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CONSTRUCTIONS FOR INFINITESIMAL GROUP SCHEMES

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Abstract. Let G be an infinitesimal group scheme over a field k of characteristic $p > 0$. We introduce the global p-nilpotent operator $\Theta_G : k[G] \to k[V(G)]$, where $V(G)$ is the scheme which represents 1-parameter subgroups of G. This operator Θ_G applied to M encodes the local Jordan type of M and leads to computational insights into the representation theory of G. For certain kG-modules M (including those of constant Jordan type), we employ Θ_G to associate various algebraic vector bundles on $\mathbb{P}(G)$, the projectivization of $V(G)$. These vector bundles not only distinguish certain representations with the same local Jordan type, but also provide a method of constructing algebraic vector bundles on $\mathbb{P}(G)$.

Contents

0. Introduction

In [25], [26], the foundations of a theory of support varieties were established for an infinitesimal group scheme G over a field k of characteristic $p > 0$, extending earlier work for elementary abelian p-groups and p-restricted finite dimensional Lie algebras ([6], [13]). These foundations relied upon cohomological calculations to identify cohomological support varieties and introduced 1-parameter subgroups to provide an alternate, representation-theoretic perspective. In contrast to the situation for finite groups, the cohomological variety for G infinitesimal is of considerable geometric complexity; partly for this reason, computations of explicit examples are

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challenging. In this present paper, we build upon this earlier work as well as more recent work of the authors ([15], [17]) to initiate a more detailed investigation of representations of G. Although representations of infinitesimal group schemes are less familiar than representations of finite groups, their importance is evident: for example, the representation theories of the family of all infinitesimal kernels $G = \mathfrak{G}_{(r)}$ of a smooth connected algebraic group $\mathfrak{G}$ is essentially equivalent to the rational representation theory of $\mathfrak{G}$.

An important structure associated to an infinitesimal group scheme G of height $\leq r$ is the scheme $V(G)$ of 1-parameter subgroups $\mathbb{G}_{a(r)} \to G$ (see [25]). In this paper we observe that the representability of $V(G)$ leads to a p-nilpotent element Θ_G in $kG \otimes k[V(G)]$, where kG is the group algebra of G. For any kG-module M, Θ_G determines a *global p-nilpotent operator* on M. The operator Θ_G encodes the local Jordan type of a kG-module M which in turn determines the support variety of M. Even though the scheme $V(G)$ was generalized to all finite group schemes in [15], [17] via the notion of π-points, the construction of Θ_G does not appear to extend to arbitrary finite groups.

The homogeneity of our global operator Θ_G enables us to associate to a kG-module M of constant Jordan type a collection of vector bundles on $\mathbb{P}(G) = \operatorname{Proj} k[V(G)]$, which we view as a family of global invariants of M. Certain modules with the same local Jordan type can be distinguished by these global invariants.

The vector bundles that we investigate are constructed directly and explicitly from kG-modules. Hence, we offer an important new method to create interesting examples of algebraic vector bundles on varieties of the form $\mathbb{P}(G)$. Varieties of this form include projective spaces, weighted projective spaces, and various singular varieties associated to algebraic groups. For example, for $G = \mathrm{GL}_{n(r)}$, $\mathbb{P}(G)$ is the projectivization of the variety of r-tuples of pairwise commuting p-nilpotent matrices. We expect that our technique of constructing algebraic vector bundles on such singular, geometrically interesting varieties will lead to insights into their algebraic K-theory.

The reader might find it instructive to contrast our use of representations of G to construct vector bundles on $\mathbb{P}(G)$ with the Borel-Weil construction which employs bundles on flag varieties for an algebraic group $\mathfrak{G}$ to construct representations of $\mathfrak{G}$. Our construction of vector bundles plays a role in the forthcoming papers by the first author, Jon Carlson, and Andrei Suslin [9], and by the second author and David Benson [5].

In this paper, we also attempt to address the lack of specific examples in the representation theory of infinitesimal groups schemes (other than those of height 1). Throughout this paper, we work with the following four fundamental, yet concrete, classes of examples.

i) p-restricted Lie algebras, $\mathfrak{g}$;

ii) infinitesimal additive group schemes, $\mathbb{G}_{a(r)}$;

iii) infinitesimal general linear groups, $\mathrm{GL}_{n(r)}$;

iv) the height 2, infinitesimal special linear group, $\mathrm{SL}_{2(2)}$.

We consistently endeavor to make our general results more concrete by applying them to our examples.

In Section 1, we recall some of the highlights from [25], [26] concerning the cohomology and theory of supports of finite dimensional kG-modules for an infinitesimal group scheme G. A key result summarized in Theorem 1.16 is the close relationship between the spectrum $\operatorname{Spec} \mathrm{H}^\bullet(G,k)$ of the cohomology of G and the scheme $V(G)$ representing (infinitesimal) 1-parameter subgroups of an infinitesimal group scheme G.

In the second section, we define the global p-nilpotent operator $\Theta_G : k[G] \longrightarrow k[V(G)]$ for an infinitesimal group scheme G. For any finite dimensional kG-module M, Θ_G determines a p-nilpotent endomorphism of the free $k[V(G)]$-module $M \otimes k[V(G)]$. We establish in Proposition 2.11 that Θ_G is homogeneous, where $k[V(G)]$ is equipped with its natural grading. We also verify that Θ_G is natural with respect to change of group.

In the third section, we verify that specializations θ_v of Θ_G at points $v \in V(G)$ determine the local Jordan type of a finite dimensional kG-module M. Theorem 3.7 can be viewed as providing an algorithm for obtaining the local Jordan type in terms of the representation $G \to \mathrm{GL}_N$ defining the kG-module M. We utilize Θ_G and its specializations to establish constraints for a kG-module M to be of constant rank (and thus of constant Jordan type). We also establish the relationship between the local Jordan type of a module and its Frobenius twists.

We envision that some of our constructions for infinitesimal group schemes may lead to analogues for a general finite group scheme. With this in mind, we begin the fourth section with a dictionary between 1-parameter subgroups for infinitesimal group schemes and π-points for general finite group schemes. Given a finite dimensional kG-module M, we consider the projectivization of the operator Θ_G,

$$\widetilde{\Theta}_G : M \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})} \to M \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}(p^{r-1}),$$

a p-nilpotent operator on the free, coherent sheaf $M \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}$ on $\mathbb{P}(\mathrm{G})$. We verify in Proposition 4.8 that $\widetilde{\Theta}_G$ determines via base change the local Jordan type of a kG-module M at any 1-parameter subgroup $\mu_v : \mathbb{G}_{a(r),k(v)} \to G_{k(v)}$. Theorem 4.13 shows that the condition that M be of constant j-rank is equivalent to the condition that the coherent sheaf $\operatorname{Im} \widetilde{\Theta}_G^j$ be locally free.

In the fifth section, we initiate an investigation of the algebraic vector bundles $\operatorname{Ker}\{\widetilde{\Theta}_G^j, \mathcal{M}\}$, $\operatorname{Im}\{\widetilde{\Theta}_G^j, \mathcal{M}\}$ on $\mathbb{P}(\mathrm{G})$ associated to kG-modules of constant Jordan type and more generally of constant j-rank. We give examples of such kG-modules in each of our four representative examples and investigate the associated vector bundles. As we see, taking kernels of powers of the global p-nilpotent power operator sends modules of constant Jordan type to vector bundles. We also obtain vector bundles by taking kernels modulo images (as inspired by a construction of M. Duflo and V. Serganova for Lie superalgebras in [11]). As an application, we prove in Proposition 5.17 a geometric characterization of endotrivial modules.

Finally, in the last section, we provide numerous explicit examples. These include the infinitesimal group scheme $G = \mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$, which has the same representation theory as the elementary abelian p-group $\mathbb{Z}/p \times \mathbb{Z}/p$, as well as the first Frobenius kernel of the reductive group SL_2. One intriguing comparison which we investigate in particularly simple examples is the relationship between the Grothendieck group of projective kG-modules and the Grothendieck group of algebraic vector bundles on $\mathbb{P}(G)$. Combined with our explicit calculations, Proposition 6.12 can be viewed both as a means to distinguish certain non-isomorphic projective kG-modules and as a means of constructing non-isomorphic algebraic vector bundles on $\mathbb{P}(\mathrm{G})$.

Throughout, k will denote an arbitrary field of characteristic $p > 0$. Unless explicit mention is made to the contrary, G will denote an infinitesimal group scheme over k. If M is a kG-module and K/k is a field extension, then we denote by M_K the KG-module obtained by base extension.

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1. Infinitesimal group schemes

The purpose of this first section is to summarize the important role played by (infinitesimal) 1-parameter subgroups of an infinitesimal group scheme as presented in [25]. The four representative examples of Example 1.5, ($\underline{\mathfrak{g}}$, $\mathbb{G}_{a(r)}$, $\mathrm{GL}_{n(r)}$, $\mathrm{SL}_{2(2)}$), and their associated schemes of 1-parameter subgroups discussed in Example 1.12 will serve as explicit models to which we will frequently return.

Definition 1.1. A finite group scheme G over k is a group scheme over k whose coordinate algebra $k[G]$ is finite dimensional over k.

Equivalently, G is a functor from commutative k-algebras to groups, $R \mapsto G(R)$, represented by a finite dimensional commutative k-algebra, the coordinate algebra $k[G]$ of G.

Associated to G, we have its group algebra $kG = \mathrm{Hom}_k(k[G], k)$; more generally, for any commutative k-algebra R, we have the R-group algebra $RG = \mathrm{Hom}_k(k[G], R)$.

Notation 1.2. If $f : G \to H$ is a map of finite group schemes, we denote by

$$f^* : k[H] \to k[G] \quad \text{ and } \quad f_* : kG \to kH$$

the induced maps on coordinate and group algebras respectively.

Observe that the R-group algebra of G consists of all k-linear homomorphisms, whereas $G(R) = \mathrm{Hom}_{k-alg}(k[G], R)$ is the subgroup of $RG^\times$ consisting of k-algebra homomorphisms.

Definition 1.3. Let G be a finite group scheme over k and M a k-vector space. Then a (left) kG-module structure on M is given by one of the following equivalent sets of data (see, for example, [22]):

- The structure $M \to M \otimes k[G]$ of a (right) $k[G]$-comodule on M.
- The structure $kG \otimes M \to M$ of a kG-module on M.
- A functorial (with respect to R) group action $G(R) \times (R \otimes M) \to (R \otimes M)$.

For most of this paper we shall restrict our consideration to infinitesimal group schemes, a special class of finite group schemes which we now define.

Definition 1.4. An infinitesimal group scheme G (over k) of height $\leq r$ is a finite group scheme whose coordinate algebra $k[G]$ is a local algebra with maximal ideal $\mathfrak{m}$ such that $x^{p^r} = 0$ for all $x \in \mathfrak{m}$.

Example 1.5. We shall frequently consider the following four examples.

(1) A finite dimensional p-restricted Lie algebra $\mathfrak{g}$ corresponds naturally with a height 1 infinitesimal group scheme which we denote $\underline{\mathfrak{g}}$ ([22, I.8.5]). The group algebra of $\underline{\mathfrak{g}}$ is the restricted enveloping algebra $\mathfrak{u}(\mathfrak{g})$ of $\mathfrak{g}$. If $\mathfrak{g}$ is the Lie algebra of a group scheme $\mathfrak{G}$, then the coordinate algebra of $\underline{\mathfrak{g}}$ is given by $k[\mathfrak{G}]/(x^p, x \in \mathfrak{m})$, where $\mathfrak{m}$ is the maximal ideal of $k[\mathfrak{G}]$ at the identity of $\mathfrak{G}$.

(2) Let $\mathbb{G}_a$ denote the additive group, so that $k[\mathbb{G}_a] = k[T]$ with coproduct defined by $\nabla(T) = T \otimes 1 + 1 \otimes T$. As a functor, $\mathbb{G}_a : (comm\ k - alg) \to (grps)$ sends an algebra R to its underlying abelian group. For any $r \geq 1$, we consider the r^{th} Frobenius kernel of $\mathbb{G}_a$,

$$\mathbb{G}_{a(r)} \ \equiv \operatorname{Ker}\{F^r : \mathbb{G}_a \to \mathbb{G}_a\}.$$

Here $F : \mathbb{G}_a \to \mathbb{G}_a$ is the (geometric) Frobenius specified by its map on coordinate algebras $k[T] \to k[T]$ given as the k-linear map sending T to T^p. The coordinate algebra of $\mathbb{G}_{a(r)}$ is given by $k[\mathbb{G}_{a(r)}] = k[T]/T^{p^r}$, whereas the group algebra of $\mathbb{G}_{a(r)}$ is given by

$$k\mathbb{G}_{a(r)} \simeq k[\mathbb{G}_{a(r)}]^{\#} \simeq k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p), \tag{1.5.1}$$

where u_i is a linear dual to T^{p^i}, $0 \leq i \leq r-1$.

(3) Let GL_n denote the general linear group, the representable functor sending a commutative algebra R to the group $\mathrm{GL}_n(R)$. For any $r \geq 1$, we consider the r^{th} Frobenius kernel of GL_n,

$$\mathrm{GL}_{n(r)} \ \equiv \operatorname{Ker}\{F^r : \mathrm{GL}_n \to \mathrm{GL}_n\},$$

where the geometric Frobenius

$$F : \mathrm{GL}_n(R) \to \mathrm{GL}_n(R)$$

is defined by raising each matrix entry to the p^{th} power. The coordinate algebra of $\mathrm{GL}_{n(r)}$ is given by

$$k[\mathrm{GL}_{n(r)}] \ = \ \frac{k[X_{i,j}]}{(X_{i,j}^{p^r} - \delta_{i,j})}_{1 \leq i,j \leq n},$$

whereas the group algebra of $\mathrm{GL}_{n(r)}$ is given as

$$k\,\mathrm{GL}_{n(r)} \ = \ \operatorname{Hom}_k(k[\mathrm{GL}_{n(r)}], k),$$

the k-space of linear functionals $k[\mathrm{GL}_{n(r)}]$ to k. The coproduct

$$\nabla : k[\mathrm{GL}_{n(r)}] \ \to \ k[\mathrm{GL}_{n(r)}] \otimes k[\mathrm{GL}_{n(r)}]$$

is given by sending $X_{i,j}$ to $\sum_k X_{ik} \otimes X_{kj}$.

(4) The height 2 infinitesimal group scheme $\mathrm{SL}_{2(2)}$ is essentially a special case of $\mathrm{GL}_{n(r)}$. This is once again defined as the kernel of the second iterate of Frobenius,

$$\mathrm{SL}_{2(2)} \ \equiv \operatorname{Ker}\{F^2 : \mathrm{SL}_2 \to \mathrm{SL}_2\}.$$

The coordinate algebra of $\mathrm{SL}_{2(2)}$ is given by

$$k[\mathrm{SL}_{2(2)}] = \frac{k[X_{1,1}, X_{1,2}, X_{2,1}, X_{2,2}]}{(X_{1,1}X_{2,2} - X_{1,2}X_{2,1} - 1, X_{i,j}^{p^2} - \delta_{i,j})},$$

whereas the group algebra of $\mathrm{SL}_{2(2)}$ is given as

$$k\,\mathrm{SL}_{2(2)} = k\langle e, f, h, e^{(p)}, f^{(p)}, h^{(p)} \rangle / \langle \text{relations} \rangle$$

with $e,\ f,\ h,\ e^{(p)},\ f^{(p)},\ h^{(p)}$ the dual basis vectors to $X_{1,2},\ X_{2,1},\ X_{1,1} - 1,\ X_{1,2}^p,$ $X_{2,1}^p,\ (X_{1,1} - 1)^p$ respectively.

We denote by $\mathbb{G}_{a(r),R}$ the base extension of $\mathbb{G}_{a(r)}$ to a commutative k-algebra R.

Definition 1.6. An (infinitesimal) 1-parameter subgroup of height r of an affine group scheme G_R over a commutative k-algebra R is a homomorphism of R-group schemes $\mathbb{G}_{a(r),R} \to G_R$.

We recall the description of height r 1-parameter subgroups of GL_n given in [25].

Proposition 1.7 ([25, 1.2])**.** *If $G = \mathrm{GL}_n$ and if R is a commutative k-algebra, then a 1-parameter subgroup of $\mathrm{GL}_{n,R}$ of height r, $f : \mathbb{G}_{a(r),R} \to \mathrm{GL}_{n,R}$, is naturally (with respect to R) equivalent to a comodule map*

$$\Delta_f : R^n \to R[T]/T^{p^r} \otimes_R R^n, \quad \Delta_f(v) = \sum_{j=0}^{p^{r-1}} T^j \otimes \beta_j(v), \quad \beta_j \in M_n(R)$$

satisfying the constraints of being counital and coassociative. This in turn is equivalent to specifying an r-tuple of matrices $\alpha_0 = \beta_0, \alpha_1 = \beta_p, \dots, \alpha_{r-1} = \beta_{p^{r-1}}$ in $M_n(R)$ such that each α_i has p^{th} power 0 and such that the α_i's pairwise commute. The other coefficient matrices β_j are given by the formula

$$\beta_j = \frac{\alpha_0^{j_0} \cdots \alpha_{r-1}^{j_{r-1}}}{(j_0)! \cdots (j_{r-1})!} \in M_n(R), \quad j = \sum_{i=0}^{r-1} j_i p^i \text{ with } 0 \le j_i < p. \tag{1.7.1}$$

As shown in [25], Proposition 1.7 implies the following representability of the functor of 1-parameter subgroups of height r.

Theorem 1.8 ([25, 1.5])**.** *For any affine group scheme G, the functor from commutative k-algebras to sets*

$$R \;\mapsto\; \mathrm{Hom}_{\mathrm{grp\,sch}}(\mathbb{G}_{a(r),R}, G_R)$$

is representable by an affine scheme $V_r(G) = \operatorname{Spec} k[V_r(G)]$. Namely, this functor is naturally isomorphic to the functor

$$R \;\mapsto\; \mathrm{Hom}_{\mathrm{k-alg}}(k[V_r(G)], R).$$

By varying r, we can associate a family of affine schemes to an affine group scheme G. In the following remark we make explicit the relationship between various $V_r(G)$ for the same G and varying r's.

Remark 1.9. For $r > s \ge 1$, let $p_{r,s} : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(s)}$ be the canonical projection given by the natural embedding of the coordinate algebras

$$p_{r,s}^* : k[\mathbb{G}_{a(s)}] = k[T]/T^{p^s} \xrightarrow{T \to T^{p^{r-s}}} k[T]/T^{p^r} = k[\mathbb{G}_{a(r)}].$$

The corresponding map on group algebras,

$$k\mathbb{G}_{a(r)} \simeq k[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p) \xrightarrow{p_{r,s,*}} k\mathbb{G}_{a(s)} \simeq k[v_0, \dots, v_{s-1}]/(v_0^p, \dots, v_{s-1}^p)\ ,$$

sends $\{u_0, \dots, u_{r-s-1}\}$ to $\{0, \dots, 0\}$ and $\{u_{r-s}, \dots, u_{r-1}\}$ to $\{v_0, \dots, v_{s-1}\}$.

Precomposition with $p_{r,s}$ determines a canonical embedding of affine schemes,

$$i_{s,r} : V_s(G) \hookrightarrow V_r(G)\,,$$

where a 1-parameter subgroup $\mu : \mathbb{G}_{a(s),R} \to G_R$ of height s is sent to the 1-parameter subgroup $\mu \circ p_{r,s} : \mathbb{G}_{a(r),R} \to \mathbb{G}_{a(s),R} \to G_R$ of height r. The construction is transitive, that is, we have $i_{s,r} = i_{s',r} \circ i_{s,s'}$ for $s \le s' \le r$. Hence, we have an inductive system

$$V_1(G) \subset V_2(G) \subset \cdots \subset V_r(G) \subset \cdots.$$

Conversely, any 1-parameter subgroup $\mathbb{G}_{a(s'),R} \to G_R$ can be decomposed as

$$\mathbb{G}_{a(s'),R} \xrightarrow{p_{s',s}} \mathbb{G}_{a(s),R} \hookrightarrow G_R$$

for some $s \leq s'$. If G is an infinitesimal group scheme of height $\leq r$, then we may choose $s \leq r$. This justifies the following definition.

Definition 1.10. Let G be an infinitesimal group scheme. Then the closed immersion $i_{r,r'} : V_r(G) \hookrightarrow V_{r'}(G)$ for $r' > r$ is an isomorphism provided the height of G is $\leq r$. We denote by $V(G)$ the stable value of $V_r(G)$,

$$V(G) \equiv \varinjlim_r V_r(G).$$

We next make explicit the construction of 1-parameter subgroups for GL_n as in Proposition 1.7. This construction can be applied to any affine group scheme of exponential type (see [25, §1] and also [23] for an extended list of groups of exponential type). We define the homomorphism

$$\exp_{\underline{\alpha}} : \mathbb{G}_{a(r),R} \to \mathrm{GL}_{n,R}$$

of R-group schemes corresponding to an r-tuple $\underline{\alpha} = (\alpha_0, \dots, \alpha_{r-1}) \in M_n(R)^{\times r}$ of pairwise commuting p-nilpotent matrices to be the natural transformation of group-valued functors on commutative R-algebras S sending any $s \in S$ with $s^{p^r} = 0$ to

$$\exp(s\alpha_0) \cdot \exp(s^p\alpha_1) \cdots \exp(s^{p^{r-1}}\alpha_{r-1}) \in \mathrm{GL}_n(S), \tag{1.10.1}$$

where for any p-nilpotent matrix $A \in \mathrm{GL}_n(S)$ we set

$$\exp(A) = 1 + A + \frac{A^2}{2} + \cdots + \frac{A^{p-1}}{(p-1)!}.$$

The following proposition proved in [25] identifies the functor of 1-parameter subgroups in the case of infinitesimal general linear groups.

Proposition 1.11 ([25, 1.2])**.** *The scheme of* 1*-parameter subgroups* $V_r(\mathrm{GL}_n)$ *is isomorphic to the scheme of* r*-tuples of pairwise commuting* p*-nilpotent* $n \times n$ *matrices* $N_p^{[r]}(gl_n)$*. The identification is given by sending* $\underline{\alpha} = (\alpha_0, \dots, \alpha_{r-1}) \in N_p^{[r]}(gl_n)(R)$ *to the* 1*-parameter subgroup* $\exp_{\underline{\alpha}} : \mathbb{G}_{a(r),R} \to \mathrm{GL}_{n,R}$.

Example 1.12. We describe $V(G)$ in each of the four examples of Example 1.5.

(1) $V(\underline{\mathfrak{g}}) \simeq N_p(\mathfrak{g})$, the closed subvariety of the affine space underlying $\mathfrak{g}$ consisting of p-nilpotent elements $x \in \mathfrak{g}$ (that is, $x^{[p]} = 0$). Let $\mathfrak{g}_a$ be the Lie algebra of the additive group $\mathbb{G}_a$. Note that $\mathfrak{g}_a$ is a one dimensional restricted Lie algebra with trivial p-restriction. Each p-nilpotent element $x \in \mathfrak{g}_R = \mathfrak{g} \otimes_k R$ determines a map of p-restricted Lie algebras over R where R is a commutative k-algebra: $\mathfrak{g}_{a,R} \to \mathfrak{g}_R$. The corresponding map of height 1 infinitesimal group schemes $\mathbb{G}_{a(1),R} \to \underline{\mathfrak{g}}_R$ is the associated 1-parameter subgroup of $\underline{\mathfrak{g}}$.

(2) $V(\mathbb{G}_{a(r)}) \simeq \mathbb{A}^r$. The r-tuple $\underline{a} = (a_0, \dots, a_{r-1}) \in R^{\times r} = \mathbb{A}^r(R)$ corresponds to the 1-parameter subgroup $\mu_{\underline{a}} : \mathbb{G}_{a(r),R} \to \mathbb{G}_{a(r),R}$ whose map on coordinate algebras $R[T]/T^{p^r} \to R[T]/T^{p^r}$ sends T to $\sum_i a_i T^{p^i}$ ([25, 1.10]).

(3) By Proposition 1.11, $V(\mathrm{GL}_{n(r)}) = N_p^{[r]}(gl_n)$, the variety of r-tuples of pairwise commuting, p-nilpotent $n \times n$ matrices. The embedding $i_{r,r+1} : V_r(\mathrm{GL}_n) \simeq N_p^{[r]}(gl_n) \subset V_{r+1}(\mathrm{GL}_n) \simeq N_p^{[r+1]}(gl_n)$ described in Remark 1.9 is given by sending an r-tuple $(\alpha_0, \dots, \alpha_{r-1})$ to the $(r+1)$-tuple $(0, \alpha_0, \dots, \alpha_{r-1})$.

Let $X_{i,j}$ be the coordinate functions of $R[\mathrm{GL}_{n(r)}] \simeq R[X_{i,j}]/(X_{i,j}^{p^r} - \delta_{i,j})$. Then $\exp_{\underline{\alpha}}^* : R[\mathrm{GL}_{n(r)}] \to R[\mathbb{G}_{a(r)}]$ is given by sending $X_{i,j}$ for some $1 \le i,j \le n$ to the (i,j)-entry of the polynomial $p_{\underline{\alpha}}(t)$ with matrix coefficients whose coefficient of t^d is computed as the multiple of s^d in the (i,j)-entry of the matrix (1.10.1).

Upon performing the indicated multiplication in (1.10.1), the coefficient of $p_{\underline{\alpha}}(t)$ multiplying s^{p^ℓ} is α_ℓ for $0 \le \ell < r$, whereas coefficients of $p_{\underline{\alpha}}(t)$ multiplying s^n for n not a power of p are determined as in formula (1.7.1). Consequently, we conclude that $\exp_{\underline{\alpha}}^*(X_{i,j})$ is a polynomial in t whose coefficient multiplying T^{p^ℓ} is $(\alpha_\ell)_{i,j}$ for $0 \le \ell < r$.

(4) Since $\mathrm{SL}_{2(2)}$ is a group scheme with an embedding of exponential type (see [25, 1.8]), its variety admits a description similar to the one of $\mathrm{GL}_{n(r)}$. Namely, $V(\mathrm{SL}_{2(2)})$ is the variety of pairs of p-nilpotent trace 0 proportional 2×2 matrices $\underline{\alpha} = (\alpha_0, \alpha_1)$. This variety is given explicitly as the affine scheme with coordinate algebra

$$k[V(\mathrm{SL}_{2(2)})] = k[x_0,y_0,z_0,x_1,y_1,z_1]/(x_iy_i - z_i^2, x_0y_1 - x_1y_1, z_0y_1 - z_1y_0, x_0z_1 - x_1z_0).$$

We give an explicit description of the map on coordinate algebras

$$\exp_{\underline{\alpha}}^* : R[\mathrm{SL}_{2(2)}] \to R[\mathbb{G}_{\mathrm{a}(2)}] \simeq R[T]/T^{p^2}$$

induced by the one-parameter subgroup $\exp_{\underline{\alpha}} : \mathbb{G}_{a(2),R} \to \mathrm{SL}_{2(2),R}$. This description follows immediately from the general discussion in the previous example. Let $\underline{\alpha} = (\begin{bmatrix} c_0 & a_0 \\ b_0 & -c_0 \end{bmatrix}, \begin{bmatrix} c_1 & a_1 \\ b_1 & -c_1 \end{bmatrix}) \in N^{[2]}(sl_2)$. Then $\exp_{\underline{\alpha}}^*$ is determined by the formulae

$$X_{1,1} \mapsto 1 + c_0T + c_1T^p, \quad X_{1,2} \mapsto a_0T + a_1T^p,$$
$$X_{2,1} \mapsto b_0T + b_1T^p, \quad X_{2,2} \mapsto 1 - c_0T - c_1T^p,$$

where $X_{i,j}$ are the standard polynomial generators of $k[\mathrm{SL}_{2(2)}] \simeq \frac{k[X_{1,1},X_{1,2},X_{2,1},X_{2,2}]}{(\det -1, X_{i,j}^{p^2} - \delta_{i,j})}$.

Remark 1.13. If $k(v)$ denotes the field of definition of the point $v \in V(G)$ for an infinitesimal group scheme G (see [26, p. 743] for a discussion of the field of definition), then we have a naturally associated map $\operatorname{Spec} k(v) \to V(G)$ and, hence, an associated group scheme homomorphism over $k(v)$ (for r sufficiently large):

$$\mu_v : \mathbb{G}_{a(r),k(v)} \longrightarrow G_{k(v)}\,.$$

Note that if K/k is a field extension and $\mu : \mathbb{G}_{a(r),K} \to G_K$ is a group scheme homomorphism, then this data defines a point $v \in V(G)$ and a field embedding $k(v) \hookrightarrow K$ such that μ is obtained from μ_v via scalar extension from $k(v)$ to K.

