PARABOLIC MANIFOLDS FOR SEMI-ATTRACTIVE ANALYTIC TRANSFORMATIONS OF $\mathbf{C}^n$

FENG RONG

ABSTRACT. We study the local dynamics of semi-attractive analytic transformations of $\mathbf{C}^n$. Under certain assumptions, Rivi showed the existence of parabolic manifolds of dimension $m + 1$, where m is the number of eigenvalues with modulus strictly less than one. Assuming moreover that certain matrix has p eigenvalues with positive real part, we show the existence of parabolic manifolds of dimension $m + p + 1$.

1. INTRODUCTION

Let Φ be an analytic transformation of $\mathbf{C}^n$ with a fixed point p . We say that Φ is *semi-attractive* at p if $d\Phi_p = \text{Diag}(I_l, \Lambda_m)$, $l + m = n$, where the eigenvalues $\{\lambda_i\}_{1 \leq i \leq m}$ of Λ_m have modulus strictly less than one. The local dynamics of such transformations have been studied by several authors ([F], [U1], [U2], [H1], [Ri], [N]).

A preliminary investigation of semi-attractive analytic transformations of $\mathbf{C}^2$ was carried out by Fatou ([F]). Much later, Ueda ([U1], [U2]) studied the two dimensional case in greater detail. The higher dimensional case was first studied by Hakim ([H1]), for $l = 1$, and then by Rivi ([Ri]), for $l > 1$. While the above mentioned works focus on the dynamics tangent to the identity, the work of Nishimura ([N]) concerns more about the dynamics in the attracting directions.

In suitable local coordinates $(w, z) \in \mathbf{C}^l \times \mathbf{C}^m$, we can write Φ as

$$(1.1) \quad \begin{cases} w_1 = f(w, z) = w + P_{1,z}(w) + P_{2,z}(w) + \cdots, \\ z_1 = g(w, z) = C(z) + Q(w, z)w, \end{cases}$$

where $P_{i,z}(w)$ are homogeneous polynomials in w of degree i whose coefficients are holomorphic functions in z , $C(z)$ is an analytic transformation of $\mathbf{C}^m$ with $C(0) = 0$ and $dC(0) = \Lambda_m$, and $Q(w, z)$ is an $l \times m$ matrix whose elements are holomorphic functions of $\mathbf{C}^n$ vanishing at $(0, 0)$.

We will always assume that $f(w, 0) \neq w$.

Let $f(w, 0) = w + P_2(w) + P_3(w) + \cdots$ be the homogeneous expansion of $f(w, 0)$. We say that Φ is of *order* ν if $P_i(w) \equiv 0$ for $i < \nu$ and $P_\nu(w) \not\equiv 0$. Consider $[v] = [v_1 : \cdots : v_n] \in \mathbf{P}^{n-1}$ with $v_i = 0$, $l < i \leq n$, and write $[v] = [\tilde{v} : 0]$. We say

Received by the editors August 8, 2008 and, in revised form, June 18, 2009.
 2010 *Mathematics Subject Classification*. Primary 32H50.

that $[v]$ is a *characteristic direction* of Φ if $P_\nu(\tilde{v}) = \lambda \tilde{v}$ for some $\lambda \in \mathbf{C}$. If $\lambda \neq 0$, then $[v]$ is said to be *non-degenerate*. Let $\hat{P}_\nu$ be the projection $\hat{P}_\nu : [X] \rightarrow [P_\nu(X)]$. The matrix associated to the linear map $\frac{1}{\nu-1}(d\hat{P}_\nu[\tilde{v}] - id) : T_{[\tilde{v}]} \mathbf{P}^{l-1} \rightarrow T_{[\tilde{v}]} \mathbf{P}^{l-1}$ is called the *matrix associated* to $[v]$.

A *parabolic manifold* of dimension d for Φ at p is an injective holomorphic map $\varphi : \Delta \rightarrow \mathbf{C}^n$ satisfying the following properties:

- (i) Δ is a simply connected domain in $\mathbf{C}^d$ with $0 \in \partial\Delta$;
- (ii) φ is continuous on $\partial\Delta$ and $\varphi(0) = p$;
- (iii) $\varphi(\Delta)$ is invariant under Φ and $\Phi^k(\varphi(\zeta)) \rightarrow p$ as $k \rightarrow \infty$ for any $\zeta \in \Delta$.

Furthermore, if $[\varphi(\zeta)] \rightarrow [v] \in \mathbf{P}^{n-1}$ as $\zeta \rightarrow 0$ (where $[\cdot]$ denotes the canonical projection of $\mathbf{C}^n \setminus \{p\}$ onto $\mathbf{P}^{n-1}$), we say that φ is *tangent to* $[v]$ at p .

In [Ri], Rivi proved the following

Theorem 1.1. *Let Φ be an analytic transformation of $\mathbf{C}^n$ with a semi-attractive fixed point p . Assume that Φ is of finite order ν and that $[v] \in \mathbf{P}^{n-1}$ is a non-degenerate characteristic direction of Φ . Then there exists at least $\nu - 1$ parabolic manifolds of dimension $m + 1$, tangent to $\mathbf{C}v \oplus E$ at p , where E is the generalized eigenspace associated to the eigenvalues of $d\Phi_p$ with modulus strictly less than one.*

Our main result is the following

Theorem 1.2. *Let Φ be an analytic transformation of $\mathbf{C}^n$ with a semi-attractive fixed point p . Assume that Φ is of finite order ν and that $[v] \in \mathbf{P}^{n-1}$ is a non-degenerate characteristic direction of Φ . Let Γ be the matrix associated to $[v]$. Assume that we can divide the eigenvalues of Γ into two sets $\{\alpha_j\}_{1 \leq j \leq p}$ and $\{\beta_k\}_{1 \leq k \leq q}$ such that for some positive real number α , we have*

$$(1.2) \quad \begin{aligned} \operatorname{Re} \alpha_j &> \alpha > 0, & j &= 1, \dots, p, \\ \operatorname{Re} \beta_k &< \alpha, & k &= 1, \dots, q. \end{aligned}$$

Then there exist at least $\nu - 1$ parabolic manifolds of dimension $m + p + 1$, tangent to $\mathbf{C}v \oplus E_1 \oplus E_2$ at p , where E_1 is the generalized eigenspace associated to α_j 's and E_2 is the the generalized eigenspace associated to the eigenvalues of $d\Phi_p$ with modulus strictly less than one.

A similar result for so-called “quasi-parabolic” analytic transformations of $\mathbf{C}^n$ has been obtained in [Ro].

In Section 2, we recall some known results and define the operator T , which is needed in the proof of our main theorem. We then prove Theorem 1.2 in Section 3.

The author would like to thank the referee for many useful comments.

2. THE OPERATOR T

Let Φ be as in (1.1). Assume that Φ is of finite order ν . By [Ri, Proposition 2.2], for any $\mu > \nu$, we can choose local coordinates such that Φ takes the form

$$\begin{cases} w_1 = f(w, z) = w + P_\nu(w) + \dots + P_{\mu-1}(w) + P_{\mu,z}(w) + \dots, \\ z_1 = g(w, z) = C(z) + Q(w, z)w, \end{cases}$$

where $P_i(w)$ (resp. $P_{i,z}(w)$) are homogeneous polynomials in w of degree i with constant (resp. holomorphic functions in z) coefficients.

Assume that Φ has a non-degenerate characteristic direction $[v]$. Up to a linear change of coordinates, we will assume that $[v] = [1 : 0 : \cdots : 0]$. Write $w = (x, y) \in \mathbf{C} \times \mathbf{C}^{l-1}$ and set $P_i(x, y) = (p_i(x, y), q_i(x, y))$ and $P_{i,z}(x, y) = (p_{i,z}(x, y), q_{i,z}(x, y))$. Making the blow-up $y = \zeta x$ with $\zeta \in \mathbf{C}^{l-1}$, we have

$$\begin{cases} x_1 = x + p_\nu(1, \zeta)x^\nu + \cdots + p_{\mu-1}(1, \zeta)x^{\mu-1} + p_{\mu,z}(1, \zeta)x^\mu + \cdots, \\ \zeta_1 = \zeta + r(\zeta)x^{\nu-1} + O(x^\nu) + O_z(x^{\mu-1}), \\ z_1 = C(z) + O(x^2, \|z\|x), \end{cases}$$

where $r(\zeta) = q_\nu(1, \zeta) - p_\nu(1, \zeta)\zeta$ and $O_z(h) = O(\|z\|h)$.

