ON QUASICONFORMAL SELF-MAPPINGS OF THE UNIT DISK SATISFYING POISSON'S EQUATION

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ABSTRACT. Let $\mathcal{QC}(K, g)$ be a family of K -quasiconformal mappings of the open unit disk onto itself satisfying the PDE $\Delta w = g$, $g \in C(\overline{\mathbb{D}})$, $w(0) = 0$. It is proved that $\mathcal{QC}(K, g)$ is a uniformly Lipschitz family. Moreover, if $|g|_\infty$ is small enough, then the family is uniformly bi-Lipschitz. The estimations are asymptotically sharp as $K \rightarrow 1$ and $|g|_\infty \rightarrow 0$, so $w \in \mathcal{QC}(K, g)$ behaves almost like a rotation for sufficiently small K and $|g|_\infty$.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

In this paper $\mathbb{U}$ denotes the open unit disk in $\mathbb{C}$, and S^1 denotes the unit circle. Also, by D and Ω we denote open regions in $\mathbb{C}$. For a complex number $z = x + iy$, its norm is given by $|z| = \sqrt{x^2 + y^2}$. For a real 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

we will consider the matrix norm $|A| = \sup\{|Az| : |z| = 1\}$ and the matrix function $l(A) = \inf\{|Az| : |z| = 1\}$.

A real-valued function u , defined in an open subset D of the complex plane $\mathbb{C}$, is harmonic if it satisfies Laplace's equation:

$$\Delta u(z) := \frac{\partial^2 u}{\partial x^2}(z) + \frac{\partial^2 u}{\partial y^2}(z) = 0 \quad (z = x + iy).$$

A complex-valued function $w = u + iv$ is harmonic if both u and v are real harmonic.

We say that a function $u : D \rightarrow \mathbb{R}$ is *ACL* (absolutely continuous on lines) in the region D , if for every closed rectangle $R \subset D$ with sides parallel to the x and y -axes, u is absolutely continuous on a.e. horizontal and a.e. vertical line in R . Such a function has, of course, partial derivatives u_x, u_y a.e. in D .

The definition carries over to complex-valued functions.

Definition 1.1. A homeomorphism $w : D \rightarrow \Omega$, between open regions $D, \Omega \subset \mathbb{C}$, is K -quasiconformal ($K \geq 1$) (abbreviated K -q.c.) if

- (1) w is *ACL* in D ,
- (2) $|w_{\bar{z}}| \leq k|w_z|$ a.e. ($k = \frac{K-1}{K+1}$).

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Here

$$w_z := \frac{1}{2}(w_x - iw_y) \quad \text{and} \quad w_{\bar{z}} := \frac{1}{2}(w_x + iw_y)$$

are complex partial derivatives (cf. [1], pp. 3, 23–24).

If by $\nabla w(z)$ we denote the formal derivative of $w = u + iv$ at z :

$$\nabla w = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

then condition (2) of Definition 1.1 can be written as

$$(1.1) \quad K^{-1}(|\nabla w|)^2 \leq J_w \leq K(l(\nabla w))^2 \quad \text{a.e. on } D,$$

where $J_w = \det(\nabla u)$ is the Jacobian of w . The above fact follows from the following well-known formulae:

$$J_w(z) = |w_z|^2 - |w_{\bar{z}}|^2, \quad |\nabla w| = |w_z| + |w_{\bar{z}}|, \quad l(\nabla w) = ||w_z| - |w_{\bar{z}}||.$$

Notice that if w is K -quasiconformal, then w^{-1} is K -quasiconformal as well (this follows from (1.1)).

Let P be the Poisson kernel, i.e. the function

$$P(z, e^{i\theta}) = \frac{1 - |z|^2}{|z - e^{i\theta}|^2},$$

and let G be the Green function of the unit disk, i.e. the function

$$(1.2) \quad G(z, \omega) = \frac{1}{2\pi} \log \left| \frac{1 - z\bar{\omega}}{z - \omega} \right| \quad z, \omega \in \mathbb{U}, \quad z \neq \omega.$$

The functions $z \mapsto P(z, e^{i\theta})$, $z \in \mathbb{U}$, and $z \mapsto G(z, \omega)$, $z \in \mathbb{U} \setminus \{\omega\}$ are harmonic.

Let $f : S^1 \rightarrow \mathbb{C}$ be a bounded integrable function on the unit circle S^1 and let $g : U \rightarrow \mathbb{C}$ be continuous. The solution of the equation $\Delta w = g$ in the unit disk satisfying the boundary condition $w|_{S^1} = f \in L^1(S^1)$ is given by

$$(1.3) \quad \begin{aligned} w(z) &= P[f](z) - G[g](z) \\ &:= \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{i\varphi}) f(e^{i\varphi}) d\varphi - \int_{\mathbb{U}} G(z, \omega) g(\omega) dm(\omega), \end{aligned}$$

$|z| < 1$, where $dm(\omega)$ denotes the Lebesgue measure in the plane. It is well known that if f and g are continuous in S^1 and in $\overline{\mathbb{U}}$, respectively, then the mapping $w = P[f] - G[g]$ has a continuous extension $\tilde{w}$ to the boundary, and $\tilde{w} = f$ on S^1 . See [9, pp. 118–120].

We will consider those solutions of the PDE $\Delta w = g$ that are quasiconformal as well and will investigate their Lipschitz character.

Recall that a mapping $w : D \rightarrow \Omega$ is said to be C -Lipschitz ($C > 0$) (c -co-Lipschitz ($c > 0$)) if

$$(1.4) \quad |w(z_1) - w(z_2)| \leq C|z_1 - z_2|, \quad z_1, z_2 \in D,$$

$$(1.5) \quad (c|z_1 - z_2| \leq |w(z_1) - w(z_2)|, \quad z_1, z_2 \in D).$$

A mapping w is bi-Lipschitz if it is Lipschitz and co-Lipschitz.

O. Martio [17] was the first who considered harmonic quasiconformal mappings on the complex plane. Recent papers [10], [12], [14], [21] and [13] bring much light on the topic of quasiconformal harmonic mappings on the plane. See also [11] for the extension of the problem on the space. In [16], the Lipschitz character of q.c. harmonic self-mappings of the unit disk was established with respect to the

hyperbolic metric and this was generalized to an arbitrary domain in [18]. See [27], [28], [26], [30] for additional results concerning the Lipschitz character of harmonic quasiconformal mappings w.r.t. the hyperbolic metric.

The following theorem is a generalization of an analogous theorem for the unit disk due to Pavlović [21] and of an asymptotically sharp version of the Pavlović theorem due to Partyka and Sakan [20] in the case of harmonic quasiconformal mappings.

The following fact is the main result of the paper.