We next recall the *rank variety* and *cohomological support variety* of a kG-module of an infinitesimal group scheme. We use the notation

$$\mathrm{H}^\bullet(G,k) = \begin{cases} \mathrm{H}^*(G,k), & \text{if } p = 2, \\ \mathrm{H}^{\mathrm{ev}}(G,k) & \text{if } p > 2. \end{cases}$$

Definition 1.14. Let G be a finite group scheme and M a finite dimensional kG-module. We define the cohomological support variety for M to be

$$|G|_M \;\equiv\; V(\operatorname{ann}_{\mathrm{H}^\bullet(G,k)} \operatorname{Ext}^*_{kG}(M,M)),$$

the reduced closed subscheme of $|G| = \operatorname{Spec} \mathrm{H}^\bullet(G,k)_{\mathrm{red}}$ given as the variety of the annihilator ideal of $\operatorname{Ext}^*_{kG}(M,M)$.

The map of R-algebras (but not of Hopf algebras for $r > 1$),

$$\epsilon : R[u]/u^p \xrightarrow{u \mapsto u_{r-1}} R[u_0, \ldots, u_{r-1}]/(u_i^p) \;\simeq\; R\mathbb{G}_{a(r)}\,, \tag{1.14.1}$$

makes its first appearance in the following definition and will recur throughout this paper.

Definition 1.15. Let G be an infinitesimal group scheme and M a finite dimensional kG-module. We define the rank variety for M to be the reduced closed subscheme $V(G)_M$ whose points are given as follows:

$$V(G)_M = \{v \in V(G) : (\mu_{v,*} \circ \epsilon)^*(M_{k(v)}) \text{ is not free as a } k[u]/u^p\text{-module}\}.$$

Proposition [26, 6.2] asserts that $V(G)_M$ is a closed subvariety of $V(G)$. A key result of [26] is the following theorem relating the scheme of 1-parameter subgroups $V(G)$ to the cohomology of G.

Theorem 1.16 ([26, 5.2, 6.8, 7.5])**.** *Let G be an infinitesimal group scheme of height $\leq r$. There is a natural homomorphism of k-algebras,*

$$\psi : \mathrm{H}^\bullet(G, k) \to k[V(G)],$$

with nilpotent kernel whose image contains the p^r-th power of each element of $k[V(G)]$. Hence, the associated morphism of schemes,

$$\Psi : V(G) \to \operatorname{Spec} \mathrm{H}^\bullet(G, k),$$

is a p-isogeny.

If M is a finite dimensional kG-module, then Ψ restricts to a homeomorphism

$$\Psi_M : V(G)_M \xrightarrow{\sim} |G|_M.$$

Furthermore, every closed conical subspace of $V(G)$ is of the form $V(G)_M$ for some finite dimensional kG-module M.

In the special case of $G = \mathrm{GL}_{n(r)}$ the isogeny Ψ has an explicitly constructed "inverse."

Theorem 1.17 ([25, 5.2])**.** *There exists a homomorphism of k-algebras,*

$$\phi : k[V(\mathrm{GL}_{n(r)})] \to \mathrm{H}^\bullet(\mathrm{GL}_{n(r)}, k),$$

such that $\psi \circ \phi$ is the r^{th} iterate of the k-linear Frobenius map. Hence, the associated morphisms of schemes,

$$\Psi : V(\mathrm{GL}_{n(r)}) \to \operatorname{Spec} \mathrm{H}^\bullet(\mathrm{GL}_{n(r)}, k), \quad \Phi : \operatorname{Spec} \mathrm{H}^\bullet(\mathrm{GL}_{n(r)}, k) \to V(\mathrm{GL}_{n(r)}),$$

are mutually inverse homeomorphisms.

Example 1.18. We investigate $V(G)_M$ for the four examples of Example 1.5.

(1) Let M be a p-restricted $\mathfrak{g}$-module of dimension m, given by the map of p-restricted Lie algebras $\rho : \mathfrak{g} \to \mathrm{End}_k(M) \simeq \mathfrak{gl}_m$. Then $V(\underline{\mathfrak{g}})_M \subset V(\underline{\mathfrak{gl}_m})$ consists of those p-nilpotent elements of $\mathfrak{g}$ whose Jordan type (as an $m \times m$-matrix in $\mathfrak{gl}_m$) has at least one block of size $< p$ (see [13]).

(2) For $G = \mathbb{G}_{a(r)}$, $kG \simeq kE$, where E is an elementary abelian p-group of rank r. The rank variety of a kE-module was first investigated in [6].

We consider directly the rank variety $V(\mathbb{G}_{a(r)})_M$ of a finite dimensional $k\mathbb{G}_{a(r)}$-module M. The data of such a module is the choice of r p-nilpotent, pairwise commuting endomorphisms $\widetilde{u}_0, \dots, \widetilde{u}_{r-1} \in \mathrm{End}_k(M)$, given as the image of the distinguished generators of $k\mathbb{G}_{a(r)}$ as in (1.5.1). A 1-parameter subgroup of $\mathbb{G}_{a(r)}$ has the form $\mu_{\underline{a}} : \mathbb{G}_{a(r),K} \to \mathbb{G}_{a(r),K}$ for some r-tuple $\underline{a} = (a_0, \dots, a_{r-1})$ of K-rational points as in Example 1.12(2). The condition that $\mu_{\underline{a}}$ be a point of $V(\mathbb{G}_{a(r)})_M$ is the condition that $(\mu_{\underline{a}} \circ \epsilon)^*(M_K)$ is not free as a $K[u]/u^p$-module, which is equivalent to the condition that M_K is not free as a $K[\widetilde{u}]/\widetilde{u}^p$-module where $\widetilde{u} = a_{r-1}\widetilde{u}_0 + a_{r-2}^p\widetilde{u}_1 + \cdots + a_0^{p^{r-1}}\widetilde{u}_{r-1} \in \mathrm{End}_K(M_K)$ (see [26, 6.5]).

(3) Let M be a finite dimensional kG-module with $G = \mathrm{GL}_{n(r)}$. By Theorem 1.16, $V(\mathrm{GL}_{n(r)})_M \subset V(\mathrm{GL}_{n(r)})$ is the closed subvariety whose set of points in a field K/k are 1-parameter subgroups $\exp_{\underline{\alpha}} : \mathbb{G}_{a(r),K} \to \mathrm{GL}_{n(r),K}$ indexed by r-tuples $\underline{\alpha} = (\alpha_0, \dots, \alpha_{r-1}) \in M_n(K)$ of p-nilpotent, pairwise commuting matrices such that $(\exp_{\underline{\alpha},*} \circ \epsilon)^*(M_K)$ is not free as a $K[u]/u^p$-module. The action of u on M_K is determined utilizing Example 1.12(3). Namely, the action of u is given by composing the coproduct $M_K \to K[\mathrm{GL}_{n(r)}] \otimes M_K$ defining the $\mathrm{GL}_{n(r)}$-module structure on M_K with the linear functional

$$u_{r-1} \circ \exp_{\underline{\alpha}}^* : \; K[\mathrm{GL}_{n(r)}] \xrightarrow{\exp_{\underline{\alpha}}^*} K[\mathbb{G}_{a(r)}] \xrightarrow{u_{r-1}} K\,.$$

In Section 3, we shall investigate this case in more detail by considering some concrete examples.

(4) A complete description of support varieties for simple modules for $\mathrm{SL}_{2(r)}$ can be found in [26, §7]. We describe the situation for $G = \mathrm{SL}_{2(2)}$. Let S_λ be irreducible modules of highest weight λ, where $0 \leq \lambda \leq p^2 - 1$. For $\lambda < p - 1$, the module S_λ has dimension less than p, and thus $V(G)_{S_\lambda} = V(G)_{S_\lambda^{(1)}} = V(G)$. Here, $S_\lambda^{(i)}$ is the i^{th} Frobenius twist of S_λ. For $\lambda = p - 1$, the restriction of S_{p-1} to $\mathrm{SL}_{2(1)} \subset \mathrm{SL}_{2(2)}$ is projective (the Steinberg module for $\mathrm{SL}_{2(1)}$), but S_{p-1} is not itself projective. Hence, $V(G)_{S_{p-1}}$ is a proper non-trivial subvariety of $V(G)$. Using the notation introduced in Example 1.12(4), we have

$$V(G)_{S_{p-1}} = \{(\alpha_0, 0) \mid \alpha_0 \in N(sl_2)\} \subset V(G)$$

and

$$V(G)_{S_{p-1}^{(1)}} = \{(0, \alpha_1) \mid \alpha_1 \in N(sl_2)\} \subset V(G)$$

(see [26, 6.10]). $V(G)_{S_{p-1}}$ can be described as the subscheme of $V(G)$ defined by the equations $x_1 = y_1 = z_1 = 0$. For $\lambda = \lambda_0 + \lambda_1 p$ where $\lambda_0, \lambda_1 \leq p - 1$ we have $S_\lambda \simeq S_{\lambda_0} \otimes S_{\lambda_1}^{(1)}$ by the Steinberg tensor product theorem. Hence, we can compute the support variety of S_λ using the tensor product property of support varieties. For $\lambda = p^2 - 1$, S_λ is the Steinberg module for $\mathrm{SL}_{2(2)}$, it is projective and, hence, $V(G)_{S_{p^2-1}} = \{0\}$. Overall, we get

$$V(G)_{S_\lambda} = \begin{cases} N^{[2]}(sl_2) & \text{if } \lambda_0, \lambda_1 \neq p-1, \\ \{(\alpha_0, 0) \mid \alpha_0 \in N(sl_2)\} & \text{if } \lambda_0 = p-1, \lambda_1 \neq p-1, \\ \{(0, \alpha_1) \mid \alpha_1 \in N(sl_2)\} & \text{if } \lambda_0 \neq p-1, \lambda_1 = p-1, \\ 0 & \text{if } \lambda = p^2 - 1. \end{cases}$$

2. Global p-nilpotent operators

In this section, we introduce in Definition 2.1 and study the global p-nilpotent operator

$$\Theta_G : k[G] \ \to \ k[V(G)],$$

a k-linear but not multiplicative map defined for any infinitesimal group scheme G. This operator, when viewed as an element of $kG \otimes k[V(G)]$, encodes all 1-parameter subgroups of G: any 1-parameter subgroup $\mu : \mathbb{G}_{a(r),K} \to G_K$ corresponds to a K-valued point of $V(G)$, and $\mu_*(u_{r-1})$ equals the specialization in KG of $\Theta_G \in kG \otimes k[V(G)]$ at this point.

If M is a rational G-module, then Θ_G determines the $k[V(G)]$-linear operator

$$\Theta_M : M \otimes k[V(G)] \ \to \ M \otimes k[V(G)]$$

as formalized in Definition 2.3.

Before giving definitions, we mention as motivation the example $G = \mathbb{G}_{a(1)}^{\times 2}$. In this case, the group algebra kG equals $k[x,y]/(x^p,y^p)$, the scheme of 1-parameter subgroups equals $V(G) = \mathbb{A}^2$, and $k[V(G)] = k[\mathbb{A}^2] = k[s,t]$. In this special case, Θ_G takes the form

$$\Theta_G \ = \ x \otimes s + y \otimes t \ \in \ k[x,y]/(x^p,y^p) \otimes k[s,t].$$

If M is a kG-module, then the $k[s,t]$-linear operator Θ_M is given by

$$\Theta_M : M \otimes k[s,t] \to M \otimes k[s,t],$$
$$m \otimes 1 \mapsto xm \otimes s + ym \otimes t.$$

"Specializing" Θ_M at some $(a,b) \in K^2$ for some field extension K/k yields the action of $ax + by$ on M_K.

To construct our global operator, we proceed as follows. Let G be an algebraic affine group scheme over k (that is, G is an affine group scheme such that the coordinate algebra $k[G]$ is finitely generated over k [27, 3.3]) and let $A = k[V_r(G)]$. The natural isomorphism of covariant functors on commutative k-algebras R,

$$\mathrm{Hom}_{\text{grp sch}}(\mathbb{G}_{a(r),R}, G_R) \ \simeq \mathrm{Hom}_{k-\mathrm{alg}}(A, R),$$

given in Theorem 1.8 implies the existence of a universal 1-parameter subgroup of height r,

$$\mathcal{U}_{G,r} : \mathbb{G}_{a(r),A} \ \longrightarrow \ G_A,$$

the subgroup corresponding to the identity map on A. The subgroup $\mathcal{U}_{G,r}$ induces a map on coordinate algebras

$$\mathcal{U}^*_{G,r} : A \otimes k[G] \longrightarrow A \otimes k[\mathbb{G}_{a(r)}].$$

Recall that $k\mathbb{G}_{a(r)} \simeq k[u_0,\dots,u_{r-1}]/(u_0^p,\dots,u_{r-1}^p)$, where u_{r-1}, the dual element to $T^{p^{r-1}}$, is a linear functional $u_{r-1} : k[\mathbb{G}_{a(r)}] \to k$.

Definition 2.1. Let G be an algebraic affine group scheme over k. We define

$$\Theta_{G,r} : k[G] \ \longrightarrow \ k[V_r(G)]$$

to be the k-linear, p-nilpotent functional defined by the composition
(2.1.1)

$$k[G] \xrightarrow{1 \otimes \mathrm{id}} k[V_r(G)] \otimes k[G] \xrightarrow{\mathcal{U}^*_{G,r}} k[V_r(G)] \otimes k[\mathbb{G}_{a(r)}] \xrightarrow{\ \mathrm{id} \otimes u_{r-1}\ } k[V_r(G)].$$

As an element of $\mathrm{Hom}(k[G], k[V_r(G)]) \equiv kG \otimes k[V_r(G)]$, $\Theta_{G,r}$ can be equivalently defined as

$$\Theta_{G,r} \ = \ \mathcal{U}_{G,r,*}(u_{r-1}) \ \in \ kG \otimes k[V_r(G)], \tag{2.1.2}$$

where

$$\mathcal{U}_{G,r,*} : k\mathbb{G}_{a(r)} \otimes k[V_r(G)] \to kG \otimes k[V_r(G)].$$

Thus $\Theta_{G,r}$, as given in (2.1.1), satisfies the property that its composition with $k[V_r(G)] \to K$ corresponding to the K-rational point $\mu : \mathbb{G}_{a(r),K} \to G_K$ equals $\mu_*(u_{r-1})$.

The following proposition justifies using the simplified notation

$$\Theta_G : k[G] \to k[V(G)], \tag{2.1.3}$$

where $V(G) = \varinjlim_r V_r(G)$ as in Definition 1.10. Namely, Θ_G is defined to be isomorphic to $\Theta_{G,r}$ provided that G is infinitesimal of height $\leq r$.

Recall the canonical projection $p_{r',r} : \mathbb{G}_{a(r')} \twoheadrightarrow \mathbb{G}_{a(r)}$, and the induced closed embedding $i_{r,r'} : V_r(G) \hookrightarrow V_{r'}(G)$, introduced in Remark 1.9, for $r' \geq r$.

Proposition 2.2. *Let G be an infinitesimal group scheme and let $r' \geq r$. Let $A_r = k[V_r(G)]$, $A_{r'} = k[V_{r'}(G)]$, and let*

$$\phi : A_{r'} \twoheadrightarrow A_r$$

be the surjective homomorphism corresponding to the canonical embedding $i_{r,r'}$: $V_r(G) \hookrightarrow V_{r'}(G)$. Consider A_r as an $A_{r'}$-module via ϕ. Then

$$\Theta_{G,r} = \Theta_{G,r'} \otimes_{A_{r'}} 1 \ \in \ kG \otimes A_{r'} \otimes_{A_{r'}} A_r \simeq kG \otimes A_r.$$

Moreover, if G is an infinitesimal group scheme of height $\leq r$, then $\Theta_{G,r}$ is thereby naturally identified with $\Theta_{G,r'}$.

Proof. Consider the composition

$$\begin{tikzcd} \mathbb{G}_{a(r'),A_r} \arrow[rr, "\mathcal{U}_{G,r}\circ p_{r',r}"] \arrow[dr, "p_{r',r}"'] & & G_{A_r} \\ & \mathbb{G}_{a(r),A_r}. \arrow[ur, "\mathcal{U}_{G,r}"'] & \end{tikzcd} \tag{2.2.1}$$

Since $\mathcal{U}_{G,r} \in V_r(G)(A_r) \simeq \mathrm{Hom}(A_r, A_r)$ corresponds to the identity map on A_r and $p_{r',r}$ is the map that induces $\phi : A_{r'} \to A_r$, we conclude that the composition $\mathcal{U}_{G,r} \circ p_{r',r} \in V_{r'}(G) \simeq \mathrm{Hom}(A_{r'}, A_r)$ corresponds to ϕ. Hence, the universality of $\mathcal{U}_{G,r'}$ implies that $\mathcal{U}_{G,r} \circ p_{r',r}$ is obtained by pushing down the universal 1-parameter subgroup $\mathcal{U}_{G,r'}$ via $\phi : A_{r'} \to A_r$. Therefore, we conclude that

$$\mathcal{U}_{G,r} \circ p_{r',r} \ = \ \mathcal{U}_{G,r'} \otimes_{A_{r'}} A_r, \tag{2.2.2}$$

which implies the equality of maps of group algebras

$$\mathcal{U}_{G,r',*} \otimes_{A_{r'}} A_r = \mathcal{U}_{G,r,*} \circ p_{r',r,*} : \ k\mathbb{G}_{a(r')} \otimes A_r \to kG \otimes A_r. \tag{2.2.3}$$

Since $p_{r',r,*}(u_{r'-1}) = u_{r-1} \in k\mathbb{G}_{a(r)}$, we conclude that

$$(\mathcal{U}_{G,r,*} \circ p_{r',r,*})(u_{r'-1}) = \mathcal{U}_{G,r,*}(u_{r-1}) = \Theta_{G,r},$$

whereas $(\mathcal{U}_{G,r',*} \otimes_{A_{r'}} A_r)(u_{r'-1}) = \mathcal{U}_{G,r',*}(u_{r'-1}) \otimes_{A_{r'-1}} 1 = \Theta_{G,r'} \otimes_{A_{r'}} 1$.

The second statement follows immediately from the fact that for G of height $\leq r$, the map $\phi : A_{r'} \to A_r$ is an isomorphism as shown in Remark 1.9. $\square$

Let G be an affine group scheme over k, M be a kG-module, and $\nabla_M : M \to M \otimes k[G]$ be the corresponding co-action. A k-linear functional with values in a commutative k-algebra A, $\Theta : k[G] \to A$, determines an action of Θ on $M \otimes A$ which is the A-linear extension

$$\Theta_M : M \otimes A \to M \otimes A \tag{2.2.4}$$

of the map

$$M \xrightarrow{\nabla_M} M \otimes k[G] \xrightarrow{\mathrm{id} \otimes \Theta_M} M \otimes A\,.$$

If G is finite, we may view $\Theta \in \mathrm{Hom}_k(k[G], A)$ as an element of $kG \otimes A$ (which we also denote by Θ). From this point of view, the action (2.2.4) is simply multiplication by Θ.

We now define the *global p-nilpotent operator* on a G-module M.

Definition 2.3. Let G be an infinitesimal group scheme. For any $k[G]$-comodule M with co-action $\nabla_M : M \to M \otimes k[G]$, we define the p-nilpotent operator

$$\Theta_M : M \otimes k[V(G)] \to M \otimes k[V(G)]$$

to be the $k[V(G)]$-linear extension of the map $(\mathrm{id}_M \otimes \Theta_G) \circ \nabla_M : M \to M \otimes k[V(G)]$.

Remark 2.4. The fact that Θ_M is p-nilpotent follows immediately from (2.1.2) since $u_{r-1}^p = 0$.

Slightly abusing notation, we shall often refer to Θ_G itself as the global p-nilpotent operator.

We reformulate the pairing (2.2.4) in a more geometric fashion as follows.

Proposition 2.5. *Let G be a group scheme over k, V be an affine k-scheme, and let M be a finite dimensional G-module. Then a k-linear functional $\Theta : k[G] \to k[V]$ determines the pairing of k-schemes*

$$V \times M \to M. \tag{2.5.1}$$

As a pairing of representable functors of commutative k-algebras A, this pairing sends $(v : k[V] \to A, m = \sum_i a_i \otimes m_i)$ to $\sum_i a_i(\sum_j v(\Theta(f_{i,j}))m_j)$, where $\nabla(m_i) = \sum_j f_{i,j} \otimes m_j$.

Example 2.6. We describe the global p-nilpotent operator Θ_G in each of the four examples of Example 1.5.

(1) Let $G = \mathrm{GL}_{m(1)} \equiv \underline{\mathfrak{gl}}_m$, with group algebra $k\underline{\mathfrak{gl}}_m = u(\mathfrak{gl}_m)$. Then the composition of

$$\Theta_G : k[G] = k[X_{i,j}]/(X_{i,j}^p - \delta_{i,j}) \to k[N_p(\mathfrak{gl}_m)]$$

with some K-rational point $x \in N_p(\mathfrak{gl}_m)$ is the evaluation of the matrix coordinate functions $X_{i,j}$ on x. In other words,

$$\Theta_{\underline{\mathfrak{gl}}_m} \;=\; \sum_{1 \le i,j \le m} x_{i,j} \otimes \overline{X}_{i,j} \;\in\; \mathfrak{u}(\mathfrak{gl}_m) \otimes k[N_p(\mathfrak{gl}_m)],$$

where $\overline{X}_{i,j}$ is the image of the i,j-matrix coefficient on $\mathfrak{gl}_m$.

For a general p-restricted Lie algebra $\mathfrak{g}$, we have

$$\Theta_{\underline{\mathfrak{g}}} \;=\; \sum x \otimes \overline{X} \;\in\; \mathfrak{u}(\mathfrak{g}) \otimes k[N_p(\mathfrak{g})], \tag{2.6.1}$$

where the sum is over basis elements $X \in \mathfrak{g}^{\#}$ with image $\overline{X} \in k[N_p(\mathfrak{g})]$ and with dual $x \in \mathfrak{g}$.

We record an explicit formula for the universal p-nilpotent operator in the case of $\mathfrak{g} = sl_2$ for future reference. We have $k[N_p(sl_2)] \simeq k[x, y, z]/(xy + z^2)$. Let e, f, h be the standard basis of the p-restricted Lie algebra sl_2. Then

$$\Theta_{\underline{s}l_2} = xe + yf + zh.$$

Observe that this formula agrees with the presentation of a "generic" π-point for $u(sl_2)$ as given in [17, 2.5].

(2) Take $G = \mathbb{G}_{a(r)}$. Then $k[\mathbb{G}_{a(r)}] \simeq k[T]/T^{p^r}$, and $k[V(\mathbb{G}_{a(r)})] \simeq k[x_0, \ldots, x_{r-1}]$ is graded in such a way that x_i has degree p^i (see Proposition 2.11 below). We compute $\Theta_{\mathbb{G}_{a(r)}} : k[\mathbb{G}_{a(r)}] \longrightarrow k[V(\mathbb{G}_{a(r)})]$ explicitly in this case (see also [26, 6.5.1]).

1-parameter subgroups of $\mathbb{G}_{a(r),K}$ are in one-to-one correspondence with the additive polynomials in $K[T]/T^{p^r}$, that is, polynomials of the form $p(T) = a_0T + a_1T^p + \cdots + a_{r-1}T^{p^{r-1}}$ (see [25, 1.10]). The map on coordinate algebras induced by the universal 1-parameter subgroup $\mathcal{U} : \mathbb{G}_{a(r),k[V(G)]} \to \mathbb{G}_{a(r),k[V(G)]}$ is given by the "generic" additive polynomial:

$$\mathcal{U}^* : k[x_0, \ldots, x_{r-1}][T]/T^{p^r} \longrightarrow k[x_0, \ldots, x_{r-1}][T]/T^{p^r},$$
$$T \mapsto x_0T + x_1T^p + \cdots + x_{r-1}T^{p^{r-1}}.$$

To determine the linear functional

$$\Theta_{\mathbb{G}_{a(r)}} = u_{r-1} \circ \mathcal{U}^* : k[T]/T^{p^r} \longrightarrow k[x_0, \ldots, x_{r-1}],$$

it suffices to determine the values of $\Theta_{\mathbb{G}_{a(r)}}$ on the linear generators $\{T^i\}, 0 \le i \le p^r - 1$. Since u_{r-1} is the dual to $T^{p^{r-1}}$, this further reduces to determining the coefficient by $T^{p^{r-1}}$ in $\mathcal{U}^*(T^i) = (x_0T + x_1T^p + \cdots + x_{r-1}T^{p^{r-1}})^i$. Computing this coefficient, we conclude that $\Theta_{\mathbb{G}_{a(r)}}$ is given explicitly on the basis elements of $k[\mathbb{G}_{a(r)}] \simeq k[T]/T^{p^r}$ by

$$T^i \mapsto \sum_{\substack{i_0+i_1+\cdots+i_{r-1}=i \\ i_0+i_1p+\cdots+i_{r-1}p^{r-1}=p^{r-1}}} \binom{i}{i_0, i_1, \ldots, i_{r-1}} x_0^{i_0} \ldots x_{r-1}^{i_{r-1}}, \tag{2.6.2}$$

where $\binom{i}{i_0,i_1,\ldots,i_{r-1}} = \frac{i!}{i_0!i_1!\ldots i_{r-1}!}$ is the multinomial coefficient. Let $\{v_0, \ldots, v_{p^r-1}\}$ be the linear basis of $k\mathbb{G}_{a(r)}$ dual to $\{1, T, \ldots, T^{p^r-1}\}$. Dualizing (2.6.2), we conclude that $\Theta_{\mathbb{G}_{a(r)}}$, as an element of $k\mathbb{G}_{a(r)} \otimes k[V(\mathbb{G}_{a(r)})]$, has the following form:

$$\Theta_{\mathbb{G}_{a(r)}} = \sum_{i=0}^{p^r-1} v_i \left[\sum_{\substack{i_0+i_1+\cdots+i_{r-1}=i \\ i_0+i_1p+\cdots+i_{r-1}p^{r-1}=p^{r-1}}} \binom{i}{i_0, i_1, \ldots, i_{r-1}} x_0^{i_0} \cdots x_{r-1}^{i_{r-1}} \right]. \tag{2.6.3}$$

By [25, 1.4], v_i can be expressed in terms of the algebraic generators u_j of $k\mathbb{G}_{a(r)}$ via the following formulae:

$$v_i = \frac{u_0^{j_0} \cdots u_{r-1}^{j_{r-1}}}{j_0! \cdots j_{r-1}!},$$

where $i = j_0 + j_1p + \cdots + j_{r-1}p^{r-1}$ $(0 \le j_\ell \le p - 1)$ is the p-adic expansion of i.

Using these formulae, it is straightforward to calculate the term of $\Theta_{\mathbb{G}_{a(r)}}$ which is linear with respect to u_i (and homogeneous of degree p^{r-1} with respect to the grading of $k[V(\mathbb{G}_{a(r)})]$):

$$u_0 x_{r-1} + u_1 x_{r-2}^p + \cdots + u_{r-1} x_0^{p^{r-1}}.$$

The "linear" term gives the entire operator $\Theta_{\mathbb{G}_{a(r)}}$ for $r = 1, 2$, but for $r \geq 3$, the non-linear terms start to appear.

(3) Let $G = \mathrm{GL}_{n(r)}$. Recall that $V(\mathrm{GL}_{n(r)})$ is the k-scheme of r-tuples of p-nilpotent, pairwise commuting matrices. For notational convenience, let A denote $k[V(\mathrm{GL}_{n(r)})] = k[M_n^{\times r}]/I$, a quotient of the coordinate algebra of the variety $M_n^{\times r}$ of r-tuples of $n \times n$ matrices. Then $\mathcal{U}_{\mathrm{GL}_{n(r)}} : \mathbb{G}_{a(r),A} \to \mathrm{GL}_{n(r),A}$ is specified by the A-linear map on coordinate algebras,

$$\mathcal{U}^*_{\mathrm{GL}_{n(r)}} : A[\mathrm{GL}_{n(r)}] \ \to \ A[T]/T^{p^r}, \quad X_{i,j} \mapsto \sum_{\ell=0}^{p^r-1} (\beta_\ell)_{i,j} T^j, \tag{2.6.4}$$

where $\{X_{i,j}; 1 \leq i, j \leq n\}$ are the matrix coordinate functions of GL_n, where β_ℓ is given as in formula (1.7.1) in terms of the matrices $\alpha_0, \ldots, \alpha_{r-1} \in M_n(A)$ and $\alpha_i = \beta_{p^i}$ have matrix coordinate functions which generate A. (Indeed, the $n^2 r$ entries of $\alpha_0, \ldots, \alpha_{r-1}$ viewed as variables generate A, with relations given by the conditions that these matrices must be p-nilpotent and pairwise commuting.)

