (\tau, y) \in \operatorname{supp}\chi_{near}$ with $t > 0$ and $\tau < 0$, we obtain

$$|t| + |\tau| + |x| + |y| \lesssim_\delta R \tag{7.11}$$

and, more importantly,

$$\begin{aligned} |t-\tau| - |x-y| &\geq |t-\tau| - (1-\delta)|t-\tau| - \tfrac{\delta}{5}(t+R) - \tfrac{\delta}{5}(|\tau|+R) \\ &\geq \tfrac{4\delta}{5}|t-\tau| - \tfrac{2\delta}{5}R \geq \tfrac{2\delta}{5}R, \end{aligned} \tag{7.12}$$

by subluminality and taking $R > R_0(u) \geq 1/\delta$. The significance of these inequalities is that they allow us to apply Proposition 2.6; see Lemma 7.5 below.

Before beginning to estimate (7.5), we first note some consequences of (7.7) in terms of our cutoffs:

Lemma 7.2. *Under the assumptions above (and taking R_0 even larger if necessary),*

$$\big\|\tilde{\chi}_{DoD}\tilde{\chi}_{far}u\big\|_{L_t^{\tilde{q}} L_x^{\frac{3p\tilde{q}}{2\tilde{q}-p}}(I\times\mathbb{R}^3)} \lesssim_{u,\tilde{q}} 1 \quad \textit{for all} \quad \tfrac{p}{2} < \tilde{q} \leq \infty,$$

uniformly for $I = [-\frac{10}{\delta}R, \frac{10}{\delta}R]$ or $I = [T, 2T] \cup [-2T, -T]$ with $T \geq \frac{10}{\delta}R$.

Proof. We will only prove the claim for positive times t. For negative times, the argument is similar.

Recall that by (7.7),

$$\sup_{T\in\mathbb{R}} \big\|u\big\|_{L_t^{\tilde{q}} L_x^{\frac{3p\tilde{q}}{2\tilde{q}-p}}(\{|x-x(T)| \geq \frac{\delta R}{20} + |t-T|\})} \lesssim_{\tilde{q}} \eta \quad \text{for all} \quad \tfrac{p}{2} < \tilde{q} \leq \infty. \tag{7.13}$$

Thus, choosing $T = 0$, we obtain

$$\big\|\tilde{\chi}_{DoD}\tilde{\chi}_{far}u\big\|_{L_t^{\tilde{q}} L_x^{\frac{3p\tilde{q}}{2\tilde{q}-p}}([0,\frac{R}{2}]\times\mathbb{R}^3)} \lesssim_{\tilde{q}} \eta \quad \text{for all} \quad \tfrac{p}{2} < \tilde{q} \leq \infty$$

since if $0 \leq t \leq \frac{R}{2}$ and $\tilde{\chi}_{DoD}(t,x) \neq 0$, then $|x| \geq \frac{4}{5}R$.

On the other hand, choosing $T \geq \frac{10}{\delta}R$ gives

$$\big\|\tilde{\chi}_{DoD}\tilde{\chi}_{far}u\big\|_{L_t^{\tilde{q}} L_x^{\frac{3p\tilde{q}}{2\tilde{q}-p}}([T,2T]\times\mathbb{R}^3)} \lesssim_{\tilde{q}} \eta \quad \text{for all} \quad \tfrac{p}{2} < \tilde{q} \leq \infty.$$

Indeed, using (4.6), for $(t,x) \in \operatorname{supp}\tilde{\chi}_{DoD}\tilde{\chi}_{far}$ and $t \geq T \geq \frac{10}{\delta}R$, we have

$$\begin{aligned} |x - x(T)| &\geq |x - x_0| - |x_0| - |x(T)| \\ &\geq (1 - \tfrac{3\delta}{10^6})|t-T| - (3 + \tfrac{8}{5})R + (1 - \tfrac{3}{10^6})\delta T \\ &\geq (1 - \tfrac{3\delta}{10^6})|t-T| + \tfrac{\delta R}{20} + \tfrac{\delta}{50}T \end{aligned}$$

and so $|x - x(T)| \geq |t - T| + \frac{\delta R}{20}$ provided $\frac{3\delta}{10^6}|t - T| \leq \frac{\delta}{50}T$, which is true when $t \in [T, 2T]$.

It remains to consider $\frac{R}{2} \leq t \leq \frac{10}{\delta}R$. For this region, we choose a mesh

$$\tfrac{R}{2} = T_0 < T_1 < \cdots < T_K = \tfrac{10}{\delta}R \quad \text{with} \quad \tfrac{\delta}{200}R \leq |T_k - T_{k-1}| \leq \tfrac{\delta}{100}R,\ 1 \leq k \leq K.$$

Note that $K \lesssim \delta^{-2}$. Then for $(t,x) \in \operatorname{supp} \tilde\chi_{DoD}\tilde\chi_{far}$ with $t \in [T_{k-1}, T_k]$,

$$\begin{aligned} |x - x(T_{k-1})| &\geq |x - x(t)| - |x(t) - x(T_{k-1})| \\ &\geq \tfrac{\delta}{20}(|t| + R) - |t - T_{k-1}| - 2C_u \geq \tfrac{\delta}{20}R + |t - T_{k-1}|, \end{aligned}$$

provided $R \geq R_0$ with R_0 sufficiently large depending on u. Thus, using (7.13) with $T = T_k$ for $0 \leq k \leq K-1$ and summing, we derive

$$\big\|\tilde\chi_{DoD}\tilde\chi_{far}u\big\|_{L_t^{\tilde q}L_x^{\frac{3p\tilde q}{2\tilde q - p}}([\frac{R}{2}, \frac{10}{\delta}R]\times\mathbb{R}^3)} \lesssim_{\tilde q} \eta\delta^{-2} \lesssim_{u,\tilde q} 1 \quad \text{for all} \quad \tfrac{p}{2} < \tilde q \leq \infty.$$

This concludes the proof of the lemma. □

After breaking up the integrals in (7.5) by introducing the cutoffs $\chi_{DoD}\chi_{near}$ and $\chi_{DoD}\chi_{far}$, the required estimate follows directly from the next three lemmas.

