

## TEST IDEALS IN NON- $\mathbb{Q}$ -GORENSTEIN RINGS

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ABSTRACT. Suppose that  $X = \operatorname{Spec} R$  is an  $F$ -finite normal variety in characteristic  $p > 0$ . In this paper we show that the big test ideal  $\tau_b(R) = \widetilde{\tau(R)}$  is equal to  $\sum_{\Delta} \tau(R; \Delta)$ , where the sum is over  $\Delta$  such that  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier. This affirmatively answers a question asked by various people, including Blickle, Lazarsfeld, K. Lee and K. Smith. Furthermore, we have a version of this result in the case that  $R$  is not even necessarily normal.

### 1. INTRODUCTION

Suppose that  $X = \operatorname{Spec} R$  is a normal  $\mathbb{Q}$ -Gorenstein variety in characteristic zero. For any ideal sheaf  $\mathfrak{a}$  on  $X$  and any positive real number  $t > 0$ , we can define the multiplier ideal  $\mathcal{J}(X; \mathfrak{a}^t)$  which reflects subtle local properties of both  $X$  and elements of  $\mathfrak{a}$ ; see for example [Laz04, Chapter 9]. Furthermore, if one reduces  $X$  and  $\mathfrak{a}$  to characteristic  $p \gg 0$ , then the multiplier ideal  $\mathcal{J}(X_p; \mathfrak{a}_p^t)$  agrees with the big test ideal  $\tau_b(R_p; \mathfrak{a}_p^t) = \widetilde{\tau(R_p, \mathfrak{a}_p^t)}$  (the big test ideal is also called the non-finitistic test ideal); see [Smi00], [Har01], [HY03], and [Tak04b]. However, while at least classically, multiplier ideals need the  $\mathbb{Q}$ -Gorenstein hypothesis in order to be defined, big test ideals do not.

In characteristic zero, one way to get around this difficulty is to consider an additional  $\mathbb{Q}$ -divisor  $\Delta$  on  $X$  such that  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier. In that setting, one can define the multiplier ideal  $\mathcal{J}(X; \Delta, \mathfrak{a}^t)$  of the triple  $(X, \Delta, \mathfrak{a}^t)$ . Unfortunately, there is no canonical choice of  $\Delta$  (although there can be quite “good” choices of  $\Delta$ ; see [DH09]). On the other hand, Hara, Yoshida and Takagi defined the big test ideal  $\tau_b(R; \Delta, \mathfrak{a}^t)$  of a triple  $(X, \Delta, \mathfrak{a}^t)$  and showed it agreed with the multiplier ideal  $\mathcal{J}(X; \Delta, \mathfrak{a}^t)$  after reduction to characteristic  $p \gg 0$ ; see [HY03] and [Tak04b].

Suppose now we work in a fixed characteristic  $p > 0$ . It is then very natural to ask how the test ideal  $\tau_b(R; \mathfrak{a}^t)$  is related to the test ideals  $\tau_b(R; \Delta, \mathfrak{a}^t)$ , where  $\Delta$  ranges over all divisors such that  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier. It is easy to see that one has containments  $\tau_b(R; \Delta, \mathfrak{a}^t) \subseteq \tau_b(R; \mathfrak{a}^t)$ . Furthermore, for several years it has been asked whether in fact one has

$$\tau_b(R; \mathfrak{a}^t) = \sum_{\Delta} \tau_b(R; \Delta, \mathfrak{a}^t),$$

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where the sum is over  $\Delta$  such that  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier. The main result of this paper is an affirmative answer to this question. This result has been shown in the toric case by [Bli04, Corollary to Theorem 3] (also see [HY03, Theorem 4.8]), and Blickle asked whether it always holds. Another version of this question is found in [LLS08, Remark 3.6]. A similar variant was asked by Lazarsfeld during the open problems session at the “ $F$ -singularities and  $D$ -modules” conference held at the University of Michigan in August 2007. In fact, we show somewhat more. We show that one can restrict to only considering  $\Delta$  so that  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier with index not divisible by  $p > 0$ . Therefore our main result is the following:

**Corollary 5.2.** *Suppose that  $(X = \operatorname{Spec} R, \mathfrak{a}^t)$  is a pair and that  $R$  is an  $F$ -finite normal domain. Then*

$$\tau_b(R; \mathfrak{a}^t) = \sum_{\Delta} \tau_b(R; \Delta, \mathfrak{a}^t),$$

where the sum is over effective  $\mathbb{Q}$ -divisors  $\Delta$  such that  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier with index not divisible by  $p > 0$ .

In fact, we even have a version of this result that works in the case that  $X = \operatorname{Spec} R$  is not necessarily normal; see Theorem 5.1. One can also obtain the following, which may be of independent interest.

**Corollary 5.4.** *Suppose that  $(X = \operatorname{Spec} R, \mathfrak{a}^t)$  is a pair and that  $R$  is an  $F$ -finite normal domain. Then there exist finitely many effective  $\mathbb{Q}$ -divisors  $\Delta_i$  such that  $K_X + \Delta_i$  is  $\mathbb{Q}$ -Cartier with index not divisible by  $p$  and also that*

$$\tau_b(R; \mathfrak{a}^t) = \sum_{\Delta_i} \tau_b(R; \Delta_i).$$

It should also be noted that changing the  $t$  in the previous corollary will change the  $\Delta_i$  in ways which we do not know how to control.

Recently de Fernex and Hacon have worked out a theory of multiplier ideals  $\mathcal{J}(X; \mathfrak{a}^t)$  for pairs  $(X, \mathfrak{a}^t)$ , where  $X$  is not necessarily  $\mathbb{Q}$ -Gorenstein; see [DH09]. In particular, it is known that  $\mathcal{J}(X; \mathfrak{a}^t) = \sum_{\Delta} \mathcal{J}(X; \Delta, \mathfrak{a}^t)$ , where the sum ranges over  $\Delta$  such that  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier (in fact, they show that there exists a *single*  $\Delta$  such that  $\mathcal{J}(X; \mathfrak{a}^t) = \mathcal{J}(X; \Delta, \mathfrak{a}^t)$ ). Therefore our result could be viewed as additional justification for their definition.

To prove our main result, we use the fact that the non-zero elements  $\phi \in \operatorname{Hom}_R(F_*^e R, R)$  induce  $\mathbb{Q}$ -divisors  $\Delta$  such that  $(K_X + \Delta)$  is  $\mathbb{Q}$ -Cartier with index not divisible by  $p$ ; see [Sch09] for details. Another key point in the proof is the construction of an element  $c \in R^\circ$  that is simultaneously a big sharp test element for a (specially chosen) infinite collection of triples  $(R, \Delta_\alpha, \mathfrak{a}^t)$ , where  $\alpha$  ranges over some indexing set; see Proposition 5.6. In proving our result, we also develop a theory of pairs  $(R, \mathcal{T})$ , where  $\mathcal{T}$  is a graded subalgebra of the non-commutative algebra  $\bigoplus_{e \geq 0} \operatorname{Hom}_R(F_*^e R, R)$  (which is Matlis-dual to the non-commutative algebra  $\mathcal{F}(E_R(k))$  as studied by Lyubeznik and Smith, [LS01]). In particular, our pairs  $(R, \mathcal{T})$  are strict generalizations of triples  $(R, \Delta, \mathfrak{a}^t)$ . See Remark 3.10.

We conclude the paper with an additional question related to the work of de Fernex and Hacon, a brief discussion of the differences between the big test ideal and the finitistic (classical) test ideal, and a comparison of the work done in this paper with some of the results of [SS09].

## 2. PRELIMINARIES AND NOTATION

In this section we recall the basic definitions and notation we will study in this paper. Throughout this paper, all rings will be assumed to be Noetherian. Unless otherwise specified, all rings considered will be assumed to contain a field of characteristic  $p > 0$ . Recall that there is the Frobenius action on any such ring  $R$ . This is the ring homomorphism  $F : R \rightarrow R$  that sends elements of  $R$  to their  $p^{\text{th}}$  powers.

We can also iterate the Frobenius morphism  $e$  times,  $R \xrightarrow{F} R \xrightarrow{F} \dots \xrightarrow{F} R$ , and we denote the composition by  $F^e : R \rightarrow R$ .

We can then view  $R$  as an  $R$ -module via the action of the  $e$ -iterated Frobenius. We will use the notation  $F_*^e R$  to denote this module (which we will still sometimes view as a ring in its own right). Note that this notation is justified, because if  $X = \text{Spec } R$  and we abuse notation and use  $F^e : X \rightarrow X$  to denote the map of schemes induced by  $F^e$ , then  $F_*^e \mathcal{O}_X$  is the module corresponding to  $F_*^e R$ . Likewise, for any  $R$ -module  $M$ , we use  $F_*^e M$  to denote  $M$  viewed as an  $R$ -module by the  $e$ -iterated Frobenius action. This notation also lets us identify the Frobenius map  $F^e$  with an  $R$ -linear map  $R \rightarrow F_*^e R$ , which we also denote by  $F^e$ .

**Definition 2.1.** We say that  $R$  is  $F$ -finite if  $F_*^e R$  is finite as an  $R$ -module.

*Remark 2.2.* For example, any ring that is essentially of finite type over a perfect field is  $F$ -finite; see [Fed83].