Theorem 1.2. *Let $K \geq 1$ be arbitrary and let $g \in C(\overline{\mathbb{U}})$ and $|g|_\infty := \sup_{w \in \mathbb{U}} |g(w)|$. Then there exist constants $N(K)$ and $M(K)$ with $\lim_{K \rightarrow 1} M(K) = 1$ such that: If w is a K -quasiconformal self-mapping of the unit disk $\mathbb{U}$ satisfying the PDE $\Delta w = g$, with $w(0) = 0$, then for $z_1, z_2 \in \mathbb{U}$:*

$$\left(\frac{1}{M(K)} - \frac{7|g|_\infty}{6} \right) |z_1 - z_2| \leq |w(z_1) - w(z_2)| \leq (M(K) + N(K)|g|_\infty) |z_1 - z_2|.$$

The proof of Theorem 1.2, given in Section 3, depends on the following two propositions:

Proposition 1.3 ([13]). *Let w be a quasiconformal C^2 diffeomorphism from a bounded plane domain D with $C^{1,\alpha}$ boundary onto a bounded plane domain Ω with $C^{2,\alpha}$ boundary. If there exist constants a and b such that*

$$(1.6) \quad |\Delta w| \leq a|\nabla w|^2 + b, \quad z \in D,$$

then w has bounded partial derivatives in D . In particular it is a Lipschitz mapping in D .

Proposition 1.4 ([5, 22, 31, Mori's Theorem]). *If w is a K -quasiconformal self-mapping of the unit disk $\mathbb{U}$ with $w(0) = 0$, then there exists a constant $M_1(K)$, satisfying the condition $M_1(K) \rightarrow 1$ as $K \rightarrow 1$, such that*

$$(1.7) \quad |w(z_1) - w(z_2)| \leq M_1(K) |z_1 - z_2|^{K^{-1}}.$$

See also [2] and [19] for some constants that are not asymptotically sharp.

The mapping $|z|^{-1+K^{-1}}z$ shows that the exponent K^{-1} is optimal in the class of arbitrary K -quasiconformal homeomorphisms.

2. AUXILIARY RESULTS

In this section, we establish some lemmas needed in the proof of the main results.

Lemma 2.1. *Let w be a harmonic function defined on the unit disk and assume that its derivative $v = \nabla w$ is bounded on the unit disk (or equivalently, according to Rademacher's theorem [7], let w be Lipschitz continuous). Then there exists a mapping $A \in L^\infty(S^1)$ defined on the unit circle S^1 such that $\nabla w(z) = P[A](z)$ and for almost every $e^{i\theta} \in S^1$ the relation*

$$(2.1) \quad \lim_{r \rightarrow 1-} \nabla w(re^{i\theta}) = A(e^{i\theta})$$

holds. Moreover the function $f(e^{i\theta}) := w(e^{i\theta})$ is differentiable almost everywhere in $[0, 2\pi]$ and the formula

$$A(e^{i\theta}) \cdot (ie^{i\theta}) = \frac{\partial}{\partial \theta} f(e^{i\theta})$$

holds.

Proof. For the proof of the first statement of the lemma, see, for example, [3, Theorem 6.13 and Theorem 6.39].

Next, since

$$\begin{aligned} \left| \frac{\partial}{\partial \theta} w(re^{i\theta}) \right| &= |r \nabla w(re^{i\theta}) \frac{\partial}{\partial \theta} e^{i\theta}| \leq |r \nabla w(re^{i\theta})| \cdot \left| \frac{\partial}{\partial \theta} e^{i\theta} \right| \\ &\leq \operatorname{ess\,sup}_{\theta} |A(e^{i\theta})| \cdot \left| \frac{\partial}{\partial \theta} e^{i\theta} \right|, \end{aligned}$$

the Lebesgue Dominated Convergence Theorem yields

$$\begin{aligned} f(e^{i\theta}) &= \lim_{r \rightarrow 1} w(re^{i\theta}) \\ &= \lim_{r \rightarrow 1-} \int_{\theta^0}^{\theta} \frac{\partial}{\partial \varphi} w(re^{i\varphi}) d\varphi + f(e^{i\theta_0}) \\ &= \int_{\theta^0}^{\theta} \lim_{r \rightarrow 1} \frac{\partial}{\partial \varphi} w(re^{i\varphi}) d\varphi + f(e^{i\theta_0}) \\ &= \int_{\theta^0}^{\theta} \lim_{r \rightarrow 1} r \nabla w(re^{i\varphi}) \frac{\partial}{\partial \varphi} e^{i\varphi} d\varphi + f(e^{i\theta_0}) \\ &= \int_{\theta^0}^{\theta} A(e^{i\varphi}) \cdot \frac{\partial}{\partial \varphi} e^{i\varphi} d\varphi + f(e^{i\theta_0}). \end{aligned}$$

Differentiating in θ we get

$$\frac{\partial}{\partial \theta} f(e^{i\theta}) = A(e^{i\theta}) \cdot \frac{\partial}{\partial \theta} e^{i\theta} = A(e^{i\theta})(ie^{i\theta})$$

almost everywhere in S^1 . □

Lemma 2.2. *If $f(e^{it}) = e^{i\psi(t)}$, $\psi(2\pi) = \psi(0) + 2\pi$, is a diffeomorphism of the unit circle onto itself, then*

$$(2.2) \quad |f(e^{it}) - f(e^{is})| \leq |\psi'|_{\infty} |e^{it} - e^{is}|,$$

where $|\psi'|_{\infty} = \max\{|\psi'(\tau)| : 0 \leq \tau \leq 2\pi\} = \max\{|\partial_{\tau} f(e^{i\tau})| : 0 \leq \tau \leq 2\pi\}$.

Proof. Take the function

$$h(t) = \frac{|f(e^{it}) - f(e^{is})|}{|e^{it} - e^{is}|}.$$

Then we have

$$(2.3) \quad h(t) = \frac{\sin \frac{\psi(t) - \psi(s)}{2}}{\sin \frac{t-s}{2}}.$$

In order to estimate the maximum of the function h , we found out that the stationary points of it satisfy the equation

$$(2.4) \quad \tan \frac{\psi(t) - \psi(s)}{2} = \tan \frac{t-s}{2} \cdot \psi'(t).$$

Substituting (2.4) into (2.3) we obtain

$$(2.5) \quad h^2(t) = \frac{\left(1 + \tan^2 \frac{\psi(t) - \psi(s)}{2}\right) \psi'^2(t)}{1 + \tan^2 \frac{\psi(t) - \psi(s)}{2} \psi'^2(t)}.$$

Now since

$$2\pi = \psi(2\pi) - \psi(0) = \int_0^{2\pi} \psi'(\tau) d\tau,$$

it follows that $|\psi'|_\infty \geq 1$. If $|\psi'(t)| \leq 1$, then from (2.5) it follows that $|h(t)| \leq 1 \leq |\psi'|_\infty$. If $|\psi'(t)| > 1$, then again employing (2.5) we obtain $|h(t)| \leq |\psi'|_\infty$. This implies the lemma. $\square$

Lemma 2.3. *If $z \in \mathbb{U}$, and*

$$I(z) = \frac{1}{2\pi} \int_{\mathbb{U}} \frac{1 - |\omega|^2}{|z - \omega| \cdot |1 - \bar{z}\omega|} dm(\omega),$$

then

$$(2.6) \quad \frac{1}{2} \leq I(z) \leq \frac{2}{3}.$$

Both inequalities are sharp. Moreover the function $z \mapsto I(z)$ is a radial function and decreasing for $|z| \in [0, 1]$.