Lemma 7.3. *Under the assumptions above, for some small $\beta = \beta(\varepsilon) > 0$ we have*

$$\begin{aligned} &\Big\|\int_0^\infty \nabla\tfrac{\sin(|\nabla|t)}{|\nabla|}\theta(i\nabla)\chi_{DoD}\chi_{far}|\nabla|^{s-1-\varepsilon}F_{\leq 1}(t)\,dt\Big\|_{L_x^2(\mathbb{R}^3)} \\ &\qquad + \Big\|\int_0^\infty \cos(|\nabla|t)\theta(i\nabla)\chi_{DoD}\chi_{far}|\nabla|^{s-1-\varepsilon}F_{\leq 1}(t)\,dt\Big\|_{L_x^2(\mathbb{R}^3)} \lesssim_u R^{-1/2-4\beta} \end{aligned}$$

and similarly,

$$\begin{aligned} &\Big\|\int_{-\infty}^0 \nabla\tfrac{\sin(|\nabla|\tau)}{|\nabla|}\theta(i\nabla)\chi_{DoD}\chi_{far}|\nabla|^{s-1-\varepsilon}F_{\leq 1}(\tau)\,d\tau\Big\|_{L_x^2(\mathbb{R}^3)} \\ &\qquad + \Big\|\int_{-\infty}^0 \cos(|\nabla|\tau)\theta(i\nabla)\chi_{DoD}\chi_{far}|\nabla|^{s-1-\varepsilon}F_{\leq 1}(\tau)\,d\tau\Big\|_{L_x^2(\mathbb{R}^3)} \lesssim_u R^{-1/2-4\beta}. \end{aligned}$$

Proof. We will only present the proof of the first inequality; for negative times, the argument is similar.

For the remainder of this proof, all spacetime norms are over the region $[0,\infty)\times\mathbb{R}^3$. Also, to ease notation we write $\chi = \sqrt{\chi_{DoD}\chi_{far}}$ and $\tilde\chi = \tilde\chi_{DoD}\tilde\chi_{far}$.

Using the L_x^2-boundedness of $\theta(i\nabla)$, then the Strichartz inequality (with $6 \le q < \infty$) followed by Hölder's inequality, we obtain

(7.14)
$$\begin{aligned}
&\Big\| \int_0^\infty \nabla \tfrac{\sin(|\nabla| t)}{|\nabla|} \theta(i\nabla) \chi^2 |\nabla|^{s-1-\varepsilon} F_{\le 1}(t)\, dt \Big\|_2 \\
&\qquad + \Big\| \int_0^\infty \cos(|\nabla| t) \theta(i\nabla) \chi^2 |\nabla|^{s-1-\varepsilon} F_{\le 1}(t)\, dt \Big\|_2 \\
&\qquad \lesssim \big\| \nabla \big[\chi^2 |\nabla|^{s-1-\varepsilon} F_{\le 1} \big] \big\|_{L_t^{\frac{2q}{q+6}} L_x^{\frac{q}{q-1}}} \\
&\qquad \lesssim \big\| \chi^2 \nabla |\nabla|^{s-1-\varepsilon} F_{\le 1} \big\|_{L_t^{\frac{2q}{q+6}} L_x^{\frac{q}{q-1}}} + \|\nabla \chi\|_{L_x^3} \big\| \chi |\nabla|^{s-1-\varepsilon} F_{\le 1} \big\|_{L_t^{\frac{2q}{q+6}} L_x^{\frac{3q}{2q-3}}} \\
&\qquad \lesssim \big\| \chi \nabla |\nabla|^{s-1-\varepsilon} F_{\le 1} \big\|_{L_t^{\frac{2q}{q+6}} L_x^{\frac{q}{q-1}}} + \big\| \chi |\nabla|^{s-1-\varepsilon} F_{\le 1} \big\|_{L_t^{\frac{2q}{q+6}} L_x^{\frac{3q}{2q-3}}} \\
&\qquad \lesssim \big\| \nabla |\nabla|^{s-1-\varepsilon} F(\tilde\chi u) \big\|_{L_t^{\frac{2q}{q+6}} L_x^{\frac{q}{q-1}}} + \big\| |\nabla|^{s-1-\varepsilon} F(\tilde\chi u) \big\|_{L_t^{\frac{2q}{q+6}} L_x^{\frac{3q}{2q-3}}} \\
&\qquad\quad + \big\| \chi \nabla |\nabla|^{s-1-\varepsilon} P_{\le 1}(\tilde\chi^c F) \big\|_{L_t^{\frac{2q}{q+6}} L_x^{\frac{q}{q-1}}} + \big\| \chi |\nabla|^{s-1-\varepsilon} P_{\le 1}(\tilde\chi^c F) \big\|_{L_t^{\frac{2q}{q+6}} L_x^{\frac{3q}{2q-3}}},
\end{aligned}$$

where $\tilde\chi^c = 1 - \tilde\chi^{p+1}$ so that $F(u) = F(\tilde\chi u) + \tilde\chi^c F(u)$.