All rings in this paper will be assumed to be  $F$ -finite. In particular, since an  $F$ -finite ring is excellent, all rings in this paper will be assumed to be excellent; see [Kun76].

Throughout this paper, we will consider the module  $\text{Hom}_R(F_*^e R, R)$ . While we will not need this directly, it may be useful for the reader to note that if  $X = \text{Spec } R$  is normal, sufficiently local, and has a dualizing complex, then

$$\text{Hom}_{\mathcal{O}_X}(F_*^e \mathcal{O}_X, \mathcal{O}_X) \cong F_*^e \mathcal{O}_X((1 - p^e)K_X)$$

as  $F_*^e \mathcal{O}_X$ -modules; see [MR85] and [HW02].

Finally, we briefly remind the reader of the definition of  $\mathbb{Q}$ -divisors. If  $X$  is normal, then a  $\mathbb{Q}$ -divisor on  $X$  is a formal sum of prime divisors on  $X$  with rational coefficients. A  $\mathbb{Q}$ -divisor  $D$  on  $X$  will be called  $\mathbb{Q}$ -Cartier if and only if there exists an integer  $n > 0$  such that  $nD$  is an integral Cartier divisor (i.e.,  $nD$  is locally trivial in the divisor class group). Abusing notation, if  $X = \text{Spec } R$  is normal, and  $D$  is any integral divisor on  $X$ , then we will use  $R(D)$  to denote the global sections of  $\mathcal{O}_X(D)$ .

## 3. AN ULTIMATE GENERALIZATION OF PAIRS

In this section we introduce a (perhaps ultimate) generalization of a pair in the characteristic  $p > 0$  setting. Our notion encompasses the triples  $(R, \Delta, \mathfrak{a}^t)$ ; see [HY03] [Tak04b], and [Tak08], and as a special case, see Remark 3.10. It also seems well behaved for any  $F$ -finite ring (we, however, restrict to the reduced case).

Suppose that  $R$  is an  $F$ -finite and reduced ring. For each  $e \geq 0$ , we can consider the module  $\text{Hom}_R(F_*^e R, R)$ . We take the direct sum of these modules together to form a non-commutative graded ring:

$$\mathcal{C}(R) = \bigoplus_{e \geq 0} \text{Hom}_R(F_*^e R, R),$$

where the multiplication for homogeneous elements

$$\phi \in \operatorname{Hom}_R(F_*^d R, R) = \mathcal{C}(R)_d \text{ and } \psi \in \operatorname{Hom}_R(F_*^e R, R) = \mathcal{C}(R)_e$$

is defined as follows:

$$\phi \cdot \psi = \phi \circ (F_*^d \psi) \in \operatorname{Hom}_R(F_*^{d+e} R, R).$$

**Definition 3.1.** Given a  $\phi \in \operatorname{Hom}_R(F_*^e R, R)$ , we define the *degeneracy locus* of  $\phi$  (denoted  $\operatorname{DegenLocus}(\phi)$ ) to be the subset of  $\operatorname{Spec} R$  made up of primes  $Q \in \operatorname{Spec} R$  such that the image of  $\phi$  in

$$\operatorname{Hom}_R(F_*^e R, R)_Q \cong \operatorname{Hom}_{R_Q}(F_*^e R_Q, R_Q)$$

does *not* generate  $\operatorname{Hom}_{R_Q}(F_*^e R_Q, R_Q)$  as an  $F_*^e R_Q$ -module.

*Remark 3.2.* If  $R$  is normal and  $\phi$  corresponds to a  $\mathbb{Q}$ -divisor  $\Delta$  as in [Sch09], then  $\operatorname{DegenLocus}(\phi) = \operatorname{Supp}(\Delta)$ .

We will now remark that the degeneracy locus is always a closed subset of  $\operatorname{Spec} R$ .

**Lemma 3.3.** *Given  $\phi \in \operatorname{Hom}_R(F_*^e R, R)$ , then  $\operatorname{DegenLocus}(\phi)$  is a closed subset of  $X = \operatorname{Spec} R$ .*

*Proof.* This is obvious since  $\operatorname{Hom}_R(F_*^e R, R)$  is a finitely generated  $F_*^e R$ -module.  $\square$

**Definition 3.4.** A *pair*  $(R, \mathcal{T})$  is the combined information of a reduced  $F$ -finite ring  $R$ , and a graded subalgebra  $\mathcal{T} \subseteq \mathcal{C}(R)$  such that  $\mathcal{T}_0 = \mathcal{C}(R)_0 = \operatorname{Hom}_R(R, R) \cong R$ . A *triple*  $(R, \mathcal{T}, \mathfrak{a}^t)$  is the combined information of a pair  $(R, \mathcal{T})$ , an ideal  $\mathfrak{a} \subseteq R$  and a positive real number  $t > 0$ .

*Remark 3.5.* Using the assumption that  $\mathcal{T}_0 \cong R$ , we immediately see that for each  $i$ ,  $\mathcal{T}_i$  has the structure of an  $F_*^i R$ -module (note that  $F_*^i R$  is also a ring in its own right).

*Remark 3.6.* When we consider a pair  $(R, \mathfrak{a}^t)$  we implicitly are referring to the triple  $(R, \mathcal{C}(R), \mathfrak{a}^t)$ . Furthermore, while it is true that for any triple,  $(R, \mathcal{T}, \mathfrak{a}^t)$ , there is a way to “absorb”  $\mathfrak{a}^t$  into  $\mathcal{T}$  (and create a smaller subalgebra), see Remark 3.10, we will find it convenient to separate the terms at times.

*Remark 3.7.* One interesting and sometimes useful property of the algebra  $\mathcal{C}(R)$  is the fact that there are natural maps

$$\rho_{i,j} : \mathcal{C}(R)_i = \operatorname{Hom}_R(F_*^i R, R) \rightarrow \operatorname{Hom}_R(F_*^j R, R) = \mathcal{C}(R)_j$$

for  $i > j$ , induced by the inclusion  $F_*^j R \subseteq F_*^i R$ . Our subalgebras are *NOT* assumed to satisfy any analog of this property. However, we still have natural maps  $\mathcal{T}_i \rightarrow \operatorname{Hom}_R(R, R) \cong R$ , for any graded subalgebra  $\mathcal{T} \subseteq \mathcal{C}(R)$ . These maps are induced by restricting the natural maps  $\mathcal{C}(R)_i \rightarrow \operatorname{Hom}_R(R, R) \cong R$  to  $\mathcal{T} \subseteq \mathcal{C}(R)$ . Alternately, these are the maps defined by evaluation at 1.

Furthermore, suppose that some  $F_*^i R$ -submodule  $\mathcal{B} \subseteq \mathcal{C}(R)_i$  is generated (as an  $F_*^i R$ -module) by elements  $\phi_1, \dots, \phi_n$ . Then notice that

$$\operatorname{Image}(\rho_{i,0}|_{\mathcal{B}}) = \sum_i \operatorname{Image}(\phi_i).$$

The containment  $\subseteq$  follows since each  $\phi \in \mathcal{B}$  can be written as a sum  $\sum_j (\phi_j \cdot x_j)$  for some  $x_j \in F_*^i R$ . Then

$$\phi(1) \in \text{Image}(\phi) \subseteq \sum_j \text{Image}(\phi_j \cdot x_j) \subseteq \sum_j \text{Image}(\phi_j).$$

The containment  $\supseteq$  follows since if  $x \in \text{Image}(\phi)$ , then there exists some  $y \in F_*^i R$  so that  $\phi(y) = x$ , which implies that  $x = (\phi \cdot y)(1)$ .

**Definition 3.8.** Suppose that  $(R, \mathcal{T})$  is a pair. Then, given any multiplicative subset  $W$  of  $R$ , we can construct a new pair  $(W^{-1}R, W^{-1}\mathcal{T})$  in the obvious way. A homogeneous element  $\phi \in \mathcal{T}_e$  will be called *non-degenerate* if its image in  $(R_\eta, \mathcal{T}_\eta)$  is non-zero for every minimal prime  $\eta$  of  $\text{Spec } R$ . We say that a pair  $(R, \mathcal{T})$  is *non-degenerate* if it contains a non-degenerate element  $\phi \in \mathcal{T}_e$  for some  $e > 0$ . We say that  $(R, \mathcal{T})$  is *non-degenerately generated* if, for every  $e > 0$  such that  $\mathcal{T}_e$  is non-zero, we have that  $\mathcal{T}_e$  is generated as an  $F_*^e R$ -module by non-degenerate elements.

We say that a triple  $(R, \mathcal{T}, \mathfrak{a}^t)$  is *non-degenerate* if  $(R, \mathcal{T})$  is non-degenerate and in addition,  $\mathfrak{a} \cap R^\circ \neq \emptyset$ . We say that  $(R, \mathcal{T}, \mathfrak{a}^t)$  is *non-degenerately generated* if  $(R, \mathcal{T})$  is non-degenerately generated and  $\mathfrak{a} \cap R^\circ \neq \emptyset$ . Note this also implies that the subalgebra  $\mathcal{T}'$  defined by  $\mathcal{T}'_e = \mathcal{T}_e \cdot (F_*^e \mathfrak{a}^{\lceil t(p^e - 1) \rceil})$  is also non-degenerately generated.