Proof. For a fixed z , we introduce the change of variables

$$\frac{z - \omega}{1 - \bar{z}\omega} = \xi,$$

or, what is the same,

$$\omega = \frac{z - \xi}{1 - \bar{z}\xi}.$$

Then

$$\begin{aligned} I &:= \frac{1}{2\pi} \int_{\mathbb{U}} \frac{1 - |\omega|^2}{|z - \omega| \cdot |1 - \bar{z}\omega|} dm(\omega) \\ &= \frac{1}{2\pi} \int_{\mathbb{U}} \frac{1 - |\omega|^2}{|\xi| \cdot |1 - \bar{z}\omega|^2} dm(\omega) \\ &= \frac{1}{2\pi} \int_{\mathbb{U}} \frac{1 - |\omega|^2}{|\xi| \cdot |1 - \bar{z}\omega|^2} \frac{(1 - |z|^2)^2}{|1 - \bar{z}\xi|^4} dm(\xi) \\ &= \frac{1}{2\pi} \int_{\mathbb{U}} \frac{(1 - |\xi|^2)(1 - |z|^2)^3}{|\xi| \cdot |1 - \bar{z}\xi|^6 |1 - \bar{z}\omega|^2} dm(\xi). \end{aligned}$$

Since

$$\begin{aligned} 1 - \bar{z}\omega &= 1 - \bar{z} \frac{z - \xi}{1 - \bar{z}\xi} \\ &= \frac{1 - |z|^2}{1 - \bar{z}\xi}, \end{aligned}$$

we see that

$$I = \frac{1}{2\pi} \int_{\mathbb{U}} \frac{(1 - |z|^2)(1 - |\xi|^2)}{|\xi| \cdot |1 - \bar{z}\xi|^4} dm(\xi).$$

In polar coordinates, we have

$$I = (1 - |z|^2) \int_0^1 (1 - \rho^2) d\rho \frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{|1 - \bar{z}\rho e^{it}|^4}.$$

By Parseval's formula (see [24, Theorem 10.22]), we get

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{|1 - \bar{z}\rho e^{it}|^4} &= \frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{|(1 - \bar{z}\rho e^{it})^2|^2} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=0}^{\infty} (n+1)(\bar{z}\rho)^n e^{nit} \right|^2 dt \\ &= \sum_{n=0}^{\infty} (n+1)^2 |z|^{2n} \rho^{2n}, \end{aligned}$$

whence

$$I = (1 - |z|^2) \sum_{n=0}^{\infty} \frac{2(n+1)^2}{(2n+1)(2n+3)} |z|^{2n}.$$

Now the desired inequality follows from the simple inequality

$$\frac{1}{2} \leq c_n := \frac{2(n+1)^2}{(2n+1)(2n+3)} \leq \frac{2}{3} \quad (n = 0, 1, 2, \dots).$$

Setting $|z|^2 = r$, and $\varphi(r) = I(z)$, we obtain

$$\varphi'(r) = \sum_{n=1}^{\infty} n(c_n - c_{n-1}) r^{n-1}.$$

Since $c_n \leq c_{n-1}$ it follows that φ is decreasing, as desired. $\square$

We need the following well-known propositions.

Proposition 2.4 ([25]). *Let X be an open subset of $\mathbb{R}$, and Ω be a measure space. Suppose that a function $F: X \times \Omega \rightarrow \mathbb{R}$ satisfies the following conditions:*

- (1) *$F(x, \omega)$ is a measurable function of x and ω jointly, and is integrable over ω , for almost all $x \in X$ held fixed.*
- (2) *For almost all $\omega \in \Omega$, $F(x, \omega)$ is an absolutely continuous function of x . (This guarantees that $\partial F(x, \omega)/\partial x$ exists almost everywhere.)*
- (3) *$\partial F/\partial x$ is "locally integrable"; that is, for all compact intervals $[a, b]$ contained in X :*

$$(2.7) \quad \int_a^b \int_{\Omega} \left| \frac{\partial}{\partial x} F(x, \omega) \right| d\omega dx < \infty.$$

Then $\int_{\Omega} F(x, \omega) d\omega$ is an absolutely continuous function of x , and for almost every $x \in X$, its derivative exists and is given by

$$\frac{d}{dx} \int_{\Omega} F(x, \omega) d\omega = \int_{\Omega} \frac{\partial}{\partial x} F(x, \omega) d\omega.$$

The following proposition is well known as well.

Proposition 2.5 ([29, pp. 24–26]). *Let ρ be a bounded (absolutely) integrable function defined on a bounded domain $\Omega \subset \mathbb{C}$. Then the potential type integral*

$$I(z) = \int_{\Omega} \frac{\rho(\omega) dm(\omega)}{|z - \omega|}$$

belongs to the space $C(\mathbb{C})$.

Lemma 2.6. *Let ρ be continuous on the closed unit disc $\overline{\mathbb{U}}$. Then the integral*

$$J(z) = \frac{1}{2\pi} \int_{\mathbb{U}} \log \left| \frac{1 - z\bar{\omega}}{z - \omega} \right| \rho(\omega) dm(\omega)$$

belongs to the space $C^1(\mathbb{U})$. Moreover

$$\nabla J(z) = \frac{1}{2\pi} \int_{\mathbb{U}} \nabla \log \left| \frac{1 - z\bar{\omega}}{z - \omega} \right| \rho(\omega) dm(\omega).$$

Proof. Straightforward calculations yield

$$(2.8) \quad \nabla_z \frac{1}{2\pi} \log \left| \frac{1 - z\bar{\omega}}{z - \omega} \right| = \frac{1}{2\pi} \left(\frac{1}{\omega - z} - \frac{\bar{\omega}}{1 - z\bar{\omega}} \right),$$

and consequently

$$(2.9) \quad |\nabla_z \frac{1}{2\pi} \log \left| \frac{1 - z\bar{\omega}}{z - \omega} \right| = \frac{1}{2\pi} \frac{1 - |\omega|^2}{|z - \omega||\bar{z}\omega - 1|}, \quad z \neq \omega.$$

(Here $\nabla_z \varphi(z, \omega)$ denotes the gradient of the function φ treated as a function of z .) Let $\Omega = \mathbb{U}$, and let μ be the Lebesgue measure of $\mathbb{U}$.