The p-nilpotent operator

$$\Theta_{\mathrm{GL}_{n(r)}} : k[\mathrm{GL}_{n(r)}] \to k[V(\mathrm{GL}_{n(r)})]$$

is given by the k-linear functional sending a polynomial in the matrix coefficients $P(X_{i,j}) \in k[\mathrm{GL}_{n(r)}]$ to the coefficient of $T^{p^{r-1}}$ of the sum of products corresponding to the polynomial P given by replacing each $X_{i,j}$ by $\sum_{\ell=0}^{p^r-1} (\beta_\ell)_{i,j} T^\ell$ (when taking products of matrix coefficients, one uses the usual rule for matrix multiplication);

$$P(X_{i,j}) \ \mapsto \ P(\sum_{\ell=0}^{p^r-1} (\beta_\ell)_{i,j} T^\ell) \ \mapsto \ \text{coeff of } \ T^{p^{r-1}}.$$

For example, the coaction $k^n \to k[\mathrm{GL}_n] \otimes k^n$ corresponding to the natural representation of GL_n on k^n determines an action of

$$\mathrm{Hom}_k(k[\mathrm{GL}_{n(r)}], A) \subset \mathrm{Hom}_k(k[\mathrm{GL}_n], A)$$

on A^n, so that $\Theta_{\mathrm{GL}_{n(r)}} : A^n \to A^n$ is given in matrix form by $(\Theta_{\mathrm{GL}_{n(r)}}(X_{i,j}))$.

(4) We consider $G = \mathrm{SL}_{2(2)}$, and assume notation and conventions of Example 1.12(4). Let $A = k[\mathrm{SL}_{2(2)}]$. Using the general discussion in Example 2.6(3) above (also compare to Example 1.12(4)), one readily computes that the map on coordinate algebras $\mathcal{U}^*_{\mathrm{SL}_{2(2)}} : A[\mathrm{SL}_{2(2)}] \to A[\mathbb{G}_{a(2)}] \simeq A[T]/T^{p^2}$ is given by

$$X_{1,1} \mapsto 1 + z_0 T + z_1 T^p, \quad X_{1,2} \mapsto x_0 T + x_1 T^p,$$
$$X_{2,1} \mapsto y_0 T + y_1 T^p, \quad X_{2,2} \mapsto 1 - z_0 - z_1 T^p.$$

By (2.1.2), $\Theta_{\mathrm{SL}_{2(2)}} = \mathcal{U}_{\mathrm{SL}_{2(2)}*}(u_1)$, which is the functional given by "reading off the coefficient of T^p".

Let $e, f, h, e^{(p)}, f^{(p)}, h^{(p)} \in k\,\mathrm{SL}_{2(2)}$ denote respectively the linear duals of the functions $X_{1,2}, X_{2,1}, X_{1,1}-1, X_{1,2}^p, X_{2,1}^p, (X_{1,1}-1)^p$ on $\mathrm{SL}_{2(2)}$, and set

$$e^{(i)} = \frac{e^i}{i!}, \quad f^{(i)} = \frac{f^i}{i!}, \quad \binom{h}{i} = \frac{h(h-1)(h-2)\dots(h-i+1)}{i!}$$

for $i < p$. Fix the linear basis of $k[\mathrm{SL}_{2(2)}]$ given by powers of $X_{1,2}, X_{2,1}, X_{1,1}-1$ (in this fixed order). Then the element of $k\,\mathrm{SL}_{2(2)}$ dual to $X_{1,2}^i X_{2,1}^j (X_{1,1}-1)^\ell$ for $i+j+\ell \le p$ is given by

$$(X_{1,2}^i X_{2,1}^j (X_{1,1}-1)^\ell)^\# \;=\; e^{(i)} f^{(j)} \binom{h}{\ell}$$

(where $\binom{h}{p}$ is identified with $h^{(p)}$ by definition).

With these conventions $\Theta_{\mathrm{SL}_{2(2)}} \in k\,\mathrm{SL}_{2(2)} \otimes k[V(\mathrm{SL}_{2(2)})]$ equals

$$x_1 e + y_1 f + z_1 h + x_0^p e^{(p)} + y_0^p f^{(p)} + z_0^p h^{(p)} + \sum_{\substack{i+j+\ell=p \\ i,j,\ell<p}} x_0^i y_0^j z_0^\ell e^{(i)} f^{(j)} \binom{h}{\ell}. \tag{2.6.5}$$

Our motivational example for $G = \mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$ from the beginning of this section is a special case of Example 2.6(1).

Example 2.7. Let $G = \mathbb{G}_{a(1)}^{\times r}$. Then G corresponds to the abelian p-nilpotent Lie algebra $g_a^{\oplus r}$, and $kG = k[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p)$. We have $V(G) \simeq \mathbb{A}^r$, and $k[V(G)] \simeq k[X_0, \dots, X_{r-1}]$, where all generators are in degree one. Then $\Theta_G \in kG \otimes k[V(G)]$ is given by the simple formula

$$\Theta_G = u_0 X_0 + \dots + u_{r-1} X_{r-1}.$$

It is useful to contrast this formula with the much more complicated result for $G = \mathbb{G}_{a(r)}$ in Example 2.6(2).

To complement Example 2.6, we make explicit the action of Θ_G on some representation of G for each of the four types of finite group schemes we have been considering in examples.

Example 2.8. (1) Let $G = \underline{\mathfrak{g}}$ and let $M = \mathfrak{g}^{ad}$ denote the adjoint representation of the p-restricted Lie algebra $\mathfrak{g}$; let $\{x_i\}$ be a basis for $\mathfrak{g}$. We identify $\Theta_{\underline{\mathfrak{g}}}$ as the $k[N_p(\mathfrak{g})]$-linear endomorphism

$$\Theta_{\underline{\mathfrak{g}}} : \mathfrak{g}^{ad} \otimes k[N_p(\mathfrak{g})] \;\to\; \mathfrak{g}^{ad} \otimes k[N_p(\mathfrak{g})], \quad x \otimes 1 \mapsto \sum_i [x_i, x] \otimes X_i,$$

where X_i is the image under the projection $S^*(\mathfrak{g}^\#) \to k[N_p(\mathfrak{g})]$ of the dual basis element to x_i.

(2) Let M denote the cyclic $k\mathbb{G}_{a(r)}$-module

$$M \;=\; k[u_0, \dots, u_{r-1}]/(u_0, u_1^p, \dots, u_{r-1}^p) \;\simeq\; k[u_1, \dots, u_{r-1}]/(u_1^p, \dots, u_{r-1}^p).$$

As recalled in Example 2.6(2), $k[V(\mathbb{G}_{a(r)})] = k[\mathbb{A}^r] = k[a_0, \dots, a_{r-1}]$, $k\mathbb{G}_{a(r)} = k[u_0, \dots, u_{r-1}]/(u_i^p)$, and

$$\Theta_{\mathbb{G}_{a(r)}} \in \; A[u_0, \dots, u_{r-1}]/(u_i^p)$$

is given by the complicated, but explicit formula (2.6.3). We conclude that

$$\Theta_{\mathbb{G}_{a(r)}} : M \otimes A \;\to\; M \otimes A$$

is the A-linear endomorphism sending u_i to $\overline{\Theta}_{\mathbb{G}_{a(r)}} \cdot u_i$, where $\overline{\Theta}_{\mathbb{G}_{a(r)}}$ is the image of $\Theta_{\mathbb{G}_{a(r)}}$ under the projection $A[u_0, \ldots, u_{r-1}]/(u_i^p) \to A[u_1, \ldots, u_{r-1}]/(u_i^p)$.

(3) Let M be the restriction to $\mathrm{GL}_{n(r)}$ of the canonical n-dimensional rational GL_n-module V_n. By Example 1.12(3), $A = k[V(\mathrm{GL}_{n(r)}]$ is the quotient of $k[\mathfrak{gl}_n]^{\otimes r}$ by the ideal generated by the equations satisfied by an r-tuple of $n \times n$-matrices with the property that each matrix is p-nilpotent and that the matrices pairwise commute. The complexity of the map

$$\Theta_{\mathrm{GL}_{n(r)}} : V_n \otimes A \ \to \ V_n \otimes A$$

is revealed even in the case $n = 2$, which is worked out explicitly below.

(4) Let M be the restriction to $\mathrm{SL}_{2(2)}$ of the rational GL_2–representation V_2. Then Example 1.12(4) gives an explicit description of $A = k[V(\mathrm{SL}_{2(2)})]$ as a quotient of $k[x_0, y_0, z_0, x_1, y_1, z_1]$ and (2.6.5) gives $\Theta_{\mathrm{SL}_{2(2)}}$ explicitly. Since V_2 is a homogeneous polynomial representation of GL_2 of degree 1, the divided powers $e^{(p)}$, $f^{(p)}$ and $h^{(p)}$ as well as all products of the form $e^{(i)} f^{(j)} \binom{h}{\ell}$ act trivially on M. Hence, the map

$$\Theta_{\mathrm{SL}_{2(2)}} : M \otimes A \ \to \ M \otimes A$$

is given by the matrix

$$A^2 \xrightarrow{\begin{bmatrix} z_1 & x_1 \\ y_1 & -z_1 \end{bmatrix}} A^2.$$

When viewing group schemes as functors, it is often convenient to think of $G_{k[V(G)]}$ as $G \times V(G)$ (i.e., $G \times V(G) = \mathrm{Spec}(k[V(G)] \otimes k[G])$). From this point of view, $\mathcal{U}_G$ has the form

$$\mathcal{U}_G : \mathbb{G}_{a(r)} \times V(G) \ \longrightarrow \ G \times V(G).$$

The following naturality property of Θ_G will prove useful when we consider $M \otimes k[V(G)]$ as a free, coherent sheaf on $V(G)$ and restrict this sheaf to $V(H) \subset V(G)$ equipped with its action of H.

Proposition 2.9. *Choose $r > 0$ and consider a closed embedding $i : H \hookrightarrow G$ of algebraic affine group schemes over k. Let $\phi : V_r(H) \to V_r(G)$ be the closed embedding of affine schemes induced by i, with an associated surjective map $\phi^* : k[V_r(G)] \to k[V_r(H)]$ on coordinate algebras. Then the following square commutes:*

(2.9.1)
$$\begin{tikzcd} \mathbb{G}_{a(r)} \times V_r(H) \arrow[r, "\mathcal{U}_{H,r}"] \arrow[d, "id \times \phi"'] & H \times V_r(H) \arrow[d, "i \times \phi"] \\ \mathbb{G}_{a(r)} \times V_r(G) \arrow[r, "\mathcal{U}_{G,r}"] & G \times V_r(G). \end{tikzcd}$$

Consequently, the following square of k-linear maps commutes:

(2.9.2)
$$\begin{tikzcd} k[G] \arrow[r, "\Theta_{G,r}"] \arrow[d, "i^*"] & k[V_r(G)] \arrow[d, "\phi^*"] \\ k[H] \arrow[r, "\Theta_{H,r}"] & k[V_r(H)]. \end{tikzcd}$$

Thus, for any rational G-module M we have a compatibility of coactions on M:

(2.9.3)
$$\begin{tikzcd}
M \arrow[r, "\nabla_M"] \arrow[dr, "\nabla_{M\downarrow H}"'] & M \otimes k[G] \arrow[r, "\mathrm{id}\otimes\Theta_{G,r}"] \arrow[d, "\mathrm{id}\otimes i^*"] & M \otimes k[V_r(G)] \arrow[d, "\mathrm{id}\otimes\phi^*"] \\
& M \otimes k[H] \arrow[r, "\mathrm{id}\otimes\Theta_{H,r}"] & M \otimes k[V_r(H)].
\end{tikzcd}$$

Proof. The fact that $\phi : V_r(H) \to V_r(G)$ induced by the closed embedding $i : H \hookrightarrow G$ is itself a closed embedding is given by [25, 1.5]. By universality of $\mathcal{U}_{G,r}$, the composition $(i \times id) \circ \mathcal{U}_{H,r} : \mathbb{G}_{a(r)} \times V_r(H) \to G \times V_r(H)$ is obtained by pull-back of $\mathcal{U}_{G,r}$ via some morphism $V_r(H) \to V_r(G)$. By comparing maps on R-valued points, we verify that this morphism must be ϕ. This implies the commutativity of (2.9.1).

The commutative square (2.9.1) gives a commutative square on coordinate algebras:

(2.9.4)
$$\begin{tikzcd}
k[V_r(G)] \otimes k[G] \arrow[r, "\mathcal{U}^*_{G,r}"] \arrow[d, "\phi^*\otimes i^*"] & k[V_r(G)] \otimes k[\mathbb{G}_{a(r)}] \arrow[d, "\phi^*\otimes \mathrm{id}"] \\
k[V_r(H)] \otimes k[H] \arrow[r, "\mathcal{U}^*_{H,r}"] & k[V_r(H)] \otimes k[\mathbb{G}_{a(r)}].
\end{tikzcd}$$

Concatenating (2.9.4) on the right with the commutative square of linear maps

$$\begin{tikzcd}
k[V_r(G)] \otimes k[\mathbb{G}_{a(r)}] \arrow[r, "u_{r-1}"] \arrow[d, "\phi^*\otimes\mathrm{id}"] & k[V_r(G)] \arrow[d, "\phi^*"] \\
k[V_r(H)] \otimes k[\mathbb{G}_{a(r)}] \arrow[r, "u_{r-1}"] & k[V_r(H)]
\end{tikzcd}$$

and with the inclusions $k[G] \to k[V_r(G)] \otimes k[G]$ and $k[H] \to k[V_r(H)] \otimes k[H]$ on the left, we obtain a commutative diagram:

$$\begin{tikzcd}
k[G] \arrow[r, hook] \arrow[d, "i^*"] & k[V_r(G)] \otimes k[G] \arrow[r, "\mathcal{U}^*_{G,r}"] \arrow[d, "\phi^*\otimes i^*"] & k[V_r(G)] \otimes k[\mathbb{G}_{a(r)}] \arrow[r, "u_{r-1}"] \arrow[d, "\phi^*\otimes\mathrm{id}"] & k[V_r(G)] \arrow[d, "\phi^*"] \\
k[H] \arrow[r, hook] & k[V_r(H)] \otimes k[H] \arrow[r, "\mathcal{U}^*_{H,r}"] & k[V_r(H)] \otimes k[\mathbb{G}_{a(r)}] \arrow[r, "u_{r-1}"] & k[V_r(H)].
\end{tikzcd}$$

Eliminating the middle square, we obtain the square (2.9.2). Hence, (2.9.2) is commutative.

Finally, the commutativity of (2.9.3) follows immediately from the commutativity of (2.9.2). □

Pre-composition determines a natural action $V_r(G) \times V_r(\mathbb{G}_{a(r)}) \to V_r(G)$ for any algebraic affine group scheme G. Recall from [25, 1.10] that $V_r(\mathbb{G}_{a(r)}) \simeq \mathbb{A}^r$: morphisms of group schemes $\mathbb{G}_{a(r),A} \to \mathbb{G}_{a(r),A}$ over A have associated maps on coordinate algebras $A[T]/T^{p^r} \to A[T]/T^{p^r}$ given by additive polynomials, that is, polynomials of the form $a_0T + a_1T^p + \cdots + a_{r-1}T^{p^{r-1}}$. Restricting the action $V_r(G) \times V_r(\mathbb{G}_{a(r)}) \to V_r(G)$ to *linear* polynomials $\mathbb{A}^1 \subset V_r(\mathbb{G}_{a(r)}) \simeq \mathbb{A}^r$ determines a natural action

(2.9.5)
$$V_r(G) \times \mathbb{A}^1 \to V_r(G),$$

which is equivalent by [25, 1.11] to a functorial grading on $k[V_r(G)]$.

Proposition 2.10. *Let G be an algebraic affine group scheme over k. Then the coordinate algebra $k[V_r(G)]$ of $V_r(G)$ is a graded algebra generated by homogeneous generators of degrees p^i, $0 \le i < r$.*

Proof. The coordinate algebra $k[V_r(G)]$ is graded by [25, 1.12]. If $G = \mathrm{GL}_N$, then an R-valued point of $V_r(\mathrm{GL}_N)$ is given by an r-tuple of $N \times N$ pairwise commuting, p-nilpotent matrices with entries in R, $(\alpha_0, \dots, \alpha_{r-1})$. The action of $c \in V(\mathbb{G}_{a(1)})(R)$ on $(\alpha_0, \alpha_1, \dots, \alpha_{r-1}) \in V(\mathrm{GL}_{N(r)})(R)$ is given by the formula

$$(\alpha_0, \alpha_1, \dots, \alpha_{r-1}) \times c = (c\alpha_0, c^p\alpha_1, \dots, c^{p^{r-1}}\alpha_{r-1}).$$

Hence, the coordinate functions of the matrix α_i have grading p^i, and, therefore, $k[V_r(GL_N)]$ is generated by homogeneous elements of degree p^i, $0 \le i < r$.

Let $i : G \to GL_N$ be a closed embedding of a finite group scheme G into some GL_N. The naturality of the grading (see [25, 1.12]) implies that the surjective map $\phi : k[V_r(\mathrm{GL}_N)] \to k[V_r(G)]$ is a map of graded algebras. □

Proposition 2.11. *For any algebraic affine group scheme G and integer $r > 0$, the k-linear map*

$$\Theta_{G,r} : k[G] \ \to \ k[V_r(G)]$$

has image contained in the homogeneous summand of $k[V_r(G)]$ of degree p^{r-1}.

If G is infinitesimal, then this is equivalent to the following:

$$\mathcal{U}_{G,r*} : k\mathbb{G}_{a(r)} \otimes k[V_r(G)] \ \to \ kG \otimes k[V_r(G)]$$

sends $u_{r-1} \otimes 1 \in k\mathbb{G}_{a(r)} \otimes k[V_r(G)]$ to some $\sum x_i \otimes a_i \in kG \otimes k[V_r(G)]$ with each $a_i \in k[V_r(G)]$ homogeneous of degree p^{r-1}.

Proof. Let $A = k[V_r(G)]$. Since $\Theta_{G,r}$ factors through the r^{th} Frobenius kernel of G, we may assume G is infinitesimal. Let $\langle \lambda_i \rangle$ be a set of linear generators of $k[G]$ and $\langle \check{\lambda}_i \rangle$ be the dual set of linear generators of kG. Then $\mathcal{U}_{G,r,*}(u_{r-1}) = \sum \check{\lambda}_i \otimes f_i$ if and only if $u_{r-1}(\mathcal{U}_G^*(\lambda_i)) = f_i$ if and only if $\mathcal{U}_{G,r}^*(\lambda_i) = \cdots + f_i T^{p^{r-1}} + \cdots$. Hence, the assertion that $\Theta_{G,r}$ is homogeneous of degree p^{r-1} is equivalent to showing that the map $k[G] \to A$ defined by reading the coefficient of

$$\mathcal{U}_{G,r}^* : k[G] \ \to \ A \otimes k[G] \ \to \ A \otimes k[\mathbb{G}_{a(r)}] \ \to \ A[T]/T^{p^r}$$

of the monomial $T^{p^{r-1}}$ is homogeneous of degree p^{r-1}.

The coordinate algebra $k[\mathbb{G}_{a(r)}] \simeq k[T]/T^{p^r}$ has a natural grading with T assigned degree 1. This grading corresponds to the monoidal action of $\mathbb{A}^1$ on $\mathbb{G}_{a(r)}$ by multiplication:

$$\mathbb{G}_{a(r)} \times \mathbb{A}^1 \xrightarrow{s \times a \mapsto sa} \mathbb{G}_{a(r)}\,.$$

We proceed to prove that this action is compatible with the action of $\mathbb{A}^1$ on $V_r(G)$ which defines the grading on A in the sense that the following diagram commutes:

(2.11.1)
$$\begin{tikzcd}
\mathbb{G}_{a(r)} \times \mathbb{A}^1 \times V_r(G) \arrow[r, "\mathrm{id} \times \mathrm{action}"] \arrow[d, "\mathrm{action} \times \mathrm{id}"] & \mathbb{G}_{a(r)} \times V_r(G) \arrow[r, "\mathcal{U}_{G,r}"] & G \times V_r(G) \arrow[d, "pr_G"] \\
\mathbb{G}_{a(r)} \times V_r(G) \arrow[r, "\mathcal{U}_{G,r}"] & G \times V_r(G) \arrow[r, "pr_G"] & G.
\end{tikzcd}$$

Commutativity of (2.11.1) is equivalent to the commutativity of the corresponding diagram of S-valued points for any choice of finitely generated commutative k-algebras S and element $a \in S$:
(2.11.2)

$$\begin{tikzcd}
\mathbb{G}_{a(r)}(S) \times V_r(G)(S) \arrow[r, "1\times a"] \arrow[d, "a\times 1"] & \mathbb{G}_{a(r)}(S) \times V_r(G)(S) \arrow[r, "\mathcal{U}_{G,r}(S)"] & G(S) \times V_r(G)(S) \arrow[d, "pr_G"] \\
\mathbb{G}_{a(r)}(S) \times V_r(G)(S) \arrow[r, "\mathcal{U}_{G,r}(S)"] & G(S) \times V_r(G)(S) \arrow[r, "pr_G"] & G(S).
\end{tikzcd}$$

Choose an embedding of G into some $\mathrm{GL}_{N(r)}$. Using Proposition 2.9 and the naturality with respect to change of G of the action of $\mathbb{A}^1$ on $V_r(G)$, we can compare the diagram (2.11.2) for G and for $\mathrm{GL}_{N(r)}$. The injectivity of $G(S) \to \mathrm{GL}_{N(r)}(S)$ implies that it suffices to assume that $G = \mathrm{GL}_{N(r)}$. Let $s \in \mathbb{G}_{a(r)}(S)$, $\underline{\alpha} = (\alpha_0, \dots, \alpha_{r-1}) \in V_r(\mathrm{GL}_N)(S)$. Then $a \circ \underline{\alpha} = (a\alpha_0, a^p\alpha_1, \dots, a^{p^{r-1}}\alpha_{r-1})$, so $\exp_{\underline{\alpha}}(s) = \exp(s\alpha_0)\exp(s^p\alpha_1)\cdots\exp(s^{p^{r-1}}\alpha_{p-1}) \in \mathrm{GL}_{N(r)}(S)$ by (1.10.1). Thus, restricted to the point $(s, \underline{\alpha}) \in (\mathbb{G}_{a(r)} \times V_r(\mathrm{GL}_N))(S)$, (2.11.2) becomes

$$\begin{tikzcd}
(s, \underline{\alpha}) \arrow[r, "1\times a"] \arrow[d, "a\times 1"] & (s, a\circ\underline{\alpha}) \arrow[r, "\mathcal{U}_{G,r}"] & (\exp_{a\circ\underline{\alpha}}(s), a\circ\underline{\alpha}) \arrow[d] \\
(as, \underline{\alpha}) \arrow[r, "\mathcal{U}_{G,r}"] & (\exp_{\underline{\alpha}}(as), \underline{\alpha}) \arrow[r] & \exp_{\underline{\alpha}}(as).
\end{tikzcd} \tag{2.11.3}$$

Commutativity of (2.11.3) is implied by the evident equality $\exp_{a\circ\underline{\alpha}}(s) = \exp_{\underline{\alpha}}(as)$.

Consequently, we have a commutative diagram on coordinate algebras corresponding to (2.11.1):

$$\begin{tikzcd}
A \otimes k[t] \otimes k[\mathbb{G}_{a(r)}] & A \otimes k[\mathbb{G}_{a(r)}] \arrow[l, "\mathrm{act}^*\otimes\mathrm{id}"'] & A \otimes k[G] \arrow[l, "\mathcal{U}^*_{G,r}"'] \\
A \otimes k[\mathbb{G}_{a(r)}] \arrow[u, "\mathrm{id}\otimes\mathrm{act}^*"] & A \otimes k[G] \arrow[l, "\mathcal{U}^*_{G,r}"'] & k[G]. \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd} \tag{2.11.4}$$

The map $\mathrm{act}^* : A \longrightarrow A \otimes k[t] = A \otimes k[\mathbb{A}^1]$ of the upper horizontal arrow is the map on coordinate algebras which corresponds to the grading on A. The left vertical map corresponds to the grading on $k[\mathbb{G}_{a(r)}] \simeq k[T]/T^{p^r}$ and is given explicitly by $T \mapsto t \otimes T$.

For $\lambda \in k[G]$, write $\mathcal{U}^*_{G,r}(1 \otimes \lambda) = \sum f_i \otimes c_i T^i \in A \otimes k[\mathbb{G}_{a(r)}]$. The composition of the lower horizontal and left vertical maps of (2.11.4) sends λ to $\sum f_i \otimes t^i \otimes c_i T^i$. On the other hand, the composition of the right vertical and upper horizontal maps of (2.11.4) sends λ to $\sum \mathrm{act}^*(f_i) \otimes c_i T^i$. We conclude that

$$f_i \otimes t^i = \mathrm{act}^*(f_i),$$

so that f_i is homogeneous of degree i. $\square$

As a corollary (of the proof) of Proposition 2.11, we see why for G infinitesimal of height $\leq r$ the homogeneous degree of $\Theta_{G,r} \in kG \otimes k[V_r(G)]$ is p^{r-1}, whereas the homogeneous degree of $\Theta_{G,r+1} \in kG \otimes k[V_{r+1}(G)]$ is p^r.

Corollary 2.12. *Let G be an infinitesimal group of height $\leq r$. Then the map $i^* : k[V_{r+1}(G)] \to k[V_r(G)]$ of Proposition 2.2 is a graded isomorphism which divides degrees by p.*

Proof. Let $\pi^* : k[V_r(G)] \to k[V_{r+1}(G)]$ be the inverse of i^*. The commutativity of (2.11.1) implies that we may compute the effect on degree of π^* by identifying the effect on degree of the map $p^* : k[\mathbb{G}_{a(r)}] = k[t]/t^{p^r} \ \to \ k[t]/t^{p^{r+1}} = k[\mathbb{G}_{a(r+1)}]$. Yet this map clearly multiplies degree by p. □

3. Local Jordan type

The purpose of this section is to exploit our universal p-nilpotent operator Θ_G to investigate the local Jordan type of a finite dimensional kG-module M. The local Jordan type of M gives much more detailed information about a kG-module M than the information which can be obtained from the support variety (or, rank variety) of M. In this section, we work through various examples, give an algorithm for computing local Jordan types, and understand the effect of Frobenius twists. Moreover, we establish restrictions on the rank and dimension of kG-modules of constant Jordan type.

Definition 3.1. Let G be an infinitesimal group scheme and $v \in V(G)$. Let $k(v)$ denote the residue field of $V(G)$ at v, and let

$$\mu_v = \mathcal{U}_G \otimes_{k[V(G)]} k(v) : \mathbb{G}_{a(r),k(v)} \ \to \ G_{k(v)}$$

be the associated 1-parameter subgroup (for $r \geq \mathrm{ht(G)}$). We define the *local p-nilpotent operator at v, θ_v,* to be

$$\theta_v = \Theta_G \otimes_{k[V(G)]} k(v) = \ \mu_{v*}(u_{r-1}) \ \in \ k(v)G.$$

Equivalently, for a $k(v)$-rational point $v : \operatorname{Spec} k(v) \to V(G)$, we define $\theta_v = \Theta_G(v)$, the evaluation of Θ_G at v:

$$\theta_v : k[G] \xrightarrow{\Theta_G} k[V(G)] \longrightarrow k(v),$$

where the second map corresponds to the point v.

In the special case that $G = \mathrm{GL}_{n(r)}$ for some $n > 0$, we use the alternate notation $\theta_{\underline{\alpha}}$ for the local p-nilpotent operator at $\underline{\alpha} = (\alpha_0, \ldots, \alpha_{r-1}) \in V(\mathrm{GL}_{n(r)}) \simeq N_p^{[r]}(gl_n)$:

$$\theta_{\underline{\alpha}} \ = \ \exp_{\underline{\alpha},*}(u_{r-1}) \ \in \ k(\underline{\alpha}) \, \mathrm{GL}_{n(r)}, \tag{3.1.1}$$

where $k(\underline{\alpha})$ is the residue field of $\underline{\alpha} \in V(\mathrm{GL}_{n(r)})$.

Let K be a field. Then a finite dimensional $K[u]/u^p$-module M is a direct sum of cyclic modules of dimension ranging from 1 to p. We may thus write $M \simeq a_p[p] + \cdots + a_1[1]$, where $[i]$ is the cyclic $K[u]/u^p$-module $K[u]/u^i$ of dimension i. We refer to the p-tuple

$$\mathrm{JType}(M, u) \ = \ (a_p, \ldots, a_1) \tag{3.1.2}$$

as the *Jordan type* of the $K[u]/u^p$-module M. We also refer to $\mathrm{JType}(M, u)$ as the Jordan type of the p-nilpotent operator u on M.

For simplicity, we introduce the following notation.

