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HIGHER BIVARIANT CHOW GROUPS AND MOTIVIC FILTRATIONS

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Abstract. The purpose of this paper is twofold: first, we extend Saito's filtration on Chow groups, which is a candidate for the conjectural Bloch Beilinson filtration on the Chow groups of a smooth projective variety, from Chow groups to the bivariant Chow groups. In order to do this, we construct cycle class maps from the bivariant Chow groups to bivariant cohomology groups. Secondly, we use our methods to define a bivariant version of Bloch's higher Chow groups.

1. Introduction

Bivariant Chow groups appear for the first time in the work of Fulton and MacPherson [7]. These groups unify the concept of Chow homology groups $CH_*(X)$ and Chow cohomology groups $CH^*(X)$ of a scheme X. For instance, if $i : X \longrightarrow Y$ is a regular imbedding of codimension d, we have a family of homomorphisms

$$i^! : CH^k(Y') \longrightarrow CH^{k-r'+d}(X') \tag{1.1}$$

for each $Y' \longrightarrow Y$, $X' = X \times_Y Y'$ and each $k \in \mathbb{Z}$, where $r' = \dim(Y') - \dim(X')$. The morphisms $i^!$ are referred to as refined Gysin morphisms. Such a family of morphisms is said to induce a class in the bivariant Chow group $CH^*(i : X \longrightarrow Y)$. In general, bivariant Chow groups are defined as a natural extension of this concept to more general morphisms, in particular to locally complete intersection (l.c.i.) morphisms $f : X \longrightarrow Y$ (see Section 2.). For each l.c.i. morphism $f : X \longrightarrow Y$, the bivariant Chow groups are denoted by $CH^p(f : X \longrightarrow Y)$ for each $p \in \mathbb{Z}$.

For a smooth scheme X over a field K of characteristic zero, it is well known that the Chow homology group $CH_p(X) \cong CH^{-p}(X \longrightarrow Spec(K))$ and the Chow cohomology group $CH^p(X) \cong CH^p(1 : X \longrightarrow X)$ can be recovered from the bivariant Chow groups, K being the ground field. It can be shown (see [7] or [6, §17]) that several standard constructions on Chow groups, such as pullback and refined Gysin morphisms, or the Chern classes of a vector bundle may be understood as classes in a bivariant Chow group. By replacing the Chow groups with cohomology groups, we can similarly define (see Definition 2.2) bivariant cohomology groups $H^p(f : X \longrightarrow Y)$.

Our objective in this paper is twofold: For our purposes, we will construct a cycle class map

$$cl^p_f : CH^p(f : X \longrightarrow Y) \longrightarrow H^{2p}(f : X \longrightarrow Y) \tag{1.2}$$

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for each morphism of schemes X and Y with nice properties (as described in Section 2). The problem of cycle class maps has been considered more deeply and in greater generality by several authors, in particular, by Brasselet-Schürmann-Yokura [4, 5], Ginzburg [8, 9] and Yokura [12, 13]. However, we shall not concern ourselves with those intricacies, since our construction in Section 3 is geared towards the following two main objectives of this paper:

(1) Saito [11] has defined a decreasing filtration on the Chow groups of a smooth projective variety X which is a candidate for the conjectural Bloch Beilinson filtration on Chow groups. In Section 4, our purpose is to define a natural extension of this filtration to bivariant Chow groups. We shall define two natural candidates for this extended filtration and check that they are equal.

(2) Our second major aim is to define a "higher bivariant Chow group", which is a bivariant version of the higher Chow groups of Bloch [3]. In this, we shall use the understanding from Section 3 (though not explicitly the cycle class map (1.2)) to define "higher refined Gysin homomorphisms", which will form the basis for our construction of "higher bivariant Chow groups" in Section 5.

Throughout this paper, the word "scheme" shall be taken to mean schemes that are smooth, projective, equidimensional and of finite type over an algebraically closed field K of characteristic zero.

2. Bivariant Chow groups and their basic properties

We shall now briefly recall the basic definitions and properties of bivariant Chow groups. The definition for bivariant Chow groups that we shall use is a slightly "restricted" version of the original definition and further, we shall be using cohomological indexing for Chow groups instead of the original homological indexing of [7]. The standard reference for the following material is [6, §17]. Since X is always equidimensional, we keep in mind that we have isomorphisms:

$$CH_p(X) \cong CH^{\dim(X)-p}(X) \qquad \forall\ p \in \mathbb{Z}. \tag{2.1}$$

We will let Sm/K denote the category of schemes that are smooth, projective, equidimensional and of finite type over the algebraically closed ground field K of characteristic 0. We will sometimes abuse notation and write $X \in Sm/K$ to denote that X is an object of Sm/K. Unless otherwise mentioned, all schemes will be assumed to lie in Sm/K. Given a fibre square in Sm/K:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array} \tag{2.2}$$

we will say that "$f' : X' \longrightarrow Y'$ is fibred over $f : X \longrightarrow Y$ via the morphism g". Recall that a morphism $f : X \longrightarrow Y$ is said to be a *locally complete intersection* (l.c.i.) morphism if f can be factored as $f = pi$:

$$f : X \xrightarrow{i} P \xrightarrow{p} Y, \tag{2.3}$$

where i is a regular imbedding of relative dimension $\dim(P) - \dim(X) = d$ and p is a smooth morphism of relative dimension $\dim(Y) - \dim(P) = -e$. We now come to our definition of a bivariant Chow group (compare [6, §17.1]).

Definition 2.1. Let $f : X \longrightarrow Y$ be a morphism of schemes in Sm/K. Consider any morphism $f' : X' \longrightarrow Y'$ in Sm/K fibred over $f : X \longrightarrow Y$ via a morphism $g : Y' \longrightarrow Y$:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y. \end{array} \tag{2.4}$$

For each such fibre square, a class c in the bivariant Chow group $CH^p(X \xrightarrow{f} Y)$ gives a family of morphisms:

$$c_g^k : CH^k(Y') \longrightarrow CH^{k-r'+p}(X'), \qquad k \in \mathbb{Z}, \tag{2.5}$$

where $r' = \dim(Y') - \dim(X')$. The morphisms c_g^k are compatible with flat pullbacks, proper pushforwards and intersection products in the following sense:

(1) If $f'' : X'' \longrightarrow Y''$ is a morphism in Sm/K fibred over the morphism $f' : X' \longrightarrow Y'$ in Sm/K via a proper morphism $h : Y'' \longrightarrow Y'$, and $f' : X' \longrightarrow Y'$ is, in turn, fibred over $f : X \longrightarrow Y$ in Sm/K via a morphism $g : Y' \longrightarrow Y$ and we have the following fibre diagram

$$\begin{array}{ccc} X'' & \xrightarrow{f''} & Y'' \\ {\scriptstyle h'}\downarrow & & {\scriptstyle h}\downarrow \\ X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & {\scriptstyle g}\downarrow \\ X & \xrightarrow{f} & Y, \end{array} \tag{2.6}$$

then, given $\alpha \in CH_k(Y'')$, we have

$$c_g^{k'}(h_*\alpha) = h'_* c_{gh}^k(\alpha), \qquad \text{where } k' = k + \dim(Y') - \dim(Y''). \tag{2.7}$$

(2) In the situation in (2.6) of condition (1), suppose that $h : Y'' \longrightarrow Y'$ is flat of relative dimension $\dim(Y') - \dim(Y'') = -n$ instead of being proper. Then, for any $\alpha \in CH_k(Y')$,

$$c_{gh}^k(h^*\alpha) = h^* c_g^k(\alpha). \tag{2.8}$$

(3) If $g : Y' \longrightarrow Y$ and $h : Y' \longrightarrow Z'$ are morphisms of schemes, $i : Z'' \longrightarrow Z'$ is a regular imbedding in Sm/K of codimension e, f, f' and f'' are morphisms in Sm/K and we have the fibre diagram

$$\begin{array}{ccccc} X'' & \xrightarrow{f''} & Y'' & \xrightarrow{h'} & Z'' \\ {\scriptstyle i''}\downarrow & & {\scriptstyle i'}\downarrow & & {\scriptstyle i}\downarrow \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{h} & Z' \\ {\scriptstyle g'}\downarrow & & {\scriptstyle g}\downarrow & & \\ X & \xrightarrow{f} & Y, & & \end{array} \tag{2.9}$$

then, for any $\alpha \in CH^k(Y')$,
(2.10)
$$i^! c_g^k(\alpha) = c_{gi'}^{k+e-(d_{Y'}-d_{Y''})}(i^!\alpha) \qquad (\text{where } d_{Y'} = \dim(Y') \text{ and } d_{Y''} = \dim(Y'')).$$

We will now define bivariant cohomology groups in an analogous manner. For that, we will need to recall the following fact: if $i : X \longrightarrow Y$ is a regular imbedding of codimension d, then, given the fibre square

$$\begin{tikzcd} W \arrow[r, "i'"] \arrow[d, "f'"'] & V \arrow[d, "f"] \\ X \arrow[r, "i"] & Y, \end{tikzcd} \tag{2.11}$$

there are the refined Gysin morphisms $i^! : CH^p(V) \longrightarrow CH^p(W)$ (see [6, §6.2] for construction). Recall that the regular imbedding i determines an "orientation class" $u_{X,Y} \in H^{2d}(Y, Y - X)$ in relative cohomology such that, if $[T]$ is a class in $CH^k(V)$, then (see [6, Theorem 19.2]):

$$cl_W^k(i^![T]) = f^*(u_{X,Y}) \cdot cl_V^k([T]). \tag{2.12}$$

Here $cl_V^k : CH^k(V) \longrightarrow H^{2k}(V)$ and $cl_W^k : CH^k(W) \longrightarrow H^{2k}(W)$ are the ordinary cycle class maps. The reader may see, for instance, [6, § 19] for the construction of the class $u_{X,Y}$. Also, $u_{X,Y}$ has the property that for any $y \in H^*(Y)$, $i^*(y) = u_{X,Y} \cdot y$. Further, given a fibre square

$$\begin{tikzcd} X' \arrow[r, "i'"] \arrow[d, "g"'] & Y' \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \end{tikzcd} \tag{2.13}$$

with i (resp. i') a regular imbedding of codimension d (resp. d'), we have (see [6, §19.2.2]):

$$f^*(u_{X,Y}) = cl_{X'}(c_{d-d'}(E)) \cdot u_{X',Y'}, \tag{2.14}$$

where $u_{X,Y}$ (resp. $u_{X',Y'}$) is the class in $H^{2d}(Y, Y - X)$ (resp. $H^{2d'}(Y', Y' - X')$) determined by the regular imbeddding i (resp. i') and E denotes the excess normal bundle in (2.13).

Definition 2.2. Let $f : X \longrightarrow Y$ be a morphism of schemes in Sm/K. A class d in the bivariant cohomology group $H^q(X \xrightarrow{f} Y)$ is an object that associates to each morphism $f' : X' \longrightarrow Y'$ in Sm/K fibred over $f : X \longrightarrow Y$ via a morphism $g : Y' \longrightarrow Y$, a family of morphisms

$$d_g^k : H^k(Y') \longrightarrow H^{k-2r'+q}(X'), \qquad k \in \mathbb{Z}, \tag{2.15}$$

satisfying the following properties:

(1)′ If $f'' : X'' \longrightarrow Y''$ is a morphism in Sm/K fibred over the morphism $f' : X' \longrightarrow Y'$ in Sm/K via a proper morphism $h : Y'' \longrightarrow Y'$, and $f' : X' \longrightarrow Y'$ is, in turn, fibred over $f : X \longrightarrow Y$ in Sm/K via a morphism $g : Y' \longrightarrow Y$ and we have the following fibre diagram;

$$\begin{array}{ccc} X'' & \xrightarrow{f''} & Y'' \\ {\scriptstyle h'}\big\downarrow & & {\scriptstyle h}\big\downarrow \\ X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ X & \xrightarrow{f} & Y, \end{array} \tag{2.16}$$

then, given $x \in H^k(Y'')$, we have

$$d_g^{k'}(h_*(x)) = h'_* d_{gh}^k(x), \qquad \text{where } k' = k + \dim(Y') - \dim(Y''). \tag{2.17}$$

$(2)'$ In the situation in (2.16) of condition (1), suppose that $h : Y'' \longrightarrow Y'$ is flat of relative dimension $\dim(Y') - \dim(Y'') = -n$ instead of being proper. Then, for any $x \in H^k(Y')$,

$$d_{gh}^k(h^*(x)) = h^* d_g^k(x). \tag{2.18}$$

$(3)'$ In the setting of the diagram (2.9) and the notation of Definition 2.1, let $u = u_{Z',Z''}$ be the orientation class corresponding to the regular imbedding $i : Z'' \longrightarrow Z'$. Then, the morphisms d_g^k of (2.15) satisfy

$$(hf')^*(u) \cdot d_g^k(x) = d_{gi'}^{k+2e-2e'}(h^*(u) \cdot x) \qquad \forall\, x \in H^k(Y'), \tag{2.19}$$

where $e' = \dim(Y') = \dim(Y'')$.