According to Lemma 2.3, condition (2.7) of Proposition 2.4 is satisfied. Now applying Proposition 2.4, and relation (2.8) together with Proposition 2.5, we obtain the desired conclusion. $\square$

Lemma 2.7. *If g is continuous on $\overline{\mathbb{U}}$, then the mapping $G[g]$ has a bounded derivative; i.e., it is Lipschitz continuous and the inequalities*

$$(2.10) \quad |\partial G[g]| \leq \frac{1}{3} |g|_{\infty}$$

and

$$(2.11) \quad |\bar{\partial} G[g]| \leq \frac{1}{3} |g|_{\infty}$$

hold on the unit disk. Moreover $\nabla G[g]$ has a continuous extension to the boundary, and for $e^{i\theta} \in S^1$ it follows that

$$(2.12) \quad \partial G[g](e^{i\theta}) = -\frac{e^{i\theta}}{4\pi} \int_{\mathbb{U}} \frac{1 - |\omega|^2}{|e^{i\theta} - \omega|^2} g(\omega) dm(\omega)$$

and

$$(2.13) \quad \bar{\partial} G[g](e^{i\theta}) = -\frac{e^{i\theta}}{4\pi} \int_{\mathbb{U}} \frac{1 - |\omega|^2}{|e^{i\theta} - \omega|^2} g(\omega) dm(\omega).$$

Finally, for $e^{i\theta} \in S^1$,

$$(2.14) \quad |\partial G[g]| \leq \frac{1}{4} |g|_{\infty}$$

and

$$(2.15) \quad |\bar{\partial} G[g]| \leq \frac{1}{4} |g|_{\infty}.$$

Proof. First of all for $z \neq \omega$ we have

$$\begin{aligned} G_z(z, \omega) &= \frac{1}{4\pi} \left(\frac{1}{\omega - z} - \frac{\bar{\omega}}{1 - z\bar{\omega}} \right) \\ &= \frac{1}{4\pi} \frac{(1 - |\omega|^2)}{(z - \omega)(z\bar{\omega} - 1)} \end{aligned}$$

and

$$G_{\bar{z}}(z, \omega) = \frac{1}{4\pi} \frac{(1 - |\omega|^2)}{(\bar{z} - \bar{\omega})(\bar{z}\omega - 1)}.$$

By Lemma 2.6 the potential-type integral

$$\partial G[g](z) = \frac{1}{4\pi} \int_{\mathbb{U}} \frac{1 - |\omega|^2}{(z - \omega)(z\bar{\omega} - 1)} g(\omega) dm(\omega)$$

exists and belongs to the space $C(\mathbb{U})$.

According to Lemma 2.3 it follows that

$$|\partial G[g]| \leq \frac{1}{4\pi} |g|_{\infty} \int_{\mathbb{U}} \frac{1 - |\omega|^2}{|z - \omega||\bar{z}\omega - 1|} dm(\omega)$$

and

$$|\partial G[g]| \leq \frac{1}{3} |g|_{\infty}.$$

The inequality (2.10) is proved. Similarly we establish (2.11).

According to Lemma 2.5 it follows that

$$(2.16) \quad \partial G[f](z) = \int_{\mathbb{U}} G_z(z, \omega) g(\omega) dm(\omega).$$

Next we have

$$(2.17) \quad \lim_{z \rightarrow e^{i\theta}, z \in \mathbb{D}} G_z(z, \omega) = -\frac{1}{4\pi} \frac{e^{-i\theta}(1 - |\omega|^2)}{|e^{i\theta} - \omega|^2}$$

and

$$(2.18) \quad \lim_{z \rightarrow e^{i\theta}, z \in \mathbb{D}} G_{\bar{z}}(z, \omega) = -\frac{1}{4\pi} \frac{e^{i\theta}(1 - |\omega|^2)}{|e^{i\theta} - \omega|^2}.$$

In order to deduce (2.12) from the last two relations, we use the Vitali theorem (see [6, Theorem 26.C]):

Let X be a measure space with finite measure μ , and let $h_n : X \rightarrow \mathbb{C}$ be a sequence of functions that is uniformly integrable, i.e. such that for every $\varepsilon > 0$ there exists $\delta > 0$, independent of n , satisfying

$$(\dagger) \quad \mu(E) < \delta \implies \int_E |h_n| d\mu < \varepsilon.$$

Now: if $\lim_{n \rightarrow \infty} h_n(x) = h(x)$ a.e., then

$$(\ddagger) \quad \lim_{n \rightarrow \infty} \int_X h_n d\mu = \int_X h d\mu.$$

In particular, if

$$\sup_n \int_X |h_n|^p d\mu < \infty, \quad \text{for some } p > 1,$$

then $(\dagger)$ and $(\ddagger)$ hold.

Hence, to prove (2.12), it suffices to prove that

$$\sup_{z \in \mathbb{U}} \int_{\mathbb{U}} \left(\frac{1 - |\omega|^2}{|z - \omega| \cdot |1 - \bar{z}\omega|} |g(\omega)| \right)^p dm(\omega) < \infty, \quad \text{for } p = 3/2.$$

In order to prove this inequality, we proceed as in the case of Lemma 2.3. We obtain

$$\begin{aligned} I_{p,g}(z) &= \int_{\mathbb{U}} \left(\frac{1 - |\omega|^2}{|z - \omega| \cdot |1 - \bar{z}\omega|} |g(\omega)| \right)^p dm(\omega) \\ &\leq |g|_{\infty}^p \int_{\mathbb{U}} \left(\frac{1 - |\omega|^2}{|z - \omega| \cdot |1 - \bar{z}\omega|} \right)^p dm(\omega) \\ &= |g|_{\infty}^p \int_{\mathbb{U}} \frac{(1 - |z|^2)^{2-p} (1 - |\omega|^2)^p}{|\xi|^p |1 - \bar{z}\xi|^4} dm(\xi) \\ &\leq |g|_{\infty}^{3/2} (1 - |z|^2)^{1/2} \int_0^1 \rho^{-1/2} (1 - \rho^2)^{3/2} d\rho \int_0^{2\pi} |1 - \bar{z}\rho e^{i\varphi}|^{-4} d\varphi \\ &\leq |g|_{\infty}^{3/2} (1 - |z|^2)^{1/2} \int_0^1 \rho^{-1/2} (1 - \rho^2)^{3/2} (1 - |z|\rho)^{-3} d\rho. \end{aligned}$$

Now the desired result follows from the elementary inequality

$$\int_0^1 \rho^{-1/2} (1 - \rho^2)^{3/2} (1 - |z|\rho)^{-3} d\rho \leq C(1 - |z|^2)^{-1/2}.$$

This proves (2.12). Similarly we prove (2.13). The inequalities (2.14) and (2.15) follow from (2.12) and (2.13) and Lemma 2.3. $\square$

A mapping $w : D \rightarrow \Omega$ is *proper* if the preimage of every compact set in Ω is compact in D . In the case where $D = \Omega = \mathbb{U}$, the mapping w is proper if and only if $|w(z)| \rightarrow 1$ as $|z| \rightarrow 1$.