*Remark 3.9.* Because  $R$  is a reduced ring,  $(R, \mathcal{T})$  is non-degenerate if and only if there exists some homogeneous  $\phi \in \mathcal{T}_{>0}$  such that  $X \setminus \text{DegenLocus}(\phi)$  is open and dense. Also note that if  $(R, \mathcal{T}, \mathfrak{a}^t)$  is reduced and  $\phi \in \mathcal{T}$  is a reduced homogeneous element, then there exists some  $c \in \mathfrak{a} \cap R^\circ$  such that  $\text{Supp}(\phi) \subseteq V(c) \subseteq \text{Spec } R$ .

*Remark 3.10.* From the point of view of  $F$ -singularities, pairs  $(R, \mathcal{T})$  (respectively triples  $(R, \mathcal{T}, \mathfrak{a}^t)$ ) are generalizations of triples  $(R, \Delta, \mathfrak{b}_\bullet)$  (respectively triples  $(R, \mathcal{T}, \mathfrak{b}_\bullet \cdot \mathfrak{a}^t)$ ) as studied by several authors (here  $\mathfrak{b}_\bullet$  is a graded system of ideals<sup>1</sup>); see for example [Tak04b], [HY03], [Har05], [BMS08], [Sch08a], [Sch09, Remark 2.8]. To construct the pair  $(R, \mathcal{T})$  associated to  $(R, \Delta, \mathfrak{b}_\bullet)$ , proceed as follows: Define  $\mathcal{T}_i$  to be

$$\text{Image} \left( \text{Hom}_R(F_*^i R((p^i - 1)\Delta), R) \rightarrow \text{Hom}_R(F_*^i R, R) \right) \cdot (F_*^i \mathfrak{b}_{p^i - 1}).$$

To see that  $\bigoplus_i \mathcal{T}_i$  is a subalgebra of  $\mathcal{C}(R)$ , simply make the following two observations.

- (1) If  $b \in \mathfrak{b}_{p^i - 1}$  and  $b' \in \mathfrak{b}_{p^j - 1}$ , then  $b^{p^j} b' \in \mathfrak{b}_{p^j(p^i - 1) + (p^j - 1)} = \mathfrak{b}_{p^{i+j} - 1}$ .
- (2) If  $\phi \in \mathcal{T}_i$  and  $\psi \in \mathcal{T}_j$ , then  $\phi \circ F_*^i \psi \in \mathcal{T}_{i+j}$  (this follows since  $p^j \lceil (p^i - 1)\Delta \rceil + \lceil (p^j - 1)\Delta \rceil \geq \lceil (p^{i+j} - 1)\Delta \rceil$ ).

This method also allows one to obtain triples  $(R, \mathcal{T}, \mathfrak{a}^t)$  from other triples  $(R, \mathcal{T}, \mathfrak{b}_\bullet \cdot \mathfrak{a}^t)$ . As we will see, changing the triple in this way does not impact the associated test ideals.

Note that if  $R$  is a normal domain and  $\mathfrak{a}_i \cap R^\circ \neq \emptyset$  for some  $i > 0$ , then the pair  $(R, \mathcal{T})$  constructed from the triple  $(R, \Delta, \mathfrak{a}_\bullet)$  is non-degenerate.

<sup>1</sup>Associated with any  $\mathfrak{a}^t$  (an ideal  $\mathfrak{a}$  formally raised to a real power  $t > 0$ ) there is a graded system defined by  $\mathfrak{b}_i = \mathfrak{a}^{\lceil ti \rceil}$ .

**Definition 3.11** ([HR74], [HW02], [Sch08b], [Sch08a]). A pair  $(R, \mathcal{T})$  is *sharply  $F$ -pure* if there exists a homogeneous  $\phi \in \mathcal{T}_e$  such that  $\phi(F_*^e R) = R$  (that is,  $\phi$  is surjective). A triple  $(R, \mathcal{T}, \mathfrak{a}^t)$  is called *sharply  $F$ -pure* if there exists a homogeneous  $\phi \in \mathcal{T}_e \cdot (F_*^e \mathfrak{a}^{\lceil t(p^e-1) \rceil})$  such that  $\phi(F_*^e R) = R$ .

**Definition 3.12** ([HH94], [HW02], [Tak04a]). We call a triple  $(R, \mathcal{T}, \mathfrak{a}^t)$  *strongly  $F$ -regular* if for every  $d \in R^\circ$ , there exists an element  $\phi \in \mathcal{T}_e \cdot (F_*^e \mathfrak{a}^{\lceil t(p^e-1) \rceil})$  such that  $1 \in \phi(F_*^e(d)) = \phi(F_*^e(dR))$  (here  $(d) = dR$  is the ideal generated by  $d$ ). For the definition for pairs  $(R, \mathcal{T})$ , set  $\mathfrak{a} = R$ .

Note that the element  $\phi$  from the above two definitions is not necessarily an element of the form  $c\phi_e$ , where  $c \in F_*^e \mathfrak{a}^{\lceil t(p^e-1) \rceil}$  and  $\phi_e \in \mathcal{T}_e$ . In general it is a sum of such elements.

*Remark 3.13.* It is clear that a strongly  $F$ -regular triple is sharply  $F$ -pure. Furthermore, if  $R$  is regular,  $\mathfrak{a} = R$  and  $\mathcal{T} = \mathcal{C}(R)$ , then  $(R, \mathcal{T}, \mathfrak{a}^t)$  is strongly  $F$ -regular by the original definition of Hochster and Huneke.

*Remark 3.14.* If  $\phi \in \mathcal{T}_e \cdot (F_*^e \mathfrak{a}^{\lceil t(p^e-1) \rceil})$  is such that  $1 \in \phi(F_*^e(dR))$ , then  $1 \in \phi^n(F_*^{ne}(dR))$  for all  $n > 0$ . This is because we have the containments:

$$\begin{aligned} 1 &\in \phi(F_*^e(dR)) \\ &\subseteq \phi(F_*^e \phi(F_*^e(dR))) \\ &= \phi^2(F_*^{2e}(dR)) \\ &\subseteq \dots \end{aligned}$$

Note that  $\phi^n \in \mathcal{T}_{ne} \cdot (F_*^{ne} \mathfrak{a}^{\lceil t(p^{ne}-1) \rceil})$ .

**Definition 3.15** ([MR85], [Sch08a]). Given a triple  $(R, \mathcal{T}, \mathfrak{a}^t)$ , we say that an ideal  $J \subseteq R$  is *uniformly  $(\mathcal{T}, \mathfrak{a}^t, F)$ -compatible* if for all  $e \geq 0$  and all homogeneous  $\phi \in \mathcal{T}_e$ , we have that  $\phi(F_*^e \mathfrak{a}^{\lceil t(p^e-1) \rceil} J) \subseteq J$ . Equivalently, for all  $e \geq 0$  and all  $\phi \in \mathcal{T}_e \cdot (F_*^e \mathfrak{a}^{\lceil t(p^e-1) \rceil})$ , we can require that  $\phi(F_*^e J) \subseteq J$ . Another equivalent definition would be to require that for all  $e \geq 0$ , all  $\phi \in F_*^e \mathcal{T}_e$  and all  $a \in \mathfrak{a}^{\lceil t(p^e-1) \rceil}$ , we have that  $\phi(F_*^e a J) \subseteq J$ . For the definition for pairs  $(R, \mathcal{T})$ , set  $\mathfrak{a} = R$ .

**Definition 3.16** ([HH90], [LS01], [Hoc07], [Sch08a]). The *big test ideal*  $\tau_b(R; \mathcal{T}, \mathfrak{a}^t)$  of a triple  $(R, \mathcal{T}, \mathfrak{a}^t)$ , if it exists, is the unique smallest ideal  $J$  that satisfies two conditions:

- (1)  $J$  is uniformly  $(\mathcal{T}, \mathfrak{a}^t, F)$ -compatible, and
- (2)  $J \cap R^\circ \neq \emptyset$ .

The *big test ideal*  $\tau_b(R; \mathcal{T})$  of a pair  $(R, \mathcal{T})$  is defined by setting  $\mathfrak{a} = R$ .

*Remark 3.17.* For a triple  $(X, \mathcal{T}, \mathfrak{a}^t)$  constructed from  $(X, \Delta, \mathfrak{a}^t)$  as in Remark 3.10, it is obvious from the definition and from [HT04, Lemma 2.1] or [Sch08a, Theorem 6.3] that the test ideal of  $(X, \Delta, \mathfrak{a}^t)$  agrees with the test ideal of  $(X, \mathcal{T}, \mathfrak{a}^t)$ , where  $\mathcal{T}$  is the algebra constructed from the pair  $(X, \Delta)$  as above.

Furthermore, given a triple  $(R, \mathcal{T}, \mathfrak{a}^t)$ , one can likewise construct a pair  $(R, \mathcal{T}')$  which has the same test ideal as the triple  $(R, \mathcal{T}, \mathfrak{a}^t)$ . Simply define  $\mathcal{T}'_i := \mathcal{T}_i \cdot (F_*^i \mathfrak{a}^{\lceil t(p^i-1) \rceil})$ .

We will show that the big test ideal exists under the assumption that  $(R, \mathcal{T})$  is non-degenerate.