Remark 2.3. From (2.12), it follows that condition $(3)'$ in Definition 2.2 is a natural adaptation of condition (3) of Definition 2.1 to cohomology.

The usual operations on Chow groups such as products, proper pushforwards, pullbacks and results such as the "projection formula" can all be defined on bivariant Chow groups. For a detailed description of these constructions, the reader may see [6, §17]. These operations can also be easily defined on the bivariant cohomlogy groups of Definition 2.2.

We now recall the following well-known facts about bivariant Chow groups (see [6, §17.4]) that will be used repeatedly throughout this paper. In the sections to follow, we will sometimes use these properties directly without referring to them.

1. If $f : X \longrightarrow Y$ is a flat morphism in Sm/K of relative dimension $\dim(Y) - \dim(X) = -n$, then f defines a class $[f]$ in $CH^{-n}(f : X \longrightarrow Y)$ determined by the pullback.

2. If $i : X \longrightarrow Y$ is a regular imbedding in Sm/K of codimension d, then i determines a class $[i] \in CH^d(i : X \longrightarrow Y)$ determined by the refined Gysin morphisms $i^!$.

3. A morphism $f : X \longrightarrow Y$ of schemes that factor as $X \xrightarrow{i} W \xrightarrow{p} Y$ with i a regular imbedding of codimension e, p a smooth morphism of relative dimension $-n$ and $d = e - n$ is referred to as a *locally complete intersection* (or l.c.i.) morphism of codimension d. If $f : X \longrightarrow Y$ is an l.c.i. morphism in Sm/K factoring as $f = pi$ as above and such that $p : P \longrightarrow Y$ is projective, then the product of the classes

$$[f] = [i] \cdot [p] \in CH^d(f : X \longrightarrow Y) \tag{2.20}$$

can be shown to be independent of the factorization of f (note that $P \in Sm/K$ because $p : P \longrightarrow Y$ is smooth and projective and $Y \in Sm/K$). We refer to $[f] \in CH^d(f : X \longrightarrow Y)$ as the *orientation class* of the morphism f.

4. Let $g : Y \longrightarrow Z$ be a smooth and projective morphism in Sm/K of relative dimension $-n$ and let $[g] \in CH^{-n}(g : Y \longrightarrow Z)$ be its orientation class. Then, for any $f : X \longrightarrow Y$ in Sm/K, the product with the orientation class determines an isomorphism

$$CH^p(f : X \longrightarrow Y) \xrightarrow{\cong} CH^{p-n}(g \circ f : X \longrightarrow Z). \tag{2.21}$$

5. (Excess intersection formula) Consider the following fibre square:

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"'] \\ X \arrow[r, "f"] & Y, \end{tikzcd} \tag{2.22}$$

where f and f' are l.c.i. morphisms in Sm/K of codimensions d and d', respectively. Then, we have

$$g^*[f] = c_e(E) \cdot [f'], \tag{2.23}$$

where $e = d - d'$, E is the excess normal bundle and $c_e(E)$ is its e-th Chern class.

Remark 2.4. We have mentioned before that Definition 2.1 is a "restricted" version of the original definition of bivariant Chow groups in the sense that it allows only morphisms in Sm/K fibred over morphisms of schemes in Sm/K. However, the properties 1–5 above continue to hold for this restricted version because the proofs of these properties for the usual bivariant Chow groups can be repeated verbatim in this situation.

As mentioned before, the usual Chow homology groups of X can be recovered from the bivariant Chow groups of Fulton and MacPherson (see [6, §17.3]). The following proposition shows that these groups can also be recovered from our "restricted" definition of bivariant Chow groups in Definition 2.1. This follows from the fact that the ground field K, of characteristic zero, admits a resolution of singularities.

Proposition 2.5. *Let X be a scheme in Sm/K and let $p_X : X \longrightarrow Spec(K)$ denote the structure map of X. Then, there exist isomorphisms:*

$$\begin{aligned} \varphi : CH^{-p}(p_X : X \longrightarrow Spec(K)) \cong CH_p(X), \\ CH^q(X) \cong CH^q(1 : X \longrightarrow X) \quad \forall p, q \in \mathbb{Z}. \end{aligned} \tag{2.24}$$

Proof. Choose any $p \in \mathbb{Z}$ and set $S = Spec(K)$. Let $d = \dim(X)$. Given any bivariant class $c \in CH^{-p}(p_X : X \longrightarrow S)$, we define a class $\varphi(c) = c[S] \in CH^{d-p}(X)$. Conversely, given $a \in CH^{d-p}(X)$, we have a class $\psi(a) \in CH^{-p}(p_X : X \longrightarrow S)$, defined as follows: given any morphism $p_Y \longrightarrow S$ for some scheme $Y \in Sm/K$ and some $\alpha \in CH^k(Y)$, $\psi(a)$ is given by the morphisms:

$$\psi(a)(\alpha) = a \times \alpha \in CH^{d-p+k}(X \times Y). \tag{2.25}$$

Since $\psi(a)([S]) = a$, it follows that $\varphi \circ \psi$ is the identity. Choose some $c \in CH^{-p}(p_X : X \longrightarrow S)$. Let $Y \in Sm/K$ and let $\alpha \in CH^k(X)$ for some k. We may assume that $\alpha = [V]$ for some irreducible, closed subscheme V of codimension k in Y. Since the ground field K has characteristic zero, it admits a resolution of singularities, and hence we have a projective birational morphism $\tilde{p} : \tilde{V} \longrightarrow V$ such that $\tilde{V}$ is smooth. Since V is projective, it also follows that $\tilde{V}$ is also projective and hence $\tilde{V} \in Sm/K$. Let $i : V \longrightarrow Y$ denote the closed immersion of V into Y. Then, $\alpha = [V] = (i\tilde{p})_*([\tilde{V}])$. Finally, let $p_{\tilde{V}}$ denote the structure morphism $p_{\tilde{V}} : \tilde{V} \longrightarrow S$. Then, we have

(2.26)
$$\begin{aligned}\psi(c[S])(\alpha) &= \psi(c[S])((i\tilde{p})_*([\tilde{V}])) = (i\tilde{p})_*\psi(c[S])([\tilde{V}]) \quad \text{(since } i\tilde{p} \text{ is proper)}\\ &= (i\tilde{p})_*\psi(c[S])(p^*_{\tilde{V}}[S]) = (i\tilde{p})_*p^*_{\tilde{V}}\psi(c[S])([S])\\ &= (i\tilde{p})_*p^*_{\tilde{V}}(c[S]) = (i\tilde{p})_*c(p^*_{\tilde{V}}[S]) = (i\tilde{p})_*c([\tilde{V}])\\ &= c((i\tilde{p})_*[\tilde{V}]) = c(\alpha).\end{aligned}$$

It follows that $\psi \circ \varphi$ is the identity. This shows that we have isomorphisms:

$$CH^{-p}(p_X : X \longrightarrow Spec(K)) \cong CH^{d-p} \cong CH_p(X). \tag{2.27}$$

Finally, the morphism $p_X : X \longrightarrow S$ being smooth, we have isomorphisms (using (2.21))

$$CH^q(1 : X \longrightarrow X) \cong CH^{q-d}(p_X : X \longrightarrow S) \cong CH_{d-q}(X) \cong CH^q(X). \tag{2.28}$$

□

3. Cycle class maps for bivariant Chow groups

For any morphism $f : X \longrightarrow Y$ in Sm/K, we intend to construct cycle class maps

$$cl^p(f) : CH^p(f : X \longrightarrow Y) \longrightarrow H^{2p}(f : X \longrightarrow Y) \qquad \forall\, p \in \mathbb{Z}. \tag{3.1}$$

Given a morphism $f : X \longrightarrow Y$ in Sm/K, we can factor f as $f = p_Y \circ i_f$, where

$$X \longrightarrow X \times Y \qquad x \mapsto (1, f) \qquad\qquad \text{and} \qquad p_Y : X \times Y \longrightarrow Y, \tag{3.2}$$

$p_Y : X \times Y \longrightarrow Y$ being the natural projection onto Y. The morphism $i_f : X \longrightarrow X \times Y$ is a closed imbedding and since X and $X \times Y$ are both smooth, it follows (see [6, Appendix B.7.2]) that $i_f : X \longrightarrow X \times Y$ is a regular imbedding of codimension $\dim(Y)$. It is also clear that the natural projection $p_Y : X \times Y \longrightarrow Y$ is a smooth morphism. Hence $f = p_Y \circ i_f$ is an l.c.i. morphism in the sense of Section 2. We also note that X being projective, $p_Y : X \times Y \longrightarrow Y$ is also a projective morphism.

Given a scheme X, we will always use $\dim(X)$ or d_X to denote its dimension. For any morphism $f : X \longrightarrow Y$ of schemes, we will use rel.dim(f) to denote its relative dimension $\dim(Y) - \dim(X)$. Additionally, for a flat morphism f, we will assume throughout this paper that the relative dimension is stable under base change (see [6, Appendix B.2.5] for details). We will also use Γ_f to denote the graph of f, i.e. the image of $i_f : X \longrightarrow X \times Y$ in (3.2) above. We start with the following lemma:

Lemma 3.1. *Let X, $Y \in Sm/K$ and let $i : X \longrightarrow Y$ be a regular imbedding of codimension d and let $[i] \in CH^d(i : X \longrightarrow Y)$ be the orientation class of i. Then, for any $c \in CH^p(i : X \longrightarrow Y)$, there exists a unique class t in $CH^{p-d}(X) = CH^{p-d}(1 : X \longrightarrow X)$ such that $c = t \cdot [i]$. In other words, the product with $[i]$ induces an isomorphism*

$$CH^{p-d}(1 : X \longrightarrow X) \xrightarrow[\cong]{\cdot[i]} CH^p(i : X \longrightarrow Y). \tag{3.3}$$

Proof. Let $\dim(Y) = d_Y$ and $\dim(X) = d_X$. Then $d_Y - d_X = d$. We know that X and Y are smooth and we use $[p_X] \in CH^{-d_X}(p_X : X \longrightarrow Spec(K))$ and $[p_Y] \in CH^{-d_Y}(p_Y : Y \longrightarrow Spec(K))$ to denote the orientation classes of the structure maps $p_X : X \longrightarrow Spec(K)$ and $p_Y : Y \longrightarrow Spec(K)$. Consider the product $c \cdot [p_Y] \in CH^{p-d_Y}(p_Y \circ i : X \longrightarrow Spec(K)) = CH^{p-d_Y}(p_X : X \longrightarrow Spec(K))$. Using (2.21), we know that there exists an isomorphism

$$CH^{p-d}(X) \cong CH^{p-d}(1 : X \longrightarrow X) \xrightarrow[\cong]{\cdot [p_X]} CH^{p-d_Y}(p_X : X \longrightarrow Spec(K)), \quad t \mapsto t \cdot [p_X]. \tag{3.4}$$

Hence, there exists a class $t \in CH^{p-d}(X)$ such that $t \cdot [p_X] = c \cdot [p_Y]$. However, since $p_X = p_Y \circ i$, we get $[p_X] = [i] \cdot [p_Y]$ and hence $t \cdot [i] \cdot [p_Y] = c \cdot [p_Y]$. Since the product with $[p_Y]$ induces an isomorphism (again using (2.21))

$$CH^p(i : X \longrightarrow Y) \longrightarrow CH^{p-d_Y}(p_X : X \longrightarrow Spec(K)) \tag{3.5}$$

we get $t \cdot [i] = c$. Finally, if there exist $t, t' \in CH^{p-d}(X)$ such that $c = t \cdot [i] = t' \cdot [i]$, then $c \cdot [p_Y] = t \cdot [p_X] = t' \cdot [p_X]$. Again, the product with $[p_X]$ induces an isomorphism from $CH^{p-d}(1 : X \longrightarrow X)$ to $CH^{p-d_Y}(i : X \longrightarrow Y)$ and hence $t = t'$. □

The next lemma extends the result of Lemma 3.1 from regular imbeddings to all morphisms in Sm/K.