Lemma 2.8 (The main lemma). *Let w be a solution of the PDE $\Delta w = g$ that maps the unit disk onto itself properly. In addition let w be Lipschitz continuous. Then there exist for a.e. $t = e^{i\theta} \in S^1$:*

$$(2.19) \quad \nabla w(t) := \lim_{r \rightarrow 1-} \nabla w(rt)$$

and

$$(2.20) \quad J_w(t) := \lim_{r \rightarrow 1-} J_w(re^{i\theta}),$$

and the following relation,

$$\begin{aligned} (2.21) \quad J_w(t) &= \psi'(\theta) \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(t) - f(e^{i\varphi})|^2}{|t - e^{i\varphi}|^2} d\varphi \\ &\quad + \psi'(\theta) \int_0^1 r \left(\frac{1}{2\pi} \int_0^{2\pi} P(re^{i\varphi}, t) \langle g(rt), f(t) \rangle d\varphi \right) dr, \end{aligned}$$

holds. Here ψ is defined by

$$e^{i\psi(\theta)} := f(e^{i\theta}) = w|_{S^1}(e^{i\theta}).$$

If w is biharmonic ($\Delta\Delta w = 0$), then we have

$$(2.22) \quad \begin{aligned} J_w(t) &= \psi'(\theta) \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(t) - f(e^{i\varphi})|^2}{|t - e^{i\varphi}|^2} d\varphi \\ &\quad + \frac{\psi'(\theta)}{2} \int_0^1 \langle g(rt), f(t) \rangle dr, \quad t \in S^1. \end{aligned}$$

For an arbitrary continuous g and $|g|_\infty = \max_{|z| \leq 1} |g(z)|$, the inequality

$$(2.23) \quad |J_w(t) - \psi'(\theta) \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(t) - f(e^{i\theta})|^2}{|t - e^{i\theta}|^2} d\theta| \leq \frac{\psi'(\theta)|g|_\infty}{2}, \quad t \in S^1$$

holds.

Proof. First of all, according to Lemma 2.7, $G[g]$ has a bounded derivative, and there exists the function $\nabla G[g](e^{i\theta})$, $e^{i\theta} \in S^1$, which is continuous and satisfies the limit relation

$$\lim_{z \rightarrow e^{i\theta}, z \in \mathbb{D}} \nabla G[g](z) = \nabla G[g](e^{i\theta}).$$

Since $w = P[f] - G[g]$ has bounded derivative, from Lemma 2.1, it follows that there exists

$$\lim_{r \rightarrow 1-} \nabla P[f](re^{i\theta}) = \nabla P[f](e^{i\theta}).$$

Thus $\lim_{r \rightarrow 1-} \nabla w(re^{i\theta}) = \nabla w(e^{i\theta})$.

It follows that the mapping $\chi: \chi(\theta) = f(e^{i\theta}) := f(t)$, $t \in S^1$, defines the outer normal vector field $\mathbf{n}_\chi$ almost everywhere in S^1 at the point $\chi(\theta) = f(e^{i\theta}) = e^{i\psi(\theta)} = (\chi_1, \chi_2)$ by the formula:

$$(2.24) \quad \mathbf{n}_\chi(\chi(\theta)) = \psi'(\theta) \cdot f(e^{i\theta}).$$

Let $\varpi(r, \theta) := w(re^{i\theta})$. According to Lemma 2.1, we obtain

$$(2.25) \quad \lim_{r \rightarrow 1-} \varpi_\theta(r, \theta) = \chi'(\theta).$$

On the other hand, for almost every $\theta \in S^1$ we have

$$\frac{\chi_j(\theta) - \varpi_j(r, \theta)}{1 - r} = \varpi_r(\rho_{j,r,\theta}, \theta),$$

where $r < \rho_{j,r,\theta} < 1$, $j = 1, 2$. Thus we have

$$(2.26) \quad \lim_{r \rightarrow 1-} \varpi_{j_r}(r, \theta) = \lim_{r \rightarrow 1-} \frac{\chi_j(\theta) - \varpi_j(r, \theta)}{1 - r}, \quad j \in \{1, 2\}.$$

Let p denote polar coordinates, i.e. $p(r, \theta) = re^{i\theta}$.

We derive

$$\begin{aligned}
 \lim_{r \rightarrow 1-} J_{w \circ p}(r, \theta) &= \lim_{r \rightarrow 1-} \left\langle \frac{\chi - P[f]}{1-r}, \psi'(\theta) \cdot f(e^{i\theta}) \right\rangle + \Lambda \\
 &= \lim_{r \rightarrow 1-} \frac{1}{2\pi} \int_0^{2\pi} \frac{1+r}{|e^{i\theta} - re^{i\varphi}|^2} \langle f(e^{i\theta}) - f(e^{i\varphi}), \psi'(\theta) \cdot f(e^{i\theta}) \rangle d\varphi + \Lambda \\
 (2.27) \quad &= \lim_{r \rightarrow 1-} \psi'(\theta) \frac{1}{2\pi} \int_0^{2\pi} \frac{1+r}{|e^{i\theta} - re^{i\varphi}|^2} \langle f(e^{i\theta}) - f(e^{i\varphi}), f(e^{i\theta}) \rangle d\varphi + \Lambda \\
 &= \psi'(\theta) \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} d\varphi + \Lambda,
 \end{aligned}$$

where

$$\Lambda = \lim_{r \rightarrow 1-} \left\langle \frac{G[g]}{1-r}, -i\chi_\theta \right\rangle.$$

In order to estimate Λ , observe first that

$$\begin{aligned}
 \lim_{z \rightarrow e^{i\theta}, z \in \mathbb{D}} \frac{G(z, \omega)}{1-|z|} &= \lim_{z \rightarrow e^{i\theta}, z \in \mathbb{D}} \frac{G(z, \omega) - G(e^{i\theta}, \omega)}{1-|z|} \\
 (2.28) \quad &= \lim_{r \rightarrow 1-} \frac{G(re^{i\theta}, \omega) - G(e^{i\theta}, \omega)}{1-r} = - \frac{\partial G(re^{i\theta}, \omega)}{\partial r} \Big|_{r=1}.
 \end{aligned}$$

Since

$$\frac{\partial G(re^{i\theta}, \omega)}{\partial r} = z_r G_z(re^{i\theta}, \omega) + \bar{z}_r G_{\bar{z}}(re^{i\theta}, \omega), \quad z_r = e^{i\theta}, \bar{z}_r = e^{-i\theta},$$

using (2.17) and (2.18) we obtain

$$(2.29) \quad \lim_{z \rightarrow e^{i\theta}, z \in \mathbb{D}} \frac{G(z, \omega)}{1-|z|} = \frac{1}{2\pi} P(e^{i\theta}, \omega).$$

On the other hand we have

$$\begin{aligned}
 (2.30) \quad &\frac{1}{2\pi} \int_{\mathbb{U}} P(\omega, e^{i\theta}) \langle g(\omega), f(e^{i\theta}) \rangle dm(\omega) \\
 &= \int_0^1 r \left(\frac{1}{2\pi} \int_0^{2\pi} P(re^{i\varphi}, e^{i\theta}) \langle g(re^{i\varphi}), f(e^{i\theta}) \rangle d\varphi \right) dr.
 \end{aligned}$$

Next, we have

$$(2.31) \quad J_{w \circ p}(r, \theta) = r J_w(re^{i\theta}).$$

Combining (2.27), (2.29), (2.30) and (2.31) we obtain (2.21). Relations (2.22) and (2.23) follow from (2.21) and (1.3). If w is biharmonic, then g is harmonic and thus

$$\frac{1}{2\pi} \int_0^{2\pi} P(re^{i\varphi}, e^{i\theta}) \langle g(re^{i\varphi}), f(e^{i\theta}) \rangle d\varphi = \langle g(r^2 e^{i\theta}), f(e^{i\theta}) \rangle.$$

This yields relation (2.22). $\square$

Lemma 2.9. *If $x \geq 0$ is a solution of the inequality $x \leq ax^\alpha + b$, where $a \geq 1$ and $0 \leq a\alpha < 1$, then*

$$(2.32) \quad x \leq \frac{a+b-\alpha a}{1-\alpha a}.$$

Observe that for $\alpha = 0$, (2.32) coincides with $x \leq a + b$, i.e. $x \leq ax^\alpha + b$.