After replacing $r(\zeta)$ by its power series expansion at 0 and changing x into λx with $\lambda^{\nu-1} = -p_\nu(1, 0)(\nu - 1)$, we have

$$\begin{cases} x_1 = x - \frac{1}{\nu-1}x^\nu + O(x^{\nu+1}, \|\zeta\|x^\nu) + O_z(x^\mu), \\ \zeta_1 = (I - \Gamma x^{\nu-1})\zeta + O(x^\nu, \|\zeta\|x^\nu, \|\zeta\|^2 x^{\nu-1}) + O_z(x^{\mu-1}), \\ z_1 = C(z) + O(x^2, \|z\|x), \end{cases}$$

where $\Gamma = r'(0)/((\nu-1)p_\nu(1, 0))$ is the matrix associated to $[v]$.

By [Ri, Proposition 3.1], the class of similarity of the matrix Γ is invariant under changes of coordinates. Therefore, the eigenvalues of Γ are invariants associated to $[v]$. We will assume, without loss of generality, that Γ is in almost diagonal Jordan form.

For simplicity, from now on we will work with $\nu = 2$. (One can deal with $\nu > 2$ similarly; cf. [H3, Section 6].)

For r, ρ, τ small enough, define

$$\Delta_{r,\rho,\tau} = \{(x, \zeta, z) \in \mathbf{C} \times \mathbf{C}^l \times \mathbf{C}^m : |x - r| < r, \|\zeta\| < \rho, \|z\| < \tau\}.$$

Choose local coordinates in $\Delta_{r,\rho,\tau}$ such that the parabolic manifold provided by Theorem 1.1 is given by $\{\zeta = 0\}$. We can then rewrite Φ as

$$\begin{cases} x_1 = x - x^2 + O_1(x^3) + O(\|\zeta\|x^2), \\ \zeta_1 = (I - \Gamma x)\zeta + O_1(\|\zeta\|x^2) + O(\|\zeta\|^2 x), \\ z_1 = C(z) + O(x^2, \|z\|x), \end{cases}$$

where $O_1(h) = O(h|\log x|^q)$ for some integer $q \geq 0$. The $\log x$ terms are introduced in some changes of coordinates needed in the proof of Theorem 1.1 (see [Ri] for details). The $\log x$ terms are well-defined, since we will only work in $\Delta_{r,\rho,\tau}$.

Divide the eigenvalues of Γ as in (1.2). Write $\zeta = (u, v) \in \mathbf{C}^p \times \mathbf{C}^q$ and set $\Gamma = \text{Diag}(A, B)$. We then have

$$(2.1) \quad \begin{cases} x_1 = f(x, u, v, z) = x - x^2 + F(x, u, v, z), \\ u_1 = \psi(x, u, v, z) = (I_p - Ax)u + G(x, u, v, z), \\ v_1 = \phi(x, u, v, z) = (I_q - Bx)v + H(x, u, v, z), \\ z_1 = g(x, u, v, z) = C(z) + I(x, u, v, z), \end{cases}$$

where $F(x, u, v, z) = O_1(x^3) + O(\|\zeta\|x^2)$, $G(x, u, v, z) = O_1(\|\zeta\|x^2) + O(\|\zeta\|^2 x)$, $H(x, u, v, z) = O_1(\|\zeta\|x^2) + O(\|\zeta\|^2 x)$ and $I(x, u, v, z) = O(x^2, \|z\|x)$.

We need the following

Proposition 2.1. *Let Φ be as in (2.1). For every integer $m, k \geq 2$, we can choose coordinates in $\Delta_{r,\rho,\tau}$ such that*

$$(2.2) \quad H(x, u, 0, z) = O(\|u\|^m x, \|u\|x^k).$$

Proof. For terms involving only x and u , we can argue as in [H3, Proposition 2.2].

Now, let $a_{s,t,j}(x, u, z) = p(z)x^s(\log x)^t u^j$ be a term involving z in the expression of $H(x, u, 0, z)$, with $p(0) = 0$. One readily checks that the transformation

$$\tilde{v} = v + q(z)x^s(\log x)^t u^j$$

cancels $a_{s,t,j}(x, u, z)$ if $q(z) = \sum_{i=0}^{\infty} p(C^i(z))$. Since C is a contraction, the series converges in a neighborhood of 0.

By the construction in the proof of Theorem 1.1, we know that for each s there are only finitely many $x^s(\log x)^t$ occurring. Therefore, starting with the smallest s and $|j|$, we can make successive transformations as above such that terms involving z in the expression of $H(x, u, 0, z)$ satisfy $O(\|u\|^m x, \|u\|x^k)$. $\square$

For γ, s, ρ, τ small enough, define

$$S_{\gamma,s,\rho,\tau} = \{(x, u, z) \in \mathbf{C} \times \mathbf{C}^p \times \mathbf{C}^m : |\operatorname{Im} x| < \gamma \operatorname{Re} x, |x| < s, \|u\| < \rho, \|z\| < \tau\}.$$

The following proposition concerns the case $p = l$, i.e. all the eigenvalues of Γ have positive real parts.

Proposition 2.2. *Let Φ be as in (2.1), with $A = \Gamma$. Then every point of the sector $S_{\gamma,s,\rho,\tau}$, with γ, s, ρ, τ small enough, is attracted to the origin in the direction $[1 : 0 : 0]$, and x_n and z_n are $O(1/n)$. Moreover, we have $|x_n| \leq \frac{2}{n}$ and*

$$(2.3) \quad \|u_n\| |x_n|^{-\alpha} \leq \|u\| |x|^{-\alpha}.$$

Proof. In [H1], Hakim showed that the transformation $x_1 = x - x^2 + F(x, 0, z)$ is conjugated to $\frac{1}{x_1} = \frac{1}{x} + 1$ in a domain of the type $\{\operatorname{Re} \frac{1}{x} > R\} \times \{\|z\| < \tau\}$, for R large enough and τ small enough. Therefore, we can choose local coordinates such that $x_1 = \frac{x}{1+x} + \tilde{F}(x, u, z)$, with $\tilde{F}(x, u, z) = O(\|u\|x^2)$.

We can then proceed as in [H3, Proposition 2.6] and [Ri, Proposition 4.1]. $\square$

As in [H3], we then have the following

Corollary 2.3. *Let Φ be as in Proposition 2.2. For $(x, u, z) \in S_{\gamma,s,\rho,\tau}$, with γ, s, ρ, τ small enough, we have*

$$\lim_{n \rightarrow \infty} \|u_n\| |x_n|^{-\alpha} = 0$$

and

$$(2.4) \quad \sum_{n=0}^{\infty} |x_n|^t |\log x_n|^q \leq C_{t,q} |x|^{t-1} |\log |x||^q,$$

where $t > 1$, $q \in \mathbf{N}$ and $C_{t,q}$ is a constant.