Lemma 3.2. *Let $f : X \longrightarrow Y$ be a morphism in Sm/K and suppose that $d = \dim(Y) - \dim(X)$. Given a class $c \in CH^p(f : X \longrightarrow Y)$, there exists a unique $t \in CH^{p-d}(X)$ such that $c = t \cdot [f]$.*

Proof. From the discussion at the beginning of this section and following (3.2), we know that any morphism $f : X \longrightarrow Y$ in Sm/K can be represented as an l.c.i. morphism

$$X \xrightarrow{1 \times f} X \times Y \xrightarrow{p} Y. \tag{3.6}$$

Let $\dim(Y) = d_Y$ and $\dim(X) = d_X$. Then $d_Y - d_X = d$. Using (2.21), we have an isomorphism

$$CH^{p+d_X}(1 \times f : X \longrightarrow X \times Y) \xrightarrow{\cong} CH^p(f : X \longrightarrow Y), \qquad \omega \mapsto \omega \cdot [p].$$

Hence, given $c \in CH^p(f : X \longrightarrow Y)$, there exists a unique $c' \in CH^{p+d_X}(1 \times f : X \longrightarrow X \times Y)$ such that $c = c' \cdot [p]$. Again, since $c' \in CH^{p+d_X}(1 \times f : X \longrightarrow X \times Y)$ and $1 \times f : X \longrightarrow X \times Y$ is a regular imbedding of codimension d_Y, it follows from Lemma 3.1 that there exists a unique $t \in CH^{p-d}(1 : X \longrightarrow X)$ such that $c' = t \cdot [1 \times f]$. Then $c = t \cdot [1 \times f] \cdot [p] = t \cdot [f]$. The uniqueness of t follows from the uniqueness of c' and the uniqueness statement in Lemma 3.1. □

Remark 3.3. Lemma 3.2 above shows that the bivariant Chow group $CH^p(f : X \longrightarrow Y)$ is actually isomorphic to the Chow group $CH^{p-d}(X)$, where $d = \dim(Y) - \dim(X)$, provided X and Y are both in Sm/K. However, this isomorphism is clearly not natural; i.e., this isomorphism is unrelated to the products on Chow groups of X. For the l.c.i. morphism $f : X \longrightarrow Y$ we will construct a class in the bivariant cohomology group $H^{2d}(f : X \longrightarrow Y)$. The cycle class map cl^p_f will therefore be the product of the ordinary cycle class of $CH^{p-d}(X)$ in $H^{2p-2d}(X)$ with the class in bivariant cohomology of the l.c.i. morphism f.

The cycle class maps $cl^p_f : CH^p(f : X \longrightarrow Y) \longrightarrow H^{2p}(f : X \longrightarrow Y)$ shall be constructed in the following proposition using a specific factorization of the morphism f; the fact that this cycle class is independent of the choice of the factorization will be shown later in Proposition 3.6. Given a scheme X, we shall use cl^*_X to denote the ordinary cycle class maps $cl^*_X : CH^*(X) \longrightarrow H^{2*}(X)$.

Proposition 3.4. *Let $f : X \longrightarrow Y$ be a morphism in Sm/K. There exist cycle class maps*

$$cl^p_f : CH^p(f : X \longrightarrow Y) \longrightarrow H^{2p}(f : X \longrightarrow Y).$$

Proof. Let $\dim(Y) = d_Y$ and $\dim(X) = d_X$. We represent f as an l.c.i. morphism:

$$X \overset{i_f = 1\times f}{\longrightarrow} X \times Y \overset{p_Y}{\longrightarrow} Y \tag{3.7}$$

and let $[i_f]$ and $[p_Y]$ denote the orientation classes of the regular imbedding i_f and the coordinate projection p_Y, respectively. From Lemma 3.2 we know that a class $c \in CH^p(f : X \longrightarrow Y)$ can be uniquely factored as $c = t \cdot [i_f] \cdot [p_Y]$, where $t \in CH^{p-d_Y+d_X}(X)$.

From the discussion above we know that $[i_f]$ is represented on the Chow groups by the refined Gysin morphism and that the smooth morphism p_Y is represented by the pullback. Let $u_{X,X\times Y}$ denote the class in $H^{2d_Y}(X\times Y, X\times Y - \Gamma_f)$ corresponding to the regular imbedding i_f (here Γ_f denotes the graph of f). Finally suppose that the morphism $f' : X' \longrightarrow Y'$ in Sm/K is fibred over f via the morphism $g : Y' \longrightarrow Y$ such that we have the fibre squares:

$$\begin{tikzcd} X' \arrow[r, "1"] \arrow[d, "g'"'] & X' \arrow[r, "i'"] \arrow[d, "g'"'] & X \times Y' \arrow[r, "p_{Y'}"] \arrow[d, "1\times g"'] & Y' \arrow[d, "g"'] \\ X \arrow[r, "1"] & X \arrow[r, "i_f"] & X \times Y \arrow[r, "p_Y"] & Y. \end{tikzcd}$$

Let $r = \dim(Y) - \dim(X)$ and $r' = \dim(Y') - \dim(X')$. Consider a class $y' \in H^k(Y')$. Then, we define the family of maps:

$$cl^p_f(c)^k_g : H^k(Y') \longrightarrow H^{k-2r'+2p}(X'),\ y' \mapsto g'^* cl^{p-r}_X(t) \cdot ((1\times g)^*(u_{X,X\times Y}) \cdot p^*_{Y'}(y')). \tag{3.8}$$

From the construction, it is clear that the maps in the family $cl^p_f(c)^k_g$ satisfy the compatibility conditions in Definition 2.2 and hence induce a class in $H^{2p}(f : X \longrightarrow Y)$, which we denote by $cl^p_f(c)$. □

We will now show that the cycle class map constructed in Proposition 3.4 is actually independent of the factorization of the morphism $f : X \longrightarrow Y$ in Sm/K into a regular imbedding followed by a smooth and projective morphism. For that, we will need the following lemma.

Lemma 3.5. *Let $X, Y \in Sm/K$ and let $j : X \longrightarrow Y$ be a regular imbedding such that j can be factored as $j = pi$ where $i : X \longrightarrow P$ is a regular imbedding and $p : P \longrightarrow Y$ is a smooth and projective morphism. Let $j' : X' \longrightarrow Y'$ be a regular imbedding in Sm/K fibred over $j : X \longrightarrow Y$ via a morphism $g : Y' \longrightarrow Y$, forming* *the fibre diagram (with $j = pi$, $j' = p'i'$):*

$$\begin{tikzcd} X' \arrow[r, "i'"] \arrow[d, "h'"'] & P' \arrow[r, "p'"] \arrow[d, "h"'] & Y' \arrow[d, "g"'] \\ X \arrow[r, "i"] & P \arrow[r, "p"] & Y. \end{tikzcd} \tag{3.9}$$

Then, if $u_j \in H^{2(\dim(Y)-\dim(X))}(Y, Y - X)$ and $u_i \in H^{2(\dim(P)-\dim(X))}(P, P - X)$ are the orientation classes induced respectively by the regular imbeddings j and i, we have:

$$h^*(u_i) \cap p'^*(y') = g^*(u_j) \cap y' \qquad \forall\ y' \in H^*(Y'). \tag{3.10}$$

Proof. Since p' is obtained by a base change from p, it follows that p' is smooth and projective. Since $Y' \in Sm/K$ and p' is smooth and projective, we have $P' \in Sm/K$. Since i' is a closed immersion of schemes in Sm/K, it follows that i' is a regular imbedding. Further, since p and p' are both smooth morphisms of the same relative dimension, we have:

$$\text{rel.dim}(i) - \text{rel.dim}(i') = \text{rel.dim}(j) - \text{rel.dim}(j') = e(\text{say}). \tag{3.11}$$

Choose $y' \in H^*(Y')$. Let E_j (resp. E_i) denote the excess normal bundle for the fibre square created by the morphism $j' : X' \longrightarrow Y'$ fibred over $j : X \longrightarrow Y$ (resp. $i' : X' \longrightarrow P'$ fibred over $i : X \longrightarrow P$). Let $u_{j'} \in H^{2(\dim(Y')-\dim(X'))}(Y', Y' - X')$ and $u_{i'} \in H^{2(\dim(P')-\dim(X'))}(P', P' - X')$ be the orientation classes induced respectively by the regular imbeddings j' and i'.

From (2.14), it follows that

$$\begin{aligned} g^*(u_j) \cap y' &= cl_{X'}(c_e(E_j)) \cdot u_{j'} \cap y' = cl_{X'}(c_e(E_j)) \cdot j'^*(y')q \\ &= cl_{X'}(c_e(E_j)) \cdot i'^* p'^*(y'), \\ h^*(u_i) \cap p'^*(y') &= cl_{X'}(c_e(E_i)) \cdot u_{i'} \cap p'^*(y') = cl_{X'}(c_e(E_i)) \cdot i'^* p'^*(y'), \end{aligned} \tag{3.12}$$

where $cl_{X'}$ denotes the ordinary cycle class map from the Chow groups to the cohomology of X'. Hence, to prove (3.10), it suffices to show that $c_e(E_j) = c_e(E_i)$. For this, we write

(3.13)

$$g^*[j] = c_e(E_j) \cdot [j'] \quad \text{and} \quad g^*[j] = g^*([i][p]) = h^*[i] g^*[p] = c_e(E_i)[i'][p'] = c_e(E_i)[j'].$$

The result now follows from the uniqueness statement in Lemma 3.2, i.e. from the fact that there exists a unique t such that $g^*[j] = t \cdot [j']$. $\square$

Indeed, suppose that we are given a morphism $f : X \longrightarrow Y$ in Sm/K, a class $c \in CH^p(f : X \longrightarrow Y)$ and a factorization $f = pi$ of f into a regular imbedding $i : X \longrightarrow P$ followed by a smooth and projective morphism $p : P \longrightarrow Y$. Then, for any morphism $f' : X' \longrightarrow Y'$ in Sm/K fibred over f via a morphism $g : Y' \longrightarrow Y$, we form the fibre squares:

$$\begin{tikzcd} X' \arrow[r, "j"] \arrow[d, "h'"'] & Q \arrow[r, "q"] \arrow[d, "h"'] & Y' \arrow[d, "g"'] \\ X \arrow[r, "i"] & P \arrow[r, "p"] & Y \end{tikzcd} \qquad f = pi,\ f' = qj. \tag{3.14}$$

In (3.14), $q : Q \longrightarrow Y'$ is obtained by a base change from p and hence q is smooth and projective. Since $Y' \in Sm/K$, this implies that $Q \in Sm/K$. Again, j being a closed immersion of smooth schemes, it follows that j is a regular imbedding. Now, using Lemma 3.2, we factor $c \in CH^p(f : X \longrightarrow Y)$ as $c = t\cdot[f]$ with $t \in CH^{p-r}(X)$, for $r = \dim(Y) - \dim(X)$. Let $u_{X,P}$ denote the class in $H^{2(\dim(P)-\dim(X))}(P, P-X)$ induced by the regular imbedding $i : X \longrightarrow P$. Then, we can alternatively define the cycle class $cl_f^p(c) \in H^{2p}(f : X \longrightarrow Y)$ as follows: for $y' \in H^k(Y')$, we can define a family of maps:

(3.15)
$$cl_f^p(c)_g^k : H^k(Y') \longrightarrow H^{k-2r'+2p}(X'), \qquad y' \mapsto h'^* cl_X^{p-r}(t) \cdot (h^*(u_{X,P}) \cdot q^*(y')),$$

which also satisfies all the conditions of Definition 2.2 for being a class in the bivariant cohomology group $H^{2p}(f : X \longrightarrow Y)$.

In order to show that the expression for the cycle class defined in (3.15) agrees with the expression for the cycle class due to (3.8), it is enough to show that for a given morphism $f : X \longrightarrow Y$ in Sm/K of relative dimension r, the cycle class $cl_f^r([f])$ of the orientation class of f defined in Proposition 3.4 is independent of the chosen factorization $f = pi$ of the l.c.i. morphism into a regular imbedding i followed by a smooth and projective morphism p. This will follow from Proposition 3.6.

Proposition 3.6. *Given a morphism $f : X \longrightarrow Y$ of relative dimension r, the cycle class $cl_f^r([f])$ of $[f]$ defined by* (3.15) *is independent of the choice of the factorization of f into $f = pi$, where i is a regular imbedding and p is a smooth and projective morphism.*

Proof. Suppose that $f = pi$ and $f = p_1 i_1$ for regular imbeddings $i : X \longrightarrow P$, $i_1 : X \longrightarrow P_1$ and smooth, projective morphisms $p : P \longrightarrow Y$, $p_1 : P_1 \longrightarrow Y$. Then, we can compare each of the factorizations to:

(3.16)
$$\begin{tikzcd} & & P_1 \arrow[dr, "p_1"] & \\ X \arrow[r, "(i_1,i)"] & P_1 \times_Y P \arrow[ur, "p'"] \arrow[dr, "p'_1"'] & & Y. \\ & & P \arrow[ur, "p"'] & \end{tikzcd}$$

The morphisms p' and p'_1 being smooth and projective, it is clear that $P_1 \times_Y P \in Sm/K$. Again, $(i_1, i) : X \longrightarrow P_1 \times_Y P$ is a closed immersion in Sm/K and hence a regular imbedding. We let $u_{X\to P_1\times_Y P}$ be the class in $H^{2(\dim(P_1\times_Y P)-\dim(X))}(P_1 \times_Y P, P_1 \times_Y P - X)$ induced by (i_1, i). Let $u_{p'\circ(i_1,i)}$ and $u_{p'_1\circ(i_1,i)}$ denote respectively the classes induced by the regular imbeddings $i_1 = p' \circ (i_1, i)$ and $i = p'_1 \circ (i_1, i)$.