To complete the proof, we have to show that each of the four terms appearing on the right-hand side of (7.14) is bounded by $R^{-1/2-4\beta}$. In order to appeal to Lemma 7.2, we will sometimes need to partition $[0,\infty)$ into the collection of intervals $I_j = [T_j, T_{j+1}]$ with $T_0 := 0$ and $T_j = \frac{10}{\delta} R 2^{j-1}$ for all $j \ge 1$.

We start with the first term in RHS(7.14). By the fractional chain and product rules together with Hölder's inequality, Sobolev embedding, Lemma 7.2, Proposition 5.1, and the combination of Proposition 6.1 and (7.10),

$$\begin{aligned}
&\big\| \nabla |\nabla|^{s-1-\varepsilon} F(\tilde\chi u) \big\|_{L_t^{\frac{2q}{q+6}} L_x^{\frac{q}{q-1}}} \\
&\lesssim \sum_{j\ge 0} \big\| \nabla |\nabla|^{s-1-\varepsilon} F(\tilde\chi u) \big\|_{L_t^{\frac{2q}{q+6}} L_x^{\frac{q}{q-1}} (I_j \times \mathbb{R}^3)} \\
&\lesssim \sum_{j\ge 0} \Big[\big\| |\nabla|^{s-\varepsilon} u \big\|_{L_t^\infty L_x^{\frac{2q}{q-1}} (I_j \times \mathbb{R}^3)} + \big\| |\nabla|^{s-\varepsilon} \tilde\chi \big\|_{L_t^\infty L_x^{\frac{3}{s-\varepsilon}} (I_j \times \mathbb{R}^3)} \|u\|_{L_t^\infty L_x^{\frac{6q}{q(3-2s+2\varepsilon)-3}} (I_j \times \mathbb{R}^3)} \Big] \\
&\qquad\qquad \times \|\tilde\chi u\|_{L_t^{\tilde q} L_x^{\frac{3p\tilde q}{2\tilde q - p}} (I_j \times \mathbb{R}^3)}^{\frac{\tilde q (q+6)}{2q}} \|\tilde\chi u\|_{L_t^\infty L_x^p (I_j \times \mathbb{R}^3)}^{\frac{3p}{2q}} \|\tilde\chi u\|_{L_t^\infty L_x^{\frac{3p}{2}} (I_j \times \mathbb{R}^3)}^{p - \frac{3p}{2q} - \frac{\tilde q (q+6)}{2q}} \\
&\lesssim_u \sum_{j\ge 0} (T_j + R)^{-(\frac{2}{3} - \frac{1}{q} - \frac{\tilde q (q+6)}{3pq})\gamma} \big\| |\nabla|^{s-\varepsilon} u \big\|_{L_t^\infty L_x^{\frac{2q}{q-1}} (I_j \times \mathbb{R}^3)} \\
&\lesssim_u R^{-(\frac{2}{3} - \frac{1}{q} - \frac{\tilde q (q+6)}{3pq})\gamma},
\end{aligned}$$

where $p/2 < \tilde q < \infty$. In the last step, we used Sobolev embedding followed by interpolation, (7.1), and Proposition 5.1. This requires $\frac{3ps}{(3p-4)\varepsilon} \le q < \frac{(3p^2+20p-16)s}{(3p^2+8p-16)\varepsilon}$. In order to make the power of R less than $-1/2$, it suffices to take q large (which in turn forces ε to be small) and $\tilde q$ close to $p/2$.

Arguing similarly, we estimate the second term in RHS(7.14) as follows:

$$\begin{aligned}
&\big\||\nabla|^{s-1-\varepsilon}F(\tilde\chi u)\big\|_{L_t^{\frac{2q}{q+6}}L_x^{\frac{3q}{2q-3}}}\\
&\lesssim \sum_{j\geq 0}\big\||\nabla|^{s-1-\varepsilon}F(\tilde\chi u)\big\|_{L_t^{\frac{2q}{q+6}}L_x^{\frac{3q}{2q-3}}(I_j\times\mathbb{R}^3)}\\
&\lesssim \sum_{j\geq 0}\Big[\big\||\nabla|^{s-1-\varepsilon}u\big\|_{L_t^\infty L_x^{\frac{6q}{q-3}}(I_j\times\mathbb{R}^3)}\\
&\qquad\qquad + \big\||\nabla|^{s-1-\varepsilon}\tilde\chi\big\|_{L_t^\infty L_x^{\frac{3}{s-1-\varepsilon}}(I_j\times\mathbb{R}^3)}\|u\|_{L_t^\infty L_x^{\frac{6q}{q(3-2s+2\varepsilon)-3}}(I_j\times\mathbb{R}^3)}\Big]\\
&\qquad\qquad\quad \times \|\tilde\chi u\|^{\frac{\tilde q(q+6)}{2q}}_{L_t^{\tilde q}L_x^{\frac{3p\tilde q}{2\tilde q-p}}(I_j\times\mathbb{R}^3)}\|\tilde\chi u\|^{\frac{3p}{2q}}_{L_t^\infty L_x^p(I_j\times\mathbb{R}^3)}\|\tilde\chi u\|^{p-\frac{3p}{2q}-\frac{\tilde q(q+6)}{2q}}_{L_t^\infty L_x^{\frac{3p}{2}}(I_j\times\mathbb{R}^3)}\\
&\lesssim_u \sum_{j\geq 0}(T_j+R)^{-(\frac{2}{3}-\frac{1}{q}-\frac{\tilde q(q+6)}{3pq})\gamma}\big\||\nabla|^{s-\varepsilon}u\big\|_{L_t^\infty L_x^{\frac{2q}{q-1}}(I_j\times\mathbb{R}^3)}\\
&\lesssim_u R^{-(\frac{2}{3}-\frac{1}{q}-\frac{\tilde q(q+6)}{3pq})\gamma},
\end{aligned}$$

which again yields the desired decay in R for q large enough and $\tilde q$ close to $p/2$.

