THE BERGMAN PROJECTION AND WEIGHTED C^k ESTIMATES FOR THE CANONICAL SOLUTION TO THE $\bar{\partial}$ PROBLEM ON NON-SMOOTH DOMAINS

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ABSTRACT. We apply integral representations for functions on non-smooth strictly pseudoconvex domains, the Henkin-Leiterer domains, to derive weighted C^k -estimates for the component of a given function, f , which is orthogonal to holomorphic functions in terms of C^k -norms of $\bar{\partial}f$. The weights are powers of the gradient of the defining function of the domain.

1. INTRODUCTION

Let X be an n -dimensional complex manifold, equipped with a Hermitian metric, and let $D \Subset X$ be a strictly pseudoconvex domain with defining function r . We allow for singularities in the boundary, ∂D , of D by permitting the possibility that dr vanishes at points on ∂D . Such domains were first studied by Henkin and Leiterer in [4], and we therefore refer to them as Henkin-Leiterer domains.

We shall make the additional assumption that r is a Morse function. Let U be a neighborhood of ∂D . Then

$$U \cap D = \{x \in U : r(x) < 0\},$$

with r having only non-degenerate critical points on U . We have

$$\partial D = \{x : r(x) = 0\},$$

and we can assume that there are finitely many critical points on bD , and none on $U \setminus bD$.

In [3], Lieb and the author studied the Bergman projection on Henkin-Leiterer domains in $\mathbb{C}^n$ and obtained weighted L^p -estimates. Set $\gamma(\zeta) = |\partial r(\zeta)|$, and let us define the weighted L^p -spaces by

$$L^{p,k}(D) = \left\{ f : \|f\|_{p,k}^p = \int_D \gamma^k |f|^p dV < \infty \right\}.$$

Theorem 1.1 (Ehsani-Lieb). *Let $D \Subset \mathbb{C}^n$ be a Henkin-Leiterer domain with a Morse defining function. The Bergman projection is continuous from $L^p(D)$ into $L^{p,k}(D)$ for $2 \leq p < \infty$ and $k = k(p)$ sufficiently large. k depends on p (with $k = 0$ for $p = 2$).*

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Our purpose in this paper is to extend the results of [3]. In one direction we look to extend the results into the setting of Henkin-Leiterer domains in complex manifolds. The integral representation used in [3] relied on integral operators constructed with the use of the Henkin-Ramírez function and is not readily adaptable to the manifold setting. In this paper we use integral operators constructed from the Levi polynomial and the geometric distance function as in [7]. One of our main results is the reproduction of the weighted L^p -estimates of [3] when D lies in a complex manifold.

We further establish C^k -estimates in the theorem

Theorem 1.2. *Let $D \Subset X$ be a Henkin-Leiterer domain with a Morse defining function. Let $f \in L^2(D)$ and let B denote the Bergman projection. Then for $\epsilon > 0$, we have*

$$\|\gamma^{(n+2)+\epsilon+k}(f - Bf)\|_{C^k} \lesssim \|\gamma^k \bar{\partial} f\|_{C^k} + \|f\|_{L^2}.$$

One of the obvious difficulties with which one comes to face in trying to establish C^k -estimates on non-smooth domains is the choice of the frame of vector fields with which one will work. In the case of smooth domains, a special boundary chart is used in which $\omega^n = \partial r$ is part of an orthonormal frame of $(1, 0)$ -forms. When ∂r is allowed to vanish, the frame needs to be modified. We get around this difficulty by defining a $(1, 0)$ -form, ω^n , by $\partial r = \gamma \omega^n$. In the dual frame of vector fields, we are then faced with factors of γ in the expressions of the vector fields with respect to local coordinates, and we deal with these terms by multiplying our vector fields by a factor of γ . This ensures that when vector fields are commuted, there are no error terms which blow up at the singularity.

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2. NOTATION

With local coordinates denoted by $\zeta_1, \dots, \zeta_n$, we equip X with a Hermitian metric

$$ds^2 = \sum_{j,k} g_{j\bar{k}}(\zeta) d\zeta_j d\bar{\zeta}_k.$$

We take $\rho(x, y)$ to be a symmetric, smooth function on $X \times X$ which coincides with the geodesic distance in a neighborhood of the diagonal, Λ , and is positive outside of Λ .

We shall write $\mathcal{E}_j(x, y)$, for $j \geq 0$, for those double forms on open sets $U \subset X \times X$ such that $\mathcal{E}_j$ is smooth on U and satisfies

$$\mathcal{E}_j(x, y) \lesssim \rho^j(x, y).$$

We follow [7] to construct kernels to be used in our integral representation. Let $V^i \Subset U^i$ be two open coverings of the boundary ∂D . δ and ε ($\varepsilon \leq \delta$) are chosen

such that

- i) V^i cover the set $\{r(x) \leq 3\delta\}$,
- ii) $x \in V^i$ and $\rho(x, y) \leq \varepsilon$ imply $y \in U^i$,
- iii) $|r(x)| \leq \delta$ and $\rho(x, y) \leq \varepsilon$ imply $|r(y)| < 2\delta$,
- iv) $|r(x)| \leq 2\delta$ and $\rho(x, y) \leq \varepsilon$ imply $|r(y)| < 3\delta$.

Let (ζ^i, z^i) be local coordinates on $U^i \times U^i$. We define the Levi polynomial F^i on $U^i \times U^i$ by

$$F^i(\zeta, z) = \sum_{j=1}^n \frac{\partial r}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) - \frac{1}{2} \sum_{j,k=1}^n \frac{\partial^2 r}{\partial \zeta_j \partial \zeta_k}(\zeta_j - z_j)(\zeta_k - z_k),$$

where, for ease of notation, we drop the superscripts, i , on the local coordinates.

We also choose a smooth partition of unity, ξ^i , such that $\text{supp } \xi^i \Subset V^i$ and $\sum \xi^i = 1$ on $\{|r(x)| \leq 3\delta\}$. We set

$$F(x, y) = \sum_i \xi^i(x) F^i(x, y).$$

We choose a smooth symmetric patching function, φ , on $X \times X$ such that $0 \leq \varphi(x, y) \leq 1$ and

$$\varphi(x, y) = \begin{cases} 1 & \text{for } \rho^2(x, y) \leq \frac{\varepsilon}{2}, \\ 0 & \text{for } \rho^2(x, y) \geq \frac{3}{4}\varepsilon, \end{cases}$$

and set

$$\phi(x, y) = \varphi(x, y)(F(x, y) - r(x)) + (1 - \varphi(x, y))\rho^2(x, y).$$

Here we write the essential properties which we use in this paper. Let $D_\delta = \{x : r(x) < \delta\}$ and $S_\delta = \{x : -\delta < r(x) < \delta\}$.

Lemma 2.1. *There exist positive constants ϵ_0 and δ_0 such that on $\overline{S}_{\delta_0} \times \overline{D}_{\delta_0}$,*

- i) $\phi = 0$ iff $x = y \in \partial D$,
- ii) $|\phi| \gtrsim 1$ if $\rho(x, y) \geq \epsilon_0$,
- iii) $2 \operatorname{Re} \phi(x, y) \gtrsim -r(x) - r(y) + \rho^2(x, y)$,
- iv) $\phi - \phi^* = \mathcal{E}_3$.

Since D is strictly pseudoconvex and r is a Morse function, we can take $r_\epsilon = r + \epsilon$ for ϵ small enough, and then r_ϵ will be defining functions for smooth, strictly pseudoconvex D_ϵ .