Definition 3.2. With notation as in Definition 3.1, we set

$$\mathrm{JType}(M,\theta_v) \;\equiv\; \mathrm{JType}((\mu_{v,*})^*(M_{k(v)}),u_{r-1}).$$

We refer to this Jordan type as the *local Jordan type* of M at $v \in V(G)$.

Remark 3.3. Essentially by definition, the rank variety $V(G)_M$ of a finite dimensional kG-module for an infinitesimal group scheme G is the closed, reduced subscheme of $V(G)$ consisting of those points $v \in V(G)$ at which the local Jordan type of M has some Jordan block of size $< p$. In other words, those $v \in V(G)$ for which $\mathrm{JType}(M,\theta_v) \;\neq\; \frac{\dim M}{p}[p]$.

The following proposition will enable us to make more concrete and explicit the local Jordan type of a kG-module M at a given 1-parameter subgroup of G.

Proposition 3.4. *Let $\underline{\alpha} = (\alpha_0,\dots,\alpha_{r-1}) \in V(\mathrm{GL}_{n(r)})$ be an r-tuple of p-nilpotent pairwise commuting matrices. Let M be a $k\,\mathrm{GL}_{(r)}$-module of dimension N, and let $\rho : \mathrm{GL}_{m(r)} \to \mathrm{GL}_N$ be the associated structure map. The (i,j)-matrix entry of the action of the local p-nilpotent operator $\theta_{\underline{\alpha}} \in k(\underline{\alpha})\,\mathrm{GL}_{n(r)}$ of* (3.1.1) *on M equals the coefficient of $T^{p^{r-1}}$ of*

$$(\exp_{\underline{\alpha}})^*(\rho^* X_{i,j}) \;\in\; k(\underline{\alpha})[\mathbb{G}_{a(r)}] \simeq k(\underline{\alpha})[T]/T^{p^r},$$

where $\{X_{i,j}, 1 \le i,j \le N\}$ are the matrix coordinate functions of GL_N.

Proof. Let $\langle m_i\rangle_{1\le i\le N}$ be the basis of M corresponding to the structure map ρ. The structure of M as a comodule for $k[\mathrm{GL}_{n(r)}]$ is given by

$$M \to M \otimes k[\mathrm{GL}_{n(r)}], \quad m_j \mapsto \sum_i m_i \otimes \rho^* X_{i,j},$$

and thus the comodule structure of $M_{k(\underline{\alpha})}$ for $k(\underline{\alpha})[\mathbb{G}_{a(r)}]$ is given by

$$M \to M \otimes k(\underline{\alpha})[\mathbb{G}_{a(r)}], \quad m_j \mapsto \sum_i m_i \otimes \exp^*_{\underline{\alpha}}(\rho^* X_{i,j}).$$

The proposition follows from the fact that $u_{r-1} : k(\underline{\alpha})[\mathbb{G}_{a(r)}] \to k(\underline{\alpha})$ is given by reading the coefficient of $T^{p^{r-1}} \in k(\underline{\alpha})[\mathbb{G}_{a(r)}]$. ☐

Example 3.5. We investigate the local Jordan type of the various representations considered in Example 2.8.

(1) Consider the adjoint representation $M = \mathfrak{g}^{\mathrm{ad}}$ of a p-restricted Lie algebra $\mathfrak{g}$ and a 1-parameter subgroup

$$\mu_X : \mathbb{G}_{a(1),K} \;\to\; \underline{\mathfrak{g}}_K, \quad \text{inducing} \;\; K[u]/u^p \;\to\; \mathfrak{u}(\mathfrak{g}_K)$$

sending u to some p-nilpotent $X \in \mathfrak{g}_K$. The local Jordan type of $\mathfrak{g}^{\mathrm{ad}}$ at μ_X is simply the Jordan type of the endomorphism $\mathrm{ad}_X : \mathfrak{g}_K^{ad} \to \mathfrak{g}_K^{\mathrm{ad}}$,

$$\mathrm{JType}(\mathfrak{g}^{\mathrm{ad}},\theta_X) \;=\; \mathrm{JType}(X).$$

(2) Let $M = k[\mathbb{G}_{a(r)}]/(u_0) \simeq k[u_1,\dots,u_{r-1}]/(u_1^p,\dots,u_{p-1}^p)$ be a cyclic $k\mathbb{G}_{a(r)} = k[u_0,\dots,u_{r-1}]/(u_0^p,\dots,u_{r-1}^p)$-module, and let $\mu_{\underline{a}} : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(r)}$ be a 1-parameter subgroup for some $\underline{a}$ of $V(\mathbb{G}_{a(r)}) = \mathbb{A}^r$. Then

$$\mathrm{JType}(M,\theta_{\underline{a}}) = \begin{cases} p^{r-2}[p], & \exists i > 0,\; a_i \neq 0, \\ p^{r-1}[1], & \text{otherwise.} \end{cases}$$

(3) Let $G = \mathrm{GL}_{n(r)}$, and let V_n be the canonical n-dimensional rational representation of $\mathrm{GL}_{n(r)}$. We apply Proposition 3.4, observing that ρ for V_n is simply the natural inclusion $\mathrm{GL}_{n(r)} \subset \mathrm{GL}_n$. Since

$$\exp^*_{\underline{\alpha}}(X_{i,j}) = \sum_{\ell=0}^{p^r-1} [\beta_\ell]_{i,j} t^\ell, \tag{3.5.1}$$

where β_ℓ are matrices determined by α_i as in Proposition 1.7, we conclude that

$$\mathrm{JType}(V_n, \theta_{\underline{\alpha}}) \ = \ \mathrm{JType}(\alpha_{r-1}).$$

Specializing to $r = 2$,

$$\mathrm{JType}(V_n, \theta_{(\alpha_0,\alpha_1)}) \ = \ \alpha_1.$$

(4) "Specializing" to $G = \mathrm{SL}_{2(2)}$, consider $\underline{\alpha} = (\begin{bmatrix} c_0 & a_0 \\ b_0 & -c_0 \end{bmatrix}, \begin{bmatrix} c_1 & a_1 \\ b_1 & -c_1 \end{bmatrix})$. Then $\mathrm{JType}(V_2, \theta_{\underline{\alpha}})$ equals the Jordan type of the matrix $\begin{bmatrix} c_1 & a_1 \\ b_1 & -c_1 \end{bmatrix}$.

We extend Example 3.5(3) by considering tensor powers $V_n^{\otimes d}$ of the canonical rational representation of GL_n restricted to $\mathrm{GL}_{n(2)}$. In this example, the role of both entries of the pair $\underline{\alpha} = (\alpha_0, \alpha_1)$ is non-trivial.

Example 3.6. Consider the $N = n^d$-dimensional rational GL_n-module $M = V_n^{\otimes d}$ where V_n is the canonical n-dimensional rational GL_n-module. Let $\rho : \mathrm{GL}_{n(r)} \to \mathrm{GL}_N$ be the representation of M restricted to $\mathrm{GL}_{n(r)}$. A basis of M is $\{e_{i_1} \otimes \cdots \otimes e_{i_d};\ 1 \leq i_j \leq n\}$, where $\{e_{i_j}; 1 \leq i_j \leq n\}$ is a basis for V_n for each $j, 1 \leq j \leq d$. Let $\{X_{i_1,j_1;\dots;i_d,j_d},\ 1 \leq i_t, j_t \leq n\}$ denote the matrix coordinate functions on GL_N, and let $\{Y_{s,t},\ 1 \leq s, t \leq n\}$ denote the matrix coordinate functions of GL_n.

Then $\rho^* : k[\mathrm{GL}_N] \to k[\mathrm{GL}_{n(r)}]$ is given by

$$X_{i_1,j_1;\dots;i_d,j_d} \mapsto \ Y_{i_1,j_1} \cdots Y_{i_d,j_d}.$$

Thus,

$$(\exp_{\underline{\alpha}})^*(\rho^*(X_{i_1,j_1;\dots;i_d,j_d})) \ = \ (\exp_{\underline{\alpha}})^*(Y_{i_1,j_1}) \cdots (\exp_{\underline{\alpha}})^*(Y_{i_d,j_d}).$$

Now, specialize to $r = 2$ so that we can make this more explicit. Then the coefficient of T^p of $(\exp_{(\alpha_0,\alpha_1)})^*(\rho^*(X_{i_1,j_1;\dots;i_d,j_d}))$ is

$$\sum_{k=1}^{d} (\alpha_1)_{i_k,j_k} + \sum_{\substack{0 \leq f_k < p \\ f_1+\cdots+f_d=p}} \frac{1}{f_1!} \cdots \frac{1}{f_d!} ((\alpha_0)^{f_1})_{i_1,j_1} \cdots ((\alpha_0)^{f_d})_{i_d,j_d}. \tag{3.6.1}$$

This gives the action of $\theta_{(\alpha_0,\alpha_1)}$ on M.

To simplify matters even further, consider the special case $(\alpha_0)^2 = 0$. For $1 \leq d < p$, $\theta_{(\alpha_0,\alpha_1)}$ on M is given by the $N \times N$-matrix

$$(i_1, j_1; \dots; i_d, j_d) \mapsto (\sum_{k=1}^{d} (\alpha_1)_{i_k,j_k}).$$

For $d = p$, the action of $\theta_{(\alpha_0,\alpha_1)}$ on M is given by the $N \times N$-matrix

$$(i_1, j_1; \dots; i_p, j_p) \mapsto (\sum_{k=1}^{p} (\alpha_1)_{i_k,j_k} + (\alpha_0)_{i_1,j_1} \cdots (\alpha_0)_{i_p,j_p}).$$

An analogous calculation applies to the d-fold symmetric product $S^d(V_n)$ and d-fold exterior product $\Lambda^d(V_n)$ of the canonical n-dimensional rational GL_n-module V_n.

The proof of Proposition 3.4 applies equally well to prove the following straightforward generalization, which one may view as an algorithmic method of computing the "local Jordan type" of a kG-module M of dimension N. The required input is an explicit description of the map on coordinate algebras ρ^* given by $\rho : G \to \mathrm{GL}_N$ determining the kG-module M.

Theorem 3.7. *Let G be an infinitesimal group scheme of height $\leq r$, and let $\rho : G \to \mathrm{GL}_N$ be a representation of G on a vector space M of dimension N. Consider some $v \in V(G)$, and let $\mu_v : \mathbb{G}_{a(r),k(v)} \to G_{k(v)}$ be the corresponding 1-parameter subgroup of height r. Then the (i,j)-matrix entry of the action of $\theta_v \in k(v)G$ on M equals the coefficient of $T^{p^{r-1}}$ of*

$$(\mu_v)^*(\rho^* X_{i,j}) \ \in \ k(v)[\mathbb{G}_{a(r)}],$$

where $\{X_{i,j}, 1 \leq i,j \leq N\}$ are the matrix coordinate functions of GL_N.

As a simple corollary of Theorem 3.7, we give a criterion for the local Jordan type of the kG-module M to be trivial (i.e., equal to $(\dim M)[1]$) at a 1-parameter subgroup $\mu_v,\ v \in V(G)$.

Corollary 3.8. *With the hypotheses and notation of Theorem 3.7,*

$$\mathrm{JType}(M,\theta_v) \ = \ \mathrm{JType}((\mu_{v,*})^*(M_{k(v)}), u_{r-1}) \ = \ (\dim M)[1]$$

if $\deg\,(\rho \circ \mu_v)^*(X_{i,j}) < p^{r-1}$ *for all* $1 \leq i,j \leq N$.

One means of constructing kG-modules is by applying Frobenius twists to known kG-modules. Our next objective is to establish (in Proposition 3.10) a simple relationship between the p-nilpotent operator $\theta_{\underline{\alpha}}$ on a $k\,\mathrm{GL}_{n(r)}$-module M and $\theta_{\underline{\alpha}}$ on the s^{th} Frobenius twist $M^{(s)}$ of M for any $0 \neq v \in V(\mathrm{GL}_{n(r)})$.

Before formulating this relationship, we make explicit the definition of the Frobenius map for an arbitrary affine group scheme over k. Let G be an affine group scheme over k and define for any $s > 0$ the s^{th} Frobenius map $F^s : G \to G^{(s)}$ given by the k-linear algebra homomorphism

$$F^{s*} : k[G^{(s)}] = k \otimes_{p^s} k[G] \ \to \ k[G], \quad a \otimes f \mapsto a \cdot f^{p^s}, \tag{3.8.1}$$

where $k \otimes_{p^s} k[G]$ is the base change of $k[G]$ along the p^s-power map $k \to k$ (an isomorphism only for k perfect). If G is defined over $\mathbb{F}_{p^s}$ (for example, if $G = \mathrm{GL}_n$), then we have a natural isomorphism

$$k[G] = k \otimes p^s \mathbb{F}_{p^s}[G] \xrightarrow{\sim} k \otimes_{p^s} k \otimes p^s \mathbb{F}_{p^s}[G] = k[G^{(s)}]$$

so that F^s can be viewed as a self-map of G.

Definition 3.9. If M is a $k[G]$-comodule, then the s^{th} Frobenius twist $M^{(s)}$ of M is the k-vector space $k \otimes_{p^s} M$ equipped with the comodule structure

$$F^{s*} \circ (k \otimes_{p^s} \nabla_M) : M^{(s)} \to M^{(s)} \otimes k[G^{(s)}] \to M^{(s)} \otimes k[G].$$

If G is a finite group scheme, then we shall view $M^{(s)}$ as a kG-module via the map $F^s_* : kG \to kG^{(s)}$ dual to (3.8.1).

To be more explicit, suppose the N-dimensional kG-module M is given by $\rho : G \to \mathrm{GL}_N$ (so that $M = \rho^*(V_N)$, where V_N is the canonical N-dimensional GL_N-module) and assume that G is defined over $\mathbb{F}_{p^s}$. Let $\mu_v : \mathbb{G}_{a(r),K} \to G_K$ be a 1-parameter subgroup, corresponding to some $v \in V(G)$. Then the identification of $M^{(s)}$ with $(\rho \circ F^s)^*(V_N)$ implies that

$$\text{(3.9.1)} \qquad \mathrm{JType}(M^{(s)}, \theta_v) \;=\; \mathrm{JType}(M, \theta_{F^s(v)}),$$

where $\theta_{F^s(v)} = (F^s \circ \mu_v)_*(u_{r-1})$.

Let $G = \mathrm{GL}_{n(r)}$, and let R be a finitely generated commutative k-algebra. The Frobenius self-map is given explicitly on the R-values of $\mathrm{GL}_{n(r)}$ by the formula

$$F : \alpha \mapsto \phi(\alpha),$$

where ϕ applied to $\alpha \in M_n(R)$ raises each entry of α to the p^{th} power. For t in $\mathbb{G}_{a(r)}(R)$, we compute

$$(F \circ \exp_{(\alpha_0,\dots,\alpha_{r-1})})(t) = F(\exp(t\alpha_0)\exp(t^p\alpha_1)\cdots\exp(t^{p^{r-1}}\alpha_{r-1}))$$
$$= \exp(t^p\phi(\alpha_0))\exp(t^{p^2}\phi(\alpha_1))\cdots\exp(t^{p^{r-1}}\phi(\alpha_{r-2})) = \exp_{(0,\phi(\alpha_0),\dots,\phi(\alpha_{r-2}))}(t).$$

Iterating s times, we obtain the following formula for $G = \mathrm{GL}_{n(r)}$:

$$\text{(3.9.2)} \qquad F^s \circ \exp_{(\alpha_0,\dots,\alpha_{r-1})} \;=\; \exp_{(0,0,\dots,0,\phi^s(\alpha_0),\dots,\phi^s(\alpha_{r-1-s}))},$$

where the first non-zero entry on the right happens at the $(s+1)^{\text{st}}$ place.

For $G = \mathbb{G}_{a(r)}$, the Frobenius map $F : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(r)}$ is given by raising an element $a \in \mathbb{G}_{a(r)}(R)$ to the p^{th} power. Let $\underline{a} = (a_0,\dots,a_{r-1})$ be a point in $V(\mathbb{G}_{a(r)}) \simeq \mathbb{A}^r$, and let $\mu_{\underline{a}} : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(r)}$ be the corresponding 1-parameter subgroup. For $t \in \mathbb{G}_{a(r)}(R)$, we have $\mu(t) = a_0 + a_1 t + \cdots + a_{r-1}t^{p^{r-1}}$ (see [25, §1]). The following formula is now immediate:

$$\text{(3.9.3)} \qquad F^s \circ \mu_{(a_0,\dots,a_{r-1})} \;=\; \mu_{(0,\dots,0,a_0^{p^s},\dots,a_{r-1-s}^{p^s})}.$$

Combining (3.9.1) and (3.9.2), we derive the following proposition.

Proposition 3.10. *Let M be a finite dimensional representation of $\mathrm{GL}_{n(r)}$ and let $\underline{\alpha} = (\alpha_0,\dots,\alpha_{r-1})$ be a point in $V(\mathrm{GL}_{n(r)})$. Then*

$$\mathrm{JType}(M^{(s)}, \theta_{\underline{\alpha}}) \;=\; \mathrm{JType}(M, \theta_{F^s\circ\underline{\alpha}}),$$

where $F^s \circ \underline{\alpha} = (0,\dots,0,\phi^s(\alpha_1),\dots,\phi^s(\alpha_{r-1-s}))$.

If M is a finite dimensional $k\mathbb{G}_{a(r)}$-module and $\underline{a} = (a_0,\dots,a_{r-1})$ is a point in $V(\mathbb{G}_{a(r)}) \simeq \mathbb{A}^r$, then

$$\mathrm{JType}(M^{(s)}, \theta_{\underline{a}}) \;=\; \mathrm{JType}(M, \theta_{F^s\circ\underline{a}}),$$

where $F^s \circ \underline{a} = (0,\dots,0,a_0^{p^s},\dots,a_{r-1-s}^{p^s})$.

Proposition 3.10 has the following immediate corollary (see Remark 3.3).

Corollary 3.11. *Let M be a finite dimensional representation of $\mathrm{GL}_{n(r)}$. Then $\underline{\alpha} = (\alpha_0,\dots,\alpha_{r-1}) \in V(\mathrm{GL}_{n(r)})_M$ if and only if*

$$F^s \circ \alpha = (0,\dots,0,\phi^s(\alpha_0),\dots,\phi^s(\alpha_{r-1-s})) \in V(\mathrm{GL}_{n(r)})_{M^{(s)}}.$$

The following definition introduces interesting classes of kG-modules which have special local behavior.

Definition 3.12. Let G be an infinitesimal group scheme and j a positive integer less than p. A finite dimensional kG-module M is said to be of constant j-rank if and only if

$$\mathrm{rk}(M, \theta_v^j) \;\equiv\; \mathrm{rk}\{\theta_v^j : M_{k(v)} \to M_{k(v)}\}$$

is independent of $v \in V(G) - \{0\}$, where θ_v is the local p-nilpotent operator at v as introduced in Definition 3.1.

M is said to be of constant Jordan type if and only if it is of constant j-rank for all j, $1 \leq j < p$. M is said to be of constant rank if it is of constant 1-rank.

As we see in the following example, one can have rational GL_n-modules of constant Jordan type when restricted to $\mathrm{GL}_{n(r)}$ of arbitrarily high degree d. This should be contrasted with Corollary 3.17.

Example 3.13. Consider the rational GL_n-module $M = \det^{\otimes d}$, the d^{th} power of the determinant representation for some $d > 0$. This is a polynomial representation of degree n^d. The restriction of M to any Frobenius kernel $\mathrm{GL}_{n(r)}$ has (trivial) constant Jordan type, for the further restriction of M to any abelian unipotent subgroup of GL_n has trivial action.

We shall see below that kG-modules of constant j-rank lead to interesting constructions of vector bundles (see Theorem 5.1). We conclude this section by establishing two constraints, Propositions 3.15 and 3.19, on kG-modules to be modules of constant rank.

We first need the following elementary lemma.

Lemma 3.14. *Let M be a $\mathbb{G}_{a(r)}$-module such that the local Jordan type at every $v \in V(\mathbb{G}_{a(r)})$ is trivial. Then M is trivial as a $k\mathbb{G}_{a(r)}$-module.*

Proof. The action of $\mathbb{G}_{a(r)}$ on M is given by the action of r commuting p-nilpotent operators $\tilde{u}_i$, $0 \leq i < p$, on M. Moreover, for $\underline{a} = (0, \ldots, 1, 0, \ldots, 0)$, with 1 at the i^{th} spot,

$$\mathrm{JType}(M, \theta_{\underline{a}}) \;=\; \mathrm{JType}(M, \tilde{u}_i),$$

as follows from the explicit description of $\Theta_{\mathbb{G}_{a(r)}}$ in Example 2.6(2). Thus, if the local Jordan type of M is trivial at each $\underline{a} = (0, \ldots, 1, 0, \ldots, 0)$, then each $\tilde{u}_i$ must act trivially on M, and M is therefore a trivial $\mathbb{G}_{a(r)}$-module. $\square$

Proposition 3.15. *Let $\mathfrak{G}$ be an algebraic group generated by 1-parameter subgroups $i : \mathbb{G}_a \subset \mathfrak{G}$. Let $\rho : \mathfrak{G} \to \mathrm{GL}_N$ determine a non-trivial rational representation M of $\mathfrak{G}$. Let D_i be the minimum of the degrees of $(\rho \circ i)^*(X_{s,t}) \in k[\mathbb{G}_a] = k[T]$ as $X_{s,t}$ ranges over the matrix coordinate functions of GL_N. Let $D = \max\{D_i, i : \mathbb{G}_a \subset \mathfrak{G}\}$.*

If $r > \log_p D + 1$, then M is not of constant Jordan type as a $\mathfrak{G}_{(r)}$-module.

Proof. Because M is non-trivial and $\mathfrak{G}$ is generated by its 1-parameter subgroups, we conclude that i^*M is a non-trivial rational $\mathbb{G}_a$ representation for some 1-parameter subgroup $i : \mathbb{G}_a \subset \mathfrak{G}$. The condition $r > \log_p D \geq \log_p D_i$ implies that i^*M is not r-twisted (i.e., of the form $N^{(r)}$). Lemma 3.14 implies that the local Jordan type of i^*M at some 1-parameter subgroup $\mu_v : \mathbb{G}_{a(r),k(v)} \to G_{k(v)}$ is non-trivial. On the other hand, Corollary 3.8 implies that the Jordan type of i^*M at the identity 1-parameter subgroup $\mathrm{id} : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(r)}$ is trivial provided that $r - 1 > \log_p D$. $\square$

As an immediate corollary, we conclude the following.

Corollary 3.16. *Let $\mathfrak{G}$ be an algebraic group generated by 1-parameter subgroups and let M be a non-trivial rational representation of $\mathfrak{G}$. Then for $r \gg 0$, M is not of constant Jordan type as a $k\mathfrak{G}_{(r)}$-module.*

As an explicit example of Proposition 3.15, we obtain the following corollary (which should be contrasted with Example 3.13).

Corollary 3.17. *Let M be a non-trivial polynomial representation of SL_n of degree D. If $r > \log_p D + 1$, then M is not a $k\,\mathrm{SL}_{n(r)}$-module of constant rank.*

The following lemma, which is a straightforward application of the Generalized Principal Ideal Theorem (see [12, 10.9]), shows that the dimension of a non-trivial module of constant rank of $\mathbb{G}_{a(r)}$ cannot be "too small" compared to r.

Lemma 3.18. *Let M be a finite dimensional $\mathbb{G}_{a(r)}$-module. If M is a non-trivial $\mathbb{G}_{a(r)}$-module of constant rank, then the following inequality holds:*

$$\dim_k M \geq \sqrt{r}. \tag{3.18.1}$$

Proof. By extending scalars if necessary we may assume that k is algebraically closed. Let $m = \dim_k M$. Let $k\mathbb{G}_{a(r)} = k[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p)$, let $K = k(s_0, \dots, s_{r-1})$ where s_i are independent variables, and let $\alpha_K : K[t]/t^p \to K\mathbb{G}_{a(r)}$ be a map of K–algebras defined by $\alpha_K(t) = s_0u_0 + \cdots + s_{r-1}u_{r-1}$. Choose a k-linear basis of M, and let $A(s_0, \dots, s_{r-1})$ be a nilpotent matrix in $M_m(k[s_0, \dots, s_{r-1}])$ representing the action of $\alpha_K(t)$ on M_K. Let $I_n(A(s_0, \dots, s_{r-1}))$ denote the ideal generated by all $n \times n$ minors of $A(s_0, \dots, s_{r-1})$. By [12, 10.9], the codimension of any minimal prime over $I_n(A(s_0, \dots, s_{r-1}))$ is at most $(m - n + 1)^2$.

Assume that (3.18.1) does not hold, that is, $m < \sqrt{r}$. Hence, $(m - n + 1)^2 < r$ for any $1 \leq n \leq m$. The variety of $I_n(A(s_0, \dots, s_{r-1}))$ is a subvariety inside $\operatorname{Spec} k[s_0, \dots, s_{r-1}] \simeq \mathbb{A}^r$ which has dimension r. Since the codimension of the variety of $I_n(A(s_0, \dots, s_{r-1}))$ is at most $(m-n+1)^2$, we conlude that the dimension is at least $r-(m-n+1)^2 \geq 1$. Hence, the minors of dimension $n \times n$ have a common non-trivial zero. Taking $n = 1$, we conclude that $A(b_0, \dots, b_{r-1})$ is a zero matrix for some non-zero specialization $b_0, \dots, b_{r-1}$ of $s_0, \dots, s_{r-1}$. Consequently, M is trivial at the π-point of $\mathbb{G}_{a(r)}$ corresponding to $b_0, \dots, b_{r-1}$. Since M is non-trivial, Lemma 3.14 implies that M is not a module of constant rank. $\square$

As an immediate corollary, we provide an additional necessary condition for a $k\mathfrak{G}_{(r)}$-module to have constant rank.

Proposition 3.19. *Let $\mathfrak{G}$ be a (reduced) affine algebraic group and M be a rational representation of $\mathfrak{G}$. Assume that $\mathfrak{G}$ admits a 1-parameter subgroup $\mu : \mathbb{G}_{a(r)} \to \mathfrak{G}$ such that $\mu^*(M)$ is a non-trivial $k\mathbb{G}_{a(r)}$-module. If $r \geq (\dim M)^2 + 1$, then M is not a $k\mathfrak{G}_{(r)}$-module of constant rank.*

4. π-points and $\mathbb{P}(\mathrm{G})$

In a series of earlier papers, we have considered π-points for a finite group scheme G (as recalled in Definition 4.1) and investigated finite dimensional kG-modules M using the "Jordan type of M" at various π-points. In particular, in [18], we verified that this Jordan type is independent of the equivalence class of the π-point provided that either the π-point is generic or the Jordan type of M at some representative of the equivalence class is maximal.

As we recall below, whenever G is an infinitesimal group scheme, the π-point space $\Pi(G)$ of equivalence classes of π-points is essentially the projectivization of $V(G)$. The purpose of the first half of this section is to relate the discussion of the previous section concerning the local Jordan type of a finite kG-module to our earlier work formulated in terms of π-points for general finite group schemes.

One special aspect of an infinitesimal group scheme G is that equivalence classes of π-points of G have canonical (up to scalar multiple) representatives.

Unless otherwise specified (as in Definition 4.1 immediately below), G will denote an infinitesimal group scheme over k, and $V(G)$ will denote $V_r(G)$ for some $r \geq \mathrm{ht(G)}$. Throughout this section we assume that $\dim V(G) \geq 1$, and we work with $\mathbb{P}(\mathrm{G}) = \operatorname{Proj} k[V(G)]$. We note that if $\dim V(G) = 0$, then Theorem 1.16 implies that the projective resolution of the trivial module k is finite. Since kG is self-injective, this further implies that k is projective. Hence, kG is semi-simple and does not have any π-points (see, for example, [15, §2]).

Definition 4.1 (see [17])**.** Let G be a finite group scheme.

(1) A π-point of G is a (left) flat map of K-algebras $\alpha_K : K[t]/t^p \to KG$ for some field extension K/k with the property that there exists a unipotent abelian closed subgroup scheme $i : C_K \subset G_K$ defined over K such that α_K factors through $i_* : KC_K \to KG_K = KG$.
(2) If $\beta_L : L[t]/t^p \to LG$ is another π-point of G, then α_K is said to be a *specialization* of β_L, written $\beta_L \downarrow \alpha_K$, provided that for any finite dimensional kG-module M, $\alpha_K^*(M_K)$ being free as a $K[t]/t^p$-module implies that $\beta_L^*(M_L)$ is free as an $L[t]/t^p$-module.
(3) Two π-points $\alpha_K : K[t]/t^p \to KG$, $\beta_L : L[t]/t^p \to LG$ are said to be *equivalent*, written $\alpha_K \sim \beta_L$, if $\alpha_K \downarrow \beta_L$ and $\beta_L \downarrow \alpha_K$.
(4) A π-point of G, $\alpha_K : K[t]/t^p \to KG$, is said to be *generic* if there does not exist another π-point $\beta_L : L[t]/t^p \to LG$ which specializes to α_K but is not equivalent to α_K.
(5) If M is a finite dimensional kG-module and $\alpha_K : K[t]/t^p \to KG$ a π-point of G, then the Jordan type of M at α_K is by definition the Jordan type of $\alpha_K^*(M_K)$ as a $K[t]/t^p$-module.

