q -CHAOS

MARIUS JUNGE AND HUN HEE LEE

ABSTRACT. We consider the L_p norm estimates for homogeneous polynomials of q -Gaussian variables ($-1 \leq q \leq 1$). When $-1 < q < 1$ the L_p estimates for $1 \leq p \leq 2$ are essentially the same as the free case ($q = 0$), whilst the L_p estimates for $2 \leq p \leq \infty$ show a strong q -dependence. Moreover, the extremal cases $q = \pm 1$ produce decisively different formulae.

1. INTRODUCTION

In classical probability theory and in analysis the Wiener chaos is well understood and fundamental. The orthogonal polynomials for the Gaussian measure in $\mathbb{R}^n$ are given by the Hermite polynomials. In this paper we are interested in norm estimates for homogeneous polynomials in q -Gaussian variables. The family of q -Gaussian variables, introduced by Bożejko and Speicher ([3]), is a natural non-commutative generalization of classical Gaussian random variables. Here q ranges between $[-1, 1]$. The case $q = 1$ corresponds to the classical situation, $q = -1$ reflects the fermionic case, and $q = 0$ comes from Voiculescu's free probability theory. Our work is motivated by the beautiful results by Haagerup/Pisier ([8]) on the operator space structure of the space of homogeneous polynomials in the free group. Similar results have been obtained by Ricard/Xu ([26]) for free products. We want to show that norm estimates for polynomials of degree 2 and higher can detect the value q .

The space of polynomials of degree one is just the linear span of the (q -) Gaussian variables and are, by now, very well understood through Khintchine type inequalities. The starting point of these results is Lust-Piquard's Khintchine inequality ([17]), later extended by Lust-Piquard/Pisier to the range $1 \leq p \leq 2$ ([18]). In the free case, Buchholz ([1]) provided very precise estimates. For general information on free chaos we refer to the work of [11], which stems from earlier work of Pisier/Parcet ([21]). For $p = \infty$, very nice estimates for polynomials can be found in [26]; the L_p versions can be found in [11].

Analyzing the known norm estimates for the span of the generators of q -Gaussian random variables, it turns out that the dependence on q disappears. Indeed, any

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suitable notion of Gaussian random variables leads to a similar expression for a fixed value of p . In this paper we provide formulas for homogeneous polynomials of higher degree and show that already for polynomials of degree 2 the results do depend on q ; however the dependence is quite subtle.

Let $-1 \leq q \leq 1$ and $(g_{q,i})_{i \geq 1}$ be q -(generalized) Gaussian random variables defined on the q -deformed Fock space with two sequence parameters $\lambda = (\lambda_i)_{i \geq 1}$ and $\mu = (\mu_i)_{i \geq 1}$ of strictly positive reals. In this case, there is a natural reference state ϕ_q (called a vacuum state) on the von Neumann algebra that the $g_{q,i}$'s generate, which satisfies $\phi_q(g_{q,i}^* g_{q,i}) = \lambda_i^2$ and $\phi_q(g_{q,i} g_{q,i}^*) = \mu_i^2$ (see section 2.1 below for the details). When $q = 0$ we simply write $(g_i)_{i \geq 1}$ and ϕ instead of $(g_{0,i})_{i \geq 1}$ and ϕ_0 , respectively, and we call $(g_i)_{i \geq 1}$ free (generalized) Gaussian random variables.

Theorem 1.1. *Let $-1 < q < 1$ and D_q (resp. D) be the density of ϕ_q (resp. ϕ). Let $(x_{ij})_{i,j=1}^m$ be $L_p(\mathcal{M})$ -valued coefficients for a von Neumann algebra $\mathcal{M}$ and $1 \leq p \leq 2$. Then*

$$\left\| \sum_{i,j} x_{ij} \otimes D_q^{\frac{1}{2p}} g_{q,i} g_{q,j} D_q^{\frac{1}{2p}} \right\|_p \sim_{c_q} \left\| \sum_{i,j} x_{ij} \otimes D^{\frac{1}{2p}} g_i g_j D^{\frac{1}{2p}} \right\|_p.$$

The good news is that the right hand side can be calculated (up to some universal constant) by a formula involving the best decomposition with respect to three norms (a generalization of the basic $\mathcal{K}$ -functional spaces in interpolation theory). Such decomposition norms should be considered classical in the theory as in so far they appear already in Lust-Piquard/Pisier's operator-valued Khintchine inequality for the range $1 \leq p \leq 2$. We refer to section 3 for a precise formulation.

It seems that Theorem 1.1 has no dependence in q (except the constant). However, we can find a strong q -dependence when $p \geq 2$. We use duality arguments starting from Theorem 1.1. These estimates are based on the previous work of Nou ([20]). Indeed, a classical tool for studying q -Gaussian variables is the Wick order. This order essentially implements the identification between the GNS-Hilbert space given by the vacuum state and the Fock space realization. In our situation the usual product $g_{q,i} g_{q,j}$ coincides with the Wick product; however, understanding $g_{q,i} g_{q,j}$ as a Wick product plays an essential role when $p \geq 2$.

Theorem 1.2. *Let $-1 < q < 1$, $(x_{ij})_{i,j=1}^m$ be $L_p(\mathcal{M})$ -valued coefficients for a von Neumann algebra $\mathcal{M}$ and $2 \leq p \leq \infty$. Let $(g_{q,i})_{i=1}^m$, D_q , $(g_i)_{i=1}^m$ and D be as in Theorem 1.1. Then*

$$\left\| \sum_{i,j} x_{ij} \otimes D_q^{\frac{1}{2p}} g_{q,i} g_{q,j} D_q^{\frac{1}{2p}} \right\|_p \sim_{c_q} \left\| \sum_{i,j} (x_{ij} + q x_{ji}) \otimes D^{\frac{1}{2p}} g_i g_j D^{\frac{1}{2p}} \right\|_p.$$

Moreover, the span of polynomials of degree 2 is completely complemented in the corresponding L_p space with a constant depending on q .

The q -dependent term $x_{ij} + q x_{ji}$ above comes from the symmetrization operator P_2 on $\mathcal{H}^2$ for a Hilbert space $\mathcal{H}$ defined by

$$P_2(f_1 \otimes f_2) = f_1 \otimes f_2 + q f_2 \otimes f_1$$

for any $f_1, f_2 \in \mathcal{H}$. See section 2.1 for the details. Moreover, in section 3 we also extend this result to polynomials of arbitrary degree. The notion becomes rather involved and the estimates depend on the degree of the polynomial.

Finally, let us move our attention to the extremal cases $q = 1$ and $q = -1$, where we can find singularities. In these special cases we have to use a decoupling technique. Indeed, it is well known from Banach space theory (see e.g. the book of Ledoux and Talagrand ([19])) that

$$\mathbb{E} \left\| \sum_{i \neq j} x_{ij} \otimes \varepsilon_i \varepsilon_j \right\|_X \sim_c \mathbb{E} \left\| \sum_{i \neq j} x_{ij} \otimes \varepsilon_i \varepsilon'_j \right\|_X$$

holds for Banach-space-valued coefficients $x_{ij} \in X$. Here $\varepsilon_i, \varepsilon'_j$ are independent coordinate functions defined on $\{-1, 1\}^n \times \{-1, 1\}^n$ and $\mathbb{E}$ is the corresponding expectation. Based on new decoupling techniques we prove the following.

Theorem 1.3. *Let $q = \pm 1$ and $(g_{q,i})_{i=1}^m, D_q, (g_i)_{i=1}^m$ and D be as in Theorem 1.1. Let $1 \leq p < \infty$ and $(x_{ij})_{i,j=1}^m$ be $L_p(\mathcal{M})$ -valued coefficients for a von Neumann algebra $\mathcal{M}$.*

i) *Let $q = -1$ and $x_{ij} = -x_{ji}$. Then*

$$\left\| \sum_{i,j} x_{ij} \otimes D_{-1}^{\frac{1}{2p}} g_{-1,i} g_{-1,j} D_{-1}^{\frac{1}{2p}} \right\|_p \sim \left\| \sum_{i,j} x_{ij} \otimes D^{\frac{1}{2p}} g_i D^{\frac{1}{2p}} \otimes D^{\frac{1}{2p}} g_j D^{\frac{1}{2p}} \right\|_p.$$

ii) *Let $q = 1$ and $x_{ij} = x_{ji}$. Then*

$$\left\| \sum_{i,j} x_{ij} \otimes D_1^{\frac{1}{2p}} g_{1,i} g_{1,j} D_1^{\frac{1}{2p}} \right\|_p \sim \left\| \sum_{i,j} x_{ij} \otimes D^{\frac{1}{2p}} g_i D^{\frac{1}{2p}} \otimes D^{\frac{1}{2p}} g_j D^{\frac{1}{2p}} \right\|_p.$$

Here again the right hand side can be calculated using martingale inequalities for the linear terms. This leads to a four-terms maximum for $p \geq 2$ and a four-terms decomposition for $1 \leq p \leq 2$. The formulae are decisively different from the three-terms expression in Theorem 1.1. Let us refer to section 4 for the concrete expressions. Again the result also holds for polynomials of higher degree, but we refer to the text for the explicit formulation. In spirit our method is closely related to similar estimates for polynomials on the free group by Haagerup and Pisier. However, in our approach the decoupling is derived from Speicher's central limit procedure combined with the ultraproduct technique from [10].

The paper is organized as follows. In section 2 we present some preliminaries we need in the sequel. That includes the Fock space realization of the q -(generalized) Gaussian random variables for $-1 \leq q \leq 1$, Wick product and some modular theory for the case $-1 < q < 1$, and matrix models for the case $q = \pm 1$. At the end of section 2 we collect some notation mainly concerned with complicated indices we will encounter. In section 3 we focus on the case $-1 < q < 1$. We first establish the free case using the result in [11] and obtain an appropriate interpolation scale. Using this interpolation scale and the approach of Nou ([20]) we can obtain the result for the general case $-1 < q < 1$. In the final section we consider the remaining cases $q = \pm 1$.

We assume that the reader is familiar with standard concepts in operator algebra ([29, 30]), operator space theory ([5, 24]), noncommutative L_p spaces ([13, 25]), and the related complex interpolation theory ([15, 31, 25]). For a von Neumann algebra $\mathcal{M}$ we denote the noncommutative L_p ($1 \leq p \leq \infty$) space with respect to $\mathcal{M}$ by $L_p(\mathcal{M})$. When $E \subseteq L_p(\mathcal{N})$ for another von Neumann algebra $\mathcal{N}$, the norm

closure of the algebraic tensor product $L_p(\mathcal{M}) \otimes E$ in $L_p(\mathcal{M} \bar{\otimes} \mathcal{N})$ will be denoted by $L_p(\mathcal{M}; E)$. For a Hilbert space H we write $S_p(H) = L_p(B(H))$ and H_r and H_c imply the row and the column Hilbert spaces on H , respectively. When $H = \ell_2$ we simply write S_p , R and C , respectively. R_p and C_p imply the linear space of the first row and column of S_p , respectively.

2. PRELIMINARIES AND NOTATION

2.1. The Fock space realization. We start with the Fock space realization of q -commutation relations for $-1 \leq q \leq 1$. Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space equipped with an orthonormal basis $(e_{\pm k})_{k \geq 1}$. We denote by $\mathcal{F}_0(\mathcal{H})$ the associated free Fock space

$$\mathcal{F}_0(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n \geq 1} \mathcal{H}^{\otimes n},$$

where Ω is a unit vector called a vacuum. We consider the operator of symmetrization P_n on $\mathcal{H}^{\otimes n}$ defined by

$$P_0\Omega = \Omega, \\ P_n(f_1 \otimes \cdots \otimes f_n) = \sum_{\pi \in S_n} q^{i(\pi)} f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)},$$

where S_n denotes the symmetric group of permutations of n elements and

$$i(\pi) = \#\{(i, j) | 1 \leq i, j \leq n, \pi(i) > \pi(j)\}$$

is the number of inversions of $\pi \in S_n$.

Now we define the q -inner product $\langle \cdot, \cdot \rangle_q$ on $\mathcal{F}_0(\mathcal{H})$ by

$$\langle \xi, \eta \rangle_q = \delta_{n,m} \langle \xi, P_n \eta \rangle \quad \text{for } \xi \in \mathcal{H}^{\otimes n}, \eta \in \mathcal{H}^{\otimes m},$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathcal{H}$. Since the P_n 's are strictly positive for $-1 < q < 1$ ([3]), $\langle \cdot, \cdot \rangle_q$ is actually an inner product. In this case, the Hilbert space $\mathcal{F}_0(\mathcal{H})$ equipped with the q -inner product is called a q -Fock space on $\mathcal{H}$, and we denote it by $\mathcal{F}_q(\mathcal{H})$. When $q = \pm 1$, the P_n 's are just positive, so we define $\mathcal{F}_q(\mathcal{H})$ as the quotient of $(\mathcal{F}_0(\mathcal{H}), \langle \cdot, \cdot \rangle_q)$ by the corresponding kernel.