**Theorem 3.18.** *Suppose that  $(R, \mathcal{T})$  is a non-degenerate pair (respectively, that  $(R, \mathcal{T}, \mathfrak{a}^t)$  is a non-degenerate triple). Then the big test ideal  $\tau_b(R; \mathcal{T})$  (respectively  $\tau_b(R; \mathcal{T}, \mathfrak{a}^t)$ ) exists.*

In order to prove this we need several preliminary results. But first we give a definition.

**Definition 3.19.** Suppose that  $(R, \mathcal{T}, \mathfrak{a}^t)$  is a non-degenerate triple. An element  $c \in \mathfrak{a} \cap R^\circ$  is called a *homogeneously defined big sharp test element* (or simply a *HDBS test element*) for the triple  $(R, \mathcal{T}, \mathfrak{a}^t)$  if for every  $d \in R^\circ$  there exists a  $\phi \in \mathcal{T}_i$ ,  $i > 0$  such that  $c \in \phi(F_*^i \mathfrak{a}^{\lceil t(p^i-1) \rceil}(d))$ . One can also make an analogous definition for pairs.

*Remark 3.20.* If  $c$  is an HDBS test element, then for any  $c' \in R^\circ$ ,  $c'c$  is also an HDBS test element.

Once we prove that HDBS test elements exist, Theorem 3.18 follows quickly.

*Proof of Theorem 3.18 modulo HDBS test elements.* Suppose that  $c \in \mathfrak{a} \cap R^\circ$  is an HDBS test element for a non-degenerate triple  $(R, \mathcal{T}, \mathfrak{a}^t)$ , and that  $J$  is uniformly  $(\mathcal{T}, \mathfrak{a}^t, F)$ -compatible with  $J \cap R^\circ \neq \emptyset$ . Then clearly  $c \in J$ . On the other hand, it is easy to see that the sum

$$I := \sum_{e \geq 0} \sum_{\phi \in \mathcal{T}_e} \phi(F_*^e \mathfrak{a}^{t(p^e-1)}(c)) \subseteq J$$

is the smallest uniformly  $F$ -compatible ideal containing  $c$ . Therefore we obtain that  $I = \tau_b(R; \mathcal{T}, \mathfrak{a}^t)$ .  $\square$

Therefore, we will prove that an HDBS test element exists. The idea is exactly the same as the usual construction of test elements; see [HH90, Section 6], [Tak04b, Lemma 2.5], [Hoc07], [Sch09, Section 6].

**Proposition 3.21.** *Suppose that  $(R, \mathcal{T}, \mathfrak{a}^t)$  is a non-degenerate triple with non-degenerate homogeneous element  $\phi \in \mathcal{T}_e$ . Then there exists an HDBS test element for  $(R, \mathcal{T}, \mathfrak{a}^t)$ .*

We first need a lemma.

**Lemma 3.22.** *Assuming the hypotheses of Proposition 3.21. Suppose that  $c \in \mathfrak{a} \cap R^\circ$  is such that  $R_c$  is strongly regular (for example, if  $R_c$  regular) and*

$$\text{DegenLocus}(\phi) \subseteq V(c) \subseteq X = \text{Spec } R.$$

*Then  $(R_c, \mathcal{T}_c, \mathfrak{a}_c^t)$  is strongly  $F$ -regular. Furthermore, for every  $d \in R^\circ$ , there exist positive integers  $m, n > 0$  such that  $c^m \in \phi^n(F_*^{ne}(d)\mathfrak{a}_c^{\lceil t(p^{ne}-1) \rceil})$ .*

*Proof.* We see that  $\bar{\phi}$  generates  $\text{Hom}_{R_c}(F_*^e R_c, R_c)$  as an  $F_*^e R_c$ -module by the hypothesis about the degeneracy locus of  $\phi$ . This then implies that  $\bar{\phi}^n$  generates  $\text{Hom}_{R_c}(F_*^{ne} R_c, R_c)$  as an  $F_*^{ne} R_c$ -module (since  $R_c$  is regular; see [Sch09, Lemma 3.8, Corollary 3.9]). In other words,  $(\mathcal{T}_c)_{ne} = (\mathcal{C}(R_c))_{ne}$ . Therefore, since  $R_c$  is strongly  $F$ -regular, there exists some  $n > 0$  and  $\psi \in \text{Hom}_{R_c}(F_*^{ne} R_c, R_c)$  such that  $1 \in \psi(F_*^{ne}(\bar{d})\mathfrak{a}_c^{\lceil t(p^{ne}-1) \rceil})$  (note  $\mathfrak{a}_c = R$ ). But then

$$1 \in \bar{\phi}^n(F_*^{ne}(\bar{d})\mathfrak{a}_c^{\lceil t(p^{ne}-1) \rceil})$$

as desired since  $\psi$  is obtained from  $\bar{\phi}^n$  by premultiplication by an element of  $R$ .

By clearing denominators, the second result is obtained.  $\square$

*Proof of Proposition 3.21.* Fix notation as in the statement of Lemma 3.22. In particular, choose some non-degenerate  $\phi$ . Choose  $c$  as in Lemma 3.22. Setting  $d = 1$  in Lemma 3.22, we see that there exist some positive integers  $m_1, n_1$  so that  $c^{m_1} \in \phi^{n_1}(F_*^{n_1 e}(1)\mathfrak{a}^{\lceil t(p^{n_1 e}-1) \rceil})$ . Furthermore, we have

$$\begin{aligned} c^{2m_1} &\in c^{m_1} \phi^{n_1} \left( F_*^{n_1 e}(1)\mathfrak{a}^{\lceil t(p^{n_1 e}-1) \rceil} \right) \\ &= \phi^{n_1} \left( F_*^{n_1 e}(c^{p^{n_1 e} m_1})\mathfrak{a}^{\lceil t(p^{n_1 e}-1) \rceil} \right) \\ &\subseteq \phi^{n_1} \left( F_*^{n_1 e}(c^{2m_1})\mathfrak{a}^{\lceil t(p^{n_1 e}-1) \rceil} \right) \\ &\subseteq \phi^{n_1} \left( F_*^{n_1 e}\mathfrak{a}^{\lceil t(p^{n_1 e}-1) \rceil} c^{m_1} \phi^{n_1} \left( F_*^{n_1 e}(1)\mathfrak{a}^{\lceil t(p^{n_1 e}-1) \rceil} \right) \right) \\ &\subseteq \phi^{2n_1} \left( F_*^{2n_1 e}(c^{2m_1})\mathfrak{a}^{\lceil t(p^{2n_1 e}-1) \rceil} \right) \\ &\subseteq \dots \subseteq \phi^{nn_1} \left( F_*^{nn_1 e}(c^{2m_1})\mathfrak{a}^{\lceil t(p^{nn_1 e}-1) \rceil} \right) \end{aligned}$$

for every integer  $n > 0$ . Therefore  $c^{2m_1} \in \phi^{nn_1}(F_*^{nn_1 e}\mathfrak{a}^{\lceil t(p^{nn_1 e}-1) \rceil})$  for every integer  $n > 0$ . We will show that  $c^{3m_1}$  is an HDBS test element.

Now fix any  $d \in R^\circ$ . Again by Lemma 3.22, we can find some positive integers  $m_d, n_d$  such that  $c^{m_d} \in \phi^{n_d}(F_*^{n_d e}(d)\mathfrak{a}^{\lceil t(p^{n_d e}-1) \rceil})$ . If  $m_d$  is less than  $3m_1$ , then we are done. Otherwise, we may assume that  $m_d = p^{nn_1 e} m_1$  for some  $n > 0$  (since making  $m_d$  larger is harmless). But then

$$\begin{aligned} c^{3m_1} &\in c^{m_1} \phi^{nn_1} \left( F_*^{nn_1 e}\mathfrak{a}^{\lceil t(p^{nn_1 e}-1) \rceil} \right) \\ &= \phi^{nn_1} \left( F_*^{nn_1 e} c^{p^{nn_1 e} m_1} \mathfrak{a}^{\lceil t(p^{nn_1 e}-1) \rceil} \right) \\ &\subseteq \phi^{nn_1} \left( F_*^{nn_1 e}\mathfrak{a}^{\lceil t(p^{nn_1 e}-1) \rceil} \phi^{n_d} (F_*^{n_d e}(d)\mathfrak{a}^{\lceil t(p^{n_d e}-1) \rceil}) \right) \\ &\subseteq \phi^{nn_1+n_d} \left( F_*^{nn_1 e+n_d e}(d)\mathfrak{a}^{\lceil t(p^{nn_1 e+n_d e}-1) \rceil} \right), \end{aligned}$$

which completes the proof.  $\square$

We now list some basic properties of test ideals.

**Proposition 3.23.** *Suppose that  $(R, \mathcal{T}, \mathfrak{a}^t)$  is a non-degenerate triple. Further suppose that  $\mathcal{T}' \subseteq \mathcal{T}$  is a graded subalgebra such that the triple  $(R, \mathcal{T}', \mathfrak{a}^t)$  is also non-degenerate. Then the following hold:*

- (i)  $\tau_b(R; \mathcal{T}, \mathfrak{a}^t) \supseteq \tau_b(R; \mathcal{T}', \mathfrak{a}^t)$ .
- (ii) For any multiplicative set  $W$ , we have

$$W^{-1}\tau_b(R; \mathcal{T}, \mathfrak{a}^t) = \tau_b(W^{-1}R; W^{-1}\mathcal{T}, W^{-1}\mathfrak{a}^t).$$

- (iii)  $(R, \mathcal{T}, \mathfrak{a}^t)$  is strongly  $F$ -regular if and only if  $\tau_b(R; \mathcal{T}, \mathfrak{a}^t) = R$ .