Now suppose that we are given a morphism $f' : X' \longrightarrow Y'$ in Sm/K fibred over $f : X \longrightarrow Y$ via a morphism $g : Y' \longrightarrow Y$. Then, we can factor f' in two ways:

(1) as $f' = q_1 j_1$ with $q_1 : Q_1 \longrightarrow Y'$ fibred over $p_1 : P_1 \longrightarrow Y$ and $j_1 : X' \longrightarrow Q_1$ fibred over $i_1 : X \longrightarrow P_1$ as in (3.14);

(2) as $f' = qj$ with $q : Q \longrightarrow Y'$ fibred over $p : P \longrightarrow Y$ and $j : X' \longrightarrow Q$ fibred over $i : X \longrightarrow P$ as in (3.14).

It follows that we have the following diagram, which is fibred over (3.16):

$$\begin{tikzcd} & & Q_1 \arrow[dr, "q_1"] & \\ X' \arrow[r, "{(j_1,j)}"] & Q_1 \times_{Y'} Q \arrow[ur, "q'"] \arrow[dr, "q'_1"'] & & Y'. \\ & & Q \arrow[ur, "q"'] & \end{tikzcd} \tag{3.17}$$

As before, it follows that $Q_1 \times_{Y'} Q \in Sm/K$ and j_1, j and (j_1, j) are all regular imbeddings. Suppose that $h' : X' \longrightarrow X$, $h_1 : Q_1 \longrightarrow P_1$, $h : Q \longrightarrow P$ and $(h_1, h) : Q_1 \times_{Y'} Q \longrightarrow P_1 \times_Y P$ are the maps connecting the diagram (3.17) to (3.16). Then, we use the following fact (which follows from Lemma 3.5):

$$\begin{aligned} (h_1,h)^*(u_{X\to P_1\times_Y P}) \cap q'^*(x_1) &= h_1^*(u_{p'\circ(i_1,i)}) \cap x_1, \qquad x_1 \in H^*(Q_1), \\ (h_1,h)^*(u_{X\to P_1\times_Y P}) \cap p_1'^*(x) &= h^*(u_{p_1'\circ(i_1,i)}) \cap x, \qquad x \in H^*(Q). \end{aligned} \tag{3.18}$$

Then, for some $y \in CH^*(Y')$, we can apply (3.18) with $x_1 = q_1^*(y)$ and $x = q^*(y)$. This shows that the two expressions for the cycle class of $[f]$ induced by the two factorizations $f = p_1 i_1$ and $f = pi$ are equal on each element of $H^*(Y')$. □

The following proposition now shows that the cycle class maps constructed in Proposition 3.4 (or in (3.15)) commute with pullbacks on bivariant Chow groups. For the sake of convenience, given a class $c \in CH^p(f : X \longrightarrow Y)$, we shall simply write $cl(c)$ for $cl_f^p(c)$ when there is no danger of confusion.

Proposition 3.7. *Let $f : X \longrightarrow Y$ be a morphism in Sm/K and let $f' : X' \longrightarrow Y'$ be another morphism in Sm/K fibred over f via $g : Y' \longrightarrow Y$. Let c be a class in $CH^p(f : X \longrightarrow Y)$. Then the cycle class maps commute with the pullback, i.e.*

$$cl_{f'}^p(g^*(c)) = g^*(cl_f^p(c)) \in H^{2p}(f' : X' \longrightarrow Y').$$

Proof. Let $f'' : X'' \longrightarrow Y''$ be a morphism in Sm/K fibred over f' via $h : Y'' \longrightarrow Y'$. Let $\dim(X) = d_X$, $\dim(X') = d_{X'}$, $\dim(X'') = d_{X''}$, $\dim(Y) = d_Y$, $\dim(Y') = d_{Y'}$ and $\dim(Y'') = d_{Y''}$. Further, let r, r' and r'' denote the relative dimensions $\dim(Y) - \dim(X)$, $\dim(Y') - \dim(X')$ and $\dim(Y'') - \dim(X'')$ respectively. Consider the following diagram, in which all squares are fibre squares:

$$\begin{tikzcd} X'' \arrow[r, "1"] \arrow[d, "h'"] & X'' \arrow[r, "i''"] \arrow[d, "h'"] & X \times Y'' \arrow[r, "p_{Y''}"] \arrow[d, "1\times h"] & Y'' \arrow[d, "h"] \\ X' \arrow[r, "1"] \arrow[d, "g'"] & X' \arrow[r, "i'"] \arrow[d, "g'"] & X \times Y' \arrow[r, "p_{Y'}"] \arrow[d, "1\times g"] & Y' \arrow[d, "g"] \\ X \arrow[r, "1"] & X \arrow[r, "i_f"] & X \times Y \arrow[r, "p_Y"] & Y. \end{tikzcd}$$

Again, we factor $f = p_Y \circ i_f$ as in (3.2). The maps $i' : X' \longrightarrow X \times Y'$, $i'' : X'' \longrightarrow X \times Y''$ are both obtained by a base change on the closed immersion i_f and since X', X'', $X \times Y'$ and $X \times Y''$ are all smooth, it follows that i' and i'' are actually regular imbeddings. Also, $p_{Y'}$ and $p_{Y''}$, both obtained by a base change on p_Y, are smooth and projective.

Choose $c \in CH^p(f : X \longrightarrow Y)$. By definition, both the classes $g^*(cl(c))$ and $cl(g^*(c))$ induce maps

$$H^k(Y'') \longrightarrow H^{k-2r''+2p}(X'') \qquad \forall\, k \in \mathbb{Z}. \tag{3.19}$$

Let $u = u_{X,X\times Y}$ be the class in $H^{2d_Y}(X\times Y, X\times Y-\Gamma_f)$ corresponding to the refined Gysin morphisms given by the regular imbedding $X \overset{i_f}{\hookrightarrow} X\times Y$ as explained before and let $u' = u_{X',X\times Y'}$ denote the class in $H^{2d_{Y'}+2d_X-2d_{X'}}(X\times Y', X\times Y'-i'(X'))$ corresponding to the refined Gysin morphisms given by the regular imbedding $X' \overset{i'}{\hookrightarrow} X\times Y'$ Let $c = t\cdot[f]$, where $t\in CH^{p-r}(X)$. For a class $y''\in H^k(Y'')$, we have

$$\begin{aligned}cl(c)^k_{g\circ h}(y'') &= cl_{X''}((g'\circ h')^*(t))\cdot((1\times gh)^*(u)\cap p^*_{Y''}(y''))\\ &= cl_{X''}(h'^*(g'^*(t)))\cdot((1\times gh)^*(u)\cap p^*_{Y''}(y'')).\end{aligned}\tag{3.20}$$

Now let $g^*(c)\in CH^p(f':X'\longrightarrow Y')$ factor as $g^*(c) = t'\cdot[f']$ with $t'\in CH^{p-r'}(X')$. From (3.15), we know that we can use the factorization $f' = p_{Y'}\circ i'$ to define the cycle class of $[f']$. Then, for $y''\in H^k(Y'')$, we get

$$cl(g^*(c))^k_h(y'') = cl_{X''}(h'^*(t'))\cdot((1\times h)^*(u')\cap p^*_{Y''}(y'')).\tag{3.21}$$

Alternatively,

$$g^*(c) = g^*(t\cdot[f]) = g'^*(t)\cdot g^*[f] = g'^*(t)\cdot c_e(E_1)\cdot[f'],\tag{3.22}$$

where $e = r-r'$ and E_1 is the excess normal bundle of the fibre square consisting of the morphism f' fibred over f. Since $g'^*(t)\cdot c_e(E_1)\in CH^{p-r'}(X')$ and $g'^*(t)\cdot c_e(E_1)\cdot[f'] = t'\cdot[f']$, it follows from the uniqueness statement in Lemma 3.2 that $g'^*(t)\cdot c_e(E_1) = t'$.

Let E_2 be the excess normal bundle of the fibre square consisting of the morphism i' fibred over i_f. Using (2.14) we get $(1\times g)^*(u) = cl_{X'}(c_e(E_2))\cdot u'$ and hence we can replace $(1\times gh)^*(u)$ in (3.20) by

$$(1\times gh)^*(u) = (1\times h)^*((1\times g)^*(u)) = cl_{X''}(h'^*(c_e(E_2)))\cdot(1\times h)^*(u'),\tag{3.23}$$

and hence in order to show that the expressions in (3.20) and (3.21) are equal, it remains to check that $c_e(E_1) = c_e(E_2)$. For this, we write

$$g^*[f] = c_e(E_1)[f']\tag{3.24}$$

and

$$g^*[f] = g^*([i_f]\cdot[p_Y]) = (1\times g)^*[i_f]\cdot g^*[p_Y] = c_e(E_2)[i']\cdot[p_{Y''}] = c_e(E_2)[f'].$$

Hence applying the uniqueness statement of Lemma 3.2 to $g^*[f]\in CH^r(f':X'\longrightarrow Y')$, it follows that $c_e(E_1) = c_e(E_2)$. □

4. Filtrations on bivariant Chow groups

In this section, our objective is to extend to bivariant Chow groups the "motivic" filtration defined by Saito [11]. In [11], Saito defines a decreasing filtration

$$CH^r(X) = F^0CH^r(X)\supseteq F^1CH^r(X)\supseteq F^2CH^r(X)\supseteq\dots\tag{4.1}$$

on the Chow groups of a smooth projective variety X. This filtration is a candidate for the conjectural Bloch-Beilinson motivic filtration on Chow groups. Recall that Beilinson has made the following important conjecture; i.e., there exists a filtration

$$CH^r(X) = F^0_{\mathcal{M}}CH^r(X)\supseteq F^1_{\mathcal{M}}CH^r(X)\supseteq F^2_{\mathcal{M}}CH^r(X)\supseteq\dots\tag{4.2}$$

on the Chow groups of any smooth projective variety X such that the graded pieces of the filtration, tensored with $\mathbb{Q}$, satisfy:

$$Gr^l_{F_{\mathcal{M}}} CH^r(X) \otimes \mathbb{Q} \cong Ext^l_{\mathcal{MM}_K}(\mathbf{1}, h^{2r-l}(X)(r)), \tag{4.3}$$

where $\mathcal{MM}_K$ is the conjectural theory of mixed motives over the field K and $\mathbf{1}$ is the trivial motive. Beilinson [1] has offered several other conjectures on the properties of this filtration. For a more detailed discussion of this filtration and its connections to the theory of mixed motives, see Beilinson [1] or Saito [10].

In [10], Saito defines a filtration to be of Bloch-Beilinson (BB) type if it satisfies the following main properties:

(a) The filtration is respected by the action of every algebraic correspondence Γ.

(b) The induced action of Γ on each graded piece of Chow groups depends only on a certain Künneth component of the cohomology class of Γ.

(c) There exists an integer $N > 0$ (depending on X) such that $F^N_{\mathcal{M}} CH^r(X) = 0$.

We mention here that Saito goes on to prove that the filtration (4.1) satisfies conditions (a) and (b). Thereafter, Saito shows that $F^r CH^k(X) = F^{k+1} CH^k(X)$ for $r \geq k+1$ and introduces the group

$$D^k(X) := \bigcap_{l \geq 0} F^l CH^k(X). \tag{4.4}$$

The quotients $CH^k_F(X) := CH^k(X)/D^k(X)$ are much more tractable, and a number of important conjectures for Chow groups are shown to be true modulo the groups $D^k(X)$. Further, Saito demonstrates that, assuming that the standard conjectures, as well as the fact that the filtration $F_{\mathcal{M}}$ comes from the conjectural theory of mixed motives, the filtration (4.1) agrees with the required motivic filtration (4.2) for every smooth projective variety X up to a tensoring with $\mathbb{Q}$.

Our objective in this section will be to extend this filtration to bivariant Chow groups. We will see that there are two natural ways of doing this: one by extending Saito's original method in detail to the bivariant case and the other by simply using the factorization of bivariant classes from Lemma 3.2. Finally, we will show in Proposition 4.4 that the two possible filtrations actually agree with each other.

We start with the following proposition, which allows us to construct a more general pushforward for classes in bivariant Chow groups.