For two sequences of strictly positive reals $\lambda = (\lambda_k)_{k \geq 1}$ and $\mu = (\mu_k)_{k \geq 1}$ we define q -(generalized) Gaussian variables (or q -generalized circular variables) by

$$g_{q,k} = \lambda_k \ell_q(e_k) + \mu_k \ell_q^*(e_{-k}),$$

where $\ell_q(h)$ is the left creation operator by $h \in \mathcal{H}$ and $\ell_q^*(h)$ is the adjoint of $\ell_q(h)$. Note that

$$\ell_q^*(h)(\Omega) = 0 \quad \text{and}$$

$$\ell_q^*(h)(f_1 \otimes \cdots \otimes f_n) = \sum_{k=1}^n q^{k-1} \langle h, f_k \rangle f_1 \otimes \cdots \otimes f_{k-1} \otimes f_{k+1} \otimes \cdots \otimes f_n.$$

When $q = \pm 1$ it is easy to check that the $g_{q,k}$'s satisfy the q -commutation relations

$$g_{q,k}^* \cdot g_{q,j} - q \cdot g_{q,j} \cdot g_{q,k}^* = \delta_{kj} (\lambda_k^2 + \mu_k^2) I \quad \text{and} \\ g_{q,k} \cdot g_{q,j} - q \cdot g_{q,j} \cdot g_{q,k} = 0,$$

which implies that the $\frac{1}{\sqrt{\lambda_k^2 + \mu_k^2}} g_{q,k}$'s are CAR and CCR sequences in the corresponding cases.

We focus on Γ_q ($-1 \leq q < 1$), the von Neumann algebra generated by $\{g_{q,k}\}_{k \geq 1}$. When $q = 1$ we define Γ_1 by the von Neumann algebra generated by $\{\exp(i \cdot g_{1,k})\}_{k \geq 1}$ since the $g_{1,k}$'s are unbounded operators in this case.

There is a canonical way to translate the above picture into the framework of the work of Shlyakhtenko and Hiai ([28, 6]). It starts with a real Hilbert space $H_{\mathbb{R}}$ and a one-parameter group $(U_t)_{t \in \mathbb{R}}$ of orthogonal operators on $H_{\mathbb{R}}$. Let $H_{\mathbb{C}} = H_{\mathbb{R}} + iH_{\mathbb{R}}$ be the complexification of $H_{\mathbb{R}}$. Then (U_t) extends linearly to a group of unitary maps on $H_{\mathbb{C}}$. Now we have $U_t = A^{it}$ for a closed unbounded operator A on $H_{\mathbb{C}}$, and we define another inner product $\langle \cdot, \cdot \rangle_U$ on $H_{\mathbb{C}}$ by

$$\langle x, y \rangle_U = \langle 2A(1 + A)^{-1}x, y \rangle$$

for any $x, y \in H_{\mathbb{C}}$. According to [28, section 4] we can associate $g_{q,k}$ with an action

$$U_t^k = \begin{pmatrix} \cos \theta_k t & -\sin \theta_k t \\ \sin \theta_k t & \cos \theta_k t \end{pmatrix},$$

where $\theta_k = \log[(\frac{\mu_k}{\lambda_k})^2]$ on $H_k (\cong \mathbb{R}^2)$. Note that the orthonormal basis $(\tilde{e}_k, \tilde{e}_{-k})$ on H_k for this matrix representation is given by

$$(2.1) \quad \begin{pmatrix} \tilde{e}_k \\ \tilde{e}_{-k} \end{pmatrix} = \frac{1}{\sqrt{\alpha_k + 1}} \begin{pmatrix} -i & \sqrt{\alpha_k} i \\ 1 & \sqrt{\alpha_k} \end{pmatrix} \begin{pmatrix} e_k \\ e_{-k} \end{pmatrix} = V \begin{pmatrix} e_k \\ e_{-k} \end{pmatrix},$$

where $\alpha_k = \lambda_k^{-2} \mu_k^2$, and the $e_{\pm k}$'s are eigenvectors for A .

Now we set $H_{\mathbb{R}} = \bigoplus_{k \geq 1} H_k$ with orthonormal basis $\{\tilde{e}_{\pm k} : k \geq 1\}$ and $U_t = \bigoplus_{k \geq 1} U_t^k$. Then, we recover $\mathcal{H}$ by $(H_{\mathbb{C}}, \langle \cdot, \cdot \rangle_U)$. Moreover, we have

$$(2.2) \quad g_{q,k} = \sqrt{\lambda_k^2 + \mu_k^2} (s_q(\tilde{e}_{-k}) - i s_q(\tilde{e}_k)),$$

where $s_q(\tilde{e}_k)$ and $s_q(\tilde{e}_{-k})$ are (q) -semi-circular variables defined by

$$s_q(\tilde{e}_{\pm k}) = \frac{1}{2}(\ell_q(\tilde{e}_{\pm k}) + \ell_q^*(\tilde{e}_{\pm k})).$$

Thus we have

$$\Gamma_q(H_{\mathbb{R}}, U_t) = \{s_q(\tilde{e}_{\pm k}) : k \geq 1\}'' = \Gamma_q.$$

Note that $A = \bigoplus_{k \geq 1} A_k$ with $U_t^k = A_k^{it}$, and by taking the conjugate with respect to the basis change matrix V in (2.1) we get the matrix representation of A_k with respect to (e_k, e_{-k}) by

$$V^{-1} A_k V = \begin{pmatrix} \lambda_k^2 \mu_k^{-2} & 0 \\ 0 & \lambda_k^{-2} \mu_k^2 \end{pmatrix},$$

which implies that

$$(2.3) \quad A e_k = \lambda_k^2 \mu_k^{-2} e_k$$

for any $k \geq 1$.

We close this section with some modular theory for Γ_q and ϕ_q ($-1 < q < 1$), the vacuum state defined by $\phi_q(\cdot) = \langle \cdot, \Omega \rangle_q$. It is well known that the modular group σ_t with respect to ϕ_q satisfies the following:

$$\sigma_t(g_{q,k}) = (\lambda_k^{-1} \mu_k)^{2it} g_{q,k}.$$

Thus, $g_{q,k}$ is an analytic element satisfying

$$(2.4) \quad D_q^{\frac{1}{2p}} g_{q,k} = (\lambda_k^{-1} \mu_k)^{\frac{1}{p}} g_{q,k} D_q^{\frac{1}{2p}}.$$

Recall that the anti-linear map S is the closure of the operator given by

$$S(x\Omega) = x^*\Omega$$

for all $x \in \Gamma_q$. Then S can be written as $S = J\Delta^{\frac{1}{2}}$, where J is the modular conjugation and Δ is the modular operator. By [6] we know that Δ is the closure of the operator $\bigoplus_{n \geq 0} (A^{-1})^{\otimes n}$, which implies that

$$(2.5) \quad S|_{\mathcal{H}^{\otimes n}} = J(A^{-\frac{1}{2}})^{\otimes n}.$$

2.2. The Wick product for the case $-1 < q < 1$. Let $-1 < q < 1$. Since it is well known that Ω is separating for Γ_q , for every $\xi \in \Gamma_q \Omega$ there exists a unique operator $W(\xi) \in \Gamma_q$ such that

$$W(\xi)\Omega = \xi.$$

This W is called the *Wick product*, and from the definition we easily get

$$W(\lambda_{i_1} e_{i_1} \otimes \cdots \otimes \lambda_{i_n} e_{i_n}) = g_{q,i_1} \cdots g_{q,i_n}$$

for any $i_1, \dots, i_n \geq 1$. There is a useful decomposition of the Wick product (known as the Wick formula) as follows (see [20, Theorem 3] and the preceding remark) when the operator A is bounded. Note that the operator A is bounded when φ is tracial (i.e. $\lambda_k = \mu_k$ for any $k \geq 1$) or $\mathcal{H}$ is finite dimensional.

$$(2.6) \quad W(\xi) = \sum_{k=0}^n U_k R_{n,k}^*(\xi)$$

for $\xi \in \mathcal{H}^{\otimes n}$, where $R_{n,k}$ is the operator on $\mathcal{H}^{\otimes n}$ given by

$$R_{n,k}(f_1 \otimes \cdots \otimes f_n) = \sum_{\pi \in S_n / S_{n-k} \times S_k} q^{i(\pi)} f_{\pi^{-1}(1)} \otimes \cdots \otimes f_{\pi^{-1}(n)}$$

and U_k is the operator defined on simple tensors by

$$U_k(h_1 \otimes \cdots \otimes h_n) = \ell(h_1) \cdots \ell(h_{n-k}) \ell^*(\overline{h_{n-k+1}}) \cdots \ell^*(\overline{h_n})$$

for any $h_1, \dots, h_n \in \mathcal{H}$. In the above, $S_n / S_{n-k} \times S_k$ means the representatives of the right cosets of $S_{n-k} \times S_k$ in S_n with minimal numbers of inversions. Note also that the conjugation of vectors in $\mathcal{H}$ is not the standard one, but the one coming from $a\tilde{e}_k + b\tilde{e}_{-k} \mapsto \bar{a}\tilde{e}_k + \bar{b}\tilde{e}_{-k}$, $a, b \in \mathbb{C}$.

It is well known that

$$(2.7) \quad P_n = R_{n,k}(P_{n-k} \otimes P_k).$$

We can estimate the norm of P_n and its inverse. In [4] it is shown that

$$P_n \leq \frac{1}{1-q} I_1 \otimes P_{n-1},$$

where I_1 is the formal identity on H ; thus by repeating the above operator inequality we have

$$(2.8) \quad \|P_n : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes n}\| \leq \left(\frac{1}{1-q} \right)^{n-1}.$$

Moreover, it is shown in [2] that $I_1 \otimes P_{n-1} \leq w(q)^{-1} P_n$ for some constant $w(q) > 0$; thus by repeating again we have

$$(2.9) \quad \|P_n^{-1} : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes n}\| \leq w(q)^{-n+1}.$$

From the definition it is clear that

$$(2.10) \quad P_n^{\frac{1}{2}} : H^{\otimes n} \rightarrow \mathcal{H}^{\otimes n}$$

is an isometry, where $H^{\otimes k}$ is the Hilbert space on $\mathcal{H}^{\otimes k}$ equipped with the q -inner product.

2.3. Matrix models for CAR and CCR generators. Now we focus on the case $q = \pm 1$. First, we fix $m \in \mathbb{N}$ and restrict our attention to $g_{q,1}, \dots, g_{q,m}$. Note that $a_k = \frac{1}{\sqrt{\lambda_k^2 + \mu_k^2}} g_{q,k}$'s are CAR and CCR sequences in the corresponding cases. Thus, we can find a matrix model for $(a_k)_{k=1}^m$ as in section 3 of [10]. Let $N = \ell_\infty^{2m}(M_2)$ and ψ be the weight on N associated to $(a_k)_{k=1}^m$ defined by

$$\psi(x) = \sum_{1 \leq |k| \leq m} [(1 - \sigma_k) x_{11}(k) + \sigma_k x_{22}(k)]$$

for $x = [(x_{ij}(\pm 1))_{i,j=1}^2, \dots, (x_{ij}(\pm m))_{i,j=1}^2] \in N$. Actually, ψ contains the information of $(a_k)_{k=1}^m$ by

$$\sigma_k = \phi_q(a_k^* a_k) = \frac{\lambda_k^2}{\lambda_k^2 + \mu_k^2}.$$

Let

$$(2.11) \quad u_n(x) = \sqrt{\frac{2m}{n}} \sum_{k=1}^n v_k \otimes \pi_k(x),$$

where $\pi_k : N \rightarrow N^{\otimes n}$ is the homomorphism which maps N into the k -th component and $v_k \in M_{2^n}$ are selfadjoint unitaries such that

$$(2.12) \quad v_k v_j = -v_j v_k \text{ (when } q = -1) \text{ or } v_k v_j = v_j v_k \text{ (when } q = 1).$$

Note that we are using the scaling factor $2m$ since $\psi(1_N) = 2m$ in this situation. Actually, we will use the following special choice of v_k 's. Let

$$v_{k,k} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and for } j < k, \quad v_{j,k} = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}.$$

Then we set

$$(2.13) \quad v_k = v_{1,k} \otimes \dots \otimes v_{k,k} \otimes 1 \otimes \dots \otimes 1 \in M_{2^n}.$$

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and

$$N_{\mathcal{U}} = \left(\prod_{n, \mathcal{U}} (M_{2^n} \otimes N^{\otimes n})_* \right)^*$$

with the ultraproduct state

$$\phi_{\mathcal{U}} = (\tau_n \otimes \left(\frac{\psi}{2m} \right)^{\otimes n})_{n, \mathcal{U}},$$

where τ_n is the normalized trace on M_{2^n} . Let $(\delta_k)_{1 \leq |k| \leq m}$ be the unit vectors in ℓ_∞^{2m} . In [10] it is shown that $(u_n(\delta_k \otimes e_{12}))_{1 \leq |k| \leq m}$ (with respect to ϕ_u) converges in $*$ -distribution to $(a_k)_{1 \leq |k| \leq m}$ (with respect to ϕ_q) as n goes infinity, so that

$$(u_n(b_k))_{1 \leq |k| \leq m} \text{ converges in } * \text{-distribution to } (g_{q,k})_{1 \leq |k| \leq m},$$

where $b_k = \sqrt{\lambda_k^2 + \mu_k^2} \delta_k \otimes e_{12}$.