We define $\phi_\epsilon(x, y)$ the same as we defined $\phi(x, y)$ above with r replaced by r_ϵ . We also define

$$\begin{aligned} P(x, y) &= \rho^2(x, y) + 2r(x)r(y), \\ P_\epsilon(x, y) &= \rho^2(x, y) + 2r_\epsilon(x)r_\epsilon(y). \end{aligned}$$

For $N \geq 0$, we let R_N denote an N -fold product, or a sum of such products, of first order vector fields applied to $r(y)$, with the notation $R_0 = 1$.

Definition 2.2. A double differential form $\mathcal{A}^\epsilon(x, y)$ on $\overline{D}_\epsilon \times \overline{D}_\epsilon$ is an *admissible* kernel, if it has the following properties:

- i) $\mathcal{A}^\epsilon$ is smooth on $\overline{D}_\epsilon \times \overline{D}_\epsilon - \Lambda_\epsilon$.

- ii) For each point $(x_0, x_0) \in \Lambda_\epsilon$ there is a neighborhood $U \times U$ of (x_0, x_0) on which $\mathcal{A}^\epsilon$ or $\overline{\mathcal{A}}^\epsilon$ has the representation

$$(2.1) \quad R_N \mathcal{E}_j P_\epsilon^{-t_0} \phi_\epsilon^{t_1} \overline{\phi}_\epsilon^{t_2} \phi_\epsilon^{*t_3} \overline{\phi}_\epsilon^{*t_4} r_\epsilon^l r_\epsilon^{*m}$$

with $N, j, t_0, \dots, m$ integers and $j, t_0, l, m \geq 0$, $-t = t_1 + \dots + t_4 \leq 0$.

The above representation is of *smooth type* s for

$$s = 2n + j + \min\{2, t - l - m\} - 2(t_0 + t - l - m).$$

We define the *type* of $\mathcal{A}^\epsilon(x, y)$ to be

$$\tau = s - \max\{0, 2 - N\}.$$

$\mathcal{A}^\epsilon$ has *smooth type* $\geq s$ if at each point (x_0, x_0) there is a representation (2.1) of smooth type $\geq s$. $\mathcal{A}^\epsilon$ has *type* $\geq \tau$ if at each point (x_0, x_0) there is a representation (2.1) of type $\geq \tau$. We shall also refer to the *double type* of an operator (τ, s) if the operator is of type τ and of smooth type s .

The definition of smooth type above is taken from [7]. Here and below, $(r_\epsilon(x))^* = r_\epsilon(y)$, the $*$ having a similar meaning for other functions of one variable.

For $\mathcal{A}_j^0$, we will simply write $\mathcal{A}_j$. We also denote by $\mathcal{A}_j^\epsilon$ operators with kernels of the form $\mathcal{A}_j^\epsilon$. $\mathcal{A}_j$ will denote the operators with kernels $\mathcal{A}_j$. We use the notation $\mathcal{A}_{(j,k)}^\epsilon$ (resp. $\mathcal{A}_{(j,k)}$) to denote kernels of double type (j, k) .

3. PROPERTIES OF OPERATORS

In this section we collect the various mapping properties of our operators. The proofs follow as in [1, 2].

A non-vanishing vector field, T , will be called tangential if $Tr = 0$ on $r = 0$. Near a boundary point, we choose a coordinate patch on which we have an orthogonal frame $\omega^1, \dots, \omega^n$ of $(1, 0)$ -forms with $\partial r = \gamma \omega^n$. Let $L_1, \dots, L_n$ denote the dual frame. $L_1, \dots, L_{n-1}, \overline{L}_1, \dots, \overline{L}_{n-1}$, and $Y = L_n - \overline{L}_n$ are tangential vector fields. $N = L_n + \overline{L}_n$ is a normal vector field. We say that a given vector field X is a smooth tangential vector field if it is a tangential field and, if near each boundary point, X is a combination of such vector fields $L_1, \dots, L_{n-1}, \overline{L}_1, \dots, \overline{L}_{n-1}, Y$, and rN with coefficients in $C^\infty(\overline{D})$.

We define the function spaces with which we will be working.

Definition 3.1. Let $0 \leq \beta$ and $0 \leq \delta$. We define

$$\|f\|_{L^{\infty, \beta, \delta}(D)} = \sup_{x \in D} |f(x)| \gamma^\beta(x) |r(x)|^\delta.$$

Definition 3.2. We set for $0 < \alpha < 1$,

$$\Lambda_\alpha(D) = \{f \in L^\infty(D) \mid \|f\|_{\Lambda_\alpha} := \|f\|_{L^\infty} + \sup \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty\}.$$

We also define the spaces $\Lambda_{\alpha, \beta}$ by

$$\Lambda_{\alpha, \beta} := \{f : \|f\|_{\Lambda_{\alpha, \beta}} = \|\gamma^\beta f\|_{\Lambda_\alpha} < \infty\}.$$

Theorem 3.3. *Let T be a smooth first order tangential differential operator on D . For A an operator of type 1 we have*

- i) $A : L^{\infty, 2+\epsilon, 0}(D) \rightarrow \Lambda_{\alpha, 2-\epsilon'}(D), \quad 0 < \epsilon, \epsilon', \quad \alpha + \epsilon + \epsilon' < 1/4,$
- ii) $A_j : L^p(D) \rightarrow L^s(D), \quad \frac{1}{s} > \frac{1}{p} - \frac{j}{2n+2},$
- iii) $\gamma^*TA : L^{\infty, 2+\epsilon, 0}(D) \rightarrow L^{\infty, \epsilon', \delta}(D), \quad 1/2 < \delta < 1, \quad \epsilon < \epsilon' < 1,$
- iv) $A : L^{\infty, \epsilon, \delta}(D) \rightarrow L^{\infty, \epsilon', 0}(D), \quad \epsilon < \epsilon', \quad \delta < 1/2 + (\epsilon' - \epsilon)/2.$

The mapping properties we need for type 0 operators are connected only with special operators, which we define as follows.

Definition 3.4. An admissible kernel $\mathcal{A}$ of type ≥ 0 is *special admissible* if, near each point $(\zeta_0, \zeta_0) \in \Lambda \subset b\Omega \times b\Omega$, it can be estimated as

$$|\mathcal{A}(\zeta, z)| \lesssim |R_j| \frac{1}{P^{t_0} |\Phi|^t},$$

with $t \neq 2$ and $2n + \min(2, t) - 2(t + t_0) - \max\{0, 2 - j\} \geq 0$, the type of $\mathcal{A}$.

Theorem 3.5. *Let A_0 be an operator of type 0 with a special admissible kernel. Then for $1 < p < \infty$,*

$$A_0 : L^p(D) \rightarrow L^p(D).$$

We let $\mathcal{E}_{j-2n}^i(x, y)$, $j \geq 1$, be a kernel of the form

$$\mathcal{E}_{j-2n}^i(x, y) = \frac{\mathcal{E}_m(x, y)}{\rho^{2k}(x, y)},$$

where $m - 2k \geq j - 2n$. We denote by E_{j-2n} the corresponding isotropic operator.

