PLÜNNECKE'S THEOREM FOR ASYMPTOTIC DENSITIES

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ABSTRACT. Plünnecke proved that if $B \subseteq \mathbb{N}$ is a basis of order $h > 1$, i.e., $\sigma(hB) = 1$, then $\sigma(A + B) \geq \sigma(A)^{1-\frac{1}{h}}$, where A is an arbitrary subset of $\mathbb{N}$ and σ represents Shnirel'man density. In this paper we explore whether σ can be replaced by other asymptotic densities. We show that Plünnecke's inequality above is true if σ is replaced by lower asymptotic density $\underline{d}$ or by upper Banach density BD but not by upper asymptotic density $\bar{d}$. The result about $\underline{d}$ has some interesting consequences such as the inequality $\underline{d}(A + P) \geq \underline{d}(A)^{2/3}$ for any $A \subseteq \mathbb{N}$, where P is the set of all prime numbers, and the inequality $\underline{d}(A + C) \geq \underline{d}(A)^{3/4}$ for any $A \subseteq \mathbb{N}$, where C is the set of all cubes of nonnegative integers. The result about BD generalizes Theorem 3 of a 2001 work of the author by reducing the requirement of B being a piecewise basis to the requirement of B being an upper Banach basis.

1. INTRODUCTION AND A BRIEF HISTORY

Let $\mathbb{N}$ be the set of all nonnegative integers and let A be a subset of $\mathbb{N}$. The *Shnirel'man density* of A is defined by

$$\sigma(A) = \inf_{n \geq 1} \frac{|A \cap [1, n]|}{n},$$

where $A(n) = |A \cap [1, n]|$. A set $B \subseteq \mathbb{N}$ is called an *essential component* if $\sigma(A+B) > \sigma(A)$, where $A + B = \{a + b : a \in A \text{ and } b \in B\}$ for any set $A \subseteq \mathbb{N}$ with $0 < \sigma(A) < 1$. Since the early part of the last century people have been interested in finding out in which set $B \subseteq \mathbb{N}$ can be an essential component (cf. [6]). By Shnirel'man's Theorem [6, page 3] it can be easily seen that if $0 \in B$ and $\sigma(B) > 0$, then B is an essential component. But even if $\sigma(B) = 0$, B can still be an essential component. A set $B \subseteq \mathbb{N}$ is called a *basis* of order h if

$$hB = \underbrace{B + B + \cdots + B}_h = \mathbb{N}.$$

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Let $h > 1$. Note that B is a basis of order h iff $\sigma(hB) = 1$. If B is a basis of some finite order, then B is an essential component, although such B may have Shnirel'man density 0. For example, $B = \{n^2 : n \in \mathbb{N}\}$ is a basis of order 4 by Lagrange's Theorem and $\sigma(B) = 0$. In 1937 Erdős proved that if B is a basis of order h , then

$$(1) \quad \sigma(A + B) \geq \sigma(A) + \frac{1}{2h} \cdot \sigma(A) (1 - \sigma(A)).$$

A short time later, Landau noticed that in Erdős' proof h can be replaced by the average order h^* (cf. [6, page 10]). Let $B \subseteq \mathbb{N}$ be a basis of order h . For each $m \in \mathbb{N}$ let $h_B(m) = \min\{h : m \in hB\}$. The *average order* h^* of B is defined by

$$h^* = \sup_{n \geq 1} \frac{1}{n} \sum_{m=1}^n h_B(m).$$

It is easy to see that $h^* \leq h \leq 2h^*$ (cf. [6, page 12]). In 1938 Rohrbach proved a theorem for lower asymptotic density parallel to Erdős–Landau's result. A set $B \subseteq \mathbb{N}$ is called an *asymptotic basis* of order h if hB contains all sufficiently large positive integers. The *average asymptotic order* h^* of an asymptotic basis is defined by

$$h^* = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n h_B(m),$$

where $h_B(m)$ is defined to be 0 if $m \notin hB$. The *lower asymptotic density* of A is defined by

$$\underline{d}(A) = \liminf_{n \rightarrow \infty} \frac{A(n)}{n}.$$

Rohrbach proved (cf. [6, page 45]) that if B is an asymptotic basis of average asymptotic order h^* , then

$$\underline{d}(A + B) \geq \underline{d}(A) + \frac{1}{2h^*} \cdot \underline{d}(A) (1 - \underline{d}(A)).$$

In 1970 Plünnecke obtained the following significant improvement of Erdős–Landau's result (cf. [15] or [14, page 225]).

Theorem 1.1 (Plünnecke, 1970). *If B is a basis of order h , then for every $A \subseteq \mathbb{N}$*

$$\sigma(A + B) \geq \sigma(A)^{1 - \frac{1}{h}}.$$

One can easily show (cf. [14, page 226]) that $f(x) \geq g(x)$ for $0 \leq x \leq 1$ by elementary differential calculus, where $f(x) = x^{1 - \frac{1}{h}}$ and $g(x) = x + \frac{1}{h} \cdot x(1 - x)$. Hence Plünnecke's Theorem implies (1) even if $2h$ in the denominator is replaced by h . Erdős–Landau's result is also a consequence of Theorem 1.1 because $h \leq 2h^*$.