Because the group algebra of a finite group scheme is always faithfully flat over the group algebra of a subgroup scheme (see [27, 14.1]), the condition on a flat map $\alpha_K : K[t]/t^p \to KG$ is equivalent to the existence of a factorization $i_* \circ \alpha_K'$ with $\alpha_K' : K[t]/t^p \to KC_K$ flat.

Definition 4.2. Let G be an infinitesimal group scheme, and let $v \in V(G)$ be the point associated to the 1-parameter subgroup $\mu_v : \mathbb{G}_{a(r),k(v)} \to G_{k(v)}$. Then the π-point of G associated to v is

$$\mu_{v,*} \circ \epsilon : k(v)[u]/u^p \;\to\; k(v)G,$$

where $\epsilon : k(v)[u]/u^p \to k(v)\mathbb{G}_{a(r),k(v)}$ is as defined in (1.14.1).

The following theorem is a complement to Theorem 1.16, revealing that spaces of (equivalence) classes of π-points are very closely related to (cohomological) support varieties.

Theorem 4.3 ([17, 7.5])**.** *Let G be a finite group scheme. Then the set of equivalence classes of π-points, denoted $\Pi(G)$, can be given a scheme structure, which is* *defined in terms of the representation theory of G. Moreover, there is an isomorphism of schemes*

$$\mathrm{Proj}\ \mathrm{H}^\bullet(G,k) \ \simeq\ \Pi(G).$$

If G is an infinitesimal group scheme so that $\mathrm{H}^\bullet(G,k)$ is related to $k[V(G)]$ as in Theorem 1.16*, then the resulting homeomorphism*

$$\mathbb{P}(G) \equiv \mathrm{Proj}\, k[V(G)] \ \longrightarrow\ \mathrm{Proj}\,\mathrm{H}^\bullet(G,k) \ \simeq\ \Pi(G) \tag{4.3.1}$$

is given on points by sending $x \in \mathbb{P}(G)$ to the equivalence class of the π-point $\mu_{v,*}\circ\epsilon$ for any $v \in V(G)\backslash\{0\}$ projecting to x. In particular, equivalence classes of generic π-points of G are represented by $(\mu_{v,*}\circ\epsilon)$ as $v \in V(G)$ runs through the (scheme-theoretic) generic points of $V(G)$.

Furthermore, for any finite dimensional kG-module M, (4.3.1) *restricts to a homeomorphism of subvarieties*

$$\mathbb{P}(G)_M \ \simeq\ \Pi(G)_M,$$

where $\mathbb{P}(G)_M = \mathrm{Proj}(k[V(G)_M])$ and $\Pi(G)_M$ consists of those equivalence classes of π-points α_K of G such that $\alpha_K^*(M_K)$ is not free (as a $K[u]/u^p$-module).

Generic π-points are particularly important when developing invariants of representations. The following corollary of Theorem 4.3 gives an explicit set of representatives of equivalence classes of generic π-points of G.

Proposition 4.4. *Let G be an infinitesimal group scheme with universal* 1*-parameter subgroup $\mathcal{U}_G : \mathbb{G}_{a(r),k[V(G)]} \longrightarrow G_{k[V(G)]}$. For each minimal prime ideal $\mathcal{P}_i$ of $k[V(G)]$, let K_i denote the field of fractions of $k[V(G)]/\mathcal{P}_i$. Then the compositions*

$$(\mathcal{U}_{G,*} \otimes_{k[V(G)]} K_i)\circ\epsilon : K_i[u]/u^p \to K_iG$$

(sending u to θ_{K_i}) are non-equivalent representatives of the equivalence classes of generic π-points of G.

To obtain vector bundles, we require the following well known, elementary observation about commutative graded algebras.

Lemma 4.5. *Let A be a finitely generated commutative, graded k-algebra with homogeneous generators whose degrees divide d and let $X = \mathrm{Proj}\,A$. Then $\mathcal{O}_X(d)$ is a locally free sheaf of rank* 1 *on X.*

Proof. Let $\{f_i\}$ be a finite set of homogeneous generators of A, set d_i equal to the degree of f_i, and choose d such that each d_i divides d. Set $U_i = X - Z(f_i)$, an affine open subset of X with coordinate algebra $(A_{f_i})_0$, the degree zero subalgebra of the localization of A obtained by inverting f_i. Then multiplication by f_i^{d/d_i} induces an isomorphism $(\mathcal{O}_X)_{|U_i} \xrightarrow{\sim} (\mathcal{O}_X(d))_{|U_i}$. Thus the restriction of $\mathcal{O}_X(d)$ to each U_i of the open covering $\{U_i\}$ is free of rank 1. $\square$

Let G be an infinitesimal group scheme of height $\le r$ and recall from Proposition 2.11 that $\Theta_{G,r} \in kG \otimes k[V_r(G)]$ is homogeneous of degree p^{r-1}.

Definition 4.6. Let G be an infinitesimal group scheme of height $\le r$ and let M be a finite dimensional kG-module. Then we denote by

$$\widetilde{\Theta}_G : \mathcal{M} \equiv M \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})} \ \longrightarrow\ \mathcal{M}(p^{r-1}) \equiv M \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}(p^{r-1}) \tag{4.6.1}$$

the homomorphism of (locally free) coherent $\mathcal{O}_{\mathbb{P}(\mathrm{G})}$-modules determined by the action of $\Theta_{G,r} \in kG \otimes k[V_r(G)]$.

We denote by

$$\widetilde{\Theta}_G(n) : \mathcal{M}(n) \longrightarrow \mathcal{M}(p^{r-1}+n) \tag{4.6.2}$$

the map obtained by tensoring (4.6.1) with $\mathcal{O}_{\mathbb{P}(G)}(n)$.

For any point $x \in \mathbb{P}(\mathrm{G})$, we use the notation

$$M_{k(x)} = \mathcal{M} \otimes_{\mathcal{O}_{\mathbb{P}(G)}} k(x)$$

for the fiber of the coherent sheaf $\mathcal{M}$ at x. Here, we have identified $k(x)$ with the residue field of the stalk $\mathcal{O}_{\mathbb{P}(\mathrm{G}),x}$.

Proposition 4.7. *Let G be an infinitesimal group scheme of height $\leq r$, and let M be a finite dimensional kG-module. For any $v,\ v' \in V(G)$ projecting to the same $x \in \mathbb{P}(\mathrm{G})$, we have*

$$\mathrm{Im}\{\theta_v : M_{k(v)} \to M_{k(v)}\} \ \simeq \ \mathrm{Im}\{\theta_{v'} : M_{k(v')} \to M_{k(v')}\},$$

and similarly for kernels.

Proof. This is essentially proved in [26, 6.1]. □

In the next section, we shall be particularly interested in kernels and images of $\widetilde{\Theta}_G$. The following proposition relates the fibers of the kernel and image of the global p-nilpotent operator $\widetilde{\Theta}_G$ at a point $x \in \mathbb{P}(G)$ with the kernel and image of the local p-nilpotent operator θ_v on $M \otimes k(v)$ for v representing x.

Proposition 4.8. *Let G be an infinitesimal group scheme of height $\leq r$, let M be a finite dimensional kG-module, let $\mathcal{M} = M \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}$, and let $s \in \Gamma(\mathbb{P}(\mathrm{G}), \mathcal{O}_{\mathbb{P}(\mathrm{G})}(p^{r-1}))$ be a non-zero global section with zero locus $Z(s) \subset \mathbb{P}(\mathrm{G})$. Set $U = \mathbb{P}(\mathrm{G}) \backslash Z(s)$. Then there is a well defined endomorphism (depending upon s)*

$$\widetilde{\Theta}_G/s : \mathcal{M}_{|U} \ \to \ \mathcal{M}_{|U}.$$

Moreover, the image and kernel of the induced map $\theta_x/s : M_{k(x)} \to M_{k(x)}$ on fibers at $x \in U \subset \mathbb{P}(\mathrm{G})$ is independent of s and satisfies

$$\mathrm{Im}\{\theta_x/s : M_{k(x)} \to M_{k(x)}\} \simeq \mathrm{Im}\{\theta_v : M_{k(v)} \to M_{k(v)}\} \tag{4.8.1}$$

and

$$\mathrm{Ker}\{\theta_x/s : M_{k(x)} \to M_{k(x)}\} \simeq \mathrm{Ker}\{\theta_v : M_{k(v)} \to M_{k(v)}\} \tag{4.8.2}$$

for any $v \in V(G) \backslash \{0\}$ that projects onto x.

Proof. Let $X = \mathbb{P}(\mathrm{G})$ and let $\frac{1}{s} \in \mathcal{O}_X(-p^{r-1})(U)$ satisfy

$$s \otimes \frac{1}{s} = 1 \ \in \mathcal{O}_X(p^{r-1})(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(-p^{r-1})(U) \ \simeq \ \mathcal{O}_X(U).$$

Then we define

$$\widetilde{\Theta}_G/s \equiv (\widetilde{\Theta}_G)_{|U} \otimes \frac{1}{s} : \mathcal{M}_{|U} \to \mathcal{M}(p^{r-1})_{|U} \otimes \mathcal{O}_X(-p^{r-1})_{|U} \ \simeq \ \mathcal{M}_{|U}.$$

Let $A = k[V(G)]$. Then U is an affine subscheme of $\mathbb{P}(\mathrm{G})$ determined by the 0-degree elements of $A\left[\frac{1}{s}\right]$, $U \simeq \operatorname{Spec} A\left[\frac{1}{s}\right]_0$. Via this identification, $\mathcal{M}_{|U}$ corresponds to the free $A\left[\frac{1}{s}\right]_0$-module $M \otimes A\left[\frac{1}{s}\right]_0$. The fiber $M_{k(x)}$ of $\mathcal{M}_{|U}$ at the point x is naturally identified with the fiber of $M \otimes A\left[\frac{1}{s}\right]_0$ at x. Since

$\Theta_G : M \otimes k[V(G)] \to M \otimes k[V(G)]$ is homogeneous of degree p^{r-1} and $s \in k[V(G)]$ is homogeneous of the same degree, the operator $\Theta_G \otimes \frac{1}{s}$ is well defined on $M \otimes A\left[\frac{1}{s}\right]_0$ and corresponds to the operator $\widetilde{\Theta}_G/s$ on $\mathcal{M}_{|U}$. Hence, $\theta_x/s : M_{k(x)} \to M_{k(x)}$ is identified with

$$(\widetilde{\Theta}_G \otimes \frac{1}{s}) \otimes k(x) = \frac{\Theta_G}{s}(x) : M_{k(x)} \to M_{k(x)}.$$

Let $v \in V(G)$ be any point projecting onto x. We have $k(x) = k(v)$. By Definition 3.1, the map $\theta_v : M_{k(v)} \to M_{k(v)}$ is given by

$$\Theta_G(v) : M_{k(v)} \to M_{k(v)}.$$

We observe that $\frac{\Theta_G}{s}(x) = \frac{\Theta_G(v)}{s(v)}$ for any v projecting onto x (and, in particular, is independent of the choice of v). Therefore,

$$\theta_x/s \;=\; \frac{\theta_v}{s(v)}. \tag{4.8.3}$$

The equalities(4.8.2) and (4.8.1) now follow. □

Remark 4.9. For a finite group G, there is no natural choice of π-point representing a typical equivalence class $x \in \Pi(G) \simeq \operatorname{Proj} \mathrm{H}^\bullet(G,k)$ of π-points. As seen in elementary examples [18, 2.3], the Jordan type of a kG-module M typically can be different for two equivalent π-points representing the same point $x \in \Pi(G)$.

Remark 4.10. Proposition 4.8 immediately generalizes to $\widetilde{\Theta}_G^j$ for any $1 \le j \le p-1$. Thus, we have the following isomorphisms for any $x \in X = \mathbb{P}(\mathrm{G})$, $v \in V(G)$ projecting onto x, and a global section s of $\mathcal{O}_X(jp^{r-1})$ such that $s(x) \neq 0$:

$$\begin{aligned}\operatorname{Im}\{(\theta_x/s)^j : M_{k(x)} \to M_{k(x)}\} &\simeq \operatorname{Im}\{\theta_v^j : M_{k(v)} \to M_{k(v)}\}\\ \simeq \operatorname{Im}\{\widetilde{\Theta}_G^j \otimes_{\mathcal{O}_X} k(x) : \mathcal{M} \otimes_{\mathcal{O}_X} k(x) &\to \mathcal{M}(jp^{r-1}) \otimes_{\mathcal{O}_X} k(x)\},\end{aligned}$$

and similarly for kernels.

In what follows, we shall use the following abbreviations:

$$\begin{aligned}
\operatorname{Im}\{\widetilde{\Theta}_G^j, \mathcal{M}\} &\equiv \operatorname{Im}\{\widetilde{\Theta}_G^j(-jp^{r-1}) : \mathcal{M}(-jp^{r-1}) \to \mathcal{M}\},\\
\operatorname{Im}\{\theta_x^j, M_{k(x)}\} &\equiv \operatorname{Im}\{(\theta_x/s)^j : M_{k(x)} \to M_{k(x)}\},\\
\operatorname{Ker}\{\widetilde{\Theta}_G^j, \mathcal{M}\} &\equiv \operatorname{Ker}\{\widetilde{\Theta}_G^j : \mathcal{M} \to \mathcal{M}(jp^{r-1})\},\\
\operatorname{Ker}\{\theta_x^j, M_{k(x)}\} &\equiv \operatorname{Ker}\{(\theta_x/s)^j : M_{k(x)} \to M_{k(x)}\},\\
\operatorname{Coker}\{\widetilde{\Theta}_G^j, \mathcal{M}\} &\equiv \mathcal{M}/\operatorname{Im}\{\widetilde{\Theta}_G^j, \mathcal{M}\},\\
\operatorname{Coker}\{\theta_x^j, M_{k(x)}\} &\equiv \operatorname{Coker}\{(\theta_x/s)^j : M_{k(x)} \to M_{k(x)}\}.
\end{aligned} \tag{4.10.1}$$

Note that both Ker and Im are subsheaves of the free sheaf $\mathcal{M}$ and that Coker is a quotient sheaf of $\mathcal{M}$.

We shall verify in Theorem 4.13 that a necessary and sufficient condition on a finite dimensional kG-module M for $\operatorname{Im}\{\widetilde{\Theta}_G^j, \mathcal{M}\}$ (and thus $\operatorname{Ker}\{\widetilde{\Theta}_G^j, \mathcal{M}\}$) to be an algebraic vector bundle on X is that M be a module of constant j-type.

The following proposition is given in [20, 5, ex. 5.8] without proof.

Proposition 4.11. *Let X be a reduced scheme and $\widetilde{M}$ a coherent $\mathcal{O}_X$-module. Then $\widetilde{M}$ is locally free if and only if $\dim_{k(x)}(\widetilde{M} \otimes_{\mathcal{O}_X} k(x))$ depends only upon the connected component of x in $\pi_0(X)$.*

Proof. Assume that the function $x \mapsto \dim_{k(x)}(\widetilde{M} \otimes_{\mathcal{O}_X} k(x))$ is constant on each connected component of X. To prove that $\widetilde{M}$ is locally free it suffices to assume that X is local so that $X = \operatorname{Spec} R$ for some reduced local commutative ring, and that M is a finite R-module (corresponding to the coherent sheaf $\widetilde{M}$) with the property that $\dim_{k(p)}(M \otimes_R k(p))$ is independent of the prime $p \subset R$. To prove that M is free, we choose some surjective R-module homomorphism $g : Q \to M$ from a free R-module $Q \simeq R^n$ with the property that $\overline{g} : Q \otimes_R R/\mathfrak{m} \ \to \ M \otimes_R R/\mathfrak{m}$ is an isomorphism where $\mathfrak{m} \subset R$ is the maximal ideal. Then g is surjective by Nakayama's Lemma. By assumption, g induces an isomorphism after specialization to any prime $\mathfrak{p} \subset R$: $Q \otimes_{R_\mathfrak{p}} k(\mathfrak{p}) \simeq M \otimes_{R_\mathfrak{p}} k(\mathfrak{p})$. Hence, $Q_\mathfrak{p}/\mathfrak{p}Q_\mathfrak{p} \simeq M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$. We conclude that if $a \in \ker g$, then $a \in \mathfrak{p}Q_\mathfrak{p} \cap Q$. Since this happens for any prime ideal, we further conclude that $\ker g \subset (\bigcap\limits_{\mathfrak{p} \in \operatorname{Spec} R} \mathfrak{p}Q_\mathfrak{p}) \cap Q$. Recall that Q is a free module so that $Q \simeq R^n$. We get $(\bigcap \mathfrak{p}Q_\mathfrak{p}) \cap Q = (\bigcap \mathfrak{p}R^n_\mathfrak{p}) \cap R^n = ((\bigcap \mathfrak{p}R_\mathfrak{p}) \cap R)^n = (\bigcap \mathfrak{p}R)^n = 0$, since R is reduced. □

We shall find it convenient to "localize" the notion of a kG-module of constant j-rank given in Definition 3.12 as follows.

Definition 4.12. Let G be an infinitesimal group scheme, and let M be a finite dimensional kG-module. For any open subset $U \subset \mathbb{P}(\mathrm{G})$, M is said to be of constant j-rank when restricted to U if $\mathrm{rk}_{k(x)}((\theta_x/s)^j : M_{k(x)} \to M_{k(x)})$ is independent of $x \in U$.

Our next theorem emphasizes the local nature of the concept of constant j-rank.

Theorem 4.13. *Let G be an infinitesimal group scheme, let M be a finite dimensional kG-module, and let $X = \mathbb{P}(G)$. Let $U \subset X$ be a connected open subset and $\widetilde{\Theta}^j_U : \mathcal{M}_{|U} \to \mathcal{M}(jp^{r-1})_{|U}$ be the restriction to U of the j^{th} iterate of $\widetilde{\Theta}_G$ on $\mathcal{M} = M \otimes \mathcal{O}_X$ as given in (4.6.1). Then the following are equivalent for some fixed j, $1 \leq j < p$:*

(1) $\mathrm{Im}\{\widetilde{\Theta}^j_U, \mathcal{M}_{|U}\}$ *is a locally free, coherent $\mathcal{O}_U$-module.*
(2) $\mathrm{Im}\{\widetilde{\Theta}^j_G, \mathcal{M}\} \otimes_{\mathcal{O}_X} k(x)$ *has dimension independent of $x \in U$.*
(3) $\mathrm{Im}\{\theta^j_x, M_{k(x)}\} \ \simeq \ \mathrm{Im}\{\widetilde{\Theta}^j_G, \mathcal{M}\} \otimes_{\mathcal{O}_X} k(x), \ \forall\ x \in U.$
(4) *M has constant j-rank when restricted to U.*

Moreover, each of these conditions implies that

(5) $\mathrm{Coker}\{\widetilde{\Theta}^j_U, \mathcal{M}_U\}$ *is a locally free, coherent $\mathcal{O}_U$-module.*
(6) $\mathrm{Coker}\{\theta^j_x, M_{k(x)}\} \ \simeq \ \mathrm{Coker}\{\widetilde{\Theta}^j_G, \mathcal{M}\} \otimes_{\mathcal{O}_X} k(x), \ \forall\ x \in U.$
(7) $\mathrm{Ker}\{\widetilde{\Theta}^j_U, \mathcal{M}_U\}$ *is a locally free, coherent $\mathcal{O}_U$-module.*
(8) $\mathrm{Ker}\{\theta^j_x, M_{k(x)}\} \ \simeq \ \mathrm{Ker}\{\widetilde{\Theta}^j_G, \mathcal{M}\} \otimes_{\mathcal{O}_X} k(x), \ \forall\ x \in U.$

Proof. Clearly, (1) implies (2), whereas Proposition 4.11 implies that (2) implies (1).

If we assume (1), we obtain a locally split short exact sequence of coherent $\mathcal{O}_U$-modules

$$0 \ \to \ \mathrm{Ker}\{\widetilde{\Theta}^j_U, \mathcal{M}_{|U}\} \ \to \ \mathcal{M}_{|U} \ \to \ \mathrm{Im}\{\widetilde{\Theta}^j_U, \mathcal{M}_{|U}\} \to 0. \tag{4.13.1}$$

In particular, $\mathrm{Ker}\{\widetilde{\Theta}_U^j, \mathcal{M}_{|U}\}$ is a locally free, coherent $\mathcal{O}_U$-module. Locally on U, $\widetilde{\Theta}_U^j$ on $\mathcal{M}_{|U}$ is isomorphic to the projection

$$pr_2 : \mathrm{Ker}\{\widetilde{\Theta}_U^j, \mathcal{M}_{|U}\} \oplus \mathrm{Im}\{\widetilde{\Theta}_U^j, \mathcal{M}_{|U}\} \ \to \ \mathrm{Im}\{\widetilde{\Theta}_U^j, \mathcal{M}_{|U}\}.$$

Since θ_x^j is the base change via $\mathcal{O}_U \to k(x)$ of $\widetilde{\Theta}_U^j$, θ_x^j can be identified with the base change of this projection, and thus we may conclude (3).

Let us now assume (3). A simple argument using Nakayama's Lemma as in the proof of Proposition 4.11 implies that the function $x \mapsto \mathrm{Im}\{\theta_x^j, M_{k(x)}\}$ is lower semi-continuous on U, whereas the function $x \mapsto \mathrm{Im}\{\widetilde{\Theta}_G^j, \mathcal{M}\} \otimes_{\mathcal{O}_X} k(x)$ is upper semi-continuous on U. Thus, we conclude that each of these functions is constant (since U is connected), thereby concluding (2).

Since $\mathrm{rk}\{(\theta_x/s)^j\} \ = \ \dim_{k(x)}(\mathrm{Im}\{\theta_x^j, M_{k(x)}\})$, (2) and (3) imply (4).

Observe that if $f : V \to V$ is an endomorphism of a finite dimensional vector space, then $\dim\{\mathrm{Coker}\, f\} = \dim\{\mathrm{Ker}\, f\}$. The assumption that the kG-module M has constant rank (i.e., (4)) implies that

$$\dim_{k(x)}(\mathrm{Coker}\{\theta_x^j, M_{k(x)}\}) \ = \ \dim_{k(x)}(\mathrm{Ker}\{\theta_x^j, M_{k(x)}\})$$

is independent of $x \in U$. Hence, Proposition 4.11 implies (5). The right exactness of $(-) \otimes_{\mathcal{O}_X} k(x)$ applied to

$$\mathcal{M}(-jp^{r-1}) \xrightarrow{\widetilde{\Theta}_G^j(-jp^{r-1})} \mathcal{M} \longrightarrow \mathrm{Coker}\{\widetilde{\Theta}_G^j, \mathcal{M}\} \longrightarrow 0$$

implies (6).

Under the assumption of (5), we obtain a locally split short exact sequence of coherent $\mathcal{O}_U$-modules,

$$0 \to \mathrm{Im}\{\widetilde{\Theta}_U^j, \mathcal{M}_{|U}\} \to \mathcal{M}_{|U} \to \mathrm{Coker}\{\widetilde{\Theta}_U^j, \mathcal{M}_{|U}\} \to 0,$$

so that $\mathrm{Im}\{\widetilde{\Theta}_U^j, \mathcal{M}_{|U}\}$ is a locally free, coherent $\mathcal{O}_U$-module. Now, using the short exact sequence of coherent $\mathcal{O}_U$-modules,

$$0 \to \mathrm{Ker}\{\widetilde{\Theta}_U^j, \mathcal{M}_{|U}\} \ \to \ \mathcal{M}_{|U} \ \to \ \mathrm{Im}\{\widetilde{\Theta}_U^j, \mathcal{M}_{|U}\} \to 0,$$

we conclude that (4) implies (7) (i.e., that $\mathrm{Ker}\{\widetilde{\Theta}_U^j, \mathcal{M}_{|U}\}$ is locally free). Since the short exact sequence (4.13.1) is locally split, applying $(-) \otimes_{\mathcal{O}_X} k(x)$ to (4.13.1) for any $x \in U$ yields a short exact sequence, thereby implying (8). $\square$

5. Vector bundles for modules of constant j-rank

In this section, we initiate the study of algebraic vector bundles associated to kG-modules of constant j-rank as defined in Definition 3.12. Our constructions have two immediate consequences. The first is that certain kG-modules with the same "local Jordan type" have non-isomorphic associated vector bundles so that the isomorphism classes of these vector bundles serve as a new invariant. The second is that our construction yields vector bundles on the highly non-trivial projective schemes $\mathbb{P}(\mathrm{G})$.

The reader will find formulas for the ranks of bundles considered, criteria for non-triviality of bundles, a criterion for producing line bundles, a relationship to duality, and another test for the projectivity of kG-modules. We also investigate the dimension of global sections of various bundles.

As in Section 4, we assume that $\dim V(G) \geq 1$ throughout this section.

The special case in which $U = \mathbb{P}(\mathrm{G})$ of Theorem 4.13 is the following assertion that kG-modules of constant j-rank determine algebraic vector bundles over $\mathbb{P}(\mathrm{G})$.

Theorem 5.1. *Let G be an infinitesimal group scheme, let M be a finite dimensional kG-module, and let $\mathcal{M} = M \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}$ be a free coherent sheaf on $\mathbb{P}(\mathrm{G})$. Then M has constant j-rank if and only if $\mathrm{Im}\{\widetilde{\Theta}^j_G, \mathcal{M}\}$ is an algebraic vector bundle on $\mathbb{P}(\mathrm{G})$.*

Moreover, if M has constant j-rank, then $\mathrm{Ker}\{\widetilde{\Theta}^j_G, \mathcal{M}\}$ is an algebraic vector bundle on $\mathbb{P}(\mathrm{G})$ as well.

Remark 5.2. Unless M is trivial as a kG-module, $\mathrm{Ker}\{\Theta_G : M \otimes k[V(G)] \to M \otimes k[V(G)]\}$ is not projective as a $k[V(G)]$-module, since the local p-nilpotent operator θ_0 at $0 \in V(G)$ is the 0-map.

We observe the following elementary functoriality of this construction.

Proposition 5.3. *Let $i : H \to G$ be an embedding of infinitesimal group schemes, let M be a finite dimensional kG-module, and let N be the restriction of M to kH. Let $\mathcal{M} = M \otimes \mathcal{O}_{\mathbb{P}(G)}$ and $\mathcal{N} = N \otimes \mathcal{O}_{\mathbb{P}(H)}$. Then for any $j, 1 \leq j < p$, there are natural isomorphisms of coherent sheaves on $\mathbb{P}(H)$, where $f : \mathbb{P}(H) \to \mathbb{P}(G)$ is induced by i:*

$$\begin{aligned} f^* \,\mathrm{Im}\{\widetilde{\Theta}^j_G, \mathcal{M}\} \ &\simeq\ \mathrm{Im}\{\widetilde{\Theta}^j_H, \mathcal{N}\}, \\ f^* \,\mathrm{Ker}\{\widetilde{\Theta}^j_G, \mathcal{M}\} \ &\simeq\ \mathrm{Ker}\{\widetilde{\Theta}^j_H, \mathcal{N}\}. \end{aligned}$$

Proof. The statement follows immediately from the commutativity of the diagram

$$\begin{tikzcd} M \otimes \mathcal{O}_{\mathbb{P}(G)} \arrow[r, "\widetilde{\Theta}^j_G"] \arrow[d, "f^*"'] & M \otimes \mathcal{O}_{\mathbb{P}(G)}(jp^{r-1}) \arrow[d, "f^*"'] \\ N \otimes \mathcal{O}_{\mathbb{P}(H)} \arrow[r, "\widetilde{\Theta}^j_H"] & N \otimes \mathcal{O}_{\mathbb{P}(H)}(jp^{r-1}). \end{tikzcd} \tag{5.3.1}$$

The diagram is commutative by Proposition 2.9. □

The following corollary will be used later in Section 6.