In order to estimate the remaining two terms in RHS(7.14), we note that

$$\operatorname{dist}(\operatorname{supp}\chi,\operatorname{supp}\tilde\chi^c)\gtrsim_\delta |t|+R.$$

Hence, by the mismatch estimate Lemma 2.5 together with Proposition 5.1,

$$\begin{aligned}
\big\|\chi\nabla|\nabla|^{s-1-\varepsilon}P_{\leq 1}(\tilde\chi^c F)\big\|_{L_t^{\frac{2q}{q+6}}L_x^{\frac{q}{q-1}}} &+ \big\|\chi|\nabla|^{s-1-\varepsilon}P_{\leq 1}(\tilde\chi^c F)\big\|_{L_t^{\frac{2q}{q+6}}L_x^{\frac{3q}{2q-3}}}\\
&\lesssim \|F(u)\|_{L_t^\infty L_x^1}\big\|(|t|+R)^{-(s-\varepsilon)-\frac{3}{q}}\big\|_{L_t^{\frac{2q}{q+6}}([0,\infty))}\\
&\lesssim_u R^{-\frac12-4\beta},
\end{aligned}$$

provided $4\beta < s-1-\varepsilon$. This finishes the proof of the lemma. ☐

Lemma 7.4. *Under the assumptions above, for any $\beta>0$ we have*

$$\begin{aligned}
&\Big\|\int_0^\infty \nabla\tfrac{\sin(|\nabla|t)}{|\nabla|}\theta(i\nabla)\chi_{DoD}\chi_{near}|\nabla|^{s-1-\varepsilon}F_{\leq 1}(t)\,dt\Big\|_{L_x^2(\mathbb{R}^3)}\\
&\qquad + \Big\|\int_0^\infty \cos(|\nabla|t)\theta(i\nabla)\chi_{DoD}\chi_{near}|\nabla|^{s-1-\varepsilon}F_{\leq 1}(t)\,dt\Big\|_{L_x^2(\mathbb{R}^3)}\lesssim_u R^{1/2+\beta}
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\Big\|\int_{-\infty}^0 \nabla\tfrac{\sin(|\nabla|\tau)}{|\nabla|}\theta(i\nabla)\chi_{DoD}\chi_{near}|\nabla|^{s-1-\varepsilon}F_{\leq 1}(\tau)\,d\tau\Big\|_{L_x^2(\mathbb{R}^3)}\\
&\qquad + \Big\|\int_{-\infty}^0 \cos(|\nabla|\tau)\theta(i\nabla)\chi_{DoD}\chi_{near}|\nabla|^{s-1-\varepsilon}F_{\leq 1}(\tau)\,d\tau\Big\|_{L_x^2(\mathbb{R}^3)}\lesssim_u R^{1/2+\beta}.
\end{aligned}$$

Proof. Again, we present the proof for positive times only.

First, recall that χ_{near} is supported in the spacetime region where $\frac{R}{2}\leq t\leq \frac{10}{\delta}R$. Thus, for the remainder of this proof, all spacetime norms will be over the region $[\frac{R}{2},\frac{10}{\delta}R]\times\mathbb{R}^3$.

Now, by the Strichartz inequality followed by Hölder's inequality, Sobolev embedding, Bernstein's inequality, and Proposition 5.1, we obtain

$$\begin{aligned}
&\Big\| \int_0^\infty \nabla \tfrac{\sin(|\nabla| t)}{|\nabla|} \theta(i\nabla) \chi_{DoD} \chi_{near} |\nabla|^{s-1-\varepsilon} F_{\leq 1}(t)\, dt \Big\|_{L_x^2(\mathbb{R}^3)} \\
&\qquad\qquad + \Big\| \int_0^\infty \cos(|\nabla| t) \theta(i\nabla) \chi_{DoD} \chi_{near} |\nabla|^{s-1-\varepsilon} F_{\leq 1}(t)\, dt \Big\|_{L_x^2(\mathbb{R}^3)} \\
&\lesssim \big\| \nabla |\nabla|^{s-1-\varepsilon} F_{\leq 1} \big\|_{L_t^{\frac{2q}{q+6}} L_x^{\frac{q}{q-1}}} + \big\| \nabla(\chi_{DoD} \chi_{far}) \big\|_{L_x^3} \big\| |\nabla|^{s-1-\varepsilon} F_{\leq 1} \big\|_{L_t^{\frac{2q}{q+6}} L_x^{\frac{3q}{2q-3}}} \\
&\lesssim \big\| |\nabla|^{s-\varepsilon} F_{\leq 1} \big\|_{L_t^{\frac{2q}{q+6}} L_x^{\frac{q}{q-1}}} \\
&\lesssim \big\| F(u) \big\|_{L_t^{\frac{2q}{q+6}} L_x^1} \\
&\lesssim_u R^{\frac{1}{2} + \frac{3}{q}},
\end{aligned}$$

for any $6 \leq q < \infty$. The claim now follows by taking q sufficiently large depending on β. ☐

We now turn to the most significant region of integration, where $(t, x(t))$ and $(\tau, x(\tau))$ may lie in the domain of dependence of B_R.