*Proof.* Part (i) is obvious. Part (ii) is easy once we observe that an HDBS test element remains an HDBS test element after localization. But this follows since if  $W$  is a multiplicative set, every element of  $(W^{-1}R)^\circ$  can be written as a fraction  $r/w$  for  $w \in W$  and  $r \in R^\circ$ ; see for example [Hoc07, Page 57].

The  $(\Rightarrow)$  direction of (iii) is obvious. Thus we prove the  $(\Leftarrow)$  direction. Since we can absorb the  $\mathfrak{a}^t$  term into  $\mathcal{T}$  as in Remark 3.10, we may assume that  $\mathfrak{a} = R$ .



Fix  $d \in R^\circ$ , choose  $c$  to be an HDBS test element and write

$$R = \tau_b(R; \mathcal{T}) = \sum_e \sum_{\phi \in \mathcal{T}_e} \phi(F_*^e(cR)).$$

First we do the case where  $R$  is local with maximal ideal  $\mathfrak{m}$ . In that case, since a sum of ideals contained in  $\mathfrak{m}$  is still contained in  $\mathfrak{m}$ , there must be some  $e > 0$  and  $\phi \in \mathcal{T}_e$  such that  $\phi(F_*^e(cR)) \not\subseteq \mathfrak{m}$ , which implies that  $1 \in \phi(F_*^e(cR))$ . Thus since for every  $d \in R^\circ$  we have that  $c \in \psi(F_*^i(dR))$  for some  $i > 0$  and  $\psi \in \mathcal{T}_i$ , we have  $1 \in (\phi \cdot \psi)(F_*^{e+i}(dR))$ , as desired.

Now assume that  $R$  is no longer necessarily local. Choose some maximal ideal  $\mathfrak{m}$ . By (ii) and the above work, we see that there exists some  $\phi \in \mathcal{T}_e$  such that  $\phi(F_*^e(dR))_{\mathfrak{m}} = R_{\mathfrak{m}}$ . Thus this also holds in a neighborhood  $U = \text{Spec } R_b$  of  $\mathfrak{m}$ . We can cover  $\text{Spec } R$  by a finite number of such neighborhoods  $U_j = \text{Spec } R_{b_j} = \text{Spec } R[b_j^{-1}]$  with associated  $\phi_j \in \mathcal{T}_{e_j}$  such that  $\bar{\phi}_j(F_*^{e_j} dR_{b_j}) = R_{b_j}$ . By replacing  $\phi_j$  with self-compositions, and using Remark 3.14, we may assume that all the  $e_j$  are equal to the same  $e$ . Consider  $I := \sum_{j=1}^m \phi_j(F_*^e(dR)) \subseteq R$ . Now  $I_{b_j} = R_{b_j}$  by construction. This implies that  $I := \sum_{j=1}^m \phi_j(F_*^e(dR)) = R$ .

Note that by Remark 3.7, there is a natural map  $\mathcal{T}_e \rightarrow R$  (the evaluation map). Thus we also have a natural map  $\Phi : \mathcal{T}_e \cdot (F_*^e(dR)) \rightarrow R$  by restriction. The image of  $\Phi$  certainly contains  $I$ . In particular, there exists a  $\phi \in \mathcal{T}_e$  such that  $1 \in \phi(F_*^e(dR))$ , as desired.  $\square$

#### 4. LOG- $\mathbb{Q}$ -GORENSTEIN PAIRS

In this section, we consider special pairs  $(R, \mathcal{T})$  that behave essentially like pairs  $(R, \Delta)$ , where  $R$  is normal and  $K_R + \Delta$  is  $\mathbb{Q}$ -Cartier with index not divisible by  $p > 0$ .

**Definition 4.1.** A non-degenerate pair  $(R, \mathcal{T})$  (or non-degenerate triple  $(R, \mathcal{T}, \mathfrak{a}^t)$ ) will be called *pseudo- $\mathbb{Q}$ -Gorenstein* if there exists an integer  $e_0 > 0$  such that the following holds:

- (i)  $\mathcal{T}_{e_0}$  is isomorphic (as an  $F_*^{e_0} R$ -module) to  $F_*^{e_0} R$ .
- (ii) For every  $e > 0$ , write  $e = ne_0 + r$  for some  $n, r \geq 0$  (note that we do not require  $r < e_0$ ). Then the natural map

$$\begin{aligned} \mathcal{T}_{ne_0} \otimes_{F_*^{ne_0} R} F_*^{ne_0} \mathcal{T}_r &\longrightarrow \mathcal{T}_e \\ \phi \otimes F_*^{ne_0} \psi &\longmapsto \phi \circ (F_*^{ne_0} \psi) \end{aligned}$$

induced by composition is an isomorphism of  $F_*^e R$ -modules.

*Remark 4.2.* Note that (i) and (ii) imply that  $\mathcal{T}_{me_0} \cong F_*^{me_0} R$  for all  $m \geq 0$ .

*Remark 4.3.* Given a non-degenerate  $\phi \in \mathcal{C}(R)_i$ , the subalgebra  $\mathcal{T} = R\langle \phi \rangle$  generated by  $\phi$  and  $R = \mathcal{C}(R)_0$  is pseudo- $\mathbb{Q}$ -Gorenstein.

*Remark 4.4.* It follows from [Sch09, Corollary 3.9] that if  $R$  is a normal domain and  $\Delta$  is an effective  $\mathbb{Q}$ -divisor such that  $R((p^{e_0} - 1)(K_R + \Delta))$  is free,<sup>2</sup> then the associated pair  $(R, \mathcal{T})$  is pseudo- $\mathbb{Q}$ -Gorenstein.

<sup>2</sup>This always happens locally for pairs  $(X, \Delta)$  such that  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier with index not divisible by  $p > 0$ . It follows because if  $(p^e - 1)\Delta$  is integral, then  $\text{Hom}_R(F_*^e R((p^e - 1)\Delta), R) \cong R((1 - p^e)(K_X + \Delta))$ .

*Remark 4.5.* Note that if  $(R, \mathcal{T})$  is pseudo- $\mathbb{Q}$ -Gorenstein, then  $\mathcal{T}$  is also finitely generated as an algebra (over  $\mathcal{T}_0$ ).

**Proposition 4.6** ([HH90, Lemma 8.16], [Tak08, Proposition 3.5], [Sch09, Proposition 4.7]). *Suppose that  $(R, \mathcal{T}, \mathfrak{a}^t)$  is pseudo- $\mathbb{Q}$ -Gorenstein with associated  $e_0 > 0$ . Consider the graded subalgebra*

$$\mathcal{T}' = \bigoplus_{n \geq 0} \mathcal{T}_{ne_0} \subseteq \mathcal{T}.$$

Then  $\tau_b(R; \mathcal{T}, \mathfrak{a}^t) = \tau_b(R; \mathcal{T}', \mathfrak{a}^t)$ .

*Proof.* We may select a  $c \in \mathfrak{a} \cap R^\circ$  that is an HDBS test element for  $(R, \mathcal{T}', \mathfrak{a}^t)$ . Choose  $d \in R^\circ$  such that

$$d\mathfrak{a}^{\lceil t(p^{ne_0+k}-1) \rceil} \subseteq (\mathfrak{a}^{\lceil t(p^{ne_0}-1) \rceil})^{[p^k]}$$

for all  $n \geq 0$  and all  $k < e_0$  (we can do this due to [Sch09, Lemma 4.6, Proposition 4.7]). Then  $dc^{p^{e_0}}$  is also an HDBS test element for both  $(R, \mathcal{T}', \mathfrak{a}^t)$  and  $(R, \mathcal{T}, \mathfrak{a}^t)$ . Fix  $\psi \in \mathcal{T}_{e_0}$  to be a generator of  $\mathcal{T}_{e_0}$  as an  $F_*^{e_0}R$ -module. Then  $\psi^n$  is a generator of  $\mathcal{T}_{ne_0}$  as an  $F_*^{ne_0}R$ -module. Furthermore, if  $e = ne_0 + k$ , then every element  $\phi \in \mathcal{T}_e$  can be written as  $\psi^n \circ F_*^{ne_0}\phi'$  for some  $\phi' \in \mathcal{T}_k$ . Therefore,

$$\begin{aligned} \tau_b(R; \mathcal{T}, \mathfrak{a}^t) &= \sum_{e \geq 0} \sum_{\phi \in \mathcal{T}_e} \phi(F_*^e(dc^{p^{e_0}})\mathfrak{a}^{\lceil t(p^e-1) \rceil}) \\ &= \sum_{n \geq 0} \sum_{k=0}^{e_0-1} \sum_{\phi \in \mathcal{T}_{ne_0+k}} \phi(F_*^{ne_0+k}(dc^{p^{e_0}})\mathfrak{a}^{\lceil t(p^{ne_0+k}-1) \rceil}) \\ &= \sum_{n \geq 0} \sum_{k=0}^{e_0-1} \sum_{\phi' \in \mathcal{T}_k} \psi^n(F_*^{ne_0}\phi'(F_*^k(dc^{p^{e_0}})\mathfrak{a}^{\lceil t(p^{ne_0+k}-1) \rceil})) \\ &\subseteq \sum_{n \geq 0} \sum_{k=0}^{e_0-1} \sum_{\phi' \in \mathcal{T}_k} \psi^n(F_*^{ne_0}\phi'(F_*^k(c^{p^k})(\mathfrak{a}^{\lceil t(p^{ne_0}-1) \rceil})^{[p^k]})) \\ &= \sum_{n \geq 0} \sum_{k=0}^{e_0-1} \sum_{\phi' \in \mathcal{T}_k} \psi^n(F_*^{ne_0}(c)(\mathfrak{a}^{\lceil t(p^{ne_0}-1) \rceil})\phi'(F_*^k(R))) \\ &\subseteq \sum_{n \geq 0} \sum_{k=0}^{e_0-1} \sum_{\phi' \in \mathcal{T}_k} \psi^n(F_*^{ne_0}(c)(\mathfrak{a}^{\lceil t(p^{ne_0}-1) \rceil})R) \\ &= \sum_{n \geq 0} \psi^n(F_*^{ne_0}(c)(\mathfrak{a}^{\lceil t(p^{ne_0}-1) \rceil})) = \tau_b(R; \mathcal{T}', \mathfrak{a}^t). \end{aligned}$$