Now, let Φ be as in (2.1), with $A \neq \Gamma$. We want to show that there exists a function $v = \varphi(x, u, z)$ in some sector $S_{\gamma,s,\rho,\tau}$ such that for

$$\begin{cases} x_1 = f(x, u, \varphi(x, u, z), z) \\ u_1 = \psi(x, u, \varphi(x, u, z), z) \\ z_1 = g(x, u, \varphi(x, u, z), z) \end{cases}$$

we have

$$(2.5) \quad \varphi(x_1, u_1, z_1) = \phi(x, u, \varphi(x, u, z), z).$$

Set $H_1(x, u, v, z) = v - x^B x_1^{-B} v_1$. One readily checks that H_1 satisfies the same estimates as H :

$$(2.6) \quad H_1(x, u, v, z) = O_1(\|\zeta\|x^2) + O(\|\zeta\|^2 x)$$

and

$$(2.7) \quad H_1(x, u, 0, z) = O(\|u\|^m x, \|u\|x^k).$$

Also, the invariance (2.5) of φ is equivalent to

$$(2.8) \quad x^{-B} \varphi(x, u, z) - x_1^{-B} \varphi(x_1, u_1, z_1) = x^{-B} H_1(x, u, \varphi(x, u, z), z).$$

Let $\{(x_n, u_n, z_n)\}$ be the iterates of the transformation

$$(2.9) \quad \begin{cases} x_1 = f(x, u, \varphi(x, u, z), z) = x - x^2 + F(x, u, \varphi(x, u, z), z), \\ u_1 = \psi(x, u, \varphi(x, u, z), z) = (I_p - Ax)u + G(x, u, \varphi(x, u, z), z), \\ z_1 = g(x, u, \varphi(x, u, z), z) = C(z) + I(x, u, \varphi(x, u, z), z), \end{cases}$$

with f, ψ, g given in (2.1) and φ analytic in some sector $S_{\gamma, s, \rho, \tau}$. Define

$$(2.10) \quad T\varphi(x, u, z) = x^B \sum_{n=0}^{\infty} x_n^{-B} H_1(x_n, u_n, \varphi(x_n, u_n, z_n), z_n).$$

By Proposition 2.2 and Corollary 2.3, we have the following proposition, whose proof is essentially the same as that of [H3, Proposition 3.2].

Proposition 2.4. *Let m, k be integers such that H_1 satisfies the estimate (2.7). There exists a positive constant K_0 such that*

$$\|\varphi(x, u, z)\| \leq K_0(\|u\|^m + \|u\|x^{k-1})$$

implies that the series (2.10) is uniformly converging on the invariant set

$$D := S_{\gamma, s, \rho, \tau} \cap \{\|u\|x^{-\alpha} \leq 1\}$$

and that

$$\|T\varphi(x, u, z)\| \leq K_0(\|u\|^m + \|u\|x^{k-1}).$$

Let B_0 be the Banach space of holomorphic functions φ on D such that

$$\|\varphi\|_0 := \sup_{x, u, z} \left\{ \frac{|\varphi(x, u, z)|}{\|u\|^m + \|u\|x^{k-1}} \right\}$$

is bounded, endowed with the norm $\|\varphi\|_0$. By Proposition 2.4, we have that T is well-defined for $\varphi \in B_0$, with $T\varphi \in B_0$.

In the next section, we will prove the following

Proposition 2.5. *There exists a closed subset B_1 of B_0 such that T restricted to B_1 is a contraction.*

Proposition 2.5 gives us a fixed point of the operator T , which is a solution to (2.8), and thus to (2.5). Therefore, Theorem 1.2 follows from Proposition 2.5.

3. PROOF OF PROPOSITION 2.5

First, let us fix some constants. Let $\delta = \min\{\alpha, 1\}$. Choose $\epsilon > 0$ such that $\epsilon < \delta/2$ and $\operatorname{Re}\alpha_j > \alpha + \epsilon$ for every eigenvalue α_j of A . Choose $\beta < \alpha$ such that $\operatorname{Re}\beta_j < \beta$ for every eigenvalue β_j of B . (Throughout this section, $\|M\|$ denotes the Euclidean norm of the entries of M .)

Lemma 3.1. *For γ, s, ρ, τ small enough, we have the following inequalities in D :*

$$(3.1) \quad \begin{aligned} \left| \frac{\partial x_n}{\partial x} \right| &\leq \left| \frac{x_n}{x} \right|^{1+\delta-2\epsilon}, & \left\| \frac{\partial x_n}{\partial u} \right\| &\leq \frac{|x_n|^{1+\delta-2\epsilon}}{|x|^{\delta-\epsilon}}, & \left\| \frac{\partial x_n}{\partial z} \right\| &\leq |x_n|^{1+\delta-2\epsilon}, \\ \left\| \frac{\partial u_n}{\partial x} \right\| &\leq \frac{\|u\| |x_n|^{\delta-\epsilon}}{|x|^{1+\delta-2\epsilon}}, & \left\| \frac{\partial u_n}{\partial u} \right\| &\leq \left| \frac{x_n}{x} \right|^{\delta-\epsilon}, & \left\| \frac{\partial u_n}{\partial z} \right\| &\leq \|u\| |x_n|^{\delta-\epsilon}, \\ \left\| \frac{\partial z_n}{\partial x} \right\| &\leq 1, & \left\| \frac{\partial z_n}{\partial u} \right\| &\leq 1, & \left\| \frac{\partial z_n}{\partial z} \right\| &\leq 1. \end{aligned}$$

Proof. First, note that given $b > a > 0$ we have

$$(3.2) \quad |x_1|^a \geq |x|^a |1 - bx|,$$

for $|x|$ small enough.

For γ, s, ρ, τ small enough, we have the following inequalities in D :

$$\begin{aligned} \left| \frac{\partial x_1}{\partial x} \right| &= |1 - 2x + o(x)| \leq \left| \frac{x_1}{x} \right|^{2-\epsilon} \leq \left| \frac{x_1}{x} \right|^{1+\delta-2\epsilon} \quad (1 + \delta - \epsilon < 2), \\ \left\| \frac{\partial x_1}{\partial u} \right\| &\leq K|x|^2 \leq \frac{|x_1|^{1+\delta-2\epsilon}}{|x|^{\delta-\epsilon}}, \\ \left\| \frac{\partial x_1}{\partial z} \right\| &\leq K(|x(\log x)^q| + \|u\|)|x|^2 \leq |x|^{1-\delta+2\epsilon}|x|^{1+\delta-2\epsilon} \leq |x_1|^{1+\delta-2\epsilon}, \\ \left\| \frac{\partial u_1}{\partial x} \right\| &\leq K\|u\| \leq \frac{\|u\| |x_1|^{\delta-\epsilon}}{|x|^{1+\delta-2\epsilon}}, \\ \left\| \frac{\partial u_1}{\partial u} \right\| &\leq |1 - \alpha x| \leq \left| \frac{x_1}{x} \right|^{\delta-\epsilon} \quad (\delta - \epsilon < \alpha), \\ \left\| \frac{\partial u_1}{\partial z} \right\| &\leq K(|x(\log x)^q| + \|u\|)\|u\||x| \leq \|u\||x|^{1-\delta+\epsilon}|x|^{\delta-\epsilon} \leq \|u\||x_1|^{\delta-\epsilon}, \\ \left\| \frac{\partial z_1}{\partial x} \right\| &\leq K(|x| + \|z\|) \leq 1, \\ \left\| \frac{\partial z_1}{\partial u} \right\| &\leq K(|x|^2 + \|z\||x|) \leq 1, \\ \left\| \frac{\partial z_1}{\partial z} \right\| &\leq c + K|x| \leq 1. \end{aligned}$$

Here K is a constant independent of $x, u, v, z, q \in \mathbf{N}$ and $c < 1$.

Therefore, (3.1) is true for $n = 1$. The last row of (3.1) for an arbitrary n is easily shown as above. We will show the other two rows of (3.1) by induction on n .

Assume that (3.1) is true for n . Using (2.3) and (3.2), we have

$$\begin{aligned} \left| \frac{\partial x_{n+1}}{\partial x} \right| &\leq |1 - 2x_n + o(x_n)| \left| \frac{\partial x_n}{\partial x} \right| + K|x_n|^2 \left\| \frac{\partial u_n}{\partial x} \right\| \\ &\quad + K(|x_n|^3 |\log x_n|^q + \|u_n\||x_n|^2) \left\| \frac{\partial z_n}{\partial x} \right\| \end{aligned}$$