If we consider the map

$$\Phi : L_p(\Gamma_{\pm 1}) \rightarrow L_p(N_{\mathcal{U}}) \quad (1 \leq p < \infty)$$

defined by

$$\Phi(D_{\pm 1}^{\frac{1}{2p}} P[(g_{q,k})_{1 \leq |k| \leq m}] D_{\pm 1}^{\frac{1}{2p}}) = D_{\phi_u}^{\frac{1}{2p}} \left(P[(u_n(b_k))_{1 \leq |k| \leq m}] \right)_{n, \mathcal{U}} D_{\phi_u}^{\frac{1}{2p}}$$

for any noncommutative polynomial P , then we know that

$$(2.14) \quad I_{\mathcal{M}} \otimes \Phi : L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1}) \rightarrow L_p(\mathcal{M} \bar{\otimes} N_{\mathcal{U}})$$

is an isometry for any von Neumann algebra $\mathcal{M}$.

When $q = 1$ we shall understand $g_{1,k} D_1^{\frac{1}{2p}}$ as $\frac{d}{idt} (\exp(itg_{1,k}) D_1^{\frac{1}{2p}})|_{t=0}$.

2.4. Notation. In the following we will frequently use the index sets

$$\mathcal{I}^d = \{\underline{i} = (i_1, \dots, i_d) \in \mathbb{N}^d : i_1 \neq \dots \neq i_d\}$$

and for $m \in \mathbb{N}$,

$$\mathcal{I}_m^d = \{\underline{i} = (i_1, \dots, i_d) \in \mathbb{N}^d : 1 \leq i_1 \neq \dots \neq i_d \leq m\},$$

where $i_1 \neq \dots \neq i_d$ means that neighboring indices are different. If we remove that restriction, we have the index sets $\mathbb{N}^d$ and

$$\mathbb{N}_m^d = \{\underline{i} = (i_1, \dots, i_d) \in \mathbb{N}^d : 1 \leq i_1, \dots, i_d \leq m\},$$

respectively

For $\underline{i} \in \mathcal{I}_m^d$ or $\mathbb{N}_m^d$ we will use the notation

$$\begin{aligned} g_{q,\underline{i}} &:= g_{q,i_1} \cdots g_{q,i_d}, \\ g_{p,q,\underline{i}} &:= D_q^{\frac{1}{2p}} g_{q,i_1} \cdots g_{q,i_d} D_q^{\frac{1}{2p}} \end{aligned}$$

and

$$e_{\underline{i}} = e_{i_1} \otimes \cdots \otimes e_{i_d}.$$

We use the symbol $a \lesssim b$ if there is a $C > 0$ such that $a \leq Cb$ and $a \sim b$ if $a \lesssim b$ and $b \lesssim a$. Similarly, $a \lesssim_{c_d} b$ (resp. $a \lesssim_{c_{q,d}} b$) if there is a constant $C > 0$ depending only on d (resp. only on q and d) such that $a \leq Cb$. We write $a \sim_{c_d} b$ if $a \lesssim_{c_d} b$ and $b \lesssim_{c_d} a$, and the meaning of $a \sim_{c_{q,d}} b$ is similar.

From now on we fix two sequences of strictly positive reals $\lambda = (\lambda_k)_{k \geq 1}$ and $\mu = (\mu_k)_{k \geq 1}$, the number of degree $d \in \mathbb{N}$ and a von Neumann algebra $\mathcal{M}$. $\mathcal{M}$ is equipped with a distinguished normal semifinite faithful weight φ which induces the trace functional $\text{tr}_{\mathcal{M}}$ on $L_1(\mathcal{M})$. We denote the unit of $\mathcal{M}$ by $1_{\mathcal{M}}$ whilst $I_{\mathcal{M}}$ implies the identity map on $\mathcal{M}$.

3. THE CASE $-1 < q < 1$

3.1. Free case: Building a model. In this section we consider the free case ($q = 0$) based on the reduction method from [11]. We will first prove results for polynomials indexed by $\mathcal{I}^d$. Then, we will get an interpolation scale for the index set $\mathcal{I}^d$, which can be extended to the interpolation scale for the general index set $\mathbb{N}^d$ by applying the same argument to the modified polynomials, which will be introduced later in this section. This extended interpolation scale enables us to establish L_p norm estimates for the general index set $\mathbb{N}^d$ even in the free case. We are grateful to the referee for pointing out that a more general result should hold for the index set $\mathbb{N}^d$ instead of $\mathcal{I}^d$.

We will simply write g_k , Γ and ϕ instead of $g_{0,k}$, Γ_0 and ϕ_0 , respectively. For $\underline{i} \in \mathcal{I}_m^d$ or $\mathbb{N}_m^d$ we will use the notation

$$g_{\underline{i}} := g_{i_1} \cdots g_{i_d} \text{ and } g_{p,\underline{i}} := D^{\frac{1}{2p}} g_{i_1} \cdots g_{i_d} D^{\frac{1}{2p}}.$$

For $x = (x_{\underline{i}})_{\underline{i} \in \mathcal{I}_m^d} \subseteq L_p(\mathcal{M})$, $0 \leq k \leq d$, and $\frac{1}{p} + \frac{1}{p'} = 1$ we denote by

$$\mathcal{RC}_p^{d,k}(\lambda, \mu; x) = \left\| \sum_{\underline{i} \in \mathcal{I}_m^d} x_{\underline{i}} \otimes f^{p,i_1} \otimes \cdots \otimes f^{p,i_k} \otimes f_{p,i_{k+1}} \otimes \cdots \otimes f_{p,i_d} \right\|_{L_p(\mathcal{M}; E_p^{d,k})},$$

where

$$f^{p,i} = \lambda_i^{\frac{1}{p'}} \mu_i^{\frac{1}{p}} e_{i1} \in C_p, \quad f_{p,i} = \lambda_i^{\frac{1}{p}} \mu_i^{\frac{1}{p'}} e_{1i} \in R_p$$

and

$$E_p^{d,k} = C_p^{\otimes k} \otimes_h R_p^{\otimes d-k} \subseteq S_p(\ell_2^{\otimes d}).$$

We define the corresponding $\mathcal{K}$ - and $\mathcal{J}$ -functional spaces $\tilde{\mathcal{K}}_p^d(\lambda, \mu)$ ($1 \leq p \leq 2$) and $\tilde{\mathcal{J}}_p^d(\lambda, \mu)$ ($2 \leq p \leq \infty$) as the closures of finite tuples in $L_p(\mathcal{M})$ indexed by $\mathcal{I}^d$ with respect to the following norms:

$$\|x\|_{\tilde{\mathcal{K}}_p^d(\lambda, \mu)} = \inf \left\{ \mathcal{RC}_p^{d,0}(\lambda, \mu; x^0) + \mathcal{RC}_p^{d,1}(\lambda, \mu; x^1) + \cdots + \mathcal{RC}_p^{d,d}(\lambda, \mu; x^d) \right\},$$

where the infimum runs over all possible $x_{\underline{i}} = x_{\underline{i}}^0 + \cdots + x_{\underline{i}}^d$ and $x^k = (x_{\underline{i}}^k)_{\underline{i} \in \mathcal{I}_m^d}$ for $0 \leq k \leq d$, and

$$\|x\|_{\tilde{\mathcal{J}}_p^d(\lambda, \mu)} = \max \left\{ \mathcal{RC}_p^{d,0}(\lambda, \mu; x), \mathcal{RC}_p^{d,1}(\lambda, \mu; x), \dots, \mathcal{RC}_p^{d,d}(\lambda, \mu; x) \right\}.$$

Note that $\tilde{\mathcal{J}}_p^d(\lambda, \mu) = (\tilde{\mathcal{K}}_p^d(\lambda, \mu))^*$ with the duality bracket

$$\langle (x_{\underline{i}})_{\underline{i} \in \mathcal{I}^d}, (z_{\underline{i}})_{\underline{i} \in \mathcal{I}^d} \rangle = \sum_{\underline{i} \in \mathcal{I}^d} \lambda_{\underline{i}} \mu_{\underline{i}} \operatorname{tr}_{\mathcal{M}}(x_{\underline{i}}^* z_{\underline{i}}).$$

Theorem 3.1. *Let $m \in \mathbb{N}$ and $x = (x_{\underline{i}})_{\underline{i} \in \mathcal{I}_m^d} \subseteq L_p(\mathcal{M})$. Then we have the following equivalence:*

$$\left\| \sum_{\underline{i} \in \mathcal{I}_m^d} x_{\underline{i}} \otimes g_{p,\underline{i}} \right\|_{L_p(\mathcal{M} \otimes \Gamma)} \sim_{c_d} \begin{cases} \|x\|_{\tilde{\mathcal{K}}_p^d(\lambda, \mu)} & \text{for } 1 \leq p \leq 2, \\ \|x\|_{\tilde{\mathcal{J}}_p^d(\lambda, \mu)} & \text{for } 2 \leq p \leq \infty. \end{cases}$$

Moreover, $\tilde{\mathcal{G}}_p^d$, the closed subspace of $L_p(\Gamma)$ spanned by

$$\{g_{p,\underline{i}} : \underline{i} \in \mathcal{I}^d\},$$

is completely complemented with the constant depending only on d .

Proof. We consider the following maps:

$$u_p : \begin{cases} \tilde{\mathcal{K}}_p^d(\lambda, \mu) & (1 \leq p \leq 2) \\ \tilde{\mathcal{J}}_p^d(\lambda, \mu) & (2 \leq p \leq \infty) \end{cases} \rightarrow L_p(\mathcal{M} \bar{\otimes} \Gamma),$$

$$(x_{\underline{i}})_{\underline{i} \in \mathcal{I}_m^d} \mapsto \sum_{\underline{i} \in \mathcal{I}_m^d} x_{\underline{i}} \otimes g_{p, \underline{i}}.$$

What we need to do is to show that the u_p 's are isomorphisms with bounded constants. By a usual density argument it is enough to check that the u_p 's are isomorphisms when they are restricted to the span of $(x_{\underline{i}})_{\underline{i} \in \mathcal{I}_m^d}$ for a fixed $m \in \mathcal{N}$ with constants independent of m .

First, we consider the case $2 \leq p \leq \infty$ and will use an induction on d . When $d = 1$ we are done by [33, Theorem 3.1] (or [11, Theorem 5.1]). Suppose we have the result for d and consider the case $d + 1$.

We will simply write $(i, \underline{i}) \in \mathcal{I}_m^{d+1}$ with $\underline{i} \in \mathcal{I}_m^d$. In this case we will use the notation $x_{i, \underline{i}}$ and $g_{i, \underline{i}}$ instead of $x_{\underline{i}'}$ and $g_{\underline{i}'}$, respectively, where

$$\underline{i}' = (i, i_1, \dots, i_d) \in \mathcal{I}_m^{d+1}.$$

For $x = (x_{i, \underline{i}})_{(i, \underline{i}) \in \mathcal{I}_m^{d+1}} \subseteq L_p(\mathcal{M})$ we consider

$$\begin{aligned} \sum_{(i, \underline{i}) \in \mathcal{I}_m^{d+1}} x_{i, \underline{i}} \otimes D^{\frac{1}{2p}} g_{i, \underline{i}} D^{\frac{1}{2p}} &= \sum_{(i, \underline{i}) \in \mathcal{I}_m^{d+1}} (x_{i, \underline{i}} \otimes D^{\frac{1}{2p}} g_{i, \underline{i}} D^{\frac{1}{2p}}) (1_{\mathcal{M}} \otimes \lambda_{\underline{i}}^{\frac{1}{p}} \mu_{\underline{i}}^{-\frac{1}{p}} g_{\underline{i}}) \\ &= \sum_{(i, \underline{i}) \in \mathcal{I}_m^{d+1}} \alpha_{i, \underline{i}} \beta_{\underline{i}}. \end{aligned}$$

Now we apply [11, Theorem B]. Let $\mathbb{E} : \mathcal{M} \bar{\otimes} \Gamma \rightarrow \mathcal{M} \bar{\otimes} \Gamma$ be the map given by

$$\mathbb{E}(x \otimes y) = x \otimes \phi(y) 1_{\Gamma}$$

for any $x \in \mathcal{M}$ and $y \in \Gamma$. It is just the composition of the conditional expectation $I_{\mathcal{M}} \otimes \phi : \mathcal{M} \bar{\otimes} \Gamma \rightarrow \mathcal{M}$ and the natural inclusion $\mathcal{M} \hookrightarrow \mathcal{M} \bar{\otimes} \Gamma$. Then, for any $1 \leq p < \infty$, $\mathbb{E}$ induces the projection $\mathbb{E}_p : L_p(\mathcal{M} \bar{\otimes} \Gamma) \rightarrow L_p(\mathcal{M} \bar{\otimes} \Gamma)$ given by

$$\mathbb{E}_p(x \otimes D^{\frac{1}{2p}} y D^{\frac{1}{2p}}) = x \otimes \phi(y) D^{\frac{1}{p}}$$

for any $x \in L_p(\mathcal{M})$ and $y \in \Gamma$ (see [13, section 2]).