Lemma 7.5. *With the assumptions above and $\beta < 1/30$, we have*

(7.15)
$$\begin{aligned}
&\Big| \int_0^\infty \int_{-\infty}^0 \big\langle \nabla \tfrac{\sin(|\nabla| t)}{|\nabla|} \theta(i\nabla) \chi_{DoD} \chi_{near} |\nabla|^{s-1-\varepsilon} F_{\leq 1}(t), \\
&\qquad\qquad \chi_R \nabla \tfrac{\sin(|\nabla| \tau)}{|\nabla|} \theta(i\nabla) \chi_{DoD} \chi_{near} |\nabla|^{s-1-\varepsilon} F_{\leq 1}(\tau) \big\rangle\, d\tau\, dt \\
&+ \int_0^\infty \int_{-\infty}^0 \big\langle \cos(|\nabla| t) \theta(i\nabla) \chi_{DoD} \chi_{near} |\nabla|^{s-1-\varepsilon} F_{\leq 1}(t), \\
&\qquad\qquad \chi_R \cos(|\nabla| \tau) \theta(i\nabla) \chi_{DoD} \chi_{near} |\nabla|^{s-1-\varepsilon} F_{\leq 1}(\tau) \big\rangle\, d\tau\, dt \Big| \lesssim_u R^{-3\beta}.
\end{aligned}$$

Proof. The claim will follow from Proposition 2.6; the hypothesis (2.5) holds by virtue of (7.11) and (7.12). Thus, using this proposition followed by Bernstein's inequality and Proposition 5.1, we obtain

$$\mathrm{LHS}(7.15) \lesssim R^{-1/10} \big\| |\nabla|^{s-1-\varepsilon} F_{\leq 1} \big\|_{L_t^\infty L_x^1}^2 \lesssim R^{-1/10} \|F\|_{L_t^\infty L_x^1}^2 \lesssim_u R^{-1/10}.$$

This completes the proof of the lemma. ☐

We now return to the proof of Theorem 7.1. Recall that it suffices to prove (7.3). This follows for $\beta < \min(\frac{1}{30}, \frac{s-1-\varepsilon}{4})$ by using Lemmas 7.3, 7.4, and 7.5 to estimate (7.5). ☐

Corollary 7.6. *Let u be a global solution to* (1.1) *that is almost periodic modulo symmetries. Recall that*

$$\big\| (u, u_t) \big\|_{L_t^\infty(\mathbb{R}; \dot{H}_x^{s_c} \times \dot{H}_x^{s_c - 1})} < \infty. \tag{7.16}$$

Also assume that

$$\inf_{t\in\mathbb{R}} N(t) \geq 1.$$

Then $\nabla_{t,x} u \in L_t^\infty L_x^2$; *in particular, the energy* $E(u)$ *of the solution is finite. Moreover, there exists* $\beta = \beta(p) > 0$ *so that*

$$\big\| \langle x - x(t)\rangle^\beta P_{\leq 1} \nabla_{t,x} u \big\|_{L_t^\infty L_x^2} \lesssim_u 1. \tag{7.17}$$

Proof. Applying Theorem 7.1 iteratively, finitely many times, we conclude that

$$\nabla_{t,x} u \in L_t^\infty \dot H_x^{s-1} \quad \text{for each} \quad 1 < s \leq s_c. \tag{7.18}$$

To pass from this to finite energy, we follow the strategy used in Theorem 7.1, indeed with some simplifications due to the local nature of the operator ∇ as opposed to $|\nabla|^{s-1-\varepsilon}$. As $P_{\leq 1}$ is also nonlocal, we replace it by $\theta(i\nabla)$, which is almost local.

Note that it suffices to prove (7.17). Indeed, using this to bound the low-frequency part of the solution and using (7.16) and Bernstein's inequality to bound the high frequencies, we deduce that $\nabla_{t,x} u \in L_t^\infty L_x^2$. This renders the first two terms in the energy (1.5) finite. Using Sobolev embedding and interpolation between $u \in L_t^\infty \dot H_x^1$ and $u \in L_t^\infty \dot H_x^{s_c}$, we also see that the potential energy term is finite. Thus, $E(u) < \infty$.

Therefore, it remains to establish (7.17). By time-translation symmetry, it suffices to prove the claim for $t = 0$. By space-translation symmetry, we may also assume $x(0) = 0$. Arguing as we did for (7.3), it suffices to show that

$$\big\| \theta(i\nabla)^2 \nabla_{t,x} u(0) \big\|_{L_x^2(B_R)}^2 \lesssim_u R^{-3\beta} \tag{7.19}$$

uniformly for $R \geq R_0(u)$.

To obtain (7.19) we use the Duhamel formulae (1.9) to write:

$$\begin{aligned} \big\| \theta(i\nabla)^2 \nabla_{t,x} u(0) \big\|_{L_x^2(B_R)}^2 &= -\int_0^\infty \int_{-\infty}^0 \big\langle \nabla \tfrac{\sin(|\nabla| t)}{|\nabla|} \theta(i\nabla)^2 F(t),\ \chi_R \nabla \tfrac{\sin(|\nabla|\tau)}{|\nabla|} \theta(i\nabla)^2 F(\tau) \big\rangle \, d\tau\, dt \\ &\quad - \int_0^\infty \int_{-\infty}^0 \big\langle \cos(|\nabla| t) \theta(i\nabla)^2 F(t),\ \chi_R \cos(|\nabla|\tau) \theta(i\nabla)^2 F(\tau) \big\rangle \, d\tau\, dt, \end{aligned} \tag{7.20}$$

where χ_R is a smooth cutoff function associated to the ball B_R, as previously.

To estimate (7.20), we decompose spacetime in exactly the same manner as in the proof of Theorem 7.1, by introducing χ_{DoD}, χ_{near}, and χ_{far} between the two copies of $\theta(i\nabla)$. Lemmas 7.2, 7.4, and 7.5 continue to hold when $s - 1 - \varepsilon = 0$ and with the replacement of $P_{\leq 1}$ by $\theta(i\nabla)$. In connection with this, we should note that the Bernstein inequalities continue to hold:

$$\|\theta(i\nabla) f\|_{L_x^q(\mathbb{R}^3)} + \|\nabla \theta(i\nabla) f\|_{L_x^q(\mathbb{R}^3)} \lesssim \|f\|_{L_x^p(\mathbb{R}^3)}$$

for all $1 \leq p \leq q \leq \infty$.