Plünnecke's approach is completely different from that of Erdős–Landau's. By analyzing the growth rates of the Plünnecke graph at different levels, Plünnecke was able to prove a powerful inequality among these growth rates, which leads to Theorem 1.1. For simplicity we will directly introduce Plünnecke's inequality on truncated sumsets instead of on the Plünnecke graph. Let $A \subseteq \mathbb{N}$ and $a, b \in \mathbb{N}$. We denote $A(a, b)$ for the number $|A \cap [a, b]|$.

Theorem 1.2 (Plünnecke, 1957). *Let $A, B \subseteq \mathbb{N}$ and $h, n \in \mathbb{N}$ be such that $A(0, n) \neq 0$. For each $1 \leq i \leq h$ define*

$$D_{A,B,n,i} = \min \left\{ \frac{(A' + iB)(0, n)}{A'(0, n)} : \emptyset \neq A' \subseteq A \cap [0, n] \right\}.$$

Then

$$D_{A,B,n,1} \geq (D_{A,B,n,2})^{1/2} \geq \cdots \geq (D_{A,B,n,h})^{1/h}.$$

Intuitively, $D_{A,B,n,i}$ can be viewed as the minimum growth rate of $A + iB$ over A truncated at n . The proof of Theorem 1.2 for the general Plünnecke graph and the verification that a truncated additive graph is a Plünnecke graph can be found in [14, Chapter 7].

Can we replace h in Theorem 1.1 by the average order h^* , similar to what Landau did to Erdős' Theorem? The following simple example dashes this hope.

Example 1.3. Let $A = \{1 + 3n : n \in \mathbb{N}\}$ and $B = \{3n : n \in \mathbb{N}\} \cup A$. Then $A + B = A \cup \{2 + 3n : n \in \mathbb{N}\}$, $\sigma(A) = \underline{d}(A) = \frac{1}{3}$ and $\sigma(A + B) = \underline{d}(A + B) = \frac{2}{3}$. It is easy to check that B is a basis of order $h = 2$, average order $h^* = \frac{3}{2}$, and average asymptotic order $h' = \frac{4}{3}$. Note that

$$\underline{d}(A)^{1-\frac{1}{h'}} = \left(\frac{1}{3}\right)^{\frac{1}{4}} \geq \sigma(A)^{1-\frac{1}{h^*}} = \left(\frac{1}{3}\right)^{\frac{1}{3}} > \frac{2}{3} = \sigma(A + B) = \underline{d}(A + B).$$

The author was informed by Georges Grekos that the above example appeared in a German article around 1953.

Can we derive theorems for other densities parallel to Theorem 1.1, similar to what Rohrbach did to the Erdős–Landau Theorem? Yes we can. In fact, we can lose the requirement of B being an asymptotic basis to the requirement of B being a lower asymptotic basis when σ is replaced by $\underline{d}$. We can also derive a parallel theorem for upper Banach density. In fact, there is already a theorem about upper Banach density parallel to Theorem 1.1 in [10]. However, the theorem in [10] will be significantly improved in this paper.

Let $A \subseteq \mathbb{N}$. The *upper Banach density* of A is defined by

$$BD(A) = \lim_{n \rightarrow \infty} \sup_{k \in \mathbb{N}} \frac{A(k, k+n)}{n+1}.$$

Upper Banach density is popular among mathematicians who work on combinatorial number theory problems using ergodic methods (cf. [1, 5]). Upper Banach density is also called uniform density in [8]. Note that $BD(A) = \alpha$ iff α is the greatest real number such that there exists a sequence of interval $\{[a_n, b_n] : n \in \mathbb{N}\}$ with

$$(2) \quad \lim_{n \rightarrow \infty} (b_n - a_n) = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{A(a_n, b_n)}{b_n - a_n + 1} = \alpha.$$

In [10] a set $B \subseteq \mathbb{N}$ is called a *piecewise basis* of order h if there exists a sequence $\{c_n : n \in \mathbb{N}\}$ of positive integers such that for each n

$$[0, n] \subseteq h((A - c_n) \cap \mathbb{N}).$$

In [10] it is proved that if B is a piecewise basis of order h , then for any set $A \subseteq \mathbb{N}$

$$BD(A + B) \geq BD(A)^{1-\frac{1}{h}}.$$

Note that a basis of order h is a piecewise basis of order at most h . Simply take $c_n = 0$. However, the definition of a piecewise basis seems tedious. In this paper we will improve this result by substituting piecewise basis with upper Banach basis.