Corollary 5.4. *Let G_1, G_2 be infinitesimal group schemes, let $G = G_1 \times G_2$, and let $f : \mathbb{P}(G_1) \to \mathbb{P}(G)$ be the natural embedding of varieties induced by the embedding of group schemes $i : G_1 \hookrightarrow G$. Let M_1, M_2 be kG_1, kG_2 modules of dimensions m_1, m_2 respectively. Then for any j, $1 \leq j \leq p$,*

$$f^*(\mathrm{Ker}\{\widetilde{\Theta}^j_G, \mathcal{M}_1 \boxtimes \mathcal{M}_2\}) \ \simeq\ \mathrm{Ker}\{\widetilde{\Theta}^j_{G_1}, \mathcal{M}_1\}^{\oplus m_2}. \tag{5.4.1}$$

Here, $\mathcal{M}_1 = M_1 \otimes \mathcal{O}_{\mathbb{P}(G_1)}$, $\mathcal{M}_2 = M_2 \otimes \mathcal{O}_{\mathbb{P}(G_2)}$, and $\mathcal{M}_1 \boxtimes \mathcal{M}_2 \simeq (M_1 \otimes M_2) \otimes \mathcal{O}_{\mathbb{P}(G)}$.

Proof. By Proposition 5.3, it suffices to observe that $(M_1 \otimes M_2)\downarrow_{G_1} \simeq M_1^{\oplus m_2}$ and that f^* and $\widetilde{\Theta}^j$ commute with direct sums. □

We have a duality for kernel and cokernel bundles. For a vector bundle $\mathcal{E}$ on a projective variety X, we denote by $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ the dual bundle.

Proposition 5.5. *Let M be a finite dimensional kG-module of constant j-rank. Let $\mathcal{N} = M^{\#} \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}$ and $\mathcal{M} = M \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}$. Then*

$$\mathrm{Ker}\{\widetilde{\Theta}_G^j, \mathcal{M}\}^\vee \simeq \mathrm{Coker}\{\widetilde{\Theta}_G^j, \mathcal{N}\}$$

as vector bundles on $\mathcal{O}_{\mathbb{P}(G)}$.

Proof. Choosing dual bases for M and $M^{\#}$, we get an isomorphism of trivial bundles $\mathcal{M}$ and $\mathcal{N}$. Hence, we may identify the dual bundle $\mathcal{M}(jp^{r-1})^\vee$ with $\mathcal{N}(-jp^{r-1})$. Under this identification, the $\mathcal{O}_{\mathbb{P}(G)}$-dual of the map

$$\widetilde{\Theta}_G^j : \mathcal{M} \to \mathcal{M}(jp^{r-1})$$

is identified with

$$\widetilde{\Theta}_G^j(-jp^{r-1}) : \mathcal{N}(-jp^{r-1}) \to \mathcal{N}.$$

Since $\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}(\mathrm{G})}}(-, \mathcal{O}_{\mathbb{P}(\mathrm{G})})$ vanishes on locally free sheaves, taking the $\mathcal{O}_{\mathbb{P}(G)}$-dual of the short exact sequence of vector bundles

$$0 \longrightarrow \mathrm{Ker}\{\widetilde{\Theta}_G^j, \mathcal{M}\} \longrightarrow \mathcal{M} \xrightarrow{\widetilde{\Theta}_G^j} \mathcal{M}(jp^{r-1}) \longrightarrow \mathrm{Coker}\{\widetilde{\Theta}^j, \mathcal{M}\}(jp^{r-1}) \longrightarrow 0,$$

we get the exact sequence

$$\begin{aligned} 0 \longleftarrow \mathrm{Ker}\{\widetilde{\Theta}_G^j, \mathcal{M}\}^\vee \longleftarrow \mathcal{N} \xleftarrow{\quad \Theta_G^j(-jp^{r-1}) \quad} \mathcal{N}(-jp^{r-1}) \\ \longleftarrow \mathrm{Coker}\{\widetilde{\Theta}^j, \mathcal{M}\}^\vee(-jp^{r-1}) \longleftarrow 0. \end{aligned}$$

Therefore, $\mathrm{Ker}\{\widetilde{\Theta}_G^j, \mathcal{M}\}^\vee \simeq \mathrm{Coker}\{\widetilde{\Theta}_G^j, \mathcal{N}\}$. □

Example 5.6. For each of our four examples of infinitesimal group schemes (initially investigated in Example 1.5), we give examples of kG-modules of constant Jordan type taken from [8].

(1) Let $\mathfrak{g}$ be a finite dimensional p-restricted Lie algebra of dimension at least 2. For any Tate cohomology class of negative dimension, $\zeta \in \widehat{H}^n(\mathfrak{u}(\mathfrak{g}), k) \simeq \mathrm{Ext}^1_{\mathfrak{u}(\mathfrak{g})}(\Omega^{n-1}(k), k)$, we consider the extension of $\mathfrak{u}(\mathfrak{g})$-modules

$$0 \longrightarrow k \longrightarrow M \longrightarrow \Omega^{n-1}(k) \longrightarrow 0$$

determined by ζ. By [8, 6.3], M is a $\mathfrak{u}(\mathfrak{g})$-module of constant Jordan type. We verify by inspection that the Jordan type of M is $(a, 0, \ldots, 0, 2)$ for some $a > 0$ if n is odd, and $(b, 1, 0, \ldots, 0, 1)$ for some $b > 0$ if n is even (see (3.1.2) for notation).

(2) Let $G = \mathbb{G}_{a(r)}$, and set I equal to the augmentation ideal of $kG \simeq k[u_0, \ldots, u_{p-1}]/(u_0^p, \ldots, u_{p-1}^p)$. As observed in [8], I^i/I^t is a module of constant Jordan type for any $t > i$. As proven in [9], the only ideals of $k\mathbb{G}_{a(2)}$ which are of constant Jordan type are of the form I^i.

(3) As observed in [8], the n^{th} syzygy module $\Omega^n(k)$, $n \in \mathbb{Z}$, is a module of constant Jordan type for any infinitesimal group scheme G. For n even, $\Omega^n(k)$ has constant Jordan type $(a, 0, \ldots, 0, 1)$ for some $a > 0$, whereas for n odd, $\Omega^n(k)$ has constant Jordan type $(b, p-1, 0, \ldots, 0)$ for some $b > 0$.

(4) For $G = \mathrm{SL}_{2(2)}$, we recall that the cohomology algebra $\mathrm{H}^\bullet(G, k)$ is generated modulo nilpotents by classes $\zeta_1, \zeta_2, \zeta_3 \in \mathrm{H}^2(G, k)$ and classes $\xi_1, \xi_2, \xi_3 \in \mathrm{H}^{2p}(G, k)$

([19]). As in [8, 6.8], the kG-module

$$M \equiv \operatorname{Ker}\{\sum \zeta_i + \sum \xi_j : (\Omega^2(k))^{\oplus 3} \oplus (\Omega^{2p}(k))^{\oplus 3} \to k\}$$

is a kG-module of constant Jordan type $(a, 0, \ldots, 0, 1)$ for some $a > 0$.

We elaborate on Example 5.6(2), constructing $\mathbb{G}_{a(r)}$-modules of constant j-rank but not of constant Jordan type.

Example 5.7. We start with the following simple observation. Let $M_1 \subset M_2 \subset M$ be a chain of k-vector spaces, and let ϕ be an endomorphism of M such that $\phi(M_1) \subset M_1$ and $\phi(M_2) \subset M_2$. If $\dim(\operatorname{Ker} \phi_{|M_1}) = \dim(\operatorname{Ker} \phi)$, then $\dim(\operatorname{Ker} \phi_{|M_1})$ $= \dim(\operatorname{Ker} \phi_{|M_2}) = \dim(\operatorname{Ker} \phi)$.

Let $G = \mathbb{G}_{a(r)}$, and set I equal to the augmentation ideal of

$$kG \simeq k[u_0, \ldots, u_{p-1}]/(u_0^p, \ldots, u_{p-1}^p).$$

Consider any ideal J of kG with the property that $I^i \subset J$ for some i, $i \leq p-1$. Note that for any $\underline{a} \in \mathbb{A}^r$ and any $j \leq p-i$,

$$\dim(\operatorname{Ker}\{\theta_{\underline{a}}^j : I^i \to I^i\}) = pj = \dim(\operatorname{Ker}\{\theta_{\underline{a}}^j : kG \to kG\}). \tag{5.7.1}$$

Indeed, since I^i is a module of constant Jordan type, it suffices to check the statement for $\theta_{\underline{a}} = u_0$ for which it is straightforward. The observation in the previous paragraph together with (5.7.1) and the inclusions $I^i \subset J \subset kG$ imply

$$\dim(\operatorname{Ker}\{\theta_{\underline{a}}^j : J \to J\}) = pj$$

for any $j \leq p-i$ and any $\underline{a} \in \mathbb{A}^r$. Hence, J has constant j-rank for $1 \leq j \leq p-i$.

In the following example, we offer a method applicable to almost all infinitesimal group schemes G of constructing kG-modules which are of constant rank but not constant Jordan type.

Example 5.8. Let G be an infinitesimal group scheme with the property that $V(G)$ has dimension at least 2. Assume that p is odd, and let $n > 0$ be an odd positive integer. Let $\zeta \in \mathrm{H}^n(G, k)$ be a non-zero cohomology class and let M denote the kernel of $\zeta : \Omega^n(k) \to k$. Then M has constant rank but not constant Jordan type. Namely, the local Jordan type of M at $0 \neq v \in V(G)$ is $(a, 0, 1, 0, \ldots, 0)$ if $\zeta(v) \neq 0$ and is $(a-1, 2, 0, \ldots, 0)$ if $\zeta(v) = 0$. These Jordan types have the same rank.

For $G = \mathrm{SL}_{2(1)}$, the restriction of any rational SL_2–module is a module of constant Jordan type (see [8]). Irreducible SL_2-modules S_λ are parameterized by their highest weight, a non-negative integer λ. Irreducible $\mathrm{SL}_{2(1)}$ modules are the restrictions of S_λ to $\mathrm{SL}_{2(1)}$ for $0 \leq \lambda \leq p-1$.

Another important family of SL_2-modules are the V_λ (also denoted $H^0(\lambda)$) defined as the subspace of $k[s,t]$ (i.e., the symmetric algebra on the natural 2-dimensional representation for SL_2) consisting of homogeneous vectors of degree λ. For $0 \leq \lambda \leq p-1$, we have an isomorphism of $\mathrm{SL}_{2(1)}$–modules: $S_\lambda \simeq V_\lambda$.

Recall that $V(G)$ is the nullcone in sl_2, and, hence,

$$A = k[V(G)] \simeq k[x, y, z]/(xy + z^2).$$

Let

$$i : \mathbb{P}^1 \to \mathbb{P}(G) \tag{5.8.1}$$

be the isomorphism given on homogeneous coordinates by

$$\frac{k[x,y,z]}{(xy+z^2)} \to k[s,t], \qquad (x,y,z) \mapsto (s^2,-t^2,st).$$

In the next proposition, we compute explicitly the kernel bundles associated to the irreducible $\mathrm{SL}_{2(1)}$-modules and to the induced modules V_λ for $p \le \lambda \le 2p-2$. For convenience, we give the answer in terms of pull-backs to $\mathbb{P}^1$ via the isomorphism i.

Proposition 5.9. *Let $G = \mathrm{SL}_{2(1)}$, and let $i : \mathbb{P}^1 \overset{\sim}{\to} \mathbb{P}(G)$ be the isomorphism defined in* (5.8.1).

(1) *For $0 \le \lambda \le p-1$,*

$$i^*(\mathrm{Ker}\{\widetilde{\Theta}_G, S_\lambda \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}\}) \simeq \mathcal{O}_{\mathbb{P}^1}(-\lambda).$$

(2) *For $p \le \lambda \le 2p-2$,*

$$i^*(\mathrm{Ker}\{\widetilde{\Theta}_G, V_\lambda \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}\}) \simeq \mathcal{O}_{\mathbb{P}^1}(-\lambda) \oplus \mathcal{O}_{\mathbb{P}^1}(\lambda - 2(p-1)).$$

Proof. We adopt the conventions of [2, §1]; in particular, we replace λ by m. Let $v_0, v_1, \ldots, v_m$ be a basis for V_m such that the generators e, f and h of sl_2 act as follows:

$hv_i = (2i-m)v_i$,
$fv_i = (m-i+1)v_{i-1}$ for $i>0$, $fv_0 = 0$,
$ev_i = (i+1)v_{i+1}$ for $i<m$, $ev_m = 0$

(see [21, 7.2] or [2, §1]). Recall that

$$\Theta_G = xe + yf + zh$$

(see Example 2.8(1)). Hence, the operator

$$\Theta_G : V_m \otimes A \simeq A^{m+1} \longrightarrow V_m \otimes A \simeq A^{m+1}$$

is represented by the matrix

$$(5.9.1) \qquad \begin{pmatrix} -mz & my & 0 & \ldots & \ldots & \\ x & -(m-2)z & (m-1)y & 0 & \ldots & \\ 0 & 2x & -(m-4)z & (m-2)y & 0 & \ldots \\ 0 & 0 & 3x & -(m-6)z & (m-3)y & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & 0 & mx & mz \end{pmatrix}.$$

Substituting $(s^2, -t^2, st)$ for (x,y,z), we get a degree two operator on $k[s,t]^{m+1}$ given by
(5.9.2)

$$B_m(s,t) = \begin{pmatrix} -mst & -mt^2 & 0 & \ldots & \ldots & \\ s^2 & (2-m)st & (1-m)t^2 & 0 & \ldots & \\ 0 & 2s^2 & (4-m)st & (2-m)t^2 & 0 & \ldots \\ 0 & 0 & 3s^2 & (6-m)st & (3-m)t^2 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & 0 & ms^2 & mst \end{pmatrix}.$$

One easily verifies that the vector

$$w_m \;=\; [t^m, -st^{m-1}, s^2t^{m-2}, \ldots, \pm s^m]$$

is annihilated by $B_m(s,t)$. For $0 \le m \le p-1$, the module V_m is irreducible and the kernel bundle has rank 1 because the (constant) Jordan type of V_m has a single block. The vector w_m generates the kernel as a graded $k[s,t]$-module (no element of smaller degree lies in the kernel). Because w_m is homogeneous of degree m in $k[s,t]^{m+1}$, we conclude for $0 \le m \le p-1$ that

$$i^*(\mathrm{Ker}\{\widetilde{\Theta}_G, V_m \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}\}) \simeq \mathcal{O}_{\mathbb{P}^1}(-m), \quad 0 \le m \le p-1.$$

For $p \le m \le 2p-2$, V_m has a decomposition series which can be represented as follows:

$$\begin{tikzcd} S_\mu \arrow[dr] & & S_\mu \arrow[dl] \\ & S_\lambda & \end{tikzcd}$$

where $\lambda = 2(p-1) - m$, $\mu = m - p = p - 2 - \lambda$, and S_λ, the irreducible module of highest weight λ, is the socle of V_m ([2, §1]). By [8], V_m has constant Jordan type. Plugging $x = 1$, $y = z = 0$ in (5.9.1), we get that the Jordan type is $[p] + [\mu + 1]$. In particular, the rank of the kernel bundle is 2.

Using the relations $\mu \equiv m \pmod p$ and $-\lambda \equiv \mu + 2 \pmod p$, we obtain that the matrix $B_m(s,t)$ has the following form:
(5.9.3)

$$B_m(s,t) = \left[\begin{array}{c|c|c} B_\mu(s,t) & 0 & \\ \hline {\scriptstyle (\mu+1)s^2} & B_\lambda(t,s)^T & {\scriptstyle (\lambda+1)t^2} \\ \hline & 0 & B_\mu(s,t) \end{array}\right].$$

Here, the top left and bottom right corners are of size $\mu + 1 \times \mu + 1$, whereas the matrix in the center is of size $\lambda + 1 \times \lambda + 1$. The only non-zero entries outside of these three square diagonal blocks are at $(\mu + 2, \mu + 1)$ and $(\mu + \lambda + 2, \mu + \lambda + 3)$, equaled to $(\mu + 1)s^2$ and $(\lambda + 1)t^2$ respectively.

In particular, the $\mu + 1 \times \lambda + 1$ blocks above and below the matrix $B_\lambda(t,s)^T$ in the middle are zero. This implies that the kernel of this operator contains a copy of the kernel of $B_\lambda(s,t)$ (since it coincides with the kernel of $B_\lambda(t,s)^T$). We therefore obtain the vector

$$w'_m \;=\; [0, \ldots, 0, t^\lambda, -st^{\lambda-1}, \ldots, \mp s^\lambda, 0, \ldots, 0]$$

in the kernel, where the non-zero entries are at the positions $(\mu + 2, \ldots, \mu + \lambda + 2)$.

One verifies that $\{w_m, w'_m\}$ generate the kernel of $B_m(s,t)$ as a graded $k[s,t]$-module. For example, one can check this by restricting to the affine pieces $U(s \ne 0)$ and $U(t \ne 0)$. Hence,

$$i^*(\mathrm{Ker}\{\widetilde{\Theta}_G, V_m \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}\}) \simeq \mathcal{O}_{\mathbb{P}^1}(-m) \oplus \mathcal{O}_{\mathbb{P}^1}(m - 2(p-1)).$$

□

One may readily determine the rank of various bundles of $\mathbb{P}(G)$ associated to modules of constant Jordan type using the next proposition.

Proposition 5.10. *Let G be an infinitesimal group scheme, let M be a kG-module of constant Jordan type $\sum_{i=1}^{p} a_i[i]$, and let $\mathcal{M} = M \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}$. Then for any $j, 1 \leq j < p$,*

$$\operatorname{rk}(\operatorname{Im}\{\widetilde{\Theta}_G^j, \mathcal{M}\}) = \sum_{i=j+1}^{p} a_i(i-j). \tag{5.10.1}$$

In particular,

$$\operatorname{Ker}\{\widetilde{\Theta}_G, \mathcal{M}\} \subset \operatorname{Ker}\{\widetilde{\Theta}_G^2, \mathcal{M}\} \subset \cdots \subset \operatorname{Ker}\{\widetilde{\Theta}_G^{p-1}, \mathcal{M}\} \subset \mathcal{M}$$

is a chain of $\mathcal{O}_{\mathbb{P}(\mathrm{G})}$-submodules with $\operatorname{rk}(\operatorname{Ker}\{\widetilde{\Theta}_G^{j-1}, \mathcal{M}\}) < \operatorname{rk}(\operatorname{Ker}\{\widetilde{\Theta}_G^j, \mathcal{M}\})$ if and only if $a_i \neq 0$ for some $1 \leq j \leq i \leq p$.

Proof. The formula (5.10.1) is the formula for the rank of u^j on the $k[u]/u^p$-module $\bigoplus_i (k[u]/u^i)^{\oplus a_i}$ of Jordan type $\sum_{i=1}^{p} a_i[i]$. This is therefore the dimension of the image of θ_v, $0 \neq v \in V(G)$ on $M_{k(v)}$, and thus the rank of the vector bundle $\operatorname{Im}\{\widetilde{\Theta}_G^j, \mathcal{M}\}$ by Theorem 4.13. $\square$

The following class of modules, of interest in its own right, is currently being studied by Jon Carlson and the authors.

Definition 5.11. Let G be an infinitesimal group scheme, M a finite dimensional kG-module, and $j < p$ a positive integer. We say that a kG-module M has the *constant j-image property* if there exists a subspace $I(j) \subset M$ such that for every $v \neq 0$ in $V(G)$, the image of $\theta_v^j : M_{k(v)} \to M_{k(v)}$ equals $I(j)_{k(v)}$. Similarly, we say that M has *constant j-kernel property* if there exists some submodule $K(j) \subset M$ such that for every $v \neq 0$ in $V(G)$, the kernel of $\theta_v^j : M_{k(v)} \to M_{k(v)}$ equals $K(j)_{k(v)}$.

We see that these modules are precisely those whose associated vector bundles are trivial vector bundles.

Proposition 5.12. *Let G be an infinitesimal group scheme, and let M be a kG-module of constant j-rank. Then the algebraic vector bundle $\operatorname{Im}\{\widetilde{\Theta}_G^j, \mathcal{M}\}$ is trivial (i.e., a free coherent sheaf) on $\mathbb{P}(\mathrm{G})$ if and only if M has the constant j-image property. Similarly, $\operatorname{Ker}\{\widetilde{\Theta}_G^j, \mathcal{M}\}$ is trivial if and only if M has the constant j-kernel property.*

Proof. If M has a constant j-image property, then $\operatorname{Im}\{\widetilde{\Theta}_G^j, \mathcal{M}\}$ is a free $\mathcal{O}_X$-module generated by $I(j)$. Conversely, assume that $\operatorname{Im}\{\widetilde{\Theta}_G^j, \mathcal{M}\}$ is a free $\mathcal{O}_X$-module. Then there exists a subspace $I(j) \subset M = \Gamma(X, \mathcal{M})$ which maps to and spans each fiber $\operatorname{Im}\{\theta_v^j, M_{k(v)}\}$, for $0 \neq v \in V(G)$. The argument for kernels is similar. $\square$

Remark 5.13. We point out that the properties of constant j-image and constant j-kernel are independent of each other. Consider the module $M^{\#}$ of Example 6.1. As shown in that example, $\operatorname{Ker}\{\widetilde{\Theta}_G, \mathcal{M}^{\#}\}$ is locally free of rank 2 but not free, since the global sections have dimension one. On the other hand, $\operatorname{Im}\{\widetilde{\Theta}_G, \mathcal{M}^{\#}\}$ is a free $\mathcal{O}_X$-module generated by the global section n_3. In particular, $M^{\#}$ has constant 1-image property but not constant 1-kernel property.

For the module M of Example 6.1, the sheaf $\operatorname{Ker}\{\widetilde{\Theta}_G, \mathcal{M}\}$ is free of rank 2, whereas $\operatorname{Im}\{\widetilde{\Theta}_G, \mathcal{M}\}$ is locally free of rank 1 but not free since it does not have any global sections. Hence, M has a constant 1-kernel property but not constant 1-image property.

We consider an analogue of the sheaf construction of Duflo-Serganova for Lie superalgebras [11]. This construction enables us to produce additional algebraic vector bundles on $\mathbb{P}(\mathrm{G})$. We implicitly use the observation $\widetilde{\Theta}_G^p = 0$.

Definition 5.14. Let G be an infinitesimal group scheme, and let M be a finite dimensional kG-module. Let $\mathcal{M} = M \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}$. For any i, $1 \leq i \leq p-1$, we define coherent $\mathcal{O}_{\mathbb{P}(\mathrm{G})}$-modules, subquotients of $\mathcal{M}$:

$$\mathcal{M}^{[i]} \ \equiv \ \mathrm{Ker}\{\widetilde{\Theta}_G^i, \mathcal{M}\}/\,\mathrm{Im}\{\widetilde{\Theta}_G^{p-i}, \mathcal{M}\}.$$

The following simple lemma helps to motivate these subquotients.

Lemma 5.15. *Let V be a finite dimensional $k[t]/t^p$-module, and let* $\mathrm{JType}(V,t) = (a_p, \dots, a_1)$ *(using the notation introduced in* (3.1.2)*). Let*

$$V^{[j]} = \mathrm{Ker}\{t^j : V \to V\}/\,\mathrm{Im}\{t^{p-j} : V \to V\}$$

for $j \leq p-1$. Then

$$\dim(V^{[j]}) \ = \ \sum_{1 \leq i \leq j} i a_i + \sum_{i > j} j a_i - \sum_{i+j>p} (i+j-p) a_i.$$

In particular, V is projective as a $k[t]/t^p$-module if and only if $V^{[1]} \ = \ 0$. Furthermore, for $j \leq p-1$, $V^{[j]} \simeq V^{[p-j]}$ as $k[t]/t^p$-modules.

As seen in the next proposition, these subquotients can provide additional examples of algebraic vector bundles over $\mathbb{P}(\mathrm{G})$.

Proposition 5.16. *Let G be an infinitesimal group scheme and let M be a finite dimensional kG-module which is of constant j-rank and constant $(p-j)$-rank for some j, $1 \leq j < p$. Then $\mathcal{M}^{[j]}$ is a locally free $\mathcal{O}_X$-module and $\mathcal{M}^{[j]} \otimes_{\mathcal{O}_X} k(x) \ \to M^{[j]}_{k(x)}$ is an isomorphism for all $x \in X \equiv \mathbb{P}(\mathrm{G})$.*

Proof. For any $x \in X$, consider the map of exact sequences

$$\begin{tikzcd}
\mathrm{Im}\{\widetilde{\Theta}_G^{p-i}, \mathcal{M}\} \otimes_{\mathcal{O}_X} k(x) \arrow[r] \arrow[d] & \mathrm{Ker}\{\widetilde{\Theta}_G^{i}, \mathcal{M}\} \otimes_{\mathcal{O}_X} k(x) \arrow[r] \arrow[d] & \mathcal{M}^{[j]} \otimes_{\mathcal{O}_X} k(x) \arrow[r] \arrow[d] & 0 \\
\mathrm{Im}\{\theta_x^{p-i}, M_{k(x)}\} \arrow[r] & \mathrm{Ker}\{\theta_x^{i}, M_{k(x)}\} \arrow[r] & M^{[j]}_{k(x)} \arrow[r] & 0.
\end{tikzcd}$$

The left and middle vertical maps are isomorphisms by Theorem 4.13. Thus, the 5-lemma implies that the right vertical arrow is also an isomorphism. $\square$

We give an application of this $(-)^{[1]}$ construction to endotrivial modules. An interested reader can compare our construction to [1]. Recall that a module M of a finite group scheme G is endotrivial if $\mathrm{End}_k(M) \simeq k + \mathrm{proj}$. It was shown in [8, §5] that an endotrivial module is a module of constant Jordan type with possible types $[1] + \mathrm{proj}$ and $[p-1] + \mathrm{proj}$.

Proposition 5.17. *Let G be an infinitesimal group scheme, and assume that G has a subgroup scheme isomorphic to $\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$ or $\mathbb{G}_{a(2)}$. Let M be a module of constant Jordan type, and set $\mathcal{M} = M \otimes \mathcal{O}_{\mathbb{P}(G)}$. Then $\mathcal{M}^{[1]}$ is a line bundle (i.e., an algebraic vector bundle of rank one) if and only if M is endotrivial.*

Proof. The sheaf $\mathcal{M}^{[1]}$ is locally free by Lemma 5.15. Let $\sum_{i=1}^{p} a_i[i]$ be the Jordan type of M. Lemma 5.15 implies that the rank of the vector bundle $\mathcal{M}^{[1]}$ equals $\sum_{i=0}^{p-1} a_i$. Hence, $\mathcal{M}^{[1]}$ is a line bundle if and only if the Jordan type of M has only one non-projective block. A theorem of D. Benson [4] states that modules of constant Jordan type with unique non-projective block must be of type $[1] + \mathrm{proj}$ or $[p-1] + \mathrm{proj}$. By [8, §5], this happens if and only if M is endotrivial. $\square$

We next give a global version of the observation in Lemma 5.15 that $V^{[j]} \simeq V^{[p-j]}$ for $j \leq p-1$. Recall that for a variety X and a coherent sheaf $\mathcal{E}$, we denote by $\mathcal{E}^\vee = Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ the dual sheaf.

Proposition 5.18. *Let G be an infinitesimal group scheme, and let M be a kG-module which is of constant j-rank and of constant $(p-j)$-rank for some j, $1 \leq j < p$. Let $\mathcal{M} = M \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}$, $\mathcal{N} = M^\# \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}$. Then*

$$\mathcal{N}^{[p-j]} \simeq (\mathcal{M}^{[j]})^\vee$$

as $\mathcal{O}_{\mathbb{P}(\mathrm{G})}$-modules.

Proof. Let $X = \mathbb{P}(\mathrm{G})$. As discussed in the proof of Proposition 5.5, the $\mathcal{O}_X$-linear dual of the complex of $\mathcal{O}_X$-modules

$$\mathcal{M}(-(p-j)p^{r-1}) \xrightarrow{\widetilde{\Theta}_G^{p-j}(-(p-j)p^{r-1})} \mathcal{M} \xrightarrow{\widetilde{\Theta}_G^{j}} \mathcal{M}(jp^{r-1})$$

is the complex

$$\mathcal{N}((p-j)p^{r-1}) \xleftarrow{\widetilde{\Theta}_G^{p-j}} \mathcal{N} \xleftarrow{\widetilde{\Theta}_G^{j}(-jp^{r-1})} \mathcal{N}(-jp^{r-1}).$$

A similar statement applies with θ_v in place of $\widetilde{\Theta}_G$.