The only part of the proof of Theorem 7.1 that needs to change is the proof of Lemma 7.3. Corollary 7.6 thus follows from the following substitute:

Lemma 7.7. *Under the hypotheses of Corollary* 7.6*,*

$$\begin{aligned}(7.21)\qquad &\Big\| \int_0^\infty \nabla \tfrac{\sin(|\nabla| t)}{|\nabla|} \theta(i\nabla) \chi_{DoD}\chi_{far}\theta(i\nabla) F(t)\,dt \Big\|_2 \\ &\qquad + \Big\| \int_0^\infty \cos(|\nabla| t) \theta(i\nabla) \chi_{DoD}\chi_{far}\theta(i\nabla) F(t)\,dt \Big\|_2, \\ &\Big\| \int_{-\infty}^0 \nabla \tfrac{\sin(|\nabla| \tau)}{|\nabla|} \theta(i\nabla) \chi_{DoD}\chi_{far}\theta(i\nabla) F(\tau) d\tau \Big\|_2 \\ &\qquad + \Big\| \int_{-\infty}^0 \cos(|\nabla| \tau) \theta(i\nabla) \chi_{DoD}\chi_{far}\theta(i\nabla) F(\tau) d\tau \Big\|_2 \\ &\qquad\qquad \lesssim_u R^{-1/2-4\beta},\end{aligned}$$

for some $\beta = \beta(p) > 0$*.*

Proof. We argue as in Lemma 7.3. Again, we only present the proof for positive times. To ease notation, we write $\chi = \sqrt{\chi_{DoD}\chi_{far}}$ and $\tilde\chi = \tilde\chi_{DoD}\tilde\chi_{far}$.

Using the Strichartz inequality (with $6 \le q < \infty$) followed by Hölder's inequality, we obtain

$$\begin{aligned}(7.22)\qquad &\Big\| \int_0^\infty \nabla \tfrac{\sin(|\nabla| t)}{|\nabla|} \theta(i\nabla) \chi^2 \theta(i\nabla) F(t)\,dt \Big\|_{L_x^2(\mathbb{R}^3)} \\ &\qquad + \Big\| \int_0^\infty \cos(|\nabla| t) \theta(i\nabla) \chi^2 \theta(i\nabla) F(t)\,dt \Big\|_{L_x^2(\mathbb{R}^3)} \\ &\qquad \lesssim \big\| \nabla \big[\chi^2 \theta(i\nabla) F\big] \big\|_{L_t^{\frac{2q}{q+6}} L_x^{\frac{q}{q-1}}} \\ &\qquad \lesssim \big\| \chi^2 \theta(i\nabla) \nabla F \big\|_{L_t^{\frac{2q}{q+6}} L_x^{\frac{q}{q-1}}} + \|\nabla\chi\|_{L_x^3} \big\| \chi\, \theta(i\nabla) F \big\|_{L_t^{\frac{2q}{q+6}} L_x^{\frac{3q}{2q-3}}} \\ &\qquad \lesssim \big\| \chi \theta(i\nabla) \nabla F \big\|_{L_t^{\frac{2q}{q+6}} L_x^{\frac{q}{q-1}}} + \big\| \chi\, \theta(i\nabla) F \big\|_{L_t^{\frac{2q}{q+6}} L_x^{\frac{3q}{2q-3}}}.\end{aligned}$$

Noting that the convolution kernel associated to $\theta(i\nabla)$ has compact support, we have

$$\big\| \chi\theta(i\nabla)\nabla F(t) \big\|_{L_x^{\frac{q}{q-1}}} \lesssim \big\| \tilde\chi^p \nabla F(t) \big\|_{L_x^{\frac{q}{q-1}}} \text{ and } \big\| \chi\theta(i\nabla) F(t) \big\|_{L_x^{\frac{3q}{2q-3}}} \lesssim \big\| \tilde\chi^p F(t) \big\|_{L_x^{\frac{3q}{2q-3}}}.$$

Thus by Sobolev embedding, Lemma 7.2, Proposition 5.1, Proposition 6.1 combined with (7.10), and (7.18),

$$\begin{aligned}&\text{LHS}(7.22) \\ &\lesssim \sum_{j\ge 0} \Big[\big\|\nabla u\big\|_{L_t^\infty L_x^{\frac{2q}{q-1}}} + \big\| u \big\|_{L_t^\infty L_x^{\frac{6q}{q-3}}} \Big] \|\tilde\chi u\|_{L_t^{\tilde q} L_x^{\frac{3p\tilde q}{2\tilde q - p}} (I_j\times\mathbb{R}^3)}^{\frac{\tilde q (q+6)}{2q}} \|\tilde\chi u\|_{L_t^\infty L_x^p}^{\frac{3p}{2q}} \|\tilde\chi u\|_{L_t^\infty L_x^{\frac{3p}{2}} (I_j\times\mathbb{R}^3)}^{p-\frac{3p}{2q}-\frac{\tilde q(q+6)}{2q}} \\ &\lesssim_u \big\| |\nabla|^{1+\frac{3}{2q}} u \big\|_{L_t^\infty L_x^2} \sum_j (T_j + R)^{-(\frac{2}{3}-\frac{1}{q}-\frac{\tilde q (q+6)}{3pq})\gamma} \\ &\lesssim_u R^{-\frac12 - 4\beta},\end{aligned}$$

where γ is as in Proposition 6.1 and $p/2 < \tilde{q} < \infty$. To obtain the stated decay in R in the last inequality, it suffices to take q sufficiently large (depending on p) and $\tilde{q}$ close to $p/2$. This proves (7.21). □

This finishes the proof of the corollary. □

8. The frequency-cascade solution

In this section, we preclude the frequency-cascade solution described in Theorem 1.8.