$$\begin{aligned}
 &\leq \left| \frac{x_n}{x} \right|^{1+\delta-2\epsilon} (|1 - 2x_n + o(x_n)| + K\|u\| |x_n|^{1+\epsilon} \\
 &\quad + K(|x_n|^{2-\delta+2\epsilon} |\log x_n|^q + \|u_n\| |x_n|^{1-\delta+2\epsilon}) |x|^{1+\delta-2\epsilon}) \\
 &\leq \left| \frac{x_n}{x} \right|^{1+\delta-2\epsilon} (|1 - 2x_n + o(x_n)| + K|x_n|(|x_n|^{1-\delta+2\epsilon} |\log x_n|^q \\
 &\quad + \|u\| |x|^{-\alpha} |x_n|^{1-\delta+2\epsilon+\alpha}) |x|^{1+\delta-2\epsilon}) \\
 &\leq \left| \frac{x_{n+1}}{x} \right|^{1+\delta-2\epsilon}, \\
 \left\| \frac{\partial x_{n+1}}{\partial u} \right\| &\leq |1 - 2x_n + o(x_n)| \left\| \frac{\partial x_n}{\partial u} \right\| + K|x_n|^2 \left\| \frac{\partial u_n}{\partial u} \right\| \\
 &\quad + K(|x_n|^3 |\log x_n|^q + \|u_n\| |x_n|^2) \left\| \frac{\partial z_n}{\partial u} \right\| \\
 &\leq \frac{|x_n|^{1+\delta-2\epsilon}}{|x|^{\delta-\epsilon}} (|1 - 2x_n + o(x_n)| + K|x_n|^{1+\epsilon} \\
 &\quad + K(|x_n|^{2-\delta+2\epsilon} |\log x_n|^q + \|u_n\| |x_n|^{1-\delta+2\epsilon}) |x|^{\delta-\epsilon}) \\
 &\leq \frac{|x_{n+1}|^{1+\delta-2\epsilon}}{|x|^{\delta-\epsilon}}, \\
 \left\| \frac{\partial x_{n+1}}{\partial z} \right\| &\leq |1 - 2x_n + o(x_n)| \left\| \frac{\partial x_n}{\partial z} \right\| + K|x_n|^2 \left\| \frac{\partial u_n}{\partial z} \right\| \\
 &\quad + K(|x_n|^3 |\log x_n|^q + \|u_n\| |x_n|^2) \left\| \frac{\partial z_n}{\partial z} \right\| \\
 &\leq |x_n|^{1+\delta-2\epsilon} (|1 - 2x_n + o(x_n)| + K\|u\| |x_n|^{1+\epsilon} \\
 &\quad + K(|x_n|^{2-\delta+2\epsilon} |\log x_n|^q + \|u_n\| |x_n|^{1-\delta+2\epsilon})) \\
 &\leq |x_{n+1}|^{1+\delta-2\epsilon}, \\
 \left\| \frac{\partial u_{n+1}}{\partial x} \right\| &\leq K\|u_n\| \left\| \frac{\partial x_n}{\partial x} \right\| + |1 - \alpha x_n| \left\| \frac{\partial u_n}{\partial x} \right\| \\
 &\quad + K(\|u_n\| |x_n|^2 |\log x_n|^q + \|u_n\|^2 |x_n|) \left\| \frac{\partial z_n}{\partial x} \right\| \\
 &\leq \frac{\|u\| |x_n|^{\delta-\epsilon}}{|x|^{1+\delta-2\epsilon}} (|1 - \alpha x_n| + K|x|^{-\alpha} |x_n|^{1-\epsilon+\alpha} \\
 &\quad + K(|x|^{-\alpha} |x_n|^{2-\delta+\epsilon+\alpha} |\log x_n|^q + \|u\| |x|^{-2\alpha} |x_n|^{1-\delta+\epsilon+2\alpha}) |x|^{1+\delta-2\epsilon}) \\
 &\leq \frac{\|u\| |x_{n+1}|^{\delta-\epsilon}}{|x|^{1+\delta-2\epsilon}}, \\
 \left\| \frac{\partial u_{n+1}}{\partial u} \right\| &\leq K\|u_n\| \left\| \frac{\partial x_n}{\partial u} \right\| + |1 - \alpha x_n| \left\| \frac{\partial u_n}{\partial u} \right\| \\
 &\quad + K(\|u_n\| |x_n|^2 |\log x_n|^q + \|u_n\|^2 |x_n|) \left\| \frac{\partial z_n}{\partial u} \right\| \\
 &\leq \left| \frac{x_n}{x} \right|^{\delta-\epsilon} (|1 - \alpha x_n| + K\|u\| |x|^{-\alpha} |x_n|^{1-\epsilon+\alpha} \\
 &\quad + K(\|u\| |x|^{-\alpha} |x_n|^{2-\delta+\epsilon+\alpha} |\log x_n|^q + \|u\|^2 |x|^{-2\alpha} |x_n|^{1-\delta+\epsilon+2\alpha}) |x|^{\delta-\epsilon}) \\
 &\leq \left| \frac{x_{n+1}}{x} \right|^{\delta-\epsilon},
 \end{aligned}$$

$$\begin{aligned}
\left\| \frac{\partial u_{n+1}}{\partial z} \right\| &\leq K \|u_n\| \left\| \frac{\partial x_n}{\partial z} \right\| + |1 - \alpha x_n| \left\| \frac{\partial u_n}{\partial z} \right\| \\
&\quad + K (\|u_n\| |x_n|^2 |\log x_n|^q + \|u_n\|^2 |x_n|) \left\| \frac{\partial z_n}{\partial z} \right\| \\
&\leq \|u\| |x_n|^{\delta-\epsilon} (|1 - \alpha x_n| + K \|u\| |x|^{-\alpha} |x_n|^{1-\epsilon+\alpha} \\
&\quad + K (|x|^{-\alpha} |x_n|^{2-\delta+\epsilon+\alpha} |\log x_n|^q + \|u\| |x|^{-2\alpha} |x_n|^{1-\delta+\epsilon+2\alpha})) \\
&\leq \|u\| |x_{n+1}|^{\delta-\epsilon}.
\end{aligned}$$

This proves the lemma. $\square$

Proposition 3.2. *Let φ be in B_0 . There are positive constants K_1 , K_2 and K_3 such that*

$$(3.3) \quad \begin{cases} \left\| \frac{\partial \varphi}{\partial x} \right\| \leq K_1 (\|u\|^m |x|^{-1} + \|u\| |x|^{k-2}), \\ \left\| \frac{\partial \varphi}{\partial u} \right\| \leq K_2 (\|u\|^{m-1} + |x|^{k-1}), \\ \left\| \frac{\partial \varphi}{\partial z} \right\| \leq K_3 (\|u\|^m + \|u\| |x|^{k-1}), \end{cases}$$

imply the same inequalities for $\left\| \frac{\partial T\varphi}{\partial x} \right\|$, $\left\| \frac{\partial T\varphi}{\partial u} \right\|$ and $\left\| \frac{\partial T\varphi}{\partial z} \right\|$.

Proof. Recall that

$$(3.4) \quad T\varphi(x, u, z) = \sum_{n=0}^{\infty} \left(\frac{x_n}{x} \right)^{-B} H_1(x_n, u_n, \varphi(x_n, u_n, z_n), z_n).$$

Note that we have the following bounds for $H_1(x, u, \varphi(x, u, z), z)$, with $\varphi \in B_0$:

$$\begin{aligned}
\|H_1(x, u, \varphi(x, u, z), z)\| &\leq C_1 (\|u\|^m |x| + \|u\| |x|^k) \\
(3.5) \quad &\quad + C_2 \|\varphi\| (|x|^2 |\log x|^q + \|\varphi\| |x| + \|u\| |x|) \\
&\leq (C_1 + 1) (\|u\|^m |x| + \|u\| |x|^k),
\end{aligned}$$

where C_1 and C_2 are constants depending only on H_1 . Therefore, we have the following bounds for the partial derivatives of H_1 :

$$\begin{aligned}
\left\| \frac{\partial H_1}{\partial x} (x, u, \varphi(x, u, z), z) \right\| &\leq C_3 (\|u\|^m + \|u\| |x|^{k-1}), \\
\left\| \frac{\partial H_1}{\partial u} (x, u, \varphi(x, u, z), z) \right\| &\leq C_3 (\|u\|^{m-1} |x| + |x|^k), \\
(3.6) \quad \left\| \frac{\partial H_1}{\partial z} (x, u, \varphi(x, u, z), z) \right\| &\leq C_3 (\|u\|^m |x| + \|u\| |x|^k), \\
\left\| \frac{\partial H_1}{\partial \varphi} (x, u, \varphi(x, u, z), z) \right\| &\leq C_4 (|x|^2 |\log x|^q + \|u\| |x|),
\end{aligned}$$

where C_3 and C_4 are constants depending only on H_1 .

Assume that the inequalities (3.3) are satisfied for arbitrarily given constants K'_1 , K'_2 and K'_3 .