Note that the final equality occurs because  $\psi^n$  generates  $\mathcal{T}_{ne}$  as an  $F_*^{ne}R$ -module.  $\square$

*Remark 4.7.* This result is closely related to the fact that if  $cz^{p^e} \in I^{[p^e]}$  for all  $e = ne_0$ , then  $z \in I^*$ ; compare with [Sch09, Proposition 4.7], [Tak08, Proposition 3.5] and [HH90, Lemma 8.16]. In fact, those previous results all had related proofs. However they all relied on (Matlis-dual) analogs of the fact that there are natural maps  $\text{Hom}_R(F_*^{e+k}R, R) \rightarrow \text{Hom}_R(F_*^eR, R)$ . We took a slightly different approach above in the proof of Proposition 4.6.

We also link the test ideals of pseudo- $\mathbb{Q}$ -Gorenstein triples  $(R, \mathcal{T}, \mathfrak{a}^t)$  with the test ideals of triples  $(R, \Delta, \mathfrak{a}^t)$ .

**Corollary 4.8.** *Suppose that  $R$  is a normal domain and that  $(R, \mathcal{T}, \mathfrak{a}^t)$  is a pseudo- $\mathbb{Q}$ -Gorenstein triple with associated  $e_0$ . Then there exists an effective  $\mathbb{Q}$ -divisor  $\Delta$  such that  $(p^{e_0} - 1)(K_R + \Delta)$  is Cartier and such that*

$$\tau_b(R; \Delta, \mathfrak{a}^t) = \tau_b(R; \mathcal{T}, \mathfrak{a}^t).$$

*Proof.* Choose  $\phi \in \mathcal{T}_{e_0}$  which generates  $\mathcal{T}_{e_0}$  as an  $F_*^{e_0}R$ -module. Then by [Sch09, Theorem 3.10],  $\phi$  corresponds to an effective  $\mathbb{Q}$ -divisor  $\Delta$ . The result then follows immediately from Definition 3.16, Proposition 4.6 and [Sch09, Proposition 4.7].  $\square$

## 5. PROOF OF THE MAIN THEOREM

In this section we prove our main result, which is stated below.

**Theorem 5.1.** *Suppose that  $(R, \mathcal{T}, \mathfrak{a}^t)$  is a non-degenerately generated triple. Then there exist graded subalgebras  $\mathcal{T}^1, \dots, \mathcal{T}^n \subseteq \mathcal{T}$  such that each  $(R, \mathcal{T}^i, \mathfrak{a}^t)$  is pseudo- $\mathbb{Q}$ -Gorenstein and such that*

$$\tau_b(R; \mathcal{T}, \mathfrak{a}^t) = \sum_{i=1}^n \tau_b(R; \mathcal{T}^i, \mathfrak{a}^t).$$

We obtain the following corollaries.

**Corollary 5.2.** *Suppose that  $(R, \mathfrak{a}^t)$  is a pair and that  $R$  is an  $F$ -finite normal domain. Then*

$$\tau_b(R; \mathfrak{a}^t) = \sum_{\Delta} \tau_b(R; \Delta, \mathfrak{a}^t),$$

where the sum is over effective  $\mathbb{Q}$ -divisors  $\Delta$  such that  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier with index not divisible by  $p > 0$  and  $X = \operatorname{Spec} R$ .

*Proof.* The containment  $\supseteq$  follows from the fact that  $\tau(R; \Delta, \mathfrak{a}^t) \subseteq \tau_b(R; \mathfrak{a}^t)$  for every effective  $\Delta$ . The containment  $\subseteq$  follows from Corollary 4.8.  $\square$

**Remark 5.3.** The actual correspondence from [Sch09] between  $\phi$ 's and  $\Delta$ 's identifies  $\phi$  (modulo some equivalence relation) with  $\Delta$  such that  $(p^e - 1)(K_X + \Delta)$  is Cartier and that  $\mathcal{O}_X((pe - 1)(K_X + \Delta)) \cong \mathcal{O}_X$ . Thus we may further restrict our sum in the previous corollary to be over such a  $\Delta$ .

**Corollary 5.4.** *Suppose that  $(R, \mathfrak{a}^t)$  is a pair and that  $R$  is an  $F$ -finite normal domain,  $X = \operatorname{Spec} R$ . Then there exist finitely many effective  $\mathbb{Q}$ -divisors  $\Delta_i$  such that  $K_X + \Delta_i$  is  $\mathbb{Q}$ -Cartier with index not divisible by  $p$  and that*

$$\tau_b(R; \mathfrak{a}^t) = \sum_{\Delta_i} \tau_b(R; \Delta_i).$$

*Proof.* Out of the pair  $(R, \mathfrak{a}^t)$ , one may create a graded subalgebra  $\mathcal{T} \subseteq \mathcal{C}(R)$  which has the same test ideal as the pair  $(R, \mathfrak{a}^t)$ ; see Remark 3.10. Then one can apply Theorem 5.1 and Corollary 4.8.  $\square$

The proof of Theorem 5.1 is rather technical. So we first outline the strategy we will use. There are three steps.

- (i) Find non-degenerate homogeneous  $\phi_i \in \mathcal{T}$ ,  $i = 1, \dots, m$  such that if  $\mathcal{T}'$  is the subalgebra of  $\mathcal{T}$  generated by  $\mathcal{T}_0$  and the  $\phi_i$ , then  $\tau_b(R; \mathcal{T}, \mathfrak{a}^t) = \tau_b(R; \mathcal{T}', \mathfrak{a}^t)$ . This can be done by the Noetherian property of  $R$ .
- (ii) Find an element  $d \in R^\circ$  that is simultaneously an HDBS test element for all triples  $(R, \mathcal{S}, \mathfrak{a}^t)$ , where  $\mathcal{S} = R\langle\psi\rangle$  ranges over all pseudo- $\mathbb{Q}$ -Gorenstein subalgebras of  $\mathcal{T}$  generated by  $\mathcal{T}_0$  and a single product  $\psi$  of the  $\phi_i$ .
- (iii) Observe that  $\tau_b(R; \mathcal{T}, \mathfrak{a}^t)$  can be generated elements which are contained in the test ideals  $\tau_b(R; \mathcal{S}, \mathfrak{a}^t)$  for various  $\mathcal{S}$  as in step (ii).

We first prove step (i).

**Proposition 5.5.** *Suppose that  $(R, \mathcal{T}, \mathfrak{a}^t)$  is a non-degenerately generated triple. Then there exists a subalgebra  $\mathcal{T}'$  generated by  $R \cong \mathcal{T}_0$  and finitely many additional homogeneous  $\phi_i$  such that  $\tau_b(R; \mathcal{T}, \mathfrak{a}^t) = \tau_b(R; \mathcal{T}', \mathfrak{a}^t)$ . Furthermore, we can assume that the  $\phi_i$ 's are non-zero at every minimal prime of  $R$  and thus that  $(R, \mathcal{T}', \mathfrak{a}^t)$  is also non-degenerately generated.*

*Proof.* Begin by choosing any homogeneous  $\phi_1$  that is non-zero at every minimal prime of  $R$  and set  $\mathcal{T}^1$  to be the subalgebra of  $\mathcal{T}$  generated by  $R$  and  $\phi_1$ . Note that  $\tau_b(R; \mathcal{T}^1, \mathfrak{a}^t)$  is the smallest ideal  $J$  such that  $J \cap R^\circ \neq \emptyset$  and such that  $\phi(F_*^e \mathfrak{a}^{\lceil t(p^e-1) \rceil}) \subseteq J$  for all  $e \geq 0$  and  $\phi \in \mathcal{T}_e^1$ . Therefore, since  $\mathcal{T}^1 \subseteq \mathcal{T}$ ,  $\tau_b(R; \mathcal{T}^1, \mathfrak{a}^t) \subseteq \tau_b(R; \mathcal{T}, \mathfrak{a}^t)$ . If we have equality, we are done; set  $\mathcal{T}' = \mathcal{T}^1$ . Otherwise choose some homogeneous  $\phi_2 \in \mathcal{T}_{e_2}$  such that  $\phi_2(F_*^{e_2} \tau_b(R; \mathcal{T}^1, \mathfrak{a}^t)) \not\subseteq \tau_b(R; \mathcal{T}^1, \mathfrak{a}^t)$ . We also assume that  $\phi_2$  is non-zero at every minimal prime at  $R$  (note that this is possible since  $(R, \mathcal{T}, \mathfrak{a}^t)$  is non-degenerately generated). Set  $\mathcal{T}_2$  to be the subalgebra of  $\mathcal{T}$  generated by  $\mathcal{T}_1$  and  $\phi_2$ .