By applying [11, Theorem B] we have

$$\begin{aligned} &\left\| \sum_{(i, \underline{i}) \in \mathcal{I}_m^{d+1}} \alpha_{i, \underline{i}} \beta_{\underline{i}} \right\|_p \\ &\sim \left\| \left(\sum_{(i, \underline{i}), (j, \underline{j}) \in \mathcal{I}_m^{d+1}} \beta_{\underline{i}}^* \mathbb{E}_{2p}(\alpha_{i, \underline{i}}^* \alpha_{j, \underline{j}}) \beta_{\underline{j}} \right)^{\frac{1}{2}} \right\|_p + \left\| \left(\sum_{(i, \underline{i}), (j, \underline{j}) \in \mathcal{I}_m^{d+1}} \alpha_{i, \underline{i}} \mathbb{E}(\beta_{\underline{i}} \beta_{\underline{j}}^*) \alpha_{j, \underline{j}}^* \right)^{\frac{1}{2}} \right\|_p \\ &= A + B. \end{aligned}$$

By (2.4) we have

$$\begin{aligned}\mathbb{E}_{2p}(\alpha_{i,\underline{i}}^* \alpha_{j,\underline{j}}) &= \mathbb{E}_{2p}(x_{i,\underline{i}}^* x_{j,\underline{j}} \otimes D^{\frac{1}{2p}} g_i^* D^{\frac{1}{p}} g_j D^{\frac{1}{2p}}) \\ &= \lambda_i^{-\frac{1}{p}} \mu_i^{\frac{1}{p}} \lambda_j^{-\frac{1}{p}} \mu_j^{\frac{1}{p}} x_{i,\underline{i}}^* x_{j,\underline{j}} \otimes \phi(g_i^* g_j) D^{\frac{2}{p}} \\ &= \delta_{ij} \lambda_i^{\frac{2}{p'}} \mu_i^{\frac{2}{p}} x_{i,\underline{i}}^* x_{i,\underline{i}},\end{aligned}$$

so that we get the following by the induction hypothesis (applied to $L_p(\mathcal{M} \bar{\otimes} \Gamma)$):

$$\begin{aligned}A &= \left\| \left(\sum_{i=1}^n \lambda_i^{\frac{2}{p'}} \mu_i^{\frac{2}{p}} \sum_{(i,\underline{i}), (j,\underline{j}) \in \mathcal{I}_m^{d+1}} x_{i,\underline{i}}^* x_{j,\underline{j}} \otimes \lambda_{\underline{i}}^{\frac{1}{p}} \mu_{\underline{i}}^{-\frac{1}{p}} \lambda_{\underline{j}}^{\frac{1}{p}} \mu_{\underline{j}}^{-\frac{1}{p}} g_{\underline{i}}^* D^{\frac{2}{p}} g_{\underline{j}} \right)^{\frac{1}{2}} \right\|_p \\ &= \left\| \sum_{(i,\underline{i}) \in \mathcal{I}_m^{d+1}} x_{i,\underline{i}} \otimes f^{p,i} \otimes \lambda_{\underline{i}}^{\frac{1}{p}} \mu_{\underline{i}}^{-\frac{1}{p}} D^{\frac{1}{p}} g_{\underline{i}} \right\|_p \\ &= \left\| \sum_{(i,\underline{i}) \in \mathcal{I}_m^{d+1}} x_{i,\underline{i}} \otimes f^{p,i} \otimes g_{p,\underline{i}} \right\|_p \\ &\sim_{c_d} \max \left\{ \mathcal{RC}_p^{d+1,1}(\lambda, \mu, x), \dots, \mathcal{RC}_p^{d+1,d+1}(\lambda, \mu, x) \right\}.\end{aligned}$$

Similarly we have

$$\begin{aligned}\mathbb{E}(\beta_{\underline{i}} \beta_{\underline{j}}^*) &= 1_{\mathcal{M}} \otimes \lambda_{\underline{i}}^{\frac{1}{p}} \mu_{\underline{i}}^{-\frac{1}{p}} \lambda_{\underline{j}}^{\frac{1}{p}} \mu_{\underline{j}}^{-\frac{1}{p}} \phi(g_{\underline{i}} g_{\underline{j}}^*) 1_{\Gamma} \\ &= \delta_{\underline{i}, \underline{j}} \lambda_{\underline{i}}^{\frac{2}{p}} \mu_{\underline{i}}^{\frac{2}{p'}} 1_{\mathcal{M}} \otimes 1_{\Gamma},\end{aligned}$$

so that

$$\begin{aligned}B &= \left\| \left(\sum_{\underline{i} \in \mathbb{N}_m^d} \lambda_{\underline{i}}^{\frac{2}{p}} \mu_{\underline{i}}^{\frac{2}{p'}} \left(\sum_{i,j=1}^m x_{i,\underline{i}} x_{j,\underline{j}}^* \otimes g_{p,i} g_{p,j}^* \right) \right)^{\frac{1}{2}} \right\|_p \\ &= \left\| \sum_{(i,\underline{i}) \in \mathcal{I}_m^{d+1}} x_{i,\underline{i}} \otimes g_{p,i} \otimes f_{p,i_1} \otimes \dots \otimes f_{p,i_d} \right\|_p \\ &\sim \max \left\{ \mathcal{RC}_p^{d+1,0}(\lambda, \mu, x), \mathcal{RC}_p^{d+1,1}(\lambda, \mu, x) \right\}.\end{aligned}$$

Consequently, we get the result for $d+1$.

Now we apply a duality argument for the case $1 \leq p \leq 2$. By the above result we know that the u_p 's ($2 \leq p \leq \infty$) are isomorphisms with bounded constants independent of m . Note that for $x = (x_{\underline{i}})_{\underline{i} \in \mathcal{I}_m^d}$, $z = (z_{\underline{i}})_{\underline{i} \in \mathcal{I}_m^d}$ and $1 \leq p \leq 2$ we have

by (2.4),

$$\begin{aligned}
 & \langle u_p(x), u_{p'}(z) \rangle \\
 &= \sum_{\underline{i}, \underline{j} \in \mathcal{I}_m^d} \operatorname{tr}_{\mathcal{M}}(x_{\underline{i}}^* z_{\underline{j}}) \operatorname{tr}_{\Gamma}(g_{p, \underline{i}}^* g_{p', \underline{j}}) \\
 (3.1) \quad &= \sum_{\underline{i}, \underline{j} \in \mathcal{I}_m^d} \lambda_{\underline{i}}^{-\frac{1}{p}} \mu_{\underline{i}}^{\frac{1}{p}} \lambda_{\underline{j}}^{-\frac{1}{p'}} \mu_{\underline{j}}^{\frac{1}{p'}} \operatorname{tr}_{\mathcal{M}}(x_{\underline{i}}^* z_{\underline{j}}) \operatorname{tr}_{\Gamma}(D^{\frac{1}{p}} g_{\underline{i}}^* g_{\underline{j}} D^{\frac{1}{p'}}) \\
 &= \sum_{\underline{i}, \underline{j} \in \mathcal{I}_m^d} \lambda_{\underline{i}}^{-\frac{1}{p}} \mu_{\underline{i}}^{\frac{1}{p}} \lambda_{\underline{j}}^{-\frac{1}{p'}} \mu_{\underline{j}}^{\frac{1}{p'}} \operatorname{tr}_{\mathcal{M}}(x_{\underline{i}}^* z_{\underline{j}}) \phi(g_{\underline{i}}^* g_{\underline{j}}) \\
 &= \sum_{\underline{i} \in \mathcal{I}_m^d} \lambda_{\underline{i}} \mu_{\underline{i}} \operatorname{tr}_{\mathcal{M}}(x_{\underline{i}}^* z_{\underline{i}}) = \langle x, z \rangle.
 \end{aligned}$$

Moreover, for any $Y \in L_p(\mathcal{M} \bar{\otimes} \Gamma)$ we have

$$(3.2) \quad \langle u_p(x), Y \rangle = \langle u_p(x), u_{p'} u_p^*(Y) \rangle.$$

Combining (3.1) and (3.2) it is enough to show that the $u_{p'} u_p^*$'s ($1 \leq p \leq 2$) are bounded with constants independent of m . It is straightforward to check that the maps

$$u_{p'} u_p^* : L_{p'}(\mathcal{M} \bar{\otimes} \Gamma) \rightarrow L_{p'}(\mathcal{M} \bar{\otimes} \Gamma)$$

are essentially the same maps for $1 \leq p \leq 2$ in the sense of interpolation theory. Thus, we have only to check the norms of $u_{\infty} u_1^*$ and $u_2 u_2^*$ by complex interpolation. Since the $p = 2$ case is trivial, it suffices to show that u_1 is bounded.

Note that

$$L_2^r(\Gamma) \widehat{\otimes} L_2^c(\Gamma) \rightarrow L_1(\Gamma), \quad a \otimes b \mapsto ab$$

is a complete contraction, so that for $0 \leq k \leq d$ we have

$$\begin{aligned}
 (3.3) \quad & \left\| \sum_{\underline{i} \in \mathcal{I}_m^d} x_{\underline{i}} \otimes D^{\frac{1}{2}} g_{\underline{i}} D^{\frac{1}{2}} \right\|_{L_1(\mathcal{M} \bar{\otimes} \Gamma)} \\
 & \leq \left\| \sum_{\underline{i} \in \mathcal{I}_m^d} x_{\underline{i}} \otimes D^{\frac{1}{2}} g_{i_1} \cdots g_{i_k} \otimes g_{i_{k+1}} \cdots g_{i_d} D^{\frac{1}{2}} \right\|_{L_1(\mathcal{M}) \widehat{\otimes} L_2^r(\Gamma) \widehat{\otimes} L_2^c(\Gamma)}.
 \end{aligned}$$

Note also that $L_2^r(\Gamma) \widehat{\otimes} L_2^c(\Gamma) \cong L_2^r(\Gamma) \otimes_h L_2^c(\Gamma)$ completely isometrically. Let

$$(3.4) \quad \underline{i}^k = (i_1, \dots, i_k) \in \mathcal{I}_m^k \text{ and } \underline{i}^{d-k} = (i_{k+1}, \dots, i_d) \in \mathcal{I}_m^{d-k}.$$

Since $\{D^{\frac{1}{2}} g_{\underline{i}^k}\}_{\underline{i}^k \in \mathcal{I}_m^k}$ and $\{g_{\underline{i}^{d-k}} D^{\frac{1}{2}}\}_{\underline{i}^{d-k} \in \mathcal{I}_m^{d-k}}$ are orthogonal families of vectors in $L_2(\Gamma)$ and

$$\operatorname{tr}_{\Gamma}(D^{\frac{1}{2}} g_{\underline{i}^k} g_{\underline{i}^k}^* D^{\frac{1}{2}}) = \phi(g_{\underline{i}^k} g_{\underline{i}^k}^*) = (\mu_{i_1} \cdots \mu_{i_k})^2$$

and

$$\mathrm{tr}_\Gamma(D^{\frac{1}{2}}g_{\underline{i}^{d-k}}^*g_{\underline{i}^{d-k}}D^{\frac{1}{2}}) = (\lambda_{i_{k+1}} \cdots \lambda_{i_d})^2,$$

we obtain complete isometries

$$\mathrm{span}\{D^{\frac{1}{2}}g_{\underline{i}^k}^*\}_{\underline{i}^k \in \mathcal{I}_m^k} (\subseteq L_2^r(\Gamma)) \cong \mathrm{span}\{f^{1,i_1} \otimes \cdots \otimes f^{1,i_k}\}_{\underline{i}^k \in \mathcal{I}_m^k} \subseteq C_1^{\otimes k}$$

and

$$\mathrm{span}\{g_{\underline{i}^{d-k}}D^{\frac{1}{2}}\}_{\underline{i}^{d-k} \in \mathcal{I}_m^{d-k}} (\subseteq L_2^c(\Gamma)) \cong \mathrm{span}\{f_{1,i_{k+1}} \otimes \cdots \otimes f_{1,i_d}\}_{\underline{i}^{d-k} \in \mathcal{I}_m^{d-k}} \subseteq R_1^{\otimes d-k}.$$

This implies that u_1 is a contraction.

Now we have isomorphisms u_p for $1 \leq p \leq \infty$ with bounded constants (depending only on d). Thus the $\tilde{\mathcal{G}}_p^d$ s are completely complemented with constants depending only on d by the following projections:

$$u_p u_{p'}^* : L_p(\mathcal{M} \bar{\otimes} \Gamma) \rightarrow L_p(\mathcal{M} \bar{\otimes} \Gamma).$$

□

- Remark 3.2.* (1) The proof of Theorem 3.1 follows closely that of [11, Theorem F], but we have to go a step further in proving the complementation result.
- (2) The theorem is still valid if we allow $*$ -polynomials of the form $g_{i_1}^{\varepsilon_1} \cdots g_{i_d}^{\varepsilon_d}$ with $\varepsilon_j \in \{\emptyset, *\}$, because the reduction method from [11] applies. In the $\mathcal{J}$ - and $\mathcal{K}$ - functional spaces, one then has to take the corresponding modified terms $f_{i_1}^{\varepsilon_1} \otimes f_{i_d}^{\varepsilon_d}$. This is canonical because g_i^* is also a free Gaussian variable.
- (3) Theorem 3.1 will be extended in Theorem 3.5 for the case of a general index set $\mathbb{N}^d$ instead of $\mathcal{I}^d$.