Differentiating (3.4) with respect to x , we have

$$\frac{\partial}{\partial x} T\varphi(x, u, z) = r_1 + r_2 + r_3 + r_4 + r_5 + r_6,$$

where

$$\begin{aligned} r_1 &= Bx^{-1} \sum_{n=0}^{\infty} \left(\frac{x_n}{x}\right)^{-B} H_1(x_n, u_n, \varphi(x_n, u_n, z_n), z_n), \\ r_2 &= -B \sum_{n=0}^{\infty} \left(\frac{x_n}{x}\right)^{-B} H_1(x_n, u_n, \varphi(x_n, u_n, z_n), z_n) x_n^{-1} \frac{\partial x_n}{\partial x}, \\ r_3 &= \sum_{n=0}^{\infty} \left(\frac{x_n}{x}\right)^{-B} \frac{\partial}{\partial x_n} H_1(x_n, u_n, \varphi(x_n, u_n, z_n), z_n) \frac{\partial x_n}{\partial x}, \\ r_4 &= \sum_{n=0}^{\infty} \left(\frac{x_n}{x}\right)^{-B} \frac{\partial}{\partial u_n} H_1(x_n, u_n, \varphi(x_n, u_n, z_n), z_n) \frac{\partial u_n}{\partial x}, \\ r_5 &= \sum_{n=0}^{\infty} \left(\frac{x_n}{x}\right)^{-B} \frac{\partial}{\partial z_n} H_1(x_n, u_n, \varphi(x_n, u_n, z_n), z_n) \frac{\partial z_n}{\partial x}, \\ r_6 &= \sum_{n=0}^{\infty} \left(\frac{x_n}{x}\right)^{-B} \frac{\partial}{\partial \varphi} H_1(x_n, u_n, \varphi(x_n, u_n, z_n), z_n) \left(\frac{\partial \varphi}{\partial x_n} \frac{\partial x_n}{\partial x} + \frac{\partial \varphi}{\partial u_n} \frac{\partial u_n}{\partial x} + \frac{\partial \varphi}{\partial z_n} \frac{\partial z_n}{\partial x} \right). \end{aligned}$$

Using (2.3), (2.4), (3.5), (3.6) and Lemma 3.1, we have

$$\begin{aligned} \|r_1\| &\leq \beta |x|^{-1} \sum_{n=0}^{\infty} \left|\frac{x_n}{x}\right|^{-\beta} (C_1 + 1) (\|u_n\|^m |x_n| + \|u_n\| |x_n|^k) \\ &\leq \beta (C_1 + 1) |x|^{-1} \sum_{n=0}^{\infty} \left|\frac{x_n}{x}\right|^{-\beta} (\|u\|^m |x|^{-m\alpha} |x_n|^{m\alpha+1} + \|u\| |x|^{-\alpha} |x_n|^{\alpha+k}) \\ &\leq C'_1 (\|u\|^m |x|^{-1} + \|u\| |x|^{k-2}), \\ \|r_2\| &\leq \beta \sum_{n=0}^{\infty} \left|\frac{x_n}{x}\right|^{-\beta} (C_1 + 1) (\|u_n\|^m |x_n| + \|u_n\| |x_n|^k) |x_n|^{-1} \left|\frac{x_n}{x}\right|^{1+\delta-2\epsilon} \\ &\leq C'_2 (\|u\|^m |x|^{-1} + \|u\| |x|^{k-2}), \\ \|r_3\| &\leq \sum_{n=0}^{\infty} \left|\frac{x_n}{x}\right|^{-\beta} C_3 (\|u_n\|^m + \|u_n\| |x_n|^{k-1}) \left|\frac{x_n}{x}\right|^{1+\delta-2\epsilon} \\ &\leq C'_3 (\|u\|^m |x|^{-1} + \|u\| |x|^{k-2}), \\ \|r_4\| &\leq \sum_{n=0}^{\infty} \left|\frac{x_n}{x}\right|^{-\beta} C_3 (\|u_n\|^{m-1} |x_n| + |x_n|^k) \frac{\|u\| |x_n|^{\delta-\epsilon}}{|x|^{1+\delta-2\epsilon}} \\ &\leq C_3 \sum_{n=0}^{\infty} \left|\frac{x_n}{x}\right|^{\delta-\epsilon-\beta} (\|u\|^m |x|^{-(m-1)\alpha} |x_n|^{(m-1)\alpha+1} + \|u\| |x_n|^k) |x|^{-1+\epsilon} \\ &\leq C'_4 (\|u\|^m |x|^{-1} + \|u\| |x|^{k-2}), \\ \|r_5\| &\leq \sum_{n=0}^{\infty} \left|\frac{x_n}{x}\right|^{-\beta} C_3 (\|u_n\|^m |x_n| + \|u_n\| |x_n|^k) \\ &\leq C'_5 (\|u\|^m |x|^{-1} + \|u\| |x|^{k-2}), \end{aligned}$$

$$\begin{aligned}
\|r_6\| &\leq \sum_{n=0}^{\infty} \left| \frac{x_n}{x} \right|^{-\beta} C_4 (|x_n|^2 |\log x_n|^q + \|u_n\| |x_n|) (K'_3 (\|u_n\|^m + \|u_n\| |x_n|^{k-1}) \\
&\quad + K'_1 (\|u_n\|^m |x_n|^{-1} + \|u_n\| |x_n|^{k-2}) \left| \frac{x_n}{x} \right|^{1+\delta-2\epsilon} \\
&\quad + K'_2 (\|u_n\|^{m-1} + |x_n|^{k-1}) \frac{\|u\| |x_n|^{\delta-\epsilon}}{|x|^{1+\delta-2\epsilon}}) \\
&\leq C_4 \sum_{n=0}^{\infty} \left| \frac{x_n}{x} \right|^{-\beta} (\|u_n\|^{m-1} + |x_n|^{k-1}) (K'_3 (\|u_n\| |x_n|^2 |\log x_n|^q + \|u_n\|^2 |x_n|) \\
&\quad + K'_1 (\|u_n\| |x_n| |\log x_n|^q + \|u_n\|^2) \left| \frac{x_n}{x} \right|^{1+\delta-2\epsilon} \\
&\quad + K'_2 (|x_n|^2 |\log x_n|^q + \|u_n\| |x_n|) \frac{\|u\| |x_n|^{\delta-\epsilon}}{|x|^{1+\delta-2\epsilon}}) \\
&\leq C'_6 (\|u\|^m |x|^{-1} + \|u\| |x|^{k-2}),
\end{aligned}$$

where C'_1, C'_2, C'_3, C'_4 are constants depending only on H_1 and where C'_5, C'_6 are small compared to C'_j , $1 \leq j \leq 4$. Set

$$K_1 = 2(C'_1 + C'_2 + C'_3 + C'_4).$$

We then have

$$\left\| \frac{\partial T\varphi}{\partial x} \right\| \leq K_1 (\|u\|^m |x|^{-1} + \|u\| |x|^{k-2}).$$

Differentiating (3.4) with respect to u , we have

$$\frac{\partial}{\partial u} T\varphi(x, u, z) = s_1 + s_2 + s_3 + s_4 + s_5,$$

where

$$\begin{aligned}
s_1 &= -B \sum_{n=0}^{\infty} \left(\frac{x_n}{x} \right)^{-B} H_1(x_n, u_n, \varphi(x_n, u_n, z_n), z_n) x_n^{-1} \frac{\partial x_n}{\partial u}, \\
s_2 &= \sum_{n=0}^{\infty} \left(\frac{x_n}{x} \right)^{-B} \frac{\partial}{\partial x_n} H_1(x_n, u_n, \varphi(x_n, u_n, z_n), z_n) \frac{\partial x_n}{\partial u}, \\
s_3 &= \sum_{n=0}^{\infty} \left(\frac{x_n}{x} \right)^{-B} \frac{\partial}{\partial u_n} H_1(x_n, u_n, \varphi(x_n, u_n, z_n), z_n) \frac{\partial u_n}{\partial u}, \\
s_4 &= \sum_{n=0}^{\infty} \left(\frac{x_n}{x} \right)^{-B} \frac{\partial}{\partial z_n} H_1(x_n, u_n, \varphi(x_n, u_n, z_n), z_n) \frac{\partial z_n}{\partial u}, \\
s_5 &= \sum_{n=0}^{\infty} \left(\frac{x_n}{x} \right)^{-B} \frac{\partial}{\partial \varphi} H_1(x_n, u_n, \varphi(x_n, u_n, z_n), z_n) \left(\frac{\partial \varphi}{\partial x_n} \frac{\partial x_n}{\partial u} + \frac{\partial \varphi}{\partial u_n} \frac{\partial u_n}{\partial u} + \frac{\partial \varphi}{\partial z_n} \frac{\partial z_n}{\partial u} \right).
\end{aligned}$$