Proof. We will use Bernoulli's inequality: $x \leq ax^\alpha + b = a(1 + x - 1)^\alpha + b \leq a(1 + \alpha(x - 1)) + b$. Relation (2.32) now easily follows. $\square$

3. THE MAIN RESULTS

Theorem 3.1. *Let $g \in C(\overline{\mathbb{U}})$. The family $\mathcal{QC}(K, g)$ of K -quasiconformal ($K \geq 1$) self-mappings of the unit disk $\mathbb{U}$ satisfying the PDE $\Delta w = g$, $w(0) = 0$, is uniformly Lipschitz; i.e., there is a constant $M' = M'(K, g)$ satisfying:*

$$(3.1) \quad |w(z_1) - w(z_2)| \leq M'|z_1 - z_2|, \quad z_1, z_2 \in \mathbb{U}, \quad w \in \mathcal{QC}(K, g).$$

Moreover $M'(K, g) \rightarrow 1$ as $K \rightarrow 1$ and $|g|_\infty \rightarrow 0$.

In Remark 3.7 below a quantitative bound of $M'(K, g)$ is given.

Proof. Combining Proposition 1.3 and Lemma 2.8, in the special case where the range of a function is the unit disk, we obtain that there exist ∇w and J_w almost everywhere in S^1 , and the following inequality,

$$(3.2) \quad J_w(t) \leq \psi'(t) \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\varphi}) - f(e^{it})|^2}{|e^{i\varphi} - e^{it}|^2} d\varphi + \frac{|g|_\infty}{2} \right),$$

holds.

Now from

$$|\nabla w(re^{i\theta})|^2 \leq K J_w(re^{i\theta}),$$

we obtain

$$(3.3) \quad \lim_{r \rightarrow 1-} |\nabla w(re^{i\theta})|^2 \leq \lim_{r \rightarrow 1-} K J_w(re^{i\theta})$$

almost everywhere in $[0, 2\pi]$. From Lemma 2.1, we deduce that

$$(3.4) \quad \lim_{r \rightarrow 1-} \frac{\partial(w(re^{i\theta}))}{\partial\theta} = \frac{\partial f(e^{i\theta})}{\partial\theta} = \psi'(t)e^{i\psi(t)}$$

almost everywhere in $[0, 2\pi]$. Since

$$\frac{\partial w \circ S}{\partial\theta}(r, \theta) = ru'(re^{i\theta})(ie^{i\theta}),$$

using (3.4) we obtain that

$$(3.5) \quad \psi'(t) \leq \lim_{r \rightarrow 1} |\nabla w(re^{i\theta})|.$$

From (3.2)-(3.5) we infer that

$$|\nabla w(e^{i\theta})|^2 \leq K |\nabla w(e^{i\theta})| \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\varphi}) - f(e^{i\theta})|^2}{|e^{i\varphi} - e^{i\theta}|^2} d\varphi + \frac{|g|_\infty}{2} \right),$$

i.e.

$$(3.6) \quad |\nabla w(e^{i\theta})| \leq K \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\varphi}) - f(e^{i\theta})|^2}{|e^{i\varphi} - e^{i\theta}|^2} d\varphi + \frac{|g|_\infty}{2} \right).$$

Let

$$M = \operatorname{ess\,sup}_{0 \leq \tau \leq 2\pi} |\nabla w(e^{i\tau})|.$$

According to Lemma 2.2 and to relation (3.5) we obtain

$$(3.7) \quad |f(e^{i\varphi}) - f(e^{i\theta})| \leq M |e^{i\varphi} - e^{i\theta}|.$$

Let $\mu = K^{-1}$, $\gamma = -1 + K^{-2}$, and let $\nu = 1 - 1/K$. Let $\varepsilon > 0$. Then there exists θ_ε such that

$$M(1 - \varepsilon) \leq |\nabla w(e^{i\theta_\varepsilon})|.$$

Now applying relation (3.6) and using (1.7), we obtain

$$\begin{aligned} (1 - \varepsilon)M &\leq K \left(M^\nu \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta_\varepsilon} - e^{i\varphi}|^\gamma \frac{|f(e^{i\theta_\varepsilon}) - f(e^{i\varphi})|^{2-\nu}}{|e^{i\theta_\varepsilon} - e^{i\varphi}|^{\mu^2+\mu}} d\varphi + \frac{|g|_\infty}{2} \right) \\ &\leq K M^\nu M_1(K)^{1+\mu} \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta_\varepsilon} - e^{i\varphi}|^\gamma d\varphi + K \frac{|g|_\infty}{2} \\ &\leq M_2(K) M^\nu + \frac{K|g|_\infty}{2}, \end{aligned}$$

where

$$M_2(K) = K M_1(K)^{1+\mu} \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta_\varepsilon} - e^{i\varphi}|^\gamma d\varphi.$$

Letting $\varepsilon \rightarrow 0$ we obtain

$$(3.8) \quad M \leq M_2(K) M^\nu + \frac{K|g|_\infty}{2}.$$

From (3.8) we obtain

$$(3.9) \quad M \leq C_0 := \left(M_2(K) + \frac{K|g|_\infty}{2} \right)^{1/(1-\nu)} = \left(M_2(K) + \frac{K|g|_\infty}{2} \right)^K.$$

From Lemma 2.9, if

$$M_1(K)^{1+\mu} \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta_\varepsilon} - e^{i\varphi}|^\gamma d\varphi < \frac{1}{K-1}$$

and $g \neq 0$, we obtain

$$(3.10) \quad M \leq C_1 := \frac{M_2(K) + K|g|_\infty/2 - \nu M_2(K)}{1 - \nu M_2(K)}.$$

Let $C_2 := \min\{C_0, C_1\}$.

If $g \equiv 0$, then by (3.9) we get

$$(3.11) \quad M \leq C_2 := (M_2(K))^{1/(1-\mu)}.$$

To continue observe that $w - G[g]$ is harmonic. Thus

$$|\nabla w(z) - \nabla G(z)| \leq \operatorname{ess\,sup}_{0 \leq \tau \leq 2\pi} |\nabla w(e^{i\tau}) - \nabla G[g](e^{i\tau})|.$$

According to Lemma 2.3 and Lemma 2.7 it follows that:

$$|\nabla w(z)| \leq \operatorname{ess\,sup}_{0 \leq \tau \leq 2\pi} |\nabla w(e^{i\tau})| + \frac{2}{3}|g|_\infty + \frac{1}{2}|g|_\infty.$$

Therefore the inequality (3.1) does hold for

$$(3.12) \quad M' = C_2 + \frac{7}{6}|g|_\infty.$$

Using (1.7), it follows that

$$\lim_{|g|_\infty \rightarrow 0, K \rightarrow 1} M'(K) = 1.$$

□

Lemma 3.2. *If w is a K -q.c. self-mapping of the unit disk satisfying the PDE $\Delta w = g$ and $w(0) = 0$, $w|_{S^1}(e^{i\theta}) = f(e^{i\theta}) = e^{i\psi(\theta)}$, $g \in C(\overline{\mathbb{D}})$, then for almost every $\theta \in [0, 2\pi]$ the relation*

$$(3.13) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} d\varphi \leq K\psi'(\theta) + \frac{|g|_\infty}{2}$$

holds.