Theorem 3.6. *Let T be a smooth tangential vector field. Set E to be an operator with kernel of the form $\mathcal{E}_{1-2n}^i(x, y)R_1(x)$ or $\mathcal{E}_{2-2n}^i(x, y)$. Then we have the following properties:*

- i) $E_{j-2n} : L^p(D) \rightarrow L^s(D), \quad j \geq 1,$
- ii) $E : L^{\infty, 2+\epsilon, 0}(D) \rightarrow \Lambda_{\alpha, 2-\epsilon'}(D), \quad 0 < \epsilon, \epsilon', \quad \alpha + \epsilon + \epsilon' < 1,$
- iii) $\gamma^*TE : \Lambda_{\alpha, 2+\epsilon}(D) \rightarrow L^{\infty, \epsilon', 0}(D), \quad \epsilon < \epsilon',$
- iv) $E : L^{\infty, \epsilon, \delta}(D) \rightarrow L^{\infty, \epsilon', 0}(D), \quad \epsilon < \epsilon', \quad \delta < 1/2 + (\epsilon' - \epsilon)/2$

for any $1 \leq p \leq s \leq \infty$ with $1/s > 1/p - j/(2n)$.

4. INTEGRAL REPRESENTATION

Here we give the basic integral representation. As in [3], the crucial property here is that certain operators in the integral representation, while of type 0, are of smooth type 1; hence multiplication by a factor of R_1 turns them into type 1 operators with smoothing properties.

We start with the differential forms

$$\beta(x, y) = \frac{\partial_x \rho^2(x, y)}{\rho^2(x, y)},$$

$$\alpha_\epsilon(x, y) = \xi(x) \frac{\partial r_\epsilon(x)}{\phi_\epsilon(x, y)},$$

where $\xi(x)$ is a smooth patching function which is equivalently 1 for $|r(x)| < \delta$ and 0 for $|r(x)| > \frac{3}{2}\delta$, and $\delta > 0$ is sufficiently small. We define

$$C^\epsilon = C(\alpha_\epsilon, \beta) = \sum_{\mu=0}^{n-2} \left(\frac{1}{2\pi i} \right)^\mu C_\mu(\alpha_\epsilon, \beta),$$

where

$$C_\mu(\alpha_\epsilon, \beta) = \alpha_\epsilon \wedge \beta \wedge (\bar{\partial}_x \alpha_\epsilon)^\mu \wedge (\bar{\partial}_x \beta)^{n-\mu-2}.$$

Denoting the Hodge $*$ -operator by $*$, we then define

$$\mathcal{L}^\epsilon(x, y) = - * \overline{C^\epsilon(x, y)}.$$

We also write

$$K^\epsilon(x, y) = \frac{1}{(2\pi i)^n} \alpha_\epsilon \wedge (\bar{\partial}_x \alpha_\epsilon)^{n-1}$$

and

$$\Gamma^\epsilon(x, y) = \frac{(n-2)!}{2\pi^n} \frac{1}{\rho^{2n-2}}.$$

The kernels in our integral representation are defined through the following:

$$\begin{aligned} \mathcal{T}_0^\epsilon(x, y) &= \vartheta_x \mathcal{L}^\epsilon(x, y) - * \overline{K^\epsilon}(x, y) + \bar{\partial}_x \Gamma^\epsilon(x, y), \\ \mathcal{P}_0^\epsilon(x, y) &= * \partial_x \overline{K^\epsilon}(x, y). \end{aligned}$$

We denote the operators with kernels $\mathcal{T}_0^\epsilon$ and $\mathcal{P}_0^\epsilon$ by $\mathbf{T}_0^\epsilon$ and $\mathbf{P}_0^\epsilon$, respectively. We also define $\mathbf{T}_0$ and $\mathbf{P}_0$ to be the operators with kernels which are the limits of the kernels $\mathcal{T}_0^\epsilon$ and $\mathcal{P}_0^\epsilon$, respectively, as $\epsilon \rightarrow 0$.

Theorem 4.1. *Let $f \in L^2(D) \cap \text{Dom } \bar{\partial}$. Then (in the sense of distributions)*

$$(4.1) \quad f = \mathbf{T}_0 \bar{\partial} f + \mathbf{P}_0 f + (A_{(0,2)} + E_{2-2n}) \bar{\partial} f + (A_{(0,1)} + E_{1-2n}) f.$$

The proof follows as in [1], up until the examination of the term (4.4) below. By carefully considering such a term, we are able to show that the resulting integral kernels in the representation are of double type $(0, 1)$, whereas the analysis in [1] would only grant us kernels of double type $(-1, 1)$. This information is not needed in the $q > 0$ case due to the difference in the definitions of the term $P_\epsilon(\zeta, z)$ in the cases $q = 0$ and $q > 0$.

Sketch of proof. We first prove the corresponding statement for the domain D_ϵ and in the case $f \in C^1(\overline{D})$.

Our starting point is the Bochner-Martinelli-Koppelman (BMK) formula for $f \in C^1(\overline{D}_\epsilon)$. Let B_0 be defined by

$$B_0 = \Omega_0(\beta) = \frac{1}{(2\pi i)^n} \beta \wedge (\bar{\partial}_\zeta \beta)^{n-1}.$$

Then for $y \in D_\epsilon$,

$$(4.2) \quad \begin{aligned} f(y) &= \int_{\partial D_\epsilon} f(x) \wedge B_0(x, y) - \int_{D_\epsilon} \bar{\partial} f(x) \wedge B_0(x, y) \\ &\quad + (f(x), \mathcal{E}_{1-2n}(x, y)) + (\bar{\partial} f(x), \mathcal{E}_{2-2n}(x, y)). \end{aligned}$$

Define the kernels $K_0^\epsilon(x, y)$ by $K_0^\epsilon(x, y) = \Omega_0(\alpha_\epsilon)$. We then proceed to replace the boundary integral in the BMK formula by

$$\int_{\partial D_\epsilon} f \wedge K_0^\epsilon.$$

Let $x_0 \in \partial D_\epsilon$ be a fixed point and U a sufficiently small neighborhood of x_0 on which ζ is a local coordinate map. $F(\zeta, z)$ vanishes on the diagonal of $\overline{U} \times \overline{U}$, so Hefer's theorem applies to give us

$$F(\zeta, z) = \sum_{j=1}^n h_j(\zeta, z)(\zeta_j - z_j).$$

We set

$$\alpha^0(\zeta, z) = \frac{\sum_{j=1}^n h_j(\zeta, z) d\zeta_j}{F}.$$

With the metric given by

$$ds^2 = \sum g_{jk}(\zeta) dz_j d\bar{z}_k,$$

we define

$$\begin{aligned} b^0(\zeta, z) &= \sum_{j,k=1}^n g_{jk}(\bar{\zeta}_k - \bar{z}_k) d\zeta_j, \\ R^2(\zeta, z) &= \sum_{j,k=1}^n g_{jk}(\zeta_j - z_j)(\bar{\zeta}_k - \bar{z}_k), \\ \beta^0(\zeta, z) &= \frac{b^0(\zeta, z)}{R^2(\zeta, z)}. \end{aligned}$$

With the use of the transition kernels C defined above, we have via Koppelman's homotopy formula,

$$\Omega_0(\beta^0) = \Omega_0(\alpha^0) - \bar{\partial}_\zeta C(\alpha^0, \beta^0).$$

On $(\partial D \cap U) \times U$ we have

$$\Omega_0(\alpha^\epsilon) = \Omega_0(\alpha_0) + \frac{\mathcal{E}_1(\zeta, z)}{\phi_\epsilon(\zeta, z)^n}$$

and on $U \times U$ we have

$$\Omega_0(\beta) = \frac{R^{2n}}{\rho^{2n}} \Omega_0(\beta^0) + \mathcal{E}_{2-2n}.$$