How about upper asymptotic density? For a set $A \subseteq \mathbb{N}$ the *upper asymptotic density* is defined by

$$\bar{d}(A) = \limsup_{n \rightarrow \infty} \frac{A(n)}{n}.$$

If $\underline{d}(A) = \bar{d}(A)$ we call $d(A) = \underline{d}(A) = \bar{d}(A)$ the *asymptotic density* of A . Clearly, we have the following inequalities among these densities. For a set $A \in \mathbb{N}$ we have

$$(3) \quad 0 \leq \sigma(A) \leq \underline{d}(A) \leq \bar{d}(A) \leq BD(A) \leq 1.$$

Although $\bar{d}$ is in the middle of $\underline{d}$ and BD in terms of their magnitudes, the behavior of $\bar{d}$ is very different from that of $\underline{d}$ and BD . The reader will see the difference when we try to generalize Plünnecke's Theorem to these densities.

Definition 1.4. Let $B \subseteq \mathbb{N}$. Then:

- B is called a *lower asymptotic basis of order h* if $\underline{d}(hB) = 1$.
- B is called an *upper asymptotic basis of order h* if $\bar{d}(hB) = 1$.
- B is called an *upper Banach basis of order h* if $BD(hB) = 1$.

Note that if $\underline{d}(hB) = 1$, then $\bar{d}(hB) = 1$. Hence the density of hB exists and $d(hB) = 1$. This shows that the lower asymptotic basis above should be called an asymptotic basis. Unfortunately, the name "asymptotic basis" is already taken. A lower asymptotic basis is sometimes also called a weak basis or Schwarz basis in some literature. Note that a lower asymptotic basis of order h must be an asymptotic basis of order at most $2h$. It was proven in [7] that for every h there is a lower asymptotic basis B of order h such that B is not an asymptotic basis of order $2h - 1$. Note that an asymptotic basis of order h is a lower asymptotic basis of order at most h , a lower asymptotic basis of order h is an upper asymptotic basis of order at most h , and an upper asymptotic basis of order h is an upper Banach basis of order at most h . Note also that a piecewise basis of order h is an upper Banach basis of order at most h .

Following are the theorems we will prove in this paper.

Theorem 1.5. Let $A, B \subseteq \mathbb{N}$ and B be a lower asymptotic basis of order h . Then

$$\underline{d}(A + B) \geq \underline{d}(A)^{1-\frac{1}{h}}.$$

We would like to mention three consequences of Theorem 1.5.

Let P be the set of all prime numbers. By a result proved independently by Estermann [4], Chudakov, and van der Corput, P is a lower asymptotic basis of order 3.¹ Hence Theorem 1.5 implies that $\underline{d}(A + P) \geq \underline{d}(A)^{2/3}$ for every $A \subseteq \mathbb{N}$.

¹ P is an asymptotic basis of order 4 by Vinogradov's Three Primes Theorem and an asymptotic basis of order 3 if Goldbach's Conjecture is true.

Let S be the set of all squares of integers. Then S is a basis of order 4 by a well-known theorem of Lagrange. Hence Theorem 1.5 implies that $\underline{d}(A + S) \geq \underline{d}(A)^{3/4}$ for every $A \subseteq \mathbb{N}$.

More interestingly, let C be the set of all cubes of nonnegative integers. Then C is a lower asymptotic basis of order 4 by a result of Davenport [3].² Hence Theorem 1.5 implies that $\underline{d}(A + C) \geq \underline{d}(A)^{3/4}$ for every $A \subseteq \mathbb{N}$.

Theorem 1.6. *There are $A, B \subseteq \mathbb{N}$ with $\bar{d}(A) = \frac{1}{2}$ and $\bar{d}(2B) = 1$ such that*

$$\bar{d}(A + B) = \bar{d}(A).$$

Theorem 1.7. *Let $A, B \subseteq \mathbb{N}$ and B be a upper Banach basis of order h . Then*

$$BD(A + B) \geq BD(A)^{1 - \frac{1}{h}}.$$

Since a piecewise basis of order h is an upper Banach basis of order at most h but not vice versa, Theorem 1.7 is clearly an improvement of [10, Theorem 3].

Theorem 1.8. *There are $A, B \subseteq \mathbb{N}$ with $\underline{d}(A) = \frac{1}{2}$ and $\bar{d}(2B) = 1$ such that*

$$\underline{d}(A + B) = \underline{d}(A).$$

We can define essential components in terms of densities besides σ . For example, we call a set B an *essential component in terms of $\bar{d}$* if $\bar{d}(A + B) > \bar{d}(A)$ for any $A \subseteq \mathbb{N}$ with $0 < \bar{d}(A) < 1$.

Theorem 1.9. *A lower asymptotic basis of a finite order is an essential component in terms of upper asymptotic density.*

As done before, we would like to use tools from nonstandard analysis in the proofs of Theorem 1.5, Theorem 1.7, and Theorem 1.9. The proofs of Theorem 1.6 and Theorem 1.8 do not involve nonstandard methods. Nonstandard tools have been proven very useful and efficient in, for example, [2, 10, 11, 12]. The reader is recommended to consult one of [2, 9, 10, 13] for the basic notation, ideas, and principles in nonstandard analysis. Other introductory texts for Robinsonian style nonstandard analysis should also be sufficient. If we work within a nonstandard universe, we always assume that the nonstandard universe is countably saturated.