Using (2.3), (2.4), (3.5), (3.6) and Lemma 3.1, we have

$$\begin{aligned}
\|s_1\| &\leq \beta \sum_{n=0}^{\infty} \left| \frac{x_n}{x} \right|^{-\beta} (C_1 + 1) (\|u_n\|^m |x_n| + \|u_n\| |x_n|^k) |x_n|^{-1} \frac{|x_n|^{1+\delta-2\epsilon}}{|x|^{\delta-\epsilon}} \\
&\leq \beta (C_1 + 1) \sum_{n=0}^{\infty} \left| \frac{x_n}{x} \right|^{\delta-\epsilon-\beta} (\|u\|^m |x|^{-m\alpha} |x_n|^{m\alpha+1} + \|u\| |x|^{-\alpha} |x_n|^{\alpha+k}) |x_n|^{-\epsilon}
\end{aligned}$$

$$\begin{aligned}
 &\leq C_1''(\|u\|^{m-1} + |x|^{k-1}), \\
 \|s_2\| &\leq \sum_{n=0}^{\infty} \left| \frac{x_n}{x} \right|^{-\beta} C_3(\|u_n\|^m + \|u_n\| |x_n|^{k-1}) \frac{|x_n|^{1+\delta-2\epsilon}}{|x|^{\delta-\epsilon}} \\
 &\leq C_3 \sum_{n=0}^{\infty} \left| \frac{x_n}{x} \right|^{\delta-\epsilon-\beta} (\|u\|^{m-1} |x|^{-m\alpha} |x_n|^{m\alpha+1} + |x|^{-\alpha} |x_n|^{\alpha+k}) \|u\| |x_n|^{-\epsilon} \\
 &\leq C_2''(\|u\|^{m-1} + |x|^{k-1}), \\
 \|s_3\| &\leq \sum_{n=0}^{\infty} \left| \frac{x_n}{x} \right|^{-\beta} C_3(\|u_n\|^{m-1} |x_n| + |x_n|^k) \left| \frac{x_n}{x} \right|^{\delta-\epsilon} \\
 &\leq C_3''(\|u\|^{m-1} + |x|^{k-1}), \\
 \|s_4\| &\leq \sum_{n=0}^{\infty} \left| \frac{x_n}{x} \right|^{-\beta} C_3(\|u_n\|^m |x_n| + \|u_n\| |x_n|^k) \\
 &\leq C_4''(\|u\|^{m-1} + |x|^{k-1}), \\
 \|s_5\| &\leq \sum_{n=0}^{\infty} \left| \frac{x_n}{x} \right|^{-\beta} C_4(|x_n|^2 |\log x_n|^q + \|u_n\| |x_n|) (K_3'(\|u_n\|^m + \|u_n\| |x_n|^{k-1})) \\
 &\quad + K_1'(\|u_n\|^m |x_n|^{-1} + \|u_n\| |x_n|^{k-2}) \frac{|x_n|^{1+\delta-2\epsilon}}{|x|^{\delta-\epsilon}} \\
 &\quad + K_2'(\|u_n\|^{m-1} + |x_n|^{k-1}) \left| \frac{x_n}{x} \right|^{\delta-\epsilon} \\
 &\leq C_4 \sum_{n=0}^{\infty} \left| \frac{x_n}{x} \right|^{-\beta} (\|u_n\|^{m-1} + |x_n|^{k-1}) (K_3'(\|u_n\| |x_n|^2 |\log x_n|^q + \|u_n\|^2 |x_n|) \\
 &\quad + K_1'(\|u_n\| |x_n| |\log x_n|^q + \|u_n\|^2) \frac{|x_n|^{1+\delta-2\epsilon}}{|x|^{\delta-\epsilon}} \\
 &\quad + K_2'(|x_n|^2 |\log x_n|^q + \|u_n\| |x_n|) \left| \frac{x_n}{x} \right|^{\delta-\epsilon}) \\
 &\leq C_5''(\|u\|^{m-1} + |x|^{k-1}),
 \end{aligned}$$

where C_1'', C_2'', C_3'' are constants depending only on H_1 and where C_4'', C_5'' are small compared to C_j'' , $1 \leq j \leq 3$. Set

$$K_2 = 2(C_1'' + C_2'' + C_3'').$$

We then have

$$\left\| \frac{\partial T\varphi}{\partial u} \right\| \leq K_2(\|u\|^{m-1} + |x|^{k-1}).$$

Differentiating (3.4) with respect to z , we have

$$\frac{\partial}{\partial z} T\varphi(x, u, z) = t_1 + t_2 + t_3 + t_4 + t_5,$$

where

$$t_1 = -B \sum_{n=0}^{\infty} \left(\frac{x_n}{x} \right)^{-B} H_1(x_n, u_n, \varphi(x_n, u_n, z_n), z_n) x_n^{-1} \frac{\partial x_n}{\partial z},$$

$$\begin{aligned}
t_2 &= \sum_{n=0}^{\infty} \left(\frac{x_n}{x}\right)^{-B} \frac{\partial}{\partial x_n} H_1(x_n, u_n, \varphi(x_n, u_n, z_n), z_n) \frac{\partial x_n}{\partial z}, \\
t_3 &= \sum_{n=0}^{\infty} \left(\frac{x_n}{x}\right)^{-B} \frac{\partial}{\partial u_n} H_1(x_n, u_n, \varphi(x_n, u_n, z_n), z_n) \frac{\partial u_n}{\partial z}, \\
t_4 &= \sum_{n=0}^{\infty} \left(\frac{x_n}{x}\right)^{-B} \frac{\partial}{\partial z_n} H_1(x_n, u_n, \varphi(x_n, u_n, z_n), z_n) \frac{\partial z_n}{\partial z}, \\
t_5 &= \sum_{n=0}^{\infty} \left(\frac{x_n}{x}\right)^{-B} \frac{\partial}{\partial \varphi} H_1(x_n, u_n, \varphi(x_n, u_n, z_n), z_n) \left(\frac{\partial \varphi}{\partial x_n} \frac{\partial x_n}{\partial z} + \frac{\partial \varphi}{\partial u_n} \frac{\partial u_n}{\partial z} + \frac{\partial \varphi}{\partial z_n} \frac{\partial z_n}{\partial z} \right).
\end{aligned}$$