Thus we write

$$\begin{aligned} \Omega_0(\beta) &= \frac{R^{2n}}{\rho^{2n}} \Omega_0(\beta^0) + \mathcal{E}_{2-2n} \\ &= \frac{R^{2n}}{\rho^{2n}} (\Omega_0(\beta^0) - \Omega_0(\alpha^0)) + \Omega_0(\alpha^0) + \frac{\mathcal{E}_{2n+1}}{\rho^{2n}} \Omega_0(\alpha^0) + \mathcal{E}_{2-2n}, \end{aligned}$$

by which it then follows from the homotopy formula and the relations between b^0 and ρ^2 and R^2 , exactly as it was obtained in [6], that we have

$$(4.3) \quad \begin{aligned} \Omega_0(\beta) = & \Omega_0(\alpha_\epsilon) - \bar{\partial}_\zeta C^\epsilon + (\Omega_0(\alpha^0) - \Omega_0(\alpha_\epsilon)) + \frac{\mathcal{E}_{2n+1}}{\rho^{2n}} \Omega_0(\alpha^0) \\ & + \mathcal{E}_{2-2n} + \bar{\partial}_\zeta \left[\left(C \left(\alpha^0, \frac{\partial \rho^2 + \mathcal{E}_2}{\rho^2} \right) - C(\alpha_\epsilon, \beta) \right) \right. \\ & \quad \left. + \sum_{\mu=0}^{n-2} \frac{\mathcal{E}_{3+2\mu}}{(\rho^2)^{1+\mu}} C_\mu \left(\alpha^0, \frac{\partial \rho^2 + \mathcal{E}_2}{\rho^2} \right) \right] \\ & + \sum_{\mu=0}^{n-2} \frac{\mathcal{E}_{4+2\mu}}{(\rho^2)^{2+\mu}} C_\mu \left(\alpha^0, \frac{\partial \rho^2 + \mathcal{E}_2}{\rho^2} \right). \end{aligned}$$

We now work with $\zeta \in \partial D_\epsilon$ so that $F = \phi_\epsilon$.

We have

$$\frac{\mathcal{E}_{2n+1}}{\rho^{2n}} \Omega_0(\alpha^0) = R_1 \frac{\mathcal{E}_{2n+1}}{\phi_\epsilon^n \rho^{2n}}$$

and

$$\Omega_0(\alpha_\epsilon) - \Omega_0(\alpha^0) = \frac{\mathcal{E}_1}{\phi_\epsilon^n}.$$

Furthermore,

$$C_\mu \left(\alpha^0, \frac{\partial \rho^2 + \mathcal{E}_2}{\rho^2} \right) = R_1 \frac{\mathcal{E}_1}{\phi_\epsilon^{1+\mu} (\rho^2)^{n-1-\mu}},$$

and thus

$$\begin{aligned} \bar{\partial}_\zeta \left(\sum_\mu \frac{\mathcal{E}_{3+2\mu}}{(\rho^2)^{1+\mu}} C_\mu \left(\alpha^0, \frac{\partial \rho^2 + \mathcal{E}_2}{\rho^2} \right) \right) &= \bar{\partial}_\zeta \left(R_1 \frac{\mathcal{E}_{4+2\mu}}{\phi_\epsilon^{1+\mu} \rho^{2n}} \right) \\ &= \frac{\mathcal{E}_{4+2\mu}}{\phi_\epsilon^{1+\mu} \rho^{2n}} + R_1 \frac{\mathcal{E}_{3+2\mu}}{\phi_\epsilon^{1+\mu} \rho^{2n}} + R_1 \frac{\mathcal{E}_{5+2\mu} + \mathcal{E}_{4+2\mu} \wedge \bar{\partial}_\zeta r_\epsilon}{\phi_\epsilon^{2+\mu} \rho^{2n}}. \end{aligned}$$

We have

$$(4.4) \quad C_\mu \left(\alpha^0, \frac{\partial \rho^2 + \mathcal{E}_2}{\rho^2} \right) - C_\mu(\alpha_\epsilon, \beta) = \frac{\mathcal{E}_2}{\phi_\epsilon^{1+\mu} (\rho^2)^{n-1-\mu}},$$

and by carefully handling this term we obtain a slight improvement, with tangible benefits, over its treatment in [6] and even in [1], where the additional information has no effect. In our relating boundary integrals to volume integrals, with the use of Stokes' Theorem, we replace ρ^2 with P_ϵ , as they are equivalent on ∂D_ϵ , in the kernels. We therefore use here the relation

$$\bar{\partial}_\zeta \left(C_\mu \left(\alpha^0, \frac{\partial \rho^2 + \mathcal{E}_2}{\rho^2} \right) - C_\mu(\alpha_\epsilon, \beta) \right) = \bar{\partial}_\zeta \left(\frac{\mathcal{E}_2}{\phi_\epsilon^{1+\mu} P_\epsilon^{n-1-\mu}} \right) + \frac{\mathcal{E}_2 r_\epsilon^* \wedge \bar{\partial} r_\epsilon}{\phi_\epsilon^{1+\mu} P_\epsilon^{n-\mu}}$$

for $\zeta \in \partial D_\epsilon$. The upshot is that the integral kernels in our final expression which stem from the term (4.4) are of double type $(0, 1)$.

(4.3) can thus be written

$$\begin{aligned}\Omega_0(\beta) &= \Omega_0(\alpha_\epsilon) - \bar{\partial}_\zeta C(\alpha_\epsilon, \beta) + \mathcal{E}_{2-2n} \\ &+ \sum_{\mu=0}^{n-2} \bar{\partial}_\zeta \left(\frac{\mathcal{E}_{4+2\mu}}{\phi_\epsilon^{1+\mu} P_\epsilon^n} \right) + r_\epsilon^* \sum_{\mu=0}^{n-2} \frac{\mathcal{E}_{2+2\mu} \wedge \bar{\partial} r_\epsilon}{\phi_\epsilon^{1+\mu} P_\epsilon^n} + \frac{\mathcal{E}_1}{\phi_\epsilon^n} \\ &+ R_1 \sum_{\mu=0}^{n-2} \frac{\mathcal{E}_{3+2\mu}}{\phi_\epsilon^{1+\mu} \rho^{2n}} + \sum_{\mu=0}^{n-2} \frac{\mathcal{E}_{4+2\mu}}{\phi_\epsilon^{1+\mu} \rho^{2n}} + R_1 \sum_{\mu=0}^{n-2} \frac{\mathcal{E}_{5+2\mu} + \mathcal{E}_{4+2\mu} \wedge \bar{\partial}_\zeta r_\epsilon}{\phi_\epsilon^{2+\mu} \rho^{2n}},\end{aligned}$$

again for $\zeta \in \partial D_\epsilon$.

Thus, after integrating by parts, we obtain

$$\begin{aligned}\int_{\partial D_\epsilon} f \wedge B_0 &= \int_{\partial D_\epsilon} f \wedge \Omega_0(\alpha_\epsilon) + \int_{\partial D_\epsilon} \bar{\partial} f \wedge C^\epsilon + \int_{\partial D_\epsilon} f \wedge \mathcal{E}_{2-2n} \\ &+ \sum_{\mu=0}^{n-2} \int_{D_\epsilon} \bar{\partial}_\zeta \left[f \wedge \left(\bar{\partial}_\zeta \left(\frac{\mathcal{E}_{4+2\mu}}{\phi_\epsilon^{1+\mu} P_\epsilon^n} \right) + r_\epsilon^* \frac{\mathcal{E}_{2+2\mu} \wedge \bar{\partial} r_\epsilon}{\phi_\epsilon^{1+\mu} P_\epsilon^n} + \frac{\mathcal{E}_1}{\phi_\epsilon^n} \right) \right] \\ &+ \sum_{\mu=0}^{n-2} \int_{\partial D_\epsilon} f \wedge \left(R_1 \frac{\mathcal{E}_{3+2\mu}}{\phi_\epsilon^{1+\mu} \rho^{2n}} + \frac{\mathcal{E}_{4+2\mu}}{\phi_\epsilon^{1+\mu} \rho^{2n}} + R_1 \frac{\mathcal{E}_{5+2\mu}}{\phi_\epsilon^{2+\mu} \rho^{2n}} \right),\end{aligned}$$

where we use an application of Stokes' Theorem in the second integral on the right-hand side.