Now we consider modified polynomials for the extension to the general index set $\mathbb{N}^d$. Let $\mathcal{J}_p^d(\lambda, \mu)$ and $\mathcal{K}_p^d(\lambda, \mu)$ be $\mathcal{J}$ - and $\mathcal{K}$ -functional spaces formed as $\tilde{\mathcal{J}}_p^d(\lambda, \mu)$ and $\tilde{\mathcal{K}}_p^d(\lambda, \mu)$, respectively, but with indices in $\mathbb{N}^d$. Note that the duality between $\mathcal{J}_p^d(\lambda, \mu)$ and $\mathcal{K}_p^d(\lambda, \mu)$ is the same as above. Let $\{(g_{il})_{i \geq 1}\}_{l=1}^d$ be a sequence of free copies of the $(g_i)_{i \geq 1}$, Γ' be the von Neumann algebra generated by the g_{il} 's, and D' be the corresponding density. Now we consider $\mathcal{G}'_p^d$, the closed subspace of $L_p(\Gamma')$ spanned by $\{g'_{p,\underline{i}} : \underline{i} \in \mathbb{N}^d\}$, where $g'_{p,\underline{i}} = D'^{1/2p} g_{i_1 1} \cdots g_{i_d d} D'^{1/2p}$. Then the above theorem also holds for $\mathcal{G}'_p^d$. Indeed, it suffices to use the map

$$u'_p[(x_{\underline{i}})] = \sum_{\underline{i} \in \mathbb{N}^d} x_{\underline{i}} \otimes g'_{p,\underline{i}}$$

now defined on the spaces $\mathcal{J}_p^d(\lambda, \mu)$ and $\mathcal{K}_p^d(\lambda, \mu)$, respectively. In the above theorem we had to avoid terms such as $g_1 g_2 g_2$, since the assumption of [11, Theorem B] is not satisfied. However, in the modified case the corresponding term is $g_{11} g_{22} g_{23}$, where the variables in the term are even free from each other. Thus, the reduction method still applies for the general index set $\mathbb{N}^d$, and we have the following theorem.

Theorem 3.3. *Let $m \in \mathbb{N}$ and $x = (x_{\underline{i}})_{\underline{i} \in \mathbb{N}_m^d} \subseteq L_p(\mathcal{M})$. Then we have the following equivalence:*

$$\left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes g'_{p,\underline{i}} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma')} \sim_{c_d} \begin{cases} \|x\|_{\mathcal{K}_p^d(\lambda, \mu)} & \text{for } 1 \leq p \leq 2, \\ \|x\|_{\mathcal{J}_p^d(\lambda, \mu)} & \text{for } 2 \leq p \leq \infty. \end{cases}$$

Moreover, $\mathcal{G}_p^d$ is completely complemented in $L_p(\Gamma')$ with the constant depending only on d .

As an application we deduce an interpolation result needed in the next section.

Corollary 3.4. $\{\mathcal{K}_p^d(\lambda, \mu) : 1 \leq p \leq 2\} \cup \{\mathcal{J}_p^d(\lambda, \mu) : 2 \leq p \leq \infty\}$ forms an interpolation scale. In particular, for $1 < p < \infty$ and $\theta = \frac{1}{p}$ we have

$$[\mathcal{J}_\infty^d(\lambda, \mu), \mathcal{K}_1^d(\lambda, \mu)]_\theta \cong \begin{cases} \mathcal{K}_p^d(\lambda, \mu) & \text{for } 1 \leq p \leq 2, \\ \mathcal{J}_p^d(\lambda, \mu) & \text{for } 2 \leq p \leq \infty \end{cases}$$

isomorphically with constants depending only on d .

3.2. The case of general q and the general index set $\mathbb{N}^d$. For $x = (x_{\underline{i}})_{\underline{i} \in \mathbb{N}_m^d} \subseteq L_p(\mathcal{M})$ we can write

$$x = \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes e_{\underline{i}}$$

as an element of $\mathcal{J}_p^d(\lambda, \mu)$. Since P_d is defined on $\mathcal{H}^{\otimes d}$, the following is well-defined:

$$(I_{\mathcal{M}} \otimes P_d)x := \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes P_d(e_{\underline{i}}).$$

It is clear that $(I_{\mathcal{M}} \otimes P_d)x$ is also an element of $\mathcal{J}_p^d(\lambda, \mu)$.

Theorem 3.5. Let $m \in \mathbb{N}$ and $x = (x_{\underline{i}})_{\underline{i} \in \mathbb{N}_m^d} \subseteq L_p(\mathcal{M})$. Then we have the following equivalence:

$$\left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes g_{p,q,\underline{i}} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_q)} \sim_{c_{d,q}} \begin{cases} \|x\|_{\mathcal{K}_p^d(\lambda, \mu)} & \text{for } 1 \leq p \leq 2, \\ \|(I_{\mathcal{M}} \otimes P_d)x\|_{\mathcal{J}_p^d(\lambda, \mu)} & \text{for } 2 \leq p \leq \infty. \end{cases}$$

Moreover, $\mathcal{G}_{p,q}^d$, the closed subspace of $L_p(\Gamma_q)$ spanned by

$$\{g_{p,q,\underline{i}} : \underline{i} \in \mathbb{N}^d\},$$

is completely complemented with the constant depending only on d and q .

Proof. We consider the maps $u_{p,q}$ ($1 \leq p \leq 2$) and $v_{p,q}$ ($2 \leq p \leq \infty$) as before with a suitable modification. For $1 \leq p \leq 2$ we define

$$\begin{aligned} u_{p,q} : \mathcal{K}_p^d(\lambda, \mu) &\rightarrow L_p(\mathcal{M} \bar{\otimes} \Gamma_q) \\ x &\mapsto \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes g_{p,q,\underline{i}}, \end{aligned}$$

and for $2 \leq p \leq \infty$ we define

$$\begin{aligned} v_{p,q} : \mathcal{J}_p^d(\lambda, \mu) &\rightarrow L_p(\mathcal{M} \bar{\otimes} \Gamma_q) \\ (I_{\mathcal{M}} \otimes P_d)x &\mapsto \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes g_{p,q,\underline{i}}. \end{aligned}$$

We observe that

$$\sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes g_{p,q,\underline{i}} = (I_{\mathcal{M}} \otimes D^{\frac{1}{2p}})(I_{\mathcal{M}} \otimes W)(\xi)(I_{\mathcal{M}} \otimes D^{\frac{1}{2p}}),$$

where

$$\xi = \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes \lambda_{\underline{i}} e_{\underline{i}}.$$

Thus, for $2 \leq p \leq \infty$ we have

$$(3.5) \quad v_{p,q}(x) = (I_{\mathcal{M}} \otimes D^{\frac{1}{2p}})(I_{\mathcal{M}} \otimes W)[(I_{\mathcal{M}} \otimes P_d^{-1})(\xi)](I_{\mathcal{M}} \otimes D^{\frac{1}{2p}}).$$

Our plan is as follows. First, we will show that $v_{\infty,q}$, $v_{2,q}$, $u_{2,q}$, and $u_{1,q}$ are bounded with constants depending only on d and q . Then by Corollary 3.4 and complex interpolation, the $u_{p,q}$'s ($1 \leq p \leq 2$) and the $v_{p,q}$'s ($2 \leq p \leq \infty$) are bounded with constants depending only on d and q , since it is clear that the $u_{p,q}$'s ($1 \leq p \leq 2$) and the $v_{p,q}$'s ($2 \leq p \leq \infty$) are essentially the same map, respectively, in the sense of interpolation theory. Then, we will apply a duality argument to show that the $u_{p,q}$'s ($1 \leq p \leq 2$) and the $v_{p,q}$'s ($2 \leq p \leq \infty$) are actually isomorphisms with constants depending only on d and q . Indeed, for $x = (x_{\underline{i}})_{\underline{i} \in \mathbb{N}_m^d}$, $z = (z_{\underline{i}})_{\underline{i} \in \mathbb{N}_m^d}$ and $1 \leq p \leq 2$,

$$(3.6) \quad \begin{aligned} & \langle u_{p,q}(x), v_{p',q}(z) \rangle \\ &= \sum_{\underline{i}, \underline{j} \in \mathbb{N}_m^d} \text{tr}_{\mathcal{M}}(x_{\underline{i}}^* z_{\underline{j}}) \text{tr}_{\Gamma_q}(D^{\frac{1}{2p}} W^*(\lambda_{\underline{i}} e_{\underline{i}}) D^{\frac{1}{2p}} D^{\frac{1}{2p'}} W(P_d^{-1} \lambda_{\underline{j}} e_{\underline{j}}) D^{\frac{1}{2p'}}) \\ &= \sum_{\underline{i}, \underline{j} \in \mathbb{N}_m^d} \lambda_{\underline{i}}^{-\frac{1}{p}} \lambda_{\underline{j}}^{-\frac{1}{p'}} \mu_{\underline{i}}^{\frac{1}{p}} \mu_{\underline{j}}^{\frac{1}{p'}} \text{tr}_{\mathcal{M}}(x_{\underline{i}}^* z_{\underline{j}}) \phi_q(W^*(\lambda_{\underline{i}} e_{\underline{i}}) W(P_d^{-1} \lambda_{\underline{j}} e_{\underline{j}})) \\ &= \sum_{\underline{i} \in \mathbb{N}_m^d} \lambda_{\underline{i}} \mu_{\underline{i}} \text{tr}_{\mathcal{M}}(x_{\underline{i}}^* z_{\underline{i}}) = \langle x, z \rangle. \end{aligned}$$

If we know that the $u_{p,q}$'s ($1 \leq p \leq 2$) and the $v_{p,q}$'s ($2 \leq p \leq \infty$) are bounded with constants depending only on d and q , then we can conclude that they are isomorphisms with constants depending only on d and q using (3.6).

Now we consider the boundedness of $v_{\infty,q}$ following the approach of Nou ([20]). For $x = (x_{\underline{i}})_{\underline{i} \in \mathbb{N}_m^d}$ we have

$$v_{\infty,q}(x) = (I_{\mathcal{M}} \otimes W)(\tilde{\xi}),$$

where $\tilde{\xi} = (I_{\mathcal{M}} \otimes P_d^{-1})(\xi)$ by (3.5). Since we are using the index set $\mathbb{N}_m^d$, the Hilbert space $\mathcal{H}$ is $2m$ -dimensional now, so that we are in the situation where the operator A is bounded. Thus, by applying [20, Theorem 3] we have

$$\begin{aligned} & \|v_{\infty,q}(x)\|_{\mathcal{M} \otimes_{\min} \Gamma_q} \\ & \leq C_q(d+1) \max_{0 \leq k \leq d} \left\| I_{\mathcal{M}} \otimes (I_k \otimes S) R_{d,d-k}^*(\tilde{\xi}) \right\|_{\mathcal{M} \otimes_{\min} (H_c^{\otimes k} \otimes_h H_F^{\otimes d-k})}, \end{aligned}$$

where $C_q = \prod_{n \geq 1} (1 - q^n)^{-1}$ and I_k is the formal identity on $\mathcal{H}^k$. Let us use the fact that $\|I_k \otimes \tilde{J}\|_{cb} \leq 1$, (2.5), (2.10) and (2.9). Then we get

$$\begin{aligned} & \left\| I_{\mathcal{M}} \otimes (I_k \otimes S) R_{d,d-k}^* (\tilde{\xi}) \right\|_{\mathcal{M} \otimes_{\min} (H_c^{\otimes k} \otimes_h H_r^{\otimes d-k})} \\ & \leq \left\| I_{\mathcal{M}} \otimes (I_k \otimes (A^{-\frac{1}{2}})^{\otimes d-k}) R_{d,d-k}^* (\tilde{\xi}) \right\|_{\mathcal{M} \otimes_{\min} (H_c^{\otimes k} \otimes_h H_r^{\otimes d-k})} \\ & = \left\| I_{\mathcal{M}} \otimes [(P_k^{\frac{1}{2}} \otimes P_{d-k}^{\frac{1}{2}}) (I_k \otimes (A^{-\frac{1}{2}})^{\otimes d-k}) R_{d,d-k}^* (\tilde{\xi})] \right\|_{\mathcal{M} \otimes_{\min} (\mathcal{H}_c^{\otimes k} \otimes_h \mathcal{H}_r^{\otimes d-k})} \\ & \sim_{c_{q,d}} \left\| I_{\mathcal{M}} \otimes [(P_k \otimes P_{d-k}) (I_k \otimes (A^{-\frac{1}{2}})^{\otimes d-k}) R_{d,d-k}^* (\tilde{\xi})] \right\|_{\mathcal{M} \otimes_{\min} \mathcal{H}_c^{\otimes k} \otimes_h \mathcal{H}_r^{\otimes d-k}} . \end{aligned}$$

Since P_{d-k} and $(A^{-\frac{1}{2}})^{\otimes d-k}$ commute we have by (2.7) that

$$\begin{aligned} (P_k \otimes P_{d-k}) (I_k \otimes (A^{-\frac{1}{2}})^{\otimes d-k}) R_{d,d-k}^* &= (I_k \otimes (A^{-\frac{1}{2}})^{\otimes d-k}) (P_k \otimes P_{d-k}) R_{d,d-k}^* \\ &= (I_k \otimes (A^{-\frac{1}{2}})^{\otimes d-k}) P_d . \end{aligned}$$

Thus, by (2.3) we have

$$\begin{aligned} & \left\| I_{\mathcal{M}} \otimes (I_k \otimes S) R_{d,d-k}^* (\tilde{\xi}) \right\|_{\mathcal{M} \otimes_{\min} (H_c^{\otimes k} \otimes_h H_r^{\otimes d-k})} \\ & \lesssim_{c_{q,d}} \left\| [I_{\mathcal{M}} \otimes (I_k \otimes (A^{-\frac{1}{2}})^{\otimes d-k})] (\xi) \right\|_{\mathcal{M} \otimes_{\min} (\mathcal{H}_c^{\otimes k} \otimes_h \mathcal{H}_r^{\otimes d-k})} \\ & = \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes \lambda_{\underline{i}^k} e_{\underline{i}^k} \otimes \mu_{\underline{i}^{d-k}} e_{\underline{i}^{d-k}} \right\|_{\mathcal{M} \otimes_{\min} (\mathcal{H}_c^{\otimes k} \otimes_h \mathcal{H}_r^{\otimes d-k})} \\ & = \mathcal{RC}_{\infty}^{d,k}(\lambda, \mu; x), \end{aligned}$$

where $\underline{i}^k$ and $\underline{i}^{d-k}$ come from (3.4). Consequently, we get

$$\|v_{\infty,q}(x)\|_{\mathcal{M} \otimes_{\min} \Gamma_q} \lesssim_{c_{q,d}} \|x\|_{\mathcal{J}_{\infty}^d(\lambda, \mu)} .$$