Proof. From (2.23) it follows that

$$(3.14) \quad \frac{J_w(e^{i\theta})}{\psi'(\theta)} \geq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} d\varphi - \frac{|g|_\infty}{2}.$$

Using Lemma 2.1 we obtain

$$(3.15) \quad \psi'(\theta) = \left| \frac{\partial f(e^{i\theta})}{\partial \theta} \right| = \left| \lim_{r \rightarrow 1-} \frac{\partial w(re^{i\theta})}{\partial \theta} \right|.$$

On the other hand,

$$(3.16) \quad \frac{\partial w(re^{i\theta})}{\partial \theta} = izw_z(re^{i\theta}) - i\bar{z}w_{\bar{z}}(re^{i\theta}) \quad (z = re^{i\theta}).$$

Therefore

$$(3.17) \quad \left| \lim_{r \rightarrow 1-} \frac{\partial w(re^{i\theta})}{\partial \theta} \right| \geq ||w_z(t)| - |w_{\bar{z}}(t)|| = l(\nabla w(t)) \quad (t = e^{i\theta}).$$

As w is K -q.c., according to (1.1) it follows that

$$(3.18) \quad \frac{J_w(t)}{(l(\nabla w(t)))^2} \leq K.$$

Combining (3.14)-(3.18) we obtain (3.13). $\square$

Lemma 3.3. *Under the conditions and notation of Lemma 3.2, there exists a function $m_1(K)$ such that $\lim_{K \rightarrow 1} m_1(K) = 1$ and*

$$(3.19) \quad m(K) := \max \left\{ m_1(K) - \frac{|g|_\infty}{4}, \frac{4 - 5|g|_\infty}{8} \right\} \leq K\psi'(\theta), \text{ for a.e. } \theta \in [0, 2\pi].$$

Proof. Applying (1.7) to the mapping w^{-1} , we obtain

$$|f(z) - f(w)| \geq M_1(K)^{-K} |z_1 - z_2|^K.$$

Now using relation (3.13) we obtain

$$(3.20) \quad \begin{aligned} K\psi'(\theta) &\geq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} d\varphi - \frac{|g|_\infty}{2} \\ &\geq M_1(K)^{-2K} \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta} - e^{i\varphi}|^{2K-2} d\varphi - \frac{|g|_\infty}{2} \\ &= m_1(K) - \frac{|g|_\infty}{2}, \end{aligned}$$

where

$$m_1(K) = M_1(K)^{-2K} \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta} - e^{i\varphi}|^{2K-2} d\varphi.$$

Let us prove the second part of the inequality (3.19). Since $w(0) = 0$ we infer that $P[f](0) = -G[g](0)$. Thus

$$P[f](0) = \int_{\mathbb{U}} G(0, \omega) g(\omega) dm(\omega);$$

i.e., in polar coordinates,

$$P[f](0) = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} r \log \frac{1}{r} g(\omega) dm(\omega).$$

Hence

$$|P[f](0)| \leq |g|_{\infty} \int_0^1 r \log \frac{1}{r} dr = \frac{|g|_{\infty}}{4}.$$

Next we have

$$\begin{aligned} K\psi'(\theta) &\geq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} d\varphi - \frac{|g|_{\infty}}{2} \\ &\geq \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \left(1 - \operatorname{Re} \overline{f(e^{i\theta})} f(e^{i\varphi})\right) d\varphi - \frac{|g|_{\infty}}{2} \\ (3.21) \quad &\geq \frac{1 - |P[f](0)|}{2} - \frac{|g|_{\infty}}{2} \\ &\geq \frac{4 - 5|g|_{\infty}}{8}. \end{aligned}$$

Combining (3.20) and (3.21) we obtain (3.19). $\square$

Theorem 3.4. *If w is a K -q.c. orientation-preserving self-mapping of the unit disk satisfying the PDE $\Delta w = g$, $w(0) = 0$, $g \in C(\overline{\mathbb{U}})$, then for*

$$m(K) = \max \left\{ M_1(K)^{-2K} \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta} - e^{i\varphi}|^{2K-2} d\varphi - \frac{|g|_{\infty}}{4}, \frac{4 - 5|g|_{\infty}}{8} \right\},$$

the inequality

$$(3.22) \quad l(\nabla w) \geq \frac{m(K)}{K^2} - \frac{7|g|_{\infty}}{6},$$

where $l(\nabla w(z)) = \min\{|\nabla w(z)t| : |t| = 1\}$, holds.

Proof. Assume, as we may, that

$$(3.23) \quad \frac{m(K)}{K^2} - \frac{|g|_{\infty}}{2} \geq \left(\frac{m(K)}{K^2} - \frac{7|g|_{\infty}}{6} \right) \geq 0.$$

From (3.19) and the definition of quasiconformality we deduce that

$$\frac{m(K)}{K^2} \leq \frac{\psi'(\theta)}{K} \leq \frac{|\nabla w(e^{i\theta})|}{K} \leq l(\nabla w);$$

i.e.,

$$\frac{m(K)}{K^2} \leq |w_z| - |w_{\bar{z}}|$$

almost everywhere on the unit circle.

According to relations (2.14) and (2.15) we obtain

$$(3.24) \quad \frac{m(K)}{K^2} - \frac{|g|_\infty}{2} \leq |P[f]_z| - |P[f]_{\bar{z}}|$$

almost everywhere on the unit circle.

To continue observe that, as w is q.c., it follows that f is a homeomorphism. Hence by the Choquet-Radó-Kneser theorem, $P[f]$ is a diffeomorphism (see [15], [4] or [23]).