We now replace all occurrences of ρ^2 in the denominators by P_ϵ , since the two are equal on ∂D_ϵ , in the remaining boundary integrals, and then change those boundary integrals to volume integrals by Stokes' Theorem:

$$\begin{aligned}\int_{\partial D_\epsilon} f \wedge B_0 &= \int_{D_\epsilon} \bar{\partial} f \wedge \Omega_0(\alpha_\epsilon) + \int_{D_\epsilon} f \wedge \bar{\partial}_\zeta \Omega_0(\alpha_\epsilon) - \int_{D_\epsilon} \bar{\partial} f \wedge \bar{\partial}_\zeta C^\epsilon + \int_{D_\epsilon} \bar{\partial} f \wedge \mathcal{E}_{2-2n} \\ &+ \int_{D_\epsilon} f \wedge \mathcal{E}_{1-2n} + \sum_{\mu=0}^{n-2} \int_{D_\epsilon} \bar{\partial} f \wedge \left(\frac{\mathcal{E}_{3+2\mu}}{\phi_\epsilon^{1+\mu} P_\epsilon^n} + R_1 \frac{\mathcal{E}_{4+2\mu}}{\phi_\epsilon^{2+\mu} P_\epsilon^n} \right. \\ &\quad \left. + \frac{\mathcal{E}_{5+2\mu}}{\phi_\epsilon^{1+\mu} P_\epsilon^{n+1}} + R_1 \frac{\mathcal{E}_{4+2\mu} r_\epsilon^*}{\phi_\epsilon^{1+\mu} P_\epsilon^{n+1}} + R_1 \frac{\mathcal{E}_{2+2\mu} r_\epsilon^*}{\phi_\epsilon^{1+\mu} P_\epsilon^n} + \frac{\mathcal{E}_1}{\phi_\epsilon^n} \right) \\ &+ \sum_{\mu=0}^{n-2} \int_{D_\epsilon} f \wedge \left(R_1 \frac{\mathcal{E}_{1+2\mu} r_\epsilon^*}{\phi_\epsilon^{1+\mu} P_\epsilon^n} + R_1 \frac{\mathcal{E}_{3+2\mu} r_\epsilon^*}{\phi_\epsilon^{1+\mu} P_\epsilon^{n+1}} + \frac{\mathcal{E}_0}{\phi_\epsilon^n} + \frac{\mathcal{E}_2 + \mathcal{E}_1 \wedge \bar{\partial} r_\epsilon}{\phi_\epsilon^{n+1}} \right) \\ &+ \sum_{\mu=0}^{n-2} \int_{D_\epsilon} f \wedge \left(\frac{\mathcal{E}_{3+2\mu}}{\phi_\epsilon^{1+\mu} P_\epsilon^n} + R_1 \frac{\mathcal{E}_{2+2\mu}}{\phi_\epsilon^{1+\mu} P_\epsilon^n} + R_2 \frac{\mathcal{E}_{3+2\mu}}{\phi_\epsilon^{2+\mu} P_\epsilon^n} \right. \\ &\quad \left. + R_1 \frac{\mathcal{E}_{4+2\mu}}{\phi_\epsilon^{1+\mu} P_\epsilon^{n+1}} + \frac{\mathcal{E}_{5+2\mu}}{\phi_\epsilon^{1+\mu} P_\epsilon^{n+1}} \right).\end{aligned}$$

Inserting this expression of the boundary integral into (4.2), and using our notation of operators of a certain type, we can write

$$\begin{aligned} f(z) = & \int_{D_\epsilon} f \wedge \bar{\partial}_\zeta \Omega_0(\alpha_\epsilon) + \int_{D_\epsilon} \bar{\partial} f \wedge \Omega_0(\alpha_\epsilon) - \int_{D_\epsilon} \bar{\partial} f \wedge \bar{\partial}_\zeta C^\epsilon \\ & - \int_{D_\epsilon} \bar{\partial} f \wedge B_0^\epsilon \\ & + (f, \mathcal{E}_{1-2n}) + (f, \mathcal{A}_{(0,1)}^\epsilon) + (\bar{\partial} f, \mathcal{E}_{2-2n}) + (\bar{\partial} f, \mathcal{A}_{(0,2)}^\epsilon). \end{aligned}$$

Then as in [7], a rearrangement of terms leads to the representation for $f \in C^1(\overline{D})$ and $z \in D_\epsilon$:

$$f = \mathbf{T}_0^\epsilon \bar{\partial} f + \mathbf{P}_0^\epsilon f + (A_{(0,2)}^\epsilon + E_{2-2n}) \bar{\partial} f + (A_{(0,1)}^\epsilon + E_{1-2n}) f.$$

We then let $\epsilon \rightarrow 0$ and the right-hand side approaches the right-hand side of (4.1). This establishes (4.1) for $f \in C^1(\overline{D})$.

To see that the theorem is valid for $f \in L^2(D) \cap \text{Dom } \bar{\partial}$, we refer to Theorems 3.3, 3.5, and 3.6 to show that all operators in (4.1) map L^p to L^p for $p \geq 2$. $\square$

5. A SYMMETRY ARGUMENT

From [3] we have the

Proposition 5.1.

$$\mathbf{P}_0 - \mathbf{P}_0^* = A_{(0,1)}.$$

We denote by $\mathcal{S}_\infty(x, y)$ a kernel which has the properties that $\forall \lambda > 0$,

$$\begin{aligned} \int_D |\mathcal{S}_\infty(x, y)|^\lambda dV(x) &< \infty, \\ \int_D |\mathcal{S}_\infty(x, y)|^\lambda dV(y) &< \infty, \end{aligned}$$

and which is also smoothing in the sense that $\forall k$,

$$\|(\gamma T)^k \mathbf{S}_\infty f\|_{L^\infty} \lesssim \|f\|_2,$$

where $\mathbf{S}_\infty$ is an operator associated with an $\mathcal{S}_\infty$ kernel. We have the mapping property

$$\mathbf{S}_\infty : L^2(D) \rightarrow L^p(D)$$

$\forall p \geq 2$ by the generalized Young's inequality [8].