Note that by hypothesis,  $\tau_b(R; \mathcal{T}^1, \mathfrak{a}^t) \subsetneq \tau_b(R; \mathcal{T}^2, \mathfrak{a}^t)$ . Again, if  $\tau_b(R; \mathcal{T}^2, \mathfrak{a}^t)$  is equal to  $\tau_b(R; \mathcal{T}, \mathfrak{a}^t)$ , we are done; set  $\mathcal{T}' = \mathcal{T}^2$ . Otherwise, we can continue the process. However, this must stop eventually since  $R$  is Noetherian.  $\square$

We now need to find an element that is simultaneously an HDBS test element for all triples  $(R, R\langle\psi\rangle, \mathfrak{a}^t)$  as in (ii).

**Proposition 5.6.** *Suppose that  $(R, \mathcal{T}', \mathfrak{a}^t)$  is a triple and that  $\mathcal{T}'$  is generated by finitely many homogeneous  $\phi_i$  each of which is non-zero at all of the minimal primes of  $R$ . We will use  $\phi_{i_1, \dots, i_n}$  to denote the product  $\phi_{i_1} \cdots \phi_{i_n}$ . Set  $\mathcal{S}^{i_1, \dots, i_n} = R\langle\phi_{i_1, \dots, i_n}\rangle$  to be the subalgebra of  $\mathcal{T}'$  generated by  $R$  and  $\phi_{i_1, \dots, i_n}$  and suppose that  $c_i \in \mathfrak{a} \cap R^\circ$  is an HDBS test element for  $(R, \mathcal{S}^i, \mathfrak{a}^t)$ . Then there exists a single  $c \in \mathfrak{a} \cap R^\circ$  that is an HDBS test element for every triple  $(R, \mathcal{S}^{i_1, \dots, i_n}, \mathfrak{a}^t)$ .*

Before proving this we need a lemma.

**Lemma 5.7.** *Suppose that  $(R, R\langle\phi\rangle, \mathfrak{a}^t)$  is a triple, where  $\phi$  is a homogeneous element of  $\text{Hom}_R(F_*^e R, R)$ . Further suppose that  $c \in \mathfrak{a} \cap R^\circ$  is an element such that  $R_c$  is regular,  $\text{DegenLocus}(\phi) \subseteq V(c)$  and that  $c \in \phi(F_*^e(c) \mathfrak{a}^{\lceil t(p^e-1) \rceil})$ . Then  $c^2$  is an HDBS test element for  $(R, R\langle\phi\rangle, \mathfrak{a}^t)$ .*

*Proof.* Choose  $d \in R^\circ$ . Since  $R_c$  is regular and  $\text{DegenLocus}(\phi) \subseteq V(c)$  we have that  $R_c$  is strongly  $F$ -regular and  $\bar{\phi}$  generates  $\text{Hom}_{R_c}(F_*^e R_c, R_c)$  as an  $F_*^e R_c$ -module; see Lemma 3.22. Therefore there exists an  $n$  so that  $1 \in \bar{\phi}^n(F_*^{ne} d R_c)$ . By clearing

denominators we see that  $c^m \in \phi^n(F_*^{ne}(d)\mathfrak{a}^{\lceil t(p^{ne}-1)\rceil})$  (note  $c$  was in  $\mathfrak{a}$ ). Roughly speaking, our goal is now to reduce  $m$  to a number which is independent of  $d$ .

On the other hand, by induction, we claim for all  $l \geq 1$  that

$$c \in \phi^l(F_*^{le}(c)\mathfrak{a}^{\lceil t(p^{le}-1)\rceil}).$$

This is because

$$\begin{aligned} & \phi^l \left( F_*^{le}(c)\mathfrak{a}^{\lceil t(p^{le}-1)\rceil} \right) \\ & \subseteq \phi^l \left( F_*^{le}\mathfrak{a}^{\lceil t(p^{le}-1)\rceil} \phi(F_*^e(c)\mathfrak{a}^{\lceil t(p^e-1)\rceil}) \right) \\ & = \phi^l \left( F_*^{le}\phi(F_*^e(c)(\mathfrak{a}^{\lceil t(p^{le}-1)\rceil})^{[p^e]}\mathfrak{a}^{\lceil t(p^e-1)\rceil}) \right) \\ & \subseteq \phi^{l+1} \left( F_*^{(l+1)e}(c)(\mathfrak{a}^{\lceil t(p^{(l+1)e}-1)\rceil}) \right). \end{aligned}$$

We now show that there exists an  $l$  such that  $c^2 \in \phi^l(F_*^{le}(c^m)\mathfrak{a}^{\lceil t(p^{le}-1)\rceil})$ . Choose  $l$  such that  $p^{le} + 1 \geq m$ . Then

$$\begin{aligned} c^2 & \in c \cdot \phi^l(F_*^{le}(c)\mathfrak{a}^{\lceil t(p^{le}-1)\rceil}) \subseteq \phi^l(F_*^{le}(c^{p^{le}+1})\mathfrak{a}^{\lceil t(p^{le}-1)\rceil}) \\ & \subseteq \phi^l(F_*^{le}(c^m)\mathfrak{a}^{\lceil t(p^{le}-1)\rceil}), \end{aligned}$$

as desired. But then we have that

$$\begin{aligned} c^2 & \in \phi^l(F_*^{le}c^m\mathfrak{a}^{\lceil t(p^{le}-1)\rceil}) \\ & \subseteq \phi^l \left( F_*^{le}\mathfrak{a}^{\lceil t(p^{le}-1)\rceil} \phi^n(F_*^{ne}(d)\mathfrak{a}^{\lceil t(p^{ne}-1)\rceil}) \right) \\ & = \phi^{l+n} \left( F_*^{(l+n)e}(d)(\mathfrak{a}^{\lceil t(p^{le}-1)\rceil})^{[p^{ne}]}\mathfrak{a}^{\lceil t(p^{ne}-1)\rceil}) \right) \\ & \subseteq \phi^{l+n} \left( F_*^{(l+n)e}(d)\mathfrak{a}^{\lceil t(p^{(l+n)e}-1)\rceil} \right), \end{aligned}$$

which completes the proof.  $\square$

*Proof of Proposition 5.6.* For each  $\phi_i$ , we claim we can choose  $c_i \in \mathfrak{a} \cap R^\circ$  such that the following conditions hold:

- (a)  $R_{c_i}$  is regular.
- (b)  $\text{DegenLocus}(\phi_i) \subseteq V(c_i)$ .
- (c)  $c_i$  is an HDBS test element for  $(R, \mathcal{S}^i, \mathfrak{a}^t)$ .
- (d)  $c_i \in \phi_i(F_*^{e_i}(c_i)\mathfrak{a}^{\lceil t(p^{e_i}-1)\rceil})$ .

Notice that finding a  $c_i$  that satisfies (a), (b) and (c) is easy. Therefore, fix a  $c'_i$  satisfying the first three conditions. To find a  $c_i$  also satisfying condition (d), first we note that there exists some  $m > 0$  such that

$$(1) \quad c_i'^m \in \phi_i(F_*^{e_i}(c'_i)\mathfrak{a}^{\lceil t(p^{e_i}-1)\rceil}).$$

To see this, localize at  $c'_i$ . Then  $R_{c'_i}$  is  $F$ -pure so that  $1 \in \psi(F_*^{e_i}R_{c'_i})$  for some  $\psi \in \mathcal{C}(R_{c'_i})_{e_i}$ . But any such  $\psi$  can be expressed as  $\bar{\phi}_i$  precomposed with some element of  $R_{c'_i}$ . Therefore  $1 \in \bar{\phi}_i(F_*^{e_i}R_{c'_i})$ . Clearing denominators proves that

equation (1) holds for some  $m$ . Note that this is a slight improvement over Lemma 3.22 since we need not raise  $\phi_i$  to a power. We then have that

$$\begin{aligned} c_i'^{2m} &\in c_i'^m \phi_i(F_*^{e_i}(c_i') \mathfrak{a}^{\lceil t(p^{e_i}-1) \rceil}) \\ &= \phi_i(F_*^{e_i}(c_i'^{1+mp^{e_i}}) \mathfrak{a}^{\lceil t(p^{e_i}-1) \rceil}) \\ &\subseteq \phi_i(F_*^{e_i}(c_i'^{2m}) \mathfrak{a}^{\lceil t(p^{e_i}-1) \rceil}). \end{aligned}$$

Set  $c_i$  to be  $c_i'^{2m}$ .