Before starting the proofs we introduce some notation dealing with infinitesimals. Let $r, s \in {}^*\mathbb{R}$. By $r \approx s$ we mean that r is infinitesimally close to s , i.e., $|r - s|$ is less than any positive standard real numbers. By $r \ll s$ we mean that $r < s$ but $r \not\approx s$. By $r \lesssim s$ we mean that $r < s$ or $r \approx s$. We define $r \gg s$ and $r \gtrsim s$ by a symmetric way.

2. PROOF OF THEOREM 1.5

We first prove a nonstandard equivalence of lower asymptotic density by the transfer principle.

Lemma 2.1. *Let $A \subseteq \mathbb{N}$. Then $\underline{d}(A) \geq \alpha$ iff for every hyperfinite integer H*

$$(4) \quad \frac{{}^*A(0, H)}{H + 1} \gtrsim \alpha.$$

² C is an asymptotic basis of order 7. The question whether C can be an asymptotic basis of order less than 7 is still open.

Proof. Suppose $\underline{d}(A) \geq \alpha$ and let ϵ be a standard positive real number. Then there is $m \in \mathbb{N}$ such that $\frac{A(0,n)}{n+1} > \alpha - \epsilon$ for any $n > m$. Hence $\frac{{}^*A(0,H)}{H+1} > \alpha - \epsilon$ for any hyperfinite integer H by the transfer principle. Since ϵ is arbitrary, we have (4).

Suppose (4) is true for every hyperfinite integer H . Let ϵ be a standard positive real number and X_ϵ be defined by

$$X_\epsilon = \left\{ n \in {}^*\mathbb{N} : \frac{{}^*A(0,n)}{n+1} > \alpha - \epsilon \right\}.$$

Then X_ϵ is an internal set containing all hyperfinite integers. Hence there is $m \in \mathbb{N}$ such that X_ϵ contains all integers above m . This means that $\frac{A(0,n)}{n+1} > \alpha - \epsilon$ for every standard $n > m$. Therefore, $\underline{d}(A) \geq \alpha$. $\square$

Let A and B be as in Theorem 1.5 such that $\underline{d}(A) = \alpha$ and $\underline{d}(hB) = 1$. Without loss of generality, we can assume $0 < \alpha < 1$. Let H be any hyperfinite integer. We want to show that

$$\frac{{}^*(A+B)(0,H)}{H+1} = \frac{({}^*A + {}^*B)(0,H)}{H+1} \gtrsim \alpha^{1-\frac{1}{h}},$$

which implies Theorem 1.5 by Lemma 2.1. Choose hyperfinite integers $N < K < H$ such that $\frac{H-K}{H} \approx 0$ and $\frac{K-N}{H-N} \approx 0$ (for example, $K = H - \lfloor \sqrt{H} \rfloor$ and $N = K - \lfloor \sqrt[4]{H} \rfloor$ satisfy the requirements). Let $C_0 = {}^*A \cap [0, K]$. Then $\frac{C_0(0,H)}{H} \gtrsim \alpha$. Next we want to trim C_0 so that the density of the trimmed set in an interval $[x, H]$ for every $x \leq K$ would not be too large. We define C_k inductively for $k = 0, 1, \dots, N$ so that $C_0 \supseteq C_1 \supseteq \dots \supseteq C_N$, $\frac{C_N(0,H)}{H+1} \approx \alpha$, and $\frac{C_N(x,H)}{H-x+1} \lesssim \alpha$ for any $x \leq K$. We have had C_0 . For each $k < N$ let

$$C_{k+1} = \begin{cases} C_k, & \text{if } \frac{C_k(N-k,H)}{H-N+k+1} \leq \alpha, \\ C_k \setminus \{N-k\}, & \text{otherwise.} \end{cases}$$

It is easy to see that $C_0, C_1, \dots, C_N$ has the desired properties. Let $A_0 = C_N$ and

$$D_{A_0, {}^*B, H, h} = \frac{(A' + h {}^*B)(0, H)}{A'(0, H)}$$

for some nonempty internal set $A' \subseteq A_0$. Let $z = \min A'$. Then $z \leq K$. Hence $H - z$ is hyperfinite, which implies $\frac{(h {}^*B)(z, H)}{H-z+1} \approx 1$. Hence by Theorem 1.2 we have

$$\begin{aligned} & \frac{(A_0 + {}^*B)(0, H)}{A_0(0, H)} \\ & \geq D_{A_0, {}^*B, H, 1} \geq (D_{A_0, {}^*B, H, h})^{1/h} = \left(\frac{(A' + h {}^*B)(0, H)}{A'(0, H)} \right)^{1/h} \\ & \gtrsim \left(\frac{(z + h {}^*B)(0, H)}{A'(0, H)} \right)^{1/h} \gtrsim \left(\frac{(h {}^*B)(0, H-z)/(H-z+1)}{A'(z, H)/(H-z+1)} \right)^{1/h} \\ & \gtrsim \left(\frac{1}{A_0(z, H)/(H-z+1)} \right)^{1/h} \gtrsim \frac{1}{\alpha^{1/h}}, \end{aligned}$$

which implies

$$\frac{{}^*(A+B)(0, H)}{H+1} \geq \frac{(A_0 + {}^*B)(0, H)}{H+1} \gtrsim \frac{A_0(0, H)}{H+1} \cdot \frac{1}{\alpha^{1/h}} \gtrsim \alpha^{1-\frac{1}{h}}.$$

Since H is an arbitrary hyperfinite integer, we have proven Theorem 1.5 by Lemma 2.1.