Lemma 5.2.

$$\bar{\partial}_x R_1 \mathcal{P}_0^* = \bar{\partial}_x \mathcal{A}_1 + \mathcal{S}_\infty.$$

Proof. $\mathcal{P}_0^*$ has the form

$$\mathcal{P}_0(x, y)^* = R_2(y) \frac{l(y)}{\phi(y, x)^{n+1}} + \mathcal{A}_{(0,1)},$$

where $l(y)$ is smooth and real-valued, and we use $R_1(y) = R_1(x) + \mathcal{E}_1$. Thus

$$\begin{aligned} \bar{\partial}_x R_1 \mathcal{P}_0^* &= \bar{\partial}_x \left(R_3(y) \frac{l(y)}{\phi(y, x)^{n+1}} \right) + \bar{\partial}_x \mathcal{A}_1 \\ &= \bar{\partial}_x \mathcal{A}_1 + \mathcal{S}_\infty, \end{aligned}$$

since

$$\bar{\partial}_x \left(R_3(y) \frac{l(y)}{\phi(y, x)^{n+1}} \right) = \mathcal{S}_\infty,$$

due to the fact that $\phi(y, x)$ is holomorphic in the x variable near $x = y$. $\square$

We let N be the $\bar{\partial}$ -Neumann operator on $(0, 1)$ -forms, and we make use of the relation

$$B = I - \bar{\partial}^* N \bar{\partial}$$

to obtain

Corollary 5.3.

$$\mathbf{P}_0^* R_1(f - Bf) = A_1(f - Bf) + \mathbf{S}_\infty N \bar{\partial} f.$$

Proof.

$$\begin{aligned} \mathbf{P}_0^* R_1(f - Bf) &= (f - Bf, R_1 \mathcal{P}_0^*) \\ &= (\bar{\partial}^* N \bar{\partial} f, R_1 \mathcal{P}_0^*) \\ &= (N \bar{\partial} f, \bar{\partial}_\zeta R_1 \mathcal{P}_0^*) \\ &= (N \bar{\partial} f, \bar{\partial}_\zeta \mathcal{A}_1 + \mathcal{S}_\infty) \\ &= (\bar{\partial}^* N \bar{\partial} f, \mathcal{A}_1) + \mathbf{S}_\infty N \bar{\partial} f. \end{aligned}$$

$\square$

6. ESTIMATES

We apply Theorem 4.1 to $f - Bf$,

$$f - Bf = \mathbf{T}_0 \bar{\partial} f + \mathbf{P}_0(f - Bf) + (A_{(0,2)} + E_{2-2n}) \bar{\partial} f + (A_{(0,1)} + E_{1-2n})(f - Bf),$$

and then we use Proposition 5.1 to obtain

$$f - Bf = \mathbf{T}_0 \bar{\partial} f + \mathbf{P}_0^*(f - Bf) + (A_{(0,2)} + E_{2-2n}) \bar{\partial} f + (A_{(0,1)} + E_{1-2n})(f - Bf).$$

Then after applying Corollary 5.3, we write

Theorem 6.1.

$$(6.1) \quad f - Bf = (\mathbf{T}_0 + A_{(0,2)} + E_{2-2n}) \bar{\partial} f + \mathbf{S}_\infty N \bar{\partial} f + (A_{(0,1)} + E_{1-2n})(f - Bf).$$

We introduce the notion of Z operators. We denote by Z_1 those operators which are of a type ≥ 1 and $\epsilon = 0$ (that is, we consider operators on spaces of the domain, D) or are $E_{1-2n} \circ R_1$ and also the sum of such operators. We further include in our notation of Z_1 those operators whose kernels conform to our definition of admissible with the factors R_1 being replaced by a factor of γ . Thus, for instance, a kernel of smooth type 1 multiplied by γ^2 will be a Z_1 operator. $E_{1-2n} \circ \gamma$ will also be a Z_1 operator.

Let $\mathcal{Z}_1(x, y)$ be a kernel of a Z_1 operator. We define Z_j operators to be those operators which have kernels of the form $\mathcal{Z}_1(x, y) \mathcal{E}_{j-1}(x, y)$. We establish mapping properties for Z_j operators.

For the proof of the properties of Z_j operators, we will refer to a lemma of Schmalz (see the proof of Lemma 3.2 in [9]) which provides a useful coordinate system in which to prove estimates.

Lemma 6.2. Let $E_\delta(y) := \{x \in D : \rho(x, y) < \delta\gamma(y)\}$ for $\delta > 0$. Then there is a constant c and a number $m \in \{1, \dots, 2n\}$ such that for all $y \in D$, there are local coordinates around each $x \in E_c(y)$ given by $\zeta_j = t_j + it_{j+n}$ for $1 \leq j \leq n$, and

$$\{-r(\zeta), t_1, \dots, \hat{t}_m, \dots, t_{2n}\},$$

where t_m is omitted, forms a coordinate system near x . We have the estimate

$$(6.2) \quad dV \lesssim \frac{1}{\gamma(z)} |dr(\zeta) \wedge dt_1 \wedge \dots \wedge \hat{t}_m \wedge \dots \wedge dt_{2n}|,$$

where dV is the Euclidean volume form on $\mathbb{R}^{2n}$.

We have the following lemma from [2].

Lemma 6.3. For $0 < \epsilon' < \epsilon$,

$$(6.3) \quad \|Z_j f\|_{L^{\infty, \epsilon, 0}} \lesssim \|f\|_{L^{\infty, j+\epsilon', 0}}.$$

Definition 6.4. An admissible kernel $\mathcal{A}$ of type ≥ 0 is *special admissible* if, near each point $(\zeta_0, \zeta_0) \in \Lambda \subset b\Omega \times b\Omega$, it can be estimated as

$$|\mathcal{A}(\zeta, z)| \lesssim |R_j| \frac{1}{P^{t_0} |\Phi|^t},$$

with $t \neq 2$ and $2n + \min(2, t) - 2(t + t_0) - \max\{0, 2 - j\} \geq 0$, the type of $\mathcal{A}$.

Proposition 6.5. Let A_0 be a special admissible operator of type 0. Then for $\epsilon > 0$,

$$A_0 : L^{\infty, \epsilon, 0}(D) \rightarrow L^{\infty, \epsilon, 0}(D).$$

Proof. We concentrate on the case that $\mathcal{A}_0(x, y)$ is estimated by

$$|\mathcal{A}_0(x, y)| \lesssim \frac{1}{P^{n-2-\mu} |\phi|^{\mu+2}}, \quad \mu \geq 1.$$

In this case we need to estimate

$$\int_{\Omega} \frac{\gamma^\epsilon(y)}{\gamma^\epsilon(x)} \frac{1}{P^{n-2-\mu} |\phi|^{\mu+2}} dV(x).$$

For such an integral, we use the calculation

$$\begin{aligned} \gamma^\epsilon(y) - \gamma^\epsilon(x) &= \frac{(\gamma^\epsilon(y) - \gamma^\epsilon(x))(\gamma^{1-\epsilon}(y) + \gamma^{1-\epsilon}(x))}{\gamma^{1-\epsilon}(y) + \gamma^{1-\epsilon}(x)} \\ &\lesssim \frac{|\gamma(y) - \gamma(x)|}{\gamma^{1-\epsilon}(y) + \gamma^{1-\epsilon}(x)} \\ &\lesssim \frac{\rho(x, y)}{\gamma^{1-\epsilon}(x)}, \end{aligned}$$

and hence estimate the integrals

$$\begin{aligned} &\int_{\Omega} \frac{1}{P^{n-2-\mu} |\phi|^{\mu+2}} dV(x), \\ &\int_{\Omega} \frac{1}{\gamma^{1-\epsilon}(x)} \frac{1}{P^{n-5/2-\mu} |\phi|^{\mu+2}} dV(x), \end{aligned}$$

which can be done in a manner similar to that used in Lemma 6.3. $\square$

Applying R_1 to (6.1) gives us

$$(6.4) \quad R_1(y)(f - Bf) = Z_1 \bar{\partial} f + \mathbf{S}_\infty N \bar{\partial} f + Z_1(f - Bf).$$

An iteration leads to

Proposition 6.6.

$$(6.5) \quad R_k(y)(f - Bf) = Z_1 \bar{\partial} f + \mathbf{S}_\infty N \bar{\partial} f + \sum_{j=1}^k \sum_{i_1+\dots+i_j=k} Z_{i_1} \circ \dots \circ Z_{i_j}(f - Bf).$$