Proposition 4.1. *Let $f : X \longrightarrow Y$ be a morphism in Sm/K and let $V \in Sm/K$ be any variety of dimension d_V. Then, there exist morphisms*

$$\begin{aligned} p^k_V &: CH^k(1 \times f : V \times X \longrightarrow V \times Y) \longrightarrow CH^{k-d_V}(f : X \longrightarrow Y), \\ q^k_V &: H^k(1 \times f : V \times X \longrightarrow V \times Y) \longrightarrow H^{k-2d_V}(f : X \longrightarrow Y) \end{aligned} \tag{4.5}$$

for all $k \in \mathbb{Z}$. When $X = Y$ and $f = id$, the p^k_V and q^k_V are the respective pushforwards from $CH^k(V \times X)$ (resp. $H^k(V \times X)$) to $CH^{k-d_V}(X)$ (resp. $H^{k-2d_V}(X)$).

Proof. Consider a morphism $g : Y' \longrightarrow Y$ and let $f' : X' \longrightarrow Y'$ be a morphism in Sm/K fibred over $f : X \longrightarrow Y$ via g. Let $\pi_{Y'} : V \times Y' \longrightarrow Y'$ and $\pi_{X'} : V \times X' \longrightarrow X'$ be the coordinate projections. Let $d \in CH^k(1 \times f : V \times X \longrightarrow V \times Y)$ and let r' denote the relative dimension $\dim(V \times Y') - \dim(V \times X') = \dim(Y') - \dim(X')$. Since $1 \times f' : V \times X' \longrightarrow V \times Y'$ is fibred over $1 \times f : V \times X \longrightarrow V \times Y$, we have the maps

$$d^t_{(1 \times g)} : CH^t(V \times Y') \longrightarrow CH^{t-r'+k}(V \times X') \qquad \forall\ t \in \mathbb{Z}. \tag{4.6}$$

Then, we define the class $p_V^k(d) \in CH^{k-d_V}(X \xrightarrow{f} Y)$ by the morphisms (for each $t \in \mathbb{Z}$)

$$p_V^k(d)_g^t : CH^t(Y') \longrightarrow CH^{t-r'+k-d_V}(X'), \qquad \alpha \mapsto \pi_{X'*} \circ d^t_{(1\times g)}(\pi^*_{Y'}(\alpha)). \tag{4.7}$$

The map q_V^k on bivariant cohomologies is defined analogously.

When $X = Y$ and $f = id$, let d be a class in $CH^k(1 : V \times X \longrightarrow V \times X)$. Then d is represented by a class, say D, in $CH^k(V \times X)$ and let $\pi_X : V \times X \longrightarrow X$ denote the coordinate projection. From the above discussion, we have a map

$$p_V^k : CH^k(1 : V \times X \longrightarrow V \times X) \longrightarrow CH^{k-d_V}(1 : X \longrightarrow X) \tag{4.8}$$

and we have to check that $p_V^k(d) \in CH^{k-d_V}(1 : X \longrightarrow X)$ is represented by $\pi_{X*}(D)$ $\in CH^{k-d_V}(X)$.

Let $Z \in Sm/K$ and consider a morphism $g : Z \longrightarrow X$. Choose $z \in CH^l(Z)$. Let $\pi_Z : V \times Z \longrightarrow Z$ denote the coordinate projection and consider the morphism $1\times g : V \times Z \longrightarrow V \times X$. Then, $p_V^k(d) \in CH^{k-d_V}(X \xrightarrow{id} X)$ determines a morphism

$$p_V^k(d)_g^l : CH^l(Z) \longrightarrow CH^{l+k-d_V}, \quad z \mapsto \pi_{Z*}((1\times g)^*(D)\cdot \pi_Z^*(z)) \in CH^{l+k-d_V}(Z). \tag{4.9}$$

From the projection formula,

$$\pi_{Z*}((1\times g)^*(D)\cdot \pi_Z^*(z)) = \pi_{Z*}(1\times g)^*(D)\cdot z. \tag{4.10}$$

From the fibre square

$$\begin{array}{ccc} V\times Z & \xrightarrow{\pi_Z} & Z \\ {\scriptstyle 1\times g}\downarrow & & \downarrow{\scriptstyle g} \\ V\times X & \xrightarrow{\pi_X} & X \end{array} \tag{4.11}$$

we get $\pi_{Z*}(1\times g)^*(D) = g^*\pi_{X*}(D)$. Hence the map from $CH^l(Z)$ to $CH^{l+k-d_V}(Z)$ is induced by $\pi_{X*}(D)$. This proves the result. □

Let $f : X \longrightarrow Y$ be a morphism in Sm/K and suppose that $V \in Sm/K$ is a variety of dimension d_V. Let $p : V \times X \longrightarrow X$ be the coordinate projection. Suppose that $\Gamma \in CH^q(V \times X \xrightarrow{1\times f} V \times Y)$, where $k \leq q \leq k + d_V$. Define a morphism

$$T_\Gamma^k : CH^{k+d_V-q}(V) \longrightarrow CH^k(f : X \longrightarrow Y), \qquad \Gamma \mapsto p_V^{k+d_V}(p^*(x)\cdot \Gamma), \tag{4.12}$$

where the morphism $p_V^{k+d_V} : CH^{k+d_V}(1\times f : V\times X \longrightarrow V\times Y) \longrightarrow CH^k(f : X \longrightarrow Y)$ is as defined in Proposition 4.1.

For any $X \in Sm/K$, recall that the coniveau filtation on the cohomology groups of X is defined as

$$N^pH^i(X) = \sum_{Y\hookrightarrow X, \mathrm{codim}_X(Y)\geq p} \mathrm{Im}(H^i_Y(X) \longrightarrow H^i(X)), \tag{4.13}$$

where the sum is taken over all closed subschemes Y in X of codimension $\geq p$. Now, the original definition of Saito [11] of a filtration on the Chow groups $CH^k(X)$ for any variety X over $\mathbb{C}$ is as follows.

Definition 4.2. For any variety $X \in Sm/\mathbb{C}$, set $F^0CH^k(X) = CH^k(X)$ for all $k \in \mathbb{Z}$. For some $\nu \geq 0$, suppose that we have already defined $F^\nu CH^k(V)$ for all varieties V. Then define

$$F^{\nu+1}CH^k(X) = \sum_{V,q,\Gamma} \operatorname{Im}(\Gamma_* : F^\nu CH^{k+d_V-q}(V) \longrightarrow CH^k(X)), \tag{4.14}$$

where V, q and Γ vary over the following data:

(1) V is smooth, projective of dimension d_V over $\mathbb{C}$.

(2) q is an integer such that $k \leq q \leq k + d_V$.

(3) $\Gamma \in CH^q(V \times X)$ satisfies the condition

$$\gamma^{2k-\nu} \in H^{2q-2k+\nu}(V) \otimes N^{k-\nu+1}H^{2k-\nu}(X), \tag{4.15}$$

$\gamma^{2k-\nu}$ being the Künneth component of the cohomology class γ of $cl^q(\Gamma)$ in $H^{2q-2k+\nu}(V) \otimes H^{2k-\nu}(X)$. The set of all $\Gamma \in CH^q(V \times X)$ whose $(2k-\nu)$-th Künneth component satisfies the condition above is denoted by $L^{k,\nu}CH^q(V \times X)$.

Given a morphism $f : X \longrightarrow Y$, we will now give an analogous definition for $L^{k,\nu}CH^q(1 \times f : V \times X \longrightarrow V \times Y)$. Given a $\Gamma \in CH^q(1 \times f : V \times X \longrightarrow V \times Y)$, we can factor Γ uniquely as $\Gamma = t_\gamma \cdot [1 \times f]$, where $t_\gamma \in CH^{q-r}(V \times X)$, where $r = \dim(Y) - \dim(X)$ and $[1 \times f]$ is the orientation class of $1 \times f$. Denote the cycle class $cl(t_\gamma)$ of t_γ in $H^{2q-2r}(V \times X)$ by γ. We will say that

$$\begin{aligned} &\Gamma \in L^{k,\nu}CH^q(1 \times f : V \times X \longrightarrow V \times Y) \\ &\qquad \Leftrightarrow \ \gamma^{2k-2r-\nu} \in H^{2q-2k+\nu}(V) \otimes N^{k-r-\nu+1}H^{2k-2r-\nu}(X), \end{aligned} \tag{4.16}$$

$\gamma^{2k-2r-\nu}$ being the Künneth component of γ lying in $H^{2q-2k+\nu}(V) \otimes H^{2k-2r-\nu}(X)$. This obviously agrees with the definition of $L^{k,\nu}(V \times X)$ in the case $X = Y$ and $f = id$.

We will now extend this filtration to bivariant Chow groups by replacing the maps Γ_* in (4.14) with the maps T^k_Γ defined in (4.12).

Definition 4.3. Let $f : X \longrightarrow Y$ be a morphism in $Sm/\mathbb{C}$. Set $G^0CH^k(f : X \longrightarrow Y) = CH^k(f : X \longrightarrow Y)$ for every $k \in \mathbb{Z}$. Let $r = \dim(Y) - \dim(X)$.

Let $\nu \geq 0$ be an integer. Assume that we have already defined $G^\nu CH^s(g : V \longrightarrow W)$ for every morphism $g : V \longrightarrow W$ of smooth projective schemes V, W. Now define

$$G^{\nu+1}CH^k(f : X \longrightarrow Y) = \sum_{V,q,\Gamma} \operatorname{Im}(T^k_\Gamma : G^\nu CH^{k+d_V-q}(V) \longrightarrow CH^k(f : X \longrightarrow Y)). \tag{4.17}$$

Here V, q and Γ range over the following data:

(1) V is a smooth projective variety of dimension d_V over $\mathbb{C}$.

(2) q is an integer such that $k \leq q \leq k + d_V$.

(3) Γ is a class in $CH^q(1 \times f : V \times X \longrightarrow V \times Y)$ such that $\Gamma \in L^{k,\nu}CH^q(1 \times f : V \times X \longrightarrow V \times Y)$.

When $X = Y$ and $f = id$, this recovers the original definition due to Saito. It is not immediately clear that this is, in fact, a filtration. Note that we could define a filtration G^ν_1, $\nu \geq 0$ on $CH^k(f : X \longrightarrow Y)$ simply by requiring that

$$c \in G^\nu_1 CH^k(f : X \longrightarrow Y) \Leftrightarrow \ c = t \cdot [f] \text{ for some } t \in F^\nu CH^{k-r}(X), \tag{4.18}$$

where F^ν is the original filtration of Saito. We will now show that $G^\nu CH^k(f : X \longrightarrow Y) = G_1^\nu(f : X \longrightarrow Y)$. This will also establish that G^ν, $\nu \geq 0$ defines a filtration.

Proposition 4.4. *Let $f : X \longrightarrow Y$ be a morphism in $Sm/\mathbb{C}$. Then, the filtrations G^ν and G_1^ν on the bivariant Chow group $CH^k(f : X \longrightarrow Y)$ coincide.*

Proof. By definition, $G^0 CH^k(f : X \longrightarrow Y) = G_1^0 CH^k(f : X \longrightarrow Y)$ and hence the result holds for $\nu = 0$. Suppose that it is true up to $\nu = \nu_0$. Let $r = \dim(Y) - \dim(X)$. Then, any $\Gamma \in L^{k,\nu} CH^q(1 \times f : V \times X \longrightarrow V \times Y)$ factor as $\Gamma = t_\gamma \cdot [1 \times f]$, where $t_\gamma \in CH^{q-r}(V \times X)$. We see directly from (4.16) that

$$\Gamma \in L^{k,\nu} CH^q(1 \times f : V \times X \longrightarrow V \times Y) \;\Leftrightarrow\; t_\gamma \in L^{k-r,\nu} CH^{q-r}(X). \tag{4.19}$$

Therefore, in order to prove the result, it suffices to show that, for any class $\alpha \in CH^{k+d_V-q}(V)$,

$$T_\Gamma^k(\alpha) = (t_\gamma)_*(\alpha) \cdot [f].$$

Suppose that $f' : X' \longrightarrow Y'$ is fibred over f via $g : Y' \longrightarrow Y$. Then we have the fibre squares:

$$\begin{array}{ccccccccc}
X' & \xrightarrow{f'} & Y' & & V \times X' & \xrightarrow{p_{X'}} & X' & & V \times Y' & \xrightarrow{p_{Y'}} & Y' \\
{\scriptstyle g'}\big\downarrow & & \big\downarrow{\scriptstyle g} & & {\scriptstyle 1\times g'}\big\downarrow & & \big\downarrow{\scriptstyle g'} & & {\scriptstyle 1\times g}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
X & \xrightarrow{f} & Y, & & V \times X & \xrightarrow{p_X} & X, & & V \times Y & \xrightarrow{p_Y} & Y.
\end{array}$$