3. PROOF OF THEOREM 1.6

In [6, Theorem 12 on page 39] a thin basis of order 2 was constructed by Cassels. Here we construct an upper asymptotic basis B following Cassels' steps and construct A at the same time so that A and B satisfy the desired properties.

Let $f_0 = 0, f_1 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for $n = 0, 1, 2, \dots$ be Fibonacci sequences. Let $g_1 = 0$ and $g_j = \sum_{i=2}^j f_{i-1}f_{i+2}$ for $j > 1$. For each $j = 1, 2, \dots$ let $G_j = \{g_j + rf_j : 0 \leq r < f_{j+3}\}$. For each $j < k$ let $G_{j,k} = \bigcup_{i=j-1}^{k+2} G_i$. As a part of a proof it is shown in [6, pages 40–41] that

$$(5) \quad [2g_j, 2g_k] \subseteq G_{j,k} + G_{j,k}.$$

Note that $\min G_{j,k} = g_{j-1}$ and $\max G_{j,k} = g_{k+3}$. Note also that for any two elements $a < b$ in $G_{j,k}$ we have $b - a \geq f_{j-1}$. We now construct A as the union of intervals $[a_n, 2a_n]$ and B as the union of G_{j_n, k_n} . We select a_n, j_n , and k_n alternately by induction on n .

Fix any positive integer a_0 . Let j_0 be large enough so that $f_{j_0-1} > 2a_0$. Let $k_0 > j_0$ be large enough so that $g_{k_0} > g_{j_0}^2$. Suppose we have constructed a_i, j_i, k_i for $i = 0, 1, \dots, n$. Let a_{n+1} be large enough so that $a_{n+1} > g_{k_n+3}^2$. Let j_{n+1} be large enough so that $f_{j_{n+1}-1} > a_{n+1}^2 + g_{k_n+3}$ and let k_{n+1} be large enough so that $g_{k_{n+1}} > g_{j_{n+1}}^2 + (2a_n)^2$. Let

$$A = \bigcup_{n=0}^{\infty} [a_n, 2a_n] \text{ and } B = \{0\} \cup \bigcup_{n=0}^{\infty} G_{j_n, k_n}.$$

Now we show that A and B constructed above have the desired properties.

Since $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 0$, we clearly have $\bar{d}(A) = \frac{1}{2}$.

For each n , by (5) we have

$$\frac{(2B)(0, 2g_{k_n})}{2g_{k_n} + 1} \geq \frac{2(g_{k_n} - g_{j_n}) + 1}{2g_{k_n} + 1} \geq 1 - \frac{g_{j_n}}{g_{k_n}}.$$

Hence $\bar{d}(2B) = 1$, i.e., B is an upper asymptotic basis of order 2.

For verifying $\bar{d}(A + B) = \frac{1}{2}$ it suffices to show that $\frac{(A+B)(0, x)}{x+1} \leq \frac{1}{2} + o(1)$ for every $x \in [a_n, a_{n+1} - 1]$. Let $a_n \leq x < g_{j_n-1}$. Note that $g_{k_{n-1}+3}^2 \leq a_n$ and $a_n + g_{k_{n-1}+3} < g_{j_n-1}$. Then

$$\begin{aligned} \frac{(A+B)(0, x)}{x+1} &\leq \frac{(A+B)(0, 2a_n + g_{k_{n-1}+3})}{2a_n + 1} \\ &\leq \frac{2a_{n-1} + g_{k_{n-1}+3} + a_n + g_{k_{n-1}+3}}{2a_n} \\ &\leq \frac{a_{n-1}}{a_n} + \frac{g_{k_{n-1}+3}}{a_n} + \frac{1}{2} = \frac{1}{2} + o(1). \end{aligned}$$

Let $g_{k_n-1} \leq x < a_{n+1}$. Note that

$$(A+B) \cap [2a_n + g_{k_{n-1}+3} + 1, a_{n+1} - 1] \subseteq ((A \cap [0, 2a_n]) + G_{j_n, k_n}).$$

Since $g_{j_n-1} > f_{j_n-1} > a_n^2 + g_{k_{n-1}+3}$ and any two consecutive elements in G_{j_n, k_n} have a distance $\geq f_{j_n-1}$, we have that

$$\begin{aligned} & \frac{(A+B)(0, x)}{x+1} \\ &= \frac{(A+B)(0, g_{j_n-1}-1)}{x+1} + \frac{(A+B)(g_{j_n-1}, x)}{x+1} \\ &\leq \left(\frac{1}{2} + o(1)\right) \cdot \frac{2a_n}{x+1} + \frac{A(2a_n) \cdot |G_{j_n, k_n} \cap [g_{j_n-1}, x]|}{x+1} \\ &\leq \left(\frac{1}{2} + o(1)\right) \cdot \frac{2a_n}{x+1} + \frac{A(2a_n)(x - g_{j_n-1})}{f_{j_n-1} \cdot (x+1)} + \frac{A(2a_n)}{x+1} \\ &\leq \left(\frac{1}{2} + o(1)\right) \cdot \frac{2a_n}{x+1} + \left(\frac{1}{2} + o(1)\right) \cdot \frac{(x - g_{j_n-1})}{x+1} + o(1) \\ &\leq \frac{1}{2} + o(1). \end{aligned}$$