Proof. The case $k = 1$ is just equation (6.4). Then, given (6.5) for a particular k , we prove the corresponding equation $R_{k+1}(y)(f - Bf)$ by multiplying (6.5) by a factor of R_1 :

$$\begin{aligned} R_{k+1}(y)(f - Bf) &= Z_1 \bar{\partial} f + \mathbf{S}_\infty N \bar{\partial} f + \sum_{j=1}^k \sum_{i_1+\dots+i_j=k+1} Z_{i_1} \circ \dots \circ Z_{i_j}(f - Bf) \\ &\quad + \sum_{j=1}^k \sum_{i_1+\dots+i_j=k} Z_{i_1} \circ \dots \circ Z_{i_j} R_1(x)(f - Bf) \\ &= Z_1 \bar{\partial} f + \mathbf{S}_\infty N \bar{\partial} f + \sum_{j=1}^k \sum_{i_1+\dots+i_j=k+1} Z_{i_1} \circ \dots \circ Z_{i_j}(f - Bf) \\ &\quad + \sum_{j=1}^k \sum_{i_1+\dots+i_j=k} Z_{i_1} \circ \dots \circ Z_{i_j} \circ Z_1(f - Bf) \\ &= Z_1 \bar{\partial} f + \mathbf{S}_\infty N \bar{\partial} f + \sum_{j=1}^{k+1} \sum_{i_1+\dots+i_j=k+1} Z_{i_1} \circ \dots \circ Z_{i_j}(f - Bf), \end{aligned}$$

where we use

$$R_1(y)Z_i = Z_i \circ R_1(x) + Z_{i+1}$$

in the first step. We also use the commutator relations below (see Theorem 6.7) to establish that $Z_j \circ \mathbf{S}_\infty = \mathbf{S}_\infty$. $\square$

When dealing with derivatives in order to establish the C^k -estimates we will need to know how certain vector fields commute with our Z_j operators. We make the important remark here that the tangential vector fields will have coefficients only in $C^0(\bar{D})$ due to factors of γ which occur in the denominators of such coefficients (see [2] for details).

Theorem 6.7. *Let X be a smooth tangential vector field. Then*

$$\gamma^* X^y Z_1 = -Z_1 \tilde{X}^x \gamma + Z_1^{(0)} + \sum_{\nu=1}^l Z_1^{(\nu)} W_\nu^x \gamma,$$

where $\tilde{X}$ is the adjoint of X , the $Z_1^{(\nu)}$ are also Z_1 operators, and $W^x \in T_x^{1,0}(\partial D) \oplus T_x^{0,1}(\partial D)$.

We need the

Proposition 6.8. *Let $p \geq 2$, $s > p$ and let $k(s)$ be an integer which satisfies*

$$\frac{1}{s} > \frac{1}{p} - \frac{k(s)}{2n+2}.$$

There are $0 < \epsilon, \epsilon'$ small enough such that

$$\begin{aligned} i) \quad & \sum_{j=1}^k \sum_{i_1+\dots+i_j=k} Z_{i_1} \circ \dots \circ Z_{i_j} : L^p(D) \rightarrow L^s(D), \\ ii) \quad & \gamma T \sum_{j=1}^{n+2} \sum_{i_1+\dots+i_j=n+2} Z_{i_1} \circ \dots \circ Z_{i_j} : L^{\infty, n+2+\epsilon, 0}(D) \rightarrow L^{\infty, \epsilon', 0}(D), \quad \epsilon < \epsilon', \\ iii) \quad & Z_j : L^{\infty, m+j+\epsilon, 0}(D) \rightarrow L^{\infty, m+\epsilon', 0}(D), \quad \epsilon < \epsilon'. \end{aligned}$$

Proof. *i)* We use Theorem 3.3 *ii)* and Theorem 3.6 *ii)* to conclude that

$$Z_{i_1} \circ \dots \circ Z_{i_j} : L^p(D) \rightarrow L^s(D),$$

where

$$\frac{1}{s} > \frac{1}{p} - \frac{i_1 + \dots + i_j}{2n+2}.$$

ii) We first consider the case of $i_1 = \dots = i_{n+2} = 1$. We use parts of Theorems 3.3 and 3.6 and part *iii)* of the current theorem to show that the composition of operators maps $L^{\infty, n+2+\epsilon, 0}$ into $L^{\infty, \epsilon', 0}$. We can see this by using the commutator relations and considering the two compositions $Z_1 \circ \gamma T A_1 \circ Z_1$ and $Z_1 \circ \gamma T E \circ Z_1$. From Theorems 3.3 and 3.6 we can find $\epsilon_1, \dots, \epsilon_4$ such that $\epsilon_{j+1} < \epsilon_j$ and in the first case we have

$$\|Z_1 \circ \gamma T A_1 \circ Z_1 f\|_{L^{\infty, \epsilon_1, 0}} \lesssim \|\gamma T A_1 \circ Z_1 f\|_{L^{\infty, \epsilon_2, \delta}} \lesssim \|Z_1 f\|_{L^{\infty, 2+\epsilon_3, 0}} \lesssim \|f\|_{L^{\infty, 3+\epsilon_4, 0}},$$

and, in the second case,

$$\|Z_1 \circ \gamma T E \circ Z_1 f\|_{L^{\infty, \epsilon_1}} \lesssim \|\gamma T E \circ Z_1 f\|_{L^{\infty, 1+\epsilon_2, 0}} \lesssim \|Z_1 f\|_{\Lambda_{\alpha, 3-\epsilon_3}} \lesssim \|f\|_{L^{\infty, 3+\epsilon_4, 0}},$$

where the second and third inequalities are proved in the same way as Theorem 3.6 *ii)* and *iii)*.

We now consider the case in which $\max\{i_1, \dots, i_j\} \geq 2$, and we show that

$$\gamma^* T Z_j : L^{\infty, j+\epsilon, 0} \rightarrow L^{\infty, \epsilon'}$$

for $\epsilon < \epsilon'$.

For $T = T^y$ a smooth first order tangential differential operator on D , with respect to the y variable, we have

$$\begin{aligned} T\gamma(y) &\lesssim 1, \\ Tr &= 0, \\ Tr^* &= \mathcal{E}_0 r, \\ TP &= \mathcal{E}_1 + \mathcal{E}_0 r r^*, \\ T\phi &= R_1 \mathcal{E}_0 + \mathcal{E}_1. \end{aligned}$$

We can therefore write for $j \geq 2$,

$$(6.6) \quad \gamma^* T Z_j = Z_{j-2} \gamma^2 + Z_{j-1} \gamma + Z_j,$$

where we use the convention that Z_0 refers to an operator of type ≥ 0 .

Noting that, as a result of the commutator relations, and our definition of Z_j , we also have for $j \geq 2$,

$$(6.7) \quad \gamma^* T Z_j = Z_{j-1} \circ \gamma + Z_j + Z_j \circ \gamma T.$$

We again use the commutator relations, (6.7), and Lemma 6.3 to reduce the proof to the examination of

$$Z_{i_1} \circ \cdots \circ \gamma T Z_{i_k} \circ \cdots \circ Z_{i_j},$$

where $i_k \geq 2$. But from (6.6) we can write

$$Z_{i_1} \circ \cdots \circ (Z_{i_k-2} \circ \gamma^2 + Z_{i_k-1} \circ \gamma + Z_{i_k}) \circ \cdots \circ Z_{i_j},$$

and since $i_1 + \cdots + i_{k-1} + i_{k+1} + \cdots + i_j = n - k + 2$, this case follows from Proposition 6.5, Lemma 6.3 and property *iii*) of the current theorem.

iii) This is just a corollary of Lemma 6.3. $\square$

We first establish the L^p -estimates. The next theorem was established in [3] in the case of strictly pseudoconvex domains in $\mathbb{C}^n$ (see Theorem 5.2 of [3]).