For any scheme Y and any complex of $\mathcal{O}_Y$-modules

$$S_1 \xrightarrow{f} S_2 \xrightarrow{g} S_3$$

with $\mathcal{O}_Y$-linear dual

$$S_1^\vee \xleftarrow{f^\vee} S_2^\vee \xleftarrow{g^\vee} S_3^\vee,$$

there is a natural pairing

$$(\mathrm{Ker}\{g\}/\,\mathrm{Im}\{f\}) \otimes (\mathrm{Ker}\{f^\vee\}/\,\mathrm{Im}\{g^\vee\}) \longrightarrow \mathcal{O}_Y, \tag{5.18.1}$$

induced by the evident pairing

$$S_2 \otimes S_2^\vee \to \mathcal{O}_Y.$$

In particular, we have a pairing

$$\mathcal{M}^{[j]} \otimes \mathcal{N}^{[p-j]} \to \mathcal{O}_X,$$

and, hence, a map

$$f : \mathcal{N}^{[p-j]} \to (\mathcal{M}^{[j]})^\vee.$$

By Proposition 5.16,

$$\mathcal{M}^{[j]} \otimes_{\mathcal{O}_X} k(x) \simeq M_{k(x)}^{[j]}, \quad \mathcal{N}^{[p-j]} \otimes_{\mathcal{O}_X} k(x) \simeq N_{k(x)}^{[p-j]}$$

for any $x \in X$. By naturality, the specialization of f at a point x corresponds to the map $f_x : N_{k(x)}^{[p-j]} \to (M_{k(x)}^{[j]})^{\#}$ induced by the pairing (5.18.1) for $Y = \operatorname{Spec} k(x)$. One readily verifies that this is a perfect pairing if $Y = \operatorname{Spec} k(x)$. Hence,

$$f \otimes_{O_X} k(x) : \mathcal{N}^{[p-j]} \otimes_{\mathcal{O}_X} k(x) \to (\mathcal{M}^{[j]})^{\vee} \otimes_{\mathcal{O}_X} k(x)$$

is an isomorphism for any $x \in X$. Therefore, $\mathcal{N}^{[p-j]} \simeq (\mathcal{M}^{[j]})^{\vee}$. $\square$

Consideration of $\mathcal{M}^{[1]}$ leads to another characterization of projective kG-modules.

Proposition 5.19. *Let G be an infinitesimal group scheme and let M be a finite dimensional kG-module. Then M is projective if and only if M has constant rank, has constant $(p-1)$-rank, and satisfies $\mathcal{M}^{[1]} = 0$.*

Proof. Assume that M is a projective kG-module. Then M has constant Jordan type (which is some multiple of $[p]$) and hence has constant rank and constant $(p-1)$-rank. For any $x \in \mathbb{P}(\mathrm{G}) = X$, $\theta_x^*(M_{k(x)})$ is a free $k(x)[t]/t^p$-module of rank equal to $\frac{\dim(M)}{p}$. If we lift a basis of this free module to $\mathcal{M}_{(x)} \equiv \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}$, then an application of Nakayama's Lemma tells us that $\mathcal{M}_{(x)}$ is free as an $\mathcal{O}_{X,x}[t]/t^p$-module. This readily implies that $(\mathcal{M}_{(x)})^{[p-1]} \equiv \operatorname{Ker}\{\widetilde{\Theta}_{G,(x)}^{p-1}, \mathcal{M}_{(x)}\} / \operatorname{Im}\{\widetilde{\Theta}_{G,(x)}, \mathcal{M}_{(x)}\}$ vanishes. Using the exactness of localization, we conclude that

$$(\mathcal{M}^{[p-1]})_{(x)} = (\mathcal{M}_{(x)})^{[p-1]}.$$

Consequently, $\mathcal{M}^{[p-1]} = 0$. By Proposition 5.18, we conclude that $\mathcal{M}^{[1]} = 0$.

Conversely, if M has constant rank and constant $(p-1)$-rank and if $\mathcal{M}^{[1]} = 0$, then Proposition 5.16 tells us that $M_{k(x)}^{[1]} \equiv \operatorname{Ker} \theta_x / \operatorname{Im} \theta_x^{p-1}$ equals 0 for all $x \in X$. Lemma 5.15 thus implies that each $M_{k(x)}$ is projective, so that the local criterion for projectivity [25] implies that M is projective. $\square$

One very simple invariant of the algebraic vector bundle $\operatorname{Ker}\{\widetilde{\Theta}_G^j, \mathcal{M}\}$ is the dimension of its vector space of global sections. The following proposition gives some understanding of $\Gamma(\mathbb{P}(\mathrm{G}), \operatorname{Ker}\{\widetilde{\Theta}_G^j, \mathcal{M}\}) \subset \Gamma(\mathbb{P}(\mathrm{G}), \mathcal{M})$.

Proposition 5.20. *Let G be an infinitesimal group scheme, and assume that $V(G)$ is reduced. Let M be a kG-module and let $\mathcal{M} = M \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}$. Then*

$$\Gamma(\mathbb{P}(\mathrm{G}), \operatorname{Ker}\{\widetilde{\Theta}_G^j, \mathcal{M}\}) \subset M$$

consists of those $m \in M$ such that $\theta_x^j(m) = 0$ for all $x \in \mathbb{P}(\mathrm{G})$.

Proof. Recall that $\mathbb{P}(\mathrm{G})$ is connected by [8, 3.4], and thus $\Gamma(\mathbb{P}(\mathrm{G}), \mathcal{M}) = M$. Under this identification, the global sections of $\operatorname{Ker}\{\widetilde{\Theta}_G^j, \mathcal{M}\}$ coincide with the subset

$$\{m \in M \mid \Theta_G^j(m \otimes 1) = 0\}.$$

Since $V(G)$ is reduced, we have $\Theta_G^j(m \otimes 1) = 0$ if and only if

$$\theta_v^j(m \otimes 1) = \Theta_G^j(m \otimes 1) \otimes_{k[V(G)]} k(v) = 0$$

for every $v \in V(G)$. Hence, $m \in \Gamma(\mathbb{P}(\mathrm{G}), \operatorname{Ker}\{\widetilde{\Theta}_G^j, \mathcal{M}\})$ if and only if $m \in \operatorname{Ker}\{\theta_v^j, M_{k(v)}\}$ for every $v \in V(G)$ if and only if $\theta_x^j(m) = 0$ for all $x \in \mathbb{P}(\mathrm{G})$. $\square$

We make Proposition 5.20 more explicit in the case of a classical Lie algebra.

Proposition 5.21. *Let $\mathfrak{G}$ be a (reduced, irreducible) algebraic group over k, let $G = \mathfrak{G}_{(1)}$, and let $\mathfrak{g} = Lie(\mathfrak{G})$. Assume that $\mathfrak{g}$ is generated by p-nilpotent elements.*

(1) If M is a rational $\mathfrak{G}$-module, then $\Gamma(\mathbb{P}(\mathrm{G}), \operatorname{Ker}\{\widetilde{\Theta}^j_G, \mathcal{M}\})$ is a rational $\mathfrak{G}$-submodule of M.

(2) If $V(G)$ is reduced, then

$$\Gamma(\mathbb{P}(\mathrm{G}), \operatorname{Ker}\{\widetilde{\Theta}_G, \mathcal{M}\}) \;=\; \mathrm{H}^0(G, M).$$

Proof. To prove that $\Gamma(\mathbb{P}(\mathrm{G}), \operatorname{Ker}\{\widetilde{\Theta}^j_G, \mathcal{M}\})$ is a $\mathfrak{G}$-submodule of M, we may base change to the algebraic closure of k, and thus we may assume k is algebraically closed. Let $g \in \mathfrak{G}$ be a k-rational point. Then

$$\theta^j_v(gm \otimes 1) = g\theta^j_{v^{g^{-1}}}(m \otimes 1), \tag{5.21.1}$$

where the action of $\mathfrak{G}$ on $V(G) = \mathcal{N}_p(\mathfrak{g})$ (the p-nilpotent cone of $\mathfrak{g}$) is via the adjoint action of $\mathfrak{G}$.

Hence, we have the following equalities:

$$\begin{aligned}\{m \in M \mid \Theta^j_G(m \otimes 1) = 0\} &= \bigcap_{0 \neq v \in V(G)} \{m \in M \mid \theta^j_v(m \otimes 1) = 0\} \\ &= \bigcap_{0 \neq \mathrm{Ad}(g^{-1})v \in V(G)} \{m \in M \mid g\theta^j_{\mathrm{Ad}(g^{-1})v}(m \otimes 1) = 0\} \\ &= \bigcap_{0 \neq v \in V(G)} \{m \in M \mid \theta^j_v(gm \otimes 1) = 0\} = \{m \in M \mid \Theta^j_G(gm \otimes 1) = 0\},\end{aligned}$$

where the first and the last equality follow from Proposition 5.20, the second equality follows from the fact that $\mathrm{Ad}(g^{-1}) : V(G) \to V(G)$ is a bijection, and the third equality from (5.21.1). We conclude that $\{m \in M \mid \Theta^j_G(m \otimes 1) = 0\}$ is a $\mathfrak{G}$-stable subspace of M.

The second assertion follows immediately from Proposition 5.20 and the fact that $v \in V(G)$ corresponds to a p-nilpotent element X_v of $\mathfrak{g}$ and that the action of θ_v is the action of X_v. □

Combining Proposition 5.10 and Proposition 5.21 in the special case $j = 1$ yields the following criterion for the non-triviality of $\operatorname{Ker}\{\widetilde{\Theta}_G, \mathcal{M}\}$.

Corollary 5.22. *Let G be an infinitesimal group scheme such that $V(G)$ is reduced and positive dimensional. Assume that for any field extension K/k, KG is generated by $\theta_v \in k(v)G$, for all $v \in V(G)$ such that $k(v) \subset K$. Let M be a finite dimensional kG-module of constant Jordan type $\sum_i a_i[i]$. If*

$$\dim \mathrm{H}^0(G, M) \;< \sum_{i=1}^{p} a_i,$$

then $\operatorname{Ker}\{\widetilde{\Theta}_G, \mathcal{M}\}$ is a non-trivial algebraic vector bundle over $\mathbb{P}(\mathrm{G})$.

Proof. By Proposition 5.10, the dimension of the fibers of $\operatorname{Ker}\{\widetilde{\Theta}_G, \mathcal{M}\}$ is $\dim M - \sum_{i=2}^p a_i(i-1) = \sum_{i=1}^p a_i$. By Proposition 5.20, the global sections of $\operatorname{Ker}\{\widetilde{\Theta}_G, \mathcal{M}\}$ equal $\mathrm{H}^0(G, M)$. Hence, the inequality $\dim \mathrm{H}^0(G, M) \;< \sum_{i=1}^p a_i$ implies that the dimension of the global sections is less than the dimension of the fibers. Therefore, the sheaf is not free. □

The following two lemmas will be applied to prove Proposition 5.25.

Lemma 5.23. *Let R be a local commutative ring with residue field k and let M be a finite $R[t]/t^p$-module which is free as an R-module. If $M \otimes_R k$ is a free $k[t]/t^p$-module, then M is free as an $R[t]/t^p$-module.*

Proof. Let $m_1, \dots, m_s \in M$ be such that $\overline{m}_1, \dots, \overline{m}_s$ form a basis for $M \otimes_R k$ as a $k[t]/t^p$-module. Let Q be a free $R[t]/t^p$-module of rank s with basis $q_1, \dots, q_s$ and consider the $R[t]/t^p$-module homomorphism $f : Q \to M$ sending q_i to m_i.

By Nakayama's Lemma, $f : Q \to M$ is surjective. Because M is free as an R-module, applying $- \otimes_R k$ to the short exact sequence $0 \to \operatorname{Ker}\{f\} \to Q \to M \to 0$ determines the short exact sequence

$$0 \to \operatorname{Ker}\{f\} \otimes_R k \ \to \ Q \otimes_R k \ \to \ M \otimes_R k \to 0.$$

Consequently, $\operatorname{Ker}\{f\} \otimes_R k = 0$, so that another application of Nakayama's Lemma implies that $\operatorname{Ker}\{f\} = 0$. Hence, f is an isomorphism, and thus M is free as an $R[t]/t^p$-module. □

Lemma 5.24. *Let G be an infinitesimal group scheme and M be a finite dimensional kG-module. Set $A = k[V(G)]$; for any $f \in A$, set $A_f = A[1/f]$. Assume that $\operatorname{Spec} A_f \subset V(G)$ has empty intersection with $V(G)_M$. Then $(\mathcal{U}_G \circ \epsilon)^*(M \otimes A_f)$ is a projective $A_f[t]/t^p$-module.*

Proof. By definition, $V(G)_M$ consists of those points $v \in V(G)$ such that $\theta_v^*(M_{(k(v)})$ is not free as a $k(v)[t]/t^p$-module. By the universal property of $\mathcal{U}_G \circ \epsilon$, the assumption that $\operatorname{Spec} A_f \cap V(G)_M = \emptyset$ implies for every point $v \in \operatorname{Spec} A_f$ that $\theta_v^*(M_{k(v)}) = (\mathcal{U}_G \circ \epsilon)^*(M \otimes A_f) \otimes_{A_f} k(v)$ is free as a $k(v)[t]/t^p$-module. Let $A_{(v)}$ denote the localization of A at v. Then Lemma 5.23 implies for every point $v \in \operatorname{Spec} A_f$ that the localization $(\mathcal{U}_G \circ \epsilon)^*(M \otimes A_f) \otimes_{A_f} A_{(v)}$ is free as a $A_{(v)}[t]/t^p$-module. This implies that $M \otimes A_f$ is projective (since projectivity of a finitely generated module over a A is determined locally). □

We conclude with a property of the (projectivized) rank variety $\mathbb{P}(\mathrm{G})_M$ of a kG-module M.

Proposition 5.25. *Let G be an infinitesimal group scheme, M be a finite dimensional kG-module, and set $\mathcal{M} = M \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}$. Then*

$$\operatorname{Supp}_{\mathcal{O}_{\mathbb{P}(\mathrm{G})}}(\mathcal{M}^{[1]}) \ \subset \ \mathbb{P}(\mathrm{G})_M,$$

where $\operatorname{Supp}_{\mathcal{O}_{\mathbb{P}(\mathrm{G})}}(\mathcal{M}^{[1]})$ is the support of the coherent sheaf $\mathcal{M}^{[1]}$ (the closed subset of points $x \in \mathbb{P}(\mathrm{G})$ at which $\mathcal{M}^{[1]}_{(x)} \neq 0$).

Proof. Let A denote $k[V(G)]$ and let X denote $\mathbb{P}(\mathrm{G})$. Consider some $x \notin X_M$ and choose some homogeneous polynomial $F \in A$ vanishing on X_M such that $F(x) \neq 0$. Thus, $x \in \operatorname{Spec}(A_F)_0 \subset X$ and $\operatorname{Spec}(A_F)_0 \cap X_M = \emptyset$, where $(A_F)_0$ denotes the elements of degree 0 in the localization $A_F = A[1/F]$. It suffices to prove that $x \notin \operatorname{Supp}_{\mathcal{O}_X}(\mathcal{M}^{[1]}) \cap \operatorname{Spec}(A_F)_0$. Since localization is exact, this is equivalent to proving that $v \notin \operatorname{Supp}_{A_F}((M \otimes A_F)^{[1]})$ for some $v \in \operatorname{Spec} A_F$ mapping to x.

The condition $\operatorname{Spec}(A_F)_0 \cap X_M = \emptyset$ implies that $\operatorname{Spec}(A_F) \cap V(G)_M = \emptyset$. Hence, by Lemma 5.24, $(\mathcal{U}_G \circ \epsilon)^*(M \otimes A_F)$ is a projective $A_F[t]/t^p$-module. This implies that $(\mathcal{U}_G \circ \epsilon)^*((M \otimes A_F)^{[1]}) = 0$ and thus that $v \notin \operatorname{Supp}_{A_F}((M \otimes A_F)^{[1]})$. □

Remark 5.26. The reverse inclusion $\mathbb{P}(\mathrm{G})_M \subset \mathrm{Supp}_{\mathcal{O}_{\mathbb{P}(\mathrm{G})}}(\mathcal{M}^{[1]})$ seems closely related to the condition that $\mathrm{Ker}\{\widetilde{\Theta}_G, \mathcal{M}\}\otimes_{\mathcal{O}_{\mathbb{P}(\mathrm{G})}} k(x) \to \mathrm{Ker}\{\theta_x, M_{k(x)}\}$ be surjective.

6. Examples and calculations with bundles

In this final section, we investigate numerous specific examples. The case in which G equals either $\mathbb{G}_{a(1)}^{\times 2}$ or $\underline{s}l_2$ (the infinitesimal group scheme associated to the restricted Lie algebra sl_2) is particularly amenable to computation for $\mathbb{P}(G)$ is isomorphic to $\mathbb{P}^1$. Specifically, we do calculations for projective kG-modules, examples of modules of constant Jordan type which are not distinguished by support varieties. We also compute bundles obtained from"zig-zag modules" and syzygies.

As we see in the following simple example, the isomorphism type of the vector bundles discussed in Theorem 5.1 can be used to distinguish certain kG-modules which have the same local Jordan type. We remind the reader that the local Jordan type of a finite dimensional kG-module M of constant Jordan type is the same as that of its linear dual $M^{\#}$.

Example 6.1. Let $G = \mathbb{G}_{a(2)}$ so that $k\mathbb{G}_{a(2)} = k[u_0, u_1]/(u_0^p, u_1^p)$, $V(\mathbb{G}_{a(2)}) = \mathbb{A}^2$, $A = k[V(\mathbb{G}_{a(2)})] = k[x_0, x_1]$ graded so that x_0 is given degree 1 and x_1 is given degree p. Then

$$\Theta_G = x_1 u_0 + x_0^p u_1 \in A[u_0, u_1]/(u_0^p, u_1^p)$$

(see Example 2.6(2)). We consider the 3-dimensional kG-module M of constant Jordan type $[2] + [1]$ and its linear dual $M^{\#}$, which we represent diagrammatically as follows:

$$\begin{tikzcd} & m_1 \arrow[dl, "u_2"'] \arrow[dr, "u_1"] & \\ m_2 & & m_3 \end{tikzcd} \qquad \begin{tikzcd} n_1 \arrow[dr, "u_1"] & & n_2 \arrow[dl, "u_2"'] \\ & n_3. & \end{tikzcd}$$

The $k\,\mathbb{G}_{\mathrm{a}(2)}$-invariant subspace of M is two dimensional, and, hence, the global sections of $\mathrm{Ker}\{\widetilde{\Theta}_G, M \otimes \mathcal{O}_{\mathbb{P}(G)}\}$ have dimension two by Proposition 5.20 (in fact, an explicit calculation shows that this is a trivial bundle of rank 2). On the other hand, $M^{\#}$ has only 1-dimensional invariant subspace and, hence, the global sections of $\mathrm{Ker}\{\widetilde{\Theta}_G, M^{\#} \otimes \mathcal{O}_{\mathbb{P}(G)}\}$ have dimension one. Thus,

$$\mathrm{Ker}\{\widetilde{\Theta}_G, M \otimes \mathcal{O}_{\mathbb{P}(G)}\} \;\not\cong\; \mathrm{Ker}\{\widetilde{\Theta}_G, M^{\#} \otimes \mathcal{O}_{\mathbb{P}(G)}\}.$$

We next give a somewhat more interesting example of pairs of modules of the same constant Jordan type with different associated bundles.

Example 6.2. As in Proposition 5.9, let S_λ be the irreducible SL_2-module of highest weight $\lambda, 0 \le \lambda \le p-1$, and consider $S^p(S_\lambda)$, the p^{th} symmetric power of S_λ. Since S_λ is self-dual, the dual of $S^p(S_\lambda)$ is $\Gamma^p(S_\lambda)$, the p^{th} divided power of S_λ. We have a short exact sequence of rational SL_2-modules:

$$0 \longrightarrow S_\lambda^{(1)} \longrightarrow S^p(S_\lambda) \longrightarrow \Gamma^p(S_\lambda) \longrightarrow S_\lambda^{(1)} \longrightarrow 0.$$

Here, $S_\lambda^{(1)}$ is the first Frobenius twist of S_λ, and thus trivial as a $\mathfrak{u}(sl_2)$-module.

Let $X = \mathrm{Proj}\, k[N(sl_2)]$. By Proposition 5.20, the space of global sections of $\mathrm{Ker}\{\widetilde{\Theta}_{\underline{s}l_2}, S^p(S_\lambda) \otimes \mathcal{O}_X\}$ equals the sl_2 invariants of $S^p(S_\lambda)$. Hence,

$$\Gamma(X, \mathrm{Ker}\{\widetilde{\Theta}_{\underline{s}l_2}, S^p(S_\lambda) \otimes \mathcal{O}_X\}) = S_\lambda^{(1)}.$$

On the other hand, $\Gamma^p(S_\lambda)$ does not have any sl_2-invariants and, hence, $\operatorname{Ker}\{\widetilde{\Theta}_{\underline{s}l_2}, \Gamma^p(S_\lambda) \otimes \mathcal{O}_X\}$ does not have any global sections. We conclude that the kernel bundles associated to the dual modules $S^p(S_\lambda)$ and $\Gamma^p(S_\lambda)$ are non-isomorphic.

We continue our consideration of $\mathrm{SL}_{2(1)} \equiv \underline{s}l_2$ in the following proposition.

Proposition 6.3. *Let $G = \underline{s}l_2$, let S_λ be the irreducible kG-module of highest weight λ, $0 \leq \lambda \leq p-1$, and let $P_\lambda \to S_\lambda$ be the projective cover of S_λ. Then:*

$$i^*(\operatorname{Ker}\{\widetilde{\Theta}_G, P_\lambda \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}\}) \simeq \begin{cases} \mathcal{O}_{\mathbb{P}^1}(1-p) & \text{if } \lambda = p-1, \\ \mathcal{O}_{\mathbb{P}^1}(\lambda - 2(p-1)) \oplus \mathcal{O}_{\mathbb{P}^1}(-\lambda) & \text{if } 0 \leq \lambda \leq p-2, \end{cases}$$

where $i : \mathbb{P}^1 \to \mathbb{P}(\mathrm{G})$ is the isomorphism (5.8.1).

Proof. For $\lambda = p-1$, $P_\lambda = \mathrm{St}$ is the Steinberg module for sl_2 and is irreducible. Hence, in this case the statement follows from Proposition 5.9.

The decomposition series of P_λ for $0 \leq \lambda < p-1$ is represented by the following diagram (see [14, 2.4]):

(6.3.1)
$$\begin{tikzcd} & S_\lambda \arrow[dl] \arrow[dr] & \\ S_{p-2-\lambda} \arrow[dr] & & S_{p-2-\lambda} \arrow[dl] \\ & S_\lambda. & \end{tikzcd}$$

Thus, we have a short exact sequence of SL_2-modules:

(6.3.2)
$$0 \to V_{2p-2-\lambda} \to P_\lambda \to S_\lambda \to 0.$$

By Proposition 5.9, it suffices to prove that $V_{2p-2-\lambda} \subset P_\lambda$ induces an isomorphism

$$\operatorname{Ker}\{\widetilde{\Theta}_G, V_{2p-2-\lambda} \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}\} \simeq \operatorname{Ker}\{\widetilde{\Theta}_G, P_\lambda \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}\}.$$

By Theorem 4.13(8) and Nakayama's Lemma, it suffices to prove that

$$\operatorname{Ker}\{\theta_x, V_{2p-2-\lambda} \otimes k(x)\} \to \operatorname{Ker}\{\theta_x, P_\lambda \otimes k(x)\}$$

is an isomorphism for all $x \in \mathbb{P}(\mathrm{G})$. This last statement follows from the observation that the Jordan decomposition of θ_x on both V_λ and P_λ consists of two blocks: on P_λ, because P_λ is projective of dimension $2p$; and on $V_{2p-2-\lambda}$, as discussed in Proposition 5.9. $\square$

If X is an algebraic variety over k (for example, $X = \mathbb{P}(G)$), then $K_0(X)$ denotes the Grothendieck group of algebraic vector bundles over X (i.e., finitely generated, locally free $\mathcal{O}_X$-modules) defined as the free abelian group on the set of isomorphism classes of such vector bundles modulo relations given by short exact sequences. We shall also consider $K_0^{\oplus}(X)$ defined as the free abelian group on the same set of generators modulo relations given by split short exact sequences. Thus, there is a canonical surjective homomorphism

$$K_0^{\oplus}(X) \to K_0(X).$$

Definition 6.4. Let G be an infinitesimal group scheme and let $K_0(kG)$ denote the Grothendieck group of finitely generated projective kG-modules. For any j, $0 \leq j \leq p-1$, the homomorphism

$$\kappa^{\oplus}_{G,j} : K_0(kG) \ \to \ K^{\oplus}_0(\mathbb{P}(G))$$

is defined by sending a projective kG-module Q to $\operatorname{Ker}\{\widetilde{\Theta}^j_G, Q \otimes \mathcal{O}_{\mathbb{P}(G)}\}$. We define $\underline{\kappa}^{\oplus}_G$ by

$$\underline{\kappa}^{\oplus}_G = (\kappa^{\oplus}_{G,1}, \ldots \kappa^{\oplus}_{G,p}) : K_0(kG) \ \to \ K^{\oplus}_0(\mathbb{P}(G))^{\oplus p}.$$

Moreover, the homomorphism

$$\kappa_{G,j} : K_0(kG) \ \to \ K_0(\mathbb{P}(G))$$

is defined to be the composition of $\tilde{\kappa}^{\oplus}_{G,j}$ with the canonical projection $K^{\oplus}_0(\mathbb{P}(G)) \to K_0(\mathbb{P}(G))$, and the homomorphism

$$\underline{\kappa}_G : K_0(kG) \ \to \ K_0(\mathbb{P}(G))^{\oplus p}$$

is defined to be the composition of $\underline{\kappa}_G$ with the canonical projection $K^{\oplus}_0(\mathbb{P}(G))^{\oplus p} \to K_0(\mathbb{P}(G))^{\oplus p}$.

We shall omit the subscript G in κ_G when the group scheme is clear from the context. Note that since $\widetilde{\Theta}^p = 0$, $\kappa_p = [Q \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}]$.

The *dimension of the global sections* function,

$$\mathcal{E} \mapsto \dim \Gamma(\mathbb{P}(\mathrm{G}), \mathcal{E}),$$

extends to a homomorphism

$$\rho : K^{\oplus}_0(\mathbb{P}(G)) \ \to \mathbb{Z}.$$

Observe that since $\Gamma(\mathbb{P}(G), -)$ is not right exact for non-split exact sequences of algebraic vector bundles on $\mathbb{P}(G)$, ρ does *not factor* through $K_0(\mathbb{P}(G))$.

Proposition 6.5. *Let $G = \underline{s}l_2$. The composition*

$$\rho \circ \underline{\kappa}^{\oplus} : K_0(kG) \ \to \ K^{\oplus}_0(\mathbb{P}(\mathrm{G}))^{\oplus p} \ \to \mathbb{Z}^{\oplus p}$$

is a rational isomorphism.

Proof. Recall that $K_0(kG) = K_0(u(sl_2)) \simeq \mathbb{Z}^{\oplus p}$, spanned by the projective indecomposable $u(sl_2)$-modules P_λ, $0 \leq \lambda \leq p-1$. By Proposition 5.21, the global sections of $\operatorname{Ker}\{\widetilde{\Theta}^j_G, P_\lambda \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}\})$ are rational SL_2-submodules of P_λ. Hence, $\Gamma(\mathbb{P}(\mathrm{G}), \operatorname{Ker}\{\widetilde{\Theta}^j_G, P_\lambda \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}\}) \neq 0$ if and only if S_λ, which is the socle of P_λ, belongs to the global sections. Hence,

$$\Gamma(\mathbb{P}(\mathrm{G}), \operatorname{Ker}\{\widetilde{\Theta}^j_G, P_\lambda \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}\}) \neq 0 \quad \Leftrightarrow \quad \widetilde{\Theta}^j(S_\lambda \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}) = 0 \quad \Leftrightarrow \quad \theta^j_v(S_\lambda) = 0$$

for any $v \in V(G) \simeq \mathcal{N}(sl_2)$. Since S_λ is a module of constant Jordan type $[\lambda+1]$, θ^j_v annihilates S_λ if and only if $j > \lambda$. Thus,

$$\Gamma(\mathbb{P}(\mathrm{G}), \operatorname{Ker}\{\widetilde{\Theta}^j_G, P_\lambda \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}\}) \neq 0 \text{ if and only if } j > \lambda.$$

Moreover, the decomposition series for P_λ (6.3.1) implies that the Jordan type of any rational submodule of P_λ that is larger than the socle S_λ has a Jordan block of size p. Hence,

$$\Gamma(\mathbb{P}(\mathrm{G}), \operatorname{Ker}\{\widetilde{\Theta}^{\lambda+1}_G, P_\lambda \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}\}) = S_\lambda.$$

Note that the last equality holds trivially for $\lambda = p-1$, since in this case the kernel bundle is the entire free sheaf $P_{p-1} \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}$, where $P_{p-1} = S_{p-1}$ is the Steinberg module for sl_2.