Further, let $p_V : V \times X \longrightarrow V$ be the projection. Choose a class $y' \in CH^l(Y')$. By definition, the bivariant class $T_\Gamma^k(\alpha) \in CH^k(f : X \longrightarrow Y)$ takes $y' \in CH^l(Y')$ to

$$(T_\Gamma^k(\alpha))_g^l(y') = p_{X'*}((1 \times g')^*(p_V^*(\alpha) \cdot t_\gamma) \cdot [1 \times f]_{1\times g}^l(p_{Y'}^*(y'))),$$

where p_V is the coordinate projection $p_V : V \times X \longrightarrow V$. Moreover, we have

$$[1 \times f]_{1\times g}^l(p_{Y'}^*(y')) = [1 \times f]_{1\times g}^l \circ [p_Y]_g^l(y') = [p_X]_{g'}^{l-r'+r}[f]_g^l(y') = p_{X'}^*([f]_g^l(y')), \tag{4.20}$$

where the latter equality follows from the fact that $(1 \times f) \circ p_Y = p_X \circ f$. It now follows that

$$\begin{aligned}
(T_\Gamma^k(\alpha))_g^l(y') &= p_{X'*}((1 \times g')^*((p_V^*(\alpha) \cdot t_\gamma) \cdot p_{X'}^*([f]_g^l(y'))) \\
&= (p_{X'*} \circ (1 \times g')^*)((p_V^*(\alpha) \cdot t_\gamma)) \cdot [f]_g^l(y') \\
&\qquad\qquad (\textit{by the projection formula}).
\end{aligned}$$

Finally, $p_{X'*} \circ (1 \times g')^* = g'^* \circ p_{X*}$ and hence

$$(T_\Gamma^k(\alpha))_g^l(y') = g'^* p_{X*}(p_V^*(\alpha) \cdot t_\gamma) \cdot [f]_g^l(y') = ((t_\gamma)_*(\alpha) \cdot [f])_g^l(y'). \tag{4.21}$$ $\square$

5. Higher bivariant Chow groups

For an algebraic scheme X, the Riemann-Roch theorem of Baum, Fulton and MacPherson applies and we have an isomorphism

$$G_0(X) \otimes \mathbb{Q} \xrightarrow{\cong} \bigoplus_{p \in \mathbb{Z}} CH^p(X) \otimes \mathbb{Q}, \tag{5.1}$$

where $G_0(X)$ denotes the Grothendieck group of the category of coherent sheaves on X. In [3], Bloch introduced the theory of higher Chow groups $CH^p(X,m)$, $p,m \geq 0$ and proved that they satisfy the following higher analogue of (5.1):

$$G_m(X) \otimes \mathbb{Q} \xrightarrow{\cong} \bigoplus_{p\in\mathbb{Z}} CH^p(X,m) \otimes \mathbb{Q} \qquad \forall m \geq 0, \tag{5.2}$$

where $G_m(X)$ denotes the m-th higher K-theory group of the category of coherent sheaves on X. For an equidimensional and quasi-projective scheme X and any $m \geq 0$, the higher Chow groups $CH^p(X,m)$ are defined as follows: let Δ^n be the usual n-simplex, given by

$$\Delta^n = Spec(K[t_0,t_1,...,t_n])/(\textstyle\sum_{i=0}^n t_i - 1) \cong \mathbb{A}^n_K \qquad n \geq 0. \tag{5.3}$$

For any $p,m \geq 0$, let $\mathcal{Z}^q(X,m)$ denote the free abelian group generated by cycles of codimension p in $X \times \Delta^n$ meeting all the faces $X \times \Delta^m$ for $m \leq n$ properly. The degeneracy maps $s_i : \Delta^n \longrightarrow \Delta^{n+1}$, $i = 0,1,...,n$ for each n, induce maps $s_{Xi} : X \times \Delta^n \longrightarrow X \times \Delta^{n+1}$, $i = 0,1,...,n$ and setting

$$\partial = \sum (-1)^i s^*_{Xi}, \qquad s^*_{Xi} : \mathcal{Z}^p(X,n+1) \longrightarrow \mathcal{Z}^p(X,n), \tag{5.4}$$

we have a complex $(\mathcal{Z}^p(X,*),\partial)$. Then, the higher Chow groups $CH^p(X,m)$ of X are defined by

$$CH^p(X,m) = H_m(\mathcal{Z}^p(X,*),\partial). \tag{5.5}$$

Bloch [3] shows that $CH^p(X,0) = CH^p(X)$ for every $p \geq 0$. If $f : X \longrightarrow Y$ is a flat morphism of schemes, then there exist pullbacks $f^* : CH^p(Y,m) \longrightarrow CH^p(X,m)$. If Y is smooth and either affine or projective (see [2, Proposition 2.5.1]), the pullback maps $f^* : CH^p(Y,m) \longrightarrow CH^p(X,m)$ always exist, regardless of whether f is flat. If $f : X \longrightarrow Y$ is proper, there exist pushforward maps $f_* : CH^p(X,m) \longrightarrow CH^{p+\dim(Y)-\dim(X)}(Y,m)$. For smooth schemes X, the higher Chow groups also carry a product

$$CH^p(X,m) \otimes CH^q(Y,n) \longrightarrow CH^{p+q}(X \times Y, m+n) \tag{5.6}$$

and hence, pulling back by the diagonal $X \xrightarrow{\Delta} X \times X$, we obtain a product

$$CH^p(X,m) \otimes CH^q(X,n) \longrightarrow CH^{p+q}(X,m+n). \tag{5.7}$$

For details on the construction and properties of higher Chow groups, see Bloch [3].

In this final section, we intend to introduce higher bivariant Chow groups $CH^p(f : X \longrightarrow Y, m)$, $m \geq 0$ for a morphism $f : X \longrightarrow Y$ in Sm/K. The formal definition will be given in Definition 5.3. First, we consider the properties that a definition higher bivariant Chow group should satisfy.

If $f : X \longrightarrow Y$ is a morphism in Sm/K, we understand that a class c in the higher bivariant Chow group $CH^p(f : X \longrightarrow Y, m)$ should be such that, given $f' : X' \longrightarrow Y'$ in Sm/K fibred over f via $g : Y' \longrightarrow Y$, for each $n \geq 0$, we obtain morphisms

$$c^k_g(n) : CH^k(Y',n) \longrightarrow CH^{k-r'+p}(X',m+n) \qquad \forall\ n,k \in \mathbb{Z} \tag{5.8}$$

(once again, $r' = \dim(Y') - \dim(X')$) satisfying certain natural compatibility conditions. For instance, consider a class $t \in CH^p(X,m)$ for some fixed m. Then, given any $g : X' \longrightarrow X$, we have a collection of morphisms $(n, p \in \mathbb{Z})$

(5.9)
$$t^q_g(n) : CH^q(X', n) \longrightarrow CH^{q+p}(X', m+n), \qquad x' \mapsto g^*(t)\cdot x' \ \forall\ x' \in CH^q(X', n),$$

which we see as a class in $CH^p(1 : X \longrightarrow X, m)$ corresponding to t. The expression (5.9) suggests that in the family (5.8) of morphisms defining a class $c \in CH^p(f : X \longrightarrow Y, m)$, the maps $c^k_g(n)$ must not only be compatible with proper pushforwards, flat pullbacks and refined Gysin morphisms, but, for different values of n, the maps $c^k_g(n)$ must also be related to each other.

Therefore, we will define a class c in the higher bivariant Chow group $CH^p(f : X \longrightarrow Y, m)$ to be a collection of maps that "raise the order of the Chow group from 0 to m":

$$c^k_g : CH^k(Y') = CH^k(Y', 0) \longrightarrow CH^{k-r'+p}(X', m) \tag{5.10}$$

for each morphism $f' : X' \longrightarrow Y'$ in Sm/K fibred over $f : X \longrightarrow Y$. We will then show that these classes can be used to define, in general, maps that "raise the order of the Chow group from n to $m+n$" for any $n \geq 0$; i.e., the maps in (5.10) can be used to define more general maps:

$$c^k_g(n) : CH^k(Y', n) \longrightarrow CH^{k-r'+p}(X', m+n). \tag{5.11}$$

Let $f : X \longrightarrow Y$ be a morphism in Sm/K. Let $f' : X' \longrightarrow Y'$ be a morphism in Sm/K fibred over $f : X \longrightarrow Y$ via $g : Y' \longrightarrow Y$. As in (3.2), the morphisms f and f' in Sm/K are also l.c.i. Let $d = \dim(Y) - \dim(X)$ and $d' = \dim(Y') - \dim(X')$ and let $[f]$ and $[f']$ denote the orientation classes of f and f' respectively. Then, we recall from (2.23) that

$$g^*[f] = c_e(E)\cdot [f'] \in CH^d(f : X \longrightarrow Y), \tag{5.12}$$

where $e = d - d'$ and E is the excess normal bundle. Here $c_e(E)$ denotes the e-th Chern class of the bundle E. We know that $c_e(E) \in CH^{d-d'}(X') = CH^e(X')$. In the following lemma, we construct "higher refined Gysin morphisms", which are key to our construction of higher bivariant Chow groups.

Lemma 5.1 (Higher refined Gysin morphisms)**.** *Let $i : X \longrightarrow Y$ be a regular imbedding in Sm/K of codimension d. Let $i' : X' \longrightarrow Y'$ be a morphism in Sm/K fibred over $i : X \longrightarrow Y$ via a morphism $g : Y' \longrightarrow Y$:*

$$\begin{array}{ccc} X' & \xrightarrow{i'} & Y' \\ {\scriptstyle g'}\downarrow & & {\scriptstyle g}\downarrow \\ X & \xrightarrow{i} & Y. \end{array} \tag{5.13}$$

Then, for any $m \geq 0$, there are higher refined Gysin morphisms

$$i^!_m : CH^k(Y', m) \longrightarrow CH^{k+d-d'}(X', m), \tag{5.14}$$

where $d' = \dim(Y') - \dim(X')$. For $m = 0$, these coincide with the usual refined Gysin morphisms.

Proof. It is clear that the closed immersion i' in Sm/K is a regular imbedding of codimension d'. Suppose that E denotes the excess normal bundle of the fibre square in (5.13) and $e = d - d'$. Then, from (5.12), we know that

$$g^*[i] = c_e(E)\cdot [i'], \tag{5.15}$$

where the Chern class $c_e(E) \in CH^e(X')$. We define the higher refined Gysin morphism by

$$i^!_m : CH^k(Y',m) \longrightarrow CH^{k+d-d'}(X',m), \qquad y' \mapsto c_e(E)\cdot i'^*(y'). \tag{5.16}$$

From the expression $g^*[i] = c_e(E)\cdot[i']$, it follows that the maps $i^!_0$ are identical to the refined Gysin maps $i^!$. □

The following proposition shows that a class in the bivariant Chow group $CH^p(f : X \longrightarrow Y)$ can be used to induce morphisms between Chow groups of any order $n \geq 0$. This is the statement of (5.11) for $m = 0$, since we will define the higher bivariant Chow groups in such a manner that $CH^p(f : X \longrightarrow Y, 0) = CH^p(f : X \longrightarrow Y)$.

Proposition 5.2. *Let $f : X \longrightarrow Y$ be a morphism in Sm/K. Let $f' : X' \longrightarrow Y'$ be a morphism in Sm/K fibred over $f : X \longrightarrow Y$ via the morphism $g : Y' \longrightarrow Y$, with $r' = \dim(Y') - \dim(X')$. Let c be a class in the bivariant Chow group $CH^p(f : X \longrightarrow Y)$. Then, for any $m \geq 0$, there exist morphisms*

$$c^k_g(m) : CH^k(Y',m) \longrightarrow CH^{k-r'+p}(X',m). \tag{5.17}$$

For $m = 0$, these are the usual maps $CH^k(Y') \longrightarrow CH^{k-r'+p}(X')$ that follow from the definition of c as a class in the bivariant Chow group $CH^p(f : X \longrightarrow Y)$.