When $p = 2$ the calculation is straightforward. Note that

$$\begin{aligned} \|g_{2,q,\underline{i}}\|_{\Gamma_q}^2 &= \text{tr}_{\Gamma_q} (D^{\frac{1}{4}} g_{q,\underline{i}}^* D^{\frac{1}{2}} g_{q,\underline{i}} D^{\frac{1}{4}}) \\ &= \text{tr}_{\Gamma_q} (D g_{q,\underline{i}}^* g_{q,\underline{i}}) \lambda_{\underline{i}}^{-1} \mu_{\underline{i}} \\ &= \phi_q(g_{q,\underline{i}}^* g_{q,\underline{i}}) \lambda_{\underline{i}}^{-1} \mu_{\underline{i}} = \lambda_{\underline{i}} \mu_{\underline{i}} \|e_{\underline{i}}\|_{H^{\otimes d}}^2 . \end{aligned}$$

Thus, we have

$$\begin{aligned} \|u_{2,q}(x)\|_{L_2(\mathcal{M} \bar{\otimes} \Gamma_q)} &= \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes \lambda_{\underline{i}}^{\frac{1}{2}} \mu_{\underline{i}}^{\frac{1}{2}} e_{\underline{i}} \right\|_{L_2(\mathcal{M}) \otimes_2 H^{\otimes d}} \\ &= \left\| (I_{\mathcal{M}} \otimes P_d^{\frac{1}{2}}) \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes \lambda_{\underline{i}}^{\frac{1}{2}} \mu_{\underline{i}}^{\frac{1}{2}} e_{\underline{i}} \right\|_{L_2(\mathcal{M}) \otimes_2 \mathcal{H}^{\otimes d}} \\ &= \left\| (I_{\mathcal{M}} \otimes P_d^{\frac{1}{2}}) x \right\|_{\mathcal{K}_2^d(\lambda, \mu)} \end{aligned}$$

and

$$\begin{aligned} \|v_{2,q}(x)\|_{L_2(\mathcal{M} \otimes \Gamma_q)} &= \|u_{2,q}[(I_{\mathcal{M}} \otimes P_d^{-1})x]\|_{L_2(\mathcal{M} \otimes \Gamma_q)} \\ &= \|(I_{\mathcal{M}} \otimes P_d^{-\frac{1}{2}})x\|_{\mathcal{J}_2^d(\lambda, \mu)}. \end{aligned}$$

Since

$$\|(I_{\mathcal{M}} \otimes P_d^{\frac{1}{2}})x\|_{\mathcal{K}_2^d(\lambda, \mu)} \sim_{c_{q,d}} \|(I_{\mathcal{M}} \otimes P_d^{-\frac{1}{2}})x\|_{\mathcal{J}_2^d(\lambda, \mu)} \sim_{c_{q,d}} \|x\|_{\mathcal{K}_2^d(\lambda, \mu)},$$

by (2.8) and (2.9) we get our desired estimate.

The boundedness of $u_{1,q}$ can be checked by a similar argument as in the proof of Theorem 3.1. We have the same inequality as (3.3), so that it is enough to show that the maps

$$(3.7) \quad C_1^{\otimes k} \rightarrow L_2^r(\Gamma_q), \quad f^{1,i_1} \otimes \cdots \otimes f^{1,i_k} \mapsto D_q^{\frac{1}{2}} g_{q,\underline{i}^k}$$

and

$$(3.8) \quad R_1^{\otimes d-k} \rightarrow L_2^c(\Gamma_q), \quad f_{1,i_{k+1}} \otimes \cdots \otimes f_{1,i_d} \mapsto g_{q,\underline{i}^{d-k}} D_q^{\frac{1}{2}}$$

are completely bounded with cb-norms depending only on q and d . Indeed, for any $\underline{i}, \underline{j} \in \mathbb{N}_m^k$ we have

$$\begin{aligned} &\langle D_q^{\frac{1}{2}} g_{q,\underline{i}}, D_q^{\frac{1}{2}} g_{q,\underline{j}} \rangle_{L_2(\Gamma_q)} \\ &= \phi_q(g_{q,\underline{i}} g_{q,\underline{j}}^*) \\ &= \mu_{\underline{i}} \mu_{\underline{j}} \langle e_{-j_k} \otimes \cdots \otimes e_{-j_1}, e_{-i_k} \otimes \cdots \otimes e_{-i_1} \rangle_{H^{\otimes k}} \\ &= \langle P_k^{\frac{1}{2}} (\mu_{j_k} e_{-j_k} \otimes \cdots \otimes \mu_{j_1} e_{-j_1}), P_k^{\frac{1}{2}} (\mu_{i_k} e_{-i_k} \otimes \cdots \otimes \mu_{i_1} e_{-i_1}) \rangle_{\mathcal{H}^{\otimes k}}. \end{aligned}$$

Thus, by rearranging the coordinates we have an isometry

$$\ell_2^{\otimes k} \rightarrow L_2(\Gamma_q), \quad P_k^{\frac{1}{2}} (\mu_{i_1} e_{i_1} \otimes \cdots \otimes \mu_{i_k} e_{i_k}) \mapsto D_q^{\frac{1}{2}} g_{q,\underline{i}^k}.$$

By composing $P_k^{-\frac{1}{2}}$ to the above isometry and the norm estimate (2.9) we have that (3.7) is completely bounded with cb-norms depending only on q and k , thus only on q and d . The argument for (3.8) is similar.

For the complementation result of $\mathcal{G}_{p,q}^d$ we can use the projections $u_{p,q} v_{p',q}^*$ when $1 \leq p \leq 2$ and $v_{p,q} u_{p',q}^*$ when $2 \leq p \leq \infty$, respectively. $\square$

Remark 3.6. (1) The projections $u_{p,q} v_{p',q}^*$ ($1 \leq p \leq 2$) and $v_{p,q} u_{p',q}^*$ ($2 \leq p \leq \infty$) are essentially the same maps. Indeed, it is clear that $u_{p,q}$ ($1 \leq p \leq 2$) and $v_{p,q}$ ($2 \leq p \leq \infty$) are compatible, respectively, in the sense of interpolation theory. Then, $u_{p,q} v_{p',q}^*$ ($1 \leq p \leq 2$) and $v_{p,q} u_{p',q}^*$ ($2 \leq p \leq \infty$) are compatible, respectively, in the same sense. Since $v_{2,q} = u_{2,q}(I_{\mathcal{M}} \otimes P_d)$ we have $u_{2,q} v_{2,q}^* = v_{2,q} u_{2,q}^*$, which implies that $u_{p,q} v_{p',q}^*$ ($1 \leq p \leq 2$) and $v_{p,q} u_{p',q}^*$ ($2 \leq p \leq \infty$) are essentially the same maps. Note finally that the above projections do depend on q .

(2) Theorem 3.5 extends Theorem 3.1 allowing polynomials of the form $g_1^{k_1} \cdots g_n^{k_n}$, i.e. allowing the index set $\mathbb{N}^d$ instead of $\mathcal{I}^d$. However, we are not allowing polynomials containing g_i^* , which means we are only looking at “analytic” polynomials.

(3) The q -dependence of the formula in Theorem 3.5 does appear in P_d , the operator of symmetrization, as well as the constant.

4. THE CASE OF CAR AND CCR GENERATORS

We will simply write a_k , $a_{p,k}$ and $\mathcal{A}_p^d$ instead of $g_{\pm 1,k}$, $g_{p,\pm 1,k}$ and $\mathcal{G}_{p,\pm 1}^d$, respectively. For any $\underline{i} \in \mathbb{N}_m^d$ we define $a_{p,\underline{i}}$ similarly. We start this section with the operator space structure of $\mathcal{A}_p^1$.

Theorem 4.1. *Let $m \in \mathbb{N}$ and $x = (x_k)_{1 \leq k \leq m} \subseteq L_p(\mathcal{M})$. Then we have*

$$\left\| \sum_{1 \leq k \leq m} x_k \otimes a_{p,k} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1})} \sim_{c_p} \begin{cases} \|x\|_{\mathcal{K}_p^1(\lambda, \mu)} & \text{for } 1 \leq p \leq 2, \\ \|x\|_{\mathcal{J}_p^1(\lambda, \mu)} & \text{for } 2 \leq p < \infty. \end{cases}$$

Moreover, $L_p(\mathcal{M}; \mathcal{A}_p)$ is complemented in $L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1})$.

In particular, we have $\mathcal{A}_p^1 \cong RC_p(\lambda, \mu)$ ($1 \leq p < \infty$) completely isomorphically, and $\mathcal{A}_p^1$ ($1 < p < \infty$) is completely complemented in $L_p(\Gamma_{\pm 1})$. All constants here depend only on p .

Proof. When $2 \leq p < \infty$ we apply the noncommutative Burkholder inequality (also see [32, Theorem 4.1]). For $1 < p < 2$ we follow the proof of [11, Theorem E] to obtain an upper bound. The lower bound follows by the duality and the previous result.

For the case $p = 1$ we can get the result with a slight modification of the proof of [10, Theorem 7.1]. (See [7] for the best constant.) Let ψ , u_n and N be as in section 2.3. Let $N_n = M_{2^n} \otimes N^{\otimes n}$ and D_ψ , D_n be the densities of ψ and $\psi_n = \tau_n \otimes (\frac{\psi}{2m})^{\otimes n}$, respectively. Then by (2.14) we have

$$\left\| \sum_{k=1}^m x_k \otimes a_{1,k} \right\|_{L_1(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1})} = \lim_n \left\| \sum_{k=1}^m x_k \otimes D_n^{\frac{1}{2}} u_n(b_k) D_n^{\frac{1}{2}} \right\|_{L_1(\mathcal{M} \bar{\otimes} N_n)},$$

where $b_k = \sqrt{\lambda_k^2 + \mu_k^2} \delta_k \otimes e_{12}$. The only difference now is that we are using b_k instead of $\delta_k \otimes e_{12}$ in [10, (7.3)]. Since

$$\mathrm{tr}_N([D_\psi^{\frac{1}{2}} b_l]^* D_\psi^{\frac{1}{2}} b_k) = \psi(b_k b_l^*) = \delta_{kl} \mu_k^2,$$

by the same argument we get

$$\begin{aligned} & \left\| \sum_{k=1}^m x_k \otimes a_{1,k} \right\|_{L_1(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1})} \\ & \sim \inf_{x_k = c_k + d_k} \left\{ \left\| \left(\sum_{k=1}^m \mu_k^2 c_k^* c_k \right)^{\frac{1}{2}} \right\|_{L_1(\mathcal{M})} + \left\| \left(\sum_{k=1}^m \lambda_k^2 d_k d_k^* \right)^{\frac{1}{2}} \right\|_{L_1(\mathcal{M})} \right\} \\ & = \|(x_k)_{k=1}^m\|_{\mathcal{K}_1^1(\lambda, \mu)}. \end{aligned}$$

□

For $x = (x_{\underline{i}})_{\underline{i} \in \mathbb{N}_m^d} \subseteq L_p(\mathcal{M})$ and $\mathbf{r} = (\mathbf{r}_1, \dots, \mathbf{r}_d) \in \{c, r\}^d$ we denote

$$\mathcal{RC}_p^{\mathbf{r}}(\lambda, \mu; x) = \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes h_{p, \mathbf{r}_1, i_1} \otimes \dots \otimes h_{p, \mathbf{r}_d, i_d} \right\|_{L_p(\mathcal{M}; F_p^{\mathbf{r}})},$$

where

$$h_{p,c,i} = \lambda_i^{\frac{1}{p'}} \mu_i^{\frac{1}{p}} e_{i1} \in C_p, \quad h_{p,r,i} = \lambda_i^{\frac{1}{p}} \mu_i^{\frac{1}{p'}} e_{1i} \in R_p,$$

$$F_p^{\mathbf{r}} = F_1 \otimes_h \cdots \otimes_h F_d \subseteq S_p(\ell_2^{\otimes d})$$

and

$$F_k = \begin{cases} C_p & \text{if } \mathbf{r}_k = c, \\ R_p & \text{if } \mathbf{r}_k = r. \end{cases}$$

We define the corresponding symmetric $\mathcal{K}$ - and $\mathcal{J}$ -functional spaces $\mathcal{SK}_p^d(\lambda, \mu)$ ($1 \leq p \leq 2$) and $\mathcal{SJ}_p^d(\lambda, \mu)$ ($2 \leq p \leq \infty$) as the closures of finite tuples in $L_p(\mathcal{M})$ indexed by $\mathbb{N}^d$ with respect to the following norms:

$$\|x\|_{\mathcal{SK}_p^d(\lambda, \mu)} = \inf_{\mathbf{r} \in \{c, r\}^d} \mathcal{RC}_p^{\mathbf{r}}(\lambda, \mu; x^{\mathbf{r}}),$$

where the infimum runs over all possible $x_{\underline{i}} = \sum_{\mathbf{r} \in \{c, r\}^d} x_{\underline{i}}^{\mathbf{r}}$ and $x^{\mathbf{r}} = (x_{\underline{i}}^{\mathbf{r}})_{\underline{i} \in \mathbb{N}_m^d}$ for $\mathbf{r} \in \{c, r\}^d$, and

$$\|x\|_{\mathcal{SJ}_p^d(\lambda, \mu)} = \max_{\mathbf{r} \in \{c, r\}^d} \mathcal{RC}_p^{\mathbf{r}}(\lambda, \mu; x).$$

The following is the tensor product extension of Theorem 4.1.