Hence $\bar{d}(A) \leq \bar{d}(A+B) \leq \frac{1}{2}$. This shows $\bar{d}(A+B) = \frac{1}{2}$.

4. PROOF OF THEOREM 1.7

First we prove a nonstandard equivalence of upper Banach density.

Lemma 4.1. *Let $A \subseteq \mathbb{N}$. Then $BD(A) \geq \alpha$ iff there is an integer $n \in {}^*\mathbb{N}$ and a hyperfinite integer H such that*

$$\frac{{}^*A(n, n+H)}{H+1} \gtrsim \alpha.$$

Proof. Suppose $\frac{{}^*A(n, n+H)}{H+1} \gtrsim \alpha$ for some hyperfinite integer H . For each standard positive real ϵ and standard $m \in \mathbb{N}$, the statement “there is an interval $[a, b] \subseteq {}^*\mathbb{N}$ such that $b-a > m$ and $\frac{{}^*A(a, b)}{b-a+1} > \alpha - \epsilon$ ” is true in the nonstandard universe. Hence the statement without $*$ is true in the standard universe by the transfer principle. This means $BD(A) \geq \alpha$.

Suppose $BD(A) \geq \alpha$. Let $\mathcal{I} = \{[a_n, b_n] \subseteq \mathbb{N} : n \in \mathbb{N}\}$ be a sequence of intervals such that $\lim_{n \rightarrow \infty} (b_n - a_n) = \infty$ and $\lim_{n \rightarrow \infty} \frac{A(a_n, b_n)}{b_n - a_n + 1} = BD(A)$. Then $\mathcal{I}$ can be extended to a nonstandard sequence ${}^*\mathcal{I} = \{[a_n, b_n] \subseteq {}^*\mathbb{N} : n \in {}^*\mathbb{N}\}$. It is easy to see that for any hyperfinite integer N we have

$$\frac{{}^*A(a_N, b_N)}{b_N - a_N + 1} \approx BD(A) \geq \alpha.$$

We can now let $n = a_N$ and $H = b_N - a_N$ for any hyperfinite integer N . $\square$

From the last line of the proof of Lemma 4.1 we see that H can be chosen arbitrarily large or arbitrarily small in ${}^*\mathbb{N} \setminus \mathbb{N}$.

Let A and B be as in Theorem 1.7. Theorem 1.7 is trivially true if $BD(A) = 0$ or $BD(A) = 1$. Suppose $0 < \alpha = BD(A) < 1$ and $BD(hB) = 1$. We can find a hyperfinite interval $[n, n+K]$ such that $[n, n+K] \subseteq (h \cdot B)$. Choose H large enough so that $\frac{n+K}{H} \approx 0$ and $\frac{{}^*A(m, m+H)}{H+1} \approx \alpha$ for some $m \in {}^*\mathbb{N}$. It suffices to show that

$$\frac{({}^*A \cap [m, m+H]) + {}^*B)(m, m+H)}{H+1} \gtrsim \alpha^{1-\frac{1}{h}}$$

by Lemma 4.1. Let $A_0 = (*A \cap [m, m + H - n - K]) - m$. By the choice of H and A_0 we have

$$\frac{A_0(0, H)}{H+1} \approx \alpha \text{ and } \frac{(A_0 + *B)(0, H)}{H+1} \lesssim \frac{(*A + *B)(m, m + H)}{H+1}.$$

It now suffices to show that $\frac{(A_0 + *B)(0, H)}{H+1} \gtrsim \alpha^{1-\frac{1}{h}}$.

Let $A' \subseteq A_0$ be nonempty such that $D_{A_0, *B, H, h} = \frac{(A' + h *B)(0, H)}{A'(0, H)}$.

Lemma 4.2.

$$\frac{(A' + h *B)(0, H)}{A'(0, H)} = D_{A_0, *B, H, h} \gtrsim \frac{1}{\alpha}.$$

Proof. Let $N = \lfloor K/2 \rfloor$ and let $I_i = [iN, (i+1)N - 1]$ for $i = 0, 1, \dots, \lfloor H/N \rfloor - 1$, and let $I_{\lfloor H/N \rfloor} = [\lfloor H/N \rfloor \cdot N, H]$. Suppose

$$\mathcal{I} = \{I_i : i = 0, 1, \dots, \lfloor H/N \rfloor \text{ and } I_i \cap A' \neq \emptyset\}.$$