Proof. As usual, we factor f as $X \xrightarrow{i_f} X \times Y \xrightarrow{p_Y} Y$ and consider the fibre squares

$$\begin{array}{ccccccc} X' & \xrightarrow{1} & X' & \xrightarrow{i'} & X \times Y' & \xrightarrow{p_{Y'}} & Y' \\ {\scriptstyle g'}\downarrow & & {\scriptstyle g'}\downarrow & & {\scriptstyle 1\times g}\downarrow & & {\scriptstyle f}\downarrow \\ X & \xrightarrow{1} & X & \xrightarrow{i_f} & X \times Y & \xrightarrow{p_Y} & Y. \end{array} \tag{5.18}$$

Let $r = \dim(Y) - \dim(X)$. Given $c \in CH^p(f : X \longrightarrow Y)$, factor c as $c = t\cdot[f] = t\cdot[i_f][p_Y]$, for some $t \in CH^{p-r}(X)$. Then the map $c^k_g(m) : CH^k(Y',m) \longrightarrow CH^{k-r'+p}(X',m)$ is defined by composing the following maps:

$$\begin{aligned} p^*_{Y'} : CH^k(Y',m) &\longrightarrow CH^k(X \times Y',m) && \text{(pullback)},\\ (i_f)^!_m CH^k(X\times Y',m) &\longrightarrow CH^{k+r-r'}(X',m) && \text{(by Lemma 5.1)},\\ g'^*(t)\cdot \_\_ : CH^{k+r-r'}(X',m) &\longrightarrow CH^{k-r'+p}(X',m) && \text{(by multiplication)}. \end{aligned}$$ □

It follows from Proposition 5.2 that the orientation class $[f] \in CH^r(f : X \longrightarrow Y)$, where $r = \dim(Y) - \dim(X)$, of a morphism $f : X \longrightarrow Y$ also induces maps

$$[f]^k_g(m) : CH^k(Y',m) \longrightarrow CH^{k-r'+p}(X',m) \tag{5.19}$$

for each morphism $f' : X' \longrightarrow Y'$ in Sm/K fibred over f via $g : Y' \longrightarrow Y$.

Finally, we come to the formal definition of higher bivariant Chow groups.

Definition 5.3. Let $f : X \longrightarrow Y$ be a morphism in Sm/K. Consider each morphism $f' : X' \longrightarrow Y'$ in Sm/K fibred over $f : X \longrightarrow Y$ via a morphism $g : Y' \longrightarrow Y$. For each such fibre square, a class c in the bivariant Chow group

$CH^p(X \xrightarrow{f} Y, n)$ gives a family of morphisms:

$$c_g^k : CH^k(Y', 0) \longrightarrow CH^{k-r'+p}(X', n), \tag{5.20}$$

where $r' = \dim(Y') - \dim(X')$, satisfying the following conditions:

(1) If $h : Y'' \longrightarrow Y'$ is proper and $g : Y' \longrightarrow Y$ is arbitrary, then, for the following diagram of fibre squares in Sm/K:

$$\begin{tikzcd} X'' \arrow[r, "f''"] \arrow[d, "h'"'] & Y'' \arrow[d, "h"] \\ X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y. \end{tikzcd} \tag{5.21}$$

Given $\alpha \in CH^k(Y'')$, we have

$$c_g^{k'}(h_*\alpha) = h'_* c_{gh}^k(\alpha), \qquad \text{where } k' = k + \dim(Y') - \dim(Y''). \tag{5.22}$$

(2) In the same diagram (5.21), if $h : Y'' \longrightarrow Y'$ is a flat morphism (instead of being proper as in condition (1)) and $g : Y' \longrightarrow Y$ is arbitrary, and we form the same diagram, then, for any $\alpha \in CH^k(Y')$,

$$c_{gh}^k(h^*\alpha) = h^* c_g^k(\alpha). \tag{5.23}$$

(3) If $g : Y' \longrightarrow Y$ and $h : Y' \longrightarrow Z'$ are morphisms, $i : Z'' \longrightarrow Z'$ is a regular imbedding in Sm/K of codimension e and we have the fibre diagram in Sm/K:

$$\begin{tikzcd} X'' \arrow[r, "f''"] \arrow[d, "i''"'] & Y'' \arrow[r, "h'"] \arrow[d, "i'"'] & Z'' \arrow[d, "i"'] \\ X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[r, "h"] \arrow[d, "g"'] & Z' \\ X \arrow[r, "f"] & Y, & \end{tikzcd} \tag{5.24}$$

then, for any $\alpha \in CH^k(Y')$ and $k \in \mathbb{Z}$,

$$i_n^! c_g^k(\alpha) = c_{gi'}^{k+e-(y'-y'')}(i^!\alpha) \qquad (\text{where } \dim(Y') = y' \text{ and } \dim(Y'') = y''). \tag{5.25}$$

If we set $n = 0$ in Definition 5.3 above, we recover Definition 2.1 and hence it follows that for any morphism $f : X \longrightarrow Y$, we have

$$CH^p(f : X \longrightarrow Y, 0) \xrightarrow{\approx} CH^p(f : X \longrightarrow Y). \tag{5.26}$$

For instance, given a line bundle $\mathcal{M}$ on S^1, for any vector bundle E of rank $e+1$ on a scheme X in Sm/K, we can construct higher bivariant classes $s_i^{\mathcal{M}}(E) \in CH^i(1 : X \longrightarrow X, e+i)$, $i \in \mathbb{Z}$, by modifying the definition of the usual Segre classes $s_i(E)$. Here S^1 is defined as follows: let Δ^1 denote the 1-simplex as in (5.3) and let $\partial\Delta = \bigcup_{i=0}^1 \Delta^0$. Then, following Bloch [3], we set $S^1 = \Delta^1 \bigcup_{\partial\Delta} \Delta^1$. Then, we choose and fix a line bundle $\mathcal{M}$ over S^1. We mention here that the Picard group $Pic(S^1) \cong CH^1(Spec(K), 1) \cong K^*$ (see [3]) is not trivial, K^* being the multiplicative group of units in the field K.

Let E be a vector bundle of rank $e+1$ on a scheme X and let $p_E : \mathbb{P}(E) \longrightarrow X$ be the projectivized bundle of E. Consider the line bundle $\mathcal{O}(1)$ on $\mathbb{P}(E)$. Then, we denote by $\mathcal{M}_E$ the line bundle $\mathcal{M}_E = p_1^*(\mathcal{O}(1)) \otimes p_2^*(\mathcal{M})$ on $\mathbb{P}(E) \times S^1$, $p_1 : \mathbb{P}(E) \times S^1 \longrightarrow \mathbb{P}(E)$ and $p_2 : \mathbb{P}(E) \times S^1 \longrightarrow S^1$ being the two coordinate projections. Then, $\mathcal{M}_E$ induces a class in $Pic(\mathbb{P}(E) \times S^1)/Pic(\mathbb{P}(E))$, which we denote by $[\mathcal{M}_E]$. From [3, §6], it follows that we have isomorphisms

$$Pic(\mathbb{P}(E) \times S^1)/Pic(\mathbb{P}(E)) \cong CH^1(\mathbb{P}(E), 1). \tag{5.27}$$

Let $c_E \in CH^1(\mathbb{P}(E), 1)$ denote the class corresponding to $[\mathcal{M}_E] \in Pic(\mathbb{P}(E) \times S^1)/Pic(\mathbb{P}(E))$ by (5.27). Now suppose that we have a morphism $g : Y \longrightarrow X$ in Sm/K and we form the fibre square

$$\begin{array}{ccc} \mathbb{P}(E) \times_X Y & \xrightarrow{p'_E} & Y \\ {\scriptstyle g'}\downarrow & & {\scriptstyle g}\downarrow \\ \mathbb{P}(E) & \xrightarrow{p_E} & X. \end{array} \tag{5.28}$$

Then, for any i, we can define the higher bivariant class $s_i^{\mathcal{M}}(E) \in CH^i(1 : X \longrightarrow X, e+i)$ by the family of morphisms:

$$s_i^{\mathcal{M}}(E)_g^k : CH^k(Y, 0) \longrightarrow CH^{k+i}(Y, e+i), \qquad \alpha \mapsto p'_{E*}(g'^*(c_E^{e+i}) \cdot p'^*_E(\alpha)), \tag{5.29}$$

where c_E^{e+i} denotes the class in $CH^{e+i}(\mathbb{P}(E), e+i)$ obtained by multiplying $c_E \in CH^1(\mathbb{P}(E), 1)$ $(e+i)$-times.

In general, given a morphism $f : X \longrightarrow Y$ in Sm/K, it is clear that if we have $c \in CH^p(f : X \longrightarrow Y, n)$, $d_1 \in CH^{q_1}(g_1 : X_1 \longrightarrow X)$ and $d_2 \in CH^{q_2}(g_2 : Y \longrightarrow Y_2)$ and the fibre diagram

$$\begin{array}{ccccccc} X'_1 & \xrightarrow{g'_1} & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'_2} & Y'_2 \\ {\scriptstyle h'''}\downarrow & & {\scriptstyle h''}\downarrow & & {\scriptstyle h'}\downarrow & & {\scriptstyle h}\downarrow \\ X_1 & \xrightarrow{g_1} & X & \xrightarrow{f} & Y & \xrightarrow{g_2} & Y_2 \end{array} \tag{5.30}$$

we can form products $c \cdot d_2$ and $d_1 \cdot c$ by the morphisms:

(5.31)

$$\begin{aligned} & c \cdot d_2 \in CH^{p+q_2}(g_2 \circ f : X \longrightarrow Y_2, n), \\ & \qquad (c \cdot d_2)_h^k(y'_2) := c_{h'}^{k-\dim(Y'_2)+\dim(Y)+q_2}((d_2)_h^k(y'_2)) \quad \forall y'_2 \in CH^k(Y'_2), \\ & d_1 \cdot c \in CH^{q_1+p}(f \circ g_1 : X_1 \longrightarrow Y, n), \\ & \qquad (d_1 \cdot c)_{h'}^k := (d_1)_{h''}^{k-\dim(Y')+\dim(X')+p}(n)(c_{h'}^k(y')) \quad \forall y' \in CH^k(Y') \end{aligned}$$

for each $k \in \mathbb{Z}$ and check that

$$d_1 \cdot c \cdot d_2 = (d_1 \cdot c) \cdot d_2 = d_1 \cdot (c \cdot d_2) \in CH^{q_1+p+q_2}(g_2 \circ f \circ g_1 : X_1 \longrightarrow Y_2, n). \tag{5.32}$$

Also, if we have fibre squares

$$\begin{array}{ccccc} X' & \xrightarrow{f'_1} & Y' & \xrightarrow{f'_2} & Z' \\ {\scriptstyle g''}\downarrow & & {\scriptstyle g'}\downarrow & & {\scriptstyle g}\downarrow \\ X & \xrightarrow{f_1} & Y & \xrightarrow{f_2} & Z, \end{array} \tag{5.33}$$

then, given $c \in CH^p(f_1 : X \longrightarrow Y, n)$ and $d \in CH^q(f_2 : Y \longrightarrow Z)$, the pullback $g^*(c \cdot d) \in CH^{p+q}(f_2 \circ f_1 : X' \longrightarrow Z', n)$ is defined in the obvious manner and, furthermore,

$$g^*(c \cdot d) = g'^*(c) \cdot g^*(d). \tag{5.34}$$

However, in order to define a more general product

$$CH^p(f : X \longrightarrow Y, m) \otimes CH^q(g : Y \longrightarrow Z, n) \longrightarrow CH^{p+q}(gf : X \longrightarrow Z, m+n), \tag{5.35}$$

we must show that any element of $CH^p(f : X \longrightarrow Y, m)$ induces maps from $CH^k(Y', n)$ to $CH^{k-r'+p}(X', n+m)$ for any integers n and k and any morphism $f' : X' \longrightarrow Y'$ in Sm/K fibred over $f : X \longrightarrow Y$ with $r' = \dim(Y') - \dim(X')$. When $m = 0$, this is already Proposition 5.2. To prove this for any $m \geq 0$, we have to start with a factorization of a class in $CH^p(f : X \longrightarrow Y, n)$ as a product of a class in $CH^{p-\dim(Y)+\dim(X)}(1 : X \longrightarrow X, n)$ and the orientation class $[f] \in CH^{\dim(Y)-\dim(X)}(f : X \longrightarrow Y, 0)$ of f as in Lemma 3.2 . We now proceed to prove the following, which is analogous to [6, Proposition 17.4.2]:

Proposition 5.4. *Let $g : Y \longrightarrow Z$ be a smooth and projective morphism in Sm/K of relative dimension $\dim(Z) - \dim(Y) = -d$ and let $[g] \in CH^{-d}(g : Y \longrightarrow Z)$ denote its orientation class. Then, for any morphism $f : X \longrightarrow Y$ in Sm/K, any integer p and any $n \geq 0$, we have an isomorphism*

$$CH^p(f : X \longrightarrow Y, n) \underset{\cong}{\xrightarrow{\cdot [g]}} CH^{p-d}(g \circ f : X \longrightarrow Z, n). \tag{5.36}$$

Proof. Consider the fibre diagram:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & & \\ \gamma\downarrow & & \delta\downarrow & & \\ X \times_Z Y & \xrightarrow{f'} & Y \times_Z Y & \xrightarrow{q} & Y \\ p'\downarrow & & p\downarrow & & g\downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z. \end{array} \tag{5.37}$$