Also notice that if  $c_i$  satisfies condition (d), then for every  $d \in R$ ,  $dc_i$  also satisfies condition (d) since

$$\begin{aligned} dc_i &\in d\phi_i(F_*^{e_i}(c_i) \mathfrak{a}^{\lceil t(p^{e_i}-1) \rceil}) \\ &= \phi_i(F_*^{e_i}(d^{p^e} c_i) \mathfrak{a}^{\lceil t(p^{e_i}-1) \rceil}) \\ &\subseteq \phi_i(F_*^{e_i}(dc_i) \mathfrak{a}^{\lceil t(p^{e_i}-1) \rceil}). \end{aligned}$$

Set  $c' = \prod_i c_i$ . Consider  $\phi_{i_1, \dots, i_n} = \phi_{i_1} \cdots \phi_{i_n}$ . Set  $U = \text{Spec } R_{c'} \subseteq \text{Spec } R$ . Note that  $U$  is regular. Also note that at each (possibly non-closed) point  $Q \in U$ ,  $\overline{\phi_i}$  generates  $\text{Hom}_{R_Q}(F_*^{e_i} R_Q, R_Q)$  as an  $F_*^{e_i} R_Q$ -module. Therefore  $\overline{\phi_{i_1, \dots, i_n}}$  also generates  $\text{Hom}_{R_Q}(F_*^{e_{i_1} + \dots + e_{i_n}} R_Q, R_Q)$  as an  $F_*^{e_{i_1} + \dots + e_{i_n}} R_Q$ -module (this follows from [Sch09, Lemma 3.8]). In particular,  $\text{DegenLocus}(\phi_{i_1, \dots, i_n}) \subseteq V(c')$ .

Finally, note that we have

$$\begin{aligned} c' &\in \phi_{i_1}(F_*^{e_{i_1}} \mathfrak{a}^{\lceil t(p^{e_{i_1}}-1) \rceil}(c')) \\ &\subseteq \phi_{i_1}(F_*^{e_{i_1}} \mathfrak{a}^{\lceil t(p^{e_{i_1}}-1) \rceil} \phi_{e_{i_2}}(F_*^{e_{i_2}} \mathfrak{a}^{\lceil t(p^{e_{i_2}}-1) \rceil}(c')))) \\ &\subseteq \phi_{i_1, i_2}(F_*^{e_{i_1} + e_{i_2}} \mathfrak{a}^{\lceil t(p^{e_{i_1} + e_{i_2}}-1) \rceil}(c')) \\ &\subseteq \dots \\ &\subseteq \phi_{i_1, \dots, i_n}(F_*^{e_{i_1} + \dots + e_{i_n}} \mathfrak{a}^{\lceil t(p^{e_{i_1} + \dots + e_{i_n}}-1) \rceil}(c')). \end{aligned}$$

Now apply Lemma 5.7 and set  $c = c'^2$ . □

We are now in a position to prove Theorem 5.1 (that is, prove (iii) in the outline).

*Proof of Theorem 5.1.* The containment  $\supseteq$  is trivial, so we will prove the other containment. Use Proposition 5.5 to find finitely many non-degenerate homogeneous  $\phi_i \in \mathcal{T}$  generating a subalgebra  $\mathcal{T}'$  such that  $\tau_b(R; \mathcal{T}, \mathfrak{a}^t) = \tau_b(R; \mathcal{T}', \mathfrak{a}^t)$ . By Proposition 5.6, we can choose a  $c \in \mathfrak{a} \cap R^\circ$  that is an HDBS test element for  $(R, \mathcal{T}^{i_1, \dots, i_n}, \mathfrak{a}^t)$  for each subalgebra  $\mathcal{T}^{i_1, \dots, i_n} = R\langle \phi_{i_1, \dots, i_n} \rangle$  of  $\mathcal{T}$ . Again we use  $\phi_{i_1, \dots, i_n}$  to denote the product  $\phi_{i_1} \cdots \phi_{i_n}$ .

Now,

$$\begin{aligned} \tau_b(R; \mathcal{T}', \mathfrak{a}^t) &= \sum_{e \geq 0} \sum_{\phi \in \mathcal{T}'_e} \phi(F_*^e(c) \mathfrak{a}^{\lceil t(p^e-1) \rceil}) \\ &= \sum_{e \geq 0} \left( \sum_{\phi_{i_1, \dots, i_n} \in \mathcal{T}'_e} \phi_{i_1, \dots, i_n}(F_*^e(c) \mathfrak{a}^{\lceil t(p^e-1) \rceil}) \right). \end{aligned}$$

Therefore we can choose generators for  $\tau_b(R; \mathcal{T}, \mathfrak{a}^t) = \tau_b(R; \mathcal{T}', \mathfrak{a}^t)$  that are elements of  $\phi_{i_1, \dots, i_n}(F_*^e(c) \mathfrak{a}^{\lceil t(p^e-1) \rceil})$  for various  $\phi_{i_1, \dots, i_n}$ . But any such generator is

also contained clearly in

$$\tau_b(R; \mathcal{S}^{i_1, \dots, i_n}, \mathfrak{a}^t) = \tau_b(R; R\langle \phi_{i_1, \dots, i_n} \rangle, \mathfrak{a}^t)$$

since  $c$  is also an HDBS test element for  $(R; \mathcal{S}^{i_1, \dots, i_n}, \mathfrak{a}^t)$ . This completes the proof.  $\square$

## 6. FURTHER REMARKS AND QUESTIONS

*Remark 6.1* (Big versus finitistic test ideals). The reader may have noticed that, in this paper, we dealt exclusively with the *big* (aka non-finitistic) test ideal  $\tau_b(R; \mathfrak{a}^t)$  and not the usual (aka finitistic) test ideal  $\tau(R; \mathfrak{a}^t)$ . Roughly speaking, the big test ideal is made up of elements which “test” tight closure for all submodules of all modules, whereas the usual test ideal is made up of elements which “test” tight closure for all submodules of *finitely generated* modules; see [Hoc07] and [Sch08a]. The big test ideal and the usual test ideal are known to agree in many situations, see for example [LS99], [LS01], [AM99], [HY03], and are conjectured to coincide in general (in particular, one always has the containment  $\tau_b(R; \mathfrak{a}^t) \subseteq \tau(R; \mathfrak{a}^t)$ ). Furthermore, if  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier, then  $\tau_b(R; \Delta, \mathfrak{a}^t) = \tau(R; \Delta, \mathfrak{a}^t)$ ; see [Tak04b, Theorem 2.8(2)], [HY03, Definition–Theorem 6.5].

The big test ideal is known to be much better behaved than the finitistic test ideal in general (for example, its formation is known to commute with localization and completion). Finally, it is believed by experts that if it is discovered that  $\tau_b(R; \mathfrak{a}^t) \subsetneq \tau(R; \mathfrak{a}^t)$  for some example, then the big test ideal is the “correct” notion in general.

Of course, it follows from this paper that if the finitistic test ideal  $\tau(R; \mathfrak{a}^t)$  is equal to the same sum

$$\sum_{K_X + \Delta \text{ } \mathbb{Q}\text{-Cartier}} \tau_b(R; \Delta, \mathfrak{a}^t),$$

then  $\tau_b(R; \mathfrak{a}^t) = \tau(R; \mathfrak{a}^t)$ .

*Remark 6.2* (Relation with de Fernex and Hacon’s multiplier ideal). Suppose that  $X = \operatorname{Spec} R$  is a normal variety in characteristic zero but  $X$  is not necessarily  $\mathbb{Q}$ -Gorenstein. It has been asked whether  $\mathcal{J}(X, \mathfrak{a}^t)$  (the multiplier ideal of de Fernex and Hacon) agrees with the (big) test ideal  $\tau_b(R_p; \mathfrak{a}_p^t)$  after reduction to characteristic  $p \gg 0$ . Initially, one might hope that the results of this paper might imply this result. However, the author believes that this paper only provides (strong) evidence that this is indeed the case. The problem is that the  $\Delta_i$  constructed in Corollary 5.2 seem to rely heavily on the particular characteristic we are working in, and so are probably not reduced from characteristic zero as well (at least not in a way in which their properties can be controlled).

Consider the following question.

**Question 6.3.** Suppose that  $(X, \mathfrak{a}^t)$  is a pair in characteristic  $p > 0$ . Then does there exist a single effective  $\mathbb{Q}$ -divisor  $\Delta$  such that  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier with index not divisible by  $p > 0$  and so that  $\tau_b(X; \Delta, \mathfrak{a}^t) = \tau(X; \mathfrak{a}^t)$ ?

The work of de Fernex and Hacon suggests this may be true. This question was also asked by the author and Karen Smith in [SS09] and was affirmatively answered in the case that  $X$  is strongly  $F$ -regular and  $\mathfrak{a} = R$  (although the version where  $\mathfrak{a} \neq R$  can also be easily obtained from the methods of this paper).

Finally, there are certain similarities between the methods employed in this paper and the methods of [SS09, Theorem 4.3(i)]. The goal in [SS09, Theorem 4.3(i)] was also to find a map  $\phi$  such that  $(R, R\langle\phi\rangle)$  was strongly  $F$ -regular (although we phrased things in terms of finding the divisor associated to  $\phi$ , instead of  $R\langle\phi\rangle$ , as in [Sch09]). One consideration that makes the setting of [SS09] easier to work in is that one must only find a single  $\phi$  such that  $1 \in \tau_b(R; R\langle\phi\rangle)$ . This  $\phi$  is obtained explicitly by composing several potential such  $\phi$  (but again, viewing them as divisors instead of maps). The proof is thus somewhat more geometric. Furthermore, much of the work we have to do in terms of keeping track of various test elements is unnecessary in the setting of [SS09].

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