Corollary 4.2. *For $x = (x_{\underline{i}})_{\underline{i} \in \mathbb{N}_m^d} \subseteq L_p(\mathcal{M})$ and $1 \leq p < \infty$ we have the following equivalence:*

$$\left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes a_{p,i_1} \otimes \cdots \otimes a_{p,i_d} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1}^{\otimes d})} \sim_{c_p, d} \begin{cases} \|x\|_{\mathcal{SK}_p^d(\lambda, \mu)} & \text{for } 1 \leq p \leq 2, \\ \|x\|_{\mathcal{SJ}_p^d(\lambda, \mu)} & \text{for } 2 \leq p < \infty. \end{cases}$$

Proof. We assume that $1 \leq p \leq 2$ and $d = 2$. The proofs for the other cases are the same. By Theorem 4.1 we have

$$\left\| \sum_{i,j=1}^m x_{ij} \otimes a_{p,i} \otimes a_{p,j} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1} \bar{\otimes} \Gamma_{\pm 1})}$$

$$\sim_{c_p} \inf_{x_{ij}=c_{ij}+d_{ij}} \left\{ \left\| \left(\sum_{j=1}^m \left(\sum_{i=1}^m c_{ij} \otimes a_{p,i} \right) \otimes \lambda_j^{\frac{1}{p'}} \mu_j^{\frac{1}{p}} e_{j1} \right)^{\frac{1}{2}} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1} \bar{\otimes} B(\ell_2))} \right.$$

$$\left. + \left\| \left(\sum_{j=1}^m \left(\sum_{i=1}^m d_{ij} \otimes a_{p,i} \right) \otimes \lambda_j^{\frac{1}{p}} \mu_j^{\frac{1}{p'}} e_{1j} \right)^{\frac{1}{2}} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1} \bar{\otimes} B(\ell_2))} \right\}.$$

If we apply Theorem 4.1 for $\mathcal{M} \bar{\otimes} B(\ell_2)$, then we get the desired result. $\square$

4.1. Polynomials of degree 2. In this section we first focus on the case $d = 2$ to get the idea for polynomials of higher degree.

Theorem 4.3. *Let $x = (x_{ij})_{i,j=1}^m \subseteq L_p(\mathcal{M})$ satisfying*

$$(4.1) \quad x_{ij} = -x_{ji} \text{ for } q = -1$$

and

$$x_{ij} = x_{ji} \text{ for } q = 1$$

for all $1 \leq i, j \leq m$. Then we have the following equivalence:

$$\left\| \sum_{i,j=1}^m x_{ij} \otimes D_{\pm 1}^{\frac{1}{2p}} a_i a_j D_{\pm 1}^{\frac{1}{2p}} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1})} \sim_{c_p} \begin{cases} \|x\|_{\mathcal{SK}_p^2(\lambda, \mu)} & \text{for } 1 \leq p \leq 2, \\ \|x\|_{\mathcal{SJ}_p^2(\lambda, \mu)} & \text{for } 2 \leq p < \infty. \end{cases}$$

Proof. Let u_n and v_k be as in (2.11) and (2.13), respectively. By (2.14) we have

$$\begin{aligned} & \left\| \sum_{i,j=1}^m x_{ij} \otimes D_{\pm 1}^{\frac{1}{2p}} a_i a_j D_{\pm 1}^{\frac{1}{2p}} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1})} \\ &= \lim_n \left\| \sum_{i,j=1}^m x_{ij} \otimes D_n^{\frac{1}{2p}} (u_n(b_i) u_n(b_j)) D_n^{\frac{1}{2p}} \right\|_{L_p(\mathcal{M} \bar{\otimes} N_n)}, \end{aligned}$$

where $b_k = \sqrt{\lambda_k^2 + \mu_k^2} \delta_k \otimes e_{12}$.

Note that we have

$$u_n(b_i) u_n(b_j) = \frac{2m}{n} \sum_{k,l=1}^n v_k v_l \otimes \pi_k(b_i) \pi_l(b_j)$$

and $\psi_n(v_k v_l \otimes \pi_k(b_i) \pi_l(b_j)) = 0$ for $k \neq l$, where $\psi_n = \tau_n \otimes (\frac{\psi}{2m})^{\otimes n}$. Thus, we have

$$\begin{aligned} & \left\| \sum_{i,j=1}^m x_{ij} \otimes D_n^{\frac{1}{2p}} (u_n(b_i) u_n(b_j)) D_n^{\frac{1}{2p}} \right\|_p \\ &= \frac{2m}{n} \left\| \sum_{i,j=1}^m x_{ij} \otimes D_n^{\frac{1}{2p}} \left(\sum_{k,l=1}^n v_k v_l \otimes \pi_k(b_i) \pi_l(b_j) \right) D_n^{\frac{1}{2p}} \right\|_p. \end{aligned}$$

We set

$$Q_n = \sum_{\substack{1 \leq i,j \leq m \\ 1 \leq k,l \leq n}} x_{ij} \otimes D_n^{\frac{1}{2p}} (v_k v_l \otimes \pi_k(b_i) \pi_l(b_j)) D_n^{\frac{1}{2p}}.$$

First, we show the lower bound by an averaging technique. Let us fix a subset $A \subseteq \{1, \dots, n\}$. We introduce

$$v_k(\varepsilon) = \begin{cases} \varepsilon_1 v_k & \text{for } k \in A, \\ \varepsilon_2 v_k & \text{for } k \notin A, \end{cases}$$

where ε_1 and ε_2 are two independent Bernoulli variables with $\text{Prob}(\varepsilon_i = \pm 1) = \frac{1}{2}$. Then

$$\begin{aligned} & \int_0^1 \int_0^1 \varepsilon_1(t_1) \varepsilon_2(t_2) \sum_{\substack{1 \leq i,j \leq m \\ 1 \leq k,l \leq n}} x_{ij} \otimes v_k(\varepsilon) v_l(\varepsilon) \otimes \pi_k(b_i) \pi_l(b_j) dt_1 dt_2 \\ &= \sum_{\substack{1 \leq i,j \leq m \\ k \in A, l \notin A}} x_{ij} \otimes v_k v_l \otimes \pi_k(b_i) \pi_l(b_j) + \sum_{\substack{1 \leq i,j \leq m \\ k \notin A, l \in A}} x_{ij} \otimes v_k v_l \otimes \pi_k(b_i) \pi_l(b_j) \\ &= 2 \sum_{\substack{1 \leq i,j \leq m \\ k \in A, l \notin A}} x_{ij} \otimes v_k v_l \otimes \pi_k(b_i) \pi_l(b_j) \end{aligned}$$

since (when $q = -1$) $x_{ij} = -x_{ji}$ and

$$\begin{aligned} \sum_{\substack{1 \leq i, j \leq m \\ k \in A, l \notin A}} x_{ij} \otimes v_k v_l \otimes \pi_k(b_i) \pi_l(b_j) &= \sum_{\substack{1 \leq i, j \leq m \\ k \notin A, l \in A}} x_{ij} \otimes v_l v_k \otimes \pi_l(b_i) \pi_k(b_j) \\ &= - \sum_{\substack{1 \leq i, j \leq m \\ k \notin A, l \in A}} x_{ij} \otimes v_k v_l \otimes \pi_k(b_j) \pi_l(b_i) \\ &= \sum_{\substack{1 \leq i, j \leq m \\ k \notin A, l \in A}} x_{ij} \otimes v_k v_l \otimes \pi_k(b_i) \pi_l(b_j). \end{aligned}$$

The case $q = 1$ is the same.

Clearly the tuple $((v_k(\varepsilon))_{k \in A}, (v_k(\varepsilon))_{k \notin A})$ has the same $*$ -distribution as the tuple $((v_k)_{k \in A}, (v_k)_{k \notin A})$. Thus, by combining the above observations we get

$$(4.2) \quad \left\| \sum_{\substack{1 \leq i, j \leq m \\ k \in A, l \notin A}} x_{ij} \otimes D_n^{\frac{1}{2p}} \left(v_k v_l \otimes \pi_k(b_i) \pi_l(b_j) \right) D_n^{\frac{1}{2p}} \right\|_p \leq \frac{1}{2} \|Q_n\|_p.$$

Now we assume that n is even and consider a specific choice of A , namely

$$A = \{1, \dots, \frac{n}{2}\}.$$

Then $k \in A$, $l \notin A$ means $1 \leq k \leq \frac{n}{2} < l \leq n$. In this case we decompose v_k and v_l as follows:

$$v_k = w_k \otimes 1, \quad v_l = u \otimes w_l,$$

where w_k, w_l and $u \in M_{2^{n/2}}$. Moreover, $\pi_k(b_i) \pi_l(b_j)$ can be understood as $\pi_k(b_i) \otimes \pi_l(b_j)$ since $\pi_k(b_i) \in N^{\otimes \frac{n}{2}} \otimes 1$ and $\pi_l(b_j) \in 1 \otimes N^{\otimes \frac{n}{2}}$. Since $D_n = I \otimes D_{\psi/2m}^{\otimes n}$ we can write $D_n = D_{n/2} \otimes D_{n/2}$. Now we have

$$\begin{aligned} &\sum_{i,j=1}^m \sum_{1 \leq k \leq \frac{n}{2} < l \leq n} x_{ij} \otimes D_n^{\frac{1}{2p}} \left[v_k v_l \otimes \pi_k(b_i) \pi_l(b_j) \right] D_n^{\frac{1}{2p}} \\ &= \sum_{i,j=1}^m \sum_{1 \leq k \leq \frac{n}{2} < l \leq n} x_{ij} \otimes D_n^{\frac{1}{2p}} \left[(w_k \otimes 1)(u \otimes w_l) \otimes \pi_k(b_i) \pi_l(b_j) \right] D_n^{\frac{1}{2p}} \\ &= \sum_{i,j=1}^m x_{ij} \otimes D_{n/2}^{\frac{1}{2p}} \left[\sum_{k=1}^{n/2} w_k u \otimes \pi_k(b_i) \right] D_{n/2}^{\frac{1}{2p}} \otimes D_{n/2}^{\frac{1}{2p}} \left[\sum_{l=n/2+1}^n w_l \otimes \pi_l(b_j) \right] D_{n/2}^{\frac{1}{2p}}. \end{aligned}$$

Since w_k and u commute, $(w_k u)_{1 \leq k \leq \frac{n}{2}}$ and $(w_l)_{\frac{n}{2} < l \leq n}$ satisfy the relations in (2.12). Thus,

$$\sqrt{\frac{2m}{n/2}} \sum_{k=1}^{n/2} w_k u \otimes \pi_k(b_i) \quad \text{and} \quad \sqrt{\frac{2m}{n/2}} \sum_{l=n/2+1}^n w_l \otimes \pi_l(b_j)$$

converge in $*$ -distribution to a_i and a_j , respectively, and consequently we have

$$\begin{aligned}
 & \left\| \sum_{i,j=1}^m x_{ij} \otimes a_{p,i} \otimes a_{p,j} \right\|_p \\
 (4.3) \quad &= \lim_n \frac{2m}{n/2} \left\| \sum_{i,j=1}^m \sum_{1 \leq k \leq \frac{n}{2} < l \leq n} x_{ij} \otimes D_n^{\frac{1}{2p}} \left(v_k v_l \otimes \pi_k(b_i) \pi_l(b_j) \right) D_n^{\frac{1}{2p}} \right\|_p \\
 &\leq 2 \lim_n \frac{2m}{n} \|Q_n\|_p = 2 \left\| \sum_{i,j=1}^m x_{ij} \otimes D_{\pm 1}^{\frac{1}{2p}} a_i a_j D_{\pm 1}^{\frac{1}{2p}} \right\|_p.
 \end{aligned}$$

The conclusion follows by Corollary 4.2.