Theorem 6.9. *For $p \geq 2$ let $k = k(p)$ satisfy*

$$\frac{1}{p} > \frac{1}{2} - \frac{k}{2n+2}.$$

Then

$$\|Bf\|_{L^{p,kp}} \lesssim \|f\|_{L^p}.$$

Proof. We start by multiplying (4.1) by R_k and subtracting (6.5):

$$(6.8) \quad R_k Bf = A_0 f + \mathbf{S}_\infty N \bar{\partial} f + \sum_{j=1}^k \sum_{i_1+\cdots+i_j=k} Z_{i_1} \circ \cdots \circ Z_{i_j} f + \sum_{j=1}^k \sum_{i_1+\cdots+i_j=k} Z_{i_1} \circ \cdots \circ Z_{i_j} Bf,$$

where A_0 above is a special admissible operator.

The L^p -norm of the first term on the right-hand side is bounded by $\|f\|_p$ by Theorem 3.5.

We use the fact that

$$\mathbf{S}_\infty : L^2 \rightarrow L^p$$

for any $p \geq 2$ and that

$$\|N \bar{\partial} f\|_{L^2} \lesssim \|f\|_{L^2} \lesssim \|f\|_{L^p}$$

(see [5]) to handle the second term.

For the last two sums in (6.8) we use Proposition 6.8 *i*) to show that

$$Z_{i_1} \circ \cdots \circ Z_{i_j} : L^2 \rightarrow L^p,$$

where p is given by

$$(6.9) \quad \frac{1}{p} > \frac{1}{2} - \frac{k}{2n+2}.$$

Hence, the L^p -norm of the last two terms in (6.8) is bounded by

$$\|f\|_{L^2} + \|Bf\|_{L^2} \lesssim \|f\|_{L^2} \lesssim \|f\|_{L^p},$$

which we use to finish the proof. $\square$

Note that the right-hand side of (6.9) is 0 for $k(p) = n + 1$. Thus, we have the

Corollary 6.10. *For $2 \leq p < \infty$ we have the weighted estimates*

$$\|Bf\|_{L^{p,(n+1)p}} \lesssim \|f\|_{L^p}.$$

For the C^k -estimates we take $k = n + 2$ in (6.5) and use Proposition 6.8 *i*) above to establish

$$(6.10) \quad \|\gamma^{n+2}(f - Bf)\|_{L^\infty} \lesssim \|\bar{\partial}f\|_{L^\infty} + \|f\|_2.$$

We use the notation D^k to denote a k th-order differential operator, which is a sum of terms which are composites of k vector fields. We define

$$Q_k(f) = \sum_{j=0}^k \|\gamma^j D^j \bar{\partial}f\|_\infty + \|f\|_2.$$

T^k will be used for a k th-order tangential differential operator, which is a sum of terms which are composites of k tangential vector fields.

Lemma 6.11. *Let T^k be a tangential operator of order k . Let $\epsilon > 0$. Then*

$$\|\gamma^{(n+2)+\epsilon+k} T^k(f - Bf)\|_{L^\infty} \lesssim Q_k(f).$$

Proof. The proof is by induction. The first step, $k = 0$, is contained in (6.10).

We start with (6.5) with $k = n + 2$:

$$R_{n+2}(y)(f - Bf) = Z_1 \bar{\partial}f + \mathbf{S}_\infty N \bar{\partial}f + \sum_{j=1}^{n+1} \sum_{i_1+\dots+i_j=n+2} Z_{r_{i_1}} \circ \dots \circ Z_{r_{i_j}}(f - Bf)$$

and apply $\gamma^\epsilon \gamma^k T^k$ to both sides. After employing the commutator relations in Theorem 6.7 and the induction step, we have

$$(6.11) \quad \begin{aligned} \gamma^\epsilon R_{n+2} \gamma^k T^k(f - Bf) &= \gamma^\epsilon \sum_{j=0}^k (Z_1 \gamma^j T^j)(\bar{\partial}f) + \gamma^\epsilon \mathbf{S}_\infty N \bar{\partial}f \\ &\quad + \gamma^\epsilon \sum_{m=1}^{k-1} \sum_{j=1}^{n+2} \sum_{i_1+\dots+i_j=n+2} Z_{r_{i_1}} \circ \dots \circ Z_{r_{i_j}} \gamma^m T^m(f - Bf) \\ &\quad + \gamma^\epsilon \gamma T \sum_{j=1}^{n+2} \sum_{i_1+\dots+i_j=n+2} Z_{r_{i_1}} \circ \dots \circ Z_{r_{i_j}} \gamma^{k-1} T^{k-1}(f - Bf). \end{aligned}$$

The L^∞ -norms of the first three terms on the right of (6.11) are bounded by $Q_k(f)$ by Proposition 6.8 *i*) and the induction step. For the last term we use Proposition 6.8 *ii*) to show its L^∞ -norm is bounded by

$$\|\gamma^{(n+2)+\epsilon'+k-1} T^{k-1}(f - Bf)\|_\infty \lesssim Q_{k-1}(f),$$

for $\epsilon' < \epsilon$, by the induction hypothesis. $\square$

In order to generalize Lemma 6.11 to include non-tangential operators, we express a normal derivative of a component of a function, f , in terms of tangential operators acting on f and components of $\bar{\partial}f$.

We have the decomposition in the following form:

$$(6.12) \quad \gamma N f = \sum_j a_j \gamma T_j f + \sum_j b_j f + \sum_j c_j \gamma (\bar{\partial} f)_j,$$

where the coefficients a_j , b_j , and c_j are all of the form $\mathcal{E}_{0,0}$. The decomposition is well known in the smooth case (see [6]), and (6.12) is verified for Henkin-Leiterer domains in [2].

It is then straightforward how to generalize Lemma 6.11. Suppose D^k is a k th-order differential operator which contains the normal field at least once. In $\gamma^k D^k$ we commute γN with terms of the form γT , where T is tangential, and we consider the operator $D^k = D^{k-1} \circ \gamma N$, where D^{k-1} is of order $k-1$. The error terms due to the commutation involve differential operators of order $\leq k-1$. From (6.12) we just have to consider $D^{k-1} \gamma T f$ and $D^{k-1} \bar{\partial} f$. The last two terms are bounded by $Q_{k-1}(f)$, and we repeat the process with $D^{k-1} \gamma T f$, until we are left with k tangential operators for which we can apply Lemma 6.11. We thereby obtain the weighted C^k estimates given in Theorem 1.2.

Theorem 6.12. For $\epsilon > 0$,

$$\|\gamma^{(n+2)+\epsilon+k}(f - Bf)\|_{C^k} \lesssim Q_k(f).$$

As an immediate consequence we obtain weighted C^k -estimates for the canonical solution, the solution of the minimal L^2 -norm, to the $\bar{\partial}$ -equation. Let v be any solution to $\bar{\partial} v = f$ with L^2 -estimates, the existence being guaranteed by Hörmander's solution [5]. Then $u = v - Bv$ is the canonical solution to $\bar{\partial} u = f$, and from above we obtain the following estimates.

Corollary 6.13. The canonical solution to $\bar{\partial} u = f$ satisfies

$$\|\gamma^{(n+2)+\epsilon+k} u\|_{C^k} \lesssim \|\gamma^k f\|_{C^k} + \|f\|_2.$$

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