Then

$$(A' + h *B)(0, H) \geq |\mathcal{I}| \cdot N$$

because $N \leq K/2$, every element in A' is less than or equal to $H - n - K$, and $N + n + I_i \subseteq (A' + h *B) \cap [0, H]$ if $A' \cap I_i \neq \emptyset$ for every $i = 0, 1, \dots, \lfloor H/N \rfloor$. Given a positive standard real ϵ , we have

$$A'(0, H) \leq |\mathcal{I}| \cdot (\alpha + \epsilon)N$$

because $\frac{|A' \cap I_i|}{|I_i|} \lesssim \alpha$ when $|I_i|$ is hyperfinite by Lemma 4.1. Because ϵ is an arbitrary standard positive real number, we have that

$$\frac{(A' + h *B)(0, H)}{A'(0, H)} \gtrsim \frac{|\mathcal{I}| \cdot N}{|\mathcal{I}| \cdot \alpha N} = \frac{1}{\alpha}.$$

□

Combining the arguments above and Theorem 1.2 we now have

$$\begin{aligned} \frac{(A_0 + *B)(0, H)}{A_0(0, H)} &\gtrsim D_{A_0, *B, H, 1} \geq (D_{A_0, *B, H, h})^{1/h} \\ &= \left(\frac{(A' + h *B)(0, H)}{A'(0, H)} \right)^{1/h} \gtrsim \frac{1}{\alpha^{1/h}}. \end{aligned}$$

Hence

$$\frac{*(A + B)(0, H)}{H+1} \gtrsim \frac{(A_0 + *B)(0, H)}{H+1} \gtrsim \frac{A_0(0, H)}{H+1} \cdot \frac{1}{\alpha^{1/h}} \approx \alpha^{1-\frac{1}{h}},$$

which implies Theorem 1.7 by Lemma 4.1.

5. PROOF OF THEOREM 1.8

Let f_i , g_i , and $G_{j,k}$ be as defined in section 3. We construct a_n, b_n, j_n, k_n inductively on n . We will let $A = \bigcup_{n \in \mathbb{N}} [a_n, b_n]$ and $B = \{0\} \cup \bigcup_{n \in \mathbb{N}} G_{j_n, k_n}$.

Let $a_0 = 0$, $j_0 = 0$, $k_0 = 0$, and $b_n = 10$ (in fact b_0 can be any fixed positive integer). Suppose we have constructed a_n, j_n, k_n , and b_n . Let $a_{n+1} = 2b_n$ (this ensures $\underline{d}(A) = \frac{1}{2}$). Let j_{n+1} be large enough so that $g_{j_{n+1}-1} > a_{n+1}$. Let k_{n+1} be large enough so that $g_{k_{n+1}} > g_{j_{n+1}}^2$ (this ensures $\overline{d}(2B) = 1$). Let $b_{n+1} > g_{k_{n+1}+3}^2 + b_n$ (this ensures that $A[0, b_n] + G_{j_{n+1}, k_{n+1}} \subseteq [a_{n+1}, b_{n+1}] \subseteq A$).

It is now similar to the proof of Theorem 1.6 to verify that $\underline{d}(A) = \frac{1}{2} = \underline{d}(A+B)$ and B is an upper asymptotic basis of order 2.

6. PROOF OF THEOREM 1.9

Similar to the proofs of Lemma 2.1 and Lemma 4.1, the following nonstandard equivalence of upper asymptotic density can be proven by the transfer principle. The proof is left to the reader.

Lemma 6.1. *For each $A \subseteq \mathbb{N}$, $\overline{d}(A) \geq \alpha$ iff there is a hyperfinite integer H such that*

$$\frac{{}^*A(0, H)}{H+1} \gtrapprox \alpha.$$

Let $A, B \subseteq \mathbb{N}$ be as in Theorem 1.9 such that $0 < \overline{d}(A) = \alpha < 1$ and $\underline{d}(hB) = 1$. By Lemma 6.1 it suffices to find a hyperfinite integer H such that $\frac{{}^*(A+B)(0, H)}{H+1} \gg \alpha$. By Lemma 6.1 again we can find a hyperfinite integer N such that $\frac{{}^*A(0, N)}{N+1} \approx \alpha$.

Case 1. For every $n \in [0, N]$, $0 \ll \frac{n}{N} \ll 1$ implies $\frac{{}^*A(n, N)}{N-n+1} \approx \alpha$.

We need to show that $H = N + \lfloor \sqrt{N} \rfloor$ is what we want. Note that $\frac{{}^*A(0, H)}{H+1} \approx \frac{{}^*A(0, N)}{N+1} \cdot \frac{N+1}{H+1} + \frac{{}^*A(N, H)}{H+1} \approx \alpha \cdot 1 + 0 = \alpha$. Since $\frac{{}^*A(0, N)}{N+1} \approx \alpha$, we have that $\frac{A(0, n)}{n+1} \approx \alpha$ when $0 \ll \frac{n}{N} \ll 1$. Similar to the construction of C_i in the proof of Theorem 1.5 we can construct $A_0 \subseteq {}^*A \cap [0, N]$ such that $\frac{A_0(0, H)}{H+1} \approx \alpha$ and $\frac{A_0(n, H)}{H-n+1} \lesssim \alpha$ for every $n \in [0, N]$. Then by Theorem 1.2 we have