Using (2.3), (2.4), (3.5), (3.6) and Lemma 3.1, we have

$$\begin{aligned}
\|t_1\| &\leq \beta \sum_{n=0}^{\infty} \left|\frac{x_n}{x}\right|^{-\beta} (C_1 + 1) (\|u_n\|^m |x_n| + \|u_n\| |x_n|^k) |x_n|^{\delta-2\epsilon} \\
&\leq C_1''' (\|u\|^m + \|u\| |x|^{k-1}), \\
\|t_2\| &\leq \sum_{n=0}^{\infty} \left|\frac{x_n}{x}\right|^{-\beta} C_3 (\|u_n\|^m + \|u_n\| |x_n|^{k-1}) |x_n|^{1+\delta-2\epsilon} \\
&\leq C_2''' (\|u\|^m + \|u\| |x|^{k-1}), \\
\|t_3\| &\leq \sum_{n=0}^{\infty} \left|\frac{x_n}{x}\right|^{-\beta} C_3 (\|u_n\|^{m-1} |x_n| + |x_n|^k) \|u\| |x_n|^{\delta-\epsilon} \\
&\leq C_3''' (\|u\|^m + \|u\| |x|^{k-1}), \\
\|t_4\| &\leq \sum_{n=0}^{\infty} \left|\frac{x_n}{x}\right|^{-\beta} C_3 (\|u_n\|^m |x_n| + \|u_n\| |x_n|^k) \\
&\leq C_4''' (\|u\|^m + \|u\| |x|^{k-1}), \\
\|t_5\| &\leq \sum_{n=0}^{\infty} \left|\frac{x_n}{x}\right|^{-\beta} C_4 (|x_n|^2 |\log x_n|^q + \|u_n\| |x_n|) (K_3' (\|u_n\|^m + \|u_n\| |x_n|^{k-1})) \\
&\quad + K_1' (\|u_n\|^m |x_n|^{-1} + \|u_n\| |x_n|^{k-2}) |x_n|^{1+\delta-2\epsilon} \\
&\quad + K_2' (\|u_n\|^{m-1} + |x_n|^{k-1}) \|u\| |x_n|^{\delta-\epsilon} \\
&\leq C_4 \sum_{n=0}^{\infty} \left|\frac{x_n}{x}\right|^{-\beta} (\|u_n\|^{m-1} + |x_n|^{k-1}) (K_3' (\|u_n\| |x_n|^2 |\log x_n|^q + \|u_n\|^2 |x_n|)) \\
&\quad + K_1' (\|u_n\| |x_n| |\log x_n|^q + \|u_n\|^2) |x_n|^{1+\delta-2\epsilon} \\
&\quad + K_2' (|x_n|^2 |\log x_n|^q + \|u_n\| |x_n|) \|u\| |x_n|^{\delta-\epsilon} \\
&\leq C_5''' (\|u\|^m + \|u\| |x|^{k-1}),
\end{aligned}$$

where C_4''' is a constant depending only on H_1 and where $C_1''', C_2''', C_3''', C_5'''$ are small compared to C_4''' . Set

$$K_3 = 2C_4'''.$$

We then have

$$\left\| \frac{\partial T \varphi}{\partial z} \right\| \leq K_3 (\|u\|^m + \|u\| \|x\|^{k-1}).$$

By definition, K_1, K_2 and K_3 are constants depending only on H_1 . Therefore the proposition is true for these constants K_1, K_2 and K_3 . $\square$

Let B_1 be the closed subset of B_0 defined by (3.3). Proposition 3.2 shows that the operator T sends B_1 into itself. To finish the proof of Proposition 2.5, we have to show that T restricted to B_1 is a contraction.

Let (x_n, u_n, z_n) and (x'_n, u'_n, z'_n) be the iterates of (x, u, z) by the transformations (2.9) associated to φ and φ' respectively, where $\varphi, \varphi' \in B_1$. Write Δx_n for $|x_n - x'_n|$, Δu_n for $\|u_n - u'_n\|$, Δz_n for $\|z_n - z'_n\|$ and $\Delta \varphi_n$ for $(\|u_n\|^m + \|u_n\| \|x_n\|^{k-1}) \|\varphi - \varphi'\|_0$. When $n = 0$, we will drop the subscript 0.

We have the following

Lemma 3.3. *For γ, s, ρ, τ small enough, we have the following inequalities in D :*

$$(3.7) \quad \frac{\Delta x_n}{\Delta \varphi} \leq \frac{|x_n|^{1+\delta-\epsilon}}{|x|^\delta}, \quad \frac{\Delta u_n}{\Delta \varphi} \leq \left| \frac{x_n}{x} \right|^\delta, \quad \frac{\Delta z_n}{\Delta \varphi} \leq \left| \frac{x_n}{x} \right|^\delta.$$

Proof. For γ, s, ρ, τ small enough, we have the following inequalities in D :

$$\begin{aligned} \frac{\Delta x_1}{\Delta \varphi} &\leq K |x|^2 \leq \frac{|x_1|^{1+\delta-\epsilon}}{|x|^\delta}, \\ \frac{\Delta u_1}{\Delta \varphi} &\leq K (|x|^2 |\log x|^q + \|u\| |x|) \leq \left| \frac{x_1}{x} \right|^\delta, \\ \frac{\Delta z_1}{\Delta \varphi} &\leq K (|x|^2 + \|z\| |x|) \leq \left| \frac{x_1}{x} \right|^\delta, \end{aligned}$$

where K is a constant independent of x, u, v, z and $q \in \mathbf{N}$.

Therefore (3.7) is true for $n = 1$. We will show (3.7) by induction on n .

Using (2.3), we have

$$\begin{aligned} \Delta \varphi_n &= (\|u_n\|^m + \|u_n\| \|x_n\|^{k-1}) \|\varphi - \varphi'\|_0 \\ &\leq (\|u\|^m \left| \frac{x_n}{x} \right|^{m\alpha} + \|u\| \|x\|^{k-1} \left| \frac{x_n}{x} \right|^{\alpha+k-1}) \|\varphi - \varphi'\|_0 \\ &\leq \left(\left| \frac{x_n}{x} \right|^{m\alpha} + \left| \frac{x_n}{x} \right|^{\alpha+k-1} \right) (\|u\|^m + \|u\| \|x\|^{k-1}) \|\varphi - \varphi'\|_0 \\ &= \left(\left| \frac{x_n}{x} \right|^{m\alpha} + \left| \frac{x_n}{x} \right|^{\alpha+k-1} \right) \Delta \varphi. \end{aligned}$$

Since $\left| \frac{x_n}{x} \right|^b \leq \left| \frac{x_n}{x} \right|^a$ for $b > a$, we then have

$$(3.8) \quad \frac{\Delta \varphi_n}{\Delta \varphi} \leq 2 \left| \frac{x_n}{x} \right|^\delta.$$

Assume that (3.7) is true for n . Using (2.3), (3.2) and (3.8), we have

$$\begin{aligned}
\frac{\Delta x_{n+1}}{\Delta \varphi} &\leq |1 - 2x_n + o(x_n)| \frac{\Delta x_n}{\Delta \varphi} + K|x_n|^2 \left(\frac{\Delta u_n}{\Delta \varphi} + \frac{\Delta \varphi_n}{\Delta \varphi} \right) \\
&\quad + K(|x_n|^3 |\log x_n|^q + \|u_n\| |x_n|^2) \frac{\Delta z_n}{\Delta \varphi} \\
&\leq \frac{|x_n|^{1+\delta-\epsilon}}{|x|^\delta} (|1 - 2x_n + o(x_n)| + 3K|x_n|^{1+\epsilon} \\
&\quad + K(|x_n|^{2+\epsilon} |\log x_n|^q + \|u_n\| |x_n|^{1+\epsilon})) \\
&\leq \frac{|x_{n+1}|^{1+\delta-\epsilon}}{|x|^\delta}, \\
\frac{\Delta u_{n+1}}{\Delta \varphi} &\leq K\|u_n\| \frac{\Delta x_n}{\Delta \varphi} + |1 - (\alpha + \epsilon)x_n| \frac{\Delta u_n}{\Delta \varphi} + K(|x_n|^2 |\log x_n|^q + \|u_n\| |x_n|) \frac{\Delta \varphi_n}{\Delta \varphi} \\
&\quad + K(\|u_n\| |x_n|^2 |\log x_n|^q + \|u_n\|^2 |x_n|) \frac{\Delta z_n}{\Delta \varphi} \\
&\leq \left| \frac{x_n}{x} \right|^\delta (|1 - (\alpha + \epsilon)x_n| + K\|u_n\| |x_n|^{1-\epsilon} + 2K(|x_n|^2 |\log x_n|^q + \|u_n\| |x_n|) \\
&\quad + K(\|u_n\| |x_n|^2 |\log x_n|^q + \|u_n\|^2 |x_n|)) \\
&\leq \left| \frac{x_{n+1}}{x} \right|^\delta, \\
\frac{\Delta z_{n+1}}{\Delta \varphi} &\leq K(|x_n| + \|z_n\|) \frac{\Delta x_n}{\Delta \varphi} + K(|x_n|^2 + \|z_n\| |x_n|) \left(\frac{\Delta u_n}{\Delta \varphi} + \frac{\Delta \varphi_n}{\Delta \varphi} \right) \\
&\quad + (c + K|x_n|) \frac{\Delta z_n}{\Delta \varphi} \\
&\leq \left| \frac{x_n}{x} \right|^\delta (c + K|x_n| + K(|x_n| + \|z_n\|) + 3K(|x_n|^2 + \|z_n\| |x_n|)) \\
&\leq \left| \frac{x_{n+1}}{x} \right|^\delta \frac{c + K|x_n| + K(|x_n| + \|z_n\|) + 3K(|x_n|^2 + \|z_n\| |x_n|)}{|1 - (1 + \epsilon)x_n|} \\
&\leq \left| \frac{x_{n+1}}{x} \right|^\delta.
\end{aligned}$$