Thus $h := P[f]$ is a harmonic diffeomorphism. According to the Heinz theorem ([8])

$$|h_z|^2 + |h_{\bar{z}}|^2 \geq \frac{1}{\pi^2},$$

which, in view of the fact that $|h_z| \geq |h_{\bar{z}}|$, implies that

$$|h_z| \geq \frac{\sqrt{2}}{2\pi}.$$

Thus the functions

$$a(z) := \frac{\overline{h_{\bar{z}}}}{h_z}, \quad b(z) := \frac{1}{h_z} \left(\frac{m(K)}{K^2} - \frac{|g|_\infty}{2} \right)$$

are holomorphic and bounded on the unit disk. As $|a| + |b|$ is bounded on the unit circle by 1 (see (3.23) and (3.24)), it follows that it is bounded on the whole unit disk by 1 because

$$|a(z)| + |b(z)| \leq P[|a|_{S^1}](z) + P[|b|_{S^1}](z), \quad z \in \mathbb{U}.$$

This in turn implies that for every $z \in \mathbb{U}$,

$$(3.25) \quad l(\nabla h) \geq \frac{m(K)}{K^2} - \frac{|g|_\infty}{2}.$$

By (2.10) and (2.11) we obtain

$$(3.26) \quad l(\nabla w) \geq \frac{m(K)}{K^2} - \frac{1}{2}|g|_\infty - \frac{2}{3}|g|_\infty.$$

□

Having in mind the fact that $l(\nabla w(z)) = |\nabla w^{-1}(w(z))|^{-1}$, and putting Theorem 3.1 and Theorem 3.4 together we obtain:

Theorem 3.5. *Let $\mathcal{QC}(K, g)$ be the family of orientation-preserving K -q.c. self-mappings of the unit disk satisfying the equation $\Delta w = g$ and $w(0) = 0$. Then for $|g|_\infty$ small enough (for example if $|g|_\infty \leq \frac{12}{15+28K^2}$) the family $\mathcal{QC}(K, g)$ is uniformly bi-Lipschitz; i.e., there exists $M_0(K, g) \geq 1$ such that*

$$M_0(K, g)^{-1} \leq \frac{|w(z_1) - w(z_2)|}{|z_1 - z_2|} \leq M_0(K, g), \quad w \in \mathcal{QC}(K, g), \quad \text{for } z_1, z_2 \in \mathbb{U}, \quad z_1 \neq z_2.$$

Moreover

$$\lim_{|g|_\infty \rightarrow 0, K \rightarrow 1} M_0(K, g) = 1.$$

Example 3.6. Let $w(z) = |z|^\alpha z$, with $\alpha > 1$. Then w is a twice differentiable $(1 + \alpha)$ -quasiconformal self-mapping of the unit disk. Moreover

$$\Delta w = \alpha(2 + \alpha) \frac{|z|^\alpha}{\bar{z}} = g.$$

Thus $g = \Delta w$ is continuous and bounded by $\alpha(2 + \alpha)$. However w is not co-Lipschitz (i.e. it does not satisfy (1.5)), because $l(\nabla w)(0) = |w_z(0)| - |w_{\bar{z}}(0)| = 0$. This means that the condition “ $|g|_\infty$ is small enough” in Theorem 3.5 cannot be replaced by the condition “ g is arbitrary”.

Remark 3.7. Let $\mathcal{QC}_K(\mathbb{U})$ be the family of K -quasiconformal self-mappings of the unit disk. Let $M_1(K)$ be Mori’s constant:

$$M_1(K) = \inf \{M : |f(z_1) - f(z_2)| \leq M|z_1 - z_2|^{1/K}, z_1, z_2 \in \mathbb{U}, f \in \mathcal{QC}_K(\mathbb{U}), f(0) = 0\}.$$

In [22] it is proved that

$$M_1(K) \leq 16^{1-1/K} \min \left\{ \left(\frac{23}{8} \right)^{1-1/K}, (1 + 2^{3-2K})^{1/K} \right\}.$$

Since for $\alpha > -1$,

$$\frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta} - e^{i\varphi}|^\alpha d\varphi = \frac{2^{\alpha+1}}{\pi} \frac{\sqrt{\pi} \Gamma[\frac{1+\alpha}{2}]}{\alpha \Gamma[\frac{\alpha}{2}]},$$

our proofs, in the case of harmonic mappings ($g \equiv 0$), yield the following estimates for the co-Lipschitz constant

$$(3.27) \quad m_2 := \frac{2^{2K-2} \Gamma[K - 1/2]}{\sqrt{\pi}(K^3 - K^2) \Gamma[K - 1] M_1(K)^{2K}},$$

which is

$$\geq \frac{1}{K^2 M_1(K)^{2K}} \geq \frac{46^2}{K^2 46^{2K}},$$

and therefore is better than the corresponding constant

$$(3.28) \quad m_1 := \frac{2^{K(1-K^2)(3+1/K)/2}}{K^{3K+1}(K^2 + K - 1)^{3K}}$$

obtained in the paper [20] for every K (see the appendix below).

Similarly we obtain the following estimate for the Lipschitz constant (see (3.9) and (3.12)):

$$M' = \left(K M_1(K)^{1+1/K} \left(\frac{2^{K-2} \Gamma[K^{-2}/2]}{\sqrt{\pi}(K^{-2} - 1) \Gamma[(K^{-2} - 1)/2]} \right) + \frac{K|g|_\infty}{2} \right)^K + \frac{7|g|_\infty}{6}.$$

The last constant (if $g \equiv 0$) is not comparable with the corresponding constant

$$K^{3K+1} 2^{5(K-1/K)/2}$$

obtained in the same paper [20] (it is better if K is large enough but it is not for K close to 1). It seems that in the proof of Theorem 3.1 there is some small place for improvement of M' (taking $\nu \neq 1 - K^{-1}$).

3.1. Appendix. Let us prove that $m_2 \geq m_1$, where m_1 and m_2 are defined in (3.27) and (3.28). Since $(3 \cdot (3^2 + 3 - 1))^{3/2} > 46$, the inequality follows directly if $K \geq 3$.

Now assume that $1 \leq K \leq 3$. First of all we have

$$\frac{46^2}{K^2 46^{2K}} - \frac{2^{K(1-K^2)(3+1/K)/2}}{K^{3K+1}(K^2 + K - 1)^{3K}} \geq \frac{1}{K^2} \left(46^{2-2K} \left(1 - \frac{46^{2K-2} \cdot 2^{2(1-K^2)}}{K^8} \right) \right).$$

Therefore, the inequality

$$(3.29) \quad 46^{2K-2} \cdot 2^{2(1-K^2)} \leq K^8$$

implies $m_2 \geq m_1$.

Let $K \leq 2$. Then $\frac{46}{2^{1+K}} < 16 = 2^4$. By Bernoulli's inequality, $2^{K-1} = (1+1)^{K-1} \leq 1+K-1 = K$ for $K \leq 2$. This yields (3.29).

Now assume that $2 \leq K \leq 3$. Then

$$\frac{46}{2^{1+K}} < e^2.$$

Thus

$$\left(\frac{46}{2^{1+K}} \right)^{K-1} \leq e^{2(K-1)}.$$

Therefore, if we prove

$$e^{K-1} \leq K^2 \text{ for } 2 \leq K \leq 3,$$

we will prove the inequality $m_2 \geq m_1$ completely.

Let $x = K - 1$. Then

$$\begin{aligned} K^2 - e^{K-1} &= 1 + 2x + x^2 - 1 - x - x^2/2 - x^3/3! - x^4/4! - \dots \\ &= x(1 + x/2 - x^2/3! - x^3/4! - \dots) \\ &\geq x(1 - x^3/4! - \dots) \\ &\geq x(1 - 0.5(e^2 - 1 - 2 - 2^2/2 - 2^3/6)) > x/2, \end{aligned}$$

as desired.

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