$$\begin{aligned} & \frac{(A_0 + {}^*B)(0, H)}{A_0(0, H)} \\ & \geq D_{A_0, {}^*B, H, 1} \geq (D_{A_0, {}^*B, H, h})^{1/h} = \left(\frac{(A' + h {}^*B)(0, H)}{A'(0, H)} \right)^{1/h} \\ & \gtrapprox \left(\frac{(z + h {}^*B)(0, H)}{A'(0, H)} \right)^{1/h} \gtrapprox \left(\frac{(h {}^*B)(0, H-z)/(H-z+1)}{A'(z, H)/(H-z+1)} \right)^{1/h} \\ & \gtrapprox \left(\frac{1}{A_0(z, H)/(H-z+1)} \right)^{1/h} \gtrapprox \frac{1}{\alpha^{1/h}}, \end{aligned}$$

where A' is a nonempty subset of A_0 and $z = \min A'$. Hence

$$\frac{{}^*(A+B)(0, H)}{H+1} \gtrapprox \frac{(A_0 + {}^*B)(0, H)}{H+1} \gtrapprox \frac{A_0(0, H)}{H+1} \cdot \frac{1}{\alpha^{1/h}} \approx \alpha^{1-\frac{1}{h}} > \alpha.$$

Case 2. There is $n \in [0, N]$ such that $0 \ll \frac{n}{N} \ll 1$ and $\frac{{}^*A(n, N)}{N-n+1} \gg \alpha$.

Let

$$\beta = \inf \left(\{1\} \cup \left\{ st\left(\frac{m}{N}\right) : m \in [0, N], \frac{n}{N} \leq \frac{m}{N} \ll 1, \text{ and } \frac{{}^*A(m, N)}{N-m+1} \lesssim \alpha \right\} \right),$$

where st is called the standard part map, i.e., $st(r)$ is the unique standard real α such that $r \approx \alpha$ for each $r \in {}^*\mathbb{R}$ with $|r| \leq \gamma$ for some positive standard real

number γ . Then $\beta \gg \frac{n}{N}$. Let $H' \in [n, N]$ be such that $\frac{H'}{N} \approx \beta$. Now we have that for every $x \in [n, H']$,

$$(6) \quad \frac{n}{H'} \leq \frac{x}{H'} \ll 1 \text{ implies } \frac{{}^*A(x, H')}{H' - x + 1} \gg \alpha$$

by the definition of β . Since $\frac{(h \cdot {}^*B)(0, H')}{H'+1} \approx 1$ we can find m such that $0 \ll \frac{m}{H'} \leq \frac{H'-n}{H'}$ and $m \in h \cdot {}^*B$. Let $m = b_1 + b_2 + \cdots + b_h$, where $b_i \in {}^*B$. Then there is $i \in [1, h]$ such that $b_i \geq \frac{m}{h}$. Hence $\frac{H'-n}{H'} \geq \frac{b_i}{H'} \gg 0$ and $\frac{n}{H'} \leq \frac{H'-b_i+1}{H'} \ll 1$. Let $H = H' + b_i$. Then by (6) we have

$$\begin{aligned} & \frac{{}^*(A+B)(0, H)}{H+1} \\ & \approx \frac{{}^*A(0, H') + (b_i + {}^*A)(H'+1, H'+b_i)}{H'+b_i+1} \\ & \approx \frac{{}^*A(0, H')}{H'+1} \cdot \frac{H'+1}{H'+b_i+1} + \frac{{}^*A(H'-b_i+1, H')}{b_i} \cdot \frac{b_i}{H'+b_i+1} \\ & \gg \alpha \frac{H'+1}{H'+b_i+1} + \alpha \frac{b_i}{H'+b_i+1} = \alpha. \end{aligned}$$

Combining the two cases above we have $\bar{d}(A+B) > \alpha$ by Lemma 6.1.

7. QUESTIONS

In Theorem 1.9 we proved that a lower asymptotic basis is an essential component in terms of upper asymptotic density. Can we improve that?

Question 7.1. Suppose B is a lower asymptotic basis of order h . Is it true that $\bar{d}(A+B) \geq \bar{d}(A)^{1-\frac{1}{h}}$ for every $A \subseteq \mathbb{N}$? If not, what should be the optimal lower bound b of $\bar{d}(A+B)$ as a function of $\bar{d}(A)$?

As a corollary of Theorem 1.5, we have $\underline{d}(A+P) \geq \underline{d}(A)^{2/3}$. Can we improve this? More specifically we ask the following question.

Question 7.2. Can we find a better lower bound of $\underline{d}(A+P)$, i.e., can we find a continuous function $f(x) \geq x^{2/3}$ for all $x \in [0, 1]$ and $f(x) > x^{2/3}$ for some $x \in [0, 1]$ such that $\underline{d}(A+P) \geq f(\underline{d}(A))$ for any $A \subseteq \mathbb{N}$?

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