We conclude that the homomorphism

$$\rho \circ \underline{\kappa}^{\oplus} : K_0(kG) \simeq \mathbb{Z}^{\oplus p} \ \to \mathbb{Z}^{\oplus p}$$

is given by a non-singular upper-triangular matrix

$$\begin{bmatrix} 1 & \cdots & & & \cdot \\ 0 & 2 & \cdots & & \cdot \\ 0 & 0 & 3 & \cdots & \cdot \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & p \end{bmatrix}.$$

Hence, $\rho \circ \underline{\kappa}^{\oplus}$ is a rational isomorphism. $\square$

In contrast with Proposition 6.5, we have the following computations for κ_1: $K_0(kG) \to K_0(\mathbb{P}(\mathrm{G}))$ and $\underline{\kappa} : K_0(kG) \to K_0(\mathbb{P}(\mathrm{G}))^{\oplus p}$ for $G = \underline{s}l_2$.

Lemma 6.6. *Let $G = \underline{s}l_2$, and denote by* St *the Steinberg module for sl_2. The image of the homomorphism*

$$\kappa_1 : K_0(kG) \to K_0(\mathbb{P}(\mathrm{G}))$$

is generated by $\kappa_1(\mathrm{St})$. Consequently, $\operatorname{rk} \kappa_1 = 1$.

Proof. The isomorphism $i : \mathbb{P}(\mathrm{G}) \simeq \mathbb{P}^1$ (5.8.1) induces an isomorphism $K_0(\mathbb{P}(\mathrm{G})) \simeq K_0(\mathbb{P}^1)$. We denote by κ_1 the composition $K_0(kG) \to K_0(\mathbb{P}(\mathrm{G})) \stackrel{\sim}{\to} K_0(\mathbb{P}^1)$. Clearly, it suffices to prove the statement of the lemma for this composition.

Let $a_n = [\mathcal{O}_{\mathbb{P}^1}(-n)] \in K_0(\mathbb{P}^1)$, $n \in \mathbb{Z}$. Then $a_0 = [\mathcal{O}_{\mathbb{P}^1}]$, $a_1 = [\mathcal{O}_{\mathbb{P}^1}(-1)]$ generate $K_0(\mathbb{P}^1) \simeq \mathbb{Z}^{\oplus 2}$. Using the short exact sequence of $\mathcal{O}_{\mathbb{P}^1}$-modules

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-(n+1)) \to \mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n) \to \mathcal{O}_{\mathbb{P}^1}(-(n-1)) \to 0,$$

we obtain the recurrence relation $a_{n+1} = 2a_n - a_{n-1}$. By induction, $a_n = na_1 - (n-1)a_0$.

By Proposition 6.3, for $0 \le \lambda < p-1$,

$$\kappa_1(P_\lambda) \ = [\mathcal{O}_{\mathbb{P}^1}(-\lambda)] + [\mathcal{O}_{\mathbb{P}^1}(\lambda - 2(p-1)].$$

Hence, $\kappa_1(P_\lambda) = a_\lambda + a_{2(p-1)-\lambda} = \lambda a_1 - (\lambda-1)a_0 + (2(p-1)-\lambda)a_1 - (2(p-1)-\lambda-1)a_0 = \ 2(p-1)a_1 - 2(p-2)a_0$. Moreover,

$$\kappa_1(\mathrm{St}) = \ [\mathcal{O}_{\mathbb{P}^1}(1-p)] \ = \ (p-1)a_1 - (p-2)a_0$$

by Proposition 5.9. Hence,

$$\kappa_1(P_\lambda) = 2\kappa_1(\mathrm{St})$$

for $0 \le \lambda \le p-2$, which proves the statement. $\square$

The second conclusion of the following proposition is in sharp contrast with Proposition 6.5.

Proposition 6.7. *Let $G = \underline{s}l_2$, and denote by* St *the Steinberg module for sl_2. Then:*

(1) *The image of $\kappa_j : K_0(kG) \to K_0(\mathbb{P}(\mathrm{G}))$ is generated by $\kappa_j(\mathrm{St})$.*

(2) *The image of*

$$\underline{\kappa} : K_0(kG) \ \to \ K_0(\mathbb{P}(\mathrm{G}))^{\oplus p}$$

is generated by $\underline{\kappa}(\mathrm{St})$ *and, consequently, has rank* 1.

Proof. As in the proof of Lemma 6.6, we identify $K_0(\mathbb{P}(\mathrm{G}))$ with $K_0(\mathbb{P}^1)$. We have a short exact sequence of bundles:

$$(6.7.1) \qquad 0 \longrightarrow \mathrm{Ker}\{\widetilde{\Theta}, P_\lambda \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}\} \longrightarrow \mathrm{Ker}\{\widetilde{\Theta}^j, P_\lambda \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}\} \xrightarrow{\widetilde{\Theta}} \mathrm{Ker}\{\widetilde{\Theta}^{j-1}, P_\lambda \otimes \mathcal{O}_{\mathbb{P}(\mathrm{G})}\}(2) \longrightarrow 0.$$

Indeed, the composition is clearly zero, and the first map is an embedding. Moreover, by Theorem 4.13, the specialization of this sequence at any point $x \in \mathbb{P}(\mathrm{G})$ looks as follows:

$$0 \longrightarrow \mathrm{Ker}\{\theta_x, P_\lambda \otimes k(x)\} \longrightarrow \mathrm{Ker}\{\theta_x^j, P_\lambda \otimes k(x)\} \xrightarrow{\theta_x} \mathrm{Ker}\{\theta_x^{j-1}, P_\lambda \otimes k(x)\} \longrightarrow 0.$$

The projectivity of P_λ implies that this sequence is exact. Hence, (6.7.1) is exact by Nakayama's Lemma.

We conclude that in $K_0(\mathbb{P}^1)$,

$$(6.7.2) \qquad \kappa_j(P_\lambda) = \kappa_1(P_\lambda) + \kappa_{j-1}(P_\lambda)(2).$$

By Lemma 6.6, $\kappa_1(P_\lambda) = 2\kappa_1(\mathrm{St})$ for $0 \le \lambda \le p-2$. Applying induction, we get $\kappa_j(P_\lambda) = 2\kappa_1(\mathrm{St}) + 2\kappa_{j-1}(\mathrm{St})(2)$. Now applying formula (6.7.2) to the Steinberg module, we get $2\kappa_1(\mathrm{St}) + 2\kappa_{j-1}(\mathrm{St})(2) = 2\kappa_j(\mathrm{St})$. To summarize,

$$\kappa_j(P_\lambda) = \begin{cases} \kappa_j(\mathrm{St}), & \lambda = p-1, \\ 2\kappa_j(\mathrm{St}), & 0 \le \lambda \le p-2, \end{cases}$$

which implies (1). Moreover, we conclude that $\underline{\kappa} : K_0(kG) \to K_0(\mathbb{P}^1)^{\oplus p}$ is given by the formula

$$\underline{\kappa}(P_\lambda) = \begin{cases} (\kappa_1(\mathrm{St}), \kappa_2(\mathrm{St}), \dots, \kappa_p(\mathrm{St})), & \lambda = p-1, \\ (2\kappa_1(\mathrm{St}), 2\kappa_1(\mathrm{St}), \dots, 2\kappa_p(\mathrm{St})) = 2\underline{\kappa}(\mathrm{St}), & 0 \le \lambda \le p-2, \end{cases}$$

which proves (2). □

Proposition 6.9 gives us some information about the behavior of κ and $\kappa^{\oplus}$ with respect to products. First, we need a trivial linear algebra lemma.

Lemma 6.8. *Let V, W be vector spaces over a field k. Let $\{v_1, \dots, v_r\}$ be a basis of V and $\{w_1, \dots, w_s\}$ be a basis of W. Let $v \in V$, $w \in W$, and let $X \subset V \oplus W$ be the span of $\{(v_1, w), \dots, (v_r, w), (v, w_1), \dots, (v, w_s)\}$. Then the dimension of X is at least $r + s - 1$.*

Proof. If $v = 0$ and $w = 0$, then we obviously have $\dim X = r + s$. We assume $v \neq 0$. Observe that

$$(6.8.1) \qquad X \twoheadrightarrow (V \oplus W)/\langle (v,0) + (0,w) \rangle$$

is surjective (thus has image of dimension $r + s - 2$). Let $v = a_1 v_1 + \cdots + a_r v_r$. Then the vector $a_1(v_1, w) + \cdots + a_r(v_r, w) = (v, w)$ is non-trivial since $v \neq 0$ and belongs to the kernel of the projection (6.8.1). Hence, $\dim X \ge r + s - 1$. □

Proposition 6.9. *Let G_1 and G_2 be infinitesimal group schemes, and let $G = G_1 \times G_2$. Then*

$$\operatorname{rk} \kappa_G^{\oplus} \geq \operatorname{rk} \kappa_{G_1}^{\oplus} + \operatorname{rk} \kappa_{G_2}^{\oplus} - 1,$$

where $\kappa^{\oplus} = \kappa_j^{\oplus}$ for any j, $1 \leq j \leq p-1$.

Proof. Let $i_\ell^* : K_0^{\oplus}(\mathbb{P}(\mathrm{G})) \to K_0^{\oplus}(\mathbb{P}(G_\ell))$ be the map induced by the pull-back of vector bundles along the embedding $\mathbb{P}(G_\ell) \hookrightarrow \mathbb{P}(\mathrm{G})$, for $\ell = 1, 2$. We consider the composition

$$K_0(kG) \xrightarrow{\kappa^{\oplus}} K_0^{\oplus}(\mathbb{P}(\mathrm{G})) \xrightarrow{i_1^* + i_2^*} K_0^{\oplus}(\mathbb{P}(G_1)) \oplus K_0^{\oplus}(\mathbb{P}(G_2)).$$

Let P be a projective kG_1-module of dimension m and Q be a projective kG_2-module of dimension n. Then $P \otimes Q$ is a projective kG-module. By Corollary 5.4,

$$\begin{aligned}(i_1^* + i_2^*) \circ \kappa_G^{\oplus}(P \otimes Q) &= (i_1^* \circ \kappa_G^{\oplus}(P \otimes Q), i_2^* \circ \kappa_G^{\oplus}(P \otimes Q)) \\ &= (n\, \kappa_{G_1}^{\oplus}(P), m\, \kappa_{G_2}^{\oplus}(Q)).\end{aligned} \tag{6.9.1}$$

Let $\{P_1, \ldots, P_r\}$ be projective kG_1-modules of dimensions $\{p_1, \ldots, p_r\}$ such that

$$\{\kappa_{G_1}^{\oplus}(P_1), \ldots, \kappa_{G_1}^{\oplus}(P_r)\}$$

are linearly independent generators of $\operatorname{Im} \kappa_{G_1}^{\oplus} \subset K_0^{\oplus}(\mathbb{P}(G_1))$, so that $\operatorname{rk} \kappa_{G_1}^{\oplus} = r$. Similarly, let $\{Q_1, \ldots, Q_s\}$ be projective kG_2-modules such that

$$\{\kappa_{G_2}^{\oplus}(Q_1), \ldots, \kappa_{G_2}^{\oplus}(Q_s)\}$$

are linearly independent generators of $\operatorname{Im} \kappa_{G_2}^{\oplus} \subset K_0^{\oplus}(\mathbb{P}(G_2))$, so that $\operatorname{rk} \kappa_{G_2}^{\oplus} = s$. Finally, let P be any projective kG_1-module, $m = \dim_k P$, and Q be any projective kG_2-module, $n = \dim_k Q$. Consider

$$S = \operatorname{Span}\{P_1 \otimes Q, P_2 \otimes Q, \ldots, P_r \otimes Q, P \otimes Q_1, P \otimes Q_2, \ldots, P \otimes Q_s\} \subset K_0(kG).$$

Then the image of $(i_1^* + i_2^*) \circ \kappa_G$ contains

$$\{(n\, \kappa_{G_1}^{\oplus}(P_1), p_1 \kappa_{G_2}^{\oplus}(Q)), (n\, \kappa_{G_1}^{\oplus}(P_2), p_2 \kappa_{G_2}^{\oplus}(Q)), \ldots, (n\, \kappa_{G_1}^{\oplus}(P_r), p_r \kappa_{G_2}^{\oplus}(Q))\}$$

and

$$\{(q_1 \kappa_{G_1}^{\oplus}(P), m\, \kappa_{G_2}^{\oplus}(Q_1)), (q_2 \kappa_{G_1}^{\oplus}(P), m\, \kappa_{G_2}^{\oplus}(Q_2)), \ldots, (q_s \kappa_{G_1}^{\oplus}(P), m\, \kappa_{G_2}^{\oplus}(Q_s))\}.$$

Since all the coefficients m, n, p_i, q_j are positive, Lemma 6.8 implies that the dimension of the image of $(i_1^* + i_2^*) \circ \kappa_G^{\oplus}$ is at least $r + s - 1$. □

Remark 6.10. As the reader can easily check, Proposition 6.9 and its proof hold for $\kappa : K_0(kG) \to K_0(\mathbb{P}(G))$ in place of $\kappa^{\oplus} : K_0(kG) \to K_0^{\oplus}(\mathbb{P}(G))$.

Observe that $\mathbb{P}(\underline{sl_2^{\oplus r}}) \simeq \mathbb{P}^{2r-1}$, the join of r copies of $\mathbb{P}(\underline{s}l_2) \simeq \mathbb{P}^1$.

Corollary 6.11. *Let $G = \underline{sl}_2^{\times r}$ be the infinitesimal group scheme corresponding to the restricted Lie algebra $sl_2^{\oplus r}$. Then*

$$\kappa_1^{\oplus} : K_0(kG) \to K_0^{\oplus}(\mathbb{P}^{2r-1})$$

has rank at least $r(p-1) + 1$.

Proof. For $r = 1$, $\kappa_{\underline{sl}_2}^{\oplus} : K_0(u(sl_2)) \to K_0^{\oplus}(\mathbb{P}^1)$ is injective by Proposition 6.3 and, hence, has rank p. The statement now follows by induction and Proposition 6.9. □

Recall that if $H \hookrightarrow G$ is a subgroup scheme of a finite group scheme G, then kG is free as a kH–module [24, 2.4]. Hence, the restriction functor $\mathrm{res} : (kG-\mathrm{mod}) \to (kH-\mathrm{mod})$ induces a well-defined map on K-groups: $\mathrm{res}^* : K_0(kG) \to K_0(kH)$. The commutativity of the diagram (5.3.1) implies that restriction commutes with κ_j. That is, we have a commutative diagram

(6.11.1)
$$\begin{tikzcd} K_0(kG) \arrow[r, "\kappa_{G,j}"] \arrow[d, "\mathrm{res}^*"] & K_0(\mathbb{P}(G)) \arrow[d, "i^*"] \\ K_0(kH) \arrow[r, "\kappa_{H,j}"] & K_0(\mathbb{P}(H)), \end{tikzcd}$$

where $i : \mathbb{P}(H) \hookrightarrow \mathbb{P}(\mathrm{G})$ is the closed embedding of projective varieties induced by the embedding of group schemes and $i^* : K_0(\mathbb{P}(G)) \to K_0(\mathbb{P}(H))$ is the pull-back via i of locally free $\mathcal{O}_{P(G)}$-modules. An analogous commutative diagram holds for $\kappa^{\oplus}$.

Proposition 6.12. *Let G be an infinitesimal group scheme, and let $H \simeq \underline{sl}_2^{\times r} \subset G$ be a closed subgroup scheme with the property that $\mathrm{res}^* : K_0(kG) \to K_0(kH)$ is rationally surjective. Then the composition*

$$K_0(kG) \xrightarrow{\kappa_1^{\oplus}} K_0^{\oplus}(\mathbb{P}(\mathrm{G})) \xrightarrow{i^*} K_0^{\oplus}(\mathbb{P}(H)) \simeq K_0^{\oplus}(\mathbb{P}^{2r-1})$$

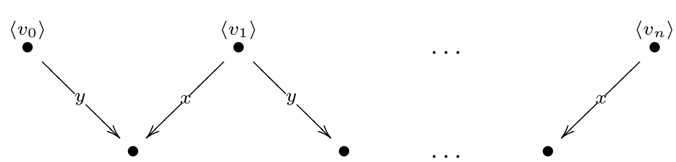
has rank at least $r(p-1)+1$.

Proof. We apply the diagram (6.11.1) for $\kappa_1^{\oplus}$:

$$\begin{tikzcd} K_0(kG) \arrow[r, "\kappa_{G,1}^{\oplus}"] \arrow[d, "\mathrm{res}^*"] & K_0^{\oplus}(\mathbb{P}(G)) \arrow[d, "i^*"] \\ K_0(kH) \arrow[r, "\kappa_{H,1}^{\oplus}"] & K_0^{\oplus}(\mathbb{P}(H)). \end{tikzcd}$$

Since res^* is assumed to be rationally surjective, $\mathrm{rk}\left(i^* \circ \kappa_{G,1}^{\oplus}\right) = \mathrm{rk}\, \kappa_{H,1}^{\oplus}$. Since $H = \underline{sl}_2^{\times r}$, $\mathrm{rk}\, \kappa_{H,1}^{\oplus} \geq r(p-1)+1$ by Corollary 6.11. This proves the statement. $\square$

For the rest of this computational section, we calculate some examples of bundles for $E = \mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$. We have $kE \simeq k[x,y]/(x^p, y^p)$. Let X_n be a $(2n+1)$-dimensional "zig-zag" module. Pictorially, we represent X_n by the following diagram:



It is straightforward to check that X_n has constant Jordan type $n[2]+[1]$ (see [8, §2]). We proceed to prove that for any integer m we can obtain the line bundle $\mathcal{O}_{\mathbb{P}^1}(m)$ on $\mathbb{P}(E) = \mathbb{P}^1$ by applying our constructions to some X_n or its linear dual $X_n^{\#}$.

Note that for X_n, the map

$$\widetilde{\Theta}_E : X_n \otimes \mathcal{O}_{\mathbb{P}^1} \longrightarrow X_n \otimes \mathcal{O}_{\mathbb{P}^1}(1)$$

as defined in (4.6.1) has nilpotentcy degree 2. Hence, there is an inclusion

$$\operatorname{Im}\{\widetilde{\Theta}_E, X_n \otimes \mathcal{O}_{\mathbb{P}^1}\} \subset \operatorname{Ker}\{\widetilde{\Theta}_E, X_n \otimes \mathcal{O}_{\mathbb{P}^1}(1)\}.$$

We, therefore, may define a subquotient sheaf of the free sheaf $X_n \otimes \mathcal{O}_{\mathbb{P}^1}$ as

$$\mathcal{X}_n := \operatorname{Ker}\{\widetilde{\Theta}_E, X_n \otimes \mathcal{O}_{\mathbb{P}^1}\}/\operatorname{Im}\{\widetilde{\Theta}_E(-1), X_n \otimes \mathcal{O}_{\mathbb{P}^1}(-1)\}.$$

Arguing as in the proof of Proposition 5.16, one verifies that $\mathcal{X}_n$ is locally free with the fiber at a point $t \in \mathbb{P}^1$ isomorphic to the 1-dimensional vector space $\frac{\operatorname{Ker}\{\theta_t: X_{n,k(t)} \to X_{n,k(t)}\}}{\operatorname{Im}\{\theta_t: X_{n,k(t)} \to X_{n,k(t)}\}}$. Hence, $\mathcal{X}_n$ is a line bundle. The linear dual $X_n^\#$ of X_n is represented by the diagram

$$\begin{tikzcd}
& \bullet \arrow[dl, "x"'] \arrow[dr, "y"] & & \bullet \arrow[dl, "x"'] & \cdots & \bullet \arrow[dr, "y"] & \\
\langle w_0 \rangle & & \langle w_1 \rangle & & \cdots & & \langle w_n \rangle.
\end{tikzcd}$$

Define the subquotient sheaf of $X_n^\# \otimes \mathcal{O}_{\mathbb{P}^1}$ as

$$\mathcal{Y}_n := \operatorname{Ker}\{\widetilde{\Theta}_E, X_n^\# \otimes \mathcal{O}_{\mathbb{P}^1}\}/\operatorname{Im}\{\widetilde{\Theta}_E(-1), X_n^\# \otimes \mathcal{O}_{\mathbb{P}^1}(-1)\}.$$

Proposition 6.13. *$\mathcal{X}_n \simeq \mathcal{O}_{\mathbb{P}^1}(-n)$, $\mathcal{Y}_n \simeq \mathcal{O}_{\mathbb{P}^1}(n)$.*

Proof. Let $k[s,t] = k[\mathbb{A}^2] \simeq k[V(E)]$. The universal p-nilpotent operator $\Theta_E \in k[x,y]/(x^p, y^p) \otimes k[s,t]$ is given by

$$\Theta_E = xs + yt$$

(see, for example, Example 2.7). We shall identify the graded $k[s,t]$-module $\operatorname{Ker}\{\Theta_E, X_n \otimes k[s,t]\}/\operatorname{Im}\{\Theta_E, X_n \otimes k[s,t]\}$, thereby determining the vector bundle $\mathcal{X}_n$. It is easy to see that $\operatorname{Im}\{\Theta_E, X_n \otimes k[s,t]\}$ is generated by the bottom row of the diagram representing X_n as a $k[s,t]$-module and that $\operatorname{Ker}\{\Theta_E, X_n \otimes k[s,t]\}$ is generated by the same bottom row and the vector $s^n v_0 + s^{n-1} t v_1 + \cdots + t^n v_n$. Hence, $\operatorname{Ker}\{\Theta_E, X_n \otimes k[s,t]\}/\operatorname{Im}\{\Theta_E, X_n \otimes k[s,t]\}$ is generated by $s^n v_0 + s^{n-1} t v_1 + \cdots + t^n v_n$ as a $k[s,t]$-module. Since the generator is in degree n, we conclude that the corresponding locally free sheaf of rank 1 is $\mathcal{O}_{\mathbb{P}^1}(-n)$.

We now compute $\mathcal{Y}_n$. The graded $k[s,t]$-module $\operatorname{Ker}\{\Theta_E, X_n^\# \otimes k[s,t]\}$ is generated by $\langle w_0, \dots, w_n \rangle$ in degree 0, and $\operatorname{Im}\{\Theta_E, X_n^\# \otimes k[s,t](-1)\}$ is generated by $\langle s w_0 + t w_1, s w_1 + t w_2, \dots, s w_{n-1} + t w_n \rangle$, also in degree 0. Hence, on $U_0 = \mathbb{P}^1 - Z(s)$, the restriction of $\mathcal{Y}_n$ is generated by w_n, with $w_0 = (-\frac{t}{s})^n w_n$. We map $\mathcal{Y}_n(U_0)$ to $K = k(t/s)$, the residue field at the generic point of $\mathbb{P}^1$, by sending w_n to 1. The image of w_0 is $(-\frac{t}{s})^n$. On the other affine piece, $U_1 = \mathbb{P}^1 - Z(t)$, the restriction of $\mathcal{Y}_n$ is generated by w_0, with the relation $w_n = (-\frac{s}{t})^n w_0$. We map this to $K = k(s/t)$ by sending w_0 to $(-\frac{t}{s})^n$. Hence, the vector bundle is given by the Cartier divisor $(U_0, 1), (U_1, (-\frac{t}{s})^n)$. This divisor is equivalent to the Cartier divisor $(U_0, 1), (U_1, (\frac{t}{s})^n)$ which corresponds to the line bundle $\mathcal{O}_{\mathbb{P}^1}(n)$. Hence, $\mathcal{Y}_n \simeq \mathcal{O}_{\mathbb{P}^1}(n)$. $\square$

In the next proposition we calculate explicitly the line bundles corresponding to the syzygies of the trivial modules, $\Omega^n k$. For convenience, we use the notation $\mathcal{H}^{[1]}(M)$ for the bundle $\mathcal{M}^{[1]}$ associated to M as defined in (5.14).

Proposition 6.14. *Let $E = \mathbb{G}_{a(1)}^{\times r}$. Then*

$$\mathcal{H}^{[1]}(\Omega^n k) \simeq \begin{cases} \mathcal{O}_{\mathbb{P}^{r-1}}(-\frac{np}{2}) & \text{if } n \text{ is even,} \\ \mathcal{O}_{\mathbb{P}^{r-1}}(-\frac{n+1}{2}p+1) & \text{if } n \text{ is odd} \end{cases}$$

for p odd and

$$\mathcal{H}^{[1]}(\Omega^n k) \simeq \mathcal{O}_{\mathbb{P}^{r-1}}(-n)$$

for $p = 2$.

Proof. Let $r = 2$, and assume $n \geq 0$. As in the proof of Proposition 6.13, the universal operator Θ_E equals $sx + ty$, where $k[V(E)] = k[s,t]$. The structure of a minimal $kE \simeq k[x,y]/(x^p, y^p)$-projective resolution $P_\bullet \to k$ is well known [10], with $P_{n-1} = kE^{\times n}$. A set of generators $a_1, \ldots, a_n$ for P_{n-1} can be chosen so that $\Omega^n(k)$ is the submodule generated by the elements

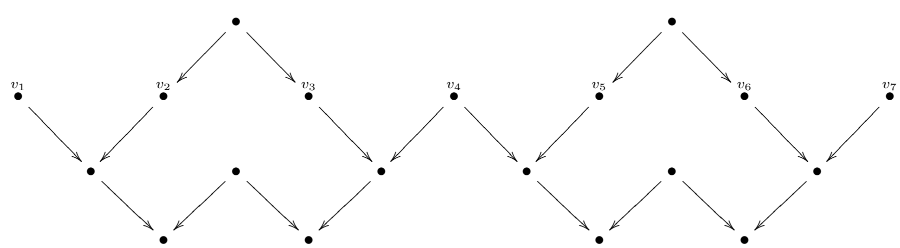
$$x^{p-1}a_1, \;\; ya_1 - xa_2, \;\; y^{p-1}a_2 - x^{p-1}a_3, \;\; ya_3 - xa_4, \;\; \ldots, \;\; ya_{n-1} - xa_n, \;\; y^{p-1}a_n$$

for n even, and

$$xa_1, \;\; ya_1 - x^{p-1}a_2, \;\; y^{p-1}a_2 - xa_3, \;\; ya_3 - x^{p-1}a_4, \;\; \ldots, \;\; y^{p-1}a_{n-1} - xa_n, \;\; ya_n$$

for n odd.

Let n be even. For illustrational purposes, we include a picture of $\Omega^4 k$ for $p = 3$



The kernel of $\Theta_E = sx+ty$ on $\Omega^n k \otimes k[s,t]$ is a submodule of a free $k[s,t]$-module generated by $v_1 = x^{p-1}a_1, v_2 = x^{p-2}(ya_1 - xa_2), v_3 = x^{p-3}y(ya_1 - xa_2), \ldots, v_p = y^{p-2}(ya_1 - xa_2), v_{p+1} = y^{p-1}a_2 - x^{p-1}a_3, \ldots, v_{\frac{np}{2}+1} = y^{p-1}a_n$. Everything below this layer which is in $\operatorname{Ker}\Theta_E$ also lies in $\operatorname{Im}\Theta_E^{p-1}$. One verifies that the quotient

$$\operatorname{Ker}\{\Theta_E : \Omega^n k \otimes k[s,t] \to \Omega^n k \otimes k[s,t]\}/\operatorname{Im}\{\Theta_E^{p-1} : \Omega^n k \otimes k[s,t] \to \Omega^n k \otimes k[s,t]\}$$

is generated by

$$s^{\frac{np}{2}} v_1 - s^{\frac{np}{2}-1} t v_2 + \cdots \pm t^{\frac{np}{2}} v_{\frac{np}{2}+1}.$$

Arguing as in the proof of Proposition 6.13, we conclude that the corresponding locally free sheaf of rank 1 is $\mathcal{O}_{P^1}(-\frac{np}{2})$. The calculation for odd positive n is similar. For negative values of n, one verifies the formula by again doing a similar calculation with dual modules.

Now let $r > 2$, let $i : F \subset E$ be a subgroup scheme isomorphic to $\mathbb{G}_{a(1)}^{\times 2}$, and let $f : \mathbb{P}(F) \to \mathbb{P}(E)$ be the map induced by the embedding i. Since $(\Omega_E^n k)\downarrow_F \simeq \Omega_F^n k \oplus \text{proj}$, we conclude that $\mathcal{H}^{[1]}((\Omega_E^n k)\downarrow_F) \simeq \mathcal{H}^{[1]}(\Omega_F^n k)$. Proposition 5.3 implies an isomorphism $f^*(\mathcal{H}^{[1]}(\Omega_E^n k)) \simeq \mathcal{H}^{[1]}((\Omega_E^n k)\downarrow_F)$. Hence,

$$f^*(\mathcal{H}^{[1]}(\Omega_E^n k)) \simeq \mathcal{H}^{[1]}(\Omega_F^n k).$$

The proposition now follows from the observation that

$$f : \mathbb{P}(F) \simeq \mathbb{P}^1 \longrightarrow \mathbb{P}(E) \simeq \mathbb{P}^{r-1}$$

induces an isomorphism on Picard groups via f^*. □

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