Theorem 8.1 (Absence of frequency-cascade solutions). *There are no frequency-cascade solutions to* (1.1) *in the sense of Theorem* 1.8.

Proof. We argue by contradiction. Assume there exists a solution $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ that is a frequency-cascade in the sense of Theorem 1.8. We will prove that this scenario is inconsistent with the conservation of energy.

Indeed, by Corollary 7.6, the energy $E(u)$ is finite. Next, let $0 < M, \eta < 1$ be small constants to be chosen later. By almost periodicity modulo symmetries, there exists $c(\eta)$ sufficiently small so that

$$\|u_{\leq c(\eta)N(t)}\|_{L_t^\infty \dot{H}_x^{s_c}} + \|P_{\leq c(\eta)N(t)} u_t\|_{L_t^\infty \dot{H}_x^{s_c-1}} \leq \eta. \tag{8.1}$$

Now decompose $u = u_{\leq M} + u_{M \leq \cdot \leq c(\eta)N(t)} + u_{\geq c(\eta)N(t)}$. To estimate the very low frequencies of u, we use the full strength of Corollary 7.6. Indeed, by Hölder's inequality and (7.17),

$$\begin{aligned}\|\nabla u_{\leq M}\|_{L_t^\infty L_x^{\frac{4}{2+\beta}}} &+ \|P_{\leq M} u_t\|_{L_t^\infty L_x^{\frac{4}{2+\beta}}} \\ &\lesssim \big\|\langle x - x(t)\rangle^\beta \nabla u_{\leq 1}\big\|_{L_t^\infty L_x^2} + \big\|\langle x - x(t)\rangle^\beta P_{\leq 1} u_t\big\|_{L_t^\infty L_x^2} \\ &\lesssim_u 1.\end{aligned}$$

Thus, by Bernstein's inequality,

$$\|\nabla u_{\leq M}\|_{L_t^\infty L_x^2} + \|P_{\leq M} u_t\|_{L_t^\infty L_x^2} \lesssim_u M^{\frac{3\beta}{4}}. \tag{8.2}$$

To estimate the medium frequencies in the decomposition of u, we use Bernstein's inequality and (8.1):

$$\begin{aligned}\|\nabla u_{M \leq \cdot \leq c(\eta)N(t)}\|_{L_t^\infty L_x^2} &+ \|P_{M \leq \cdot \leq c(\eta)N(t)} u_t\|_{L_t^\infty L_x^2} \\ &\lesssim M^{1-s_c}\big[\|u_{\leq c(\eta)N(t)}\|_{L_t^\infty \dot{H}_x^{s_c}} + \|P_{\leq c(\eta)N(t)} u_t\|_{L_t^\infty \dot{H}_x^{s_c-1}}\big] \\ &\lesssim M^{1-s_c}\eta. \end{aligned} \tag{8.3}$$

We estimate the high frequencies in the decomposition of u similarly:

$$\begin{aligned}\|\nabla u_{\geq c(\eta)N(t)}\|_{L_t^\infty L_x^2} &+ \|P_{\geq c(\eta)N(t)} u_t\|_{L_t^\infty L_x^2} \\ &\lesssim [c(\eta)N(t)]^{1-s_c}\big[\|u_{\geq c(\eta)N(t)}\|_{L_t^\infty \dot{H}_x^{s_c}} + \|P_{\geq c(\eta)N(t)} u_t\|_{L_t^\infty \dot{H}_x^{s_c-1}}\big] \\ &\lesssim_u [c(\eta)N(t)]^{1-s_c}. \end{aligned} \tag{8.4}$$

Putting together (8.2), (8.3), and (8.4), we get

$$\|\nabla u\|_{L_t^\infty L_x^2} + \|u_t\|_{L_t^\infty L_x^2} \lesssim_u M^{\frac{3\beta}{4}} + M^{1-s_c}\eta + [c(\eta)N(t)]^{1-s_c}. \tag{8.5}$$

By Sobolev embedding and interpolating between (8.5) and the fact that $u \in L_t^\infty \dot{H}_x^{s_c}$, we also obtain

$$\|u\|_{L_t^\infty L_x^{p+2}} \lesssim_u \left[M^{\frac{3\beta}{4}} + M^{1-s_c}\eta + [c(\eta)N(t)]^{1-s_c}\right]^{\frac{2}{p+2}}. \tag{8.6}$$

Combining (8.5) and (8.6), we thus get

$$E(u) \lesssim_u \left[M^{\frac{3\beta}{4}} + M^{1-s_c}\eta + [c(\eta)N(t)]^{1-s_c}\right]^2.$$

Taking M small, and then η small depending on M, and then t sufficiently large depending on η (and recalling that for a frequency-cascade solution, $\limsup_{t\to\infty} N(t) = \infty$), we may deduce that the energy, which is conserved, is smaller than any positive constant. Thus $E(u) = 0$ and so $u \equiv 0$. This contradicts the fact that u is a blow-up solution. □

9. The soliton-like solution

In this section, we preclude the soliton-like solution described in Theorem 1.8.

Theorem 9.1 (Absence of solitons)**.** *There are no soliton-like solutions to* (1.1) *in the sense of Theorem* 1.8.

Proof. We argue by contradiction. Assume there exists a solution $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ that is soliton-like in the sense of Theorem 1.8. We will show that this scenario is inconsistent with the Morawetz inequality (1.11).