In the diagram above, δ is the diagonal imbedding and p and q are the projections on the first and second coordinates respectively. We note that δ and γ are both regular imbeddings of the same codimension d. Since g is smooth and projective, it is clear that all schemes in the diagram above lie in Sm/K. Further, since g is flat, it also follows, from (2.23), that $g^*[g] = [q]$. Define the inverse homomorphism

$$L : CH^{p-d}(gf : X \longrightarrow Z, n) \longrightarrow CH^p(f : X \longrightarrow Y, n) \tag{5.38}$$

by $L(c') = [\gamma] \cdot g^*(c')$ for $c' \in CH^{p-d}(gf : X \longrightarrow Z, n)$. For any $c' \in CH^{p-d}(gf : X \longrightarrow Z, n)$, we check that

$$\begin{aligned} L(c') \cdot [g] &= [\gamma] \cdot g^*(c') \cdot [g] \\ &= [\gamma] \cdot (g^*(c') \cdot [g]) \\ &= [\gamma] \cdot ([p'] \cdot c') \qquad \text{(using (5.23))} \\ &= [p' \circ \gamma] \cdot c' \\ &= [id] \cdot c' = c'. \end{aligned}$$

For any $c \in CH^p(f : X \longrightarrow Y, n)$, we check that

$$\begin{aligned} L(c \cdot [g]) &= [\gamma] \cdot g^*(c \cdot [g]) \\ &= [\gamma] \cdot p^*(c) \cdot g^*[g] \qquad \text{(using (5.34))} \\ &= (p \circ \delta)^*(c) \cdot [\delta] \cdot [q] \quad \text{(using (5.25))} \\ &= c \cdot [q \circ \delta] = c. \end{aligned}$$

□

Proposition 5.5. *(1) Let $i : X \longrightarrow Y$ be a regular imbedding in Sm/K of codimension d and let c be a class in the bivariant higher Chow group $CH^p(i : X \longrightarrow Y, n)$ for some integer $n \geq 0$. Then, there exists a class $t \in CH^{p-d}(1 : X \longrightarrow X, n)$ such that $c = t \cdot [i]$, where $[i] \in CH^d(i : X \longrightarrow Y, 0)$ is the orientation class of i.*

(2) Let $f : X \longrightarrow Y$ be a morphism in Sm/K and suppose that c is a class in $CH^p(f : X \longrightarrow Y, n)$ for some integer $n \geq 0$. Then, if $r = \dim(Y) - \dim(X)$, there exists a class $c' \in CH^{p-r}(1 : X \longrightarrow X, n)$ such that $c = c' \cdot [f]$, where $[f] \in CH^{p-r}(f : X \longrightarrow Y, 0)$ is the orientation class of f.

Proof. (1) We proceed as in the proof of Lemma 3.1. We have the regular imbedding $i : X \longrightarrow Y$ of codimension $d = \dim(Y) - \dim(X)$ and the class $c \in CH^p(i : X \longrightarrow Y, n)$. Let $[p_Y] \in CH^{-\dim(Y)}(p_Y : Y \longrightarrow Spec(K), 0)$ be the orientation class of the structure morphism $p_Y : Y \longrightarrow Spec(K)$ and consider the product $c \cdot [p_Y] \in CH^{p-\dim(Y)}(p_X : X \longrightarrow Spec(K), n)$. From Proposition 5.4 above, it follows that multiplication by $[p_X]$ induces an isomorphism

$$CH^{p-d}(1 : X \longrightarrow X, n) \xrightarrow[\cong]{\cdot [p_X]} CH^{p-\dim(Y)}(p_X : X \longrightarrow Spec(K), n). \tag{5.39}$$

Hence, there exists $t \in CH^{p-d}(1 : X \longrightarrow X, n)$ such that $c \cdot [p_Y] = t \cdot [p_X]$. But $[p_X] = [i] \cdot [p_Y]$ and again we know from Proposition 5.4 that multiplication by $[p_Y]$ induces an isomorphism

$$CH^p(i : X \longrightarrow Y, n) \xrightarrow[\cong]{\cdot [p_Y]} CH^{p-\dim(Y)}(p_X : X \longrightarrow Spec(K), n). \tag{5.40}$$

Since $c \cdot [p_Y] = t \cdot [i] \cdot [p_Y]$, we have $c = t \cdot [i]$.

(2) Again, the proof of (2) is the same as the proof of Lemma 3.2. □

Given a scheme X and some $n \geq 0$, any class $t \in CH^p(X, n)$ induces a class in $CH^p(1 : X \longrightarrow X, n)$ as mentioned in (5.9). We will now check that for each scheme $X \in Sm/K$, there are natural isomorphisms

$$CH^p(X, n) \xrightarrow{\cong} CH^p(1 : X \longrightarrow X, n) \qquad \forall\ p \in \mathbb{Z}, n \geq 0. \tag{5.41}$$

Proposition 5.6. *Let $X \in Sm/K$. Then, there exist natural isomorphisms*

$$CH^p(X, n) \xrightarrow{\cong} CH^p(1 : X \longrightarrow X, n) \qquad \forall\ p \in \mathbb{Z}, n \geq 0. \tag{5.42}$$

Proof. A given class $c \in CH^p(1 : X \longrightarrow X, n)$ induces a map

$$c_1^0 : CH^0(X, 0) \longrightarrow CH^p(X, n). \tag{5.43}$$

Hence, we can define a morphism

$$F : CH^p(1 : X \longrightarrow X, n) \longrightarrow CH^p(X, n), \qquad c \mapsto c_1^0([X]), \tag{5.44}$$

where $[X] \in CH^0(X, 0)$ is the class of X. Conversely, given a class $t \in CH^p(X, n)$, we can multiply by pullbacks of this class to give maps (as already explained in (5.9))

$$CH^k(X', 0) \longrightarrow CH^{k+p}(X', n), \qquad x' \mapsto x' \cdot g^*(t) \ \forall x' \in CH^k(X', 0) \tag{5.45}$$

for any $g : X' \longrightarrow X$ in Sm/K, thus defining a class in $CH^p(1 : X \longrightarrow X, n)$. Let us denote this class by $G(t)$. This gives us a map

$$G : CH^p(X, n) \longrightarrow CH^p(1 : X \longrightarrow X, n). \tag{5.46}$$

It is clear that $(F \circ G)(t) = t$ for any $t \in CH^p(X, n)$. On the other hand, consider $F(c) \in CH^p(X, n)$ for some $c \in CH^p(1 : X \longrightarrow X, n)$ and choose some morphism $g : X' \longrightarrow X$ in Sm/K. Then, we have morphisms

$$c_g^k : CH^k(X', 0) \longrightarrow CH^{k+p}(X', n) \tag{5.47}$$

for each $k \in \mathbb{Z}$. The group $CH^k(X', 0)$ is generated by classes $[Y']$, where Y' is a subscheme of codimension k in X'. We choose a subscheme Y' of codimension k in X' and let $i' : Y' \longrightarrow X'$ be the inclusion. Since resolution of singularities holds over the ground field K, there exists a smooth variety Y'' and projective birational morphism $p'' : Y'' \longrightarrow Y'$. Since X, X' and Y'' are all smooth, it follows that $i'p''$ and g are both l.c.i. morphisms as mentioned in (3.2). In what follows, we shall, by abuse of notation, use $(i'p'')^!$, $g^!$ and $(gi'p'')^!$ to denote the morphisms induced by the orientation classes of the l.c.i. morphisms $i'p''$, g and $gi'p''$ respectively. Since the higher bivariant classes are compatible with flat pullbacks and higher refined Gysin morphisms induced by regular imbeddings, it follows that they are compatible with $(i'p'')^!$, $g^!$ and $(gi'p'')^!$. Then, we have:

$$\begin{aligned} c_g^k([Y']) &= c_g^k((i'p'')_*(gi'p'')^!([X])) \\ &= (i'p'')_* c^0_{gi'p''}((gi'p'')^!([X])) \\ &= (i'p'')_*(gi'p'')^! c_1^0([X]) = (i'p'')_*(gi'p'')^!(F(c)) \\ &= (i'p'')_*(gi'p'')^!([X] \cdot F(c)) \\ &= (i'p'')_*((gi'p'')^!([X]) \cdot (gi'p'')^*(F(c))) \\ &= (i'p'')_*([Y''] \cdot (i'p'')^*(g^*(F(c)))) = [Y'] \cdot g^*(F(c)). \end{aligned} \tag{5.48}$$

Hence, the bivariant higher Chow class given by multiplication with pullbacks of $F(c)$ is identical to the original class c. Hence, $(G \circ F)(c) = c$. This completes the proof. □

Proposition 5.7. *Let $f : X \longrightarrow Y$ be a morphism in Sm/K and suppose that c is a class in $CH^p(f : X \longrightarrow Y, n)$ for some integer $n \geq 0$. Suppose that $f' : X' \longrightarrow Y'$ is a morphism in Sm/K fibred over $f : X \longrightarrow Y$ via a morphism $g : Y' \longrightarrow Y$. Let $r' = \dim(Y') - \dim(X')$. Then, for any $m \geq 0$, there exist maps*

$$c_g^k(m) : CH^k(Y', m) \longrightarrow CH^{k-r'+p}(X', m+n). \tag{5.49}$$

Proof. From Proposition 5.5(2), we know that there exists a class $c' \in CH^{p-r}(1 : X \longrightarrow X, n)$ such that $c = c' \cdot [f]$, where $[f] \in CH^{p-r}(f : X \longrightarrow Y, 0)$ is the orientation class of f. From Proposition 5.6, we know that $c' \in CH^{p-r}(1 : X \longrightarrow X, n)$ corresponds to a class t in $CH^{p-r}(X, n)$. Again, we factor f as $X \xrightarrow{i_f}$

$X \times Y \xrightarrow{p_Y} Y$ and consider the fibre squares

$$\begin{array}{ccccccc} X' & \xrightarrow{1} & X' & \xrightarrow{i'} & X \times Y' & \xrightarrow{p_{Y'}} & Y' \\ {\scriptstyle g'}\downarrow & & {\scriptstyle g'}\downarrow & & {\scriptstyle 1\times g}\downarrow & & {\scriptstyle f}\downarrow \\ X & \xrightarrow{1} & X & \xrightarrow{i_f} & X \times Y & \xrightarrow{p_Y} & Y. \end{array} \tag{5.50}$$

Then the map $c_g^k(m) : CH^k(Y', m) \longrightarrow CH^{k-r'+p}(X', m+n)$ is defined by composing the following maps:

$$\begin{aligned} p_{Y'}^* : CH^k(Y', m) &\longrightarrow CH^k(X \times Y', m) \qquad \text{(pullback)}, \\ (i_f)^!_m : CH^k(X \times Y', m) &\longrightarrow CH^{k+r-r'}(X', m) \qquad \text{(by Lemma 5.1)}, \\ g'^*(t) \cdot \_\_ : CH^{k+r-r'}(X', m) &\longrightarrow CH^{k-r'+p}(X', m+n) \qquad \text{(by multiplication)}. \end{aligned}$$

□

Finally, we can construct the product of higher bivariant Chow groups.

Corollary 5.8. *Let f and g be composable morphisms of schemes $X \xrightarrow{f} Y \xrightarrow{g} Z$ in Sm/K. Then, for any integers $m, n, p, q \geq 0$, there exists a product*

$$CH^p(f : X \longrightarrow Y, m) \otimes CH^q(g : Y \longrightarrow Z, n) \longrightarrow CH^{p+q}(g \circ f : X \longrightarrow Z, m+n). \tag{5.51}$$

Proof. Let $c \in CH^p(f : X \longrightarrow Y, m)$ and $d \in CH^q(g : Y \longrightarrow Z, n)$ and let the squares in the following diagram be Cartesian:

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \\ {\scriptstyle h''}\downarrow & & {\scriptstyle h'}\downarrow & & {\scriptstyle h}\downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z. \end{array} \tag{5.52}$$

Then the class $(c \cdot d) \in CH^{p+q}(g \circ f : X \longrightarrow Z, m+n)$ is defined by composing the following series of maps:

$$\begin{aligned} & d_h^k : CH^k(Z', 0) \longrightarrow CH^{k-\dim(Y')+\dim(Z')+q}(Y', n), \\ & c_{h'}^{k-\dim(Y')+\dim(Z')+q}(n) : CH^{k-\dim(Y')+\dim(Z')+q}(Y', n) \\ & \qquad \longrightarrow CH^{k-\dim(X')+\dim(Z')+q+p}(X', m+n). \end{aligned}$$

□

Remark 5.9. We note that our definition of "higher bivariant Chow groups" $CH^p(f : X \longrightarrow Y, n)$ only incorporates the situation in which $n \geq 0$. The case $n < 0$ has not been treated; this is because it is not clear whether such a group would be nonempty.

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