For the upper bound we consider the discrete probability space Ω of

$$A \subseteq \{1, \dots, n\} \text{ with } |A| = \frac{n}{2}.$$

Then for fixed $1 \leq k \neq l \leq n$ we have

$$\begin{aligned}
 \text{Prob}(k \in A, l \notin A) &= \frac{\#\{A \subseteq \{1, \dots, n\} : |A| = \frac{n}{2}, k \in A, l \notin A\}}{\#\{A \subseteq \{1, \dots, n\} : |A| = \frac{n}{2}\}} \\
 &= \frac{\binom{n-2}{n/2-1}}{\binom{n}{n/2}} = \frac{n}{4(n-1)}.
 \end{aligned}$$

Thus, by the symmetry we have

$$\begin{aligned}
 \|Q_n\|_p &= \frac{4(n-1)}{n} \left\| \sum_{k,l=1}^n \mathbb{E}_A(1_{k \in A, l \notin A}) \sum_{i,j=1}^m x_{ij} \otimes D_n^{\frac{1}{2p}} \left(v_k v_l \otimes \pi_k(b_i) \pi_l(b_j) \right) D_n^{\frac{1}{2p}} \right\|_p \\
 &\leq 4 \mathbb{E}_A \left\| \sum_{k \in A, l \notin A} \sum_{i,j=1}^m x_{ij} \otimes D_n^{\frac{1}{2p}} \left(v_k v_l \otimes \pi_k(b_i) \pi_l(b_j) \right) D_n^{\frac{1}{2p}} \right\|_p \\
 &= 4 \left\| \sum_{1 \leq k \leq n/2 < l \leq n} \sum_{i,j=1}^m x_{ij} \otimes D_n^{\frac{1}{2p}} \left(v_k v_l \otimes \pi_k(b_i) \pi_l(b_j) \right) D_n^{\frac{1}{2p}} \right\|_p,
 \end{aligned}$$

where $\mathbb{E}_A$ implies the expectation with respect to Ω . Consequently, by (4.3) we get

$$\left\| \sum_{i,j=1}^m x_{ij} \otimes D_{\pm 1}^{\frac{1}{2p}} a_i a_j D_{\pm 1}^{\frac{1}{2p}} \right\|_p \leq 8 \left\| \sum_{i,j=1}^m x_{ij} \otimes a_{p,i} \otimes a_{p,j} \right\|_p,$$

and the conclusion follows by Corollary 4.2 again. $\square$

4.2. Polynomials of higher degree. Now we consider the general case $d \geq 2$. For a permutation $\rho \in S_d$ and $\underline{i} \in \mathbb{N}_m^d$ we denote $(i_{\rho(1)}, \dots, i_{\rho(d)}) \in \mathbb{N}_m^d$ by $\underline{i}(\rho)$.

Theorem 4.4. *Let $x = (x_{\underline{i}})_{\underline{i} \in \mathbb{N}_m^d} \subseteq L_p(\mathcal{M})$ satisfying*

$$\begin{aligned}
 (4.4) \quad & x_{\underline{i}} = \text{sgn}(\rho) x_{\underline{i}(\rho)} \text{ for } q = -1, \\
 & x_{\underline{i}} = x_{\underline{i}(\rho)} \text{ for } q = 1
 \end{aligned}$$

for any $\rho \in S_d$ and $\underline{i} \in \mathbb{N}_m^d$. Then we have the following equivalence:

$$\left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes a_{p, \underline{i}} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1})} \sim_{c_{p,d}} \begin{cases} \|x\|_{\mathcal{SK}_p^d(\lambda, \mu)} & \text{for } 1 \leq p \leq 2, \\ \|x\|_{\mathcal{SJ}_p^d(\lambda, \mu)} & \text{for } 2 \leq p < \infty. \end{cases}$$

Proof. As in the proof of Theorem 4.3 we have

$$\begin{aligned} & \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes a_{p, \underline{i}} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1})} = \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes D_{\pm 1}^{\frac{1}{2p}} a_{\underline{i}} D_{\pm 1}^{\frac{1}{2p}} \right\|_p \\ &= \lim_n \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes D_n^{\frac{1}{2p}} (u_n(b_{i_1}) \cdots u_n(b_{i_d})) D_n^{\frac{1}{2p}} \right\|_{L_p(\mathcal{M} \bar{\otimes} N_n)} \\ &= \lim_n \left(\frac{2m}{n} \right)^{\frac{d}{2}} \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes \sum_{\underline{j} \in \mathbb{N}_n^d} v_{\underline{j}} \otimes D_n^{\frac{1}{2p}} (\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d})) D_n^{\frac{1}{2p}} \right\|_p, \end{aligned}$$

and we set

$$Q_n = \sum_{\underline{i} \in \mathbb{N}_m^d, \underline{j} \in \mathbb{N}_n^d} x_{\underline{i}} \otimes v_{\underline{j}} \otimes D_n^{\frac{1}{2p}} (\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d})) D_n^{\frac{1}{2p}}.$$

First, we consider the lower bound. Let $(A_1, \dots, A_d)$ be a partition of $\{1, \dots, n\}$. For $1 \leq k \leq d$ we introduce

$$v_j(\varepsilon) = \varepsilon_k v_j \text{ for all } j \in A_k,$$

where $(\varepsilon_k)_{k=1}^d$ is a collection of independent Bernoulli variables with $\text{Prob}(\varepsilon_i = \pm 1) = \frac{1}{2}$. Now we observe that

$$\begin{aligned} & \int_{[0,1]^d} \varepsilon_1 \cdots \varepsilon_d \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes \sum_{\underline{j} \in \mathbb{N}_n^d} v_{\underline{j}}(\varepsilon) \otimes D_n^{\frac{1}{2p}} (\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d})) D_n^{\frac{1}{2p}} \\ &= \sum_{\rho \in S_d} \sum_{j_{\rho(1)} \in A_1, \dots, j_{\rho(d)} \in A_d} \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes v_{\underline{j}} \otimes D_n^{\frac{1}{2p}} (\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d})) D_n^{\frac{1}{2p}} \end{aligned}$$

since the integration kills all the terms where there exists a partition containing more than two j_k 's.

On the other hand, for a fixed $\rho \in S_d$, we have (when $q = -1$)

$$\begin{aligned} & \sum_{j_{\rho(1)} \in A_1, \dots, j_{\rho(d)} \in A_d} \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes v_{\underline{j}} \otimes \pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d}) \\ &= \sum_{j_1 \in A_1, \dots, j_d \in A_d} \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes v_{\underline{j}(\rho^{-1})} \otimes \pi_{j_{\rho^{-1}(1)}}(b_{i_1}) \cdots \pi_{j_{\rho^{-1}(d)}}(b_{i_d}) \\ &= \sum_{j_1 \in A_1, \dots, j_d \in A_d} \text{sgn}(\rho) \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes v_{\underline{j}} \otimes \pi_{j_1}(b_{i_{\rho^{-1}(1)}}) \cdots \pi_{j_d}(b_{i_{\rho^{-1}(d)}}) \\ &= \sum_{j_1 \in A_1, \dots, j_d \in A_d} \sum_{\underline{i} \in \mathbb{N}_m^d} \text{sgn}(\rho) x_{\underline{i}(\rho)} \otimes v_{\underline{j}} \otimes \pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d}) \\ &= \sum_{j_1 \in A_1, \dots, j_d \in A_d} \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes v_{\underline{j}} \otimes \pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d}). \end{aligned}$$

The case $q = 1$ is similar.

Since $(v_j(\varepsilon))_{j=1}^n$ has the same $*$ -distribution with $(v_j)_{j=1}^n$, by combining the above two observations we get

$$d! \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} \sum_{j_1 \in A_1, \dots, j_d \in A_d} x_{\underline{i}} \otimes v_{\underline{j}} \otimes D_n^{\frac{1}{2p}}(\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d})) D_n^{\frac{1}{2p}} \right\|_p \leq \|Q_n\|_p.$$

Now we assume that n is a multiple of d and choose

$$A_k = \left\{ \frac{(k-1)n}{d} + 1, \dots, \frac{kn}{d} \right\} \text{ for } 1 \leq k \leq d.$$

As in the case $d = 2$ we can write $v_j = u^{\otimes k-1} \otimes w_{j,k} \otimes 1$ for $j \in A_k$ and $u, w_{j,k} \in M_{2^{n/d}}$. Then

$$v_{\underline{j}} = w_{j_1,1} u^{d-1} \otimes v_{j_2,2} u^{d-2} \otimes \cdots \otimes v_{j_d,d}.$$

Moreover, we have $\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d}) = \pi_{j_1}(b_{i_1}) \otimes \cdots \otimes \pi_{j_d}(b_{i_d})$ for $j_1 \in A_1, \dots, j_d \in A_d$ and $D_n = D_{n/d} \otimes \cdots \otimes D_{n/d}$, so that

$$\begin{aligned} & \sum_{\underline{i} \in \mathbb{N}_m^d} \sum_{j_1 \in A_1, \dots, j_d \in A_d} x_{\underline{i}} \otimes v_{\underline{j}} \otimes D_n^{\frac{1}{2p}}(\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d})) D_n^{\frac{1}{2p}} \\ &= \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes \bigotimes_{k=1}^d D_{n/d}^{\frac{1}{2p}} \left[\sum_{j_k \in A_k} v_{j_k,k} u^{d-k} \otimes \pi_{j_k}(b_{i_k}) \right] D_{n/d}^{\frac{1}{2p}}. \end{aligned}$$

Since $v_{j_k,k}$ and u commute, the $(v_{j_k,k} u^{d-k})_{j_k \in A_k}$'s satisfy the relations in (2.12) and

$$\sqrt{\frac{2m}{n/d}} \sum_{j_k \in A_k} v_{j_k,k} u^{d-k} \otimes \pi_{j_k}(b_{i_k})$$

converges in the $*$ -distribution to a_{i_k} by the central limit procedure. Thus, we have

$$\begin{aligned} & \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes a_{p,i_1} \otimes \cdots \otimes a_{p,i_d} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1}^{\otimes d})} \\ &= \lim_n \left(\frac{2m}{n/d} \right)^{\frac{d}{2}} \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes \bigotimes_{k=1}^d D_{n/d}^{\frac{1}{2p}} \left[\sum_{j_k \in A_k} v_{j_k,k} u^{d-k} \otimes \pi_{j_k}(b_{i_k}) \right] D_{n/d}^{\frac{1}{2p}} \right\|_{L_p(\mathcal{M} \bar{\otimes} N_n)} \\ &= d^{\frac{d}{2}} \lim_n \left(\frac{2m}{n} \right)^{\frac{d}{2}} \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} \sum_{j_1 \in A_1, \dots, j_d \in A_d} x_{\underline{i}} \otimes v_{\underline{j}} \otimes D_n^{\frac{1}{2p}}(\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d})) D_n^{\frac{1}{2p}} \right\|_p \\ &\leq \frac{d^{\frac{d}{2}}}{d!} \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes a_{p,\underline{i}} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1})}. \end{aligned}$$

Then by Corollary 4.2 we get the lower bound.

For the upper bound we consider a fixed $(j_1, \dots, j_d) \in \mathbb{N}_m^d$ and the discrete probability space Ω of partitions $\tilde{A} = (A_1, \dots, A_d)$ of $\{1, \dots, n\}$ with $|A_k| = \frac{n}{d}$.

Then we have

$$\begin{aligned} \text{Prob}(j_1 \in A_1, \dots, j_d \in A_d) &= \frac{\#\{(A'_1, \dots, A'_d) : A'_1 \cup \dots \cup A'_d = \{1, \dots, n-d\}\}}{\#\{(A_1, \dots, A_d) : A_1 \cup \dots \cup A_d = \{1, \dots, n\}\}} \\ &= \frac{\binom{n-d}{n/d-1, \dots, n/d-1}}{\binom{n}{n/d, \dots, n/d}} = \frac{n^d(n-d)!}{d^d n!}. \end{aligned}$$

This implies that

$$\begin{aligned} &\|Q_n\|_p \\ &= \frac{d^d n!}{n^d(n-d)!} \left\| \sum_{\substack{\underline{j} \in \mathbb{N}_n^d \\ \underline{i} \in \mathbb{N}_m^d}} \mathbb{E}_{\tilde{A}}(1_{j_1 \in A_1, \dots, j_d \in A_d}) x_{\underline{i}} \otimes v_{\underline{j}} \otimes D_n^{\frac{1}{2p}}(\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d})) D_n^{\frac{1}{2p}} \right\|_p \\ &\leq d^d \mathbb{E}_{\tilde{A}} \left\| \sum_{j_1 \in A_1, \dots, j_d \in A_d} \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes v_{\underline{j}} \otimes D_n^{\frac{1}{2p}}(\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d})) D_n^{\frac{1}{2p}} \right\|_p, \end{aligned}$$

where $\mathbb{E}_{\tilde{A}}$ is the expectation with respect to Ω , and consequently,

$$\begin{aligned} &\left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes a_{p, \underline{i}} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1})} = \lim_n \left(\frac{2m}{n} \right)^{\frac{d}{2}} \|Q_n\|_p \\ &\leq d^d \lim_n \left(\frac{2m}{n} \right)^{\frac{d}{2}} \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} \sum_{j_1 \in A_1, \dots, j_d \in A_d} x_{\underline{i}} \otimes v_{\underline{j}} \otimes D_n^{\frac{1}{2p}}(\pi_{j_1}(b_{i_1}) \cdots \pi_{j_d}(b_{i_d})) D_n^{\frac{1}{2p}} \right\|_p \\ &= d^{\frac{d}{2}} \left\| \sum_{\underline{i} \in \mathbb{N}_m^d} x_{\underline{i}} \otimes a_{p, i_1} \otimes \cdots \otimes a_{p, i_d} \right\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_{\pm 1}^{\otimes d})}. \end{aligned}$$

We get the desired conclusion by Corollary 4.2 again. □

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, 273 ALT-GELD HALL, 1409 W. GREEN STREET, URBANA, ILLINOIS 61801

E-mail address: `junge@math.uiuc.edu`

DEPARTMENT OF MATHEMATICS, CHUNGBUK NATIONAL UNIVERSITY, 410 SUNGBONG-RO, HEUNGDUK-GU, CHEONGJU 361-763, KOREA

E-mail address: `hhlee@chungbuk.ac.kr`