By Corollary 7.6, the soliton has finite energy; hence, the right-hand side in the Morawetz inequality is finite and so

$$\int_0^T \int_{\mathbb{R}^3} \frac{|u(t,x)|^{p+2}}{|x|}\, dx\, dt \lesssim E(u) \lesssim_u 1, \tag{9.1}$$

for any $T > 0$. On the other hand, by Corollary 3.5 we have concentration of potential energy; that is, there exists $C = C(u)$ so that

$$\int_{t_0}^{t_0+1} \int_{|x-x(t)|\le C} |u(t,x)|^{p+2}\, dx\, dt \gtrsim_u 1,$$

for any $t_0 \in \mathbb{R}$. Translating space so that $x(0) = 0$ and employing finite speed of propagation in the sense of (4.2), we obtain that for $T \ge 1$,

$$\text{LHS(9.1)} \ge \int_0^T \int_{|x-x(t)|\le C} \frac{|u(t,x)|^{p+2}}{|x|}\, dx\, dt \gtrsim_u \int_0^T \frac{dt}{1+t} \gtrsim_u \log(T).$$

Choosing T sufficiently large depending on u, we derive a contradiction to (9.1). □

10. The finite-time blow-up solution

In this section, we preclude the finite-time blow-up scenario described in Theorem 1.8 by showing that such solutions are inconsistent with the conservation of energy.

Theorem 10.1 (Absence of finite-time blow-up solutions)**.** *There are no finite-time blow-up solutions to* (1.1) *in the sense of Theorem* 1.8.

Proof. We argue by contradiction. Assume that there exists a solution $u : I \times \mathbb{R}^3 \to \mathbb{R}$ that is a finite-time blow-up solution in the sense of Theorem 1.8. By the time-reversal and time-translation symmetries, we may assume that the solution blows up as $t \searrow 0 = \inf I$.

First note that $N(t) \to \infty$ as $t \to 0$, for otherwise a subsequential limit of the normalizations $u^{[t]}$ would blow up instantaneously, in contradiction of the local theory. Combining this with (4.2), we deduce that $\lim_{t\to 0} x(t)$ exists. By space-translation symmetry, we may assume that $\lim_{t\to 0} x(t) = 0$.

Next we show that

$$\operatorname{supp} u(t) \cup \operatorname{supp} u_t(t) \subseteq B(0,t) \quad \text{for all} \quad t \in I, \tag{10.1}$$

where $B(0,t)$ denotes the closed ball in $\mathbb{R}^3$ centered at the origin of radius t. Indeed, it suffices to show that

$$\lim_{t\to 0} \int_{t+\varepsilon \le |x| \le \varepsilon^{-1}-t} \tfrac12 \big|\nabla_{t,x} u(t,x)\big|^2 + \tfrac{1}{p+2} |u(t,x)|^{p+2}\, dx = 0 \quad \text{for all} \quad \varepsilon > 0, \tag{10.2}$$

because the energy on the annulus $\{x : t+\varepsilon \le |x| \le \varepsilon^{-1} - t\}$ is finite and does not decrease as $t \to 0$. To obtain (10.2), fix $\varepsilon > 0$. As the parameters $N(t)$ and $x(t)$ satisfy

$$\lim_{t\to 0} N(t) = \infty \quad \text{and} \quad |x(t)| \le |t| + C_u N(t)^{-1} \text{ for all } t \in I,$$

we deduce that for all $\eta > 0$ there exists $t_0 = t_0(\varepsilon, \eta)$ such that for $0 < t < t_0$ we have

$$\{x \in \mathbb{R}^3 :\ t + \varepsilon \le |x| \le \varepsilon^{-1} - t\} \subseteq \{x \in \mathbb{R}^3 :\ |x - x(t)| \ge C(\eta)/N(t)\},$$

where $C(\eta)$ is as in (1.8). Thus by Hölder's inequality and (1.8),

$$\begin{aligned}
&\int_{t+\varepsilon \le |x| \le \varepsilon^{-1}-t} \tfrac12 \big|\nabla_{t,x} u(t,x)\big|^2 + \tfrac{1}{p+2} |u(t,x)|^{p+2}\, dx \\
&\quad \lesssim \varepsilon^{\frac4p - 1} \Big[\big\|\nabla_{t,x} u(t)\big\|^2_{L_x^{\frac{3p}{p+2}}(\{|x-x(t)|\ge C(\eta)/N(t)\})} + \|u(t)\|^{p+2}_{L_x^{\frac{3p}{2}}(\{|x-x(t)|\ge C(\eta)/N(t)\})} \Big] \\
&\quad \lesssim \varepsilon^{\frac4p - 1} \eta^2
\end{aligned}$$

for all $0 < t < t_0$. As η can be made arbitrarily small, this proves (10.2) and hence (10.1).

To continue, by (10.1), Hölder's inequality, and Sobolev embedding we obtain

$$\begin{aligned}
E(u(t)) &= \int_{B(0,t)} \Big(\tfrac12 |\nabla_{t,x} u(t,x)|^2 + \tfrac{1}{p+2} |u(t,x)|^{p+2} \Big)\, dx \\
&\lesssim \Big(\|\nabla_{t,x} u(t)\|^2_{L_x^{\frac{3p}{p+2}}} + \|u(t)\|^{p+2}_{L_x^{\frac{3p}{2}}} \Big) t^{1-\frac4p} \\
&\lesssim_u t^{1-\frac4p}
\end{aligned}$$

for all $t \in I$. In particular, the energy of the solution is finite and converges to zero as the time t approaches the blow-up time 0. Invoking the conservation of energy, we deduce that $u \equiv 0$. This contradicts the fact that u is a blow-up solution. □

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