This proves the lemma. $\square$

We can write the difference

$$\begin{aligned}
S &:= \sum_{n=0}^{\infty} \left(\frac{x_n}{x} \right)^{-B} H_1(x_n, u_n, \varphi(x_n, u_n, z_n), z_n) - \sum_{n=0}^{\infty} \left(\frac{x'_n}{x} \right)^{-B} H_1(x'_n, u'_n, \varphi'(x'_n, u'_n, z'_n), z'_n) \\
&\text{as}
\end{aligned}$$

$$S = S_1 + S_2 + S_3,$$

where

$$\begin{aligned}
S_1 &= \sum_{n=0}^{\infty} \left(\frac{x_n}{x} \right)^{-B} (H_1(x_n, u_n, \varphi(x_n, u_n, z_n), z_n) - H_1(x'_n, u'_n, \varphi(x'_n, u'_n, z'_n), z'_n)), \\
S_2 &= \sum_{n=0}^{\infty} \left(\left(\frac{x_n}{x} \right)^{-B} - \left(\frac{x'_n}{x} \right)^{-B} \right) H_1(x'_n, u'_n, \varphi(x'_n, u'_n, z'_n), z'_n), \\
S_3 &= \sum_{n=0}^{\infty} \left(\frac{x'_n}{x} \right)^{-B} (H_1(x'_n, u'_n, \varphi(x'_n, u'_n, z'_n), z'_n) - H_1(x'_n, u'_n, \varphi'(x'_n, u'_n, z'_n), z'_n)).
\end{aligned}$$

Using (2.4), (3.3), (3.5), (3.6) and Lemma 3.3, we have

$$\begin{aligned}
 \|S_1\| &\leq \sum_{n=0}^{\infty} \left| \frac{x_n}{x} \right|^{-\beta} \{ [C_3(\|u_n\|^m + \|u_n\||x_n|^{k-1}) \\
 &\quad + C_4(|x_n|^2 |\log x_n|^q + \|u_n\||x_n|) K_1(\|u_n\|^m |x_n|^{-1} + \|u_n\||x_n|^{k-2})] \Delta x_n \\
 &\quad + [C_3(\|u_n\|^{m-1} |x_n| + |x_n|^k) \\
 &\quad + C_4(|x_n|^2 |\log x_n|^q + \|u_n\||x_n|) K_2(\|u_n\|^{m-1} + |x_n|^{k-1})] \Delta u_n \\
 &\quad + [C_3(\|u_n\|^m |x_n| + \|u_n\||x_n|^k) \\
 &\quad + C_4(|x_n|^2 |\log x_n|^q + \|u_n\||x_n|) K_3(\|u_n\|^m + \|u_n\||x_n|^{k-1})] \Delta z_n \} \\
 &\leq \sum_{n=0}^{\infty} \left| \frac{x_n}{x} \right|^{-\beta} \{ (\|u_n\|^m + \|u_n\||x_n|^{k-1}) [C_3 \\
 &\quad + C_4 K_1(|x_n| |\log x_n|^q + \|u_n\|)] \frac{|x_n|^{1+\delta-\epsilon}}{|x|^\delta} \\
 &\quad + (\|u_n\|^{m-1} |x_n| + |x_n|^k) [C_3 + C_4 K_2(|x_n| |\log x_n|^q + \|u_n\|)] \left| \frac{x_n}{x} \right|^\delta \\
 &\quad + (\|u_n\|^m |x_n| + \|u_n\||x_n|^k) [C_3 + C_4 K_3(|x_n| |\log x_n|^q + \|u_n\|)] \left| \frac{x_n}{x} \right|^\delta \} \Delta \varphi \\
 &\leq \sum_{n=0}^{\infty} \left| \frac{x_n}{x} \right|^{\delta-\beta} \{ (\|u_n\|^{m-1} |x_n| + |x_n|^k) [C_3 + C_4 K_2(|x_n| |\log x_n|^q + \|u_n\|) \\
 &\quad + (C_3 + C_4 K_1(|x_n| |\log x_n|^q + \|u_n\|)) \|u_n\||x_n|^{-\epsilon} \\
 &\quad + (C_3 + C_4 K_3(|x_n| |\log x_n|^q + \|u_n\|)) \|u_n\||x_n|] \} \Delta \varphi \\
 &\leq C_5 (\|u\|^{m-1} + |x|^{k-1}) (\|u\|^m + \|u\||x|^{k-1}) \|\varphi - \varphi'\|_0, \\
 \|S_2\| &\leq \beta \sum_{n=0}^{\infty} \left| \frac{x_n}{x} \right|^{-\beta} (C_1 + 1) (\|u_n\|^m |x_n| + \|u_n\||x_n|^k) |x_n|^{-1} \Delta x_n \\
 &\leq (C_1 + 1) \beta \sum_{n=0}^{\infty} \left| \frac{x_n}{x} \right|^{\delta-\beta} (\|u_n\|^{m-1} |x_n| + |x_n|^k) \|u_n\||x_n|^{-\epsilon} \Delta \varphi \\
 &\leq C_6 (\|u\|^{m-1} + |x|^{k-1}) (\|u\|^m + \|u\||x|^{k-1}) \|\varphi - \varphi'\|_0, \\
 \|S_3\| &\leq \sum_{n=0}^{\infty} \left| \frac{x'_n}{x} \right|^{-\beta} K(|x'_n|^2 |\log x'_n|^q + \|u'_n\||x'_n|) \Delta \varphi'_n \\
 &\leq C_7 (|x| |\log x|^q + \|u\|) (\|u\|^m + \|u\||x|^{k-1}) \|\varphi - \varphi'\|_0,
 \end{aligned}$$

where C_5, C_6, C_7 are constants depending only on H_1 .

The above inequalities show that T restricted to B_1 is a contraction. Therefore, the proof of Proposition 2.5 is complete.

REFERENCES

- [F] Fatou, P.; *Substitutions analytiques et équations fonctionnelles à deux variables*, Ann. Sc. Ec. Norm. Sup. 40 (1924), 67-142.
- [H1] Hakim, M.; *Attracting domains for semi-attractive transformations of $\mathbf{C}^p$* , Publ. Mat. 38 (1994), 479-499. MR1316642 (96a:32050)
- [H2] Hakim, M.; *Analytic transformations of $(\mathbf{C}^p, 0)$ tangent to the identity*, Duke Math. J. 92 (1998), 403-428. MR1612730 (99a:32036)

- [H3] Hakim, M.; *Transformations tangent to the identity. Stable pieces of manifolds*, Preprint, 1998.
- [N] Nishimura, Y.; *Automorphismes analytiques admettant des sous-variétés de points fixés attractives dans la direction transversale*, J. Math. Kyoto Univ. 23 (1983), 289-299. MR706166 (85g:32040)
- [Ri] Rivi, M.; *Parabolic manifolds for semi-attractive holomorphic germs*, Michigan Math. J. 49 (2001), 211-241. MR1852300 (2002k:32031)
- [Ro] Rong, F.; *Quasi-parabolic analytic transformations of $\mathbf{C}^n$. Parabolic manifolds*, Ark. Mat. 48 (2010), no. 2, 361-370. MR2672615
- [U1] Ueda, T.; *Local structure of analytic transformations of two complex variables, I*, J. Math. Kyoto Univ. 26 (1986), 233-261. MR849219 (87k:58145)
- [U2] Ueda, T.; *Local structure of analytic transformations of two complex variables, II*, J. Math. Kyoto Univ. 31 (1991), 695-711. MR1127094 (92i:58166)

DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY, SYRACUSE, NEW YORK 13244

Current address: Department of Mathematics, Shanghai Jiao Tong University, 800 Dong Chuan Road, Shanghai 200240, People's Republic of China